TITLE

THE DESIGN AND ANALYSIS OF RELIABLE COMMUNICATION NETWORKS

by

TREFOR EVANS B.A., C.ENG., M.I.MECH.E.

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Department of Mathematics and Computer Science,
Polytechnic of Wales,
Pontypridd,
Mid-Glamorgan.
CF37 1DL

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Certificate of Research

This is to certify that, except where specific reference is made, the work described in this thesis is the result of the investigation of the candidate.

T. Evans

Candidate

Supervisors

D. H. Smith

(Director of Studies)

R. Wiltshire

Declaration

This is to certify that an advanced course of reading has been undertaken and that this thesis has neither been presented, nor is being concurrently submitted, for any degree at any other academic or professional institution.

T. Evans

Candidate

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I would like to express my thanks to those who provided assistance during the course of this study particularly my supervisors Dr. D. H. Smith and Dr. R. J. Wiltshire for their valuable advice and guidance.

To all the above and for the understanding attitude of my family I am most grateful.

Abstract

The design and analysis of reliable communication networks

by T. Evans

In this study, a communication network is represented by a graph, and the problem of designing a network that will operate as reliably as possible is investigated. Failure of the network is associated with the removal of a set of nodes which disconnects the graph or by the removal of a set of edges which disconnects the graph. We are interested in finding reliable graphs for which the probability of disconnection is as small as possible.

We survey various reliability measures and then deal with the design of a reliable communication network based on the construction of graphs with the smallest number of minimum cut sets. The number of minimum size vertex cut sets may give a much better indication of the reliability of the graph than the connectivity alone, at least where the probability of failure of a vertex is close to 0. The determination of the number of minimum size vertex cut sets of such a graph is therefore of interest and we describe a construction of infinite **families** of such graphs in various cases. These cases are spread through the range $\frac{3}{8} \leq \frac{k}{|V|} < 1$, (where k = connectivity = degree of the graph, $|V|$ = th number of vertices of the graph).

Graphs with the smallest number of minimum cut sets are compared with other graphs of optimal connectivity, to assess their reliability when other values of the probability are considered. These values of the probability are; when the probability of failure of a vertex is close to 1, when the probability of failure of an edge is close to 0, and when the probability of failure of an edge is close to 1. In many cases the graphs proved to be highly reliable. Consideration is also given to the expected number of vertices disconnected from the largest component of a graph.

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CHAPTER 1

CHAPTER 1

Introduction

Many real world situations can conveniently be described by means of a diagram consisting of a set of points together with lines joining certain pairs of these points. Some practical illustrations of these diagrams are communication networks, transportation networks, and electrical networks. In a communication network for example the points might be communication centres with lines representing communication links. In such diagrams one is mainly interested in whether or not two given points are joined by a line; the manner in which they are joined is immaterial. A mathematical abstraction of situations of this type gives rise to the concept of a graph.

Discrete systems or organized collections of objects are frequently encountered, for instance in the networks mentioned above and graph theory provides simple techniques for constructing models of systems of this kind, and powerful methods for their analysis and optimization with respect to their ability to function as reliably as possible.

Technology today poses a great number of problems that require the construction of complex systems through specific arrangements of their components. The applications are numerous (for example, a railway network or a telephone network) and the reliability and availability of communication paths between all pairs of centres is a primary consideration in such applications.

In this thesis we shall be studying the reliability of communication networks but before discussing the problems to be dealt with we describe some of the notation and definitions from graph theory necessary to describe the mathematical model.

1.1 NOTATION

V	Vertex set.
E	Undirected edge set.
$G=G(V,E)$	Connected graph with vertex set V, and edge set E.
$\|B\|$	The number of elements in the set B.
e	Number of elements in E; $e=\|E\|$.
v	Number of elements in V; $v=\|V\|$.
$\rho(v)$	Degree of a vertex (or valency).
$k(G)$	Vertex connectivity.
$\lambda(G)$	Edge connectivity.
d	Diameter of a graph.
$\Gamma(v)$	Neighbour set of a vertex v.
$K_{\|V\|}$	Complete graph.
$G(V_1V_2)$	Bipartite graph.
$K_{\|V_1\|\|V_2\|}$	Complete bipartite graph.
T	Spanning tree.
$T_{\|V\|-1}$	Complexity of a graph.
ρ_m	**Minimum degree of a vertex.**
S_k	Number of vertex cut sets each with k vertices.
R_λ	Number of edge cut sets each with λ edges.

1.2 Definitions

A finite graph G consists of a finite set V of vertices v_1, v_2, v_3 ... v_n, together with a finite set E of unordered pairs of vertices. The elements of E are called edges.

If e is the edge containing vertices v_i and v_j, then we write $e = v_i v_j$ and say that v_i and v_j are adjacent and that vertex v_i and edge e are incident.

A subgraph of G consists of subsets of V and E which form a graph.

A spanning subgraph of G has the same set V of vertices as G.

A simple graph is a graph with no loops or multiple edges i.e. there are no edges joining vertices to themselves and there is at most one edge joining each pair of vertices.

A path joining two vertices v_i and v_j of G is the set of vertices and edges in a sequence of succeeding incident vertices and edges beginning with v_i and terminating with v_j, in which all vertices are distinct.

e.g. v_i, e_p, v_r ... e_s, v_j

The length of a path is the number of edges in it.

A circuit in a graph G is defined in the same way as a path except that the initial and terminal vertices coincide.

The complete graph $K_{|V|}$ with $|V|$ vertices has every two distinct vertices adjacent.

The distance between two vertices is the length of a shortest path joining them.

The diameter of a graph is the maximum distance between any two vertices.

The degree or valency of a vertex is the number of edges incident with that vertex.

A graph is regular of degree k if all vertices have the same degree k.

A graph is connected if there is a path existing between any two vertices.

A graph G is called k-connected if G has at least k+1 vertices and it is impossible to disconnect G by removing k-1 or fewer vertices.

The connectivity of G, is defined to be k if G is k-connected but not (k+1)-connected.

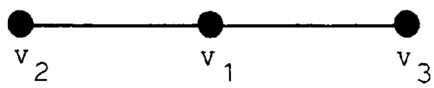
The vertex connectivity (or just connectivity) of a non-complete graph G is the minimum number of vertices whose removal together with the edges incident to those vertices results in the graph being disconnected.

The edge connectivity (or cohesion) of a graph G is the minimum number of edges whose removal results in a disconnected graph.

A spanning tree in G is an edge-subgraph of G which has $|V|-1$ edges and contains no circuits. In a spanning tree every vertex in G is incident with at least one edge of the tree.

The neighbour set of a vertex v is the set of vertices $\Gamma(v)$ such that all the vertices of $\Gamma(v)$ are adjacent to v.

A set $X \subset V$ is called a vertex cut set of G if G-X is disconnected.

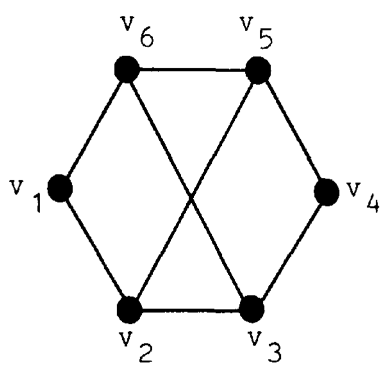


(a) connected graph

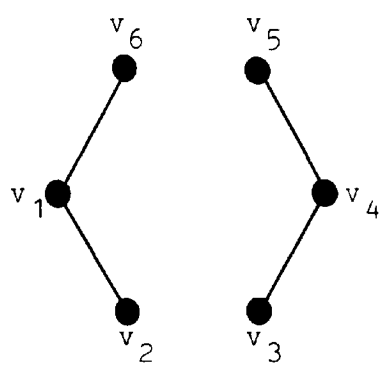
(b) disconnected graph

FIG. 1.1 The vertex cut set is $\{v_1\}$.

A set $Y \subset E$ is called an edge cut set of G if G-Y is disconnected.



(a) connected graph



(b) disconnected graph

FIG. 1.2 The edge cut set is $\{v_6v_5,\ v_6v_3,\ v_5v_2,\ v_2v_3\}$

A directed graph, G, consists of a set of vertices and a set of ordered pairs of vertices called directed edges. The edge may be represented by a line connecting vertices v_i and v_j, with an arrowhead pointing from v_i to v_j. FIG. 1.3.

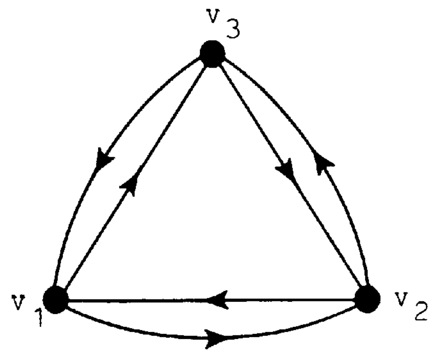


FIG. 1.3 A directed graph.

Thus our original definition of a graph can be referred to as an undirected graph. There sometimes exist 'mixed graphs'.

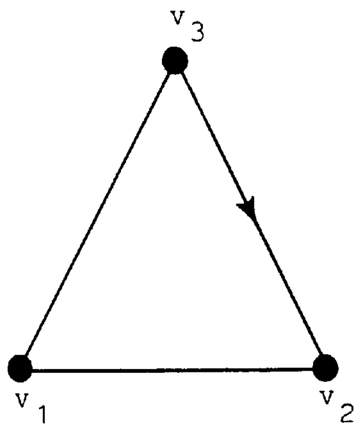


FIG. 1.4 A mixed graph.

A weighted graph, G, is a graph in which numbers, called weights, are associated with the edges or vertices.

The minimum length of any circuit in G is known as the "girth" of the graph and is denoted by g.

The adjacency matrix of a graph G is the $|V| \times |V|$ matrix $A(G)$ whose entries a_{ij} are given by

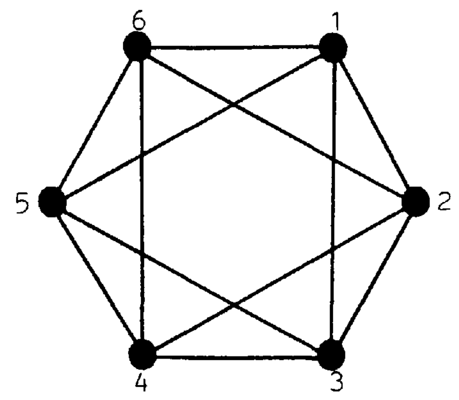
$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

A circulant matrix $A_c(G)$ is a $|V| \times |V|$ matrix such that row i of $A_c(G)$ is obtained from the first row of $A_c(G)$ by a cyclic shift of i-1 steps, so any circulant matrix is determined by its first row. FIG 1.5 (a).

A circulant graph is a graph G whose adjacency matrix A(G) is a circulant matrix. FIG 1.5 (b).

$$A(G)=S=\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

(a)



(b)

FIG 1.5

Any two paths between vertices v_i and v_j are said to edge disjoint if they have no edges in common and vertex disjoint if they have no vertices in common except for v_i and v_j.

1.3 General Discussion of the Problem

Topologically, a communication network may be represented by a connected graph, where stations (nodes) and links of the network correspond to vertices and edges, respectively, of the graph. By N(G), we denote a network represented by a graph G.

If we think of a graph G as representing a communication network, the vertex connectivity (or edge connectivity) becomes the smallest number of communication stations (nodes) or communication links whose breakdown would disrupt communication in the system. The higher the vertex connectivity and the edge connectivity, the more reliable the network.

For example, if a network is modelled by the graph in FIG.1.6 it is reasonable to conclude that the system is not reliable since removal of the single station (node) represented by vertex v_1 breaks all communications.

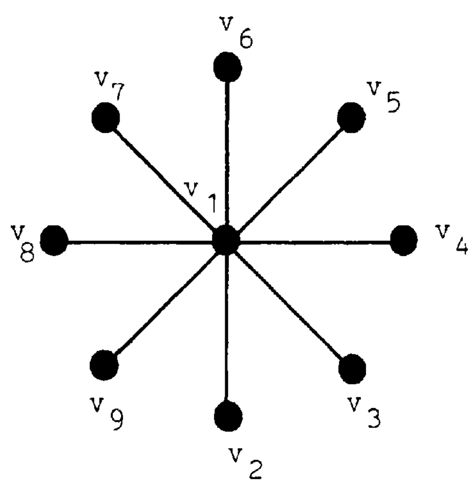


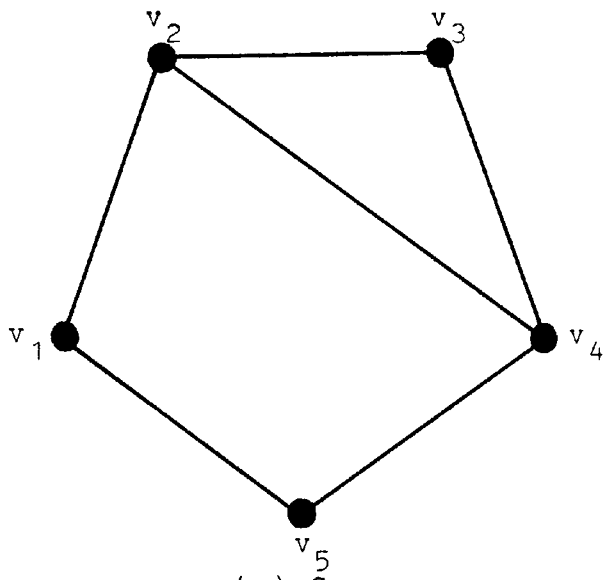
FIG.1.6

In many applications, the vertices of a graph may be the unreliable elements. For example, in an airline network, the vertices represent aiports and the edges represent air routes. Suppose the network is the subject of an attack aimed at disrupting service between various airports. It will often be much easier to destroy airports than to achieve air superiority to close air routes.

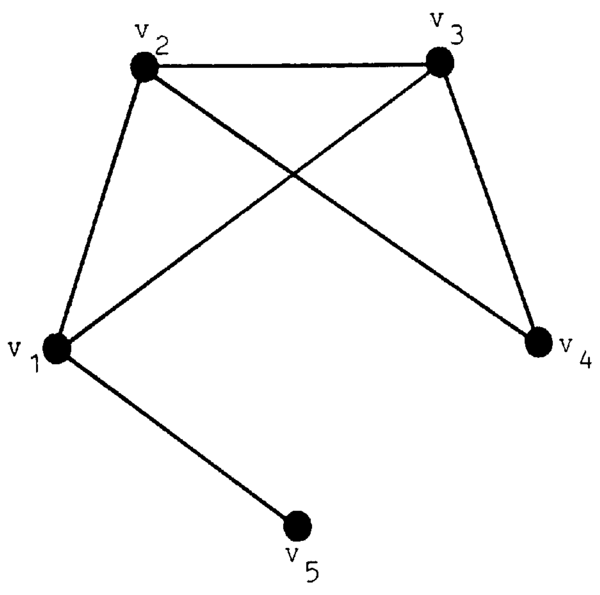
In the design and analysis of communication networks, one of the fundamental considerations is the reliability, in particular that the stations or centres can communicate in case of link or node failure. It is clearly more serious for link or node failures to isolate half the nodes in the network from the other half than one node in the network from all the others. Consequently we would expect a highly reliable network to reflect this property.

In the design of networks that are best with respect to node and link failure, the aim is to maximize the number of nodes or links that must fail in order to disconnect the operation of the network. The function of a communication network is to communicate between pairs of terminals or stations and it is usually required to do this as reliably as possible. The most simplistic approach is to regard the network as reliable if a communication entered into the network at one terminal can actually be routed to the required destination. This corresponds to the requirement that the underlying graph of the network be connected and it is one of the designers' objectives to maximize the number of vertices and edges that have to be removed to disconnect the graph.

For example, for G_1 FIG.1.7(a), $k(G_1)=2$ and for G_2 FIG.1.7(b), $k(G_2)=1$, although G_1 and G_2 have the same number of vertices and edges.



(a) G_1



(b) G_2

FIG. 1.7

The difficulty of finding the minimum number of vertices and edges of a graph G which if removed would disconnect G can easily be seen by considering FIG.1.8(a) and FIG.1.8(b). The answer is readily seen once G is redrawn. The removal of either v_3 or v_{10} disconnects G.

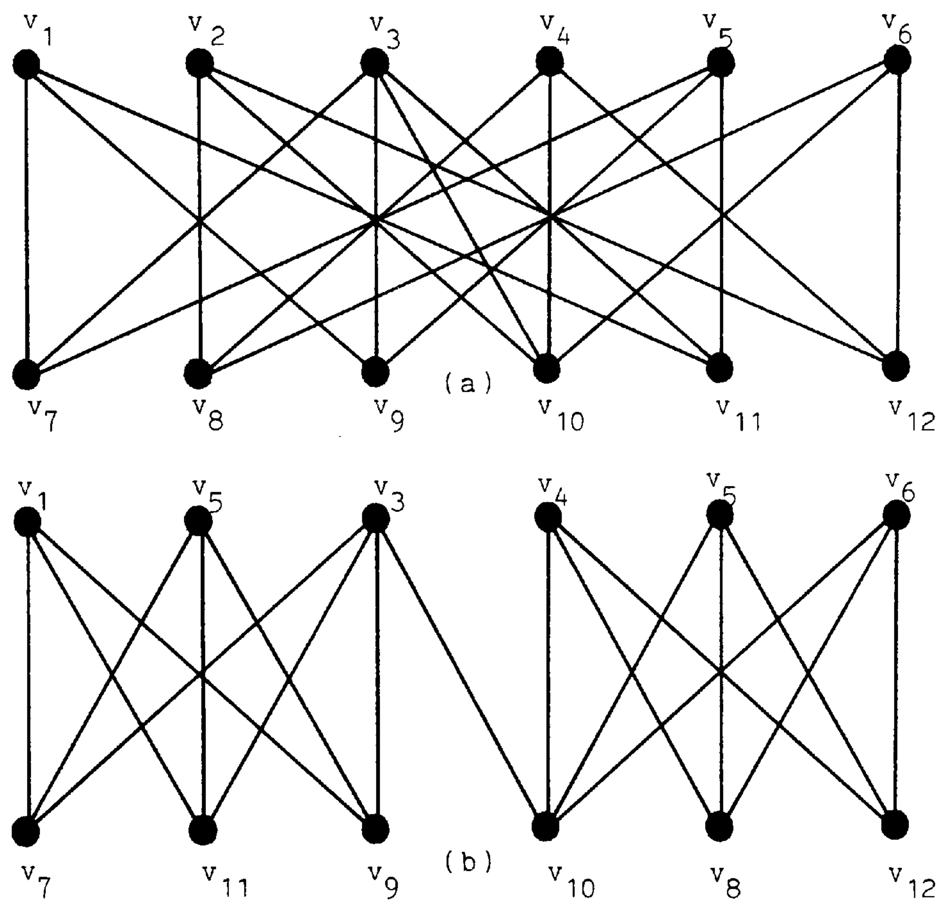


FIG. 1.8 Two drawings of the same graph.

One objection to using the parameters of edge and vertex connectivity is that they fail to differentiate between the different types of disconnected graphs which result from removing k vertices or λ edges. It is not the same in practice to isolate a single vertex or to divide the graph into two equal components.

If we consider regular graphs in which all vertices are of degree ρ, then such graphs are maximally connected if they are ρ-connected and ρ-edge-connected. If all nodes in a communication or computer network are of equal importance, a maximally reliable network corresponds to a maximally connected regular graph.

Maximally connected graphs having the property that their diameters increase very rapidly with the number of vertices are undesirable since in large networks of this type, the shortest route between several pairs of nodes would have to pass through many intermediate communication stations. This might cause processing and queuing delay associated with each node.

More realistically, one might consider that a communication in a network system has some probability p of accurate reception after passage over a single link of the network. Hence if the length of the shortest path from start to finish in the network is d, then the commodity will be received accurately with probability p^d. Furthermore in data networks, the delay is proportional to the length of the path being used. From these points of view, it seems natural to describe the reliability of the network in terms of the diameter of the corresponding graph.

Different research workers have suggested different reliability measures in their attempts to realize maximally reliable communication networks. R. S. Wilkov [51] has surveyed these reliability criteria and discussed their relevance to different applications by pointing out the difficulties and limitations associated with each of these reliability measures. H. Frank and I. T. Frisch [24] have also comprehensively treated these reliability measures.

A recent work by J. C. Bermond, J. Bond, M. Paoli and C. Peyrat [2] surveys the results concerning diameter and connectivity in graphs and hypergraphs, in particular those of some importance for communication networks. F. T. Boesch [4] considers that the notion of connectivity is one of the most important graph theoretic concepts that is useful in applications, and investigates the properties of some new and important generalizations of connectivity.

1.4 Graph Theoretic Models

A graph theoretic model G of a system can yield many significant properties of the system. Thus, if the graph represents a power system, it can be be used to determine such factors as the possible routes over which power can be sent and the number of sub-stations that must be out of use before power transmission is interrupted for some users.

Suppose we are given a telephone communication system in which there are three stations, S_1, S_2, S_3 with wire connections between them. Furthermore assume that the wire between S_i and S_j is of length $\ell(i,j)$, has a cost $C(i,j)$, and a probability $p(i,j)$ of normal operation. Also assume that each station S_j has capacity $C(j)$ and probability $p(j)$ of normal operation. The system can be represented by a weighted graph G shown in FIG. 1.9 where vertex v_i corresponds to station S_i and edge (i,j) corresponds to the wire between S_i and S_j.

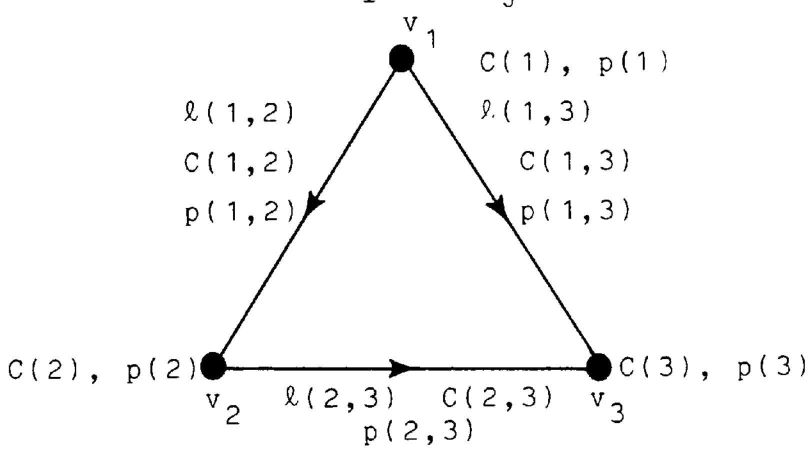


FIG. 1.9 A weighted graph.

The purpose of the edge and vertex weights is to include non-structural information into the graph theoretic model of a system. The modelling of some physical systems by graphs is quite natural. The edges of the graph can represent roads, telephone wires, railway lines, airline routes, water, gas or oil pipes, in general channels through which commodities are transmitted. The vertices of the graph can represent communities, road junctions, telephone stations, railway depots, airline terminals, water reservoirs, in general, points where flow starts, is relayed, or terminates. The following examples further illustrate these points.

1.4.1 A traffic network. Let each vertex of a graph represent a city. Two vertices are connected by an edge if there is a road between the corresponding cities. A number is associated with each edge to indicate the length of the corresponding roads. A second weight represents the maximum number of cars that can be accommodated per unit length per unit time, and a third edge weight could be the speed limit.

1.4.2 An airline system. Let each vertex of a graph represent an airport. Two vertices are connected by an edge if there is a direct air link between the airports. Each vertex of the graph has a weight indicating the number of aeroplanes that the airport can handle in a given interval of time. This vertex weight could be a fixed number if the traffic handling capacity of the airport is assumed to be constant or it could be a random variable if it depends on unpredictable elements such as weather.

1.4.3 A telephone system. Let each vertex of a graph represent a telephone exchange. Two vertices are connected if there is a telephone wire between the corresponding telephone exchanges. That is, there will be an edge between two vertices if the exchange can communicate directly, without any intermediate relay exchange. At each telephone exchange there will be a limit to the maximum number of messages which can be simultaneously transmitted and received. This can be included in the model by an appropriate vertex weight. The maximum number of messages on an edge is determined by the number of telephone wires. Each edge could be weighted by the maximum number of simultaneous messages that can be handled.

1.4.4 An economic model. Suppose we are given a system of factories, warehouses, and outlets connected by a set of roads, railways, and canals. This system can be structurally modelled by a graph with the edges representing transportation channels and the vertices representing factories, warehouses and outlets. In the graph, the factories are source vertices, the outlets are terminal vertices. For the relay vertices, a single weight representing the storage space might be sufficient. A terminal vertex could be weighted with numbers which indicate the types of commodity which are sold at that vertex or the price of each commodity. Typical edge weights could be maximum volume per unit time or cost of transportation.

The use of graphs as models depends on the nature of the physical problems to be solved. Weighted graphs are considered when the existence of a path between a pair of vertices implies that some amount of flow can be transmitted between these two vertices. The problem of finding the maximum amount of a given quantity which can be conveyed between two points is known as the Maximum Flow Problem. L. R. Ford and D. R. Fulkerson [22] have been among the most original and prolific contributors to the development of theory of flows in networks. Their work is of interest in reliability studies because it can be used to determine, for example, the edge and vertex connectivity of a graph.

The examples given are sufficient to point out the wide range of applications of graph theoretic models and the nature of the problems that can be posed. Connectivity and maximum flow problems are related to a problem of "reliability" and it is the problem of calculating and optimizing the reliability of a communication network in terms of probability that will be the basis for the work in this thesis.

1.5 Reliability and the Model

Reliability analysis of communication networks is concerned with the dependence of the reliability of the network on the reliability of its nodes and links.

Node failures can affect network reliability in two ways. First, if a node fails, clearly it cannot communicate with any other node in the network. Thus if there are N nodes in the network and one fails, a minimum of N-1 node pairs cannot communicate independent of the network structure. Changing the network configuration has no effect on this component of network reliability. Another effect of node failures is that the failed nodes disrupt some otherwise useable communication paths between other pairs of nodes. Link failures also affect network reliability in the second way.

R. Van Slyke and H. Frank [50] consider networks with randomly failing links and nodes and give a combinational analysis when all links have equal reliabilities, two general simulation methods are compared both involving sampling techniques. In communication networks randomly distributed natural disruptive forces are not ruled out. Therefore, a measure of interest is overall reliability, rather than the terminal pair reliability, because one is interested in knowing the probability of successful communication or disconnection between any pair of nodes. A number of problems arising in the analysis and sythesis of communication networks lead to a mathematical model representing a probabilistic network. By probabilistic network is meant a finite, simple, undirected graph G each of whose edges (or vertices) can fail with a given probability p (or q), the failure in different edges or vertices are assumed to be independent. The assumption of independent failures is important because a network with links that are topologically separated might nevertheless share a common duct in the ground for part of their extent, and this would conflict with the assumption.

A tree network is the most economical way to connect a set of nodes together, if for example we wish to build a railway network connecting N given cities in such a way that a passenger can travel from any city to any other, and we assume for economical reasons that the amount of track used must be a minimum, then it is clear that the graph formed by taking the N cities as vertices and the connecting rails as edges must be a tree. FIG. 1.10 shows a spanning tree which uses the least amount of track, assuming that the distances between the various cities are known.

Failures will happen both in nodes and links. Careful design and duplication of equipment can largely eliminate failure in the nodes, if the cost is warranted. Failure of individual links is more difficult to avoid, for example, cables can be damaged by digging operations.

The function P(G) of a network is defined to be the probability of the network being disconnected or connected as a function of the probability p of a link or q of a node failing. Calculations of network reliability should distinguish between node and link failures and be based on different failure probabilities for these two equipments.

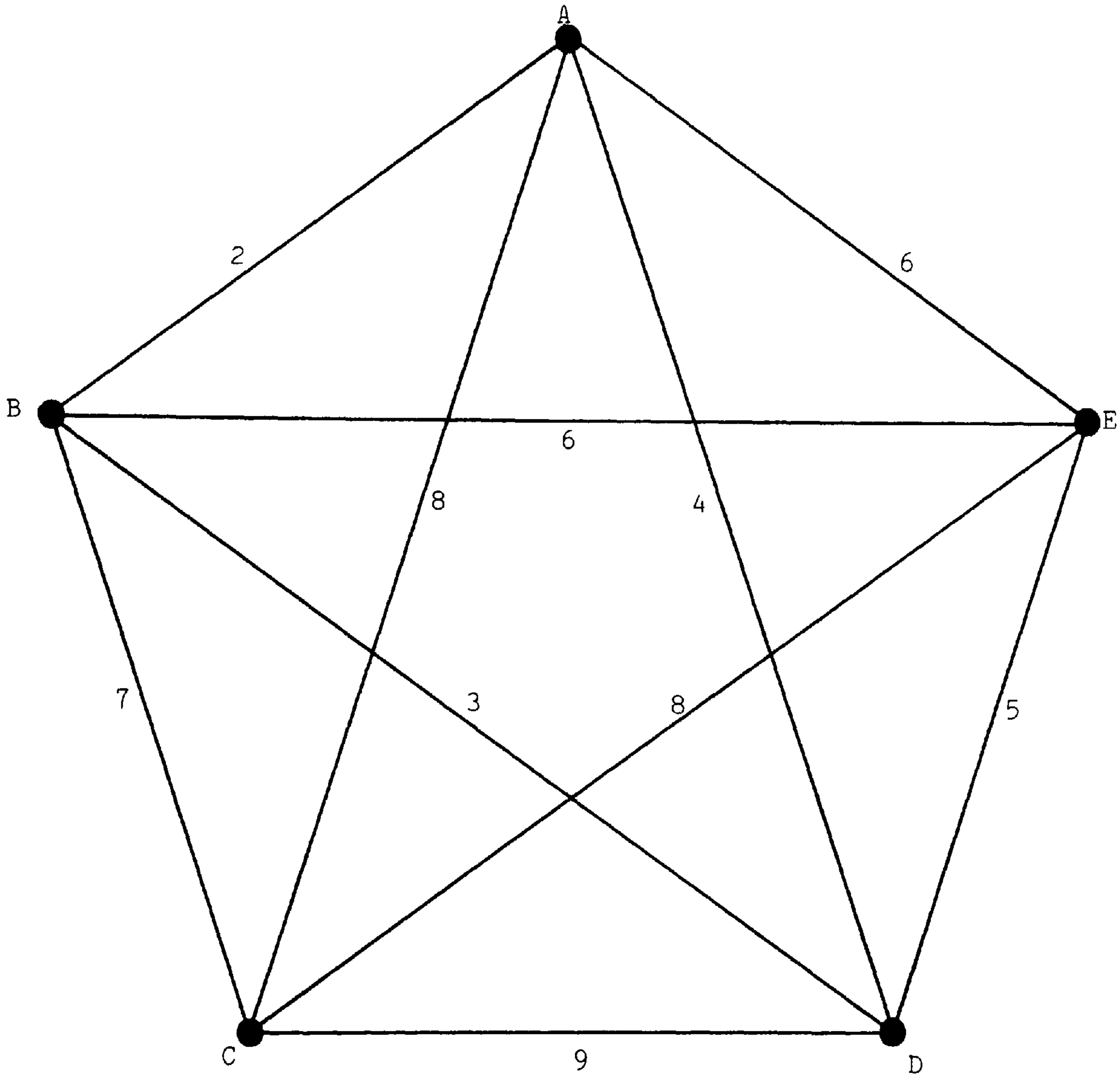


FIG. 1.10. Spanning tree T of G is the subgraph whose edges are e_1=AB, e_2=BD, e_3=DE, e_4=BC.

In general if the graph modelling the network has $|E|$ edges and $|V|$ vertices with probability p of each edge failing, there are $\binom{|E|}{i}$ ways that exactly i of the edges can fail, and each such event has probability $p^i(1-p)^{|E|-i}$.

Thus, if R_i is the number of ways exactly i edges can result in a disconnected graph, then

$$P_d(G) \text{ (probability of disconnection)} = \sum_{i=0}^{|E|} R_i p^i (1-p)^{|E|-i} \quad (1a)$$

Where $R_i = \binom{|E|}{i} = 0$ for $i = 0, 1, \ldots \lambda-1$, and λ is the minimum number of edges which must be removed to disconnect the graph. If the graph has $|V|$ vertices, it takes at least $|V|-1$ edges to connect them. Thus the maximum value of i to be looked at is $|E|-(|V|-1)$, (to leave a tree).

Similarly the probability of disconnection of the graph with probability q of each vertex failing is given by,

$$P_d(G) \text{ (probability of disconnection)} = \sum_{i=k}^{|V|} S_i q^i (1-q)^{|V|-i} \quad (1b)$$

Where S_i denotes the number of ways exactly i vertices can result in a disconnected graph, and k is the minimum number of vertices which must be removed to disconnect the graph, the maximum value of i is generally $|V|-2$.

In the analysis of communication networks the designers will often be interested in two particular solutions:

(a) Perfectly reliable nodes and failing links, equation (1a).

(b) Perfectly reliable links and failing nodes, equation (1b).

Case (a) will be of interest in calculating required link redundancy in the network, that is, when attention is restricted to the reliability of the links and their structure.

Case (b) is of interest in calculating the link structure required to provide a satisfactory protection against node failure, that is, when attention is restricted to the reliability of the nodes and their communication function.

The analysis of networks in the study will be modelled by a probabilistic graph and the graphs in the model are such that the nodes (or edges) are chosen independently with the same probability $p(0<p<1)$. No reference to cost or the relationship between the cost and the sum of the lengths of the edges are included in the mathematical model.

The network study is based on the work of a number of researchers. F. Boesch and R. Tindell [11] present results for circulant graphs and their connectivities and define super-λ circulants to be graphs in which every edge cut set with λ edges isolates a vertex of degree ρ. F. T. Boesch and J. F. Wang [12] determine a lower bound R_i for such graphs for certain values of i, and point out that in order to minimise the probability of disconnection (edge failure), one must first maximise λ and then minimise all the R_i.

H. Frank [23] has described the problem of finding graphs with the minimum probability of disconnection if the probability of failure of any vertex is close to 0. S. L. Hakimi and A. T. Amin [26] have constructed regular graphs of valency k and connectivity k in which the vertex cut sets with k vertices are vertex neighbour sets. They also show that these graphs do not have the smallest number of vertex cut sets with k vertices. Further work by D. H. Smith [45] has shown how to construct infinite families of graphs with the minimum number of vertex cut sets with k vertices, spread through the range $\frac{3}{8} \leq \frac{k}{|V|} < 1$. Also dealt with are cases in which k is small and cases with $|V|-k$ small.

In this study, the non-zero dominant terms in equations (1a) and (1b) are of interest for either vertex or edge failure. For example, for p close to 0, later work will show that R_λ determines the behaviour of P(G). Similarly if p is close to 1, the last non-trivial term is of interest. This term is simply $\binom{|E|}{|V|-1}$ minus the number of spanning trees in the graph. With this motivation the investigation aims to identify and compare networks which will operate as reliably as possible in the presence of vertex and edge failure.

CHAPTER 2

CHAPTER 2

Reliability Measures

The design of large scale networks, particularly communication networks, usually involves some type of reliability considerations. In most cases, the network is considered to be failed if it is no longer possible to communicate between two nodes. In this Chapter we survey some of the important reliability measures.

2.1 Connectivity (Vertex Connectivity)

A graph G is connected if there exists a path in G between any pair of distinct vertices of G; otherwise it is disconnected. A connected graph has only one component (the graph itself), while a disconnected graph has at least two components.

The complete graph $K_{|V|}$ with $|V|$ vertices has every two distinct vertices adjacent; thus the degree of the regular complete graph is $|V|-1$ and it follows that the number of edges will be equal to $\frac{|V|(|V|-1)}{2}$. The vertex connectivity or simply connectivity k(G) of a graph with $|V|$ vertices is $|V|-1$ if G is the complete graph and otherwise is the minimum number of vertices whose removal results in a disconnected graph.

A tree is a connected graph with the minimum number of edges. The number of edges is equal to $|V|-1$ and for $|V|>1$, $k(G) = 1$. Consider the three

connected graphs of FIG. 2.1. G_1 is a tree, a minimal connected graph; G_2 has no single edge cut set or single vertex cut set but even so G_2 is clearly not as well connected as G_3 the complete graph. Thus intuitively, each successive graph is more strongly connected than the previous one.
A graph G in which $k(G) \geqq k$ is said to be k-connected.

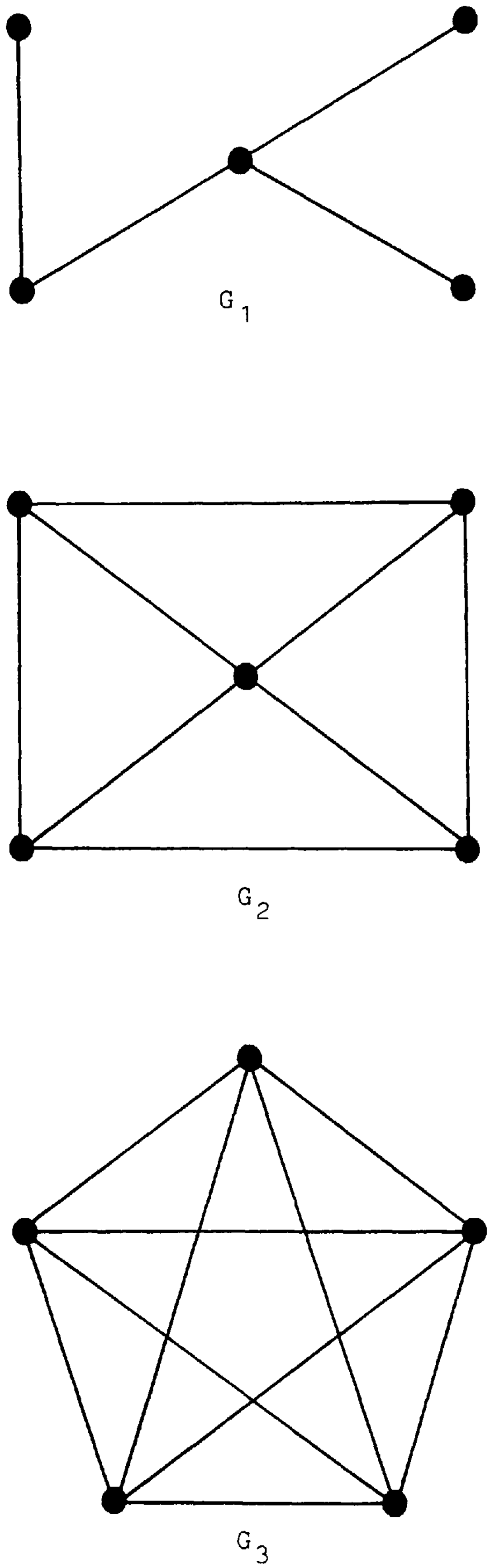


FIG. 2.1. Connected graphs.

It is well known that for a graph G with $|V|$ vertices and $|E|$ edges, since every edge in G has two end vertices the sum of the degrees of the vertices in G equals $2|E|$. Thus the average degree $\bar{\rho} = \frac{2|E|}{|V|}$.

<u>Lemma</u>. The connectivity of a connected graph is at most the minimum of the degrees of its vertices.

<u>Proof</u>: If ρ_m is the minimum degree and v is a vertex of degree ρ_m, adjacent to vertices $v_1 \ldots v_n$, then on removing the vertices $v_i (i = 1 \ldots n)$ from G, v becomes an isolated vertex. ■

Given positive integers $|V|$ and $|E|$, a graph G is said to be of optimal connectivity if k(G) is a maximum over all graphs with $|V|$ vertices and $|E|$ edges. Essential results of interest are contained in the work of F. Harary [29] who solved the problem of finding the maximum connectivity of any graph with a given number of vertices and edges.

