Some Mathematical Problems in Robotics

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1 Introduction

1.1 Motivation

In May 2000 there was a meeting at the NSF in Arlington Virginia on "The Interplay between Mathematics and Robotics". The report of this meeting can be found at [1]. Many leading experts in the U.S. discussed the importance of Mathematics in Robotics and also the role that robotic problems could play in the development of Mathematics. The experts gave a broad overview of the problems they saw as important and worth studying. Their list was long and touched on many branches of Mathematics and many areas in Robotics.

This work is my contribution to the debate. Below I will outline four problems, or rather areas which I believe deserve further study. The focus is quite narrow since this is only intended as a personal opinion. The list is not intended as a list of 'millennium' problems more a list of my personal research problems. Hence, I would not be surprised if these problems are quite straightforward to someone with a good background in Mathematics.

1.2 History

Before beginning, it is worth recalling a little history. In modern times there has been a distinct separation between practical disciplines like Engineering and theoretical ones, in particular Mathematics. This was not always the case. Much modern Mathematics was developed in response to the practical needs of Engineers. In particular the beginnings of the modern theory of Algebraic Geometry seems to have been related in part at least to problems in the design of mechanisms. Famous Mathematicians such as Chebyshev, Koenigs, Darboux, Sylvester and many other worked in this area and wrote about the curves described by mechanical linkages. Today's Mechanical Engineers do not know much about these researches but the Mathematicians have forgotten about them almost completely.

Another example comes from control theory. Almost all introductory texts on control theory credit James Clarke Maxwell with writing the first paper in the subject, on the control of the Watt Steam Governor. But few realise that this paper treats the system as a non-linear system and describes it using elliptic functions. On the other hand no modern mathematical text on elliptic functions reports this application, possibly because applications are hardly ever given in such books.

One possible explanation for this separation is the growth of Applied Mathematics as a subject in itself. Pure Mathematicians can concentrate on the Mathematics and let the Applied mathematicians worry about possible applications. In most colleges and Universities Pure Mathematicians do not teach undergraduate Engineers. Modern Engineers don't usually have any other contact with Mathematicians and hence know as much about current research in Mathematics as any educated layman. At the same time, Applied Mathematicians do not see it as their function to bridge this gap. They have their own discipline and follow their own research priorities.

These difficulties have been recognised in the UK by the EPSRC. In September 2001 a meeting was organised in Manchester to explore the possibility of collaboration between Mathematicians and Engineers. However, the topics suggested for collaboration seemed to be very heavily biased towards continuum mechanics.

1.3 Theoretical Robotics?

Robotics is a practical discipline. It grew out of Engineers ability to build very sophisticated machines which combine computer control with electro-mechanical actuators and sensors. Any theory in the subject must take account of what is practically possible with real machines. Nevertheless, I believe that there is a place for a theoretical side to the subject.

Of course, by definition theory is always useless, otherwise it wouldn't be theory! But surely all disciplines recognise the need for sound theoretical underpinnings. The question really is whether the theoretical underpinnings of Robotics are distinct or just a part of the general theory used in the disciplines that make up Robotics. One cannot sensibly separate say, a theory of Robot mechanisms from the general theory of mechanisms. However, there is something special about Robotics and that is the central importance of the group of rigid body motions $SE(3)$. That is not to say that theory not involving this group is not Robotics nor that other disciplines can't profitably use this group. Its just that I see this as a major theme running through much of Robotics: The links of a robot are not really rigid, but to a first approximation they are. The motions allowed by the joints of the robot are rigid body motions. The payload carried by the robot's end-effector is more often than not a rigid body. Standard analysis of the kinematics, dynamics and control of these robots all reflect this rigid body approach. In Robot vision a central problem is to find the rigid motion undergone by the camera using information derived from the images.

With all this in mind I offer the following problems.

2 Geometry of Curves in $SE(3)$

This problem concerns the differential geometry of curves in the group of rigid-body motions, $SE(3)$. In particular, what are the differential invariants for such curves? For a curve in space the speed, curvature and torsion are differential invariants. Given these three functions, or more usually, just the curvature and torsion the speed being assumed to be unity, it is in principle possible to reconstruct the curve uniquely up to an overall rigid transformation. What are the equivalent functions for curves in the group?

