Half-turns and Line Symmetric Motions

J.M. Selig
Faculty of Business, Computing and Info. Management,
London South Bank University, London SE1 0AA, U.K.
(e-mail: seligjm@lsbu.ac.uk)

and

M. Husty
Institute of Basic Sciences in Engineering
Unit Geometry and CAD,
Leopold-Franzens-Universität Innsbruck, Austria.
(e-mail: manfred.husty@uibk.ac.at)

Abstract

A line symmetric motion is the motion obtained by reflecting a rigid body in the successive generator lines of a ruled surface. In this work we review the dual quaternion approach to rigid body displacements, in particular the representation of the group $SE(3)$ by the Study quadric. Then some classical work on reflections in lines or half-turns is reviewed. Next two new characterisations of line symmetric motions are presented. These are used to study a number of examples one of which is a novel line symmetric motion given by a rational degree five curve in the Study quadric. The rest of the paper investigates the connection between sets of half-turns and linear subspaces of the Study quadric. Line symmetric motions produced by some degenerate ruled surfaces are shown to be restricted to certain 2-planes in the Study quadric. Reflections in the lines of a linear line complex lie in the intersection of a the Study-quadric with a 4-plane.

1 Introduction

In this work we revisit the classical idea of half-turns using modern mathematical techniques. In particular we use the dual quaternion representation of rigid-body motions.

We use these methods to study line symmetric motions, recovering and extending some of the classically known results. Line symmetric motions were first studied by Krames in a series of papers [13] - [17] using mainly synthetic methods. Line symmetric motions have become important once more as it been realised that some self motions of Stewart-Gough parallel manipulators are line symmetric. This is not very surprising, because Krames had already noticed that motions having spherical paths are almost always line symmetric. Motions having spherical paths were studied by Emile Borel [2] and R. Bricard [4] in two award winning papers (see also [7]). Another motivation to resume these investigations is that most of the literature on this topic is in German. The classical papers of Krames have been dealt with in Bottema and Roth [3], using analytic methods. More recent papers like [8], [9], [19] or [23] and [24] have never been translated to English. These papers generalise line symmetric motions to Non-Euclidean settings or multidimensional spaces.

We give two characterisations of line symmetric motions which we believe are novel. Some well know examples are briefly studied and a new example is introduced. These demonstrate the use of dual quaternions in proving the motions to be line symmetric and finding the base surface for the motions.

Next we introduce and solve a new problem concerning the reflections in the space of lines in a linear line complex. This produces a 3-parameter family of rigid-motions.

Finally we look at plane symmetric motions and find an interesting connection between these motions an the B-planes in the Study quadric.

The first section deals with a short review of the algebra of dual quaternions.

2 Dual Quaternions and The Study Quadric

The dual quaternions were invented by Clifford to describe the geometry of space. This work seems to have appeared first in Clifford's 1871 paper, "Preliminary sketch of biquaternions" [5] but see also [20] for more details on the history. In Clifford's paper several different 'biquaternions' are considered, these are characterised by the properties of the extra generator (ω), introduced. The term 'dual' for the case where the generator squares to zero ($\omega^2 = 0$) seems to have come later.

Study claims credit for associating the dual quaternions with a six dimensional quadric, actually an open set in the quadric, [22]. This 'Study quadric' represents the group of rigid-body transformations using eight homogeneous parameters. He introduced what he called "soma" which are orthogonal coordinate frames, the dual quaternions were used to represent transformations between the soma. It was also probably Study who first used ε for the dual unit, rather than Clifford's ω, and it is this convention that is followed below.

In the latter half of the 20th century mathematicians seems to forget about dual quaternions, preferring matrix methods. However, they were kept alive in Kinematics, notable by Dimentberg [6], Freudenstein and his student Yang, [25] and [26].

2.1 Quaternions

We begin however, with Hamilton's quaternions and their connection with rotations. A rotation of angle θ about a unit vector $\mathbf{v} = (v_x,\, v_y,\, v_z)^T$ is represented by the quaternion,

$$r = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(v_x i + v_y j + v_z k).$$

The action of such a quaternion on a point $p = xi + yj + zk$ in space is given by the conjugation:

$$p' = rpr^-,$$

where the quaternion conjugate r^-, is given by,

$$r^- = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}(v_x i + v_y j + v_z k).$$

Notice here that quaternions representing rotations satisfy $rr^- = 1$ and also that r and $-r$ represent the same rotation. The set of unit quaternions, those satisfying $rr^- = 1$, comprise the group Spin(3), which is the double cover of the group of rotations $SO(3)$. A detailed treatment on the use of quaternions in kinematics can be found in Blaschke [1] or Husty et.al.[10].

2.2 Dual Quaternions

Now to include translations the dual unit ε is introduced. This dual unit satisfies the relation $\varepsilon^2 = 0$ and commutes with the quaternion units i, j and k. A general dual quaternion has the form,

$$h = q_0 + \varepsilon q_1$$

where q_0 and q_1 are ordinary quaternions. A dual quaternion representing a rigid transformation is given by,

$$g = r + \frac{1}{2}\varepsilon t r,$$

where r is a quaternion representing a rotation as above and t is a pure quaternion representing the translational part of the transformation, that is $t = t_x i + t_y j + t_z k$.

In this description points in space are represented by dual quaternions of the form,

$$\hat{p} = 1 + \varepsilon p,$$

where p is a pure quaternion as above. The action of a rigid transformation on a point is given by,

$$\hat{p}' = (r + \frac{1}{2}\varepsilon t r)\hat{p}(r^- + \frac{1}{2}\varepsilon r^- t).$$

That is,

$$
\begin{aligned}
\hat{p}' &= (r + \frac{1}{2}\varepsilon tr)(1 + \varepsilon p)(r^- + \frac{1}{2}\varepsilon r^- t) \\
&= rr^- + \varepsilon(rpr^- + \frac{1}{2}rr^- t + \frac{1}{2}trr^-) \\
&= 1 + \varepsilon(rpr^- + t).
\end{aligned}
$$

Notice that, as with the pure rotations, g and $-g$ represent the same rigid transformation.

