Motion Interpolation in Lie Subgroups and Symmetric Subspaces

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Abstract. We show that a map defined by Pfurner, Schröcker and Husty, mapping points in 7-dimensional projective space to the Study quadric, is equivalent to the composition of an extended inverse Cayley map with the direct Cayley map, where the Cayley map in question is associated to the adjoint representation of the group $SE(3)$. We also verify that subgroups and symmetric subspaces of $SE(3)$ lie on linear spaces in dual quaternion representation of the group. These two ideas are combined with the observation that the Pfurner-Schröcker-Husty map preserves these linear subspaces. This means that the interpolation method proposed by Pfurner *et al* can be restricted to subgroups and symmetric subspaces of $SE(3)$.

Key words: Motion interpolation, Lie triple systems, symmetric subspaces, Cayley map.

1 Introduction

There is a very large number of methods and procedures for interpolating rigid-body motion as this is an important problem not only in robotics but also in computer graphics. See the review paper by Röschel [6].

Recently [5], a new simple method was presented for interpolating motions based on the embedding of the group of rigid-body displacements $SE(3)$ in the seven dimensional projective space $\mathbb{P}^7$ as a non-singular quadric, known as the Study quadric Q_S. This method is reminiscent of a method proposed by Belta and Kumar where interpolation was carried out on matrices and then the matrices mapped back to the group of rigid-body displacements, see [9] and references therein.

Here we show that this new interpolation method is equivalent to using the Cayley map associated to the adjoint representation of the group. That is, the rational map, used by Pfurner, Schröcker and Husty to map points in $\mathbb{P}^7$ to Q_S is equivalent to performing the composition of the inverse Cayley map, extended to all of $\mathbb{P}^7$, followed by the Cayley map sending points in $\mathbb{P}^6$ to the Study quadric.

In another recent work [13], a method for interpolating motions in symmetric subspaces of $SE(3)$ was given. Symmetric subspaces of $SE(3)$ are important in many practical applications. In [14], it was observed that these symmetric subspaces lie in the intersection of the Study quadric with some linear space. Below we make this statement precise. We also verify that the map defined in [5] respects these linear subspaces; a point in $\mathbb{P}^7$, not on the Study quadric but lying in a linear subspace which defines a subgroup or symmetric subspace, will be mapped to a point in the intersection of the subspace and Q_S. So the interpolation method of Pfurner, Schröcker and Husty (PSH method) is ideally suited as an interpolation method on the symmetric subspaces. Finally, we give a couple of simple examples of the method.

2 Cayley Maps

Given a matrix representation of $se(3)$, the Lie algebra to the group of rigid-body displacements, we can map the Lie algebra to the group itself using the map,

$$\mathrm{Cay}(A) = (I+A)(I-A)^{-1}, \tag{1}$$

where A is the matrix representing an element in $se(3)$ and I is the identity matrix. The result is a group element in $SE(3)$ represented as a matrix of the same dimension. These maps, unlike the exponential map, depend on the particular representation used. Here, the adjoint representation of $SE(3)$ is used and the corresponding Cayley map will be written Cay_6.

Now a general dual quaternion is given by,

$$g = (a_0 + a_1 i + a_2 j + a_3 k) + \varepsilon(c_0 + c_1 i + c_2 j + c_3 k) \tag{2}$$

where i, j and k are the unit quaternion generators and ε is the dual unit which commutes with the quaternions and squares to zero, $\varepsilon^2 = 0$.

A rigid-body displacement is given by a dual quaternion with elements satisfying the equation,

$$a_0c_0 + a_1c_1 + a_2c_2 + a_3c_3 = 0. \tag{3}$$

Taking $(a_0 : a_1 : a_2 : a_3 : c_0 : c_1 : c_2 : c_3)$ as homogeneous coordinates in a $\mathbb{P}^7$, the above quadratic equation determines the Study quadric Q_S.