<u>Theorem</u>. Among all graphs with $|V|$ vertices and $|E|$ edges, the maximum connectivity is 0 when $|E| < |V| - 1$ and is $\left\lfloor \frac{2|E|}{|V|} \right\rfloor$ when $E \geq |V| - 1$.

<u>Outline of proof</u>.

To show that the maximum connectivity is $\left\lfloor \frac{2|E|}{|V|} \right\rfloor$ when $|E| \geq |V|$ it is necessary to prove the following,

(a) the connectivity of a graph cannot exceed $\left\lfloor \frac{2|E|}{|V|} \right\rfloor$

(b) there exists a graph G whose connectivity is $\left\lfloor \frac{2|E|}{|V|} \right\rfloor$

The proof of (a) uses the lemma that the minimum degree of all the vertices of a graph G is an upper bound to the connectivity. Hence for a graph G which is not regular of average degree $\bar{\rho}$ the connectivity $k<\bar{\rho}$. If on the other hand, G is a regular graph of degree ρ then $k \leqq \rho$.

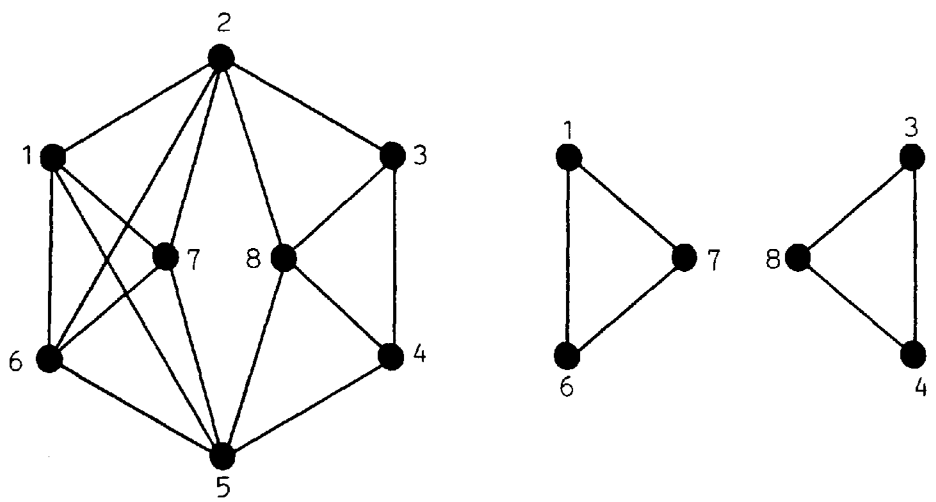
The proof of (b) is by construction and begins by drawing a polygon and labelling its vertices by the integers 0, 1, 2, ... $|V|-1$. Two cases are then dealt with, the first considers the average degree $\frac{2|E|}{|V|}$ to be an integer S and gives the construction separately for even and odd values of S. The second case assumes $\frac{2|E|}{|V|}$ is not an integer and begins by constructing a regular graph G_S with $|V|$ vertices and $\frac{S|V|}{2}$ edges.

The proof is completed using the properties that the connectivity of a connected graph is at most the minimum of the degrees of its vertices and if G_1 is a spanning subgraph of G, then $k(G_1) \leqq k(G)$.

In many design problems, one is interested in the reliability between a specific pair of nodes. For example, some pairs of nodes may have more critical communication needs than other pairs of nodes in the network, and hence require a higher degree of reliability.

The idea of local connectivity measures for a vertex pair is well known. The local vertex connectivity or simply the local connectivity k_{ij} is defined as the minimum number of vertices whose removal breaks all paths between vertices i and j (results in v_i in one component and v_j in the other).

Considering the graph of FIG. 2.2, v_2 and v_5 form a vertex cut set since their removal results in a graph with $\{v_1, v_6, v_7\}$ in one component and $\{v_3, v_4, v_8\}$ in the other. Thus the connectivity between non adjacent vertices v_1 and v_8 is 2 because the vertex cut set results in v_1 in one component and v_8 in the other.



(a) connected (b) disconnected

FIG. 2.2 A graph with k_{18} equal to 2

The connectivity of a non complete graph is the minimum value of the local connectivity over all pairs of vertices. The classic theorem of K. Menger [35] states that k_{ij} is equal to the maximum number of vertex disjoint paths joining i and j.

The starred polygon (or circulant graph) of FIG. 2.3, is 4-connected. There are four vertex disjoint paths between any pair of vertices.

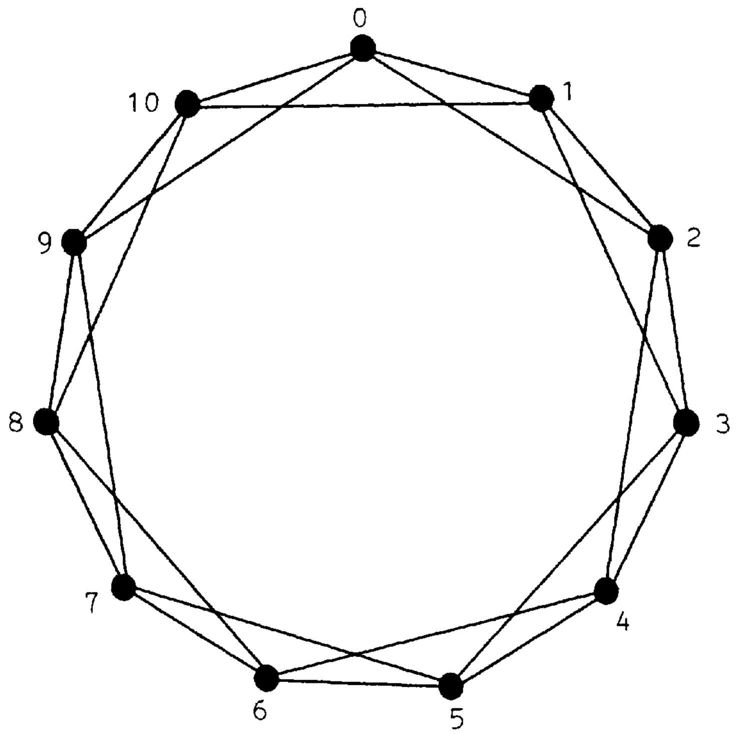


FIG. 2.3 Starred polygon (or circulant graph) that is 4-connected, A set of four vertex disjoint paths between v_1 and v_4 is $\{v_1\ v_3\ v_4\}$, $\{v_1\ v_{10}\ v_8\ v_6\ v_4\}$, $\{v_1\ v_0\ v_9\ v_7\ v_5\ v_4\}$, $\{v_1\ v_2\ v_4\}$.

2.2 Edge Connectivity (or Cohesion)

The edge connectivity (or cohesion) of a graph G is denoted by λ and is defined as the minimum number of edges whose removal results in a disconnected graph. Of interest therefore are graphs in which the smallest edge cut set is as large as possible. In general the economical design of reliable communication networks requires the construction of k-connected or k-edge-connected graphs with a minimum number of edges for a given number of vertices. It is known that for any graph the connectivity is less than or equal to the minimum vertex degree.

Theorem. In any graph G, $k \leqq \lambda \leqq \rho_m$.

Proof:

(a) If G is trivial $\lambda = 0 \leqq \rho_m$.

(b) If G is not trivial then the set of edges incident with a vertex of degree ρ_m is a ρ_m edge cut set of G. Thus $\lambda \leqq \rho_m$.

(c) We prove $k \leqq \lambda$ by induction on λ. The result is true if $\lambda = 0$, since then G must be either trivial or disconnected. Suppose that it holds for all graphs with $\lambda < r$, let G be a graph with $\lambda = r > 0$, and let e be an edge in a r-edge cut set of G. Setting $G_1 = G - e$, we have $\lambda(G_1) = r-1$ and so, by the induction hypothesis, $k(G_1) \leqq r-1$.

(d) If G_1 contains a complete graph as a spanning subgraph, then so does G and $k(G) = k(G_1) \leqq r-1$.

(e) Otherwise, let C be a vertex cut set of G_1 with $k(G_1)$ elements. Since $G_1 - C$ is disconnected, either G-C is disconnected, and then $k(G) \leqq k(G_1) \leqq r-1$.

(f) Or else G-C is connected and e is an edge of a cut set of G-C. In this latter case, either $|V(G-C)| = 2$ and $k(G) \leqq |V(G)| - 1 = k(G_1) + 1 \leqq r$.

(g) Or G-C has a 1- vertex cut $\{v\}$, implying that $C \cup \{v\}$ is a vertex cut set of G and $k(G) \leqq k(G_1) + 1 \leqq r$.

Thus in each case we have $k(G) \leqq r = \lambda(G)$. The result follows by the principle of induction. ■

We now recall that in any graph G having $|E|$ edges and $|V|$ vertices and minimum degree ρ_m.

$$\sum_{i=1}^{|V|} \rho_i = 2|E|.$$

Hence $2|E| \geqq |V|\rho_m$.

or $\rho_m \leqq \frac{2|E|}{|V|}$.

Since $\lambda \leqq \rho_m$.

It follows that $k \leqq \lambda \leqq \rho_m \leqq \frac{2|E|}{|V|}$.

Therefore a graph with $\lambda = \frac{2|E|}{|V|}$ has a maximum value of λ. Furthermore if $k = \frac{2|E|}{|V|}$ then both k and λ are maximum.

A maximally connected graph is one in which $k = \lambda = \rho = \frac{2|E|}{|V|}$.

The design of reliable communication networks is based on the node or edge connectivity of the corresponding graphs and for this reason we give some examples of classes of graphs which are maximally connected or have maximum connectivity, i.e. for given values of $|E|$ and $|V|$ we have maximum connectivity if the smallest vertex cut set is as large as possible i.e. $k = \frac{2|E|}{|V|}$.

A class of maximally connected regular bipartite graphs has been introduced by F. T. Boesch and R. E. Thomas [10]. These graphs are such that for $|V|$ even, vertex i is adjacent to vertex $i+2j-1$ (Mod $|V|$) where $i \leq j \leq \rho$. We note that a bipartite graph is one in which the set of vertices V can be partitioned into two disjoint sets V_1 and V_2 and each edge of the graph joins a vertex in V_1 with a vertex in V_2. The regular bipartite graph just mentioned have a girth of four and a diameter that has been shown by R. S. Wilkov [52] to be approximately $\left\lceil \frac{|V|+1}{2(\rho-1)} \right\rceil$ where $\lceil X \rceil$ is the smallest integer greater than or equal to X. For example the 26 vertex regular bipartite graph is shown in FIG. 2.4.

F. Harary [29] and S. L. Hakimi [25] have introduced a class of maximally connected regular graphs such that for even vertex connectivity, vertex i is adjacent to vertices $i \pm j$ (Mod $|V|$), where $1 \leq j \leq \frac{\rho}{2}$.

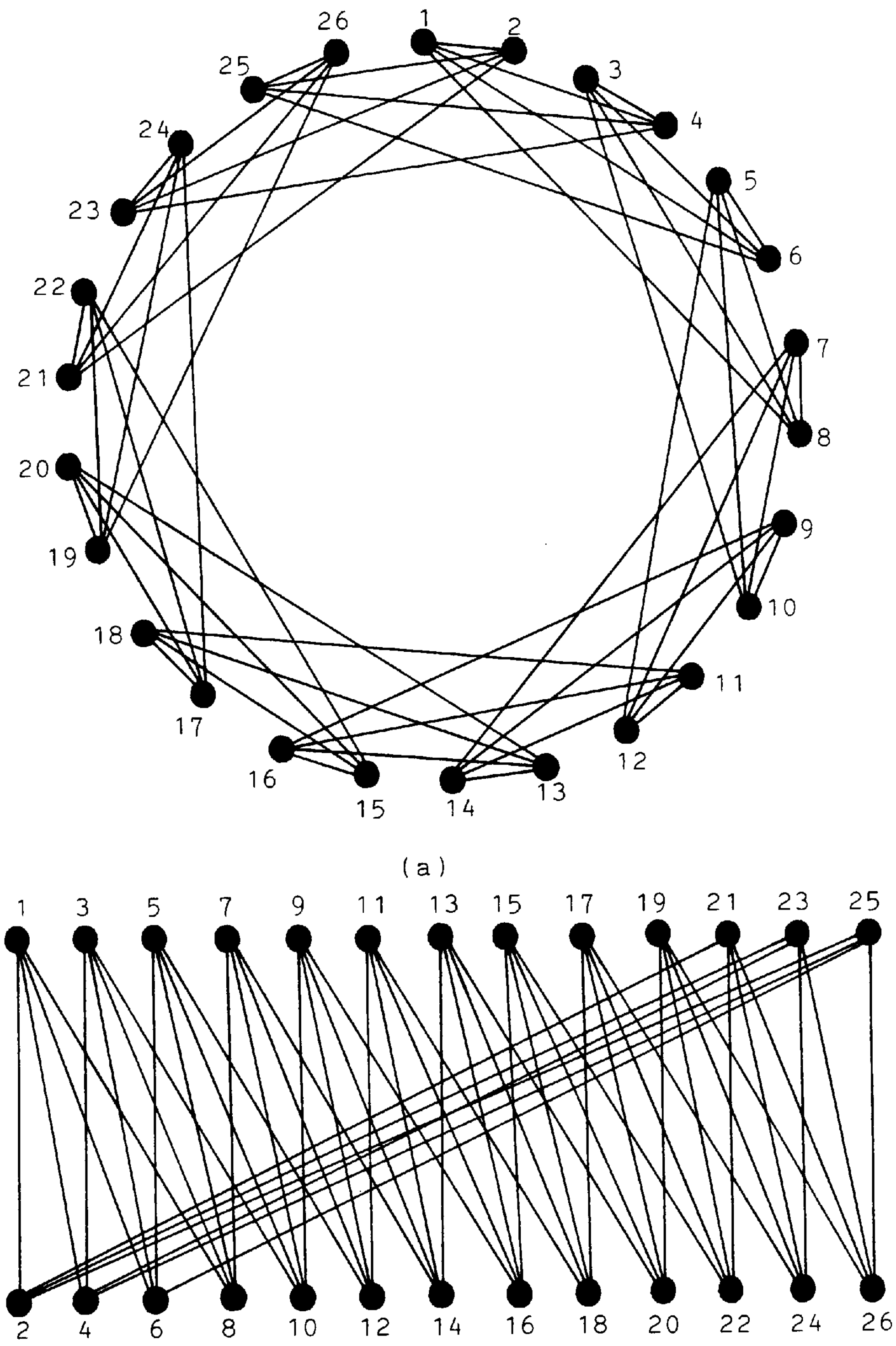


FIG. 2.4. Two constructions of the same maximally connected regular bipartite graph with 26 vertices, $k = 4$ and $d = 5$.

For ρ odd and $|V|$ even vertex i is also adjacent to vertex $i+\frac{|V|}{2}$ (Mod $|V|$) and the diameter has been found to be $\left(\frac{|V|}{2(\rho-1)}\right)$. This class of graphs is constructed by placing the $|V|$ vertices around the circumference of a circle and joining each vertex to the ρ (for ρ even) or $\rho-1$ (for ρ odd) other vertices nearest to it. If ρ is odd, each vertex is also joined to the one furthest from it.

S. L. Hakimi [25] considered the problem of constructing a graph with $|V|$ vertices and $|E|$ edges that has maximum connectivity and in essence restated the complete results of F. Harary [29].

T. Sasaki [41] proposed a method of constructing graphs with connectivity k $\left(2 \leq k = \frac{2|E|}{|V|} \leq |V|-1\right)$. The method is based on the work of S. L. Hakimi [25] maximum connectivity graph construction, and the k-connected bipartite construction of F. T. Boesch and R. E. Thomas [10]. T. Sasaki [41] produced a graph which is maximally connected and contained a total number of spanning trees which for a large scale graph ($|V| \geq 20$) is much larger than those of the graph constructed by the method of S. L. Hakimi [25].

S. L. Hakimi and A. T. Amin [26] show how to construct graphs with $|V|$ vertices and $|E|$ edges, whose connectivity $k = \left(\frac{2|E|}{|V|}\right) \geq 3$ and have no more than $|V|$ minimum vertex cut sets.

D. H. Smith [45] shows how to construct maximally connected graphs with the minimum number of vertex cut sets with k vertices ($\rho = k$). The various infinite families of these graphs have $\frac{k}{|V|}$ in the range $\frac{3}{8} \leq \frac{k}{|V|} < 1$ and also deal with the cases $k = 3$, $k = 4$ and $|V| - k$ small.

A graph which has a minimum number of vertex cut sets with k vertices and a graph with a maximum number of spanning trees are of interest in the design of reliable communication networks, in the first case the graph G corresponds to a maximally reliable network (with respect to vertex failures) when the probability q of a vertex failure is small, and in the second case the graph G corresponds to a maximally reliable network (with respect to edge failures) when the probability of an edge failure is large.

Examples of the maximally connected graphs derived by S. L. Hakimi [25], T. Sasaki [41], S. L. Hakimi and A. T. Amin [26], D. H. Smith [45] are illustrated in FIGS. 2.5, 2.6, 2.7 and 2.8.

2.3 Network Diameter for Graphs with Optimal Connectivity

A graph in which $k = \lambda = \left(\frac{2|E|}{|V|}\right)$ corresponds to a maximally connected network. However, many of these graphs have a very large diameter (d), which we recall from Chapter one is the maximum of the lengths of the shortest paths between all pairs of vertices

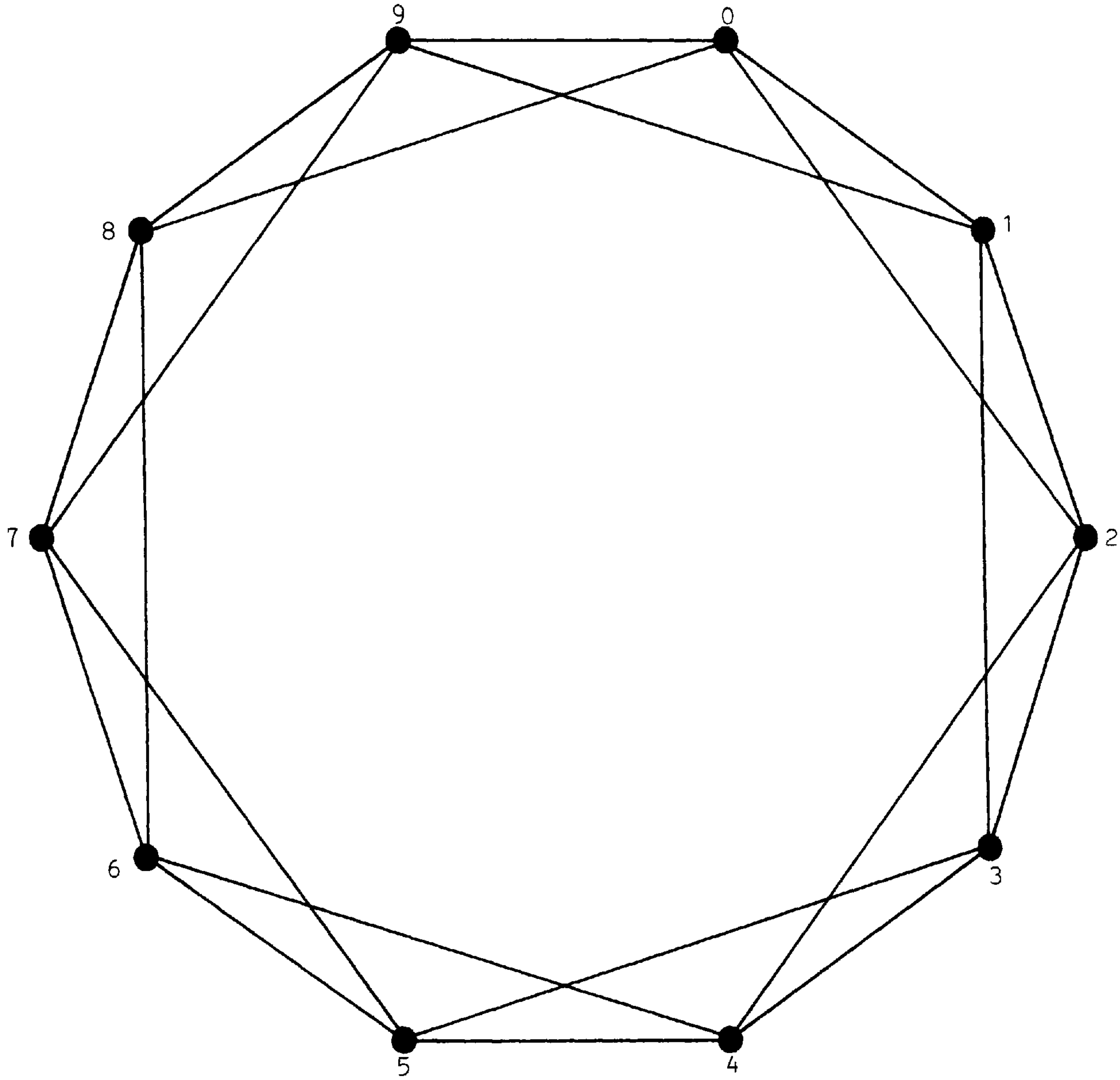


FIG. 2.5. A maximally connected graph with $|V| = 10$, $|E| = 20$, $k = 4$, $\lambda = 4$. The number of spanning trees = 30250.

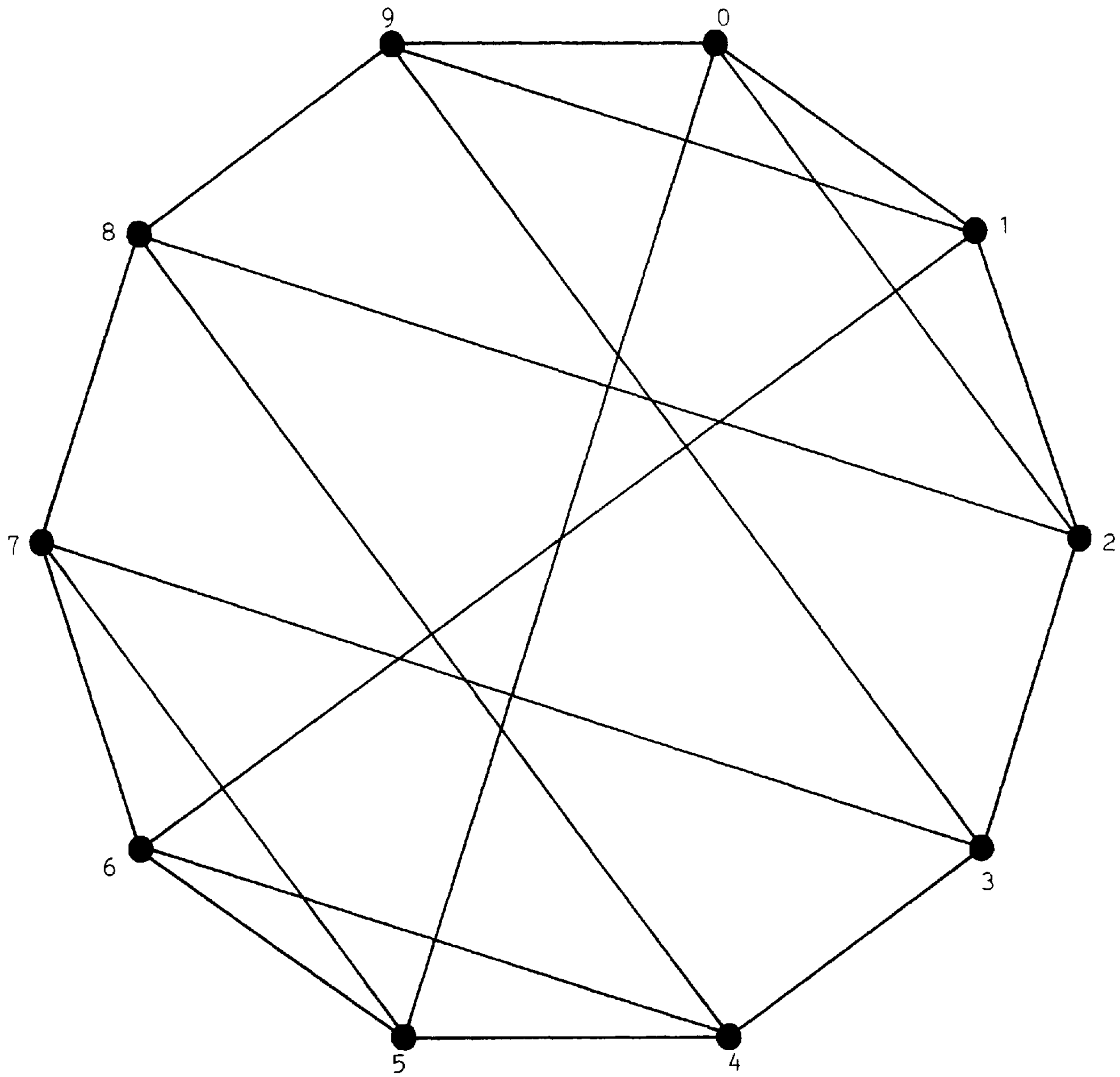


FIG. 2.6. A maximally connected graph with $|V| = 10$, $|E| = 20$, $k = 4$, $\lambda = 4$. The number of spanning trees = 36860.

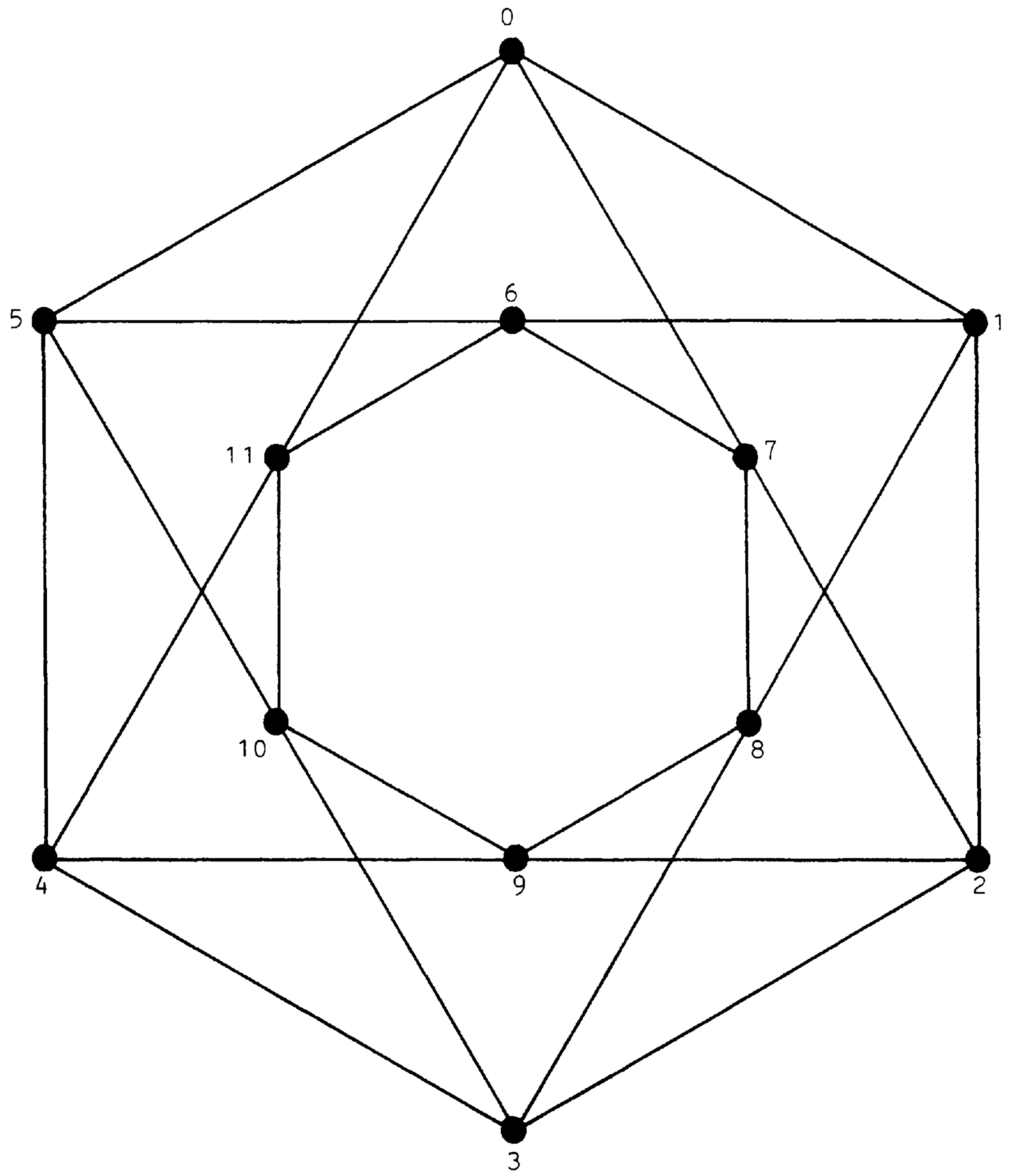


FIG. 2.7. A maximally connected graph with $|V| = 12$, $|E| = 24$, $k = 4$, $\lambda = 4$. Number of minimum size vertex cut sets = 9.

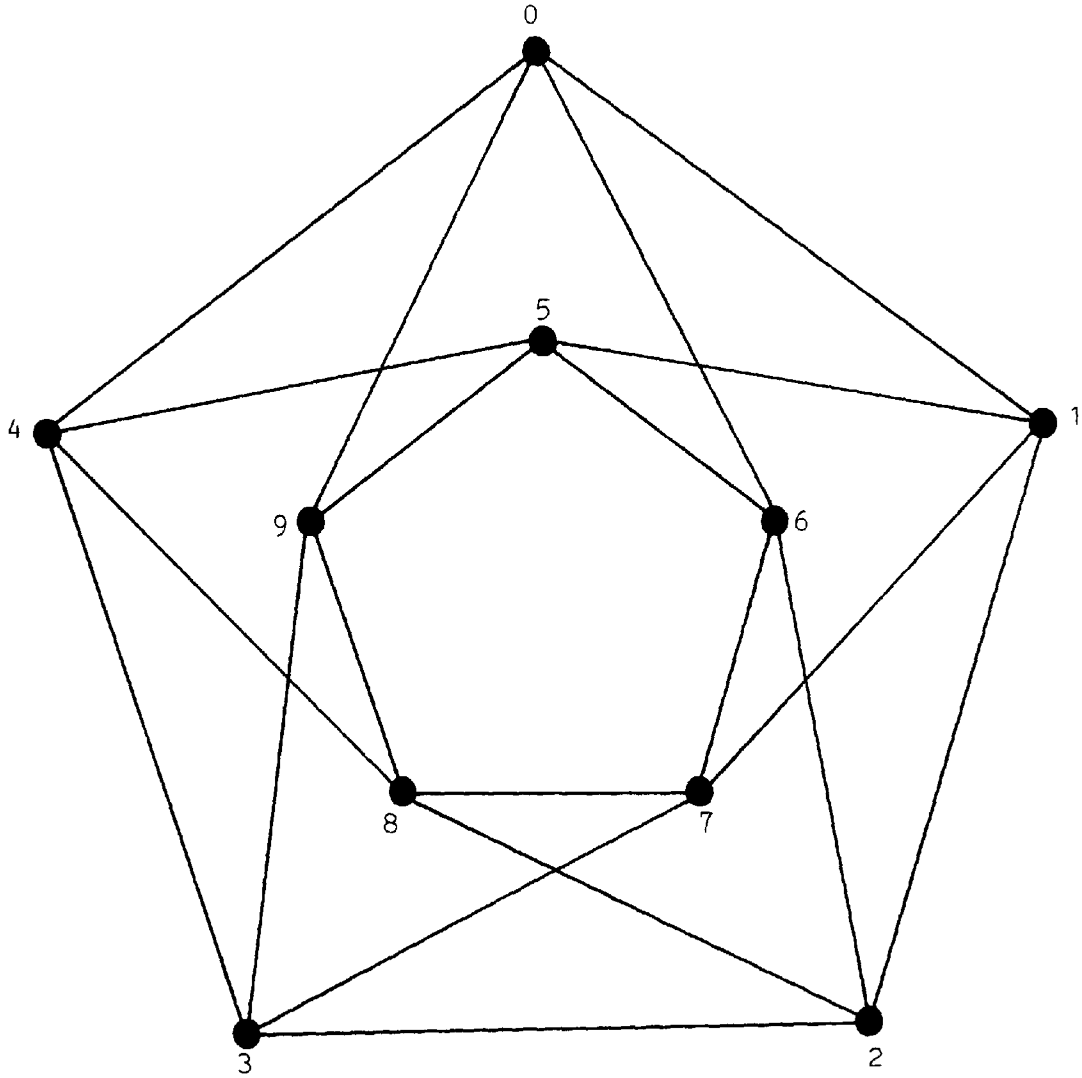


FIG. 2.8. A maximally connected graph with $|V| = 10$, $|E| = 20$, $k = 4$, $\lambda = 4$. Number of minimum size vertex cut sets = 5.

in the graph. In large networks when N becomes much larger than d, the diameter of the graph increases very rapidly and the shortest route between several pairs of nodes would have to pass through many intermediate nodes. This is undesirable because of the processing and queuing delay at each node. A communication network may be connected after some edges fail, but the paths between some vertices may be too long to allow adequate communication. For example, the attenuation may be too large in an analogue voice network, or the delay time might be too large in a digital data network. Therefore, it is desirable in practice for the graph of a communication network to have a reasonably small diameter.

Regular graphs having a minimum number of vertices $|V|$ and specified girth g, where the girth of a graph G is the minimum length of any circuit, have been studied by W. T. Tutte [49] who has shown that for any regular graph of degree ρ, girth g, and diameter d, $g \leq 2d + 1$ and the minimum number of vertices $|V|$ is a function of ρ and g.

The existence of regular graphs of degree ρ, diameter d, and girth $g = 2d$ has been investigated by R. R. Singleton [43]. These graphs, which exist for only certain values of ρ and g, are referred to as "Singleton Graphs". Those regular graphs of diameter d and girth $g = 2d + 1$ are known as

"Moore Graphs". They exist for very few combinations of values of ρ and g, as shown by R.M. Damerell [20] and E. Bannai and T. Ito[55] . Singleton and Moore graphs meeting the bounds of equations (2a) and (2b) have an underlying tree like structure as shown in FIG. 2.9.

$$\text{Minimum (Singleton) graphs } |V| = \frac{2(\rho-1)^{d}-2}{\rho-2}, \quad g = 2d \qquad (2a)$$

$$\text{Minimum (Moore) graphs } |V| = \frac{\rho(\rho-1)^{d}-2}{\rho-2}, \quad g = 2d+1 \qquad (2b)$$

We now describe the graphs satisfying equations (2a) and (2b) and show how the equations are obtained.

The vertex at the top of the tree, denoted by U_0, can be any node in the graph. It is level 0 in the tree. Level 1 consists of the ρ vertices adjacent to vertex U_0, which we will denote by $U_1, U_2, \ldots U_\rho$. Level 2 consists of $\rho(\rho-1)$ vertices derived from the vertices in level 1 at a distance of 2 from vertex U_0. Tier $i(i<d)$ consists of $\rho(\rho-1)^{i-1}$ vertices at a distance of i from vertex U_0.

For $g = 2d$, there are $(\rho-1)^{d-1}$ vertices at level d. Each of these vertices must be adjacent to ρ vertices in level d-1 in order for the $\rho(\rho-1)^{d-2}$ vertices in level d-1 to be of degree ρ. Furthermore, any vertex in level d cannot be adjacent to two vertices in level d-1 that are derived from the same vertex U_i,

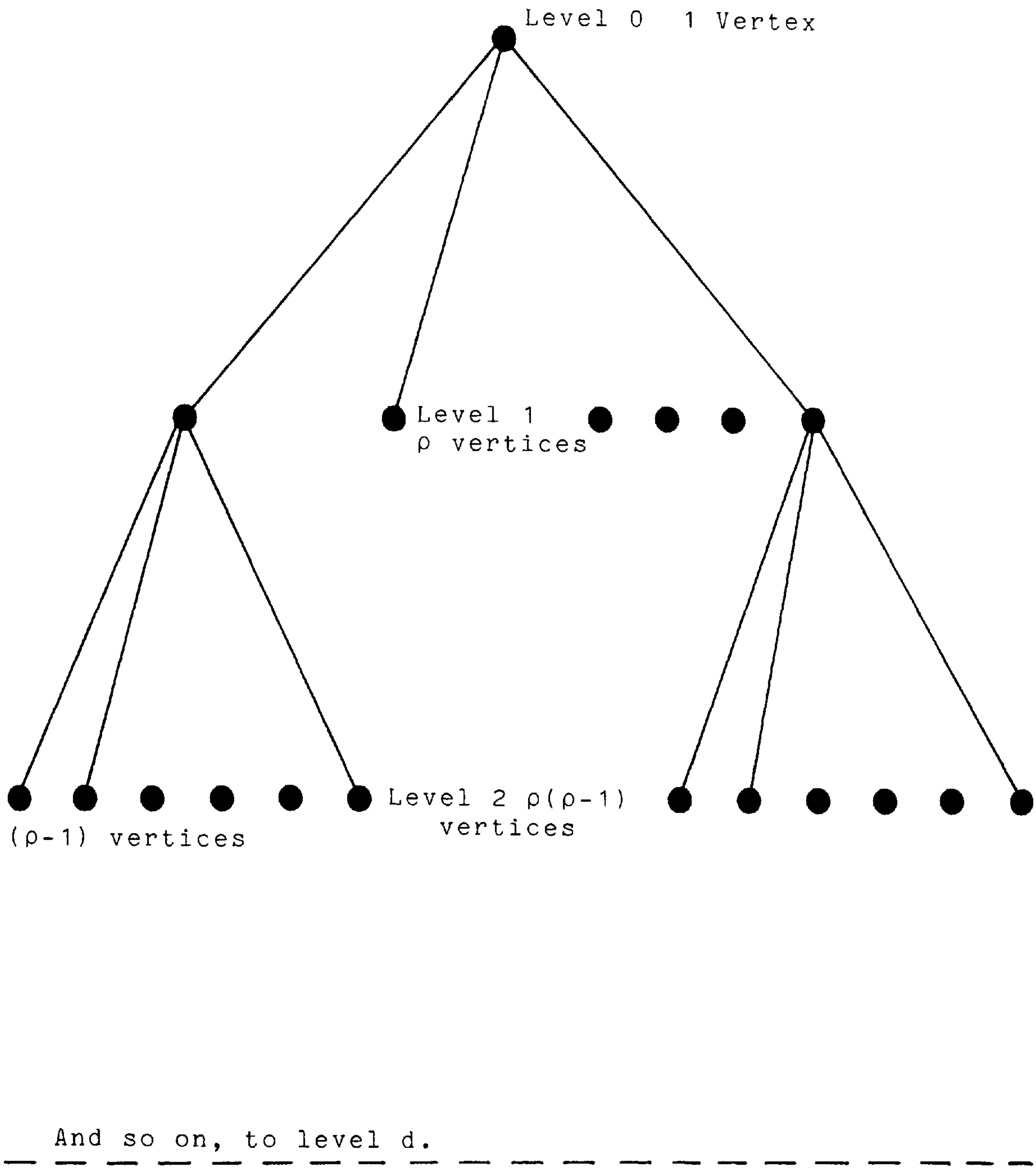


FIG. 2.9 Diagram showing the underlying tree like structure of Moore and Singleton graphs.

otherwise a circuit of length 2d-2 would be formed. Therefore, every vertex in level d must be adjacent to exactly one vertex of level d-1 derived from each of the vertices in level 1.