2.3 Lines

Lines in space can also be represented by dual quaternions, the line joining the points p and q will have Plücker coordinates,

$$
\begin{aligned}
p_{01} &= p_x - q_x, & p_{23} &= p_y q_z - p_z q_y, \\
p_{02} &= p_y - q_y, & p_{31} &= p_z q_x - p_x q_z, \\
p_{03} &= p_z - q_z, & p_{12} &= p_x q_y - p_y q_x,
\end{aligned}
$$

the corresponding dual quaternion will be,

$$\ell = (p_{01}i + p_{02}j + p_{03}k) + \varepsilon(p_{23}i + p_{31}j + p_{12}k).$$

Notice that for dual quaternions of this form we have,

$$\ell\ell^- = p_{01}^2 + p_{02}^2 + p_{03}^2,$$

normally in line geometry, we take the Plücker coordinates to be homogeneous coordinates in a 5-dimensional projective space. The Plücker coordinates then satisfy the quadratic relation,

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0.$$

In the dual quaternion representation this is equivalent to the fact that the product $\ell\ell^-$ has no dual part. In 5-dimensional projective space the points satisfying the Plücker relation above form a 4-dimensional quadric known as the Klein quadric (or Plücker quadric).

The effect of a rigid transformation g, on a line ℓ is given by,

$$\ell' = g\ell g^-.$$

This is easily verified from the transformation of the Plücker coordinates. Notice that this action of the group on lines does not have the same form as the action on points. That is, if $g = r + (1/2)\varepsilon tr$ then $g^- = r^- - (1/2)\varepsilon r^- t$ not $r^- + (1/2)\varepsilon r^- t$ the term which appears in the transformation of points.

Suppose $\mathbf{v}$ is a unit vector and $\mathbf{p}$ is a point in space, then a line in the direction of $\mathbf{v}$ and passing through the point $\mathbf{p}$ can be written as,

$$\ell = \mathbf{v} + \varepsilon\mathbf{p} \times \mathbf{v}.$$

A rotation of θ radians about such a line is given by the dual quaternion,

$$g = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\ell = (\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{v}) + \varepsilon \sin\frac{\theta}{2}\mathbf{p}\times\mathbf{v}.$$

To see this, notice that the rotational part of this transformation is simply the quaternion, $r = \cos(\theta/2) + \sin(\theta/2)(v_x i + v_y j + v_z k)$ as above. If the line passes through the origin, that is if $\mathbf{p} = \mathbf{0}$ then we are done, otherwise we can produce the rotation about the line by first translating it to the origin, rotating and then translating back,

$$g = (1 + \frac{1}{2}\varepsilon p)r(1 - \frac{1}{2}\varepsilon p) = r + \frac{1}{2}\varepsilon(pr - rp),$$

finally a simple computation confirms that the quaternion $\frac{1}{2}(pr - rp)$ corresponds to the vector $\sin\frac{\theta}{2}\mathbf{p}\times\mathbf{v}$.

2.4 The Study Quadric

Notice that not all dual quaternions represent rigid transformations. In fact the condition for a dual quaternion g, to be a rigid transformation is just,

$$gg^- = 1.$$

This is easily checked using the form $g = r + (1/2)\varepsilon tr$ given above and remembering that the rotation r satisfies $rr^- = 1$ and since the translation t is a pure quaternion $t^- = -t$. It is a little harder to see that all dual quaternions satisfying this equation are rigid transformations.

If we write a general dual quaternion as,

$$g = (a_0 + a_1 i + a_2 j + a_3 k) + \varepsilon(c_0 + c_1 i + c_2 j + c_3 k),$$

then the equation above can be separated into its dual and quaternion parts,

$$\begin{aligned} a_0^2 + a_1^2 + a_2^2 + a_3^2 &= 1, \\ a_0c_0 + a_1c_1 + a_2c_2 + a_3c_3 &= 0. \end{aligned}$$

Now suppose that the eight variables $(a_0, a_1, a_2, a_3, c_0, c_1, c_2, c_3)$ are actually homogeneous coordinates for a 7-dimensional projective space $\mathbb{P}^7$. This has the effect of identifying g and $-g$ so that points of this space correspond to elements of the group of rigid transformations, not the double cover of the group. The first equation above is no longer relevant, but the second is homogeneous and so applies to homogeneous coordinates,

$$a_0c_0 + a_1c_1 + a_2c_2 + a_3c_3 = 0,$$

is the equation of a 6-dimensional quadric in $\mathbb{P}^7$, this is the Study quadric. Every rigid transformation corresponds to a single point on the quadric. On the other hand some points on the quadric do not correspond to rigid transformations. There is a 3-plane of 'ideal points', the points satisfying $a_0 = a_1 = a_2 = a_3 = 0$ do not correspond to and rigid transformation.

Next we consider the possible 3-planes which lie entirely within the Study quadric. To do this let us group the homogeneous coordinates together as 4-vectors, $\mathbf{a} = (a_0, a_1, a_2, a_3)^T$ and $\mathbf{c} = (c_0, c_1, c_2, c_3)^T$. The 3-planes in the Study quadric are then given by the linear equations,

$$(I_4 - M)\mathbf{a} + (I_4 + M)\mathbf{c} = \mathbf{0},$$

where I_4 is the 4×4 identity matrix and M is a 4×4 orthogonal matrix. This works because if we change the variables to $\mathbf{a} = (\mathbf{x} + \mathbf{y})$ and $\mathbf{c} = (\mathbf{x} - \mathbf{y})$ the equation for the Study quadric becomes,

$$\mathbf{a} \cdot \mathbf{c} = \mathbf{x} \cdot \mathbf{x} - \mathbf{y} \cdot \mathbf{y} = 0$$

and the equations for the 3-planes are then,

$$\mathbf{x} = M\mathbf{y}.$$

There are two kinds of 3-planes in the Study quadric, they are distinguished by the sign of the determinant of M. If $\det(M) = 1$ we call the 3-plane an A-plane, while if $\det(M) = -1$ the 3-plane is called a B-plane. Notice that the 3-plane of ideal points $a_0 = a_1 = a_2 = a_3 = 0$, introduced above, is determined by the orthogonal matrix $M = -I_4$. Since $\det(-I_4) = 1$ this is an A-plane, below it will be referred to as the A-plane at infinity. As another example, consider the space of all translations, dual quaternions of the form, $1 + (1/2)\varepsilon(t_x i + t_y j + t_z k)$. That is, $\mathbf{a} = (1, 0, 0, 0)^T$ and $\mathbf{c} = (0, t_x, t_y, t_z)^T$. It is not difficult to see that these group elements lie on the 3-plane determined by the diagonal matrix $M = \mathrm{diag}(1, -1, -1, -1)$, moreover the determinant of this matrix is clearly -1, so this is a B-plane in the Study quadric.