In [10] the map Cay_6 and its inverse were described in terms of dual quaternions. An element of $se(3)$ can be written as a pure dual quaternion; $s = (w_1 i + w_2 j + w_3 k) + \varepsilon(v_1 i + v_2 j + v_3 k)$. In the algebra of dual quaternions the Cayley map based on the adjoint representation can be written as,

$$\mathrm{Cay}_6(s) = \frac{1}{2(w_0^2+\mu^2)^{3/2}}\big((2w_0^2+3\mu^2)w_0 + (2w_0^2+3\mu^2)s + w_0 s^2 + s^3\big), \tag{4}$$

where $\mu^2 = w_1^2 + w_2^2 + w_3^2$. The variable w_0 has been included to make the equation homogeneous. In this way the map can be viewed as a map from the projective space $\mathbb{P}^6$, with homogeneous coordinates, $(w_0 : w_1 : w_2 : w_3 : v_1 : v_2 : v_3)$ to Q_s. In order to give a compact but explicit formula for this map we introduce the following notation. Let,

$$\mathbf{a} = (a_0, a_1, a_2, a_3)^T, \quad \mathbf{c} = (c_0, c_1, c_2, c_3)^T,$$

and also

$$\mathbf{w} = (w_0, w_1, w_2, w_3)^T, \quad \mathbf{v} = (0, v_1, v_2, v_3)^T.$$

Now, the powers of s can be expanded in equation (4) and in terms of the above notation the Cayley map can be written,

$$\mathbf{a} = \mathbf{w}(\mathbf{w}\cdot\mathbf{w}), \qquad \mathbf{c} = \mathbf{v}(\mathbf{w}\cdot\mathbf{w}) - \mathbf{w}(\mathbf{v}\cdot\mathbf{w}). \tag{5}$$

Note that, since the codomain of the map lies in a projective space, any common factors can be ignored. It is simple to check, using equation (3), that the image of the transformation is indeed the Study quadric. The map clearly has degree 3 in the homogeneous coordinates of $\mathbb{P}^6$. The exceptional set for the map consists of the 2-plane $w_0 = w_1 = w_2 = w_3 = 0$ and the 4-dimensional intersection of the 2 quadrics $w_0^2 + w_1^2 + w_2^2 + w_3^2 = 0$ and $w_1v_1 + w_2v_2 + w_3v_3 = 0$.

The inverses of these maps were also found in [10]. Given a group element g satisfying (3), the inverse map is,

$$\mathrm{Cay}_6^{-1}(g) = \frac{-1}{2a_0^2}\big(g^2 - 4a_0 g + (3a_0^2 + a_1^2 + a_2^2 + a_3^2)\big). \tag{6}$$

If we assign $w_0 = -2a_0^2$, the common denominator, then the other coordinates of $s = \mathrm{Cay}_6^{-1}(g)$ are given by expanding the polynomial in g and simplifying using the Study quadric, (3) and cancelling common factors:

$$\mathbf{w} = a_0\mathbf{a}, \qquad \mathbf{v} = a_0\mathbf{c} - c_0\mathbf{a}. \tag{7}$$

This is a quadratic transformation with exceptional set consisting of the 5-plane $a_0 = c_0 = 0$ and the 3-plane $a_0 = a_1 = a_2 = a_3 = 0$. The 3-plane is the A-plane of ideal elements in the Study quadric, the points which do not correspond to any rigid-body transformation. The intersection of the 5-plane with the Study quadric is the set of half-turns, that is rotations by π radians about some axis. This pair of maps can be viewed as a birational transformation between the six-dimensional projective space $\mathbb{P}^6$ and the Study quadric Q_S in $\mathbb{P}^7$.

Notice that the definition of the inverse of Cay_6, can be extended to all points of $\mathbb{P}^7$, using the same definition as above. The extended map will be denoted $\widetilde{\mathrm{Cay}}_6^{-1}$.

In [5] Pfurner *et al* introduced a simple method for interpolating rigid-body motions. The algorithm consists of writing the control points of the motion as dual quaternions and then performing the interpolation in the ambient $\mathbb{P}^7$. Finally, the motion is found by projecting the curve into the group. The map given in [5] takes an arbitrary point of $\mathbb{P}^7$ to Q_S in $\mathbb{P}^7$, so let us write the coordinates in the first $\mathbb{P}^7$ as $\bar{\mathbf{a}}$ and $\bar{\mathbf{c}}$, so the map can be written as,

$$\mathbf{a} = \bar{\mathbf{a}}(\bar{\mathbf{a}}\cdot\bar{\mathbf{a}}), \qquad \mathbf{c} = \bar{\mathbf{c}}(\bar{\mathbf{a}}\cdot\bar{\mathbf{a}}) - \bar{\mathbf{a}}(\bar{\mathbf{a}}\cdot\bar{\mathbf{c}}). \tag{8}$$

It is straightforward to check that the image of this map satisfies the equation defining the Study quadric and hence the image of the map is indeed Q_S. Notice that this map is the analogue in $\mathbb{P}^7$ of the map in $\mathbb{P}^5$ which maps a screw to its axis: a line in the Klein quadric.