For $g = 2d + 1$ there are $\rho(\rho-1)^{d-1}$ vertices in level d connected to the vertices in level d-1 in the usual tree like manner. The remaining edges in the graph are drawn between pairs of vertices in level d in such a way that each vertex derived from vertex U_i is adjacent to exactly one vertex of level d derived from each of the other vertices of level 1.

Graphs with $g = 2d$ and $g = 2d + 1$ are illustrated in FIG. 2.10(a) and (b).

<u>Moore Graphs</u>. $g = 2d + 1$.

Let U_i $(i=0, 1,...d)$ be the number of vertices at a distance i from vertex U_0 at level 0.

Then $U_0 = 1, U_1 = \rho$

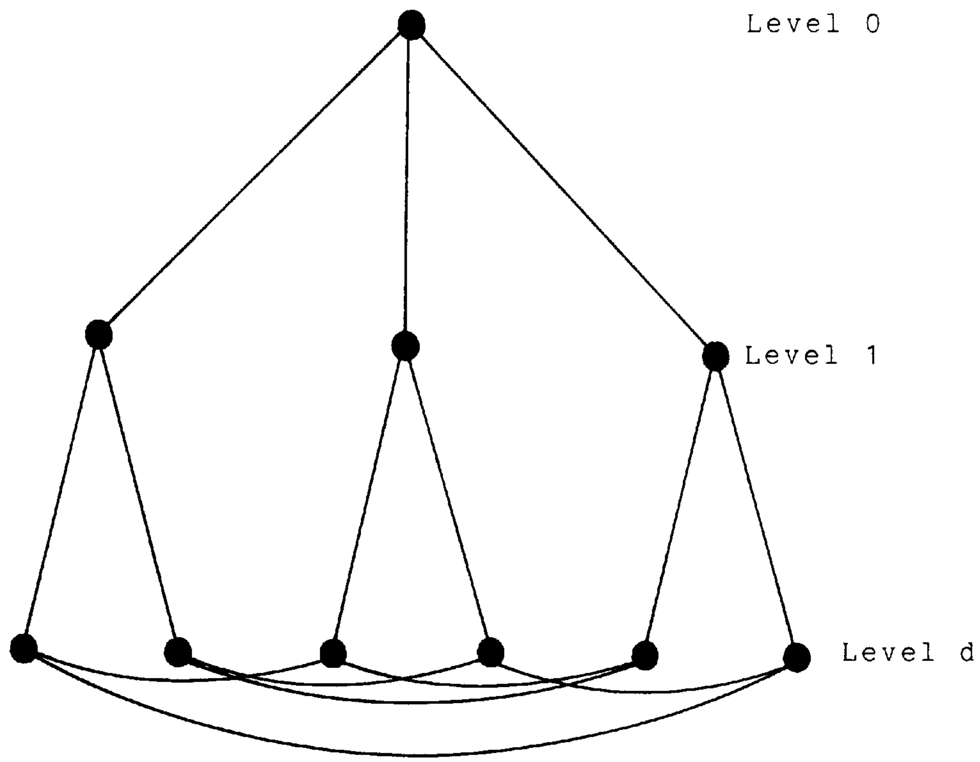
and $U_i = \rho(\rho-1)^{i-1} \qquad i \geq 1$

Hence

$$\sum_{i=0}^{d} U_i = |V| = 1 + \rho \sum_{i=1}^{d} (\rho-1)^{i-1}$$

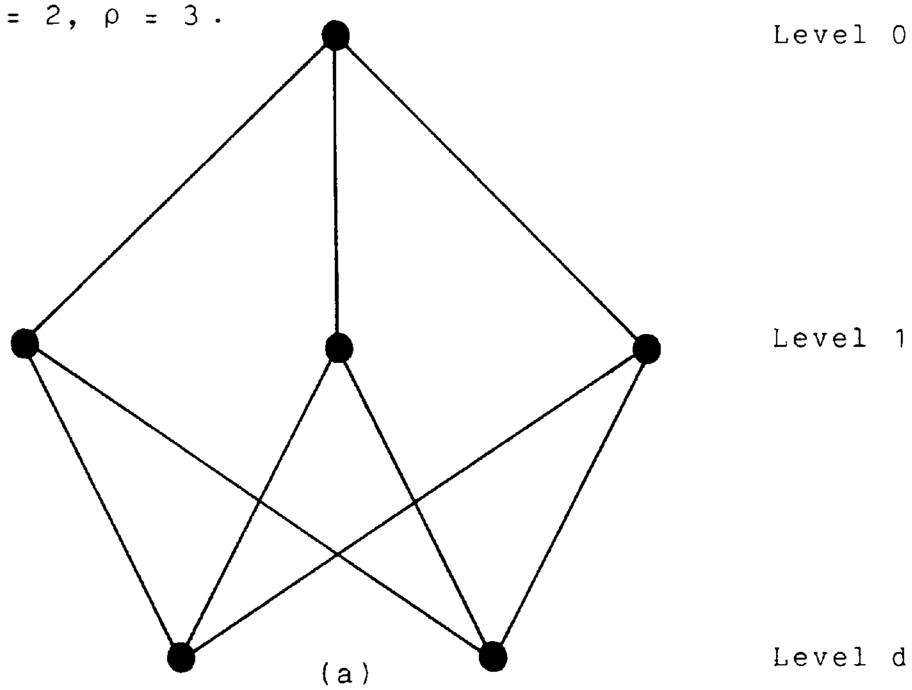
Thus we can write $|V| = 1+\rho S$ where $S = \frac{a(1-r^n)}{1-r}$, the sum of a geometric progression, with $a = 1$, $r = \rho-1$. Therefore $|V| = 1 + \rho S = 1 + \frac{\rho-\rho(\rho-1)^d}{2-\rho} = \frac{\rho(\rho-1)^d-2}{\rho-2}$.

$g = 2d + 1, d = 2, \rho = 3.$



(b)

$g = 2d, d = 2, \rho = 3.$



(a)

FIG. 2.10 Graphs with girth $g = 2d$, and $g = 2d + 1$.

Singleton Graphs. $g = 2d$.

At level d, $U_d = (\rho-1)^{d-1}$

Therefore $|V| = 1 + \ldots \rho(\rho-1)^{i-1} + \ldots(\rho-1)^{d-1}$ $i<d$

$|V|-1-(\rho-1)^{d-1} = \rho+\rho(\rho-1)+\rho(\rho-1)^2+\ldots\rho(\rho-1)^{d-2}$ (a)

Multiplying (a) by $(\rho-1)$ gives,

$\{|V|-1(\rho-1)^{d-1}\}(\rho-1) = \rho(\rho-1)+\rho(\rho-1)^2+\ldots\rho(\rho-1)^{d-1}$ (b)

Subtracting (a) and (b) we have,

$$\{|V|-1-(\rho-1)^{d-1}\}(2-\rho) = \rho-\rho(\rho-1)^{d-1}$$

$$|V|-1-(\rho-1)^{d-1} = \frac{\rho-\rho(\rho-1)^{d-1}}{(2-\rho)}$$

$$|V| = \frac{\rho-\rho(\rho-1)^{d-1}}{(2-\rho)} + (\rho-1)^{d-1}+1$$

$$|V| = \frac{(\rho-1)^{d-1}(-\rho+2-\rho)+2}{(2-\rho)}$$

$$|V| = \frac{(\rho-1)^{d-1}2(1-\rho)+2}{(2-\rho)}$$

$$|V| = \frac{2(\rho-1)^{d}-2}{(\rho-2)}.$$

R. S. Wilkov [53] has demonstrated that Moore and Singleton graphs are maximally connected regular graphs of minimum diameter. R. S. Wilkov [52] also shows that the known graphs illustrated by F. T. Boesch and R. E. Thomas [9] and S. L. Hakimi [25] with number of vertices $|V|$ and a maximum even connectivity of $k = \rho$ have a diameter $d = \frac{|V|}{\rho}$ whereas from equation (2b)

$|V| = \frac{\rho(\rho-1)^{d}-2}{\rho-2}$ and for ρ greater than 2 this is approximately ρ^d.

Thus d is approximately $\frac{\log |V|}{\log \rho}$ which grows much slower with $|V|$ than does $\frac{|V|}{\rho}$.

Unfortunately, Moore and Singleton graphs constitute only a small class of regular graphs of maximum node connectivity and minimum diameter. The 26 vertex Singleton graph of degree 4 and girth 6 obtained by R. S. Wilkov [52] is shown in FIG. 2.11 together with a table of results comparing the diameters of differently constructed graphs FIG. 2.12. The known diameter graphs in the table are those constructed by F. T. Boesch and R. E. Thomas [9] and S.L. Hakimi [25], the improved diameter graphs are those constructed by R. S. Wilkov [52].

Recent work by U. Schumacher [42] has utilized the tree like structure of Moore and Singleton graphs. In this work k-connected graphs are generated which have a minimum number of edges and a diameter which is twice as large as the theoretical minimum. This problem in terms of a communication network corresponds to designing a network with minimum costs and minimum transmission delay in which switching nodes are equally weighted.

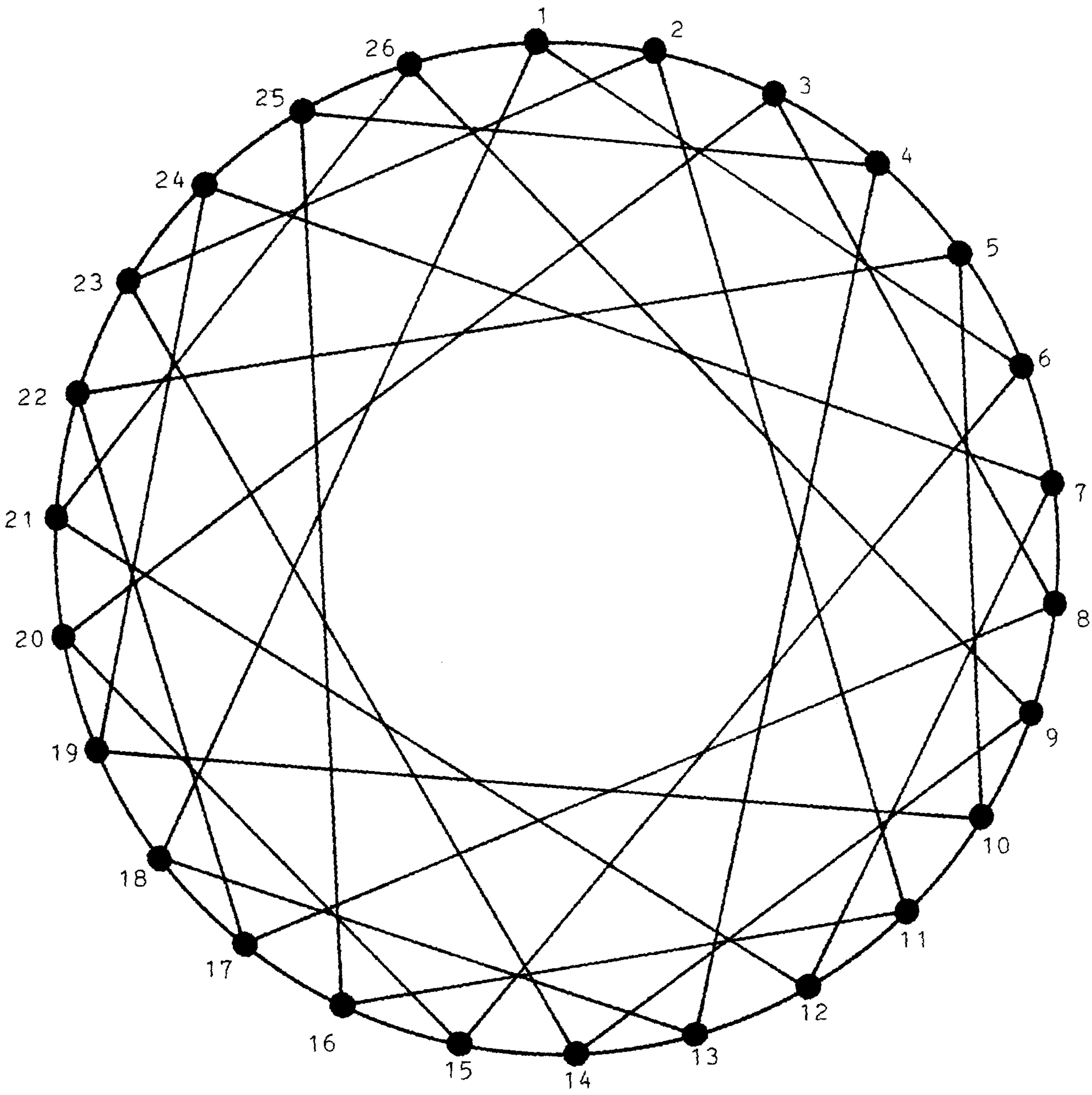


FIG. 2.11 Singleton graph with a connectivity of 4, diameter of 3, girth 6 and degree 4.

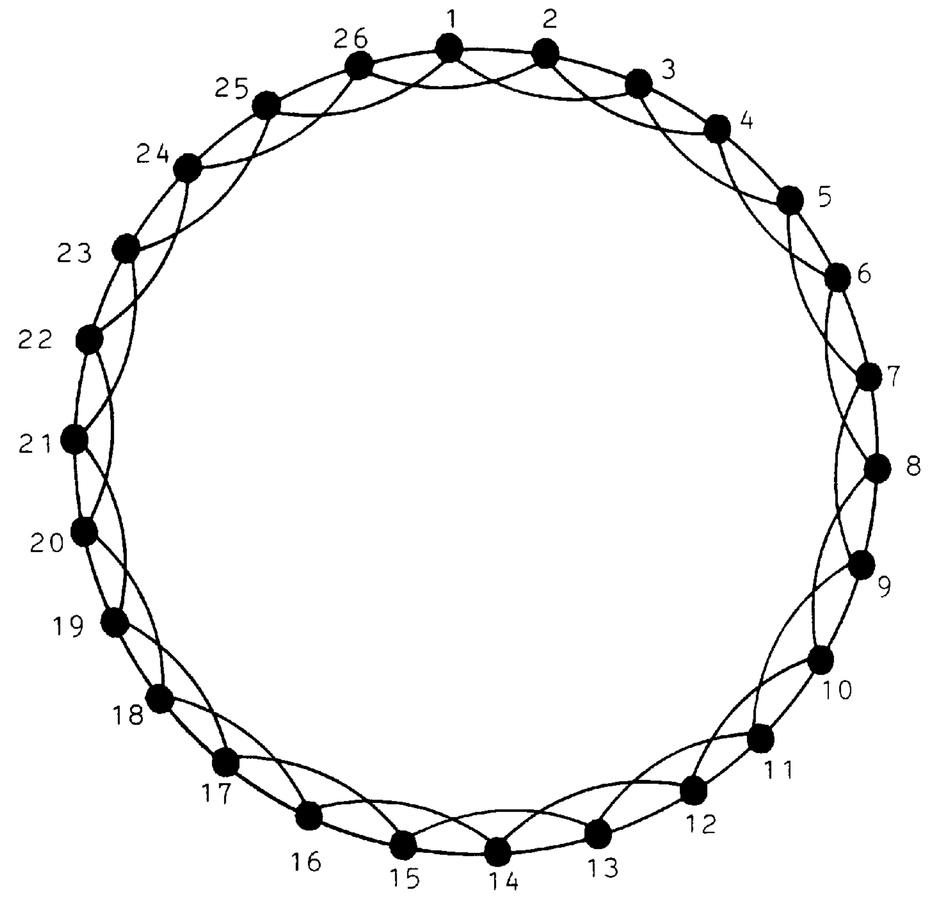
Vertices	Connectivity	Edges	Diameter (Known)	Diameter (Improved)
10	3	15	3	2
14	3	21	4	3
20	3	30	5	3
24	3	36	6	4
28	3	42	8	4
30	3	45	8	4
10	4	20	3	2
16	4	32	4	3
18	4	36	5	3
20	4	40	5	3
22	4	44	6	3
26	4	52	7	3
32	4	64	8	4
42	5	105	6	3
40	6	120	7	3
62	6	186	11	3
50	7	175	5	2
114	8	456	15	3

FIG. 2.12 Table comparing the diameters of maximally reliable graphs.

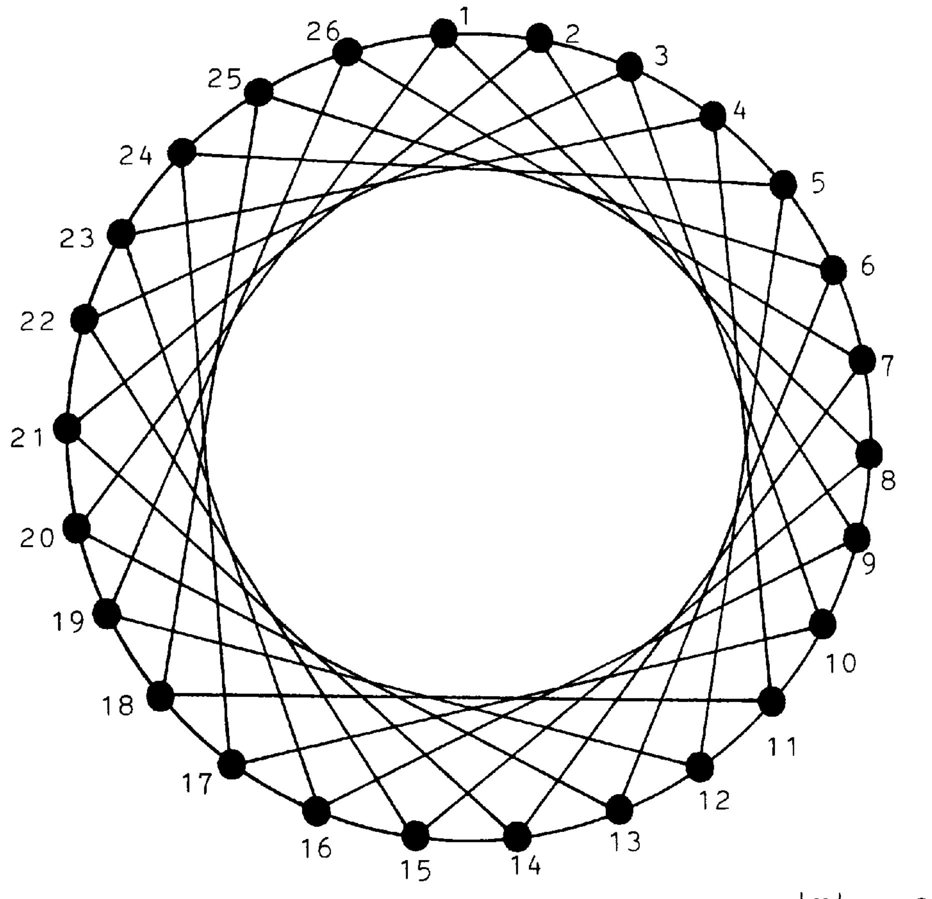
S. V. Trufanov [48] has given a method for reducing the diameter of the maximally connected graphs introduced by F. Harary [29] and S. L. Hakimi [25]. In the construction of such graphs of even degree ρ, S. V. Trufanov [48] suggests that the ρ-2 vertices on the circumference of a circle connected to each vertex i be chosen in such a way that the maximum distance from vertex i to any other vertex is minimised. When ρ is odd, this same rule is to be followed after vertex i has been joined to vertex $i+\frac{|V|}{2}$ (Mod |V|).

For the graph with 26 vertices of connectivity 4, vertex i is furthest from vertex i+13 on the circumference of the circle. For all i, the distance between vertices i and i+13 is minimised by also connecting vertex i to vertices i $\pm$ 7. The resulting graph is shown in FIG. 2.13(b). It is vertex symmetric and is found to have a diameter of only 4, compared with a diameter of 7 for the equivalent graph in FIG. 2.13(a).

A minimum (d, k, ρ) graph is one which is regular of diameter d, connectivity k, degree ρ and contains a minimum number of vertices |V|. The study of this class of graphs has been pioneered by V. Klee and H. Quaife [33] who have noted that such graphs have application in the design of reliable communication networks. When the value of ρ is 3, the graphs are called cubic graphs.



(a) Maximally connected graph |V| = 26, |E|= 52, d = 7, k = 4



(b) Modified maximally connected graph with |V| = 26, |E|= 52, d = 4, k = 4.

FIG. 2.13 Comparison of graphs with 26 vertices and diameters equal to 7 and 4.

V. Klee and H. Quaife [33] classified and enumerated all minimum (d, 1, 3) graphs and minimum (d, 2, 3) graphs. B. Myers [36] subsequently reviewed their work through systematic mathematical theorems and lemmas and provided some new insights into the general (d, k, ρ) graph problem.

The minimum number of vertices in a (d, 3, 3) graph with specified diameter $d<5$ has been determined by B. Myers [37] who also shows how to construct all such minimum graphs. The complete graph K_4 is the unique (1, 3, 3) graph, it has a diameter of 1 and therefore each pair of its points must be adjacent i.e. the graph must be complete. The only complete graph that is regular of degree 3 is K_4 shown in FIG. 2.14, the number of vertices $|V|$ being equal to $3d + 1 = 4$.

The only minimum (2, 3, 3) graphs are shown in FIG. 2.15 and the only minimum (3, 3, 3) graphs are shown in FIG. 2.16.

The minimum number of vertices in a (4, 3, 3) graph is $3d = 12$. There are thirty-one such graphs of diameter $d = 4$, each with $|V| = 3d = 12$, examples of these graphs are shown in FIG. 2.17(a), (b), (c) and (d).

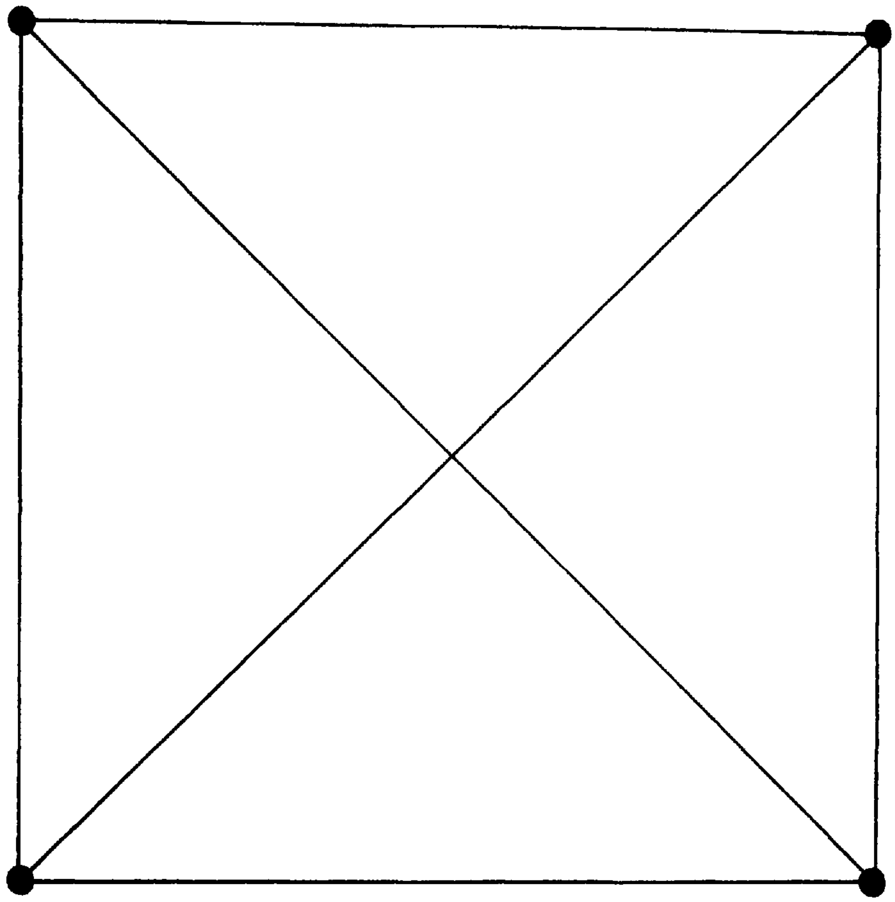


FIG. 2.14 The K_4 graph with $|V| = 3d + 1$, this is the unique (1, 3, 3) graph.

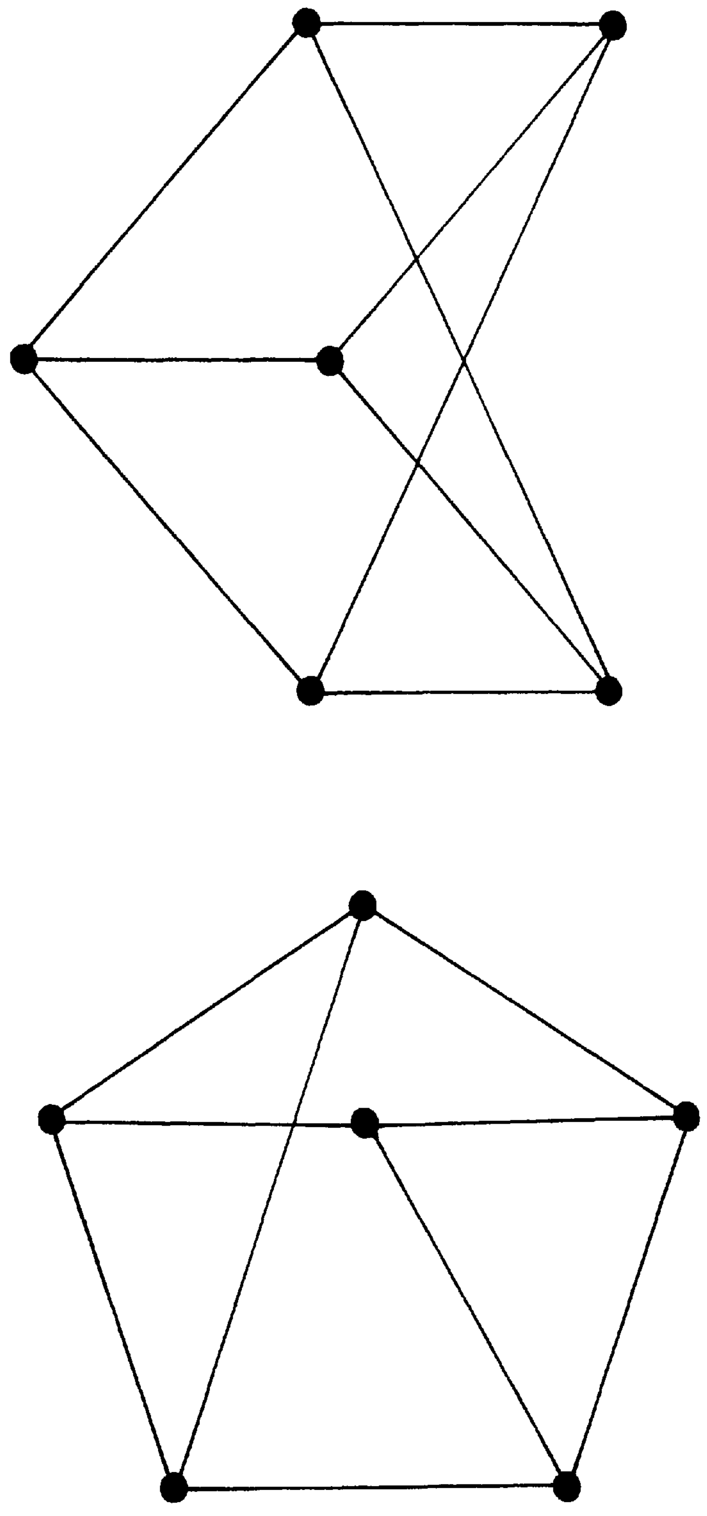


FIG. 2.15 Two minimum (2, 3, 3) graphs with a minimum number of vertices |V| equal to at least 3d when d is even.

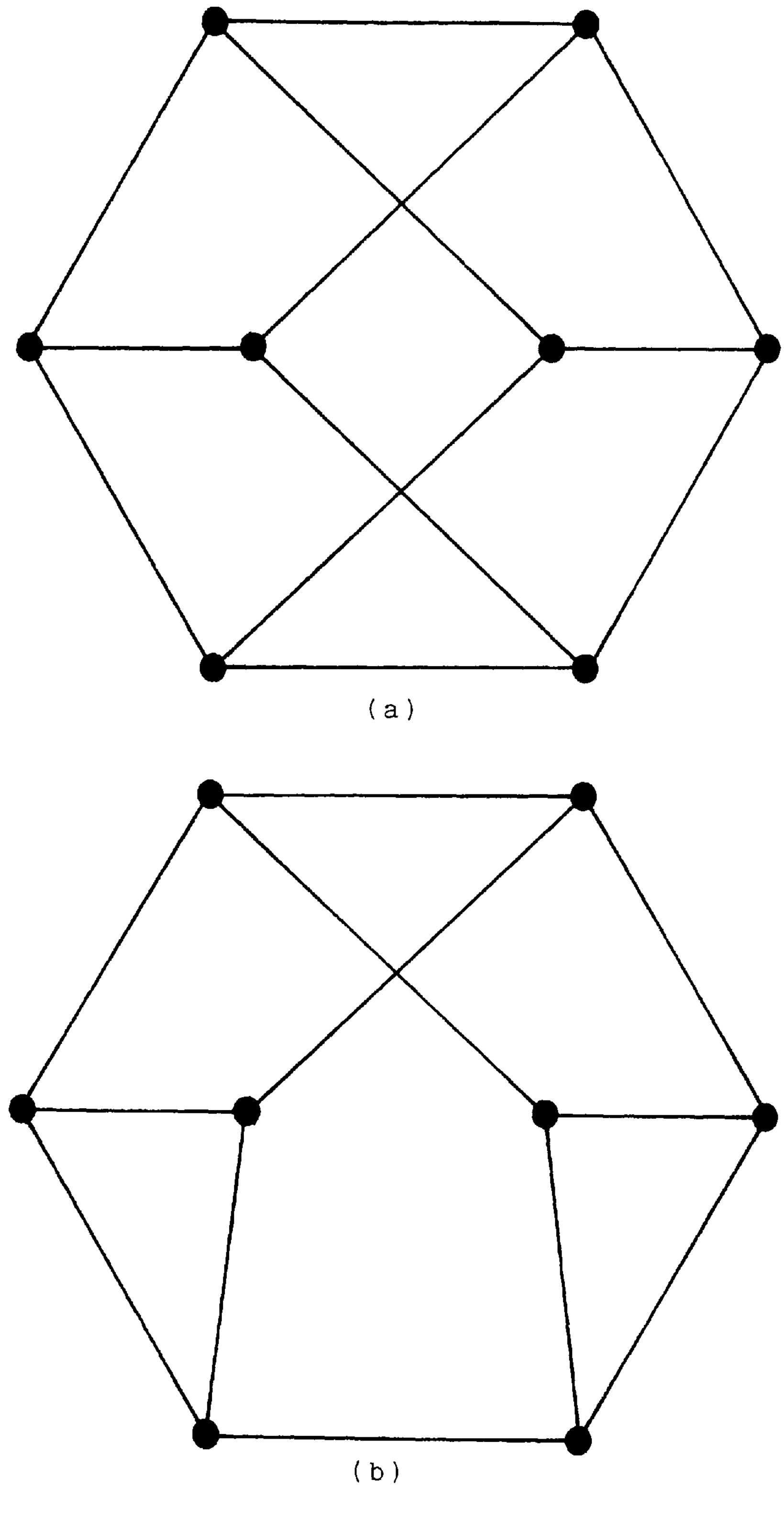
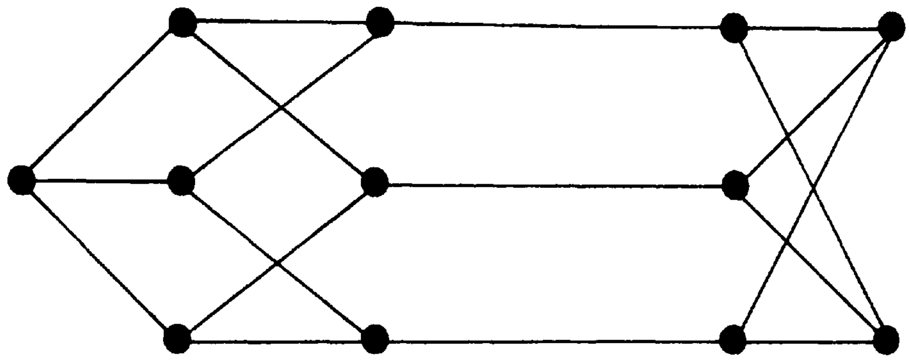
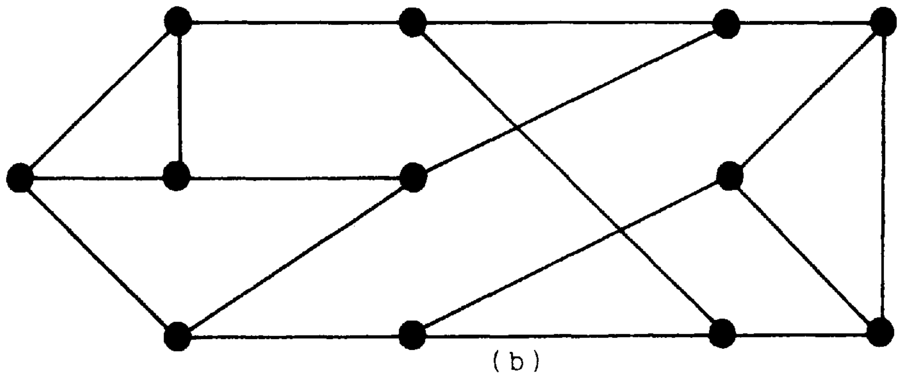


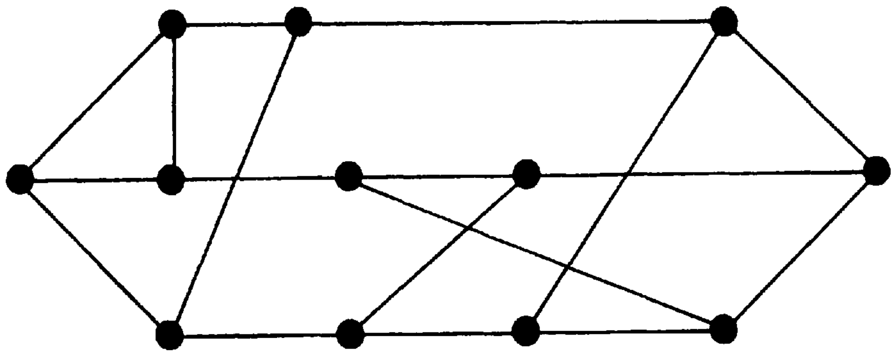
FIG. 2.16 Two minimum (3, 3, 3) graphs with minimum $|V| = 3d - 1$.



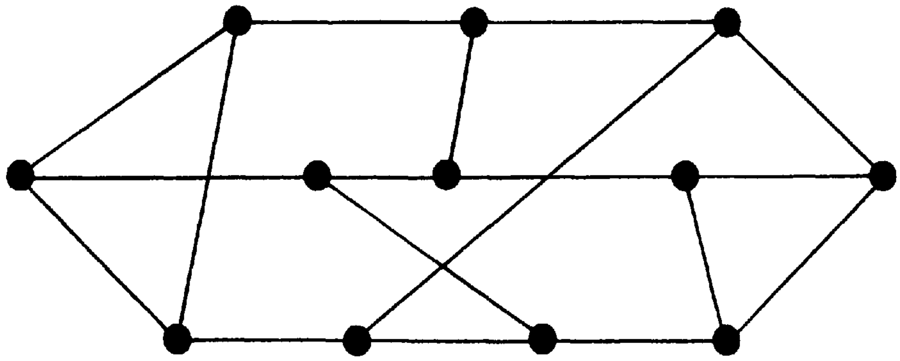
(a)



(b)



(c)



(d)

FIG. 2.17 Examples of the structure of minimum (4, 3, 3) graphs.

2.4 Total Isolation or Independence Number

Another measure of reliability that has been suggested by A. T. Amin and S. L. Hakimi [1] is the total number of vertices which must fail to give a graph consisting of only isolated vertices. The relevant graph parameter is called the independence number which we now define.

A set $S \subset V$ is called an independent set of G if no two vertices in S are adjacent in G. An independent set S_0 is a maximum independent set of G if $|S_0| \geq |S|$, where S is any independent set of G. The number of vertices in a maximum independent set of a graph G is called the independence number of G and is denoted by $\beta(G)$. Consider the graph shown in FIG. 2.18.

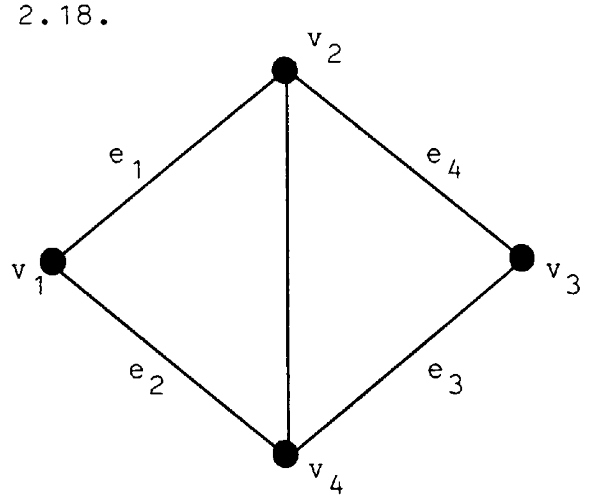


FIG. 2.18

Let $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{e_1, e_2, e_3, e_4\}$

$Y \subset V$ separates two non-adjacent vertices in G if in $V - Y$ there exists no path from v_i to v_j. Let $Y = \{v_2, v_4\}$, then $V - Y = \{v_1, v_3\}$.

i.e. $V - Y$ consists of isolated vertices.

Let $S = V - Y$, $Y \subset V$ denote the maximum subgraph of G containing vertices not in Y. Let $Y \subset V$ be such that $V - Y$ consists of isolated vertices. Thus a set $Y \subset V$ is a minimum set with the property that $V - Y$ consists of isolated vertices if and only if $V - Y$ is a maximum independent set of G. A network N(G) is optimally reliable in their sense if the independence number $\beta(G)$ is the minimum possible over all graphs with $|V|$ vertices and $|E|$ edges.

It may be noted that in a communication network N(G), failure of such a set Y results in complete disruption of communication. In this sense, the reliability of network N(G) corresponds to the size of the minimum such set in the graph G.

Consider the graphs G_1 and G_2 in FIG. 2.19. Each of the graphs has the same number of vertices and edges.

Moreover $k(G_1) = k(G_2) = 3$ while $\beta(G_1) = 3$ and $\beta(G_2) = 2$; thus the corresponding network $N(G_2)$ is more reliable than the network $N(G_1)$.

A. T. Amin and S. L. Hakimi [1] considered the problem of finding the minimum value of the independence number $\beta(G)$ given the optimal connectivity $k \geqq a$, number of vertices $|V|$, number of edges $|E| = \frac{|V| a}{2}$ for given values of a and $|V|$. They give a complete solution for the case where $|E|$ is odd. For the case of $|E|$ even, they give a partial solution.