There are many potential applications for this. In robotics the trajectory of the robot's end-effector is a curve in $SE(3)$, people often want to know what is the 'best' curve for the robot to choose. There is some existing work on this, an early suggestion was, that given a pair of positions (with orientations) for the start and finish of a move, there is almost always a unique screw motion which transforms one position into the other, that is a rotation about a fixed axis followed by a translation in the direction of the same axis. A more sophisticated approach has been to move the robot so that the acceleration or jerk is minimised. There seems to be some evidence that humans and animals move in such a way as to minimise jerk, which is roughly the third derivative of position, see [2]. The curves with stationary arc-length are called geodesics. There are no bi-invariant positive definite metrics on the group but there are indefinite ones. If we use one of these bi-invariant metrics then the geodesics turn out to be the one-parameter subgroups, the screw motions again. On the other hand we could use a positive definite metric that was only right-invariant with respect to the group action. In this case the geodesics are the solutions to the equations of motion of a rigid body with no external forces, the metric is the inertia matrix for the body. What are the differential invariants for these curves? It may not be possible to solve these equations and find closed form expressions for the curves, but it may be possible to say something about the differential invariants. Moreover, there will be special points on these curves defined by the invariants. For example, on plane curves we have inflections and vertices, these are characterised by the curvature function vanishing or having a stationary point respectively.

Suppose that a desired trajectory for a 6R robot's end-effector is given, how can we control the robot to move along this curve? When the desired trajectory is a screw motion it is possible to design a non-linear feedback control law such that the closed-loop dynamics of the whole system are the geodesic equations, see [3]. Using this law the robot's end-effector cannot help but follow a screw motion. Notice that such a scheme doesn't need to compute the inverse kinematics of the robot and hence could be applied to designs of robot for which the inverse kinematics are not known in closed form. As we saw above these screw paths are not very practical so can these ideas be extended to stationary acceleration and stationary jerk paths? More generally we might also require this type of control along general paths specified by their invariants.

Another application of these ideas is to interpolation. Suppose we wanted to simulate numerically the dynamics of a robot or other system of rigid-bodies, perhaps as part of the robot's control system or simply to create a realistic animation. Rather than solve the dynamic equations at every time-step we could solve at "key-frames" and then interpolate between the key frames ("inbetweening" in the jargon of animation). There have been several schemes suggested for this, unfortunately most of them seem to be as computationally intensive as solving the dynamic equations at every step. Assuming simple forms for the invariants provide a basis for a simple solution.

These ideas even extend to more abstract numerical methods, interpolation on a Stiefel manifold has been investigated in connection with solutions of problems with orthogonality constraints. That is finding eigenvalues of symmetric matrices.

The final application I want to mention here is to elastic rods and beams. These are often used in robotics to simulate the links/members of the robot which are always slightly compliant. There is also a growing area of of continuum robotics, that is machines with a flexible spine controlled by tensioning cables. These are tentacle, serpentine or elephant's trunk type robots. At first sight there doesn't seem to much connection with the curves in $SE(3)$ discussed above, but the standard theory of these slender rods assumes that successive cross-sections along the beam only undergo rigid transformations. Hence the deformed state of the beam can be described by curve in the group $SE(3)$, the parameter along the curve now is the length along the beam. In fact under a static load the equations for the deflection of such a beam are precisely the equations of motion for a rigid body, with external forces this time. The inertia of the rigid body corresponds to the stiffness matrix of the beam, more precisely the stiffness density. And the time corresponds to the arc-length along the beam. In other words, the differential invariants might even have something to contribute to classical beam theory.

One way of approaching the problem of determining the differential invariants might be to look at the contact between the curves and one-parameter subgroups, (and their translates). The guess is that the geodesic curves will have invariants which are constant. As supporting evidence for this consider space curves again. The curves with constant curvature and torsion are helices. To generate a helix, choose a point in space and sweep it around a one-parameter subgroup, a screw motion. Now turn to the theory of ruled surfaces, these are on-parameter families of lines. In other words curves in the space of all lines, this space of all possible lines in 3D is usually identified with the Klien quadric. For ruled surfaces there are three differential invariants, a kind of curvature called the distribution parameter and two sorts of torsion, see [4]. When these invariants are constants the surface is a ruled helichoid. To get such a surface, take a line in space and sweep it around a one-parameter subgroup.

3 Solvability of Robot Kinematics

It is well known in Robotics that only certain arrangements of joints in a six-joint serial manipulator have solvable inverse kinematics. If the joints are all revolute joints then if any three consecutive joints axes meet at a common point or are parallel then the inverse kinematics can be expressed in terms of inverse sines and inverse cosines of the joint angles. The first part of this result is due to Peiper [5] and the part about the parallel axes seems to be due to Duffy [6].

Many people wonder if there are other geometric conditions on the joint axes which lead to solvable inverse kinematics. However, a more interesting question is: is it possible to show that for a general (generic) arrangement of joints the inverse kinematics are not solvable?