Theorem 1. *The Pfurner-Schröcker-Husty map (PSH map), given in (8), is equivalent to the composite map* $\mathrm{Cay}_6 \circ \widetilde{\mathrm{Cay}}_6^{-1}$.

Proof. The proof is by direct computation. First the effect of the extended inverse Cayley map will be,

$$\mathbf{w} = \bar{a}_0\bar{\mathbf{a}}, \qquad \mathbf{v} = \bar{a}_0\bar{\mathbf{c}} - \bar{c}_0\bar{\mathbf{a}} \tag{9}$$

Now we can easily substitute into the definition for the Cayley map to get,

$$\mathbf{a} = \mathbf{w}(\mathbf{w}\cdot\mathbf{w}) = \bar{a}_0^3\bar{\mathbf{a}}(\bar{\mathbf{a}}\cdot\bar{\mathbf{a}}) \tag{10}$$

and

$$\mathbf{c} = \mathbf{v}(\mathbf{w}\cdot\mathbf{w}) - \mathbf{w}(\mathbf{v}\cdot\mathbf{w}) = \bar{a}_0^3\bar{\mathbf{c}}(\bar{\mathbf{a}}\cdot\bar{\mathbf{a}}) - \bar{a}_0^3\bar{\mathbf{a}}(\bar{\mathbf{a}}\cdot\bar{\mathbf{c}}). \tag{11}$$

Clearly, apart from the common factor $\bar{a}_0^3$, which is irrelevant in a projective space, this gives the same result as the PSH map. $\square$

3 Subgroups and Symmetric Subspaces

Loos [3], defines symmetric spaces as spaces with a multiplication defined on the points of the space. The map defined by left-multiplication by a particular point x, is an involutive automorphism of the space with isolated fixed point x. Loos also shows that any Lie group, with Lie multiplication xy, becomes a symmetric space when the multiplication is modified to $\sigma(x,y) = xy^{-1}x$. Here, by a symmetric subspace of $SE(3)$, we mean a proper subspace of $SE(3)$ closed under σ. There is a correspondence between Lie triple system (LTS) of the Lie algebra and symmetric subspaces of the group.

In $SE(3)$ linear subspaces of the Lie algebra are known as screw systems. Screw systems were classified up to rigid-body transformations by Gibson and Hunt [2], see also [1]. The Gibson-Hunt type (GH type) of a screw system distinguishes between type II systems, which contain screws with the same pitch, and type I systems which contain screws with different pitches. The number of infinite pitch screw in the system is given by a letter, A for no infinite pitch screws, B for 1 infinite pitch screw, C for a line of infinite pith screws, and so forth. These basic classes split into finer classes, characterised by invariants, often a characteristic pitch or set of pitches.

The Lie triple systems of $se(3)$ were classified in [4, 7, 12], details of symmetric subspaces of $SE(3)$ can also be found in [14]. It was observed in [4], that most of the symmetric subspaces of $SE(3)$ are linear spaces or the intersection of the Study quadric $Q_s$with a linear subspace. We state this as a theorem here.

Theorem 2. *Algebraic subgroups of $SE(3)$ and algebraic symmetric subspaces of $SE(3)$ lie on linear spaces contained in the Study quadric or on the intersection of the Study quadric with a linear subspace of $\mathbb{P}^7$.*

By an algebraic subgroup or symmetric subspace we mean a subspace that can be generated by exponentiating linear combinations of only zero pitch or infinite pitch twists.

Proof. The theorem can be proved by straightforward inspection of all possible cases. All possibilities were found in [12, 4] and [7]. To find points in the symmetric subspaces we need to be able to exponentiate elements of the Lie triple system. This can be done using the Rodrigues-like formula,

$$e^s = \frac{1}{2}\big(2\cos\theta + \theta\sin\theta\big) - \frac{1}{2\theta}\big(\theta\cos\theta - 3\sin\theta\big)s \\ + \frac{1}{2\theta}\big(\sin\theta\big)s^2 - \frac{1}{2\theta^3}\big(\theta\cos\theta - \sin\theta\big)s^3. \quad (12)$$

where s is a dual quaternion of the form, $s = (\theta_x i + \theta_y j + \theta_z k) + \varepsilon(u_x i + u_y j + u_z k)$ and $\theta^2 = \theta_x^2 + \theta_y^2 + \theta_z^2$. A derivation of this formula can be found in [10][1].

