Introduction to Polynomial Invariants of Screw Systems

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Abstract

Screw systems describe the infinitesimal motion of multi–degree-of-freedom rigid bodies, such as end-effectors of robot manipulators. While there exists an exhaustive classification of screw systems, it is based largely on geometrical considerations rather than algebraic ones. Knowledge of the polynomial invariants of the adjoint action of the Euclidean group induced on the Grassmannians of screw systems would provide new insight to the classification, along with a reliable identification procedure. However many standard results of invariant theory break down because the Euclidean group is not reductive.

We describe three possible approaches to a full listing of polynomial invariants for 2–screw systems. Two use the fact that in its adjoint action, the compact subgroup $SO(3)$ acts as a direct sum of two copies of its standard action on $\mathbb{R}^3$. The Molien–Weyl Theorem then provides information on the primary and secondary invariants for this action and specific invariants are calculated by analyzing the decomposition of the alternating 2–tensors. The resulting polynomials can be filtered to find those that are $SE(3)$ invariants and invariants for screw systems are determined by considering the impact of the Plücker relations. A related approach is to calculate directly the decomposition of the symmetric products of alternating tensors. Finally, these approaches are compared with the listing of invariants by Selig based on the existence of two invariant quadratic forms for the adjoint action.

Keywords: Euclidean group, polynomial invariant, representation theory, screw system

1 Introduction

Screws and screw systems are the fundamental mathematical representations of single and multi-variate infinitesimal motion of rigid bodies. Hence they are widely used in robotics and mechanism theory [2,12]. The position of a rigid body in three dimensions with respect to some reference position can be represented by an element of the Euclidean group $SE(3)$, which is a semi-direct product of the group of orientation-preserving rotations $SO(3)$ and the translation group $\mathbb{R}^3$. $SE(3)$ is thus a 6–dimensional Lie group. Its Lie algebra $\mathfrak{se}(3)$ is the sum of $\mathfrak{so}(3)$, the algebra of 3×3 antisymmetric matrices (infinitesimal rotations) and $\mathfrak{t}(3)$ (infinitesimal translations). Each subalgebra can be identified with $\mathbb{R}^3$, but in the first instance the bracket is equivalent to vector product while in the second the bracket is zero, so it is abelian. The twisted product in the group, $(B, \mathbf{b}).(A, \mathbf{a}) = (BA, B\mathbf{a} + \mathbf{b})$ means that only the translations form a normal subgroup and the infinitesimal translations an ideal of the Lie algebra.

A *screw* is an element of the projective Lie algebra, i.e. a 1–dimensional subspace of $\mathfrak{se}(3)$ while a *screw system of degree* k (or k–system) is a k–dimensional subspace. As an example, the motion of the end–effector of a serial robot arm with k joints is given by a kinematic mapping $\lambda : M^{(k)} \to SE(3)$, where $M^{(k)}$ is a k–dimensional manifold representing the joint spaces; its infinitesimal capabilities in a given configuration $x \in M$ are represented by the image of the derivative of λ at x. Away from singularities of the kinematic mapping and with appropriate choice of coordinate systems so that $\lambda(x)$ is the identity in $SE(3)$, this image will be a k–system.

The classification of screw systems due to Hunt and Gibson [7, 12] is based on various geometric considerations, but its mathematical foundation is the adjoint action of $SE(3)$,which simply describes the effect of a simultaneous change of orthonormal coordinate frame in the moving rigid body and the ambient space. Donelan and Gibson [3,4] give a direct, essentially elementary proof that the ring of polynomial invariants for the adjoint action of $SE(n)$ is finitely generated, exhibiting explicit generators. For $n = 3$, these are the two classical quadratic invariants, the Klein and Killing forms, whose ratio h is the pitch of a screw. This distinguishes all orbits except those for which $h = \infty$. There is an induced action on the spaces of screw systems (Grassmannians) and the Hunt–Gibson classes are unions of orbits of this action, generally displaying moduli such as principal pitches.

The typical way of presenting a screw system is by means of a basis for the corresponding subspace in $\mathfrak{se}(3)$, consisting of 6–vectors in Plücker or screw coordinates. A fundamental problem is to determine the Hunt–Gibson class of a system effectively from this information. One approach is to deterrmine polynomial invariants for the induced action. The only results we are aware of concerning polynomial invariants for screw systems are those first given by Selig [19], who deduces certain quadratic invariants for screw systems and shows that the Hunt–Gibson classification can be partially recovered from them. That work is the foundation for this paper. However it is also worth mentioning, since it provides an early model for the application of invariant theory in this field, the work of von Mises [24] on invariants of tensor products and symmetric products (which he termed dyads) of motors (see also [20]).

Semi-direct products often do not fit the standard invariant theory; for example, as a Lie group, $SE(3)$ is not semisimple, as the infinitesimal translations in the Lie algebra form a non-trivial abelian ideal. Consequently $SE(3)$ is not completely reducible (equivalently, linearly reductive in the context of algebraic groups), that is, invariant subspaces of a representation do not necessarily possess invariant complements, and the representation theory cannot be based on a study of irreducible representations alone [8]. In particular, the algebra of polynomial invariants of a representation is not automatically known to be finitely generated (Hilbert's Finiteness Theorem) [21].

There is nevertheless an extensive literature on semi-direct products and the Euclidean group in particular. In the mathematical physics literature the relevant groups are usually called *inhomogeneous* groups. A theorem of Gel'fand [6] shows that the invariant polynomials of the coadjoint action are determined by the elements of the centre of the universal enveloping algebra. For semisimple groups, the non-degeneracy of the Killing form enables the same to be said for the adjoint action. Guillemin and Sternberg [9]) analyze the co-adjoint orbits for semi-direct products and Perroud [17] deduces invariants for a large range of inhomogeneous classical groups. A different approach arises from Inönü–Wigner contraction [13], which gives $SE(n)$ as a contraction of $SO(n+1)$. Rosen [18] and Takiff [23] use this to derive invariants for the Euclidean groups. Panyushev [16] gives a quite general theoretical treatment of Lie algebra semi-direct products from an algebraic-geometric viewpoint. In particular, for semi-direct products $G \ltimes V$ with G reductive, the ring of polynomial invariants for the adjoint action is finitely generated.

The purpose of this paper is to describe approaches to deriving a complete list of polynomial invariants for screw–systems and to begin such a programme in the case of 2–sytems. These form a Grassmannian variety in the space of alternating 2–forms on the Lie algebra $\mathfrak{se}(3)$, defined by the Plücker relations. We follow an approach similar to that of King and Welsh [14] for invariants of 2-qubit systems and use the fact that the adjoint representation contains as a subrepresentation a twofold copy of the standard representation V of $SO(3)$, which is reductive. Section 2 reviews relevant aspects of representation and invariant theory, in particular for $SO(3)$. Polynomial invariants for this action correspond to the trivial components of the symmetric powers of the dual representation. In Section 3 this is applied to the adjoint action of $SE(3)$ and its invariants deduced. This approach is developed in Section 4, where the Molien–Weyl Theorem gives a prediction of the number and degree of polynomial invariants for the induced action of $SO(3)$ on the alternating 2-forms.