This, of course, begs the question, what do we mean by solvable. One answer might be, solvable in the sense of Galois theory. That is, the inverse kinematics will be considered solvable if there are formulas for the sines and cosines of the joint angles which lie in an algebraic extension of the field determined by the design parameters.

If we want to include helical joints into the discussion then the above definition will not work. This is because the equation will contain the joint angles themselves as well as their sines and cosines. So a better definition might be something similar to the familiar 'integrable in terms of elementary functions'. This leads us to the realm of differential rings and ideals. This area has also been suggested for studying non-holonomic control problems, for instance trajectory planning for wheeled robots, see [7].

The kinematic equations for a 6 joint robot are usually a system of algebraic equations, or transcendental equations if helical joints are used. However it is not too difficult to turn these into a system of differential equations. There are probably many ways to do this but a simple method is to assume that the desired position and orientation of the end-effector is a member of a one-parameter subgroup and then differentiate with respect to the parameter. The result is a system of equations where the joint rates are multiplied by the Jacobian matrix of the manipulator to give a constant element in the Lie algebra of the group. Of course the Jacobian matrix will be a fairly complicated function of the joint variables making the equations non-linear.

Notice that the differential equations discussed above give a good method of solving the inverse kinematics numerically. In most problems where the inverse kinematics is required the robot is in some particular configuration and needs to move a nearby configuration. If we set up the problem as an initial value problem starting in the known present configuration then standard numerical methods could be used to find the joint variables for the target configuration. In this way, so long as we avoid singularities, we expect a unique answer. So this method circumvents a lot of the usual problems arising from the multiple solutions to the algebraic equations. Newton-Raphson is the numerical method most commonly chosen for such problems but this would require evaluating the Jacobian matrix at configurations between the time-steps used. So perhaps a predictor-corrector method could be used, then a good estimate of the Jacobian will be produced automatically for the end of one time-step that can be used at the beginning of the next step.

A better understanding of the structure of the differential ideal determined by the equations will lead to better, that is faster and more robust methods of computing the inverse kinematics. Even if the equations are not solvable there may be semi-numerical methods that work. It is even possible that such methods will be better than simply implementing the closed-form solutions for robots whose structure does have solvable inverse kinematics. Perhaps this could also be extended to the forward kinematics of parallel manipulators such as the Stewart platform.

4 Invariant Theory of $SE(3)$

The theory of invariants, that is algebraic invariants, has a very long and distinguished history. Classically, the problem was to find invariants of polynomials or systems of polynomial in n-variables. These were invariants in the sense that they were unaffected by general linear coordinate changes in the n-variables. A modern interpretation would be that we have a linear action of a Lie group on a vector space, a representation, and we seek invariant functions in the coordinates of the vectors which are invariant with respect to the action of the group, see [8].

There is a large amount of modern invariant theory with very general and beautiful results. Unfortunately, most of it only applies to semi-simple groups and $SE(3)$ is not semi-simple.

Nevertheless, in his classic work [9] Weyl worked out the invariant theory for the Euclidean group in any dimension. However, this was only done for the standard representation. For $SE(3)$ this would be the 4-dimensional representation that in robotics is called the homogeneous representation. More recently Donelan and Gibson [10] worked out the invariants for the adjoint representation of $SE(3)$.

In [11] von Mises claimed to have found a set of fifteen invariants that generate all the invariants of a 6×6 symmetric matrix. This would be the invariant theory for the symmetric square of the adjoint representation. These symmetric matrices can be thought of as stiffness matrices for a rigid body in equilibrium. Although I have no doubt that the claim is correct it would be nice to see a modern proof of this result. Some progress has been made in connecting these invariants with more prosaic properties of the stiffness matrix, see [12, 13]. One might hope that given the full set of these invariants it might be possible to reconstruct the stiffness matrix, up to an overall rigid transformation. That is, any pair of stiffness matrices which are not related by a rigid transformation should have different values for their invariants. Unfortunately this is not the case, however, the cases where it fails may be quite minor and quite degenerate. So there is some hope that for most practical stiffness matrices the invariants do indeed determine the structure of the stiffness matrix.

Screw systems have an important place in robotics and the theory of mechanisms. They relate to instantaneous freedoms and are also important when studying constrained motion. Essentially a screw system is a linear subspace in the Lie algebra of the group $SE(3)$. If we think of the elements of the group as points in a manifold the Lie algebra can be thought of as the tangent space at the identity element. An older way of looking at the Lie algebra is as the space of infinitesimal group elements. We can represent the Lie algebra elements as 6-dimensional vectors combining the angular velocity of the rigid body with a linear velocity. The Lie algebra has the structure of a vector space so we can look at linear subspaces. A one dimensional subspace is a 1-system and corresponds to a line through the origin in the Lie algebra. There is just one invariant for a 1-system, the pitch of the Lie algebra element or screw.