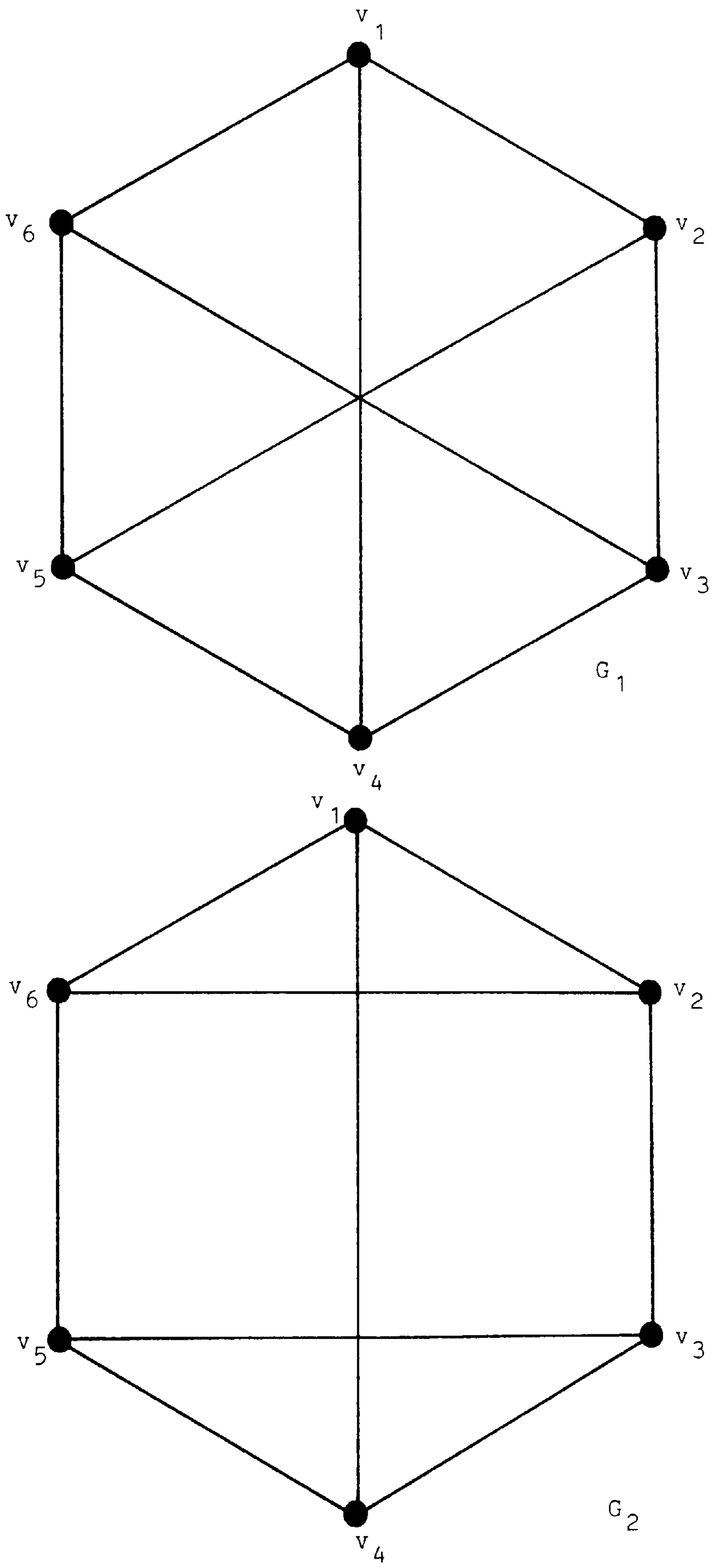


FIG. 2.19 Two graphs with different independence numbers $\beta(G_1) = 3$, $\beta(G_2) = 2$.

2.5 Probability of Connection

One common definition of the reliability of a graph is the probability that the graph is connected given the probability that an edge or a vertex is operating or failed under the assumption that edges or vertices fail independently. A further assumption is that either all vertices or edges are perfectly reliable. We firstly consider graphs with perfectly reliable vertices and unreliable edges and then graphs with perfectly reliable edges and unreliable vertices.

A comprehensive study has been made by A. K. Kelmans [32] of maximally reliable networks whose links fail independently with equal probability. In this study it was assumed that the vertices of the graph G were perfectly reliable and all edges failed independently with the same probability p. As a measure of reliability, he considered the probability $P_c(G)$ that the graph G was a connected graph. The connection probability $P_c(G)$ for a graph G with $|E|$ edges and $|V|$ vertices is given by:-

$$\underset{\text{(Probability of Connection)}}{P_c(G)} = \sum_{i=|V|-1}^{|E|} T_i (1-p)^i p^{|E|-i}$$

Where T_i denotes the number of connected spanning subgraphs of G consisting of exactly i edges.

Alternatively,

$$\underset{\text{Connection}}{\overset{P_c(G)}{(\text{Probability of})}} = 1 - \sum_{i=\lambda}^{|E|} R_i p^i (1-p)^{|E|-i}$$

where R_i denotes the number of disconnecting subgraphs containing exactly $|E|-i$ edges and λ is equal to the edge connectivity.

Thus,

$$\underset{(\text{Probability of Disconnection})}{P_d(G)} = \sum_{i=\lambda}^{|E|} R_i p^i (1-p)^{|E|-i}$$

For values of p close to 1,

$$\underset{(\text{Probability of Connection})}{P_c(G)} \sim T_{|V|-1} (1-p)^{|V|-1} p^{|E|-|V|+1},$$

where $T_{|V|-1}$ is the number of spanning trees in the graph.

When p is close to 0,

$$\underset{(\text{Probability of Connection})}{P_c(G)} \sim 1 - R_\lambda p^\lambda,$$

where R_λ is the number of cut sets containing λ edges.

It follows that for values of p close to 1, since $P_c(G)$ the probability of connection $\sim T_{|V|-1}(1-p)^{|V|-1} p^{|E|-|V|+1}$, a maximally reliable network is one with a maximum number of trees.

For p close to 0, $P_c(G)$ the probability of connection $\sim 1 - R_\lambda p^\lambda$, and the best graph with $|E|$

edges and $|V|$ vertices has a minimum number of cut sets R_λ of size λ.

A. K. Kelmans [32] has shown that there exists two graphs G_1 and G_2 such that $P_c(G_1) < P_c(G_2)$ for values of p close to 0 and $P_c(G_1) > P_c(G_2)$ for values of p close to 1. Thus the structure of maximally reliable graphs depends on the value of the edge failure probability, this will be illustrated in section 2.6, FIG. 2.23.

In the event that all edges are perfectly reliable and all vertices are likely to fail independently with probability q, then the probability of disconnection $P_d(G)$ is given by:

$$\underset{\text{(Probability of Disconnection)}}{P_d(G)} = \sum_{i=k}^{|V|-2} S_i q^i (1-q)^{|V|-i}$$

where S_i denotes the number of disconnected subgraphs of G resulting from the removal of exactly i vertices, k = vertex connectivity.

The probability of connection $P_c(G)$ is given by:

$$\underset{\text{(Probability of Connection)}}{P_c(G)} = 1 - \sum_{i=k}^{|V|-2} S_i q^i (1-q)^{|V|-i}$$

When q is close to 0,

$$\underset{\text{(Probability of Connection)}}{P_c(G)} \sim 1 - S_k q^k ,$$

and a maximally reliable graph has a maximum vertex connectivity. Therefore for small values of q, the best graph with $|V|$ vertices and $|E|$ edges has a minimum number of vertex cut sets of size k.

It has been shown by H. Frank [23] that for q close to 0, the complete bipartite graph with $|V_1|$ vertices in one subset and $|V_2| > |V_1|$ vertices in the other has a larger connection probability than any other graph of connectivity $|V_1|$ having no more than $|V_1||V_2|$ edges. The bipartite class of optimal graphs also have the additional property that the minimum size cut sets can only isolate a single vertex.

Consider the regular graph G with $|V|$ vertices divided into r classes such that $|V| = ru$. Each class contains exactly $(|V| - k)$ vertices and k the vertex connectivity $= (r-1)u$. Then two vertices are adjacent in G if and only if they belong to distinct classes. Such a graph is called a complete r-partite graph. F. T. Boesch and A. Felzer [5] generalizing a result of H. Frank [23], have shown that such graphs have the minimum number of distinct vertex cut sets among all regular graphs of degree $\rho = k$ with the number of vertices $|V| = (|V| - k)r$. Ching-Shui Cheng [18] has also shown that regular complete multipartite graphs have the maximum number of spanning trees among all the simple graphs with the same numbers of vertices and edges. The networks of such graphs are therefore desirable from probabilistic reliability considerations.

A. K. Kelmans [32] has shown that in a graph G given the edge failure probability p, number of edges $|E|$, number of vertices $|V|$, and specified degree of reliability R_m, then:

$|E|$ the number of edges required $\geq \frac{|V| \ln |V|}{2 |\ln p|}$

and there exists a value $|V_m|$ for which $R(G) \geq R_m$ whenever $|V| \geq |V_m|$. In addition the average degree of each vertex must be greater than or equal to $\frac{\ln |V|}{|\ln p|}$ otherwise the probability that the graph is connected will decrease as the number of vertices increase, regardless of the structure of the graph.

The probability of disconnection of a graph G is minimised over all graphs with $\frac{k|V|}{2}$ edges if G is regular with degree $\rho = k$ and S_k (the number of vertex cut sets with k vertices) is minimised.

D. H. Smith [45] has shown that in many cases it is possible to construct a graph with the minimum number of vertex cut sets with k vertices. From a practical point of view this solution is open to the criticism that although the probability of disconnection is minimised, when disconnections do occur a rather large number of vertices may be isolated. Depending on the application, it might be more sensible to require that the expected number of vertices disconnected from the largest remaining component of the graph (or isolated if all components are isolated vertices) be minimised. This will be dealt with in Section 2.8 and Chapter 8.

2.6 Complexity (or the number of spanning trees)

Given any connected graph G, we can choose a circuit and remove one of its edges, the resulting graph remaining connected. We repeat this procedure with one of the remaining circuits, continuing until there are no circuits left. The graph which remains will be a tree which connects all the vertices of G; it is called a spanning tree of G. The total number of spanning trees in a graph is called the complexity denoted by $T_{|V|-1}$. An example of a graph and three of its spanning trees is shown in FIG. 2.20.

It is evident that a spanning tree represents a minimum set of edges which preserves the connectedness of a graph. This concept is in a sense complementary to that of a proper cut set of edges (which is a minimum set of edges whose removal disconnects some vertices from others). These notions are related by the following theorem,

Theorem: In a connected graph, every cut set of edges has at least one edge in common with every spanning tree.

Proof: Let Y be a cut set of edges of a graph G and let T be a spanning tree of G. Then, if Y did not contain at least one edge from T, the removal of Y from G would not separate G into two or more components. ■

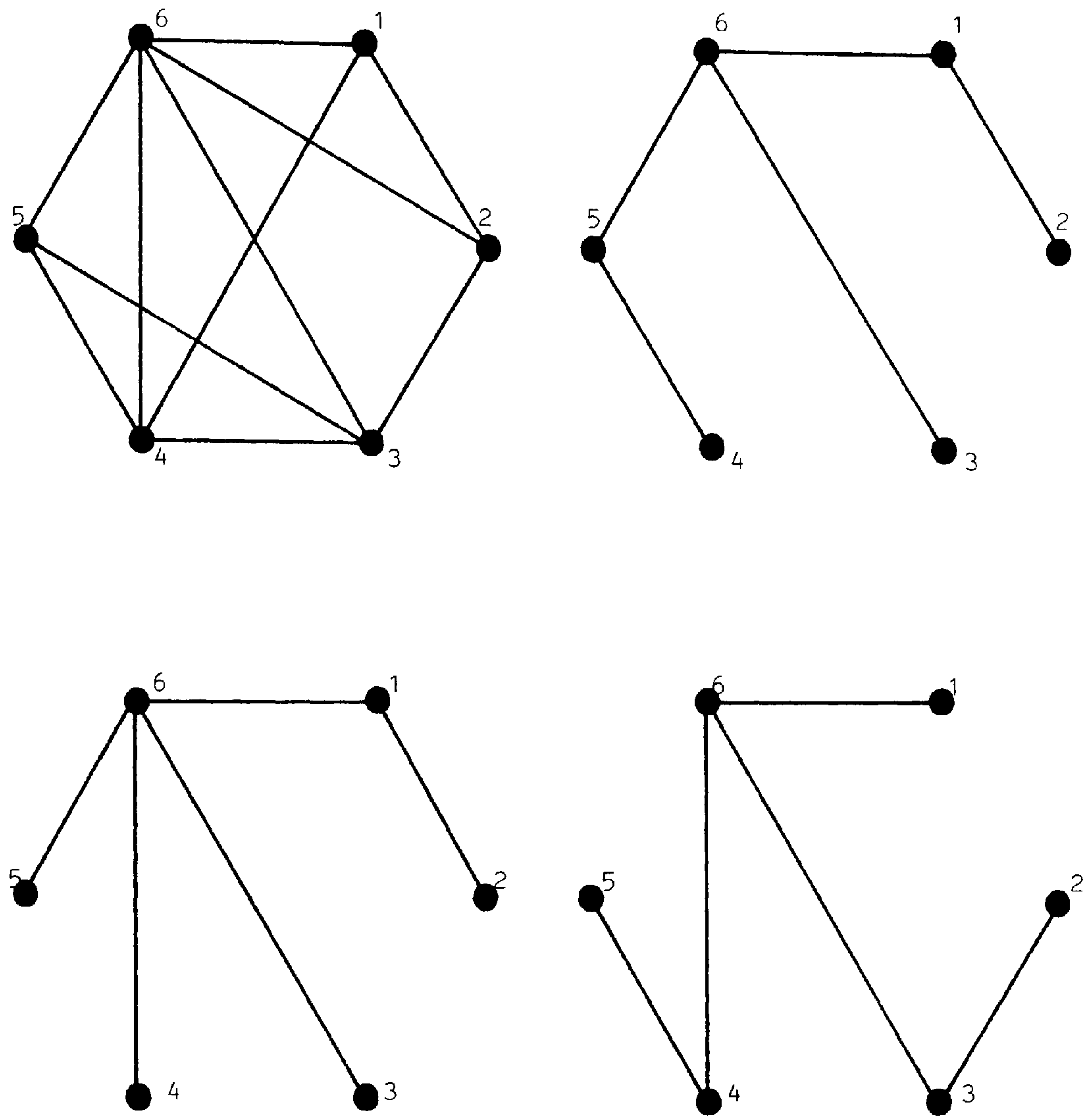


FIG. 2.20. Illustration of a graph G and three of its spanning trees.

The suggestion to classify the "connectedness" of a graph by the complexity i.e. the number of spanning trees is of interest because the number can be calculated for any graph. Many methods are known for calculating the number of spanning trees (or complexity $T_{|V|-1}$) of a regular graph, but for a given number $|V|$ of vertices and a given degree ρ it appears to be a rather difficult question to find which graphs have maximum complexity. It is known that $\frac{1}{|V|}\left(\frac{|V|\rho}{|V|-1}\right)^{|V|-1}$ is an upper bound for the complexity of a regular graph, N. Biggs [3] pp36-38.

For the complete graph $K_{|V|}$, the complexity $T_{|V|-1}$ equals $|V|^{|V|-2}$ and its proof may be found in R. J. Wilson [54] pp 50-52. Further examples of complexity calculations are given below.

Let G be a regular graph with degree ρ and $|V|$ vertices. The spectrum of a graph G is the set of numbers which are eigenvalues of A(G), together with their multiplicities (m_r) as eigenvalues (λ_r) of A(G).

$$\text{Spectrum (G)} = \begin{pmatrix} k & \lambda_1 & \cdots & \lambda_{S-1} \\ 1 & m_1 & \cdots & m_{S-1} \end{pmatrix}$$

Then the complexity of G is given by N. Biggs [3] as,

$$T_{|V|-1} = \frac{1}{|V|} \prod_{r=1}^{S-1} (k-\lambda_r)^{m_r}. \qquad k \neq \lambda_r$$

An important result which can be used to calculate the number of spanning trees in any connected simple graph is stated below, it is known as the matrix tree theorem, and its proof may be found in F. Harary [29].

<u>Theorem</u> Let G be a connected simple graph with vertex set V and let $A_m = (a_{ij})$ be the $|V| \times |V|$ matrix in which $a_{ii} = \rho(v_i)$, $a_{ij} = -1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. Then the number of spanning trees of G is equal to the cofactor of any element of A_m.

We now show how to calculate the complexity of a simple graph G using each of the methods mentioned previously. The graph G used in the example is shown in FIG. 2.21.

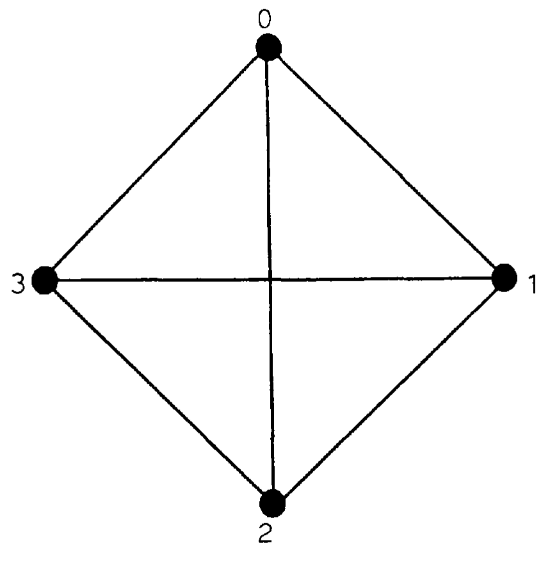


FIG. 2.21

(a) The graph is complete, therefore

$$T_{|V|-1} = |V|^{|V|-2} = 4^2 = \underline{16}$$

(b) This method uses the eigenvalues of the adjacency matrix A(G).

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Characteristic equation $|A(G) - \lambda I| = 0$

giving $$\begin{vmatrix} -\lambda & 1 & 1 & 1 \\ 1 & -\lambda & 1 & 1 \\ 1 & 1 & -\lambda & 1 \\ 1 & 1 & 1 & -\lambda \end{vmatrix} = 0$$

From which $\lambda = -1, -1, -1, 3$

Thus $$T_{|V|-1} = \frac{1}{|V|} \prod_{r=1}^{|V|-1} (k - \lambda_r)^{m_r}$$
$$= \frac{1}{4} \{ (3+1)(3+1)(3+1) \} = \underline{16}$$

(c) Using the matrix tree theorem, we have

$$A_m = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

$T_{|V|-1}$ = cofactor of any element of A_m = $\underline{16}$

The sixteen spanning trees calculated in this example are shown in FIG. 2.22.

In Section 2.5 we have indicated that when considering probabilistic reliability criteria the calculation of the complexity of a graph is of

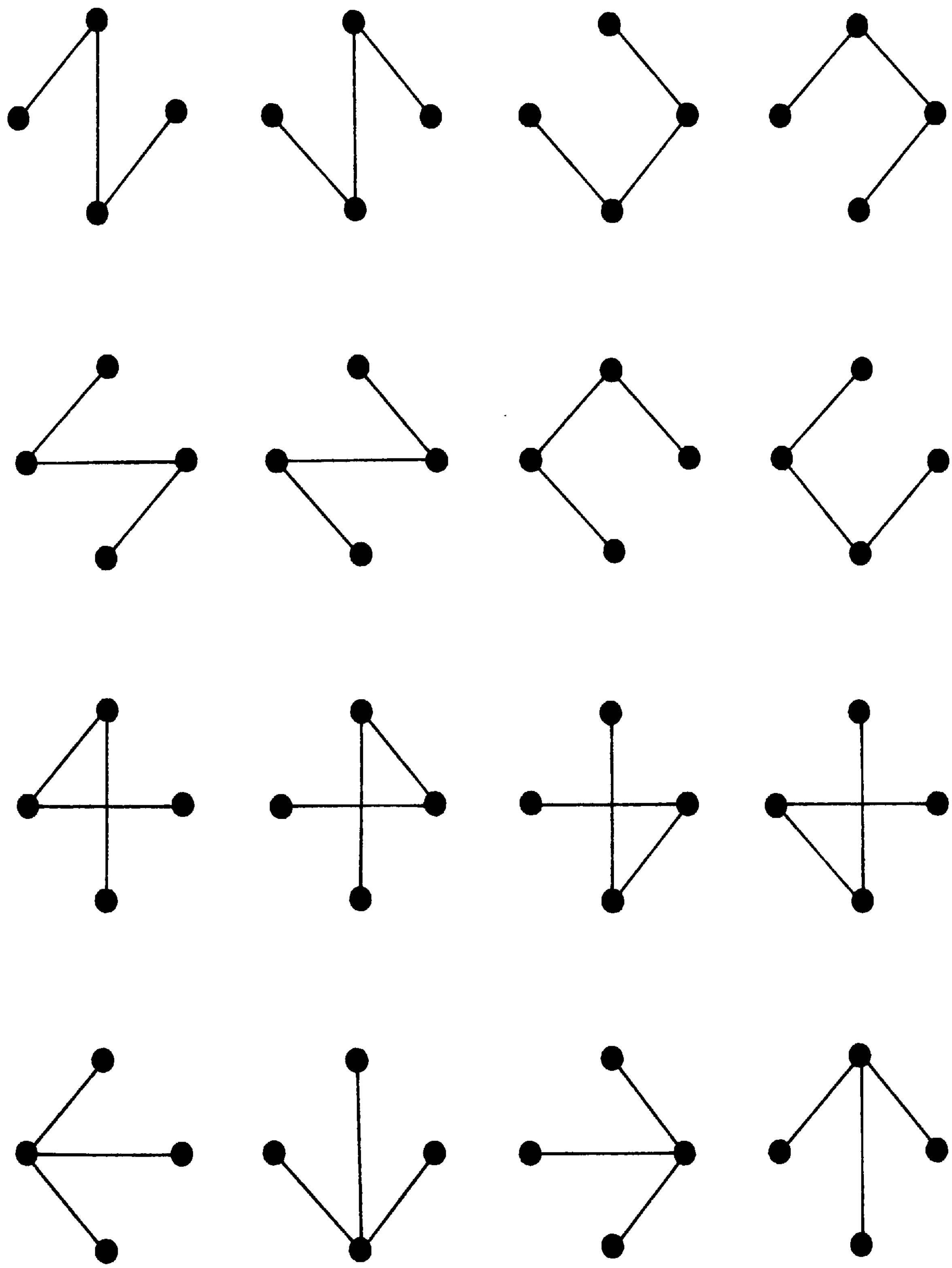
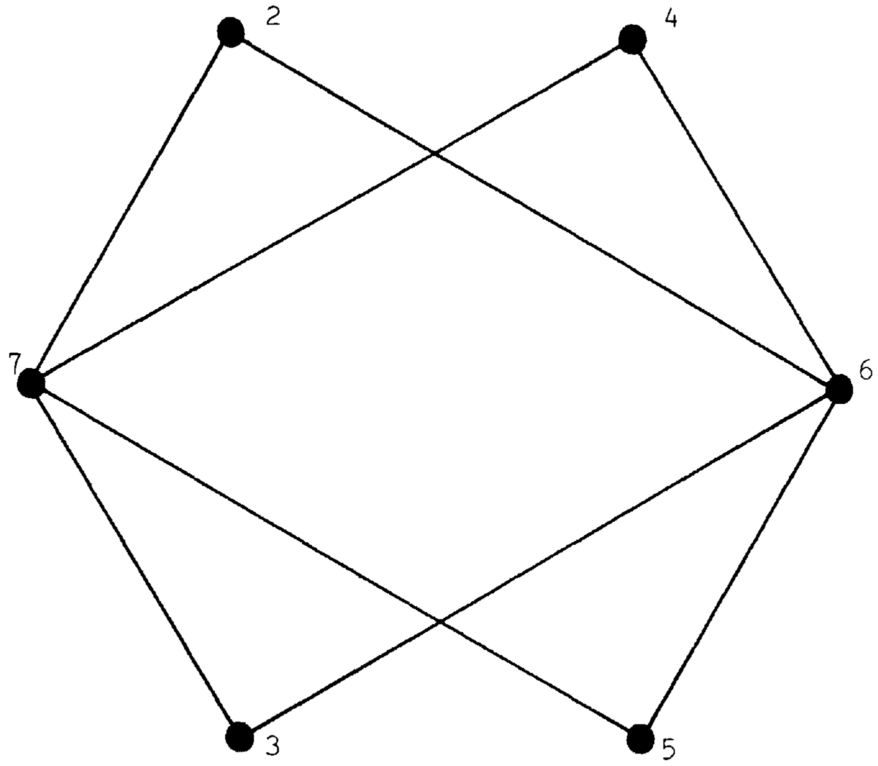


FIG. 2.22. Illustration of the 16 spanning trees of the graph G of FIG. 2.21.

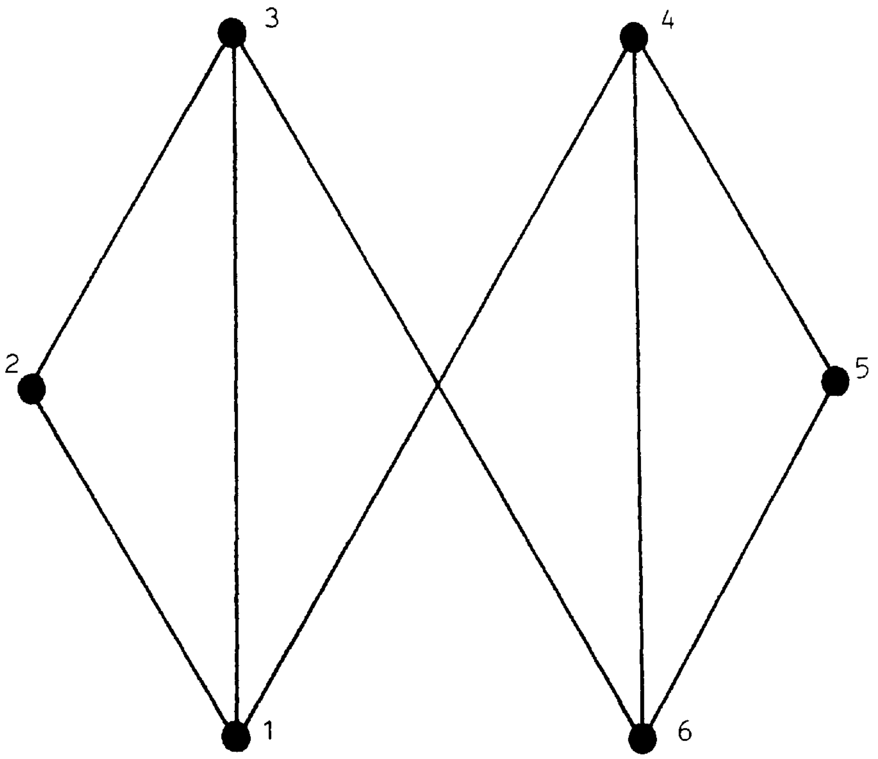
interest when p the probability of failure of an edge is close to 1 (assuming the edges fail independently). For p close to 0 the smallest number of minimum size edge cut sets is important. FIG. 2.23(a) and FIG. 2.23(b) compares two graphs with the same number of vertices $|V|$ and edges $|E|$ and shows by calculating the complexity and listing the edge cut sets that one graph is better than the other for p close to 0, and worse when p is close to 1.



(a)

better for large p

($T_{|V|-1}$ = 32, λ = 2, R_{λ} = 4)



(b)

better for small p

($T_{|V|-1}$ = 30, λ = 2, R_{λ} = 3)

FIG. 2.23. Two graphs with different reliability characteristics, both having 6 vertices and 8 edges.

Calculation of the complexity ($T_{|V|-1}$) of the graph in FIG. 2.23(a)

$$A_m = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 & 0 \\ -1 & 2 & 0 & 0 & 0 & -1 \\ -1 & 0 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 & 0 & -1 \\ -1 & 0 & 0 & 0 & 2 & -1 \\ 0 & -1 & -1 & -1 & -1 & 4 \end{bmatrix}$$

$T_{|V|-1}$ = cofactor of any element of A_m = $\underline{32}$

For FIG. 2.23(a),

R_λ = Number of edge cut sets with λ edges = {14,46}, {12,26}, {13,36}, {15,56}.

$$\lambda = \underline{2}, \quad R_\lambda = \underline{4}$$

Calculation of the complexity ($T_{|V|-1}$) of the graph in FIG. 2.23(b)

$$A_m = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ -1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

$T_{|V|-1}$ = cofactor of any element of A_m = $\underline{30}$

For FIG. 2.23(b),

R_λ = Number of edge cut sets with λ edges = {12,23}, {45,56}, {14,36}.

$$\lambda = \underline{2}, \quad R_\lambda = \underline{3}$$

2.7 (k, k+1)-Connected Graphs

The neighbour set of a vertex v is the set of vertices $\Gamma(v)$ that are adjacent to v. A k-connected graph is said to be (k, k+1)-connected if every vertex cut set with k vertices is the neighbour set of a vertex. A k-edge-connected graph is (k, k+1)-edge-connected if the set of edge cut sets of size k is the set of all sets of edges incident with a single vertex.

We note that the above definition of a (k, k+1)-connected graph is neither a necessary nor a sufficient condition for a graph to have the minimum number of vertex cut sets which are neighbour sets of vertices which is very much less than $|V|$. (k, k+1)-connected graphs have been constructed for each $(|V|,k)$ by S. L. Hakimi and A. T. Amin [26].

The number of vertex cut sets of size k in a graph of connectivity k has been used as a measure of network reliability. Let G be a regular graph with $|V|$ vertices, degree $\rho = k$, connectivity k, and with the minimum number of vertex cut sets with k vertices. D. H. Smith [45] has shown how to construct infinite families of such graphs in various cases.

We note that since, for any graph G, $k \leqq \lambda$ and G is said to be k-connected if the connectivity $\geqq$ k and

G is said to be k-edge-connected if $\lambda \geqq k$, also $\rho = k$, then k-connected $\Rightarrow$ k-edge-connected.

Theorem If G has degree $\rho = k > 3$ and is $(k, k+1)$-connected then G is $(k, k+1)$-edge-connected.

Note If the graph G has $\rho = k = 3$ then there is a graph shown in FIG. 2.24 which is a counter example, it is (3,4)-connected but not (3,4)-edge-connected.

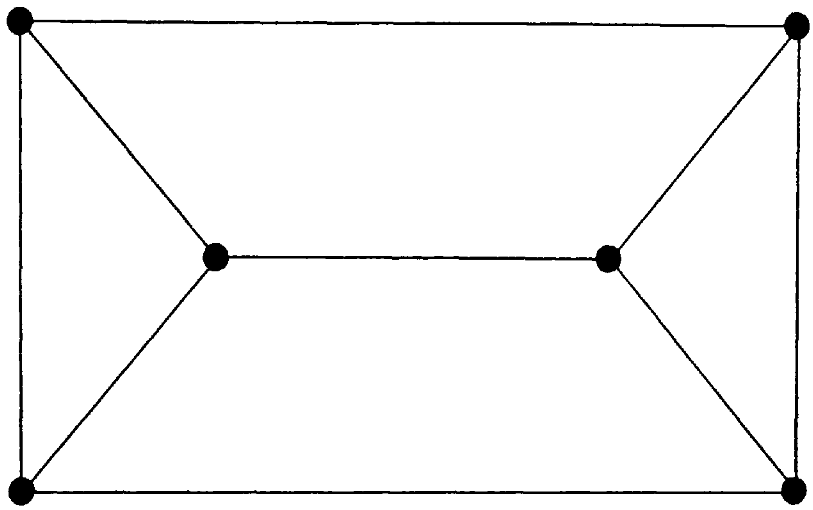


FIG. 2.24. A graph which is (3,4)-connected but not (3,4)-edge-connected.

Proof Since connectivity $\leqq$ edge connectivity $\leqq$ minimum valency G has edge connectivity k. Let E be an edge cut set of G with k edges. Let V_1 be the set of vertices of one component of G-E and let V_2 be the set of vertices of G that are not in V_1. Suppose that E is not the set of edges incident with a single vertex. Then the ends of the edges of E are all distinct or if two edges share an end v, then v together with a set consisting of one end from each remaining edge, will be a vertex cut set with less than k vertices. Now choose two vertices of V_1 that are ends of edges of E and the k-2 vertices that are ends of edges of E and in V_2 but not adjacent to either of the two vertices chosen from V_1. This is a vertex cut set that is not the neighbour set of a vertex unless G is the graph of FIG. 2.24. ■

Examples of (k, k+1)-connected graphs are shown in FIG. 2.25(a) and FIG. 2.25(b).

Using a famous result of L. R. Ford and D. R. Fulkerson [22] known as the maximum-flow, minimum-cut theorem we describe and illustrate how to make use of a flow algorithm for measuring in a graph G,

(i) edge connectivity.

(ii) vertex connectivity.

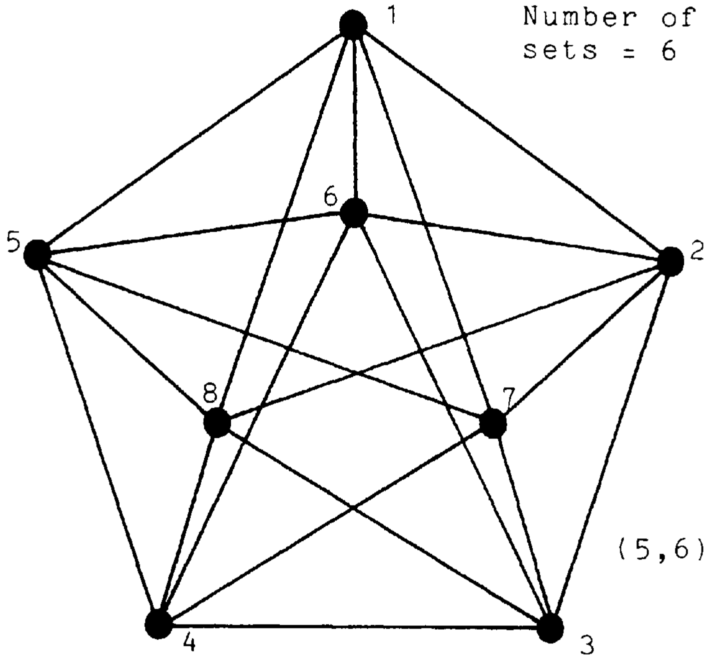
(iii) (k, k+1)-edge-connectivity.

Before dealing with these cases we briefly discuss the notion of network flows.

Given a digraph G = G(V,E) we define a flow in G to be a function Ø which assigns to each arc e of G a non-negative real number f (called the flow in e), in such a way that (i) for any arc e, $f \leqq c$ (where c is a non-negative real number called the capacity of the arc , i.e. the maximum permissible value of the flow in the edge), (ii) with respect to the

$\frac{k}{|V|} = \frac{5}{8}$

Number of vertex cut sets = 6

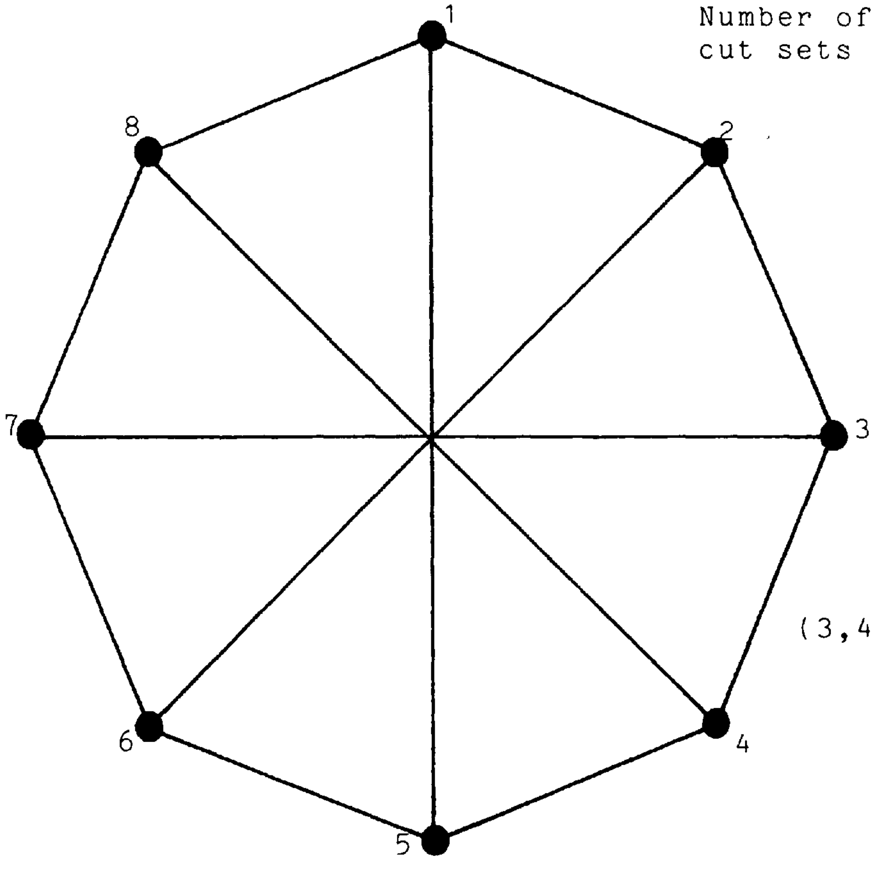


(5,6)-connected

(a)

$\frac{k}{|V|} = \frac{3}{8}$

Number of vertex cut sets = 8



(3,4)-connected

(b)

FIG. 2.25. Two diagrams showing (k, k+1)-connected graphs.

graph G the out- flow and in- flow of any vertex (other than the source S or sink t) are equal. This means that the flow in any arc cannot exceed its capacity, and that the total flow into any vertex (other than S or t) is equal to the total flow out of it. It follows that the amount flowing out of S equals the amount flowing into t and this is called the value of the flow. The capacity of a cut is defined to be the sum of the capacities of the forward arcs of the cut. We shall be concerned with those cuts whose capacity is as small as possible i.e. the minimal cuts. The maximal flow is defined as a flow whose value is as large as possible.

Theorem In any digraph, the value of any maximal flow is equal to the capacity of any minimal cut. The proof of this maximum-flow, minimum-cut theorem is found in R. J. Wilson [54] pp 133-134.

A displacement graph G(f) associated with a flow f on a graph G is the graph with the vertices as in G and arcs determined as follows. For each arc e_i of G, G(f) has (i) a normal arc e_i^+ which has the same initial and terminal endpoints as e_i and (ii) a reverse edge e_i^- which has the same endpoints as e_i but the opposite orientation. The capacities of e_i^+ and e_i^- which are denoted by c_i^+ and c_i^- respectively are defined by,

$$\left.\begin{aligned} c_i^+ &= c_i - f_i \\ c_i^- &= f_i \end{aligned}\right) \qquad (i = 1, 2, \ldots |E|)$$

A flow-augmenting circuit f_{ac} of a displacement graph G(f) is defined as a circuit which travels along $e_r = (v_t v_S)$ but not $e_r^- = (v_S v_t)$ and whose edges all have non-zero capacities. The arc $e_r = (v_t v_S)$ is called the return arc of G and has infinite capacity. FIG. 2.26(a),(b),(c) and (d) illustrates a graph G, its displacement graph G(f) and a flow-augmenting circuit associated with G(f). The significance of flow-augmenting circuits is given in the following theorem.

<u>Theorem</u> A flow f on a digraph G is a maximal flow if and only if G(f) does not contain any flow-augmenting circuits.

The proof of the theorem is given by B. Carre [16] pp 207-208.