Three approaches to calculating explicit lists of invariants are described. The first uses coefficients of the characteristic polynomial of products of the components of a partition of the antisymmetric matrix representation of alternating 2–forms. The second is based on an explicit decomposition of the symmetric powers of the dual representation into irreducible components. The third, following Selig [19], makes use of the invariant quadratic forms of the adjoint action. Some comparison of the results is attempted.

It s then be necessary to check which of the invariants is in fact an $SE(3)$ invariant and then to work modulo the Plücker relations to obtain invariants in the coordinate ring of the relevant Grassmannian variety of 2–systems. However this final part of the programme has not yet been attempted.

2 Representations of $SO(3)$ and their invariants

We first briefly review general notation from multilinear algebra and representation theory (see, for example, [5]). For any finite-dimensional vector space V (over $\mathbb{R}$ or $\mathbb{C}$), let V^* denote its dual space, $\otimes^k V = V \otimes \cdots \otimes V$ the k-fold tensor product of V, $\mathrm{S}^k V$ and $\bigwedge^k V$ the subspaces of symmetric and alternating tensors respectively. If $\{e_1, \ldots, e_n\}$ is a basis for V, the canonical dual basis is $\{f_1, \ldots, f_n\}$ where $f_j(e_i) = \delta_{ij}$. In terms of these bases, if $\mathbf{x} \in V$ and $\mathbf{y} \in V^*$ then the dual pairing is given by $\mathbf{y}^t\mathbf{x}$. The elements $e_{i_1} \otimes \cdots \otimes e_{i_k}$, $1 \leq i_j \leq n$ for $j = 1, \ldots, k$ form a basis for $\otimes^k V$. Bases for the symmetric and alternating powers consist of elements of the form

$$\begin{aligned} e_{i_1} \cdots e_{i_k} &= \sum_{\sigma \in \mathfrak{S}_k} e_{i_{\sigma(1)}} \otimes \ldots \otimes e_{i_{\sigma(k)}} \qquad 1 \leq i_1 \leq \cdots \leq i_k \leq n \\ e_{i_1} \wedge \cdots \wedge e_{i_k} &= \sum_{\sigma \in \mathfrak{S}_k} \mathrm{sgn}(\sigma) e_{\sigma(i_1)} \otimes \ldots \otimes e_{\sigma(i_k)} \qquad 1 \leq i_1 < \cdots < i_k \leq n \end{aligned} \tag{1}$$

where $\mathfrak{S}_k$ denotes the permutation group on $\{1, \ldots, k\}$ and $\mathrm{sgn}(\sigma)$ the sign of a permutation $\sigma \in S_k$. We will use the abbreviation e^k for $e.e \ldots e \in \mathrm{S}^k V$. Note that $\dim \otimes^k V = n^k$, $\dim \mathrm{S}^k V = \binom{k+n-1}{k}$ and $\dim \bigwedge^k V = \binom{n}{k}$. The following direct sum decompositions will be used:

$$\mathrm{S}^k(V \oplus W) \cong \bigoplus_{r=0}^{k} \mathrm{S}^r V \otimes \mathrm{S}^{k-r} W, \qquad \textstyle\bigwedge^k (V \oplus W) \cong \bigoplus_{r=0}^{k} \bigwedge^r V \otimes \bigwedge^{k-r} W \tag{2}$$

The k-fold symmetric power $\mathrm{S}^k V^*$ can be identified in a natural way with the space of homogeneous polynomials on V of degree k since for $\mathbf{x} = \sum_{i=1}^n x_i e_i$, $(f_{i_1} \cdots f_{i_k})(\mathbf{x}) = x_{i_1} \cdots x_{i_k}$.

If G is a Lie group acting on V, so there is a homomorphism $\rho : G \to GL(V)$ (we will usually say simply that V is a representation of G, without specifying ρ), then there is a dual representation ρ^* on V^* defined to ensure duality is preserved; it follows that $\rho^*(g) = \rho(g^{-1})^t$ with respect to any basis for V and its dual basis in V^*. Also, there are induced representations on $\otimes^k V$, consequently also $\mathrm{S}^k V$, $\bigwedge^k V$, defined by $g \cdot (v_1 \otimes \cdots \otimes v_k) = g(v_1) \otimes \cdots \otimes g(v_k)$. In particular, polynomial invariants of the group action are fixed points of the induced action of G on $\mathrm{S}(V^*) = \bigoplus_{k=0}^\infty \mathrm{S}^k V^*$.

There is an associated representation of the Lie algebra $\mathfrak{g}$ of G, obtained by differentiating and evaluating at the identity in G, which provides infinitesimal information about the representation. Since the derivative of a constant is zero, a polynomial invariant of G is therefore killed by the corresponding action of $\mathfrak{g}$, though we will still describe it as an invariant of the action of $\mathfrak{g}$. Associated representations of the Lie algebra satisfy $X^* = -X^t$ and

$$X.(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = X.v_1 \otimes v_2 \otimes \cdots \otimes v_k + v_1 \otimes X.v_2 \otimes \cdots \otimes v_k + \cdots + v_1 \otimes v_2 \otimes \cdots \otimes X.v_k \quad (3)$$

Similar expressions apply to symmetric and alternating powers. For a connected, simply connected group G, there is a one-to-one correspondence between representations of G and those of $\mathfrak{g}$.

The *adjoint* representation Ad of a Lie group G on its Lie algebra $\mathfrak{g}$ is given by differentiating conjugation $h \mapsto ghg^{-1}$ in the group at the identity. Thus $\mathrm{Ad}(g)(X) = gXg^{-1}$. The corresponding action of $\mathfrak{g}$ is denoted ad and it determines the Lie bracket operation in the algebra $\mathrm{ad}(X)Y = [X, Y]$. The *coadjoint* representation is the dual Ad^*.

The group $SO(3)$ is the subset of the group $GL(3)$ of (real) non-singular 3×3 matrices that preserve a given symmetric positive-definite bilinear form Q, the Euclidean inner product on $V = \mathbb{R}^3$. In terms of the standard basis for $\mathbb{R}^3$, $Q(\mathbf{x}_1, \mathbf{x}_2) = x_1 x_2 + y_1 y_2 + z_1 z_2$. The action of $SO(3)$ on $\mathbb{R}^3$ is its *standard* representation. The universal propoerty of the symmetric product means that Q can be regarded as an element of $(\mathrm{S}^2 V)^* \cong \mathrm{S}^2 V^*$ and, by definition, it is a basic quadratic invariant of the standard representation. Let $\{e_x, e_y, e_z\}$ denote the standard basis for V and $\{f_x, f_y, f_z\} \subset V^*$ its canonical dual basis, then Q has the form $\frac{1}{2}(f_x^2 + f_y^2 + f_z^2)$. (The factor $\frac{1}{2}$ arises because our definition of the symmetric product (1) does not include an averaging factor.)