Two-systems correspond to planes in the Lie algebra. These planes can be coordinatised using Plücker coordinates and in turn we can think of these coordinates as elements of a 6×6 anti-symmetric matrix. Imitating von Mises' construction, we can produce several invariants of the 2-system. It is not known if the invariants produced in this way generate all the invariants of the 2-systems.

There is a classification of all possible screw systems due to Gibson and Hunt [14]. But given an arbitrary screws system it is quite a long process to identify the Gibson-Hunt type of the system. It should be possible to do this quite easily by evaluating the invariants of the system and comparing the results to those for the normal forms given by Gibson and Hunt. So we need to know the invariants for the 3, 4 and 5-systems. In terms of the representation theory, we are seeking the invariants for the exterior powers of the adjoint representation of the group.

The dual of the Lie algebra is the space of wrenches. These are also 6-dimensional vectors, this time composed of the torque and the force acting on a rigid body. The representation of the group on this six dimensional vector space is called the coadjoint representation. In robotics it is common to think of the forces exerted by the fingers in a gripper as wrenches. Given a multi-fingered gripper and an object what is the best grasp? Attempts at answering this question have introduced grasp metrics, these are functions of the finger wrenches. Unfortunately most of the grasp metrics studied have not been invariant, hence an arbitrary change of coordinates can affect the result of an optimisation procedure to find the best grasp. A sensible grasp metric must be an invariant for the coadjoint representation. The fingers will have to supply different forces in order to balance different loads on the grasped object. So, the amplitudes of the finger forces should not affect the grasp metric, we are looking for a function that depends only on the relative positions and orientations of the fingers. We are really seeking some kind of projective invariant, perhaps simply the ratio of a pair of straightforward invariants. Moreover, this grasp metric, if it exists, should generalise to different numbers of fingers.

Although invariants have been mentioned above the covariants are important too. Much of the work on invariants may be expressed much more succinctly if we also consider the covariants.

5 Statistics on $SE(3)$

A large number of important problems in Robotics, both Robot vision and kinematics seem to come down to finding a rigid body transformation given some data describing its effects on points or lines and so forth. In many cases the geometrical problem is quite simple if the data is accurate. In reality the data comes from measurements and all measurements are subject to noise. The real question that these geometrical problems pose is thus: What is the best estimate of the rigid transformation given noisy data?

Here is just a small selection of problems which fall under this category.

For mobile robots there are several navigation problems. A common method of finding the position and orientation of a mobile robot is to set up some beacons in its work area. The robot has sensors which can locate and distinguish the beacons, for example sonar or radar sensors. In the simplest case these sensors will only measure the bearing of the beacons, that is the angle between a fixed direction on the robot and the line to beacon. It is reasonable to assume that the error in the measured angle obeys a von Mises' distribution. This distribution is defined on a circle, but has many properties in common with the Gaussian distribution on a line. The distribution has a single mode and a single parameter called the concentration parameter which determines the 'width' of the distribution. Given several measurements from the beacons how should the robot best estimate its position and orientation?

Another method of navigation for mobile robots is 'dead-reckoning' — measuring the movement of the wheels and then inferring the position of the robot. Here we might demand a model of how the errors in position and orientation develop over time. If this were known, it might be possible to predict when the robot will get lost! For this method of navigation it is usually necessary to find the position and orientation of the robot by a more accurate method every so often, a better understanding of the errors could lead to an optimisation of this procedure.

There are also related problems for industrial robot arms. The forward kinematics of the arm relates the angular positions at each of the joints to the position and orientation of the robot's end-effecter. Suppose that the errors at each joint are distributed according to a von Mises' distribution. What will the errors at the end-effector look like? That is, what is the distribution of errors for the position and orientation of the robot's end-effector? Clearly we need to convolve the von Mises' distributions in some sense, however it is not entirely obvious what to do since we expect the end-effector errors to die away very sharply outside the workspace of the robot. It is unlikely that we can find a definitive answer to this problem but it may be possible to find out enough about the error distribution to use in applications concerning the accuracy of these robots.