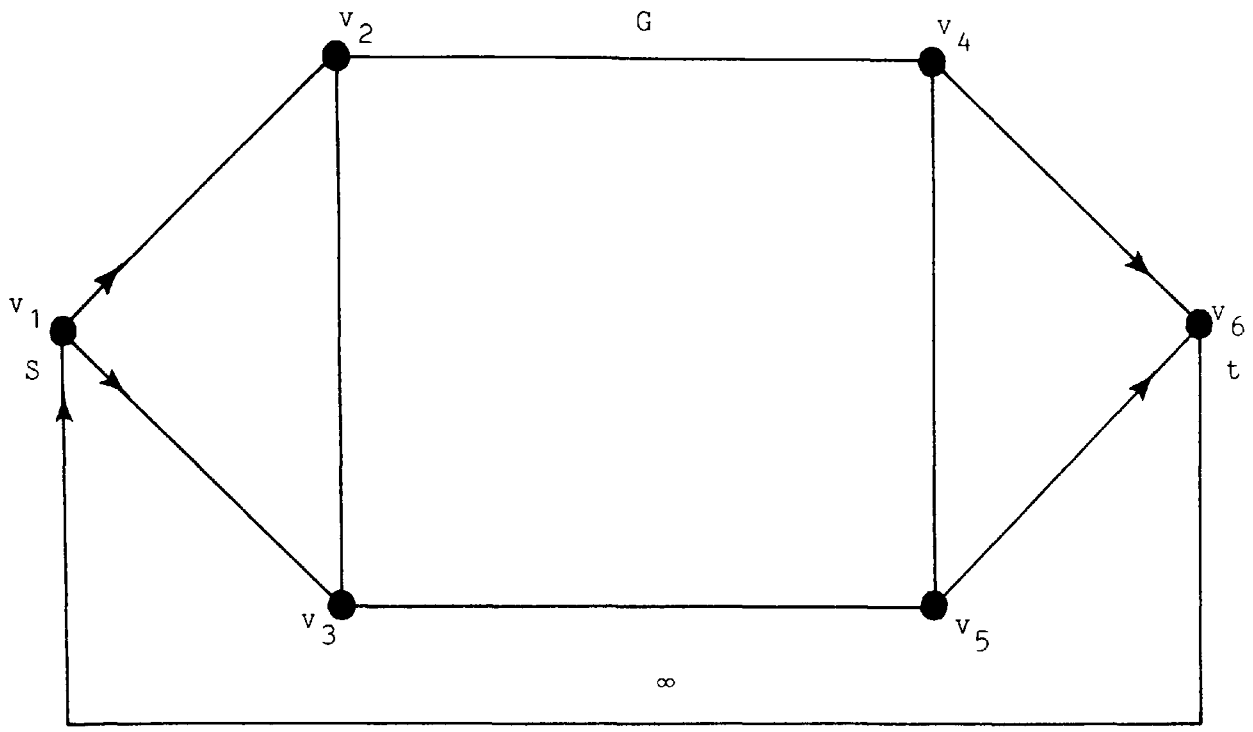
To obtain the augmented flow g_{af} if G(f) contains flow-augmenting circuits, we modify the flow f as follows: for each normal edge e_i^+ on f_{ac} we increase the flow in the corresponding edge e_i on G by the capacity of f_{ac}, and for each reverse edge e_i^- of f_{ac} we decrease the flow in e_i on G by the capacity of f_{ac}, FIG. 2.26(d).

In our discussion on flows, we have shown that for any flow f on a digraph G, it is possible to determine from G(f) whether or not the flow is maximal. It has also been demonstrated that if f is not a maximal flow, we can construct a flow of larger value by repetition of our flow-augmentation graph to yield a maximal flow in a finite number of steps. We illustrate this by examining the three cases previously mentioned.

1) Edge Connectivity Apply the maximum flow algorithm to pairs of vertices S, t in a graph G with $c_i = 1$ on all edges. Value of maximum flow = capacity of minimum cut = edge connectivity with respect to S, t.

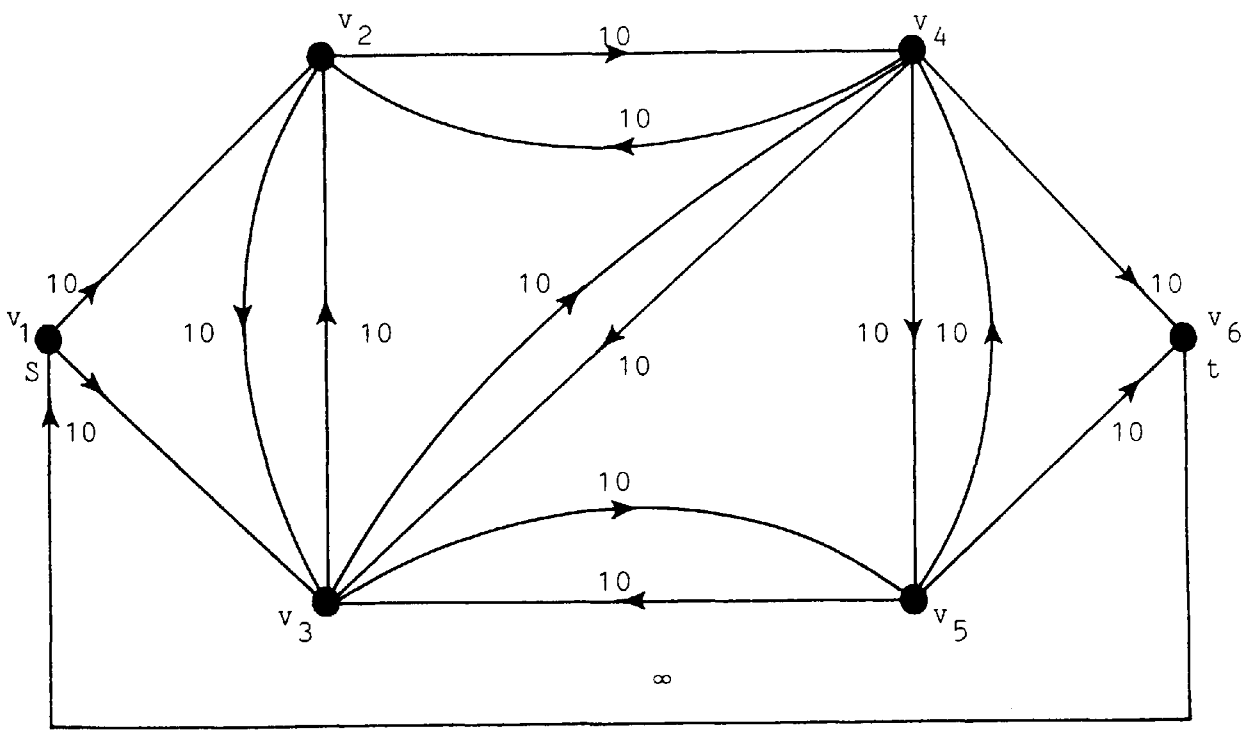
$$\text{Edge connectivity of } G = \min_{S,\,t} \left(\begin{array}{c} \text{edge connectivity} \\ \text{with respect to} \\ S,\ t \end{array} \right)$$

D. J. Kleitman [34] indicates that $\binom{|V|}{2}$ separate verifications of maximum flow values is highly inefficient, and is impractical in large graphs. In his correspondence, he points out several results that greatly reduce the number of verifications necessary to solve the problem just described. In our cases we limit the work to a demonstration in each case to one pair of vertices S and t.



(a)

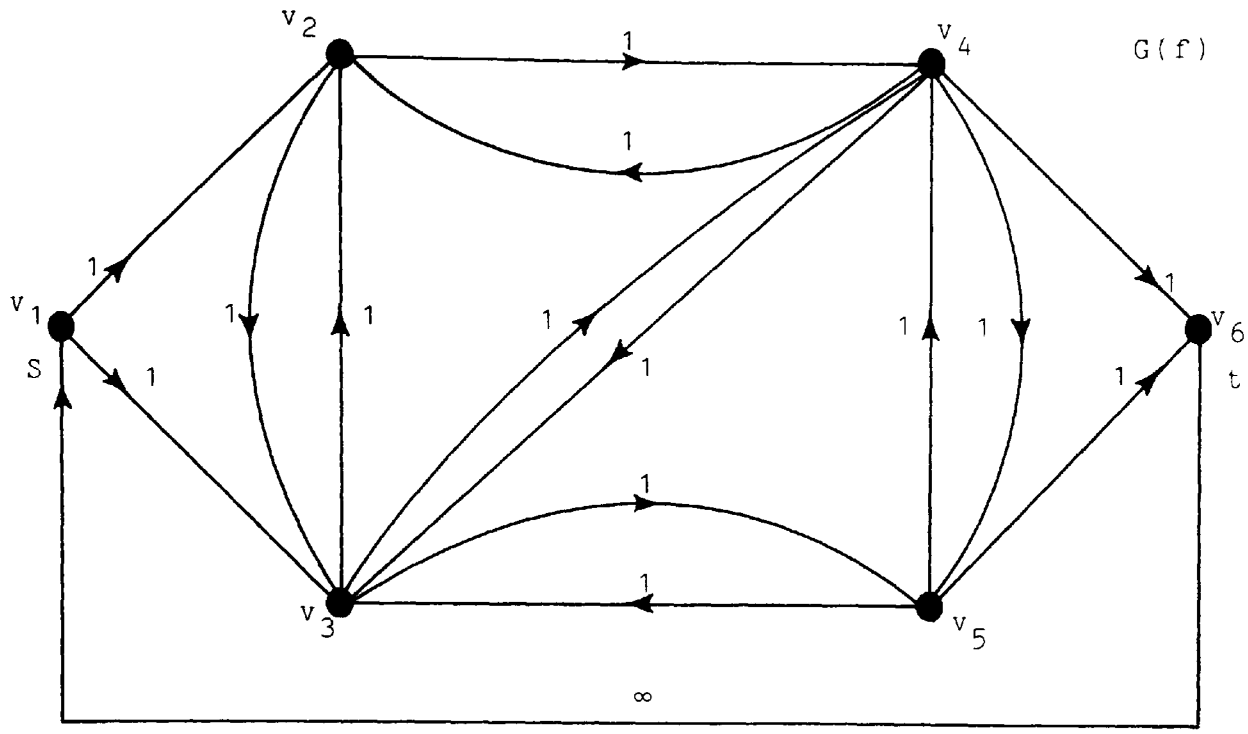
Graph G with return edge e_r = (t,S) having c = ∞



(b)

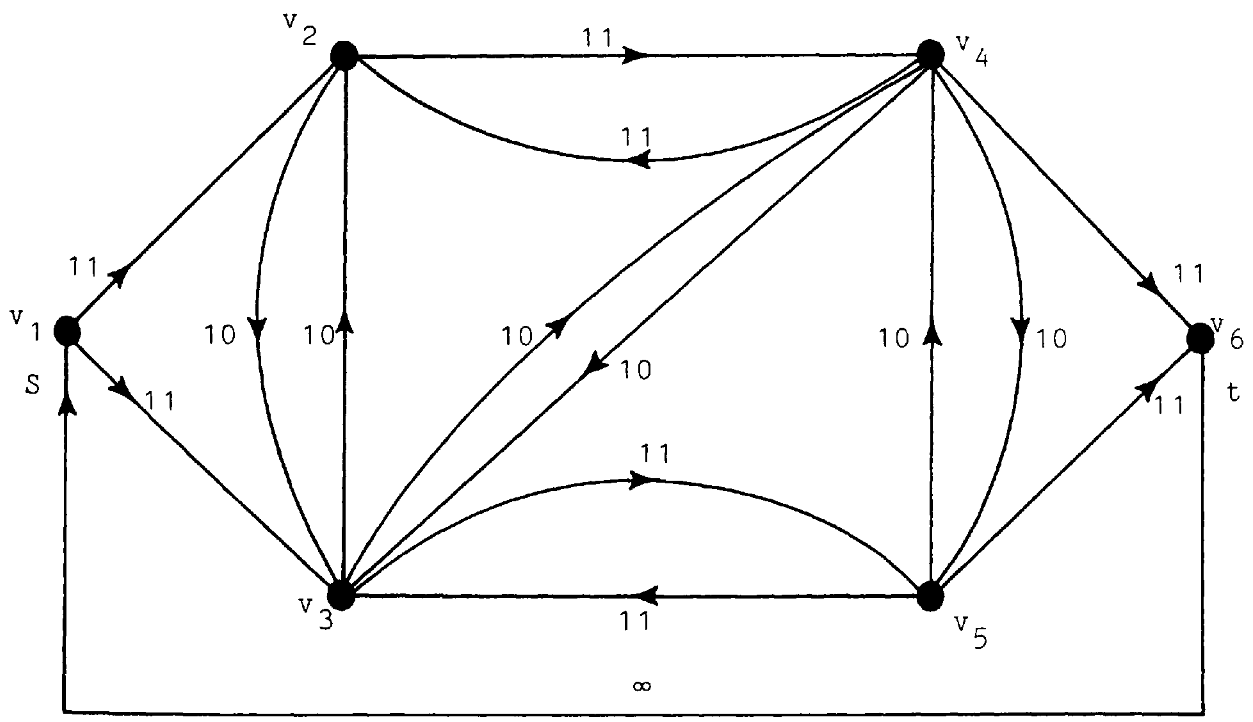
Capacity/flow graph, each edge of G except those at S and t is replaced by two edges as shown each edge having a capacity of 1 and zero flow.

FIG. 2.26



(c)

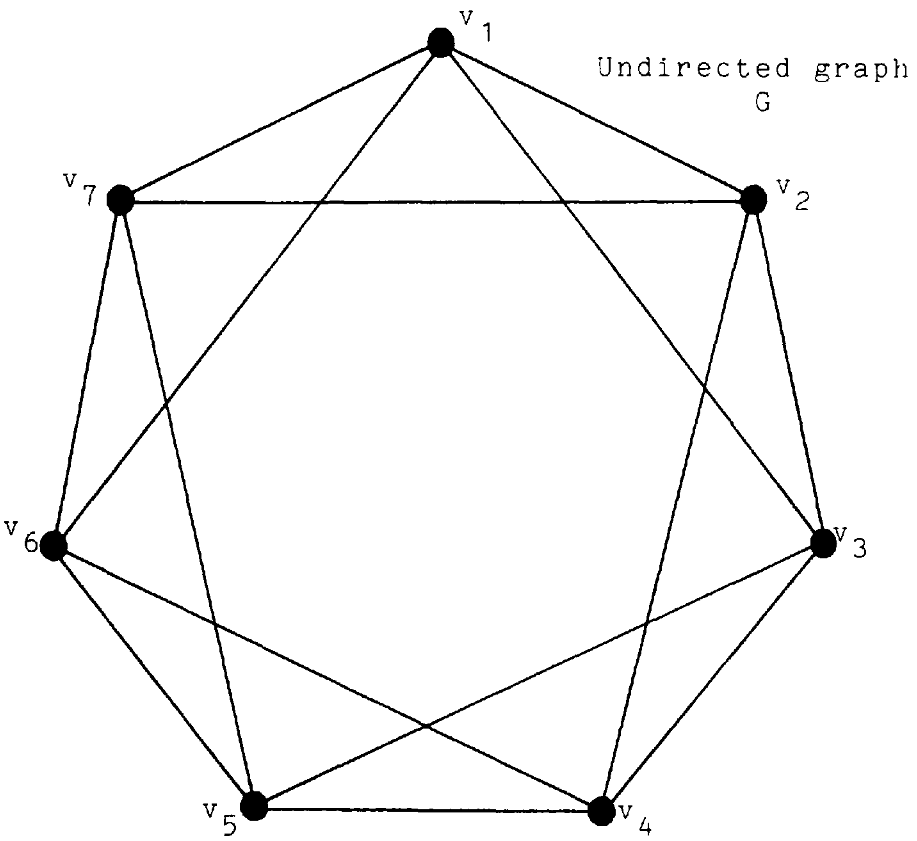
Displacement graph G(f) for simplicity zero flows for each edge have been omitted, i.e. $c_i^+ = c_i - f_i = 1$, $c_i^- = f_i = 0$ (omitted). G(f) has flow-augmenting circuits (f_{ac}). $f_{ac} = (v_1v_2), (v_2v_4), (v_4v_6), (v_6v_1)$, or $f_{ac} = (v_1v_3), (v_3v_5), (v_5v_6), (v_6v_1)$.



(d)

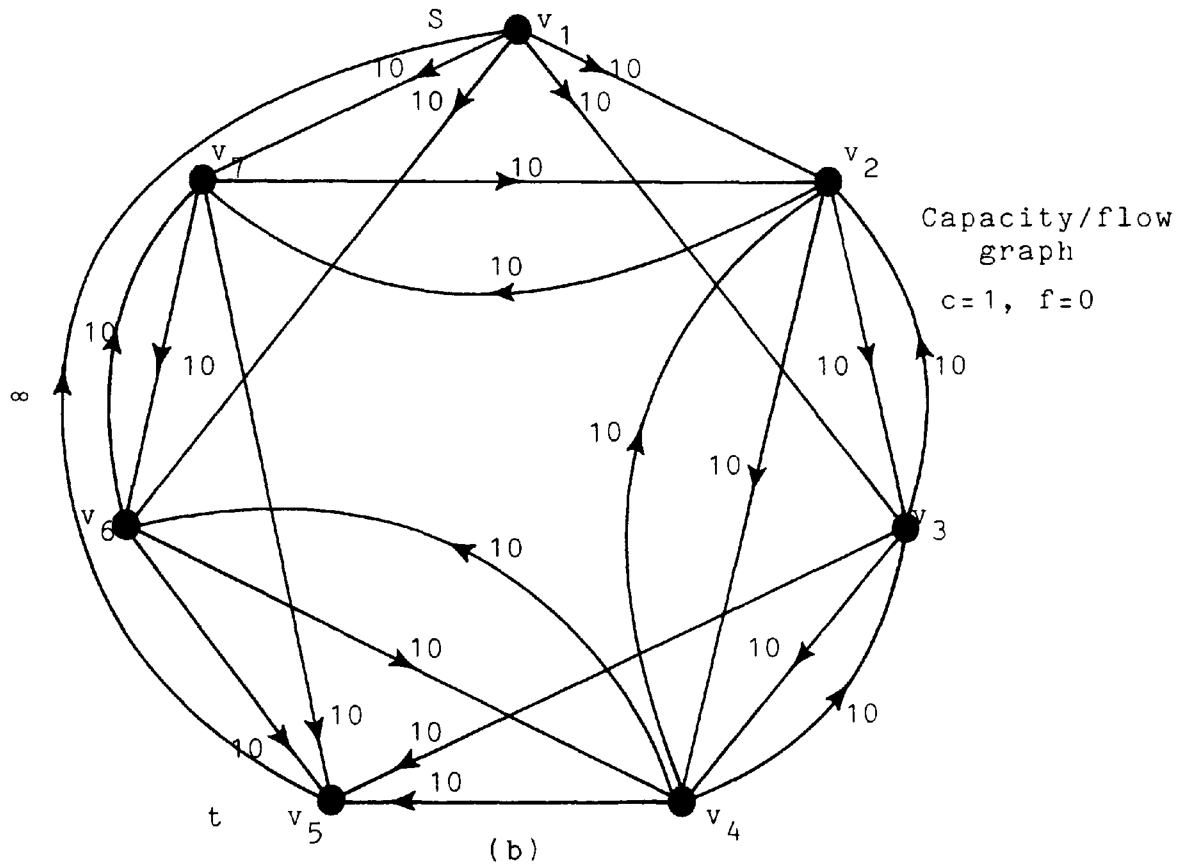
Augmented flow graph g_{af}

FIG. 2.26 (Cont'd)



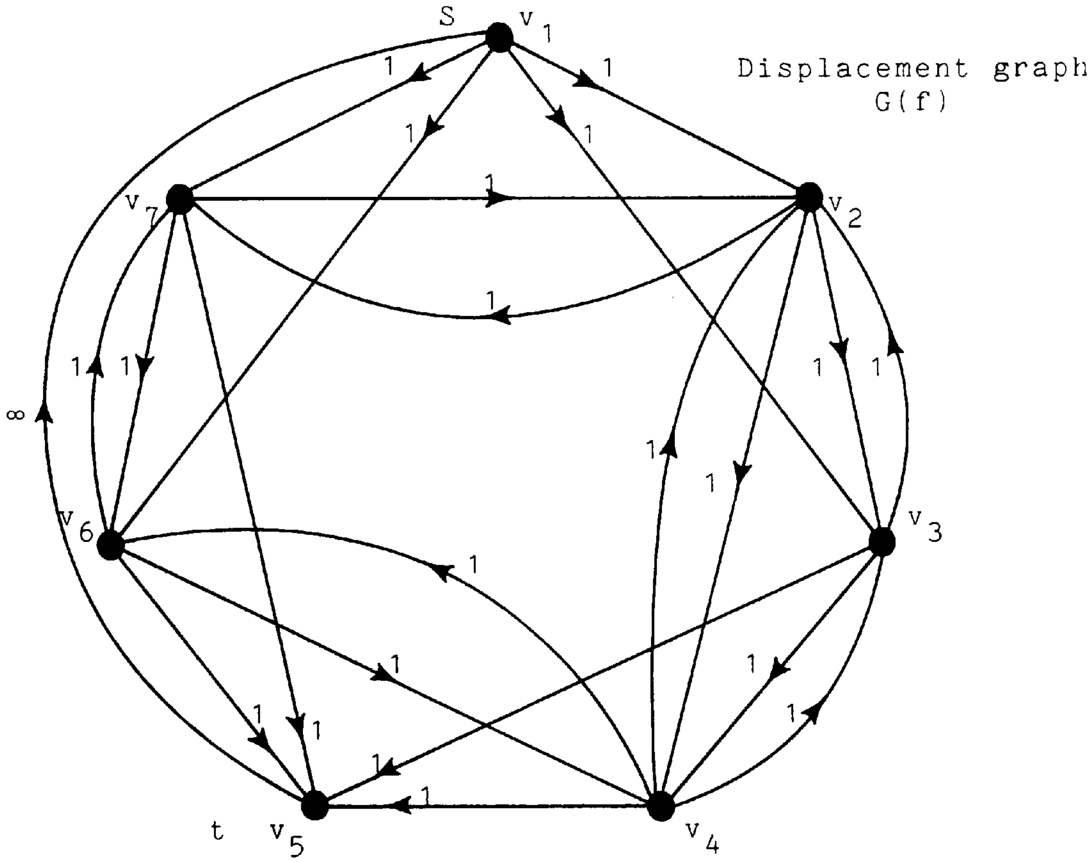
(a)

Let $v_1 v_5$ = S,t

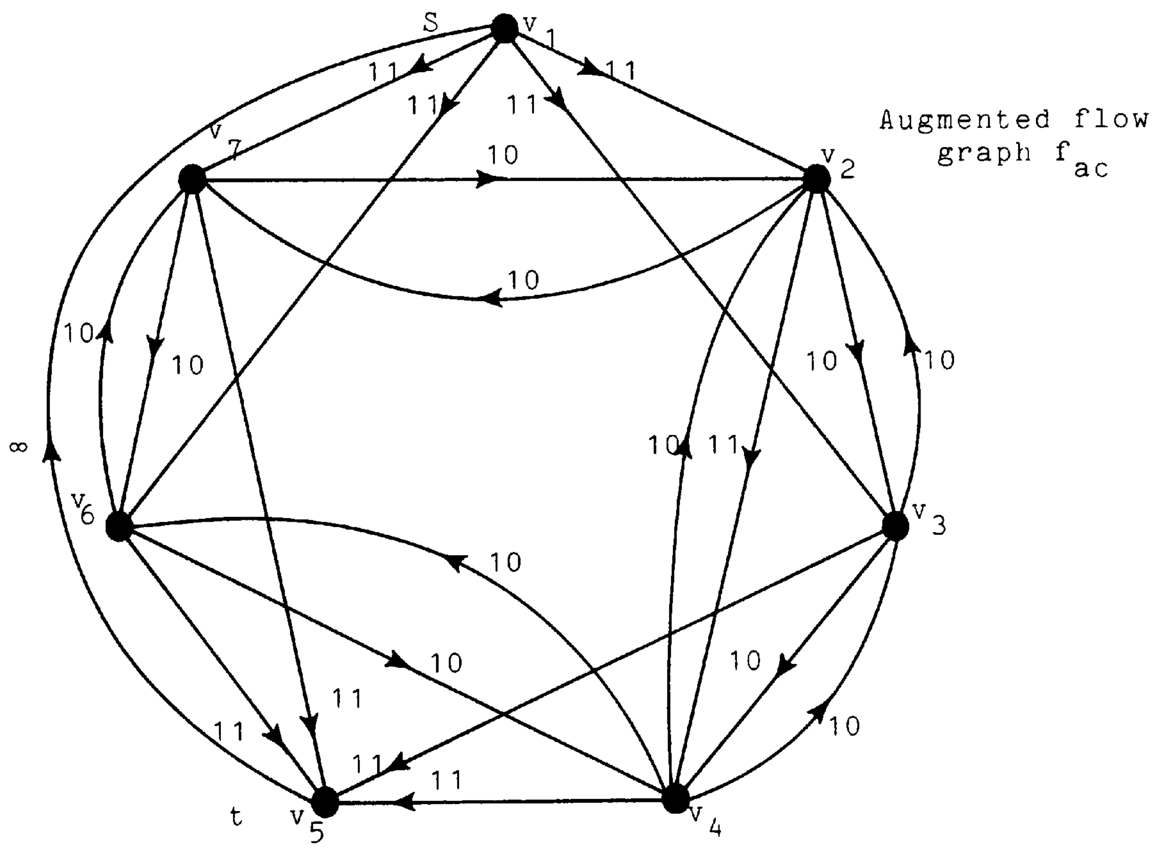


(b)

FIG. 2.27



(c)



(d)

FIG. 2.27 (Cont'd)

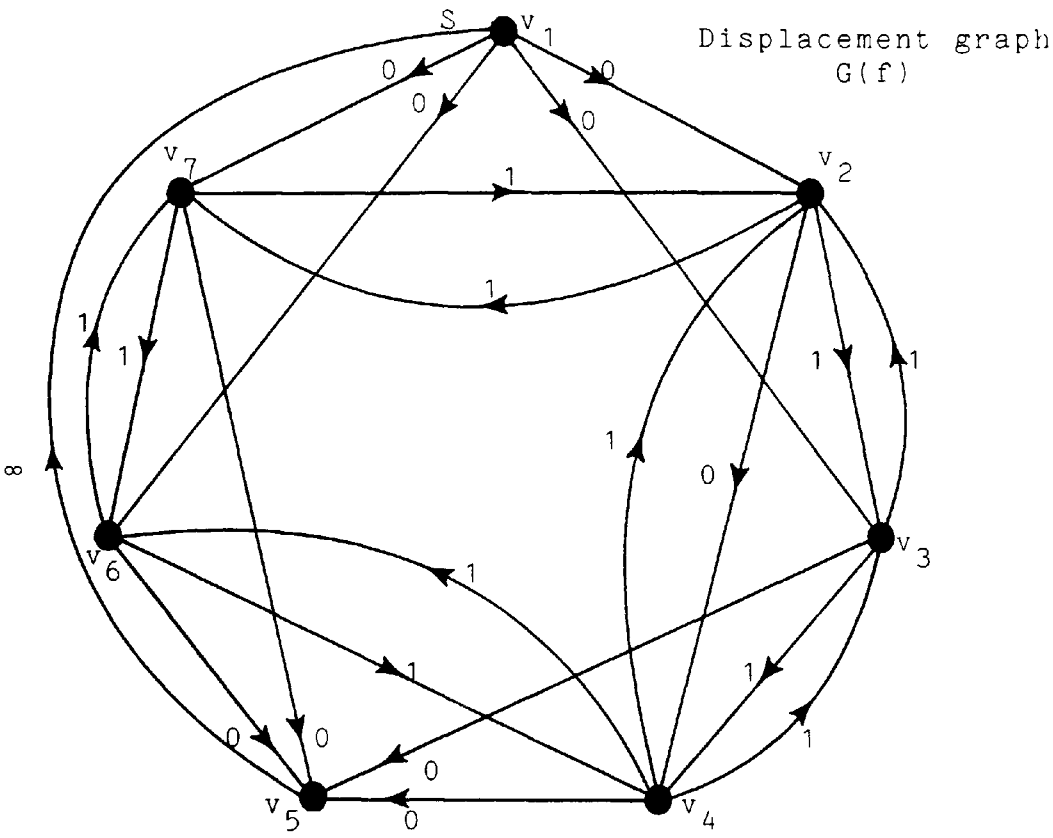


FIG. 2.27 (Cont'd). Diagrams showing the flow-graphs required to find a maximal flow between a pair of vertices S,t in an undirected graph G.

The displacement graph shown in FIG. 2.27(e) does not contain any flow-augmenting circuits, the flow depicted in FIG. 2.27(d) is a maximal flow = 4 (values of flow in edges incident at S or t) therefore edge connectivity with respect to S,t = 4.

2) <u>Vertex Connectivity</u> We describe a basic method for finding the vertex connectivity of a graph G, by testing each vertex pair v_S, v_t and obtaining the minimum value of k_{St}. We also use the following theorem.

<u>Theorem</u> The number of vertices k_{St} in the smallest S-t vertex cut set is equal to the maximum number of vertex disjoint S-t paths.

The proof of this theorem is given by H. Frank and I. T. Frisch [24] Chapter 7 pp 304.

Our method proceeds as follows:

(a) Change each undirected edge into two directed edges which will have equal capacity FIG. 2.28(a).

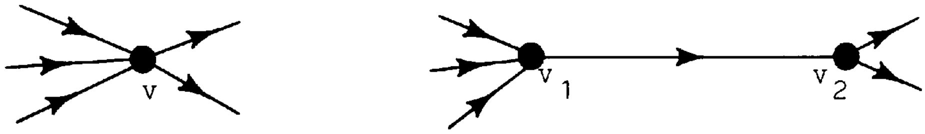
(b) Change each vertex into an edge joining two vertices (other than S and t) as shown in FIG. 2.28(b).

(c) Put a capacity of 1 on each edge such as $(v_1 v_2)$ and ∞ on all the other edges, connect vertices as illustrated in the graph of FIG. 2.28(c). We note that in a graph G modified as given every vertex disjoint path is edge disjoint.

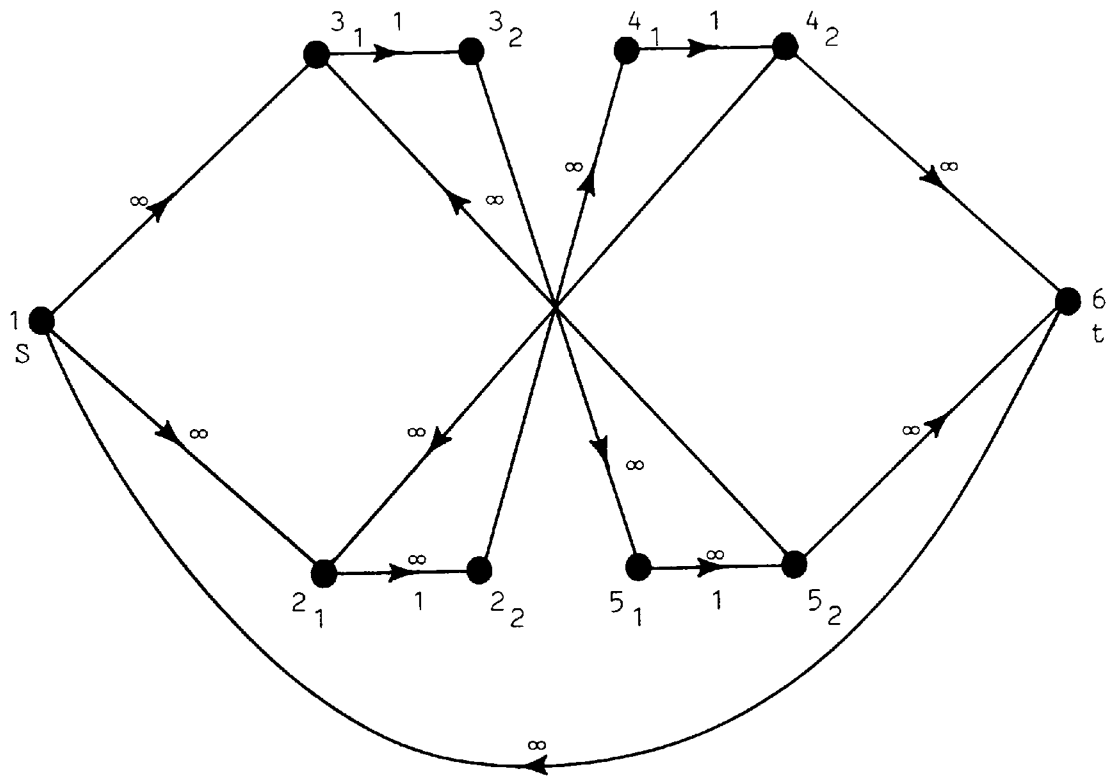
The maximum flow S to t = capacity of minimum cut = number of edges such as $(v_1 v_2)$ which disconnect S,t = k_{St}, the number of vertices in a minimum vertex cut set disconnecting S,t.

$$\text{Vertex connectivity of } G = \min_{S,t} \left(\begin{array}{c}\text{vertex connectivity}\\ \text{with respect to } S,t\end{array}\right)$$

(a)



(b)



(c)

FIG. 2.28 Diagrams illustrating the modified undirected graph required when finding k_{St}

3) <u>(k, k+1)-edge-connected</u>

Let S,t be a pair of vertices in a graph G. We define a pair of S-t bypass edges as follows:

Let v,w be vertices adjacent to S,t respectively and let e_1 be an edge incident with v but not S, and e_2 an edge incident with w but not t. We construct a pair of S-t bypass edges by inserting new vertices v_1, u_1 in the middle of e_1, e_2 respectively and joining S to v_1 and u_1 to t respectively FIG. 2.29.

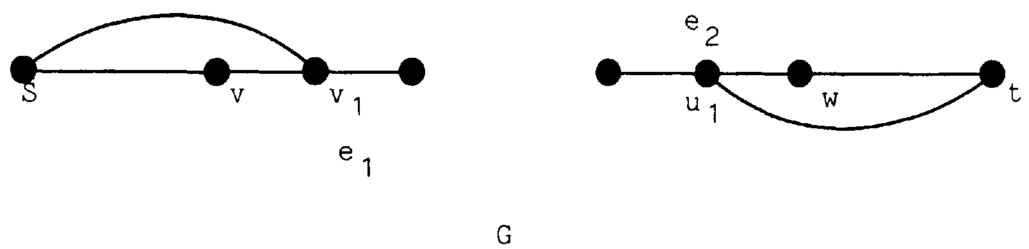


FIG. 2.29. Diagram illustrating the introduction of two bypass edges Sv_1 and tu_1 in a graph G.

<u>Theorem</u>. Let G be a simple undirected regular graph with degree $\rho = k$ that is k-connected and k-edge-connected. Then G is (k, k+1)-edge-connected if and only if for every pair S,t of vertices and every pair of S,t bypass edges the graph we obtain by adding these bypass edges has the property that S,t are (k+1)-edge-connected (i.e. we need to remove at least k+1 edges to disconnect S and t).

Proof: Insert a capacity of 1 on each edge. For any pair S,t of vertices the maximum S-t flow in G is k.

We must show,

A) G is not (k, k+1)-edge-connected $\Rightarrow$ there is a pair of bypass edges which do not increase flow.

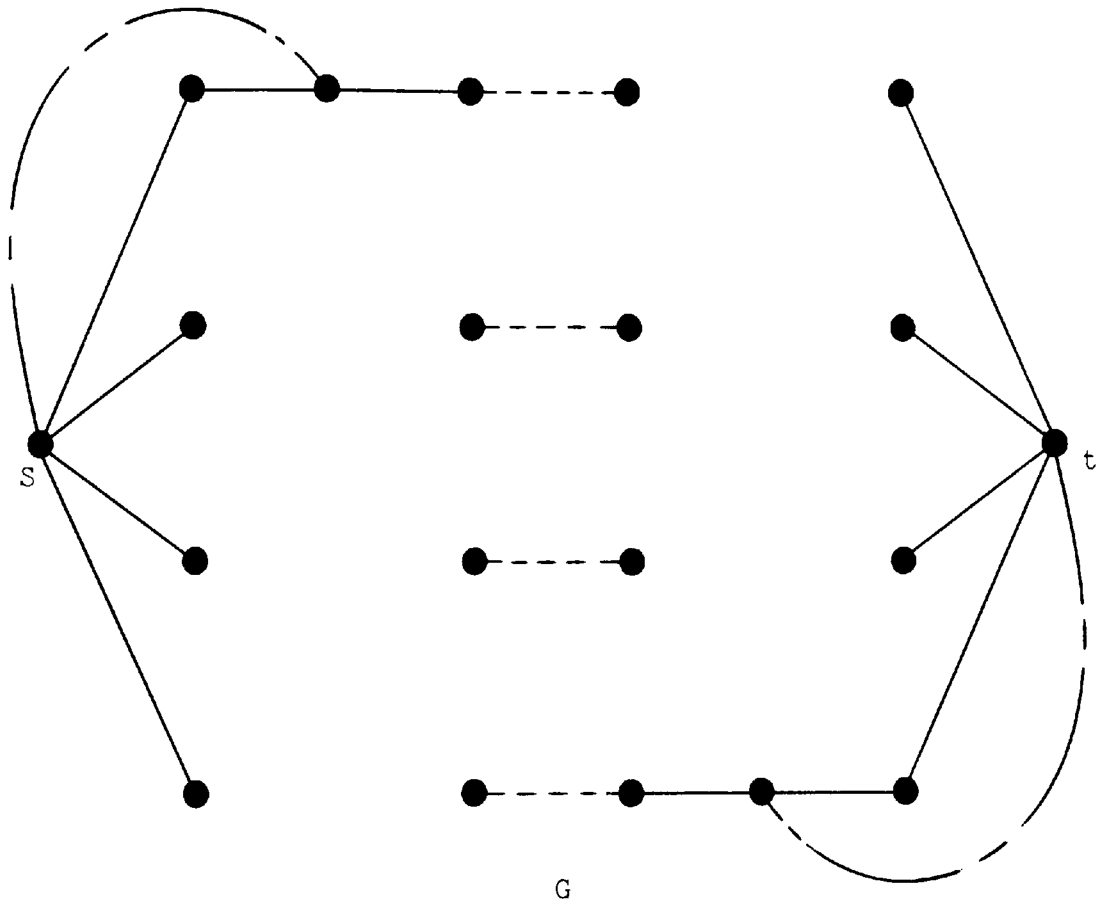
B) G is (k, k+1)-edge-connected $\Rightarrow$ every pair of bypass edges increases flow.

A(i) Let C_k (λ=k) be a k-edge cut set meeting neither S nor t nor any vertex adjacent to either S or t, FIG. 2.30.

Let v_1, v_2, ... v_n be a path with v_1 = S, v_n = t.

Any path from S to t must contain some edge of every cut set.

Thus if all edges of a cut set were deleted from the graph, there would be no path from S to t and the maximal flow value for the new graph (now in its component parts) would be zero. This is true in the case A(i) with the bypass edges at S and t, i.e. bypass edges cannot increase flow. Therefore G is not (k, k+1)-edge-connected $\Rightarrow$ there exist bypass edges which do not increase flow.