The Lie algebra $\mathfrak{so}(3)$ has as a basis the infinitesimal rotations X, Y, Z derived from the one-parameter subgroups of rotations with angular velocity 1 about the axes e_x, e_y, e_z, respectively. Thus in matrix form, for example

$$Z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this basis, the standard action of the Lie algebra corresponds to vector product in $\mathbb{R}^3$:

$$(uX + vY + wZ)(xe_x + ye_y + ze_z) = (u, v, w) \times (x, y, z).$$

The adjoint action coincides with this:

$$[u_1X + v_1Y + w_1Z, u_2X + v_2Y + w_2Z] = (u_1, v_1, w_1) \times (u_2, v_2, w_2).$$

The representation theory for $SO(3)$ can be derived from that of its simply connected double cover $SU(2)$, their Lie algebras being isomorphic. Every complex representation of $\mathfrak{su}(2)$ restricts to a real one [5, 19] and one can check which of the corresponding representations of $SU(2)$ project to representations on $SO(3)$.

The element $J_3 = Z$ (or equally any infinitesimal rotation) spans a Cartan subalgebra, giving rise to the decomposition of the Lie algebra into its root spaces; the roots are $\pm i$, with root spaces spanned by $J_\pm = Y \pm iX$ respectively. In matrix form

$$J_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ -1 & -i & 0 \end{pmatrix}.$$

The commutation relations for this basis are

$$[J_3, J_+] = iJ_+, \qquad [J_3, J_-] = -iJ_-, \qquad [J_+, J_-] = 2iJ_3. \tag{4}$$

In any representation L of $\mathfrak{so}(3, \mathbb{C})$, there exists an eigenvector of J_3, $u \in L$, which is in the kernel of J_+. The maximal sequence of non-zero vectors $u, J_-(u), J_-^2(u), \ldots, J_-^m(u)$ spans an irreducible invariant subspace L_m spanned by eigenvectors of J_3 with highest weight $mi/2$. Irreducible representations are thus isomorphic to some $L_{m/2}$, $m = 0, 1, 2, \ldots$, of dimension $m + 1$. These only correspond to representations of $SO(3)$ for m even. As real representations, $L_0 \cong \mathbb{R}$ is the trivial representation; the standard representation is $L_1 \cong \mathbb{R}^3$. The adjoint and coadjoint representations are also isomorphic to L_1 and from now on we identify X, Y, Z with e_x, e_y, e_z respectively. In fact Q gives rise to a canonical $SO(3)$–invariant isomorphism $V \to V^*$, given by $u \mapsto Q(u, \cdot)$, which ensures that structural expressions involving V remain true when V is replaced by V^*.

For the adjoint (standard) representation, the relevant highest weight eigenvector may be taken as $e_+ = (i, 1, 0)^t$, with eigenvalue i, which together with $e_0 = J_-(e_+)$, $e_- = J_-(e_0)$ form a basis for L_1. In terms of e_x, e_y, e_z:

$$\begin{array}{rclcrcl} e_+ &=& ie_x + e_y & \qquad & e_x &=& -\frac{1}{4}i(2e_+ - e_-) \\ e_0 &=& -2ie_z & & e_y &=& \frac{1}{4}(2e_+ + e_-) \\ e_- &=& -2ie_x + 2e_y & & e_z &=& \frac{1}{2}ie_0 \end{array} \tag{5}$$

There is a dual basis f_+, f_0, f_- for the coadjoint action but with subscripts denoting the sign of the corresponding eigenvalue, so that $f_+(e_-) = 1$ etc, since J_3 dualises to $-J_3^t$. The invariant form becomes $Q = 4f_+f_- - 2f_0^2$ in this basis [19]. Note that, for example $Q(e_+, e_+) = 0$.

In order to determine all the polynomial invariants, we can analyze the structure of the symmetric powers of the dual or, equivalently, the symmetric powers of $V = L_1$ itself. This is well known but we include it here for completeness and because it provides a simple example of the approach we use to to find invariants for screw systems.

Lemma 2.1. *The symmetric powers of the standard representation of $SO(3)$ for $k \geq 0$ may be decomposed as:*

$$\mathrm{S}^k L_1 = \begin{cases} L_{2m} \oplus L_{2m-2} \oplus \cdots \oplus L_0 & k = 2m \\ L_{2m+1} \oplus L_{2m-1} \oplus \cdots \oplus L_1 & k = 2m+1 \end{cases} \tag{6}$$

Proof. The result is trivial for $k = 0, 1$. We proceed by induction. For each $k \geq 2$, Q gives rise to a (surjective) linear contraction $\Psi^{(k)} : \mathrm{S}^k L_1 \to \mathrm{S}^{k-2} L_1$:

$$\Psi^{(k)}(v_1 v_2 \cdots v_k) = \sum_{i<j} Q(v_i, v_j) v_1 \cdots \hat{v}_i \cdots \hat{v}_j \cdots v_k \tag{7}$$

(where $\hat{}$ denotes omission). $\Psi^{(k)}$ is $SO(3)$-equivariant so its kernel is an invariant subspace of dimension $\binom{k+2}{k} - \binom{k}{k-2} = 2k+1$, and invariant complement, so $\mathrm{S}^k L_1 = \ker \Psi^{(k)} \oplus \mathrm{S}^{k-2} L_1$. It is clear that e_+^k is in the kernel of $\Psi^{(k)}$ and by (3) eigenvalues are additive, so its eigenvalue is ki. Thus the kernel must be the irreducible representation L_k of $SO(3)$. The result follows by induction. $\square$

We will also make use of the following identity:

$$L_m \otimes L_n \cong L_{m+n} \oplus L_{m+n-1} \oplus \cdots \oplus L_{|m-n|}. \tag{8}$$

This is readily derived by observing that eigenvalues of elements of the tensor product are sums of eigenvalues of the components and then counting eigenvectors.

Since V^* is isomorphic to L_1, polynomial invariants can only arise from trivial components of the decompositions in Lemma 2.1. Up to scalar multiples therefore there can only be one invariant of each even degree. For degree 2, we know this is $Q = \frac{1}{2}(f_x^2 + f_y^2 + f_z^2)$; powers of this account for all the other components so Q spans the ring of polynomial invariants.