In robot control it is common to compare several different control laws and try to claim that one is more accurate than another. Since it is well known that there is no bi-invariant positive definite metric on the group these comparisons tend to be rather *ad hoc*. Often the comparisons are simply in terms of a particular set of coordinates. Given a particular probability distribution on the group of rigid body motions we could compare control laws which attempt to place the end-effector at a given position and orientation, we could even compare rates of convergence. The success of this approach would depend on there being 'obvious' choices for the probability distribution, other wise the free choice of coordinates is simply replaced with a free choice of distributions.

In robot vision it is common to try to estimate the rigid motion undergone by the camera from data about points in the image. If the absolute positions of some collection of points is known then this problem is almost identical to the beacon navigation problem above. The differences are, first that it is the position in 3D that we are trying to find here. Secondly, in the beacon navigation problem it seems quite natural to choose the von Mises' distribution for the errors in the bearing of the beacons. In 3D the bearing to a fixed point is given by two angles but is essentially a point on a 2-dimensional sphere. The analogue of the von Mises' distribution here is the Fisher distribution, but this might not be the appropriate distribution to choose, it depends on the main source of errors. Indeed when this problem has been considered in the robot vision literature the error distribution has usually been assumed to obey Gaussian statistics in the image plane.

If the position of the image point is not known beforehand then the problem changes radically in character. Suppose the camera undergoes a rigid transformation and we have images before and after the move. If points correspondences can be found, that it images points in both images which correspond to the same object point, then what is the best estimate for the rigid motion experienced by the camera? In fact, this problem cannot have a unique answer since a small rotation when the points are near the camera cannot be distinguished from a large rotation viewing distant points. Usually this difficulty is overcome by demanding that the translation part of the motion should have unit length. A better approach might be to work in a space where points correspond to equivalence classes of the solutions. Mathematically speaking we can quotient by the ambiguity.

The final example here is camera calibration. The general idea here is to deduce the camera parameters from information about the image points. The camera parameters are such things as scale factors for the x and y-directions and so forth. One model for the camera parameters represents them as a matrix of numbers belonging to the special linear group $SL(3)$. Once again, if the data is free from error then there are algorithms which can reconstruct the calibration matrix of the camera. When the data is noisy, however, it should be possible to estimate the calibration matrix.

The idea that unifies these examples is the notion that we should be considering probability distribution which are defined on a Lie group or possibly a quotient space of a group. In the mobile robot examples the group is $SE(2)$ the planar rigid motion group. For the robot arms and vision problems the group is $SE(3)$ the rigid motions in space. In the camera calibration problem the group could be $SL(3)$.

A large amount is known about distributions on $SO(2)$ since the manifold of this group is a circle. A fair amount is known about distribution on the rotation group $SO(3)$, the manifold in this case is the projective space $\mathbb{PR}^3$, this space is double covered by the 3-dimensional sphere. So any distribution on the 3-sphere that has antipodal symmetry can be considered as a distribution on $SO(3)$, see also [15].

I am not aware of any work directly related to distributions on $SE(3)$. However, since the manifold $SE(3)$ is the Cartesian product of $SO(3)$ with $\mathbb{R}^3$, the group of translations, we can construct distributions on $SE(3)$ as products of distributions on the components. Of course this is equivalent to treating the rotations and translations separately. This may not always be appropriate, in most problems we should expect that the errors in the rotation and translation are not independent.

It would be useful to be able to employ Bayesian methods in these problems. As usual we have the difficulty of choosing a prior probability on the group. Since the rotation group $SO(3)$ is a compact manifold there is a uniform distribution on this space. But $SE(3)$ is non-compact so has no uniform distribution. Using maximum the entropy principle is also problematic, I am not aware of any well-founded definitions of entropy on group manifolds, and again the non-compactness may be a problem here. Nevertheless there may be sensible choices for prior distributions on these spaces perhaps using group invariance to define them.

There are some advantages to working with these groups however. There is a well established theory of Fourier transforms on Lie groups. The groups we are interested in $SO(3), SE(3)$ and for the mobile robot $SE(2)$ are often standard examples used in text-books. Hence there is some hope that computations could be carried out in practice.

6 Concluding Remarks

I feel rather protective of these problems, there are essentially a list of my personal research problems. I'm not sure that I should publicise them since its very likely that other will solve most of them before I do. Then again there is more than enough to go around here, and of course it is not who solves them but the consequence of their solution that are important.

When all these problems have been solved the effect on the discipline of robotics may not be earth-shattering. Moreover, if the problems turn out to be good ones then they will give rise to more problems. However, I believe that working on these problems will produce a deeper understanding of what is possible and practical in robotics. When people come to build new machines in the future I hope that they can be guided by the knowledge generated by work on these problems. So I think the problems are important for robotics but as I said in the introduction I don't think they are very hard.

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