So for example, if we take a general twist from a IIB $(p = 0)$ 3-system, $s = ai + bj + c\varepsilon k$ the exponential of this is,

$$e^s = \cos\theta + \frac{a}{\theta}\sin\theta i + \frac{b}{\theta}\sin\theta j + \frac{c}{\theta}\sin\theta\varepsilon k \quad (13)$$

where $\theta^2 = a^2 + b^2$. Clearly, whatever the values of the parameters a, b and c, the exponential lies in the 3-plane $a_3 = c_0 = c_1 = c_2 = 0$. This 3-plane is a generator plane of the Study quadric.

In this way, all possible subalgebras and Lie triple systems can be examined. Tables of canonical forms for the possible subalgebras and Lie triple systems can be found in Table 1 and 2 respectively, together with the linear equations satisfied by the subspaces they generate. □

[1] Note, reference [10] contains a couple of errors. Equation (8.6) for the log of a dual quaternion should read,

$$\log(g) = \frac{1}{4\sin^3(\theta)}\Big(\big(2\theta - \sin(2\theta)\big)g^3 + \big(2\sin(3\theta) - 6\theta\cos(\theta)\big)g^2 \\ - \big(6\theta\cos(\theta) - 2\sin(3\theta)\big)g - \big(3\theta\cos(\theta) - \theta\cos(3\theta) + \sin(\theta) - \sin(3\theta)\big)\Big).$$

Thanks to J. Bookshire for pointing this out. The formula in §5 for the quasi-pitch of the dual quaternion Cayley map should read,

$$h_q = \frac{a\cdot b}{a\cdot a} = \frac{\theta/2}{\sin\theta/2}\left(\frac{p}{2\pi}\right)$$

Table 1 Canonical Forms for the Connected Subgroups of $SE(3)$. GH type denotes the class of the screw system in the Gibson-Hunt classification of screw systems.

Dim	GH type	Subgroup	Sub. Alg. basis	Linear equations	Description
1	IA $(p=0)$	$SO(2)$	$\{i\}$	$a_2=a_3=c_0=c_1=c_2=c_3=0$	line in Q_S
1	IA $(p\neq 0)$	H_p	$\{i+p\varepsilon i\}$	not algebraic	-
1	IIB	$\mathbb{R}$	$\{\varepsilon i\}$	$a_1=a_2=a_3=c_0=c_2=c_3=0$	line in Q_S
2	IB^0	$SO(2)\times\mathbb{R}$	$\{i,\varepsilon i\}$	$a_2=a_3=c_2=c_3=0$	3-plane
2	IIC	$\mathbb{R}^2$	$\{\varepsilon i,\varepsilon j\}$	$a_1=a_2=a_3=c_0=c_1=0$	2-plane in Q_S
3	IIA $(p=0)$	$SO(3)$	$\{i,j,k\}$	$c_0=c_1=c_2=c_3=0$	A-plane
3	IIC $(p=0)$	$SE(2)$	$\{i,\varepsilon j,\varepsilon k\}$	$a_2=a_3=c_0=c_1=0$	A-plane
3	IIC $(p\neq 0)$	$H_p\ltimes\mathbb{R}^2$	$\{i+p\varepsilon i,j,k\}$	not algebraic	-
3	IID	$\mathbb{R}^3$	$\{\varepsilon i,\varepsilon j,\varepsilon k\}$	$a_1=a_2=a_3=c_0=0$	B-plane
4	$\overline{\mathrm{IIC}}$	$SE(2)\times\mathbb{R}$	$\{i,\varepsilon i,\varepsilon j,\varepsilon k\}$	$a_2=a_3=0$	5-plane

Table 2 Canonical Forms for the Connected Symmetric Subspaces of $SE(3)$. LTS basis denotes a basis for the Lie triple system.

Dim	GH type	LTS basis	Linear equations	Description
2	IIA $(p=0)$	$\{i,j\}$	$a_3=c_0=c_1=c_2=c_3=0$	2-plane in Q_S
2	IIB $(p=0)$	$\{i,\varepsilon j\}$	$a_2=a_3=c_0=c_1=c_3=0$	2-plane in Q_S
2	IIB $(p\neq 0)$	$\{i+p\varepsilon i,\varepsilon j\}$	not algebraic	-
3	IIB $(p=0)$	$\{i,j,\varepsilon k\}$	$a_3=c_0=c_1=c_2=0$	B-plane
3	IC^0	$\{i,\varepsilon i,\varepsilon j\}$	$a_2=a_3=c_3=0$	4-plane
4	$\overline{\mathrm{IB}^0}$	$\{i,j,\varepsilon i,\varepsilon j\}$	$a_3=c_3=0$	5-plane
5	$\overline{\mathrm{IIB}}$	$\{i,j,\varepsilon i,\varepsilon j,\varepsilon k\}$	$a_3=0$	hyperplane