●------● denotes edge of k-edge cut set C_k

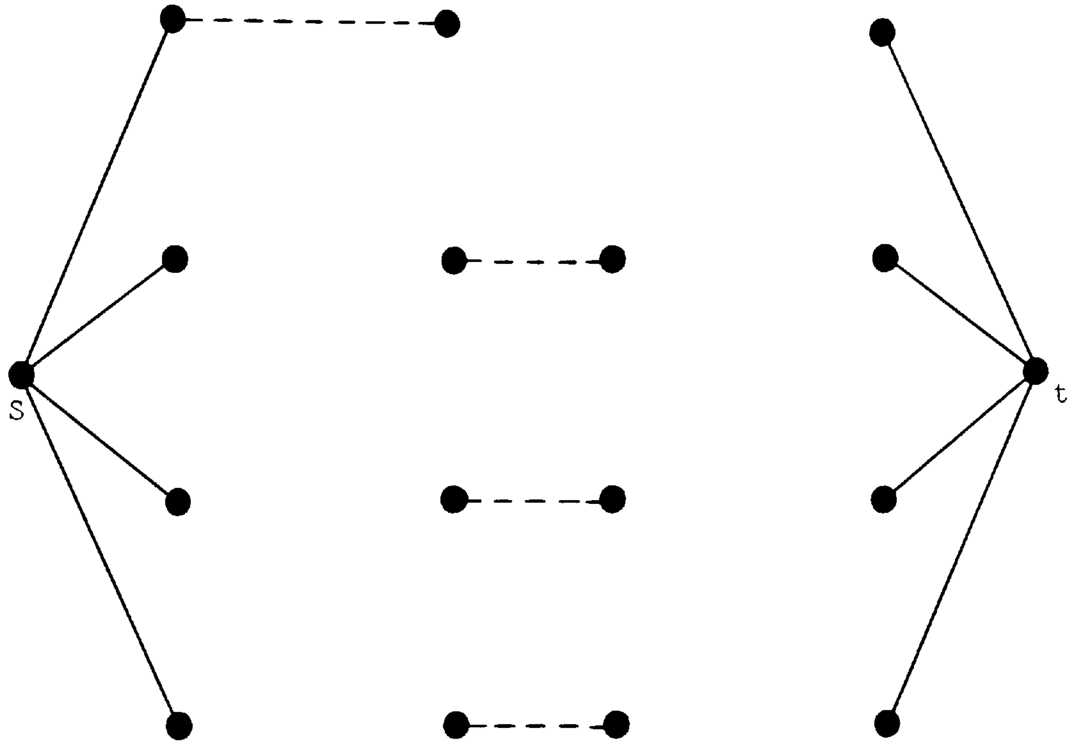
●—— - ——● denotes bypass edge

FIG. 2.30. Diagram illustrating the construction of a pair of S-t bypass edges.

A(ii) Let C_k be a k-edge cut set meeting neither S nor t but with an edge incident with a vertex adjacent to S or t, FIG. 2.31(a).

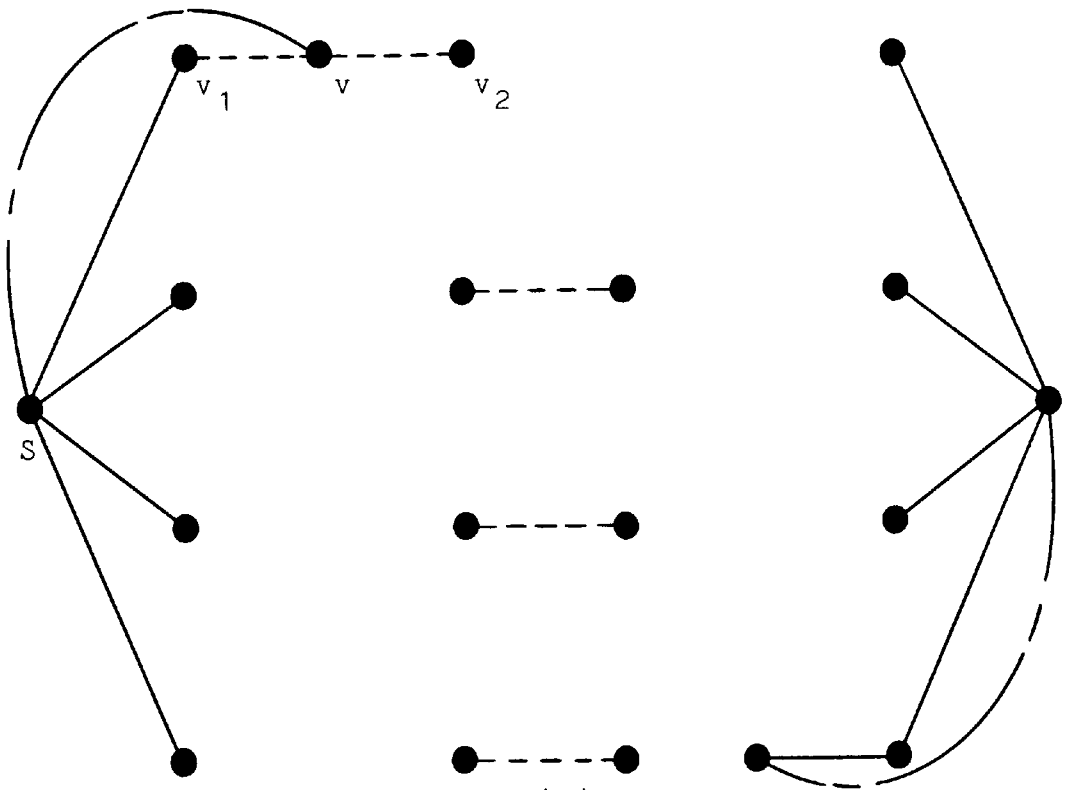
If $v_1v_2 \in C_k$, Sv does not increase flow and Sv, vv_2 are two edges of a path from S to t.

Since any path from S to t must contain some edge of every cutset, the path containing



(a)

Consider the introduction of a bypass edge Sv where v is a new vertex and also a bypass edge at t, Fig. 2.31 (b)



(b)

denotes edge of a k-edge cut set C_k

denotes bypass edge

FIG. 2.31

Sv, vv_2 also contains one of the edges $C_k - \{v_1 v\}$ i.e. removal of C_k disconnects G.

Therefore bypass edge cannot increase flow. Similarly the bypass edge to t does not increase flow.

Therefore G is not (k, k+1)-edge-connected $\Rightarrow$ there is a pair of bypass edges which do not increase flow.

A(iii) Suppose that some but not all edges incident with S are in the k-edge cut set C_k, FIG. 2.32.

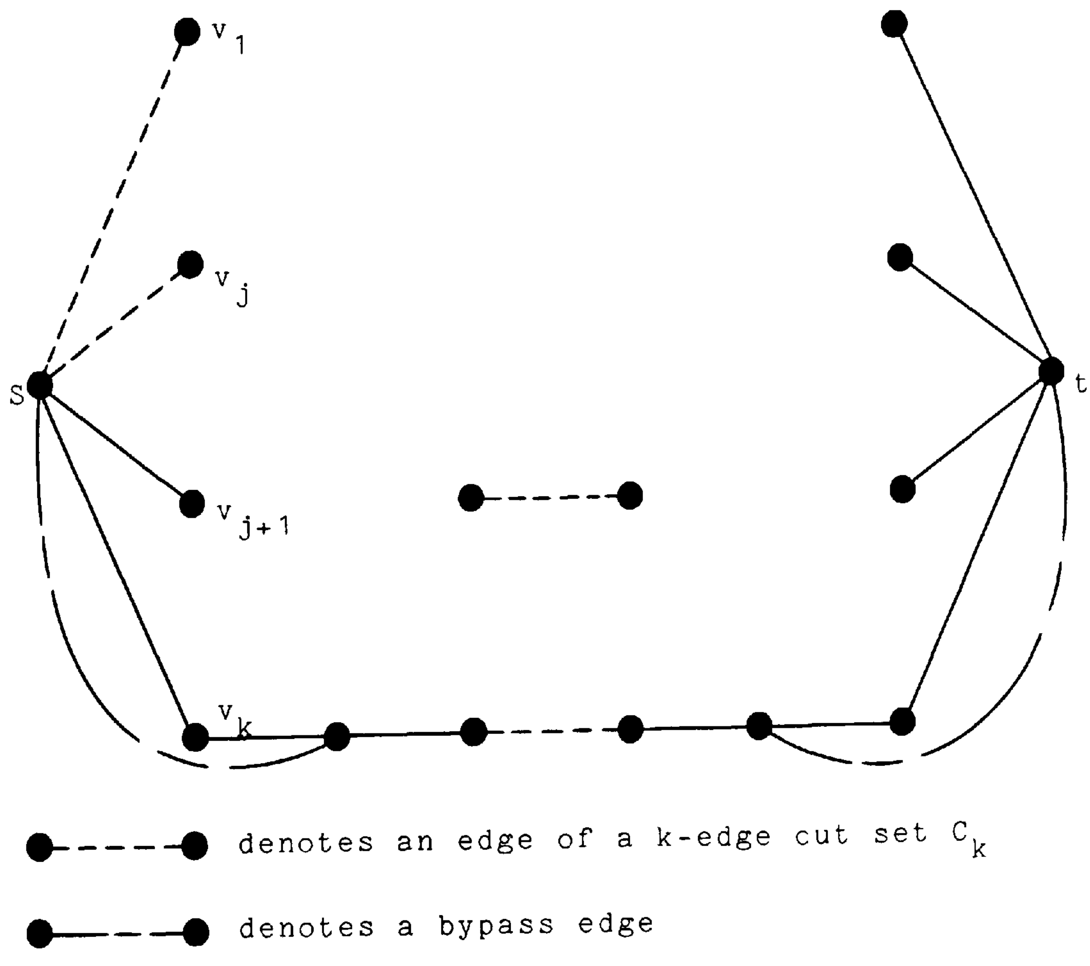


FIG. 2.32

Let the edges of the cut set C_k incident with S be denoted by Sv_1, Sv_2, ... Sv_j and the edges not in the cut set but incident with S be denoted by Sv_{j+1}, Sv_{j+2}, ... Sv_k. Then Sv_{j+1}, Sv_{j+2}, ... Sv_k are edges of a path containing some edges of $C_k-\{Sv_1, Sv_2, \ldots Sv_j\}$. The bypass edge introduced at S also lies on a path containing $C_k-\{Sv_1, Sv_2, \ldots Sv_j\}$. If the cut set edges are removed then no path exists and there can be no flow. Thus the introduction of bypass edges at S and t will not increase the flow.

Therefore G is not (k, k+1)-edge-connected $\Rightarrow$ there is a pair of bypass edges which do not increase flow.

B) Suppose the graph G is (k, k+1)-edge-connected FIG. 2.33.

There are only two minimum cut sets at S or t represented by Sv_1, Sv_2, ... Sv_k or tu_1, tu_2, ... tu_k respectively.

Let the bypass edges introduced at S and t be denoted by Sv_{k+1} and tu_{k+1}. If the cut set of C_k edges is removed there exists a path Sv_{k+1}, v_{k+1} v_{k+2}, ... $u_{k+1}t$ joining S and t and such a path must contain some edge of every cut set. Therefore, by the maximum flow, minimum cut theorem the

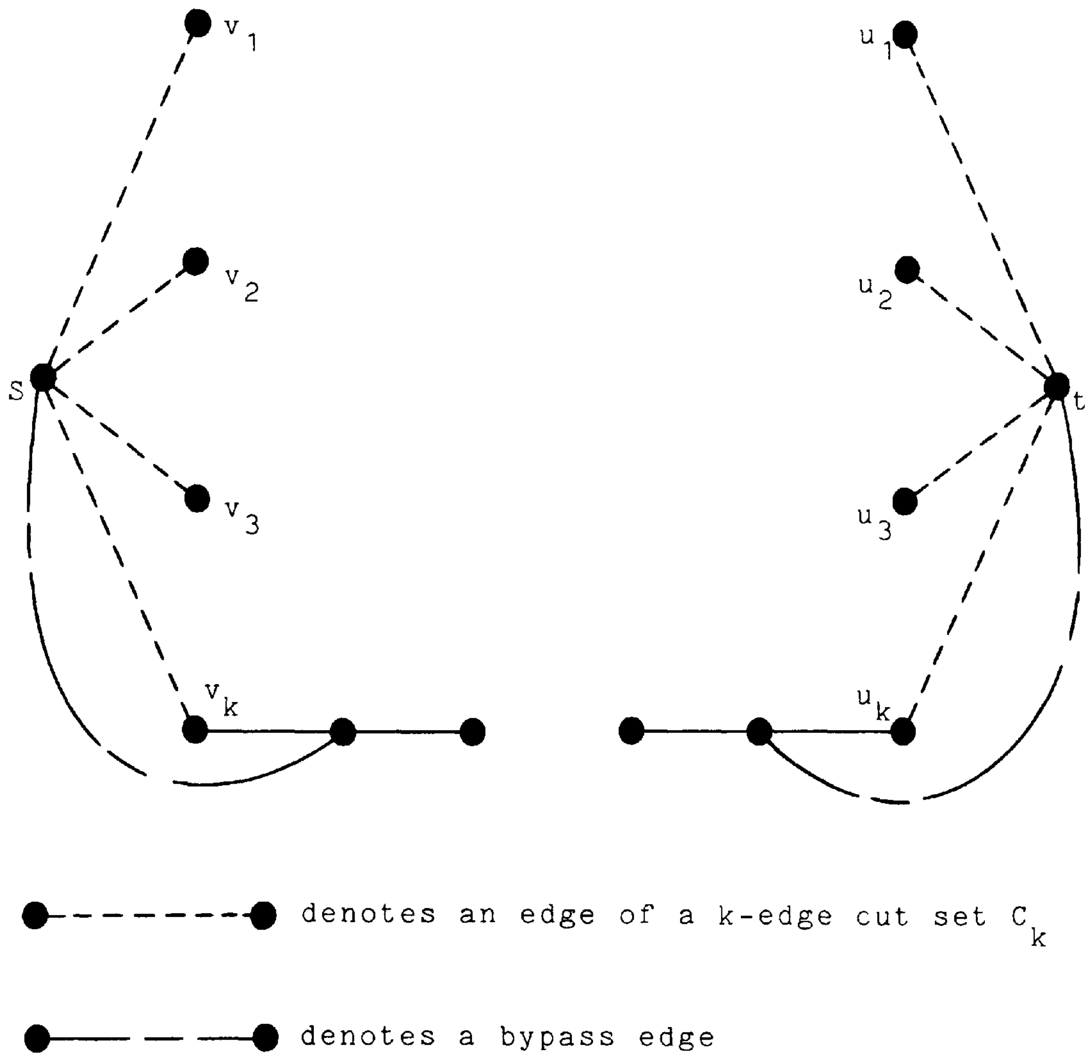


FIG. 2.33

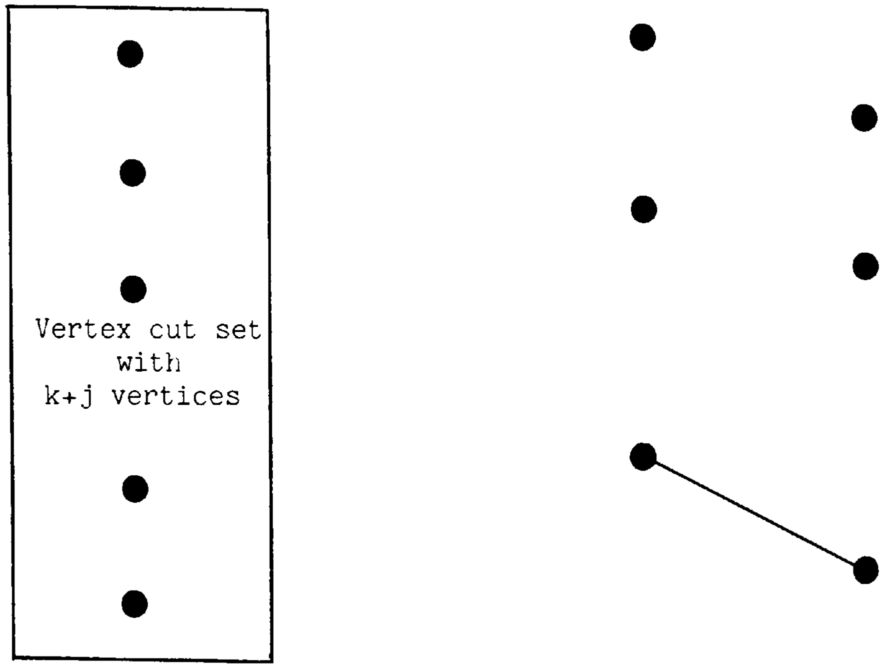
bypass edges must be members of the cut set otherwise the graph is disconnected. Thus the bypass edges increase the flow.

Therefore G is (k,k+1)-edge-connected

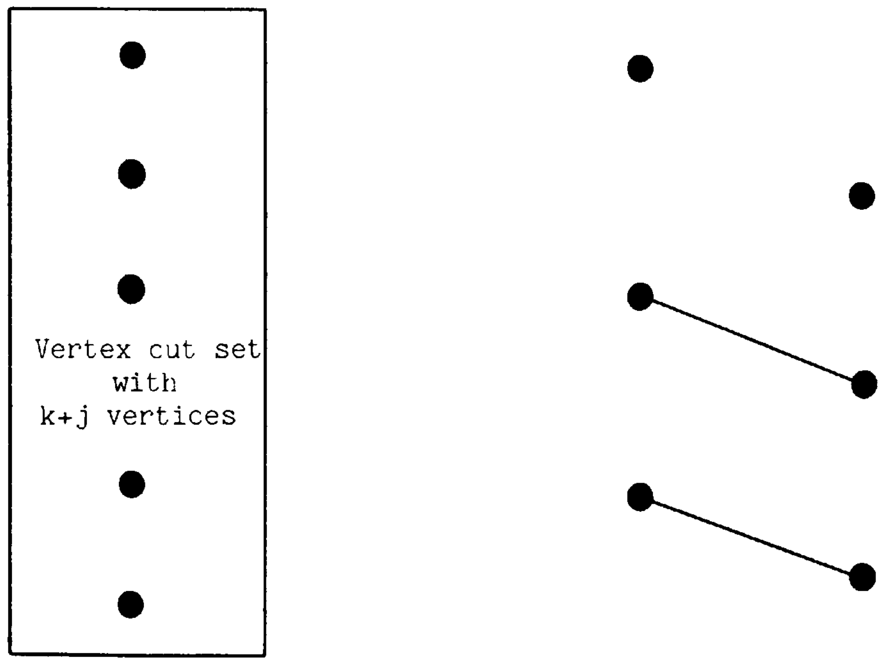
$\Rightarrow$ every pair of bypass edges increases flow. ■

A graph is (k, k+j)-connected if it has connectivity k, has a vertex cut set with k+j vertices and all vertex cut sets W with $|W| < k+j$ have the property that G-W has at most one component which is not an isolated vertex. FIG. 2.34(a) and (b) illustrates diagramatically a graph which is (k, k+j)-connected and a graph which is not (k, k+j)-connected.

A graph is (k, k+j)-edge-connected if it has edge connectivity k, has an edge cut set with k+j edges and all edge cut sets P with $|P| < k+j$ have the property that G-P has at most one component which is not an isolated vertex. These definitions are discussed further in Chapter six.



(a) (k, k+j)-connected graph



(b) graph is not (k, k+j)-connected

FIG. 2.34

2.8 The Expected Number of Vertices Disconnected

The probability of disconnection of a finite, simple, undirected graph G is given by,

$$\underset{\text{(Probability of Disconnection)}}{R_d(G)} = \sum_{i=k}^{|V|-2} S_i q^i (1-q)^{|V|-i}$$

Where S_i denotes the number of vertex cut sets with i vertices, q is the probability of failure (removal with its incident edges) of each vertex. Assume that failure of vertices are independent. For small values of q the probability $R_d(G)$ is minimised over all graphs with $\frac{k|V|}{2}$ edges if G is regular of degree $\rho = k$ and S_k is minimised. The work of S. L. Hakimi and A. T. Amin [26] gives the construction of regular graphs of degree $\rho = k$ and connectivity = k in which the vertex cut sets with k vertices are vertex neighbour sets. These constructions however do not have the smallest number of vertex cut sets with k vertices. D. H. Smith [45] has shown that in many cases it is possible to construct a graph with the minimum number of vertex cut sets with k vertices.

The application of the above solution to communication networks although desirable from the point of view of minimising $R_d(G)$ the probability of disconnection, does have the disadvantage that when in practice node failures do occur a rather large number of nodes may become isolated. It might therefore be more appropriate to require that the expected number of vertices disconnected from the largest remaining component of the graph (or isolated if all components are isolated vertices) be minimised.

Let G be a graph with degree $\rho = k$, connectivity $= k$ and suppose there are C_r vertex cut sets with r vertices. Let N_{ri} be a vertex cut set with r vertices ($i = 1, 2, \ldots C_r$) and suppose that N_{ri} disconnects exactly V_{ri} vertices from the largest component of $G-N_{ri}$ (or V_{ri} = the number of isolated vertices in $G-N_{ri}$ if all components are isolated vertices). Then the expected number of vertices disconnected form the largest component (or left isolated if all components are isolated vertices) is E_V where,

$$E_V = \sum_{r=k}^{|V|} \left(\sum_{i=1}^{C_r} V_{ri} \right) q^r (1-q)^{|V|-r}$$

Since each vertex can be disconnected by at least one vertex cut set with k vertices the minimum value of the coefficient of $q^k(1-q)^{|V|-k}$ is $|V|$ but to attain this minimum we require not just that all vertex cut sets of size k be vertex neighbour sets but also the stronger condition that if V is a vertex cut set of size k then G-V has at most one component that is not an isolated vertex.

D. H. Smith [44] indicates that the graphs constructed by S. L. Hakimi and A. T. Amin [26] have the smallest value of E_V for some sufficiently small probability q of vertex failure. However this result does not tell us how small q must be. For practical values of q it may be better to attempt to minimise the first few coefficients of

$$\sum_{i=1}^{C_r} V_{ri}, \quad \text{say for } r = k, k+1, \ldots k+j.$$

D. H. Smith [44] attempts to solve this problem by constructing (k, 2k-2)-connnected graphs (i.e. (k, k+j)-connected graphs with j = k-2), and states that the graphs of S. L. Hakimi and A. T. Amin [26] are not (k, 2k-2)-connected. We require regular graphs with $|V|$ vertices, degree $\rho = k$, connectivity = k, which have vertex cut sets with 2k-2 vertices and such that the only vertex cut sets V with less than 2k-2 vertices have the property that G-V has at most one component that is not an isolated vertex. The definition of (k, 2k-2)-connected requires $k \leqq \frac{|V|}{2}$.
This will be dealt with further in Chapter eight.

CHAPTER 3

CHAPTER 3

Some standard graphs and their importance for network reliability.

The problems discussed in this thesis involve the construction of graphs with a given number of vertices and edges which are optimal with respect to some measure of reliability. Such problems are normally easier if the number of edges is such that the graph can be regular in which case the optimal graph is normally regular.

For practical purposes the best way to proceed when the number of edges does not allow a regular graph is to construct an optimal regular graph using as many edges as possible and to insert the remaining edges afterwards. It appears that it is not normally too difficult to insert these edges in such a way that an optimal or near optimal graph is obtained. For this reason the emphasis of this work is on the construction of regular optimal graphs.

In this chapter we consider regular graphs which are of interest in the field of network reliability because of their connectivity properties. The three classes of graphs dealt with are circulant graphs, graphs obtained using Construction A, and bipartite graphs.

All have regularity, not only in that the vertices of the graph have the same degree, but in the pattern of edge connection; for example in the bipartite graph shown in FIG. 3.1 any pair of vertices in $|V_1|$ has common adjacency with exactly one vertex in $|V_2|$.

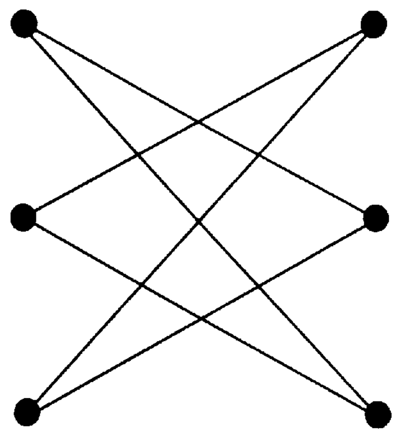


FIG. 3.1

This principle is used in practical telecommunications; we can think of $|V_1|$ as a set of switches each of which has access to only two out of three switches $|V_2|$ (or by a simple extension, to two out of four).

3.1 Circulant Graphs (or Circulants)

Circulant graphs (or circulants) are a class of graphs that have desirable connectivity and edge connectivity properties and are consequently important in network reliability studies.

We recall from Chapter one that a circulant graph is a graph G whose adjacency matrix $A_c(G)$ is a circulant matrix, and a $|V| \times |V|$ matrix $A_c(G)$ is obtained from the first row of $A_c(G)$ by a cyclic shift of i-1 steps and so any circulant matrix is determined by its first row.

From the fact the adjacency matrix is a symmetric matrix with zero entries on the main diagonal, if the first row of the adjacency matrix of a circulant graph is $[a_1, a_2, \ldots a_{|V|}]$, then $a_1 = 0$ and

$a_i = a_{|V|-i+2}$ $(2 \leq i \leq |V|)$. If the first row of the adjacency matrix of a circulant graph G is $[0, a_2, \ldots a_{|V|}]$, then the eigenvalues of G are,

$$\lambda_r = \sum_{j=2}^{|V|} a_j w^{(j-1)r} \qquad r = 0, 1, \ldots |V|-1$$

where $w = \exp\left(\frac{2\pi i}{|V|}\right)$ N. Biggs [3] pp 15-16

Let G be a graph with degree $\rho = k$ with $|V|$ vertices and let

$$\text{Spec. } G = \begin{pmatrix} k & \lambda_1 & \ldots & \lambda_{S-1} \\ 1 & m_1 & \ldots & m_{S-1} \end{pmatrix}$$

Then the complexity of G is given by,

$$\text{Complexity } (T_{|V|-1}) = \frac{1}{|V|} \prod_{r=1}^{S-1} (k-\lambda_r)^{m_r}$$

where $\lambda_r \neq k$

The complexity is of interest in our study because it enables us to compare graphs when the probability of edge failure p is large.

$R_{|V|-1} = \binom{|E|}{|V|-1} - T_{|V|-1}$, this is explained as follows:

Number of spanning trees = Number of sets of $|V|-1$ edges which leave the graph connected.

= Number of sets of $|V|-1$ edges - number of sets of $|V|-1$ edges which disconnect the graph.

$= \binom{|E|}{|V|-1}$ - number of edge cut sets of size $|V|-1$

Thus the number of edge cut sets of size $|V|-1$
$= \binom{|E|}{|V|-1}$ - complexity. □

We recall that the probability of disconnection of a graph $P_d(G)$, where p is the probability of edge failure is given by,

$$\underset{\text{(Probability of disconnection)}}{P_d(G)} = \sum_{i=\lambda}^{|E|} R_i p^i (1-p)^{|E|-i}$$

where R_i = the number of edge cut sets of size i.

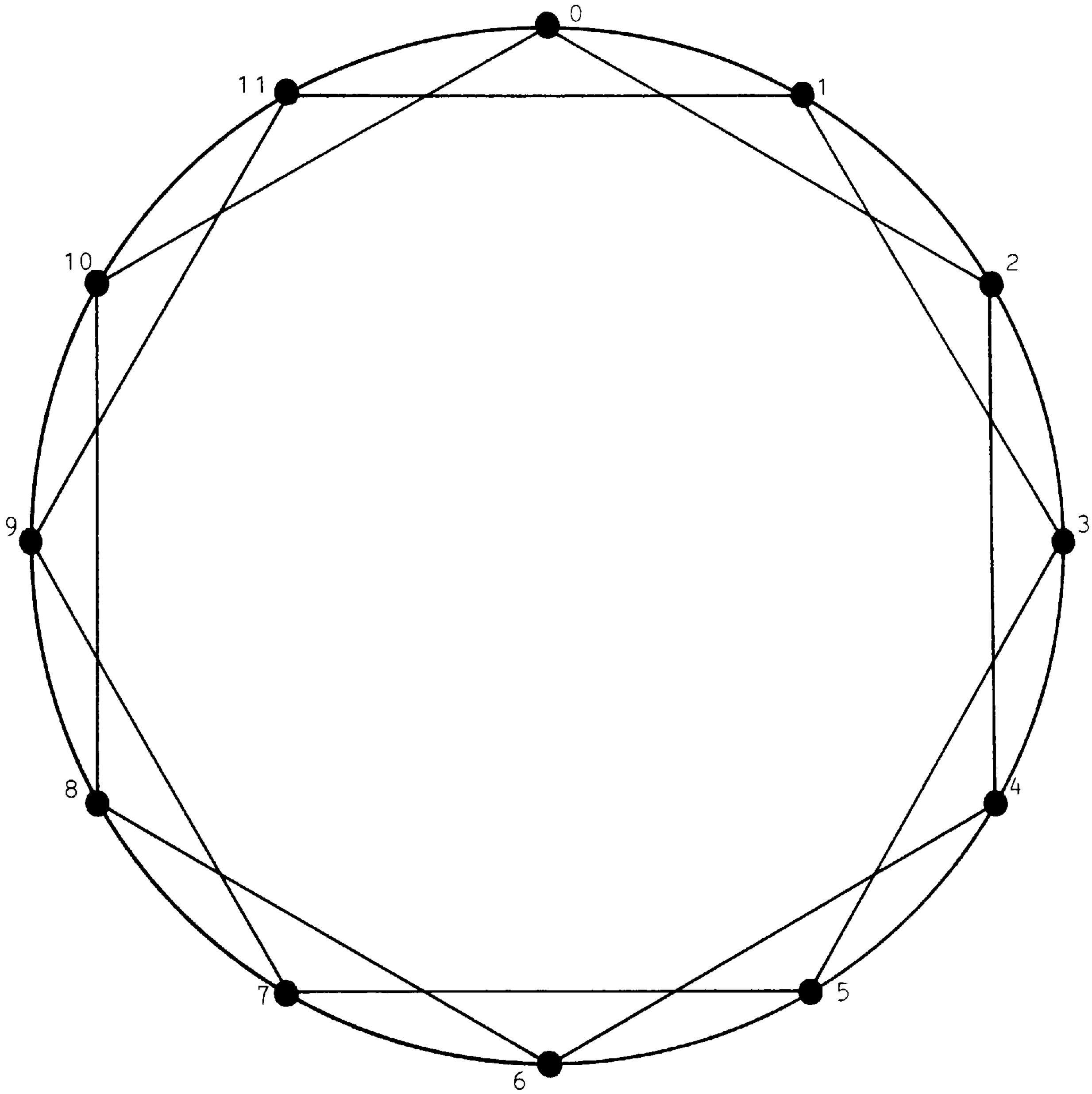
i.e.
$$\underset{\text{(Probability of disconnection)}}{P_d(G)} =$$
$$R_\lambda p^\lambda (1-p)^{|E|-\lambda} + R_{\lambda+1} p^{\lambda+1} (1-p)^{|E|-(\lambda+1)} + \ldots$$
$$R_{|V|-1} p^{|V|-1} (1-p)^{|E|-|V|+1} + \ldots R_{|E|-|V|+2} p^{|E|-|V|+2} (1-p)^{|V|-2} +$$
$\ldots R_{|E|} p^{|E|}$. We note that $R_{|E|-|V|+2} = \binom{|E|}{|E|-|V|+2}$, $R_{|E|-|V|+3} = \binom{|E|}{|E|-|V|+3}$, $R_{|E|} = \binom{|E|}{|E|}$, do not depend on the choice of graph.

Thus the term $R_{|E|-|V|+1} \, p^{|E|-|V|+1} (1-p)^{|V|-1}$ is significant and for p large we require to make this term as small as possible in attempting to minimise $P_d(G)$ the probability of disconnection of the graph.

FIG. 3.2, FIG. 3.3, and FIG. 3.4 give examples of circulant graphs with different constuctions. The figures illustrate the varying calculated values obtained for the eigenvalues, complexity and number of edge cut sets with $|E|-|V|+1$ edges respectively. By comparing the values of $R_{|E|-|V|+1}$ for the three examples given, choosing the smallest value of

$R_{|E|-|V|+1}$ we are able to say approximately that the graph in FIG. 3.4 is the more reliable graph in the sense that the probability of disconnection $P_d(G)$ is minimised when p the probability of edge failure is close to 1.

Construction $|V| = 12$, $k = 4$, $k = 2r$, $(v_i v_j) \varepsilon E(G)$ if $(i-j) \equiv m \pmod{|V|}$ where $2 \leqq m \leqq r$.



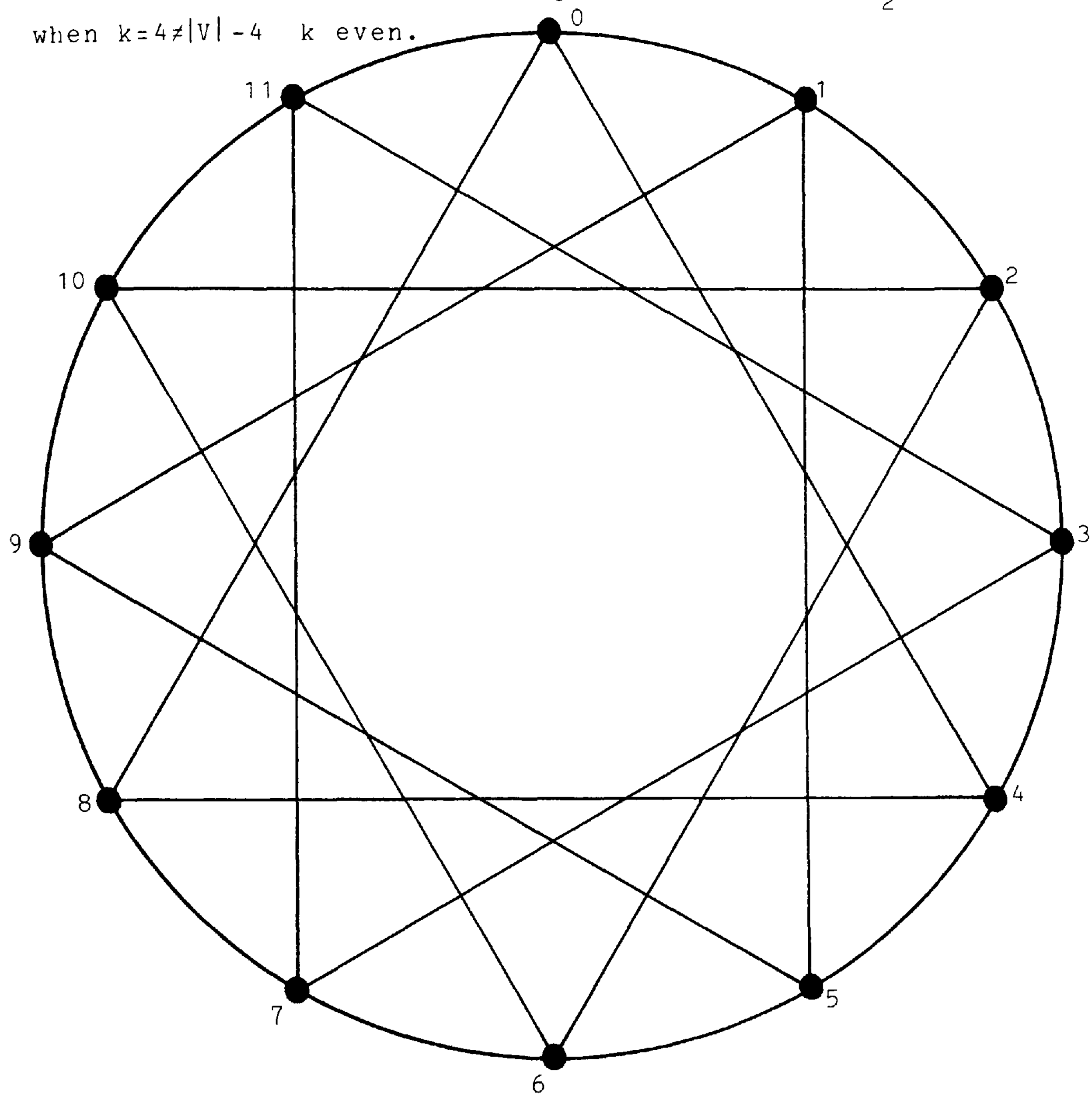
$A_c(G) = [0\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1]$

Spectrum of G = $\begin{pmatrix} 4 & 1+\sqrt{3} & 0 & -\sqrt{3}+1 & -2 \\ 1 & 2 & 3 & 2 & 4 \end{pmatrix}$

$T_{|V|-1} =$ 248832, $R_{|E|-|V|+1} =$ 2247312.

FIG. 3.2. A graph G with the maximum connectivity, F. Harary [29].

Construction $|V| = 12$, $k = 4$, $(v_i v_j) \varepsilon E(G)$ if $j = i+p(\mathrm{mod}|V|)$ $p = 1, 2, \ldots r_1$, except when $|V|-4 \geq 6$ and is even, in which case $p = 1, 2, \ldots r_1-1$. Let $k = 2r_1+2$ for k even, and $k = 2r_1+1$ if k is odd $(v_i v_j) \varepsilon G$ if $j = i + \frac{(|V|-3)}{2}(\mathrm{mod}|V|)$ when $k=4 \neq |V|-4$ k even.



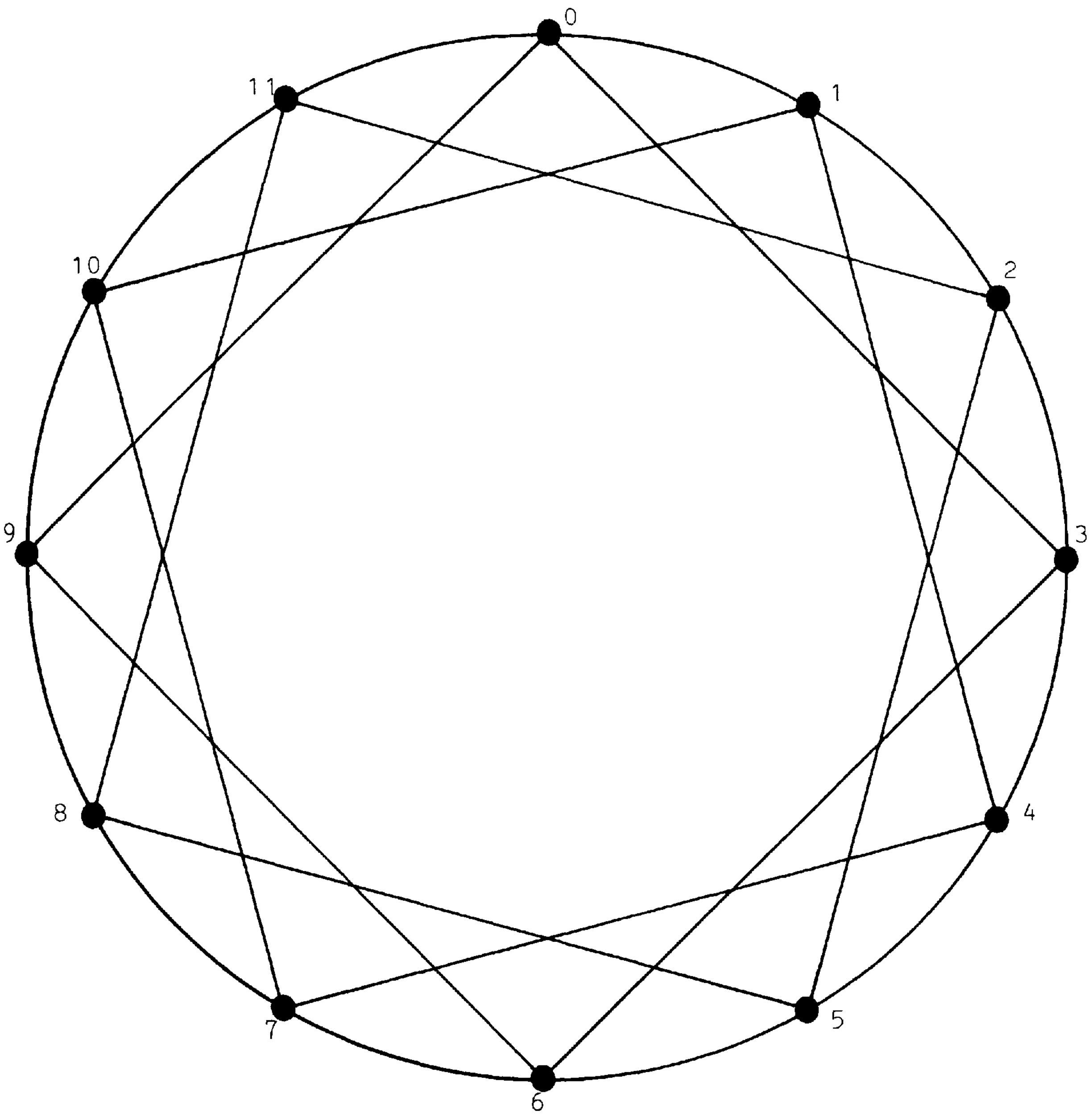
$A_c(G) = [0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1]$

$$\text{Spectrum of } G = \begin{pmatrix} 4 & \sqrt{3}-1 & 0 & 2 & -2 & -1-\sqrt{3} \\ 1 & 2 & 3 & 2 & 2 & 2 \end{pmatrix}.$$

$T_{|V|-1} = 3.71718 \times 10^5$, $R_{|E|-|V|+1} = 2.124426 \times 10^6$.