An alternative approach to the structure of the invariant ring is provided by the following classical theorem. Let V be a representation of a Lie group G, $k[V]$ the ring of polynomial functions on V where $k = \mathbb{R}, \mathbb{C}$. Denote by $k[V]^G$ the ring of invariant polynomials under the action of G. The ring is graded by degree d, the homogeneous components of an invariant polynomial being themselves homogeneous: $k[V]^G = \oplus_{d=0}^{\infty} k[V]_d^G$. As we have seen, the direct sum components can be realised as the trivial components of the representations $\mathrm{S}^d V$. The *Hilbert series* of the representation is the formal power series

$$H(k[V]^G, t) = \sum_{d=0}^{\infty} \dim k[V]_d^G t^d$$

Further, if $V = \oplus_{i=1}^m V_i$, let $\mathbf{t} = (t_1, \ldots, t_m)$ and for a multi-degree $d = (d_1, \ldots, d_m) \in \mathbb{N}^m$, let $\mathbf{t}^d = \prod_{i=1}^m t_i^{d_i}$. The direct sum decomposition enables the ring S^G to also be graded over the multidegrees $d \in \mathbb{N}^m$, say $k[V]^G = \oplus_{d \in \mathbb{N}^m} k[V]_d^G$. The *multivariable Hilbert series* is

$$H(k[V]^G, \mathbf{t}) = \sum_{d \in \mathbb{N}^m} \dim k[V]_d^G \mathbf{t}^d.$$

Theorem 2.2 (Molien–Weyl). *(a) Suppose that G is compact and $d\mu$ is normalised Haar measure on G. Then for $|t| < 1$,*

$$H(k[V]^G, t) = \int_G \frac{d\mu}{\det(1 - tg)}.$$

(b) Suppose G is semi-simple and T a maximal torus of a maximal compact subgroup of G, $\alpha_1, \ldots \alpha_d$ roots of G with respect to T, and W its Weyl group. If $d\nu$ is normalised Haar measure on T then for $|t| < 1$,

$$H(k[V]^G, t) = \frac{1}{|W|} \int_T \frac{(1 - \alpha_1(g)) \cdots (1 - \alpha_d(g))}{\det(1 - tg)} d\nu$$

(c) If $V = \oplus_{i=1}^m V_i$, then with notation as in (b) and $|t_i| < 1$, $i = 1, \ldots, m$

$$H(k[V]^G, t) = \frac{1}{|W|}\int_T \frac{(1-\alpha_1(g))\cdots(1-\alpha_d(g))}{\prod_{i=1}^m \det(1-t_i g)} d\nu$$

Since the components of the maximal torus T can be parametrized by a unit complex number $z \in S^1$, it is generally possible to evaluate the integrals using the Residue Theorem from complex analysis. In the case of $G = SO(3)$, the group is already compact and a maximal torus is given by the subgroup of rotations about the z-axis. These may be diagonalized over the complex numbers with eigenvalues $1, e^{i\phi}, e^{-i\phi}$. The normalized Haar measure on T is given by $d\nu = \frac{1}{\pi}(1-\cos\phi)d\phi$, and setting $z = e^{i\phi}$ one obtains:

$$d\nu = -\frac{(1-z)^2}{4\pi i z^2} dz.$$

Applying Theorem 2.2(b), gives

$$H(k[L_1]^{SO(3)}, t) = -\frac{1}{4\pi i}\oint_{|z|=1} \frac{(1-z)^2\, dz}{z^2(1-t)(1-tz)(1-tz^{-1})}$$

Given $|t| < 1$, the integral is equal to $2\pi i$ times the sum of the residues at $z = 0$ and $z = t$. the Hilbert series then evaluates to $1/(1-t^2)$. Expanding, this shows that there is one invariant of each even power, as observed before.

For a reductive group such as $G = SO(3)$, the ring $k[V]^G$ satisfies the Cohen–Macaulay property [11] and we may obtain more precise information about the invariants. In addition to the ring being finitely generated, there exist sets of *primary* invariants $\theta_1, \ldots, \theta_r$ and *secondary* invariants $\eta_1, \ldots, \eta_s$ such that $k[V]^G = \oplus_{i=0}^s \eta_i k[\theta_1, \ldots, \theta_r]$ where $\eta_0 = 1$. In particular, there are syzygies of the form

$$\eta_i^2 = g_i(\theta_1, \ldots, \theta_r) \tag{9}$$

for some polynomials g_i. It can also be shown (see, for example, Sturmfels [22]) that

$$H(k[V]^G, t) = \frac{\sum_{i=0}^s t^{\deg \eta_i}}{\prod_{j=1}^r (1-t^{\deg \theta_j})}. \tag{10}$$

Note however that there will be more than one way to write a given Hilbert series in this form, so obtaining such a form does not automatically determine the degrees of primary and secondary invariants—see [14] for a simple counterexample. An alternative eligible form can be obtained by multiplying numerator and denominator by a term of the form

$(1+t^{\deg\theta_j}+\cdots+t^{k\deg\theta_j})$ for some θ_j and $k\geq 1$. This has the effect of replacing θ_j by a primary invariant with a multiple of its degree, increasing by one the number of secondary invariants of $\deg\theta_j$ and adding further secondary invariants of degree $\geq 2\deg\theta_j$. Sturmfels describes a methodology for computing primary and secondary invariants using Gröbner bases but that may prove computationally intensive. While the number and degrees of the invariants is well defined, the invariants themselves do not have a canonical form.

3 Invariants of the adjoint action of $SE(3)$

Our primary concern is with the proper Euclidean group $SE(3)\cong SO(3)\ltimes\mathbb{R}^3$ and in particular its adjoint action. See Selig [19] for a more detailed description. The standard action of an element $g=(R,\mathbf{t})\in SE(3)$ on $\mathbf{x}\in\mathbb{R}^3$ is by rotation and translation $\mathbf{x}\mapsto R\mathbf{x}+\mathbf{t}$. Its 6–dimensional Lie algebra $\mathfrak{se}(3)$ is, as a vector space, the direct sum of the infinitesimal rotations $\mathfrak{so}(3)$ and infinitesimal translations $\mathfrak{t}(3)$. However the commutation relations are twisted; the translations form an abelian subalgebra but $0\neq[\mathfrak{so}(3),\mathfrak{t}(3)]\subseteq\mathfrak{t}(3)$. Elements $\mathbf{s}\in\mathfrak{se}(3)$ can be written as a pair of 3–vectors $(\boldsymbol{\omega}^t,\mathbf{v}^t)^t$. Here $\boldsymbol{\omega}=\omega_xe_x+\omega_ye_y+\omega_ze_z\in\mathfrak{so}(3)$ denotes an infinitesimal rotation or angular velocity vector (recall that we identified the infinitesimal rotation X with its axis e_x) while $\mathbf{v}=v_xe'_x+v_ye'_y+v_ze'_z\in\mathfrak{t}(3)$ is an infinitesimal translation (velocity vector) with respect to the basis e'_x,e'_y,e'_z of unit infinitesimal translations along the axes.