The A and B-planes can be further classified according to how they meet the A-plane at infinity. Notice that the A-plane at infinity is invariant with respect to multiplication by dual quaternions. The intersection of an arbitrary A or B-plane with the A-plane at infinity is given by the solutions to the following system of eight homogeneous linear equations,

$$\begin{pmatrix} I_4 - M & I_4 + M \\ I_4 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{c} \end{pmatrix} = \mathbf{0}.$$

The dimension of the solution set is determined by the rank of the coefficient matrix, this in turn is given by the determinant of the matrix. Clearly, the determinant is proportional to $\det(I_4 + M)$. Setting this equal to zero we see that the dimension of the solution set is determined by the number of eigenvalues of M equal to -1. That is, there are no solutions, the solution set is empty if M has no -1 eigenvalues. If M has a -1 eigenvalue then the planes meet at a point. If M has two -1 eigenvalues then the intersection is a line, and so forth.

Now any element of $SO(4)$ is conjugate to a matrix of the form,

$$M = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix},$$

this is the maximal torus of $SO(4)$. So, in general an A-plane will have no -1 eigenvalues and hence will not meet the A-plane at infinity. These A-planes will be called A_0-planes. Exceptionally, if θ or ϕ is $\pm\pi$ the matrix M will have two -1 eigenvalues and hence will intersect the A-plane at infinity in a line. These will be called A_2-planes. There is only a single A-plane determined by a matrix M with four -1 eigenvalues, this of course is the A-plane at infinity itself.

If M is a reflection in $O(4)$, that is if M has determinant -1, then a conjugation can reduce the matrix to the general form,

$$M = \begin{pmatrix} -\cos\theta & \sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}.$$

In general such a matrix has a single -1 eigenvalue, and hence these B_1-planes meet the A-plane at infinity at a single point. Exceptionally, M can have three -1 eigenvalues, resulting in a B_3-plane that intersects the A-plane at infinity in a 2-plane.

To understand the geometry of these A and B-planes a little better it is useful to look at examples where the planes pass through the identity element of the group, the dual quaternion 1. Notice that any A or B-plane can be translated to an A or B-plane through the identity by multiplying every element on the left (or right) by the inverse of some dual quaternion in the plane. There is only one exception to this; the A-plane at infinity.

The A_0-planes through the identity are comprised of the rotations about a single point. For example, rotations about the origin are given by dual quaternions of the form $\mathbf{a} + \varepsilon\mathbf{0}$, and these clearly lie in an A_0-plane given by the matrix $M = I_4$. Any A_0-plane containing the identity is isomorphic to the rotation group $SO(3)$.

The A_2-planes through the identity consist of rotations about lines parallel to a given line. For example, consider the rotations parallel to the z-axis, these are represented by the dual quaternions of the form, $(\cos(\theta/2) + \sin(\theta/2)k) + \varepsilon(c_x \sin(\theta/2)i + c_y \sin(\theta/2)j)$. The matrix determining this A_2-plane is given by the diagonal matrix $M = \mathrm{diag}(1, -1, -1, 1)$. Notice that this A_2-plane also contains the translations perpendicular to the z-axis, these could be thought of as rotations about lines at infinity. We see that any A_2-plane containing the identity is isomorphic to the group of planar motions $SE(2)$.

The B_1-planes through the identity are not isomorphic to any subgroup of $SE(3)$. Rather they consist of the set of rotations about axes lying in a fixed plane. For example, consider the rotations about lines in the xy-plane.

These consist of dual quaternions of the form, $(\cos(\theta/2) + \sin(\theta/2)\cos\alpha i + \sin(\theta/2)\sin\alpha j) - \varepsilon d\sin(\theta/2)k$. This plane is determined by the diagonal matrix $M = \mathrm{diag}(1, 1, 1, -1)$. Notice that this B_1-plane also contains translations parallel to the xy-plane, again these could be thought of as rotations about lines at infinity.

There is only one B_3-plane containing the identity. This is the B-plane we met above, the set of all translations, determined by the matrix, $M = \mathrm{diag}(1, -1, -1, -1)$. This B_3-plane is clearly isomorphic to the group $\mathbb{R}^3$.

2.5 Matrix Representations of Quaternions

Consider the product of two arbitrary quaternions, $ab = c$,

$$(a_0 + a_1 i + a_2 j + a_3 k)(b_0 + b_1 i + b_2 j + b_3 k) = (c_0 + c_1 i + c_2 j + c_3 k)$$

where,

$$\begin{aligned} c_0 &= a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3, \\ c_1 &= a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2, \\ c_2 &= a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1, \\ c_3 &= a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0. \end{aligned}$$

This can be written as a vector-matrix equation in two ways, either as,

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

or as,

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & b_3 & -b_2 \\ b_2 & -b_3 & b_0 & b_1 \\ b_3 & b_2 & -b_1 & b_0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

These relations give us two 4×4 representations of the quaternion product. These are the left multiplication,

$$L(a) = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix},$$

and the right multiplication,

$$R(b) = \begin{pmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & b_3 & -b_2 \\ b_2 & -b_3 & b_0 & b_1 \\ b_3 & b_2 & -b_1 & b_0 \end{pmatrix}.$$

These 4×4 representations extend to 8×8 representations of the dual quaternions, let $h = (p_0 + \varepsilon p_1)$ and $f = (q_0 + \varepsilon q_1)$ be two arbitrary dual quaternions, then the representations for left and right multiplication can be written in partitioned form as,

$$\overline{L}(h) = \begin{pmatrix} L(p_0) & 0 \\ L(p_1) & L(p_0) \end{pmatrix} \quad \text{and} \quad \overline{R}(f) = \begin{pmatrix} R(q_0) & 0 \\ R(q_1) & R(q_0) \end{pmatrix}.$$

3 Half-turns

A half-turn is a rotation by π radians about some line. Half-turns can be represented by dual quaternions of the form $\ell = (a_1 i + a_2 j + a_3 k) + \varepsilon(c_1 i + c_2 j + c_3 k)$. They can be thought of as reflections in the line.

Clearly for each line in space there is exactly one possible half-turn, so there is a correspondence between half-turns and lines. Intersecting the Study quadric with the 5-plane $a_0 = 0$, $c_0 = 0$, gives a 4-dimensional quadric which is essentially the Klein quadric,

$$a_1 c_1 + a_2 c_2 + a_3 c_3 = 0.$$

The action of the group by conjugation, the adjoint action of the group, preserves the set of half-turns. That is, for any group element h and half-turn ℓ the conjugation $h\ell h^- = \ell'$ is another half-turn. To see this notice that these lines are the dual equivalent of the pure quaternions, that is $\ell^- = -\ell$ for half-turns. Moreover, the half-turns are the only dual quaternions that satisfy this relation. Now the quaternion conjugate of $h\ell h^- = \ell'$ is,

$$(\ell')^- = (h\ell h^-)^- = (h^-)^- \ell^- h^- = -h\ell h^- = -\ell'.$$

So the 5-plane $a_0 = c_0 = 0$, is preserved by the adjoint action of $SE(3)$.