FIG. 3.3. A graph with no more than $|V|$ minimum vertex cut sets, S. L. Hakimi and A. T. Amin [26].

Construction $|V| = 12$, $k = 4$, $v_i v_j \varepsilon E(G)$ if $|i-j| = 1$ or 3.



$A_c(G) = [0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 1]$

$$\text{Spectrum of } G = \begin{pmatrix} 4 & 1 & \sqrt{3} & 0 & -1 & -\sqrt{3} & -4 \\ 1 & 2 & 2 & 2 & 2 & 2 & 1 \end{pmatrix}$$

$T_{|V|-1} = 405600$, $R_{|E|-|V|+1} = 2090544$.

FIG. 3.4. A graph G which is (k, k+1)-connected, D. H. Smith [45].

Recently J. Provan and M. Ball [39] have shown that the calculation of $P_d(G)$, belongs to the class of problems for which there is no known efficient algorithm. We mention the importance of finding graphs which maximise λ for a given value of $|V|$ and $|E|$ and also minimise R_λ the number of edge cut sets of size λ, thus enabling $P_d(G)$ to be minimised.

A class of graphs which achieve maximum connectivity was found by F. Harary [29]. The example shown in FIG. 3.5 illustrates that the shortest path length between some vertices in a Harary graph can be rather large. This, as indicated previously may result in intolerable queuing delays at certain nodes in a communication or computer network and is undesirable. We note also that the example contains a large number of minimum size vertex cut sets.

At this point to be more concise in our description of circulant graphs we assume the vertices of a graph are labelled 0, 1, 2, ...$|V|-1$ and refer to the circulant graph as $C_{|V|}(v_1, v_2, \dots v_S)$ or briefly $C_{|V|}(v_i)$ where $0 < v_1 < \dots v_S < \frac{|V|+1}{2}$ has $i \pm v_1$, $i \pm v_2$, ... $i \pm v_S (\bmod |V|)$ adjacent to each vertex i.

It is of practical interest to note that the graphs of F. Harary [29] can have quite large diameters if $|V|$ is large and S is small. It is possible that other circulants can be used to construct smaller diameter graphs for the same value of S. An example is the circulant $C_{18}\langle 1,8,\rangle$ shown in FIG. 3.6. In addition to the diameter it is of interest in the design of reliable networks to investigate the connectivity of circulants, FIG. 3.8 shows a circulant which is regular of degree $\rho = 8$, but the vertex connectivity k is equal to 6.

The determination of simple, necessary and sufficient conditions for a circulant to have maximum vertex connectivity is complex and is generalized by F. Boesch and R. Tindell [11] who give results when the vertex connectivity k is equal to ρ the degree of a regular graph and when $k < \rho$ or $\lambda < \rho$. The results are given in a main theorem which we state below,

<u>Theorem</u>. The circulant $C_{|V|}\langle v_i\rangle$, $1 \leqq i \leqq S$, satisfies $k < \rho$ if and only if for some proper divisor m of $|V|$, the number of distinct positive residues of the numbers $v_1, v_2, \ldots v_S, |V|-v_S, |V|-v_{S-1}, \ldots |V|-v_1$ is less than the minimum of m-1 and $\frac{\rho m}{|V|}$. The proof is given by F. Boesch and R. Tindell [11].

A simple condition which is known to be sufficient but not necessary for a circulant to have maximum connectivity is given by F. Boesch and A. Felzer [6]. They define a convex circulant to be one in which $v_{i+1} - v_i \leqq v_{i+2} - v_{i+1}$ $(1 \leqq i \leqq S-1)$, and that when $v_1 = 1$ and $\langle v_i\rangle$ is convex then $k = \rho$. However convexity is not necessary as illustrated by the graph in FIG. 3.7.

<u>Theorem</u> A convex circulant $C_{|V|} \langle 1, v_2, \ldots v_S \rangle$ is regular of degree $\rho = 2S$ and as $k = \lambda = 2S$.

The proof is given by F. Boesch and A. Felzer [6].

The earliest results on the connectivity of circulant graphs is due to F. Harary [29] who showed that $C_{|V|}(1, 2, \ldots S)$, has $k = \lambda = \rho = \frac{2|E|}{|V|}$.

Recent work by F. T. Boesch and J. F. Wang [12] give the conditions for a circulant to be (k, k+1)-edge-connected, they also determine R_i, the number of edge cut sets of size i (where $i > \lambda$) for the graphs of F. Harary [29].

<u>Theorem</u> Let $G = C_{|V|}(1, 2, \ldots S)$, $2 \leqq S < \frac{|V|}{2}$, and R be an edge cut set. If $|R| = i$ and $\lambda \leqq i \leqq 4S-3$ then R isolates exactly one vertex and $R_i = \binom{|E|-2S}{i-2S}|V|$, where $|E|$ is the number of edges, $|V|$ the number of vertices in a graph G, and R_i is the number of edge cut sets of size i.

The proof of this theorem is given by F. T. Boesch and J. F. Wang [12] who conclude that any regular graph with degree $\rho = 2S$, and with $|V|$ vertices will have $R_i \geqq \binom{|E|-2S}{i-2S}|V|$ as this lower bound counts only those edge cut sets of size i which are obtained from edge incident sets at vertices.

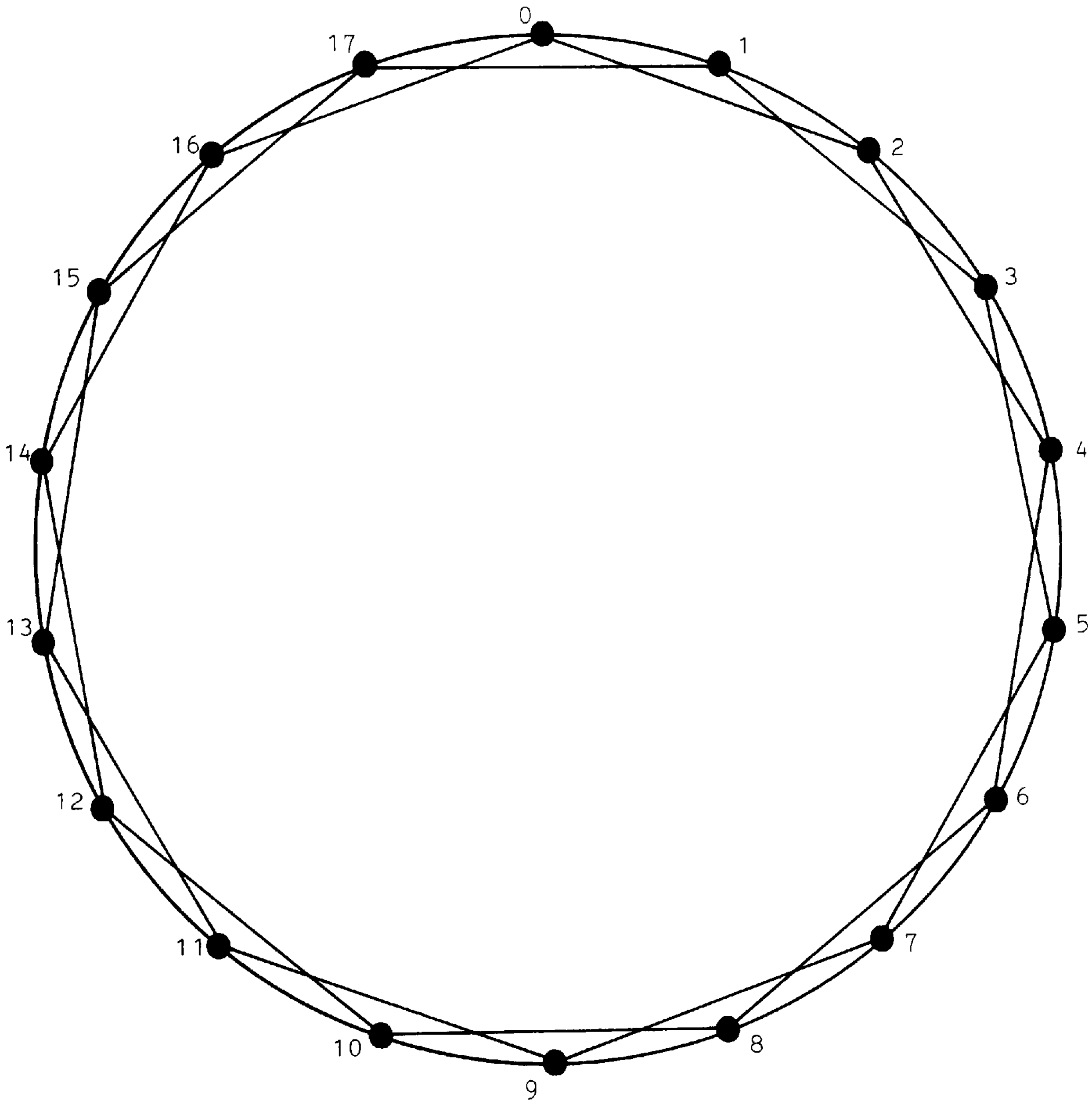


FIG. 3.5. Illustration that the shortest path length between some points in a Harary graph can be rather large.

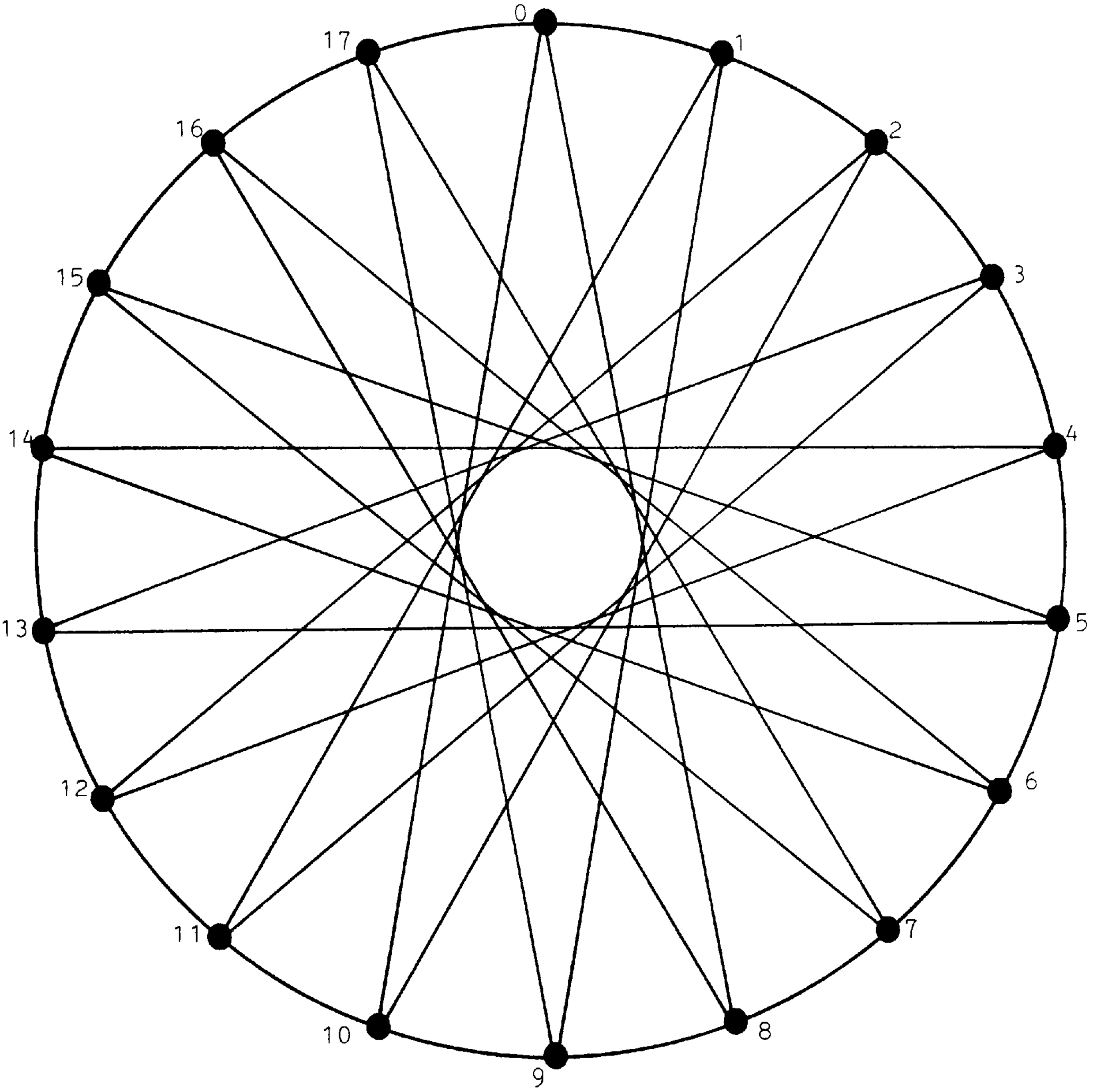


FIG. 3.6. Illustration of a circulant graph with $|V| = 18$, $|E| = 36$, $k = 4$ and diameter = 4.

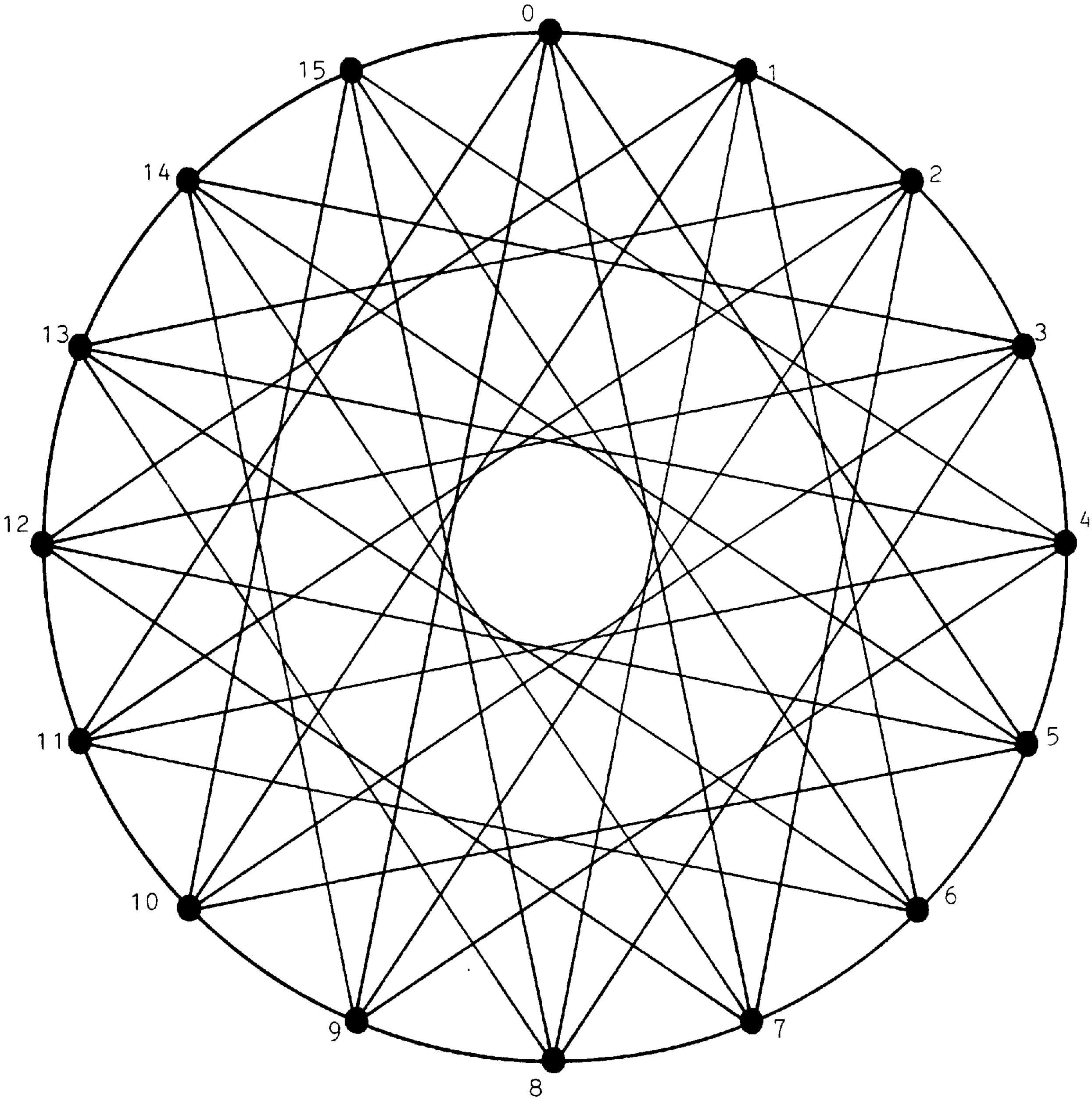


FIG. 3.7. Example of a graph $C_{16}\langle 1, 5, 7\rangle$ with $k = \rho$, which is not a convex circulant.

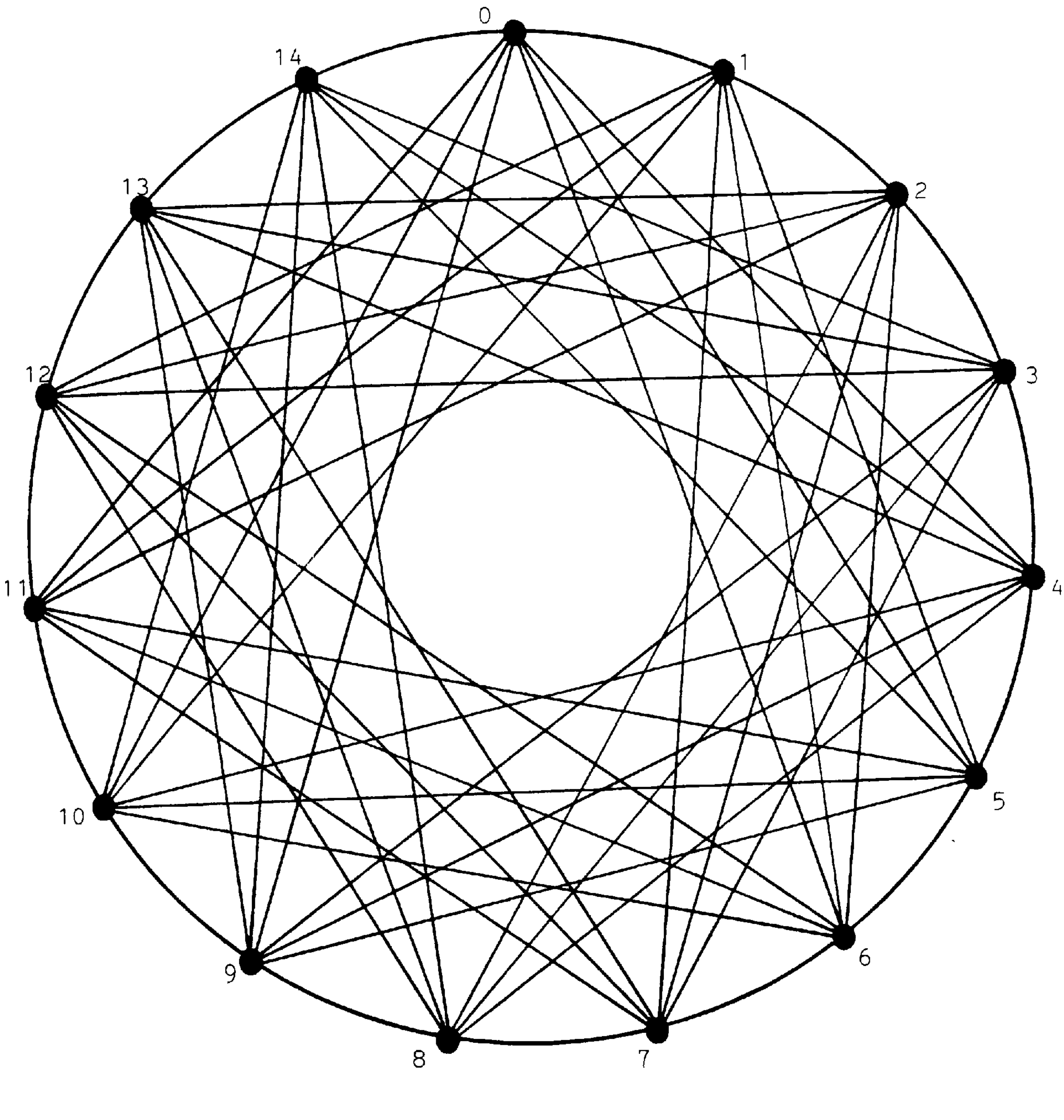


FIG. 3.8 Example of a circulant $C_{15}\langle 1, 4, 5, 6\rangle$ with $\rho = 8$, $k = 6$, e.g. $\{1, 4, 6, 9, 11, 14\}$ is a vertex cut set.

3.2 A General Construction

Construction A

Let G_1 be a graph with vertices $v_1, v_2, \ldots v_{|V|}$. Replace each vertex v_i by m vertices v_{ij} (j = 1, 2, ... m). Vertices v_{ij}, v_{gh} are adjacent if and only if vertices v_i, v_g are adjacent in G_1. This new graph G_2 has mn vertices.

FIG. 3.9(a) and (b) shows a graph G_1 and the graph G_2 obtained by applying Construction A.

We recall that a k-connected graph is said to be (k, k+1)-connected if every vertex cut set with k vertices is the neighbour set of a vertex.

Lemma If the graph G_1 is (k, k+1)-connected then the graph G_2 of Construction A is (mk, mk+1)-connected. If G_1 has S vertex cut sets with k vertices then G_2 has S vertex cut sets with mk vertices.

Proof If G_1 has minimum size vertex cut sets of the form $A = \{v_\alpha, \ldots v_\delta\}$ then the minimum size vertex cut sets of G_2 are of the form $B = \{v_{\alpha 1}, \ldots v_{\alpha m}, \ldots v_{\delta 1}, \ldots v_{\delta m}\}$. If A is the neighbour set of v_i then B is the neighbour set of v_{ij} for each j (j = 1, 2, ... m). ■

We now make a general comparison between the graphs with $|V|$ vertices which are (k, k+1)-connected and

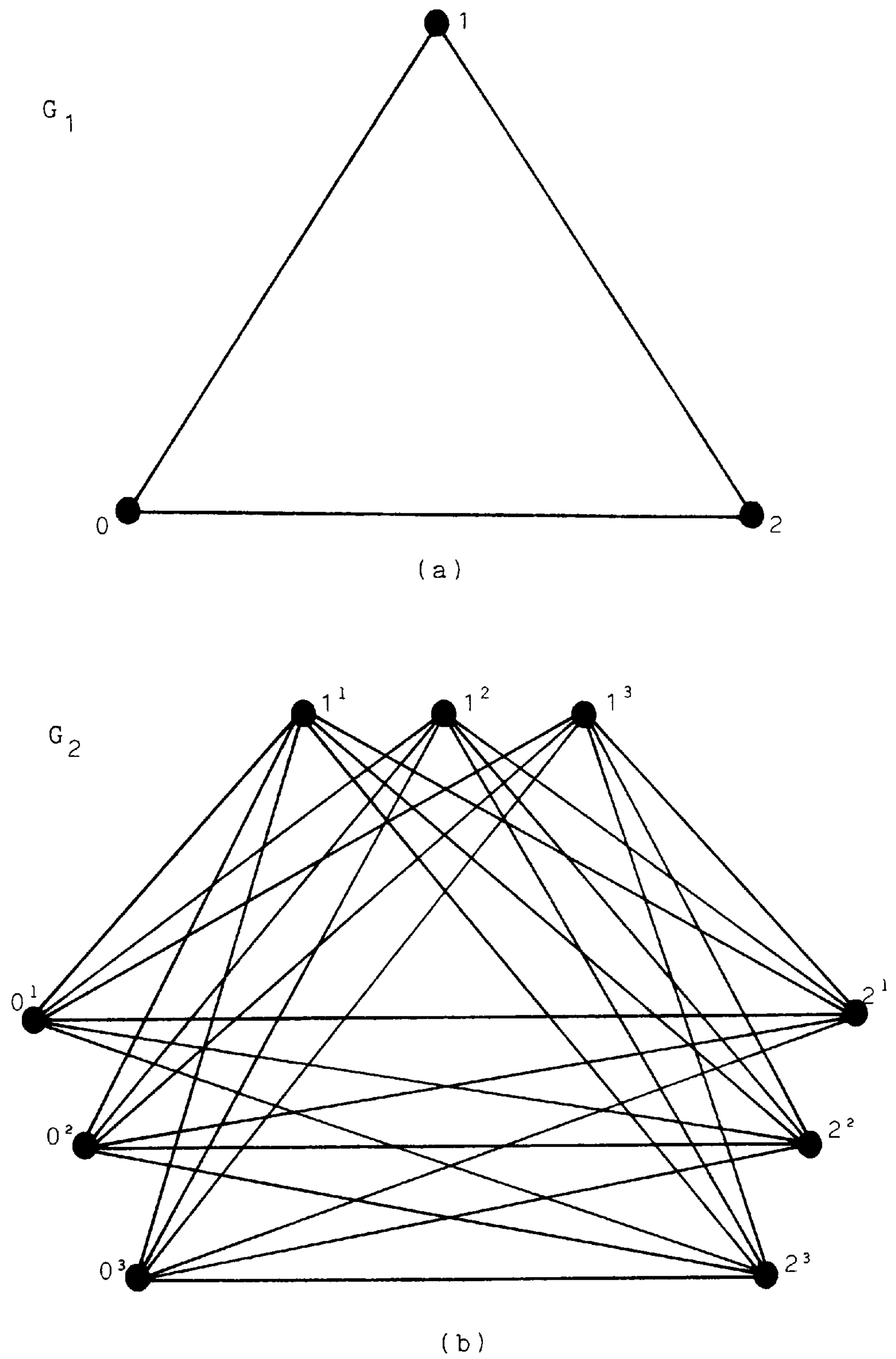


FIG. 3.9. Diagrams illustrating the application of Construction A.

graphs constructed with $m|V|$ vertices which are $(mk, mk+1)$-connected (i.e. using Construction A). Also given are numerical examples illustrating the calculation of the eignevalues, complexity and edge cut sets of size $|E|-|V|+1$.

A general comparison between graphs with $|V|$ vertices which are $(k, k+1)$-connected denoted by G_1, and graphs obtained using Construction A which are $(mk, mk+1)$-connected denoted by G_2.

Let $A(G_2)$ be the adjacency matrix of the graph G_2 and $A(G_1)$ the adjacency matrix of the graph G_1.

Then $A(G_2)$ has blocks $A(G_1)$ $\quad i = 1, 2, 3, \ldots m$
$\quad j = 1, 2, 3 \ldots m$

$$A(G_2) = \begin{bmatrix} A(G_1) & - & - & - & A(G_1) \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ A(G_1) & - & - & - & A(G_1) \end{bmatrix}$$

Let λ_r be the eigenvalues of $A(G_2)$ where $r = 1, 2, 3, \ldots m \times |V_1|$.

Now if X_r is an eigenvector of $A(G_1)$,

$$A(G_1)X_r = \lambda_r X_r$$

λ_r = eigenvalues of the matrix $A(G_1)$,
$r = 1, 2, 3, \ldots |V_1|$

X_r = the column vector $\begin{pmatrix} x_1 \\ \vdots \\ x_{|V_1|} \end{pmatrix}$

Partitioning of $A(G_2)$ gives,

$$\begin{bmatrix} A(G_1) & - & - & - & A(G_1) \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ A(G_1) & & & & A(G_1) \end{bmatrix} \begin{bmatrix} X_r \\ - \\ - \\ - \\ X_r \end{bmatrix} = \begin{bmatrix} mA(G_1)X_r \\ - \\ - \\ - \\ mA(G_1)X_r \end{bmatrix}$$

$$= m\lambda_r \begin{bmatrix} X_r \\ - \\ - \\ - \\ X_r \end{bmatrix} \qquad r = 1, 2, 3, \ldots |V_1|$$

i.e. to each eigenvector of $A(G_1)$ corresponding to λ_r there is an eigenvector of $A(G_2)$ corresponding to $m\lambda_r$. $A(G_1)$ has eigenvectors corresponding to $|V_1|$ eigenvalues.

Similarly $A(G_2)$ has $m|V_1|$ eigenvalues, we know $|V_1|$ eigenvalues and require the remaining $|V_1|(m-1)$ eigenvalues of $A(G_2)$. We look at the independent eigenvectors of $A(G_2)$ which exist for $\lambda_r = 0$.

$$\begin{bmatrix} A(G_1) & - & - & - & A(G_1) \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ A(G_1) & - & - & - & A(G_1) \end{bmatrix} \begin{bmatrix} X_r \\ -X_r \\ - \\ - \\ 0 \end{bmatrix} = 0 \begin{bmatrix} X_r \\ -X_r \\ - \\ - \\ 0 \end{bmatrix}$$

$$\text{Where } X_r = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0_{|V_1|} \end{bmatrix} \quad \text{and } -X_r = \begin{bmatrix} -1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0_{|V_1|} \end{bmatrix}$$

In general the eigenvectors corresponding to $\lambda_r = 0$ can be found from,

$$\begin{matrix} 1 & 2 & \cdot \quad \cdot \quad \cdot & |V_1|(m-1) \\ \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0_{|V_1|} \\ -1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0_{2|V_1|} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0_{m|V_1|} \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0_{|V_1|} \\ 0 \\ -1 \\ 0 \\ \cdot \\ \cdot \\ 0_{2|V_1|} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0_{m|V_1|} \end{bmatrix} & \begin{matrix} - \; - \; - \\ - \; - \; - \\ - \; - \; - \\ - \; - \; - \\ - \; - \; - \\ - \; - \; - \\ - \; - \; - \\ - \; - \; - \\ - \; - \; - \\ - \; - \; - \\ - \; - \; - \\ - \; - \; - \\ - \; - \; - \\ - \; - \; - \\ - \; - \; - \\ - \; - \; - \\ - \; - \; - \end{matrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 1_{|V_1|} \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0_{2|V_1|} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ -1_{m|V_1|} \end{bmatrix} \end{matrix}$$

Therefore λ_r of $G_2 = m\lambda_r$ of G_1 $(r = 1, 2, \ldots |V_1|)$ and $0, |V_1|(m-1)$ times.

	(k, k+1)-connected	Construction A
	G_1	G_2
Number of Vertices	$\lvert V_1 \rvert$	$m\lvert V_1 \rvert$
Number of Edges	$\lvert E_1 \rvert = \frac{k\lvert V_1 \rvert}{2}$	$\lvert E_2 \rvert = \frac{m^2 k \lvert V_1 \rvert}{2}$
Vertex Connectivity	$(k, k+1)$	$(mk, mk+1)$
Edge Connectivity	$(k, k+1)$	$(mk, mk+1)$
Minimum Vertex Cut Set	k	mk
Number of Minimum Vertex Cut Sets	$S_i \quad i = k$	$S_i \quad i = mk$
Minimum Edge Cut Set	k	mk
Number of Minimum Edge Cut Sets	$R_i \quad i = k$	$R_i \quad i = mk$
Eigenvalues	λ_r $(r = 1, 2, \ldots \lvert V_1 \rvert)$	λ_r $(r = 1, 2, \ldots m\lvert V_1 \rvert)$ $= \begin{pmatrix} m\lambda_r, & r = 1, 2, \ldots \lvert V_1 \rvert \\ 0, & \lvert V_1 \rvert (m-1) \text{ times.} \end{pmatrix}$

FIG. 3.10. Some general results between (k, k+1)-connected graphs and those graphs obtained using Construction A.

	(k, k+1)-connected	Construction A
	G_1	G_2
Adjacency Matrix	$A(G_1) = \|V_1\| \times \|V_1\|$	$A(G_2) = m\|V_1\| \times m\|V_1\|$
Complexity	$T_{\|V_1\|-1} = \frac{1}{\|V_1\|} \prod_{r=1, k\neq\lambda_r}^{\|V_1\|} (k-\lambda_r)$	$T_{\|V_2\|-1} = \frac{1}{m\|V_1\|} \prod_{r=1, mk\neq\lambda_r}^{m\|V_1\|} (mk-\lambda_r)$
Edges in Cut Sets of Size $\|E\|-\|V\|+1$	$\frac{\|V_1\|(k-2)+2}{2}$	$\frac{m\|V_1\|(mk-2)+2}{2}$
Edge Cut Sets of Size $\|E\|-\|V\|+1$	$\frac{\left(\frac{k\|V_1\|}{2}\right)!}{(\|V_1\|-1)!\left(\frac{\|V_1\|(k-2)+2}{2}\right)!} - T_{\|V_1\|-1}$	$\frac{\left(\frac{m^2k\|V_1\|}{2}\right)!}{(m\|V_1\|-1)!\left(\frac{m\|V_1\|(mk-2)+2}{2}\right)!} - T_{\|V_2\|-1}$

FIG. 3.10 (Cont'd). Some general results between (k, k+1)-connected graphs and those graphs obtained using Construction A.

Calculations of the eigenvalues, complexity and edge cut sets of size $|E| - |V| + 1$ for the graphs in FIG. 3.9.

a)

$$A(G_1) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Characteristic equation $|A(G_1) - \lambda I| = 0$

giving
$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

from which $\lambda^3 - 3\lambda - 2 = 0$

$$\underline{\lambda_r = 2, -1, -1} \qquad r = (1, 2, \ldots |V|)$$

Complexity of G_1 $= T_{|V|-1} = \frac{1}{|V_1|} \prod_{r=1}^{|V_1|} (k - \lambda_r) \quad k \neq \lambda_r$

$$T_{|V|-1} = \frac{1}{3} \times 3 \times 3 = 3.$$

Number of edge cut sets of size $|E| - |V| + 1$

$$= \binom{|E_1|}{|E_1| - |V_1| + 1} - T_{|V|-1} = 3 - 3 = 0$$

b)

$$A(G_2) = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Partitioning $A(G_2)$ gives,

$$A(G_2) = \begin{bmatrix} A(G_1) & A(G_1) & A(G_1) \\ A(G_1) & A(G_1) & A(G_1) \\ A(G_1) & A(G_1) & A(G_1) \end{bmatrix}$$

where

$$A(G_1) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} A(G_1) & A(G_1) & A(G_1) \\ A(G_1) & A(G_1) & A(G_1) \\ A(G_1) & A(G_1) & A(G_1) \end{bmatrix} \begin{bmatrix} x_r \\ x_r \\ x_r \end{bmatrix} = \begin{matrix} 3A(G_1)x_r \\ 3A(G_1)x_r \\ 3A(G_1)x_r \end{matrix} = \begin{matrix} 3\lambda_r x_r \\ 3\lambda_r x_r \\ 3\lambda_r x_r \end{matrix}$$

The eigenvalues of $A(G_1)$ are $\lambda_r = 2, -1, -1,$

$(r = 1, \ldots |V_1|)$

Some of the eigenvalues of $A(G_2)$ are $3\lambda_r$,

i.e. λ_r of $A(G_2)$ $(r = 1, 2, \ldots |V_1|)$

$= 3(2), 3(-1), 3(-1)$

$= 6, -3, -3.$

$A(G_2)$ has $m|V_1|$ eigenvalues and we know $|V_1|$ eigenvalues of $A(G_2)$. To find the remaining $|V_1|(m-1)$ eigenvalues for $A(G_2)$ we look at the independent eigenvectors of $A(G_2)$ corresponding to $\lambda_r = 0$.

$$A(G_2)X_r = \lambda_r X_r$$

$$\begin{bmatrix} 0&1&1&0&1&1&0&1&1\\ 1&0&1&1&0&1&1&0&1\\ 1&1&0&1&1&0&1&1&0\\ 0&1&1&0&1&1&0&1&1\\ 1&0&1&1&0&1&1&0&1\\ 1&1&0&1&1&0&1&1&0\\ 0&1&1&0&1&1&0&1&1\\ 1&0&1&1&0&1&1&0&1\\ 1&1&0&1&1&0&1&1&0 \end{bmatrix} \begin{bmatrix} 1\\0\\0\\-1\\0\\0\\0\\0\\0 \end{bmatrix} = 0 \begin{bmatrix} 1\\0\\0\\-1\\0\\0\\0\\0\\0 \end{bmatrix}$$

We are able to find 6 independent eigenvectors corresponding to $\lambda_r = 0$, these are,

$$\begin{bmatrix} 1\\0\\0\\-1\\0\\0\\0\\0\\0 \end{bmatrix} \begin{bmatrix} 0\\1\\0\\0\\-1\\0\\0\\0\\0 \end{bmatrix} \begin{bmatrix} 0\\0\\1\\0\\0\\-1\\0\\0\\0 \end{bmatrix} \begin{bmatrix} 1\\0\\0\\0\\0\\0\\-1\\0\\0 \end{bmatrix} \begin{bmatrix} 0\\1\\0\\0\\0\\0\\0\\-1\\0 \end{bmatrix} \begin{bmatrix} 0\\0\\1\\0\\0\\0\\0\\0\\-1 \end{bmatrix}$$

The eigenvalues of $A(G_2)$ are,

6, -3, -3, 0(6 times).

$$\text{Complexity of } G_2 = T_{|V_2|-1} = \frac{1}{|V_2|} \prod_{r=1}^{|V_2|} (k-\lambda_r) \quad k \neq \lambda_r$$

where $|V_2| = 9$, $k = 6$.

$$T_{|V_2|-1} = \frac{1}{9} \times 9 \times 9 \times (6)^6 = \underline{419904}.$$

Number of edge cut sets of size $|E_2| - |V_2| + 1$

$$= \binom{|E_2|}{|E_2| - |V_2| + 1} - T_{|V_2|-1} = \binom{27}{19} - 4.19904 \times 10^5$$

$$= \underline{1800171}.$$

Calculations of the eigenvalues, complexity and edge cut sets of size $|E| - |V| + 1$ for the graphs shown in FIG. 3.11.

a)

$$A(G_3) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Characteristic equation $|A(G_3) - \lambda I| = 0$

giving
$$\begin{vmatrix} -\lambda & 1 & 1 & 1 \\ 1 & -\lambda & 1 & 1 \\ 1 & 1 & -\lambda & 1 \\ 1 & 1 & 1 & -\lambda \end{vmatrix} = 0$$

from which $\lambda_r = -1, -1, -1, 3 \quad (r = 1, 2, \ldots |V|)$

$$\text{Complexity of } G_3 = T_{|V|-1} = \frac{1}{|V|} \prod_{r=1}^{|V|} (k - \lambda_r) \quad k \neq \lambda_r$$

$$T_{|V|-1} = \frac{1}{4} \times 4^3 = \underline{16}$$

Number of edge cut sets of size $|E| - |V| + 1$

$$= \binom{|E|}{|E| - |V| + 1} - T_{|V|-1} = \binom{6}{3} - 16 = \underline{4}$$

b)

$$A(G_4) = \begin{bmatrix} 0&1&1&1&0&1&1&1&0&1&1&1 \\ 1&0&1&1&1&0&1&1&1&0&1&1 \\ 1&1&0&1&1&1&0&1&1&1&0&1 \\ 1&1&1&0&1&1&1&0&1&1&1&0 \\ 0&1&1&1&0&1&1&1&0&1&1&1 \\ 1&0&1&1&1&0&1&1&1&0&1&1 \\ 1&1&0&1&1&1&0&1&1&1&0&1 \\ 1&1&1&0&1&1&1&0&1&1&1&0 \\ 0&1&1&1&0&1&1&1&0&1&1&1 \\ 1&0&1&1&1&0&1&1&1&0&1&1 \\ 1&1&0&1&1&1&0&1&1&1&0&1 \\ 1&1&1&0&1&1&1&0&1&1&1&0 \end{bmatrix}$$

Partitioning $A(G_4)$ gives,

$$A(G_4) = \begin{bmatrix} A(G_3) & A(G_3) & A(G_3) \\ A(G_3) & A(G_3) & A(G_3) \\ A(G_3) & A(G_3) & A(G_3) \end{bmatrix}$$

where

$$A(G_3) = \begin{bmatrix} 0&1&1&1 \\ 1&0&1&1 \\ 1&1&0&0 \\ 1&1&1&0 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} A(G_3) & A(G_3) & A(G_3) \\ A(G_3) & A(G_3) & A(G_3) \\ A(G_3) & A(G_3) & A(G_3) \end{bmatrix} \begin{bmatrix} x_r \\ x_r \\ x_r \end{bmatrix} = \begin{matrix} 3A(G_3)x_r \\ 3A(G_3)x_r \\ 3A(G_3)x_r \end{matrix} = \begin{matrix} 3\lambda_r x_r \\ 3\lambda_r x_r \\ 3\lambda_r x_r \end{matrix}$$

Thus some of the eigenvalues of $A(G_4)$ are $3\lambda_1$, $3\lambda_2$, $3\lambda_3$, $3\lambda_4$ giving $\lambda_r = -3, -3, -3, -9$.