Let $R\in SO(3)$ be a rotation matrix and define

$$T=\begin{pmatrix}0&-t_z&t_y\\t_z&0&-t_x\\-t_y&t_x&0\end{pmatrix}$$

to be the antisymmetric 3×3 'translation' matrix corresponding to $\mathbf{t}=(t_x,t_y,t_z)^t\in\mathbb{R}^3$. This correspondence is a natural isomorphism $\bigwedge^2L_1\cong L_1$, which we will use again in Section 4. Then in terms of the basis $(e_x,e_y,e_z,e'_x,e'_y,e'_z)$ for $\mathfrak{se}(3)$, the adjoint action is given by the 6×6 matrix

$$\mathrm{Ad}(g)=\begin{pmatrix}R&0\\TR&R\end{pmatrix}\tag{11}$$

The adjoint action of the Lie algebra (Lie bracket) is given by

$$ad(\mathbf{s}_1)\mathbf{s}_2=[\mathbf{s}_1,\mathbf{s}_2]=\begin{pmatrix}\boldsymbol{\omega}_1\times\boldsymbol{\omega}_2\\\boldsymbol{\omega}_1\times\mathbf{v}_2-\boldsymbol{\omega}_2\times\mathbf{v}_1\end{pmatrix}\tag{12}$$

As for $\mathfrak{so}(3)$, the complex adjoint representation of $\mathfrak{se}(3)$ can be diagonalized. A suitable basis is given by

$$J_3 = e_z, \quad J_+ = e_y + ie_x, \quad J_- = e_y - ie_x$$
$$P_3 = e'_z, \quad P_+ = e'_y + ie'_x, \quad P_- = e'_y - ie'_x$$

The commutation relations include $[J_+, P_3] = -iP_+$, $[J_-, P_3] = iP_-$. It follows that an $\mathfrak{so}(3)$ invariant of a representation is an $\mathfrak{se}(3)$ invariant provided it is killed by P_3, since these commutation relations then insure it is also killed by $P_\pm$.

The adjoint action of P_3 is given by

$$P_3(e_+) = ie'_+,\ P_3(e_0) = 0,\ P_3(e_-) = -ie'_-,\ P_3(e'_+) = P_3(e'_0) = P_3(e'_-) = 0. \tag{13}$$

As before, we select eigenvectors e_+, e_0, e_- of J_3 (with respect to its adjoint action) and similarly e'_+, e'_0, e'_- for P_3, where $J_+(e_+) = J_+(e'_+) = 0$ and J_- acts as a ladder operator down each sequence. Likewise there is a dual basis $f_+, f_0, f_-, f'_+, f'_0, f'_-$ with $f_i(e_{-j}) = f'_i(e'_{-j}) = \delta_{ij}$, $i, j = +, 0, -$. The adjoint and coadjoint actions are equivalent under the isomorphism between $\mathfrak{se}(3)$ and $\mathfrak{se}(3)^*$ that takes $e_i \mapsto f'_i$, $e'_i \mapsto -f_i$. The coadjoint action of P_3 is then

$$P_3(f_+) = P_3(f_0) = P_3(f_-) = 0,\ P_3(f'_+) = -if_+,\ P_3(f'_0) = 0,\ P_3(f'_-) = if_-. \tag{14}$$

It is clear from (11) and (12) that the adjoint representation contains a subrepresentation of $SO(3)$ (equivalently $\mathfrak{so}(3)$), corresponding to two copies of its standard representation $L_1 \oplus L_1$. Any $SE(3)$ invariant must be an invariant of the $SO(3)$ subrepresentation. Evaluating the Hilbert series using the Molien–Weyl Theorem gives $1/(1-t^2)^3$, so there are three primary quadratic invariants. Combining the identities (2) and (8) we have

$$\mathrm{S}^2(L_1 \oplus L_1) \cong \mathrm{S}^2 L_1 \oplus (L_2 \oplus L_1 \oplus L_0) \oplus \mathrm{S}^2 L_1$$

Since the coadjoint representation is also L_1, trivial components correspond to polynomial invariants. The first and last components give rise to the basic invariants of each copy of L_1, that is $\boldsymbol{\omega}.\boldsymbol{\omega}$ and $\mathbf{v}.\mathbf{v}$ (in terms of coordinates), while the trivial component in the middle term corresponds to $\boldsymbol{\omega}.\mathbf{v}$. These are the 2-fold joint invariants of the standard representation [5], Appendix F. In terms of the dual basis the invariants are

$$f_0^2 - 2f_+f_-, \quad f_0'^2 - 2f'_+f'_-, \quad f_0f'_0 - f_+f'_- - f_-f'_+.$$

It is clear, in this form, that each of the invariants has eigenvalue 0.

We now determine which of these invariants is additionally preserved by the action of the translation subgroup. Directly, the subgroup action is given by:

$$\begin{pmatrix} I & 0 \\ T & I \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} + \mathbf{t} \times \boldsymbol{\omega} \end{pmatrix} \tag{15}$$

Clearly $\boldsymbol{\omega}.\boldsymbol{\omega}$, $\boldsymbol{\omega}.\mathbf{v}$ (the Killing and Klein forms respectively) remain invariant, while $\mathbf{v}.\mathbf{v}$ does not. Alternatively, we can apply P_3 to each form using (14) to get the same result.

We conclude by giving a proof of the standard fact that these two forms generate the ring of invariant polynomials.

Theorem 3.1. *Every polynomial invariant of the adjoint action of $SE(3)$ belongs to $k[\boldsymbol{\omega}.\boldsymbol{\omega}, \boldsymbol{\omega}.\mathbf{v}]$.*

Proof. Suppose $F(\boldsymbol{\omega}, \mathbf{v})$ is an $SE(3)$ invariant polynomial of degree $2k$, then it is an invariant for the subrepresentation of $SO(3)$ and hence can be written as a polynomial G in $\boldsymbol{\omega}.\boldsymbol{\omega}$, $\boldsymbol{\omega}.\mathbf{v}$, $\mathbf{v}.\mathbf{v}$. Write this in the form

$$F(\boldsymbol{\omega}, \mathbf{v}) = G(\boldsymbol{\omega}.\boldsymbol{\omega}, \boldsymbol{\omega}.\mathbf{v}, \mathbf{v}.\mathbf{v}) = \sum_{r=0}^{k} g_r(\boldsymbol{\omega}.\boldsymbol{\omega}, \boldsymbol{\omega}.\mathbf{v})(\mathbf{v}.\mathbf{v})^r$$

where g_r is a polynomial of degree $\leq k - r$. Under the action of a translation (15) the g_r are fixed. For any choice of $\boldsymbol{\omega}, \mathbf{v}$ in the Zariski open dense subset U defined by $\boldsymbol{\omega} \neq 0$, let $g_r(\boldsymbol{\omega}.\boldsymbol{\omega}, \boldsymbol{\omega}.\mathbf{v}) = \gamma_r$ and note that $\mathbf{v}.\mathbf{v}$ can take arbitrary non-negative values $(\mathbf{v} + \mathbf{t} \times \boldsymbol{\omega}).(\mathbf{v} + \mathbf{t} \times \boldsymbol{\omega})$ under the action of elements $\mathbf{t} \in \mathfrak{t}(3)$. But the polynomial $\sum_{r=0}^{k} \gamma_r(\mathbf{v}.\mathbf{v})^r$ must be constant so $\gamma_r = 0$ for $r \geq 1$. It follows that $F(\boldsymbol{\omega}, \mathbf{v}) = g_0(\boldsymbol{\omega}.\boldsymbol{\omega}, \boldsymbol{\omega}.\mathbf{v})$ on U and hence everywhere. □

4 Invariants of Alternating 2–Forms

The set of 2–systems may be regarded as a subvariety of the 2-fold alternating power of the adjoint representation of $SE(3)$. Hence, to determine screw system invariants we will want first to consider the symmetric powers $\mathrm{S}^k(\bigwedge^2 \mathrm{Ad})^*$. The classical description of $\bigwedge^2(\mathrm{Ad})^*$ is in terms of its Plücker coordinates, used for describing Grassmannian varieties and therefore ideally adapted for screw systems. Label the standard basis vectors $e_x, e_y, e_z, e'_x, e'_y, e'_z$ as $e_1, \ldots, e_6$, then for each $1 \leq i < j \leq 6$, define p_{ij} to be the coordinate of $e_i \wedge e_j$ in the basis of 2–forms for $\bigwedge^2(\mathrm{Ad})$. In other words, $p_{ij} = f_i \wedge f_j$ in the obvious notation for the canonical dual basis.