A classical theorem states that any proper rigid motion can be written as the product of two half-turns, see [3]. As an example, consider a finite screw motion about the z-axis, this can be written as the dual quaternion,

$$g = (\cos\frac{\theta}{2} + \sin\frac{\theta}{2}k) + \varepsilon(-\frac{d}{2}\sin\frac{\theta}{2} + \frac{d}{2}\cos\frac{\theta}{2}k).$$

It is easy to see that this can be decomposed as, $g = \ell_1 \ell_2$ where the two half-turns are,

$$\ell_1 = i \qquad \text{and} \qquad \ell_2 = (-\cos\frac{\theta}{2}i + \sin\frac{\theta}{2}j) + \varepsilon(\frac{d}{2}\sin\frac{\theta}{2}i + \frac{d}{2}\cos\frac{\theta}{2}j)$$

There are many other possible solutions since if g_0 is any transformation which commutes with g, that is any other screw motion with the same axis as g, then since $gg_0 = g_0 g$,

$$g = \ell_1' \ell_2', \qquad \text{where} \quad \ell_1' = g_0 \ell_1 g_0^-, \quad \text{and} \quad \ell_2' = g_0 \ell_2 g_0^-.$$

In fact this is enough to prove the theorem since any motion can be brought to this standard form using the adjoint action of the group. Notice that the factors, ℓ_1 and ℓ_2 are perpendicular to the axis of the original screw transformation, the angle between the lines is half the rotation angle of the transformation and the perpendicular distance between the lines is half the translation along the axis of the screw.

4 Line Symmetric Motions

A rigid body motion can be thought of as a curve in the Study quadric. Bottema and Roth [3], define several types of special rigid body motions. Among these are the line symmetric motions.

Line symmetric motions are defined as follows: Take a ruled surface $\ell(\mu)$ and a fixed coordinate frame (or soma), now a line symmetric motion is given by reflecting the fixed frame in consecutive generating lines of the ruled surface, to give a 1-parameter family of frames.

This can be seen as a curve in the Study quadric by choosing a starting line in the ruled surface, say $\ell_0 = \ell(0)$. Now the rigid motion from the frame given by this line to any subsequent line will be,

$$g(\mu) = \ell(\mu)\ell_0^{-1} = \ell(\mu)\ell_0,$$

since half-turns are self-inverse.

It can be seen that such a curve will satisfy the relation:

$$g(\mu)\ell_0^- + \ell_0 g^-(\mu) = 0, \tag{1}$$

since any line satisfies $\ell^- = -\ell$ and ℓ^2 is a real number. On the other hand, suppose that $g(\mu)$ is a curve in the Study quadric which satisfies the above equation for some line ℓ_0, then

$$g(\mu)\ell_0 = \ell_0 g^-(\mu),$$

and hence

$$(g(\mu)\ell_0)^- = -\ell_0 g^-(\mu) = -(g(\mu)\ell_0).$$

So $(g(\mu)\ell_0)$ is a line and the motion is line-symmetric.

Notice that the above assumes that the motion passes through the identity in the group. In the Sudy quadric the identity is the point with coordinates $(1, 0, 0, 0, 0, 0, 0, 0)$, that is the dual quaternion 1. A motion which doesn't pass through the identity might still be line symmetric, the motion can always be translated to a path through the identity, that is the motion may have the form $g(\mu) = \ell(\mu)\ell_0 g_0$ where g_0 is some fixed group element. Such a path will clearly satisfy the equation,

$$g(\mu)\gamma_0^- + \gamma_0 g^-(\mu) = 0, \tag{2}$$

where $\gamma_0 = \ell_0 g_0$.

These equations represent a set of homogeneous linear equations in the coordinates for g, that is we can think of g as a 8-dimensional vector $g = (a_0, a_1, a_2, a_3, c_0, c_1, c_2, c_3)^T$. Then we can use the 8×8 representation of the dual quaternion product to write equation (1) as,

$$\left(\overline{R}(\ell_0^-) + \overline{L}(\ell_0)C\right)g = 0,$$

where C is the 8×8 diagonal matrix representing dual quaternion conjugation, $C = \mathrm{diag}(+1, -1, -1, -1, +1, -1, -1, -1)$. To see how many of these 8 homogeneous equations are independent we can look at a particular case, assume here that the line ℓ_0 is the z-axis, $\ell_0 = k$. In this case we can compute the matrices,

$$\left(\overline{R}(-k) + \overline{L}(k)C\right) = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix},$$

where,

$$X = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So there are just two independent equations, $2a_3 = 0$ and $2c_3 = 0$ and hence a curve representing a line symmetric motion will lie in the intersection of the Study quadric with a 5-plane.

The line symmetric motions can be characterised in another way. If the motion $g(\mu)$ can be factored into a product of two half-turns one of which is fixed then the screw axis of $g(\mu)$ will meet the fixed line at right-angles. Hence the set of all axes of the motion will lie in the congruence of lines meeting and perpendicular to, a fixed line. In fact it can be seen that the ruled surface formed by the screw axes of the motion will form a right conoid.

These two characterisations are, of course, equivalent. To see this first recall that two lines ℓ_1 and ℓ_2 will be coincident and perpendicular if and only if they satisfy $\ell_1\ell_2^- + \ell_2\ell_1^- = 0$. Now a screw motion about a line ℓ can be written as,

$$g = (1 + \frac{1}{2}\varepsilon\frac{\theta p}{2\pi}\ell)(\cos\frac{\theta}{2} + \ell\sin\frac{\theta}{2}) = (\cos\frac{\theta}{2} + \ell\sin\frac{\theta}{2}) + \frac{\theta p}{4\pi}\varepsilon(\ell^2\sin\frac{\theta}{2} + \ell\cos\frac{\theta}{2}),$$

where p is the pitch of the screw motion. Hence after a little computation we get that,

$$g\ell_0^- + \ell_0 g^- = (\sin\frac{\theta}{2} + \frac{\theta p}{4\pi}\varepsilon\cos\frac{\theta}{2})(\ell\ell_0^- + \ell_0\ell^-).$$

Now if the axis of the motion ℓ is coincident and perpendicular to a line ℓ_0 then clearly (1) is satisfied. On the other hand if (1) is satisfied the either the lines are coincident and perpendicular or $\theta = 0$, that is the motion is a pure translation.