4 Interpolation

Finally the two parts can be combined. The idea is to interpolate the motion in the subgroup or symmetric subspace using the Study coordinates. The result may not lie in the Study quadric but will lie in a linear subspace defining the subgroup or symmetric subspace. Now use the PSH map to send the curve back to the Study quadric. For this to work we must check that the PSH map preserves the linear spaces. A point in the linear space, not on the Study quadric must be mapped to a point on Q_S but still in the linear space.

If the linear spaces lies entirely within the Q_s there is nothing to check since points in Q_S are not changed by the map. This leaves 5 cases to check, the cylindrical subgroup (IB^0), the Schönflies subgroup ($\overline{\mathrm{IIC}}$), and the last three rows in Table 2, (IC^0, $\overline{\mathrm{IB}^0}$ and $\overline{\mathrm{IIB}}$). The checks are not difficult and all do satisfy the required condition. For example, the linear space for the canonical Schönflies group is given by $\bar{a}_2=\bar{a}_3=0$, after the PSH map points satisfying these equations will satisfy $a_2 = \bar{a}_2(\bar{\mathbf{a}}\cdot\bar{\mathbf{a}}) = 0$ and similarly for a_3. For the canonical cylindrical subgroup $\bar{a}_2 = \bar{a}_3 = \bar{c}_2 = \bar{c}_3 = 0$ and after the PSH map, $a_2 = a_3 = 0$ and also $c_2 = \bar{c}_2(\bar{\mathbf{a}}\cdot\bar{\mathbf{a}}) - \bar{a}_2(\bar{\mathbf{a}}\cdot\bar{\mathbf{c}}) = 0$ and similar for c_3. Since the results hold for the canonical forms of the subgroups and symmetric subspaces, they hold for all subgroups and symmetric subspaces by symmetry.

As a first example, consider linearly interpolating between the identity in the group and a 2θ screw motion with pitch p, note that linear interpolation was also considered in [5]. We can choose coordinates so that the screw axis is the x-axis and then the group element will be,

$$g_1 = c + si - p\theta s\varepsilon + p\theta c\varepsilon i \tag{14}$$

where $s = \sin(\theta)$ and $c = \cos(\theta)$. Now the interpolated motion will be given in terms of a parameter t as,

$$g(t) = (1-t) + tg_1. \tag{15}$$

Then the PSH map takes this to a twisted cubic curve in the group,

$$\begin{aligned} PSH\big(g(t)\big) = {}& \big(1 + 3(c-1)t + 2(c-1)(c-2)t^2 - 2(c-1)^2t^3\big) \\ & + st\big(1 + 2(c-1)t - 2(c-1)t^2\big)i \\ & - p\theta st^2\big(c + (1-c)t\big)\varepsilon + p\theta t\big(c + (c-1)^2t - (c-1)^2t^2\big)\varepsilon i \end{aligned} \tag{16}$$

The two group elements lie in a cylindrical subgroup, hence so does the twisted cubic curve. So the result is a vertical Darboux motion, in agreement with the results of [10].

Next we look at a slightly more complicated example, we interpolate between three group elements with a conic. The three group elements will be,

$$g_0 = 1, \quad g_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}k - \frac{\pi}{4\sqrt{3}}\varepsilon + \frac{\pi}{12}\varepsilon k, \quad g_2 = \frac{\sqrt{3}}{2} + \frac{1}{2}k + \frac{3}{2}\varepsilon j \tag{17}$$

so that the points lie in the symmetric subspace generated by the IC^0 3-system. A conic through these points can be given by,

$$\bar{g}(t) = \frac{1}{2}(1-t)(2-t)g_0 + t(2-t)g_1 - \frac{1}{2}t(1-t)g_2. \tag{18}$$

Note that there are many other conics through these three points, however, this is the unique conic passing through the knot points at time $t = .0$, 1 and 2 respectively. The PSH map then gives a degree 6 curve in the Study quadric.

5 Conclusion

We were not able to find an algebraic proof that the subgroups and symmetric subspaces lie on linear subspaces in $\mathbb{P}^7$. However, it is clear that such an explanations should exist, this fact cannot be a coincidence.

Acknowledgements Many thanks to the anonymous reviewers whose suggestions have greatly improved this work.

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