Let P denote the antisymmetric matrix whose upper triangular entries are the Plücker coordinates p_{ij}. The action induced from the adjoint action for an element $g \in SE(3)$ with $\Omega = \mathrm{Ad}(g)$ written in the form (11) is $P \mapsto \Omega P \Omega^t$. The corresponding $\mathfrak{se}(3)$ action is

$$P \mapsto \Phi P + P\Phi^t, \qquad \Phi \in \mathfrak{se}(3) \tag{16}$$

The aim of this section is to identify $SE(3)$ invariants of the alternating 2–form representation using a variety of approaches.

4.1 Invariants via the Killing and Klein forms

Selig [19] adapts the method of von Mises [24] to directly find invariants as coefficients of the determinant of a pencil of quadratics constructed from the basic Killing and Klein forms of Section 3. Explicitly, these quadratic forms can be represented by 6×6 symmetric matrices

$$Q_0 = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}, \qquad Q_\infty = \begin{pmatrix} -2I_3 & 0 \\ 0 & 0 \end{pmatrix}$$

Given a 2–system $\mathcal{S}$, let $\mathbf{s}_1, \mathbf{s}_2$ be basis of screws. Then we may form 2×2 symmetric matrices

$$\Upsilon_\epsilon = (\mathbf{s}_i^t Q_\epsilon \mathbf{s}_j), \quad \epsilon = 0, \infty$$

Suppose that

$$\det(\alpha\Upsilon_0 + \beta\Upsilon_\infty) = \alpha^2 j_1 + \alpha\beta j_2 + \beta^2 j_3$$

then the expressions j_1, j_2, j_3, which are functions of the coordinates of the $\mathbf{s}_k$, are invariants of pairs of screws under the adjoint action of $SE(3)$. Moreover they are relative invariants of the action on 2–screws, allowing for a change of basis. The same invariants may also be found from the coefficients of

$$\det(A - \lambda Q_0 - \mu Q_\infty^*)$$

where $Q_\infty^* = Q_0 Q_\infty Q_0$ is the dual of Q_0 and $A = \mathbf{s}_1\mathbf{s}_2^t - \mathbf{s}_2\mathbf{s}_1^t$. (This differs slightly from [19].) These three invariants are sufficient to distinguish all the broad classes of 2–systems in the Hunt–Gibson classification except IIB and IIC, though not the orbits. Selig gives explicit formulae in terms of the Plücker coordinates.

4.2 $SO(3)$ invariants

In view of the Section 3, an alternative approach would be to find first $SO(3)$ invariants for the representation $\bigwedge^2(L_1 \oplus L_1)$. Since $\dim(L_1 \oplus L_1) = 6$, this is a 15–dimensional space. From the identity (2) and $\bigwedge^2 L_1 \cong L_1$,

$$\begin{aligned} \bigwedge^2(L_1 \oplus L_1) &= (\bigwedge^2 L_1 \otimes \bigwedge^0 L_1) \oplus (\bigwedge^1 L_1 \otimes \bigwedge^1 L_1) \oplus (\bigwedge^0 L_1 \otimes \bigwedge^2 L_1) \\ &= L_1 \oplus (L_2 \oplus L_1 \oplus L_0) \oplus L_1. \end{aligned} \tag{17}$$

The decomposition tells us that the roots are $\pm 2i$ once each, $\pm i$ four times, and 0 five times. The Molien–Weyl Theorem can be used to determine the Hilbert series:

$$\begin{aligned} H(k[\bigwedge^2(L_1 \oplus L_1)]^{SO(3)}, t) &= -\frac{1}{4\pi i} \oint_{|z|=1} \frac{(1-z)^2\, dz}{z^2(1-t)^5(1-tz)^4(1-tz^{-1})^4(1-tz^2)(1-tz^{-2})} \\ &= -\frac{1}{4\pi i(1-t)^5} \oint_{|z|=1} \frac{z^4(1-z)^2\, dz}{(1-tz)^4(z-t)^4(1-tz^2)(z^2-t)} \end{aligned} \tag{18}$$

Maple was used to evaluate and sum the residues at $z = t, \pm\sqrt{t}$ and hence, after some manipulation of the denominator to express it in a correct form, to obtain a rational form and its series expansion:

$$\begin{aligned} &\frac{1 + 4t^3 + 14t^4 + 8t^5 + 4t^6 + 8t^7 + 14t^8 + 4t^9 + t^{12}}{(1-t)(1-t^2)^7(1-t^3)^4} \\ =&1 + t + 8t^2 + 16t^3 + 58t^4 + 122t^5 + 334t^6 + 678t^7 + 1536t^8 + 2960t^9 \\ &+ 5932t^{10} + 10772t^{11} + 19820t^{12} + O(t^{13}). \end{aligned} \tag{19}$$

The rational form indicates that there are 12 primary and ≥ 57 secondary invariants (though not necessarily of the degrees indicated by the rational form) while the coefficients of the series expansion give definitive dimensions for the space of homogeneous invariants in each degree.

4.3 Invariants via explicit decomposition of symmetric powers

A basis for $\bigwedge^2(L_1 \oplus L_1)^*$ can be constructed from the dual eigenvector bases for the two copies of L_1: f_+, f_0, f_- and f'_+, f'_0, f'_-. Explicitly, denote the basis elements by

$$\begin{array}{lll} a_+ = f_+ \wedge f_0, & a_0 = f_+ \wedge f_-, & a_- = f_0 \wedge f_0 \\ b_{++} = f_+ \wedge f'_+, & b_{+0} = f_+ \wedge f'_0, & b_{+-} = f_+ \wedge f'_- \\ b_{0+} = f_0 \wedge f'_+, & b_{00} = f_0 \wedge f'_0, & b_{0-} = f_0 \wedge f'_- \\ b_{-+} = f_- \wedge f'_+, & b_{-0} = f_- \wedge f'_0, & b_{--} = f_- \wedge f'_- \\ c_+ = f'_+ \wedge f'_0, & c_0 = f'_+ \wedge f'_-, & c_- = f'_0 \wedge f'_0 \end{array} \tag{20}$$

It is straightforward to determine the action of elements of $\mathfrak{se}(3)$ on this basis. However the basis elements b_{ij} do not respect the decomposition into irreducible components. Such a basis can be found by choosing a highest weight eigenvector in each component and applying the ladder operator J_- as given in Table 1. Conversions between expressions in terms of a_i, b_{ij}, c_i and those in terms of Plücker coordinates can be made using (5).