This second condition leads to a small but useful result, that a motion about a fixed axis is always line symmetric[1]. This is easily seen since any line coincident and perpendicular to the fixed axis of the motion can be taken as ℓ_0.

[1]This result is also stated, but without proof, in Krames [13], p. 395.

5 Examples

5.1 Vertical Darboux Motion

In [3] the, so called, vertical Darboux motion is given as an example of a line-symmetric motion. Here this will be verified using the methods developed above.

Writing the vertical Darboux motion, given in [3], as a dual quaternion we have,

$$g(\phi) = (\cos\frac{\phi}{2} + \sin\frac{\phi}{2}k) + \varepsilon(\beta\sin\phi + \gamma(1-\cos\phi))(-\sin\frac{\phi}{2} + \cos\frac{\phi}{2}k).$$

The axis of this motion is always the z-axis, and so by the 'small but useful' result at the end of section 4 this must be a line symmetric motion. Also we can see that any line perpendicular to the z-axis, for example,

$$\ell_0 = (\cos\delta\, i + \sin\delta\, j) + \varepsilon\lambda(-\sin\delta\, i + \cos\delta\, j)$$

where δ and λ are arbitrary constants, will satisfy (1).

We can also derive the ruled surface which produces this motion. For simplicity choose the constants $\delta = 0$ and $\lambda = 0$ so that $\ell_0 = i$, this gives a parameterisation of the ruled surface as,

$$\ell(\phi) = g(\phi)i = -(\cos\frac{\phi}{2}i + \sin\frac{\phi}{2}j) + \varepsilon(\beta\sin\phi + \gamma(1-\cos\phi))(\sin\frac{\phi}{2}i + \cos\frac{\phi}{2}j).$$

The points $\mathbf{p}$, on a line are given by $\mathbf{p} = (\mathbf{v}\times\boldsymbol{\omega})/|\omega|^2 + \nu\boldsymbol{\omega}$, where $\boldsymbol{\omega}$ is the direction and $\mathbf{v}$ the moment of the line and ν is an arbitrary parameter, see [21, §6.5.2]. So the points on the surface can be parameterised as,

$$\begin{aligned} x &= -\nu\cos\frac{\phi}{2}, \\ y &= -\nu\sin\frac{\phi}{2}, \\ z &= \beta\sin\phi + \gamma(1-\cos\phi), \\ &= 2\beta\cos\frac{\phi}{2}\sin\frac{\phi}{2} + 2\gamma\sin^2\frac{\phi}{2}. \end{aligned}$$

The last relation here is found by using the half-angle formulas for $\sin\phi$ and $(1-\cos\phi)$. Finally eliminating the parameters ϕ and ν gives the equation of the surface,

$$(x^2+y^2)z = 2\beta xy + 2\gamma y^2.$$

This can be recognised as a cylindroid in agreement with the results in [3].

5.2 The Borel-Bricard Motion

This special motion is also given in Bottema and Roth [3] but see also [12] for the relevance of this motion for self motions of Griffis-Duffy parallel manipulators. As a dual quaternion the motion is given by,

$$g(\phi) = (\cos\phi + \sin\phi\, k) - \varepsilon(\sin\phi\sqrt{\rho^2 - d^2\sin\phi} - \cos\phi\sqrt{\rho^2 - d^2\sin\phi}k),$$

where ρ and d are constants. This curve is in fact an elliptic quartic curve. Certainly we can easily see that it lies in the 3-plane $a_1 = a_2 = c_1 = c_2 = 0$. Also we see that if $0 \leq (d^2/\rho^2) \leq 1$ then we can set $m^2 = d^2/\rho^2$ and parameterise the curve in terms of Jacobi elliptic functions with parameter m, and amplitude $\phi = \mathrm{am}(u)$,

$$g(u) = (\mathrm{cn}(u) + \mathrm{sn}(u)\, k) - \varepsilon\rho(\mathrm{sn}(u)\,\mathrm{dn}(u) - \mathrm{cn}(u)\,\mathrm{dn}(u)k).$$

Such curves are generally given as the intersection of a pair of quadric surfaces, and that is indeed the case here. One of the quadrics is the restriction of the Study quadric to the 3-plane, $a_0c_0 + a_3c_3 = 0$, and the other is given by,

$$\rho^2 a_0^2 + (\rho^2 - d^2)a_3^2 - c_0^2 - c_3^2 = 0.$$

Again this motion is clearly line symmetric by the 'small but useful' result at the end of section 4. Any line of the form,

$$\ell_0 = \cos\alpha\, i + \sin\alpha\, j$$

can be taken to satisfy (1). For definiteness let's take $\ell_0 = j$ so that the ruled surface defining the motion is,

$$\begin{aligned}\ell(\phi) = g(\phi)\ell_0 = \\ (-\sin\phi\, i + \cos\phi\, j) - \varepsilon(\cos\phi\sqrt{\rho^2 - d^2\sin\phi}\, i + \sin\phi\sqrt{\rho^2 - d^2\sin\phi}\, j),\end{aligned}$$

This surface is well known to be a spherical coniod, that is the set of lines meeting a central axis at right-angles and tangent to a given sphere. In this case the central axis is the z-axis and the sphere has radius ρ and its centre is located at the point $(0, d, 0)$. Treating this ruled surface in the same way as for the vertical Darboux motions we get the equation of the surface as,

$$(x^2 + y^2)z^2 = (\rho^2 - d^2)x^2 + \rho^2 y^2.$$

5.3 The Bennett Motion

It is well known that the motion of the coupler bar of a Bennett mechanism is a conic in the Study quadric. This led to the definition of a generalised Bennett motion as any motion determined by a conic in the Study quadric, [3]. Any such motion will necessarily lie in the intersection of a 2-plane with the Study quadric. Here we show that any such motion is line symmetric. Without loss of generality we may assume that the plane containing the motion passes through the identity, and let g_1 and g_2 be two other points on the curve defining the motion, we require that 1, g_1 and g_2 be linearly independent. Then a general point on the 2-plane containing the motion, not necessarily lying in the Study quadric, can be written as a dual quaternion, $g(\alpha, \beta, \gamma) = \alpha + \beta g_1 + \gamma g_2$, where α, β and γ are arbitrary real parameters. Now as we saw above, any element of the group can be written as a pair of half-turns with axes perpendicular and

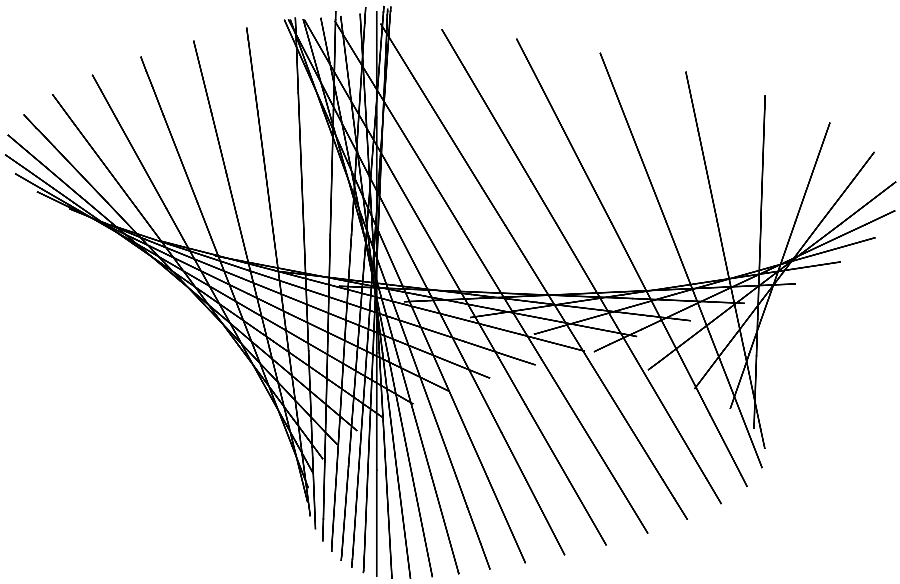

Figure 1: A degree 5 rational ruled surface generating a line symmetric motion.

coincident with the axis of the displacement. So let $g_1 = \ell_1\ell_0$ and $g_2 = \ell_2\ell_0$ where ℓ_0 is the common perpendicular to the axes of g_1 and g_2. Any point on the 2-plane $g(\alpha,\beta,\gamma)$ satisfies the equation,

$$g(\alpha,\beta,\gamma)\ell_0^- + \ell_0 g^-(\alpha,\beta,\gamma) = \alpha(\ell_0 - \ell_0) + \beta(\ell_1 - \ell_1)(\ell_0)^2 + \gamma(\ell_2 - \ell_2)(\ell_0)^2 = 0,$$

and hence any conic is a line symmetric motion.

The ruled surface generating such a motion can be found by multiplying $g(\alpha,\beta,\gamma)$ by ℓ_0^- on the right. The conic of group elements of the form $g = \ell\ell_0$ will be transformed to lines but general points in the 2-plane containing the curve will be transformed to points of the form $-\alpha\ell_0 + \beta\ell_1 + \gamma\ell_2$. The lines in the generating surface will be the intersection of this new 2-plane with the Klein quadric. That is a conic in the Klein quadric. Such a conic of lines is well known to represent a regulus of a hyperboloid, see for example[21]. This result is in agreement with the original results of Krames, [17].

5.4 Rational Motions of Degree Five

A rational normal curve is a rational curve of degree n lying in an n-plane. Such curves lie on several quadric varieties. Hence the results of the previous sections suggest that it should be possible to find a rational normal curve of degree five in the study quadric representing a line symmetric motion.

Such a motion can be produced by combining a vertical Darboux motion as above with a rotation about a line perpendicular to the axis of the Darboux motion. Let us write $c = \cos(\phi/2)$ and $s = \sin(\phi/2)$, then a vertical Darboux motion about the x-axis can be written,

$$g_D(\phi) = \big(c(c^2+s^2) + s(c^2+s^2)i\big) + \varepsilon\big((2\beta sc + 2\gamma s^2)(-s+ci)\big).$$

The rotation is simply a rotation about the z-axis; $g_k(\phi) = (c+sk)$. The combination is a conjugation,

$$\begin{aligned} g_k(\phi)g_D(\phi)g_k^-(\phi) \quad = \quad & (c^2+s^2)(c(c^2+s^2) + s(c^2-s^2)i + 2s^2cj) \\ & +\varepsilon(2\beta sc + 2\gamma s^2)(-s(c^s+s^2) + c(c^2-s^2)i + 2sc^2j). \end{aligned}$$

This is clearly a rational curve of degree 5 and by construction it lies on the Study quadric. As we saw above in section 4, the fact that is lies on the 5-plane $a_3 = c_3 = 0$ ensures that it is a line symmetric motion. All that remains is to check that the curve doesn't lie in a smaller linear subspace, this is easily done by checking that the six coefficients of the dual quaternion generators are linearly independent polynomial. This involves computing a 6×6 determinant, the result is proportional to $\gamma(\beta^2-\gamma^2)$. This shows that the curve lies in a 5-plane and not in any 4-plane unless $\gamma = 0$ or $\gamma = \pm\beta$. This motion is believed to be novel, a diagram of the ruled surface producing this motion is shown in Figure 1, in the case illustrated the parameters have been set to $\beta = 0$ and $\gamma = 1$. Of course there may be other such motions, for example we could have used $g_k(\phi) = (c+sk) + \varepsilon(-ds + dck)$ where d is a constant.

6 Some Degenerate Cases

In this section we study some cases where the ruled surface that we are reflecting in simplifies in some way. These simplifications place more constraints on the possible motions that can be generated from such a surface.

6.1 Cones

Suppose that the ruled surface is a cone. That is a set of lines all passing through a common point, the apex of the cone. If, as usual, we take one line to be our fixed line ℓ_0 then reflecting in this line and any other line of the surface will produce a rotation, since the lines meet at the apex of the cone. The axes of these rotations will all be perpendicular to ℓ_0 and will pass through the apex of the cone. The set of all possible rotations about a point in space form a A_0-plane in the Study quadric as mention in section 2. Also all rotations about axes which lie in a plane form a B_1-plane in the Study quadric. So the possible displacements generated by a line symmetric motion based on the lines of a cone must lie in the intersection of an A_0-plane and a B_1-plane which must be a 2-plane lying in the Study quadric.

6.2 Cylinders

Here suppose that the ruled surface is a cylinder with all generating lines parallel. Reflecting in a fixed line and any parallel line will produce a translation in a direction perpendicular to the axis of the cylinder. The set of rigid body displacements in a plane form a subgroup of $SE(3)$ namely the planar motion group $SE(2)$. These subgroups are represented by A_2-planes in the Study quadric. A planar motion group will also contain the set of rotations about axes perpendicular to the plane under consideration. On the other hand the set of all possible translations in space form another subgroup $\mathbb{R}^3$ represented by a particular B_3-plane in the Study quadric. Clearly the displacements generated by a line symmetric motion based on a cylindrical ruled surface lie in the intersection of an A_2-plane and the B_3-plane. This will necessarily be a 2-plane in the Study Quadric.