eigenvalue	L_1	L'_2	L'_1	L'_0	L''_1
$2i$		b_{++}			
i	a_+	$b_{+0} + b_{0+}$	$b_{+0} - b_{0+}$		c_+
0	a_0	$b_{00} + b_{+-} + b_{-+}$	$b_{+-} - b_{-+}$	$b_{00} - b_{+-} - b_{-+}$	c_0
$-i$	a_-	$3b_{0-} + 3b_{-0}$	$b_{0-} - b_{-0}$		c_-
$-2i$		$6b_{--}$			

Table 1: Generators for the components of $\bigwedge^2(L_1 \oplus L_1)$

The decomposition (17) shows that there is a linear $SO(3)$ invariant. From Table 1, we see that this is

$$\begin{aligned} i_1 = b_{00} - b_{+-} - b_{-+} \\ &= (-2ie_z) \wedge (-2ie'_z) - (ie_x + e_y) \wedge (-2ie'_x + 2e'_y) - (-2ie_x + 2e_y) \wedge (ie'_x + e'_y) \\ &= -4(e_x \wedge e'_x + e_y \wedge e'_y + e_z \wedge e'_z) \\ &= p_{14} + p_{25} + p_{36} \end{aligned}$$

In principle, the Theorem 2.2(c) together with the decomposition (17) could be used to give more precise information about where the invariants come from. However it has not, to date, proved possible to obtain a usable form of the series using Maple. It is possible to look for higher degree invariants by decomposing the symmetric powers $\mathrm{S}^d(\bigwedge^2(L_1 \oplus L_1))$. In general:

$$\begin{aligned}\mathrm{S}^d \textstyle\bigwedge^2(L_1 \oplus L_1) &\cong \mathrm{S}^d(L_1 \oplus (L_2 \oplus L_1 \oplus L_0) \oplus L_1) \\ &= \bigoplus_{\sum_{j=1}^{5} i_j = d} \mathrm{S}^{i_1} L_1 \otimes \mathrm{S}^{i_2} L_2 \otimes \mathrm{S}^{i_3} L_1 \otimes \mathrm{S}^{i_4} L_0 \otimes \mathrm{S}^{i_5} L_1 \end{aligned} \tag{21}$$

The decomposition of the symmetric powers of L_1 was noted above (6). We are not aware of a comparable formula for L_2 but explicit enumeration in terms of a basis for L_2 gives:

$$\begin{aligned}\mathrm{S}^2 L_2 &\cong L_4 \oplus L_2 \oplus L_0 \\ \mathrm{S}^3 L_2 &\cong L_6 \oplus L_4 \oplus L_3 \oplus L_2 \oplus L_0 \\ \mathrm{S}^4 L_2 &\cong L_8 \oplus L_6 \oplus L_5 \oplus 2L_4 \oplus 2L_2 \oplus L_0 \\ \mathrm{S}^5 L_2 &\cong L_{10} \oplus L_8 \oplus L_7 \oplus 2L_6 \oplus L_5 \oplus 2L_4 \oplus L_3 \oplus 2L_2 \oplus L_0 \\ \mathrm{S}^6 L_2 &\cong L_{12} \oplus L_{10} \oplus L_9 \oplus 2L_8 \oplus L_7 \oplus 3L_6 \oplus L_5 \oplus 3L_4 \oplus L_3 \oplus 2L_2 \oplus 2L_0 \end{aligned} \tag{22}$$

So for $d = 2$, equations (8), (21) and (22) give:

$$\begin{aligned}\mathrm{S}^2 \textstyle\bigwedge^2(L_1 \oplus L_1) &\cong \mathrm{S}^2 L_2 \oplus 3\mathrm{S}^2 L_1 + 3(\mathrm{S}^1 L_1 \otimes \mathrm{S}^1 L_1) + \mathrm{S}^2 L_0 \\ &\cong (L_4 \oplus L_2 \oplus L_0) + 3(L_2 \oplus L_0) + 3(L_2 \oplus L_1 \oplus L_0) \oplus L_0\end{aligned}$$

from which it is evident that there are exactly 8 quadratic invariants, one of which, corresponding to the final component, is i_1^2. The remaining 7 can be found by taking linear combinations $\sum a_i \beta_i$ of appropriate subsets of the basis elements from Table 1 and solving for $J_-(\sum a_i \beta_i) = 0$. This gives:

$$\begin{aligned} i_2 &= a_0^2 - 2a_+ a_- \\ i_3 &= (b_{+-} - b_{-+})^2 - 2(b_{+0} - b_{0+})(b_{0-} - b_{-0}) \\ i_4 &= c_0^2 - 2c_+ c_- \\ i_5 &= a_0(b_{+-} - b_{-+}) - a_-(b_{+0} - b_{0+}) - a_+(b_{0-} - b_{-0}) \\ i_6 &= a_0 c_0 - a_+ c_- - a_- c_+ \\ i_7 &= (b_{+-} - b_{-+})c_0 - (b_{+0} - b_{0+})c_- - (b_{0-} - b_{-0})c_+ \\ i_8 &= (2b_{00} + b_{+-} + b_{-+})^2 - 6(b_{+0} + b_{0+})(b_{0-} + b_{-0}) + 12 b_{++} b_{--} \end{aligned} \tag{23}$$

A similar decomposition for $\mathrm{S}^3\bigwedge^2(L_1 \oplus L_1)$ shows that it has 16 trivial factors. Since there are 8 (i_1^3 and $i_1 i_m$, $m = 2, \ldots, 8$) derived from lower degree invariants, this concurs with there being 4 primary and 4 secondary cubic invariants.

However, the decomposition is *not* preserved by the action of the translations in $SE(3)$. Using the additional elements P_+, P_3, P_- in $\mathfrak{se}(3)$ (sending e_is to e'_js), clearly the components L''_1, $(L'_2 \oplus L'_1 \oplus L'_0) \oplus L''_1$ are invariant but do not have invariant complements. In particular, there is no linear $SE(3)$ invariant. Invariants of $SE(3)$ are explored further below.

4.4 Invariants via characteristics of the Plücker matrix

Partition P using 3×3 matrices A, B, C with A, C antisymmetric—this reflects the decomposition (17). Then the actions of the rotations and translations are given by:

$$\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} \begin{pmatrix} R^t & 0 \\ 0 & R^t \end{pmatrix} = \begin{pmatrix} RAR^t & RBR^t \\ -RB^tR^t & RCR^t \end{pmatrix} \tag{24}$$

$$\begin{pmatrix} I & 0 \\ T & I \end{pmatrix} \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} \begin{pmatrix} I & -T \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & -AT + B \\ TA - B^t & -TAT + B^tT + TB + C \end{pmatrix} \tag{25}$$

It is clear from (24), using the defining property $RR^t = I$, that we can derive $SO(3)$ polynomial invariants from the coefficients of the characteristic polynomials of A, B, C and their combinations. The following procedure is followed at each degree d:

(a) determine the set Λ_d of all degree d products of invariants of degree $< d$

(b) determine any syzygies (linear dependencies) among these invariant products; in view of the theoretical form (9) this is only required for d even

(c) determine any linear dependencies among the syzygies and hence the dimension λ_d of the space of invariants spanned by Λ_d

(d) the difference between the coefficient $\dim k[V]_d^G$ of the Hilbert series and λ_d is the number of new invariants of degree D required; search for these among the coefficients of characteristic polynomials of products of $X = A, B, B^t, C$. Note that by the Cayley–Hamilton Theorem, any invariant involving X^3 can be written in terms of lower degree invariants.