6.3 Developables of Plane Curves

There do not seem to be any algebraic constraints giving the possible motions generated by line symmetric motions based on torsal surfaces. A torsal surface, or torse, is the developable surface of a curve, that is the surface formed from the tangent lines to a curve. There will however, be differential constraints on such motions.

If we restrict our attention to plane curves, then a developable of a plane curve will generate planar motions; translations from reflections in parallel lines and rotations about perpendicular lines from reflections in lines that meet. However, fixing one tangent line to the curve we can see that all possible displacements in such a motion will be either rotations about axes intersecting the fixed line and perpendicular to the plane or translations in the plane perpendicular to the fixed line. Again the set of possible displacements forms a 2-plane in the Study quadric, this time the intersection of the A_2-plane of planar displacements with the B_1-plane of rotations in the plane containing the fixed line and the perpendicular to the original plane.

6.4 Conoids

Above, in section 4, we saw that the screw axes of the displacements comprising a line symmetric motion will generally form a conoid. Here we consider the case where the surface we are reflecting in is a conoid. Notice that the vertical Darboux motion and the Borel-Bricard motion studied in examples 5.1 and 5.2 are examples of this type of motion. Let ℓ_0 be a fixed line in the conoid and ℓ some other line in the conoid. The common perpendicular for such a pair of lines will be the axis of the conoid, since by definition the lines in a conoid meet and are perpendicular to a fixed line. Reflecting in ℓ_0 and then ℓ will produce a finite screw motion whose screw axis is the axis of the conoid. Hence the displacements comprising such a line symmetric motion will all be screw motions about a fixed axis. In fact such a set of motions, the screw motions of arbitrary pitch and angle about a fixed axis form a subgroup of the group of rigid-body displacements, namely cylinder group $SO(2) \times \mathbb{R}$. As subvarieties of the Study quadric, it is known that these subgroups are the intersection of Study quadric with a 3-plane, see [21, p. 255].

7 Reflections in a Line Complex

Suppose we take a coordinate frame or soma and reflect it in every line in a line complex. The result will be a 3-parameter family of soma and hence a 3-dimensional subspace of the Study quadric.

As a subspace of $\mathbb{P}^7$ a general line complex can be viewed as the intersection of a hyperplane with the two hyperplanes $a_0 = 0$ and $c_0 = 0$, and the Study quadric.

These hyperplanes can be written in the form $\mathbf{v}^T\mathbf{h} = 0$, where $\mathbf{h}$ is an arbitrary point in the space, $\mathbf{h}^T = (a_0, a_1, a_2, a_3, c_0, c_1, c_2, c_3)$ and we have three different $\mathbf{v}$s,

$$\begin{aligned}
\mathbf{v}_c^T &= (0, \gamma_1, \gamma_2, \gamma_3, 0, \alpha_1, \alpha_2, \alpha_3), \\
\mathbf{v}_1^T &= (1, 0, 0, 0, 0, 0, 0), \\
\mathbf{v}_\varepsilon^T &= (0, 0, 0, 0, 1, 0, 0, 0),
\end{aligned}$$

where $\mathbf{v}_c$ produces the standard equation of a line complex and $\mathbf{v}_1$, $\mathbf{v}_\varepsilon$ and the Study quadric give the Klein quadric of lines as above.

Now fix one line in the complex, say ℓ_0. Then any rigid motion in the subspace is given, by a reflection in ℓ_0 followed by a reflection in any other line ℓ in the complex. As we saw above, we can represent this quaternion product as a matrix product, $g = \overline{R}(\ell_0)\ell$. Notice that $\overline{R}(\ell_0)$ is self-inverse since ℓ_0 is a line. The elements of this form now clearly satisfy the three linear equations,

$$\mathbf{v}_c^T\overline{R}(\ell_0)g = 0, \quad \mathbf{v}_1^T\overline{R}(\ell_0)g = 0, \quad \mathbf{v}_\varepsilon^T\overline{R}(\ell_0)g = 0.$$

This defines a 4-plane in $\mathbb{P}^7$ and so the 3-dimensional subspace of these motions are given by the intersection of this 4-plane with the Study quadric.

If the line complex in question is a singular line complex then the situation is only a little different. The linear space defining the complex is tangent to the Klein quadric, and clearly this extends to the linear space determined by $\mathbf{v}_c$ above. So the set of displacements is still the intersection of the Study quadric with a 4-plane. There is another way to look at this subspace of the Study quadric though. A singular line complex consists of all lines that meet (or are parallel to) a given line. So fix the home frame as the image of the given frame in the central line of the complex. The rigid motions that transform this home frame to the other frames in the subspace will be given by a reflection in the central line followed by a reflection in some other line of the complex. Since these lines meet the motion will be a rotation, moreover the axis of rotation will be perpendicular to the central line in the complex. This will be true for any line in the complex that meets the central line, hence the subspace in the Study quadric will contain the space of all rotations about axes that are perpendicular to, and meet the central axis of the complex. The subspace will also contain translations perpendicular to the central line resulting from the lines parallel to the central line in the complex.

It is easy to see that this space of displacements can be generated by a linkage consisting of a cylinder joint in series with a revolute joint where the axes of the two joints meet and are perpendicular.

Notice that these ideas extend easily to spaces of lines. For example, the set of displacements generated by reflection in all the lines of a quadratic line complex. A quadratic line complexes is the 3-parameter space of lines determined by the intersection of the Klein quadric with a another 4-dimensional quadric. The set of displacements generated by reflection in all the lines of a quadratic line complex clearly lies in the intersection of the Study quadric with a 4-dimensional quadric.

Another example would be the displacements generated by all reflections in a linear line congruence. A linear line congruence consists of the lines in the intersection of a 3-plane with the Klien quadric. So the 2-parameter family of displacements generated by reflections in all the lines of a linear line congruence will lie in the intersection of the Study quadric with a 3-plane. These will include the motions generated by a cylindrical joint mentioned in section 6.4 above, since this set is generated by all reflections in the linear congruence of lines meeting and perpendicular to a fixed line. However, there are sets of rigid displacements which are given by the intersection of the Study quadric and a 3-plane which are not the result of reflecting in the lines of a linear line congruence. For example the set of possible displacements generated by an R-R linkage is of this form, [21, p.256].

8 Point and Plane Symmetric Motions

Finally here we look briefly at plane symmetric and point symmetric motions which have very similar definitions to line symmetric motions. Just as a rigid body motion can be specified by reflections in a sequence of lines we can also specify a rigid motion by reflecting in a sequence of points or planes, the difference is that these reflections are not simply rotations by π but orientation reversing reflections.