Explicitly we can generate the following linear and quadratic invariants (where χ_1 denotes the coefficient of the linear term in the characteristic polynomial and tr the trace):

$$
\begin{aligned}
\mathcal{I}_1 = \quad & \mathrm{tr}\,(B) && = p_{14} + p_{25} + p_{36} \\
\mathcal{I}_2 = \quad & \chi_1(A) && = p_{12}^2 + p_{13}^2 + p_{23}^2 \\
\mathcal{I}_3 = \quad & \chi_1(B) && = p_{14}p_{25} + p_{14}p_{36} + p_{25}p_{36} - p_{26}p_{35} - p_{24}p_{15} - p_{34}p_{16} \\
\mathcal{I}_4 = \quad & \chi_1(C) && = p_{45}^2 + p_{46}^2 + p_{56}^2 \\
\mathcal{I}_5 = \quad & \mathrm{tr}\,(AB) && = p_{12}p_{24} + p_{13}p_{34} + p_{23}p_{35} - p_{12}p_{15} - p_{13}p_{16} - p_{23}p_{36} \\
\mathcal{I}_6 = \quad & \mathrm{tr}\,(AC) && = -2p_{12}p_{45} - 2p_{13}p_{46} - 2p_{23}p_{56} \\
\mathcal{I}_7 = \quad & \mathrm{tr}\,(BC) && = p_{24}p_{45} + p_{34}p_{46} + p_{35}p_{56} - p_{15}p_{45} - p_{16}p_{46} - p_{26}p_{56} \\
\mathcal{I}_8 = \quad & \mathrm{tr}\,(BB^t) && = p_{14}^2 + p_{15}^2 + p_{16}^2 + p_{24}^2 + p_{25}^2 + p_{26}^2 + p_{34}^2 + p_{35}^2 + p_{36}^2
\end{aligned}
\tag{26}
$$

To the quadratics we can add the following linearly independent higher-order invariants:

Degree 3 $\mathcal{I}_9 = \det B$,

$\mathcal{I}_{10} = \mathrm{tr}\,(AB^2),\ \mathcal{I}_{11} = \mathrm{tr}\,(ABC), \mathcal{I}_{12} = \mathrm{tr}\,(CBA),\ \mathcal{I}_{13} = \mathrm{tr}\,(B^2C)$

$\mathcal{I}_{14} = \mathrm{tr}\,(A^2B),\ \mathcal{I}_{15} = \mathrm{tr}\,(B^2B^t),\ \mathcal{I}_{16} = \mathrm{tr}\,(BC^2)$

Degree 4 $\mathcal{I}_{17} = \chi_1(AB),\ \mathcal{I}_{18} = \chi_1(BC),\ \mathcal{I}_{19} = \chi_1(BB^t)$

$\mathcal{I}_{20} = \mathrm{tr}\,(A^2BB^t),\ \mathcal{I}_{21} = \mathrm{tr}\,(BB^tC^2)$

$\mathcal{I}_{22} = \mathrm{tr}\,(AB^2B^t),\ \mathcal{I}_{23} = \mathrm{tr}\,(ABB^tB),\ \mathcal{I}_{24} = \mathrm{tr}\,(B^2B^tC),\ \mathcal{I}_{25} = \mathrm{tr}\,(BB^tBC)$

$\mathcal{I}_{26} = \mathrm{tr}\,(A^2BC),\ \mathcal{I}_{27} = \mathrm{tr}\,(ABC^2)$

$\mathcal{I}_{28} = \mathrm{tr}\,(ABB^tC),\ \mathcal{I}_{29} = \mathrm{tr}\,(ABCB),\ \mathcal{I}_{30} = \mathrm{tr}\,(ABCB^t)$

Degree 5 $\mathcal{I}_{31} = \mathrm{tr}\,(AB^2B^tA),\ \mathcal{I}_{32} = \mathrm{tr}\,(CB^2B^tC),\ \mathcal{I}_{33} = \mathrm{tr}\,(AB^2B^tB)$

$\mathcal{I}_{34} = \mathrm{tr}\,(B^2B^tBC),\ \mathcal{I}_{35} = \mathrm{tr}\,(AB^2B^tC),\ \mathcal{I}_{36} = \mathrm{tr}\,(ABB^tBC)$

$\mathcal{I}_{37} = \mathrm{tr}\,(ABABC),\ \mathcal{I}_{38} = \mathrm{tr}\,(ABCBC)$

Note that there are no syzygies of degree 4 so that the linear and quadratic invariants are all primary. However for $d = 6$, there are 340 invariants generated by products of I_1–I_{38}, and step (b) in the procedure reveals 16 syzygies with one linear relation among them, so that $\lambda_6 = 340 - 16 + 1 = 325$. Therefore, one would expect from the Molien–Weyl series (19) to find a further 9 new invariants of degree 6.

However it should be noted that a slightly different approach yielded an apparently incompatible result. By building the invariants from traces of products alone, a similar set of 38 independent invariants was found, with 16 syzygies having *no* linear relations among them. This conflict remains to be resolved.

4.5 $SE(3)$ invariants

It is now possible to determine which of the $SO(3)$ invariants are also $SE(3)$ invariants, by applying P_3 as a filter. There is of course a relation between the characteristic polynomial invariants of Section 4.4 and those found by direct decomposition in Section 4.3. Explicitly, the invariants $i_1, \ldots, i_8$ are, up to scalar multiples, respectively:

$$\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_8 + 2\mathcal{I}_3 + \mathcal{I}_1^2, \mathcal{I}_4, \mathcal{I}_5, \mathcal{I}_6, \mathcal{I}_7, \mathcal{I}_8 - 2\mathcal{I}_3 + \mathcal{I}_1^2$$

Similarly, the invariants can be related to those of Selig in Section 4.1, which are already known to be $SE(3)$ invariants, as follows:

$$j_1 = 2\mathcal{I}_3 - 2\mathcal{I}_6 - \mathcal{I}_1^2, \quad j_2 = 4\mathcal{I}_5, \quad j_3 = -2\mathcal{I}_2.$$

This shows that some care is needed in determining which $SO(3)$ invariants are$SE(3)$ invariants—they may arise as linear combinations of a given basis of $SO(3)$ invariants of a fixed degree.

Application of the P_3 filter suggest that there are in fact 4 quadratic $SE(3)$ invariants. Further work is required to clarify these results.

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