Suppose we have a one parameter sequence of planes, take a left-handed soma and reflect in the successive planes of the sequence, this will give a one parameter sequence of right-handed somas. As usual, one plane in the sequence can be fixed and the right-handed soma associated with this plane can be taken as the fixed coordinate frame. Now the rigid motions in the sequence can be described as a reflection in the fixed plane followed by a reflection in a plane in the sequence. Such a motion is clearly a rotation about the line of intersection of the two planes. That is these plane symmetric motions are rotations about lines lying in a fixed plane. Since the set of all rotations about lines lying in some fixed plane form a B_1-plane in the Study quadric, a plane symmetric motion can be thought of as a curve in a B_1-plane of the Study quadric.

It is easy to see that a pair of successive reflections in two different points will produce a translation. Point-symmetric motions, that is motion given by reflections in successive points along a curve, thus gives only translations. That is, the set of possible rigid displacements produced by a point symmetric motion lie in a B_3-plane.

9 Conclusions

It is clear from the above that reflections are closely related to linear subspaces of the Study quadric.

References

[1] W. Blaschke, *Kinematik und Quaternionen*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1960.

[2] Borel E., "Mémoire sur les déplacements à trajectoires sphérique", Mém. présentés par divers savants, Paris (2), 33, pp. 1–128, 1908.

[3] O. Bottema and B. Roth. *Theoretical Kinematics.* Dover Publications, New York, 1990.

[4] Bricard R., "Mémoire sur les déplacements à trajectoires sphérique", Journ. École Polyt.(2), 11,pp. 1–96, 1906.

[5] W. K. Clifford. "Preliminary sketch of biquaternions", in Proc. London Math. Soc., 1871 s1-4(1):381–395; doi:10.1112/plms/s1-4.1.381

[6] F.M. Dimentberg. "The screw calculus and its applications in mechanics", Published by Wright-Patterson Air Force Base (Ohio). Foreign Technology Division. Translation Division, 1968. Translation of: Vintovoye ischisleniye i yego prilozheniya v mekhanike. Izdatel'stvo "Nauka", Glavnaya Redaktsiya, Fiziko-Matematicheskoy Literatury. Moskva, 1965.

[7] M. Husty. "E. Borel's and R. Bricard's Papers on Displacements with Spherical Paths and their Relevance to Self-motions of Parallel Manipulators", in: International Symposium on History of Machines and Mechanisms-Proceedings HMM 2000, Ed. M. Ceccarelli, Kluwer Acad. Pub., 163 – 172, ISBN 0-7923-6372-8, 2000.

[8] M. Husty. "Symmetrische Schrotungen im einfach isotropen Rau", Sitzungberichte d. österr. Akad. d. Wiss., math.-nw. Kl., **195**: 291–306, 1986.

[9] M. Husty. "Über eine symmetrische Schrotung mit einer Cayley-Fläche als Grundfläche", Stud. Sci. Math. Hungarica, **22**:463–469, 1987.

[10] M. Husty, A. Karger, H. Sachs, W. Steinhilper,: Kinematik und Robotik, Springer Verlag, Berlin - Heidelberg - New York, ISBN 3-540-63181-X, 633 S, 1997.

[11] M. Husty and A. Karger. "Self motions of Stewart-Gough platforms, an overview", Proceedings of the workshop on "Fundamental Issues and Future Research Directions for Parallel Mechanisms and Manipulators", Quebec City, pp. 131–141, 2002.

[12] A. Karger and M. Husty. "Classification of all self-motions of the original Stewart-Gough platform", Computer-Aided Design, **30**(3):205–215, 1998.

[13] J. Krames. "Über Fußpunktkurven von Regelflächen und eine besondere Klasse von Raumbewegungen (ber symmetrische Schrotungen I)", Mh. f. Math. u. Phys., Bd. 45, pp. 394–406, 1937.

[14] J. Krames. "Zur Bricardschen Bewegung, deren sämtliche Bahnkurven auf Kugeln liegen (ber symmetrische Schrotungen II)", Mh. f. Math. u. Phys., Bd. 45, pp. 407–417, 1937.

[15] J. Krames. "Zur aufrechten Ellipsenbewegung des Raumes (Über symmetrische Schrotungen III)", Mh. f. Math. u. Phys., Bd. 46, pp. 38–50, 1937.

[16] J. Krames. "Zur kubischen Kreisbewegung des Raumes, (Über symmetrische Schrotungen IV)", Sitzungberichte d. österr. Akad. d. Wiss. math.-nw. Kl., Abt. IIa, **146**, pp. 145–158, 1937.

[17] J. Krames. "Zur Geometrie des Bennett'schen Mechanismus, (Über symmetrische Schrotungen V)", Sitzungberichte d. österr. Akad. d. Wiss. math.-nw. Kl., Abt. IIa, **146**, pp. 159–173, 1937.

[18] J. Krames. "Die Borel-Bricard-Bewegung mit punktweise gekoppelten orthogonalen Hyperboloiden (ber symmetrische Schrotungen VI)", Mh. f. Math. u. Phys., Bd. 46, pp. 172–195, 1937.

[19] J. Krames. "Über eine konoidale Regelfläche fünften Grades und die darauf gegründete symmetrische Schrotung", Sitzungberichte d. österr. Akad. d. Wiss. math.-nw. Kl., Abt. IIa, **190**:221–230, 1981.

[20] J. Rooney. "William Kingdon Clifford (1845–1879)" in *Distinguished Figures in Mechanism and Machine Science: Their Contributions and Legacies Series: History of Mechanism and Machine Science*, Vol. 1, Ed. M. Ceccarelli, Springer Verlag, New York, 2007.

[21] J.M. Selig. *Geometric Fundamentals of Robotics.* Springer Verlag, New York, 2005.

[22] E. Study. "Von den Bewegungen und Umlegungen", Math. Ann. **39**:441–566, 1891.

[23] J. Tölke. “Elementare Kennzeichnungen der symmetrischen Schrotung”, manuscripta mathematica, **15**(4):309–321, 1975.

[24] H. Wresnik “Symmetrische Schrotungen an Verallgemeinerten Regelflächen des E^n”, Journal of Geometry, **36**(1-2):189–200, 1989.

[25] A. T. Yang. “Application of Quaternion Algebra and Dual Numbers to the Analysis of Spatial Mechanisms” Thesis Columbia University, New York. No. 64-2803 (University Microfilm, Ann Arbor, Michigan), 1963.

[26] Yang, A.T. and Freudenstein, F. “Application of dual number quaternion algebra to the analysis of spatial mechanism”, J. of Applied Mechanisms, **86**:300–308, 1964.