$A(G_4)$ has $m|V_3|$ eigenvalues, we know $|V_3|$ eigenvalues of $A(G_4)$. To find the remaining $|V_3|(m-1)$ eigenvalues for $A(G_4)$ we find the independent eigenvectors of $A(G_4)$ corresponding to $\lambda_r = 0$.

$$A(G_4)X_r = \lambda_r X_r$$

We are able to find $|V_3|(m-1)$ i.e. 8 independent eigenvectors corresponding to $\lambda_r = 0$, these are,

$$\begin{bmatrix}1\\0\\0\\0\\-1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\\0\\-1\\0\\0\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\\0\\0\\0\\-1\\0\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\\0\\0\\0\\-1\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\\-1\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\\0\\0\\0\\0\\0\\-1\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\-1\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\-1\end{bmatrix}$$

The eigenvalues of $A(G_4)$ are,

-3, -3, -3, -9, 0(8 times).

Complexity of G_4 $= T_{|V|-1} = \dfrac{1}{|V|}\displaystyle\prod_{r=1}^{|V_4|}(k-\lambda_r) \quad k \neq \lambda_r$

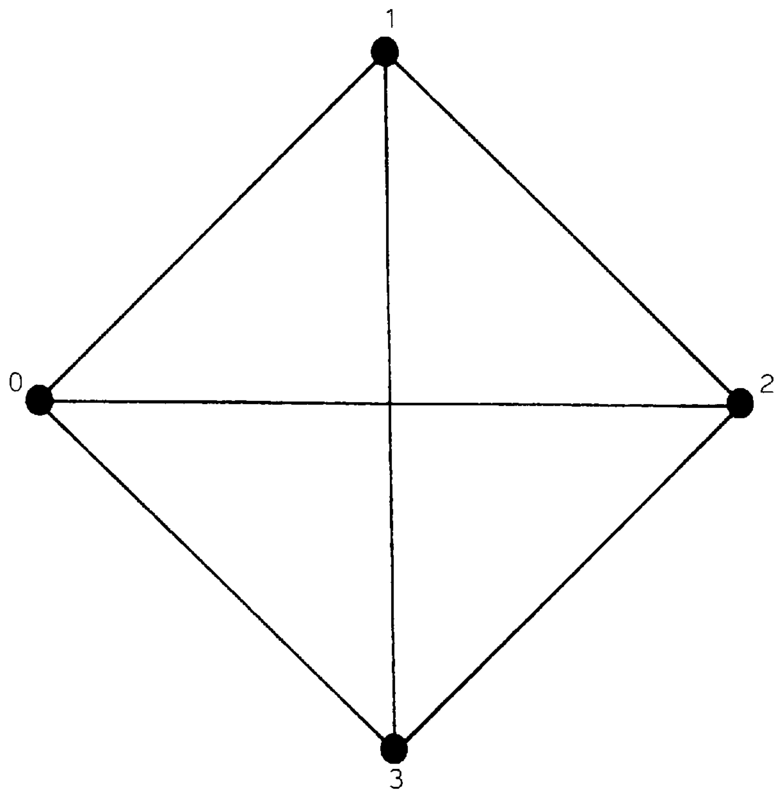
Where $|V_4| = 12$, $k = 9$.

Complexity of G_4 $= \dfrac{1}{12} \times 12^3 \times 9^8 = \underline{6.1987278 \times 10^9}$

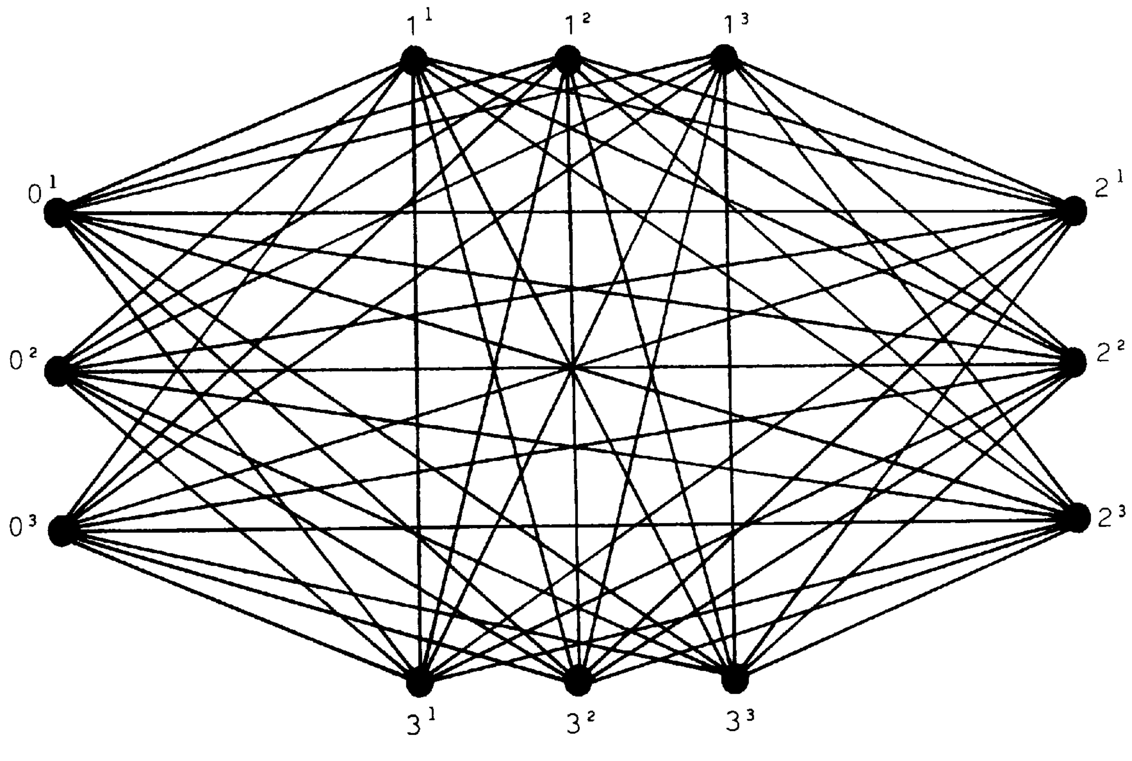
Number of edge cut sets of size $|E| - |V| + 1$

$$= \binom{|E|}{|E|-|V_4|+1} - T_{|V|-1} = \binom{54}{43} - 6.1987278 \times 10^9$$

$= \underline{8.9524125 \times 10^{10}}$



G_3



G_4

FIG. 3.11

3.3 Bipartite graphs of optimal complexity

A bipartite graph is defined in Chapter 2, Section 2.2; this class of graphs is important in our discussion because the bipartite graphs have optimal connectivity also the complexity of a bipartite regular graph is an upper bound if the graph is a bipartite distance regular graph of diameter three. This class of graphs is therefore of interest in our reliability studies when p the probability of edge failure is close to 1.

Before dealing with distance regular graphs we now give a number of general properties of bipartite graphs. In the complete bipartite graph, each vertex V_1 is adjacent to every vertex in V_2 and vice versa. FIG. 3.12 illustrates a complete bipartite graph.

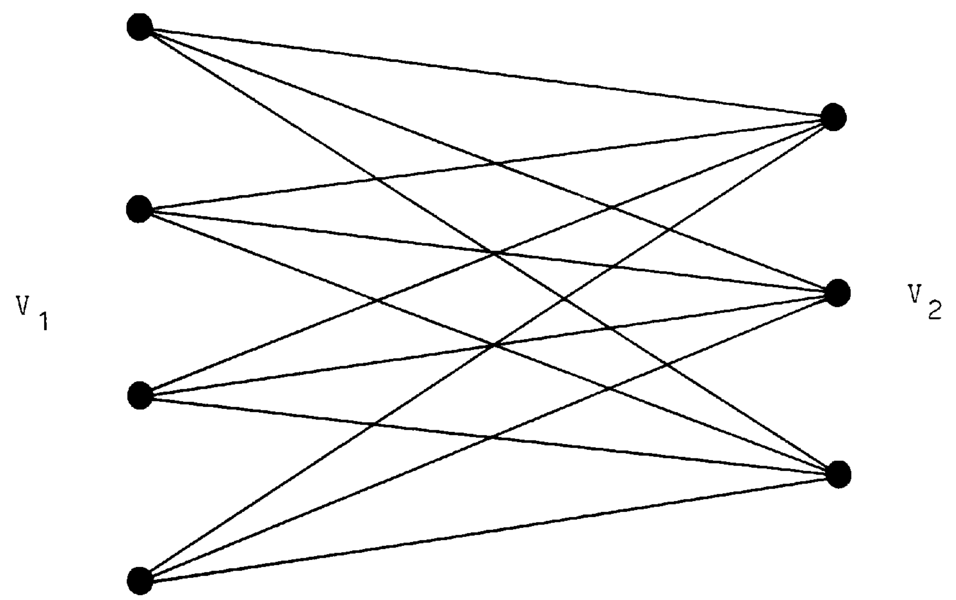


FIG. 3.12

The number of edges in the complete bipartite graph is $|E| = |V_1||V_2|$. F. T. Boesch and R. E. Thomas [10] show that this graph is of optimal connectivity if and only if the vertex sets V_1 and V_2 are equal. K. W. Cattermole [17] applies the optimal connectivity concept to bipartite networks and exhibits a rather large class of optimal bipartite graphs and gives a number of examples of the relationship between the graph and practical telecommunications networks. For example a telephone exchange contains a large number of switches, at least some tens of thousands of contacts arranged in some hundreds of blocks or groups. The interconnection of these blocks is known as trunking. Exchange trunking can be represented by the bipartite graph.

A bipartite graph contains no odd circuits and the spectrum of the complete bipartite graph $K_{|V_1||V_2|}$ is:-

$$\text{Spectrum } K_{|V_1||V_2|} = \begin{pmatrix} \sqrt{|V_1||V_2|} & 0 & -\sqrt{|V_1||V_2|} \\ 1 & |V_1| + |V_2| - 2 & 1 \end{pmatrix}$$

N. Biggs [3] p. 50

N. Biggs [3] p. 50 shows that if the bipartite graph G has an eigenvalue λ of multiplicity $m(\lambda)$, then $-\lambda$ is also an eigenvalue of G and $m(-\lambda) = m(\lambda)$. It follows that in a regular graph of degree $\rho = k$ that since k is an eigenvalue, then we have the eigenvalues of the adjacency matrix of the bipartite regular graph G are $k, -k, \lambda_1, -\lambda_1 \ldots \lambda_r, -\lambda_r$ where $r = \frac{|V|-2}{2}$

A distance regular graph, with diameter d is defined as a regular connected graph with degree $\rho = k$ with the following property. There are natural numbers

$b_0 = k, b_1, \dots b_{d-1} \quad C_1 = 1, C_2, \dots C_d,$

such that for each pair (u,v), of vertices satisfying $Z(u,v) = j$ (where $Z(u,v)$ is called the distance between u and v)

we have,

(1) the number of vertices in $G_{j-1}(v)$ adjacent to u is C_j $(1 \leqq j \leqq d)$.

(2) the number of vertices in $G_{j+1}(v)$ adjacent to u is b_j $(0 \leqq j \leqq d-1)$.

where we define $G_i(v)$ for any connected graph G, and each v in V to be

$$G_i(v) = \{u\varepsilon V | Z(u\ v) = i\}$$

where $0 \leqq i \leqq d$, and d is the diameter of G.

$G_0(v) = \{v\}$, and V is partitioned into the disjoint subsets $G_0(v), \dots G_d(v)$, for each v in V. Small graphs can be partitioned in this manner by arranging their vertices in columns, according to distance from an arbitrary vertex v. For example, K_{33} is displayed in this way in FIG. 3.13

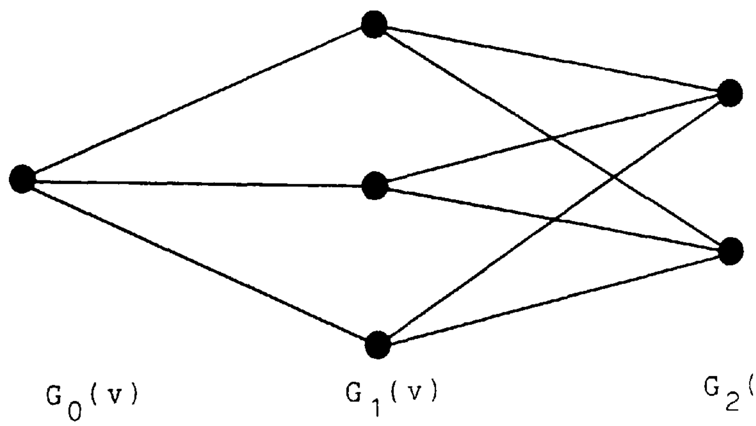


FIG. 3.13. Example of a distance regular graph, K_{33}.

We now find an upper bound for the complexity of a bipartite regular graph and show that this bound is attained by bipartite distance regular graphs of diameter 3.

For a regular graph,

$$\text{Complexity, } T_{|V|-1} = \frac{1}{|V|} \prod_{r=1}^{|V|-1} (k-\lambda_r) \qquad k \neq \lambda_r$$

for the regular bipartite graph this gives,

$$\text{Complexity, } T_{|V|-1} = \frac{1}{|V|} (k+k)(k+\lambda_1)(k-\lambda_1)\ldots(k+\lambda_S)(k-\lambda_S)$$

$$T_{|V|-1} = |V|^{-1} 2k \prod_{r=1}^{S} (k^2-\lambda_r^2)$$

Now $\sum \lambda_r = 0$, and $\sum \lambda_r^2 = 2|E|$ N. Biggs [3] p. 13

for the regular bipartite graph, we have

$$\begin{aligned} 2|E| &= \sum \lambda_r^2 \\ &= k^2 + k^2 + \lambda_1^2 + \lambda_1^2 + \ldots \lambda_S^2 + \lambda_S^2 \\ &= 2k^2 + 2\sum_{r=1}^{S} \lambda_r^2 \end{aligned}$$

i.e.

$$|E| = k^2 + \sum_{r=1}^{S} \lambda_r^2$$

<u>Theorem</u> The complexity, $T_{|V|-1}$ of a finite, simple, connected, undirected, bipartite regular graph with $|V| = 2S + 2$ vertices, $|E|$ edges, degree $\rho = k$ is given by,

$$T_{|V|-1} \leq \frac{2k}{|V|}\left(\frac{(S+1)k^2 - |E|}{S}\right)^S$$ with equality if and only if G has exactly 4 distinct eigenvalues $k, -k, \gamma, -\gamma$.

<u>Proof</u> Applying the arithmetic-geometric mean equality to

$$T_{|V|-1} = |V|^{-1} 2k \prod_{r=1}^{S} (k^2 - \lambda_r^2)$$

gives,

$$T_{|V|-1} \leq \frac{2k}{|V|}\left(\frac{1}{S} \sum_{r=1}^{S} (k^2 - \lambda_r^2)\right)^S$$

Now $|E| = k^2 + \sum_{r=1}^{S} \lambda_r^2$

we can write,

$$T_{|V|-1} \leq \frac{2k}{|V|}\left(\frac{Sk^2 - \sum_{r=1}^{S} \lambda_r^2}{S}\right)^S$$

Substituting for $\sum_{r=1}^{S} \lambda_r^2$ gives,

$$T_{|V|-1} \leq \frac{2k}{|V|}\left(\frac{(S+1)k^2 - |E|}{S}\right)^S$$

with equality if and only if all λ_r's are equal. ■

If G is a distance regular graph with diameter d then G has just d+1 distinct eigenvalues [20]. Consequently we have equality in the theorem on complexity if G is a bipartite distance-regular graph of diameter three. Two infinite families of bipartite distance-regular graphs of diameter three are,

1) G has vertices V_{1i} $(i=1,2,\ldots k+1)$ and V_{2i} $(i=1,2,\ldots k+1)$ with V_{1i} joined to all V_{2j} $(j \neq i)$.

2) G is the point-line graph of a projective plane of order k-1.

As a consequence of the theorem on complexity we conclude that in terms of reliability the bipartite distance regular graphs of diameter three have the smallest probability of disconnection in the presence of edge failures, (when the probability of failure of an edge is sufficiently large), of all bipartite graphs with the same number of vertices and edges.

We now give a definition of a symmetric balanced incomplete block design. This corresponds to a bipartite distance regular graph of diameter three. We give an example in which we can see that a bipartite distance regular graph of diameter three with $|V|$ vertices and degree $\rho = k$ corresponds to a symmetric block design with parameters v,v,k,k,ℓ where $|V| = 2v$, $k = k$.

Definition A balanced incomplete block design is an arrangement of v distinct objects into b blocks such that each block contains exactly k distinct objects, each object occurs in exactly r different blocks, and every pair of distinct objects a_i, a_j occurs together in exactly ℓ blocks.

A block design is called symmetric if $v = b$ (and so, also $k = r$).

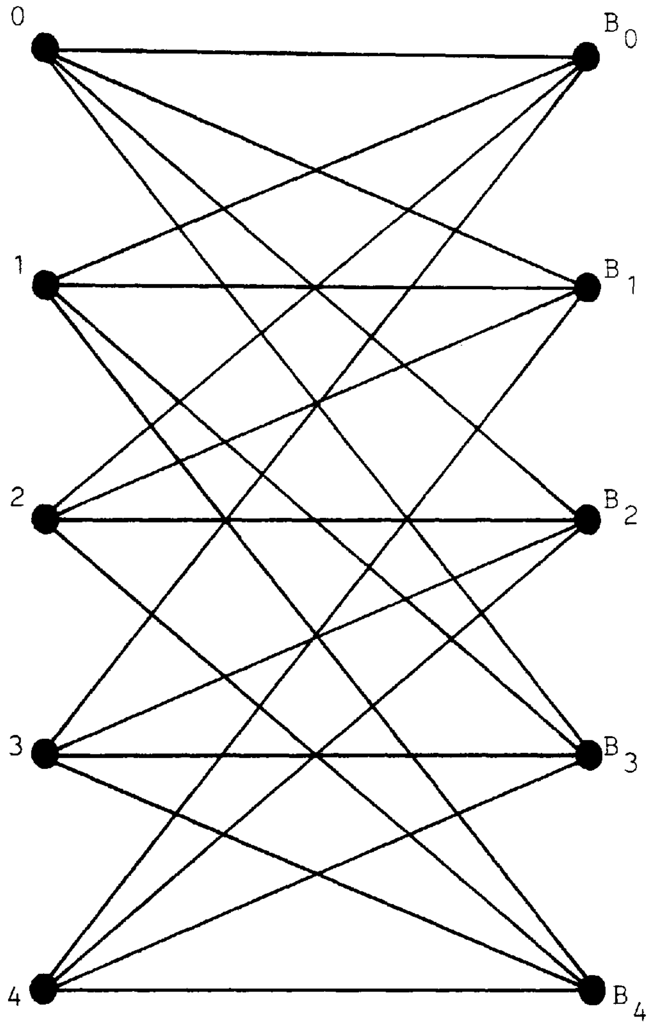
<u>Definition</u> The point-block graph has vertices corresponding to the points and to the blocks of a symmetric balanced incomplete block design. A point vertex is joined to a block vertex if and only if the point is contained in the block.

FIG. 3.14(a) and FIG. 3.14(b) show examples of a symmetric block design and the corresponding bipartite distance regular graph of diameter three. Necessary conditions for the existence of symmetric balanced incomplete block designs and a table containing sporadic examples is given by M. Hall [27]. Many later examples of symmetric block designs are known.

For $v = b = 5$, $r = k = 4$, $\ell = 3$:

B_0	:	0	1	2	3
B_1	:	0	1	2	4
B_2	:	0	2	3	4
B_3	:	0	3	4	1
B_4	:	1	2	3	4

(a)



$|V| = 2v$
$k = k$

(b)

FIG. 3.14

CHAPTER 4

CHAPTER 4

Graphs Which Are Optimal With Respect To Small Probability Of Vertex Failure

We consider finite, simple, undirected, regular graphs with $|V|$ vertices, degree $\rho = k$ and connectivity $= k$. Such graphs exist for all $|V|$, k and constructions are given by F. Harary [29]. In particular we show how to generate various infinite families of maximally reliable graphs as given by D. H. Smith [45] with $\frac{k}{|V|}$ in the range $\frac{3}{8} \leq \frac{k}{|V|} < 1$, which are optimal in the sense that they contain the smallest number of minimum vertex cut sets. This means that they are optimal when q (the probability of failure of a vertex) is close to 0.

We define an equivalence relation $\sim$ on the vertices of a graph G_1 by $u \sim v$ if and only if $\Gamma(u) = \Gamma(v)$. If E_u denotes the equivalence class containing vertex u then we construct a graph G in which the vertices represent the equivalence classes and vertices representing E_u, E_v are adjacent if and only if there exists $a \varepsilon E_u$, $b \varepsilon E_v$ such that a, b are adjacent in G_1. We then label the vertices of G, the label of a vertex being the number of vertices in the equivalence class. We shall refer to G as the base graph of G_1 as shown in FIG. 4.1.

Given a graph G we shall study the question of when the vertices can be labelled with integers in the range 1 to k so that properties (a) to (e) on page 140 can be satisfied. The graph G has the following properties:

(a) For any vertex u_L of G, the sum of the labelled vertices which form a vertex neighbour set is equal to k.

(b) The sum of the labelled vertices u_L in any vertex cut set of G is $\geq k$.

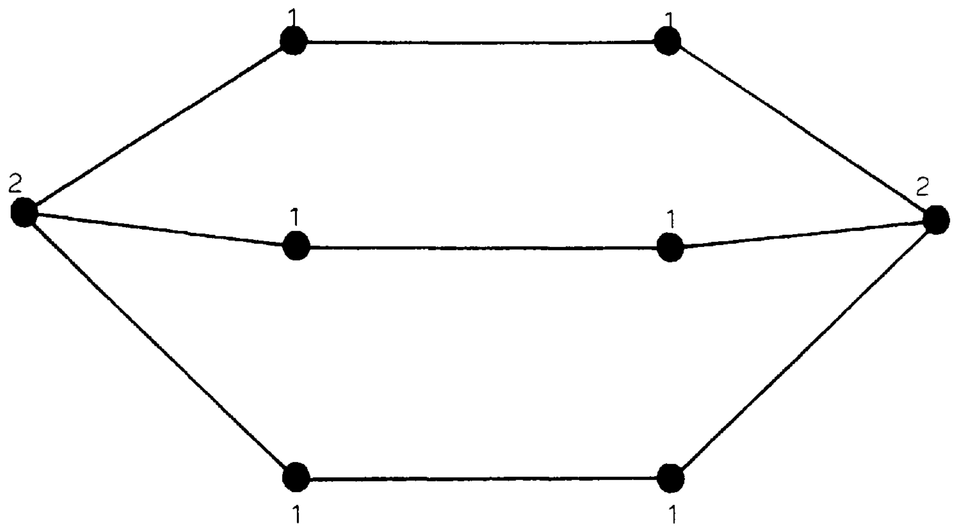
(c) G is connected.

(d) If the neighbour set of a labelled vertex u_L is equal to the neighbour set of a labelled vertex v_L, then $u_L = v_L$.

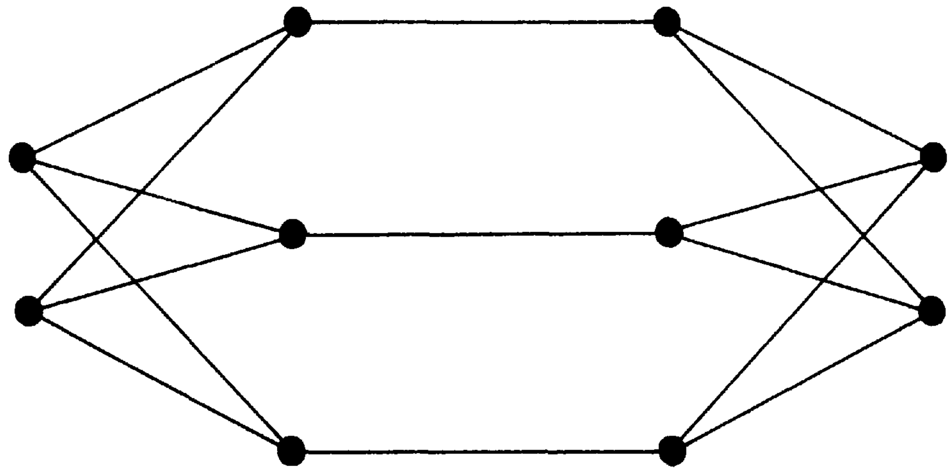
(e) The labelled graph G has no vertices of degree 1 unless G_1 is K_{kk} (the complete bipartite graph).

We now give a further discussion of the properties (a) to (e).

The sum of the labelled vertices in any vertex neighbour set $\Gamma(v_L)$ of a vertex v_L is equal to k. This is illustrated in FIG. 4.2 and any graph G_1 generated will have degree $\rho = k$.



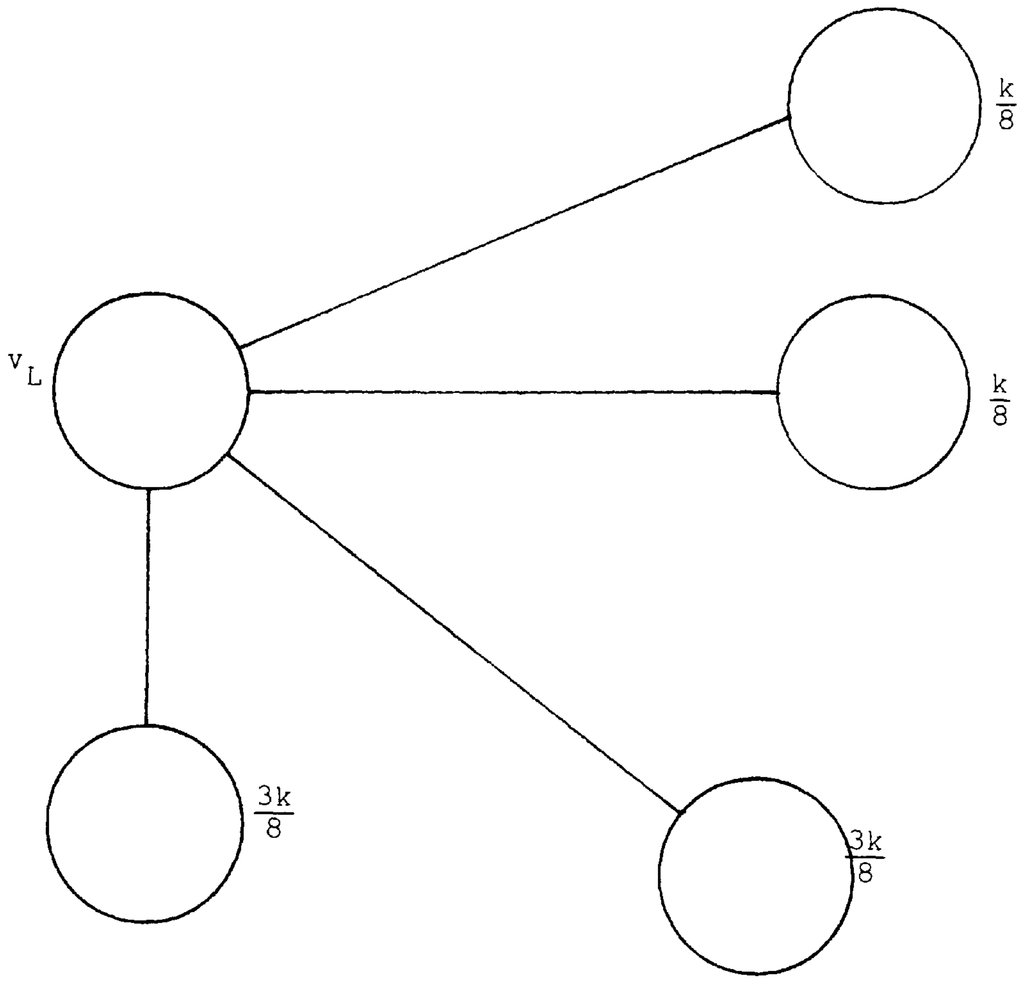
G



G_1

u~v if and only if $\Gamma(u) = \Gamma(v)$

FIG. 4.1



The sum of the labelled vertices of the vertex neighbour set $\Gamma(v_L) = \frac{3k}{8} + \frac{3k}{8} + \frac{k}{8} + \frac{k}{8} = k$

FIG. 4.2

It follows that in a labelled graph and the regular graphs generated that the minimum number of vertices in a vertex cut set is equal to k, the degree of a vertex and hence the connectivity is equal to k. For any vertex cut set the number of vertices in that cut set is greater than or equal to k.

The diagram in FIG. 4.3 illustrates that if $\Gamma(u_L) = \Gamma(v_L)$ then $u_L = v_L$.

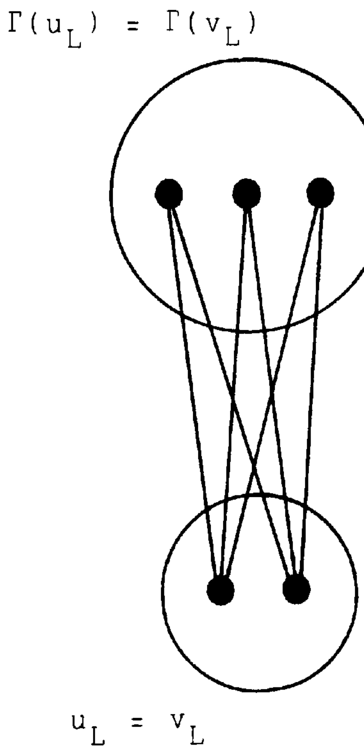
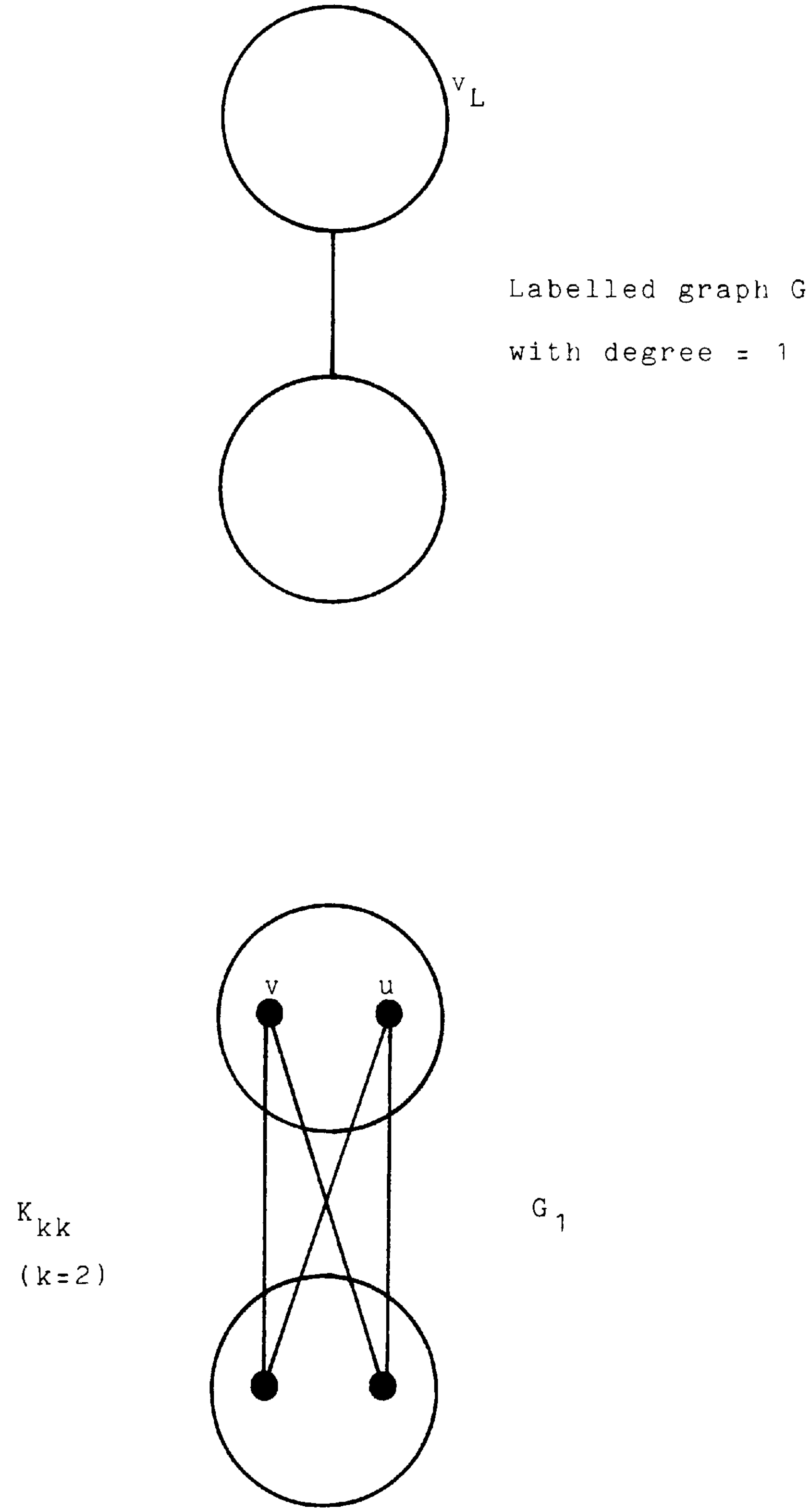


FIG. 4.3

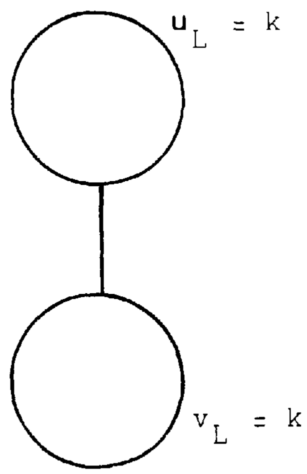
If a labelled vertex v_L has valency 1 then the number of vertices adjacent to a vertex v in G_1 is equal to k, and any other vertex u in G_1 adjacent to $\Gamma(v)$ has $\Gamma(u) = \Gamma(v)$. Hence G_1 is K_{kk}. FIG. 4.4 illustrates the final property that G has no vertices of valency 1 unless G_1 is K_{kk}.



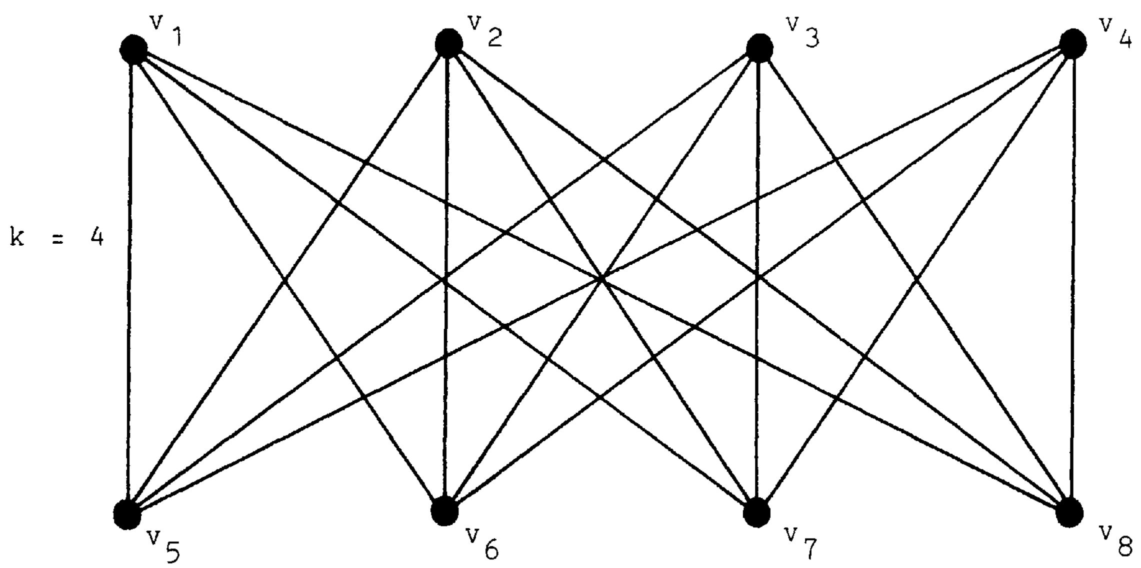
$\Gamma(u) = \Gamma(v)$

FIG. 4.4

An example of a graph G_1 obtained from a labelled graph G is given in FIG. 4.5.



Labelled graph G



G_1

FIG. 4.5

The lists of graphs available have been examined by D.H. Smith [45] e.g. graphs with at most six vertices, illustrated by F. Harary [29] .

Each graph in the list is checked to see if it can be labelled in accordance with the conditions (a) to (e).

The class of graphs under consideration in this chapter have R_k minimum vertex cut sets with k vertices. The values of R_k dealt with are,

$$2 \leqslant R_k \leqslant 6.$$

$$R_k = 7.$$

$$R_k = 8.$$

If a graph is found that satisfies the labelling conditions then the graph is constructed and the actual number R_k of minimum vertex cut sets recorded. If the graph that is found has $R_k \leqq 6$ and no graph with the same number of vertices and the same degree with a smaller value of R_k has appeared elsewhere in the search then the graph has the minimum number of minimum vertex cut sets. For the cases $R_k = 7$ and $R_k = 8$ some short cut methods were used to avoid considering large numbers of graphs and the list is not complete for $R_k = 8$.

The graphs shown in FIG. 4.6 and FIG. 4.7 are graphs taken from those illustrated by F. Harary [30] and they are used as examples to show the procedure adopted for each graph in the list.

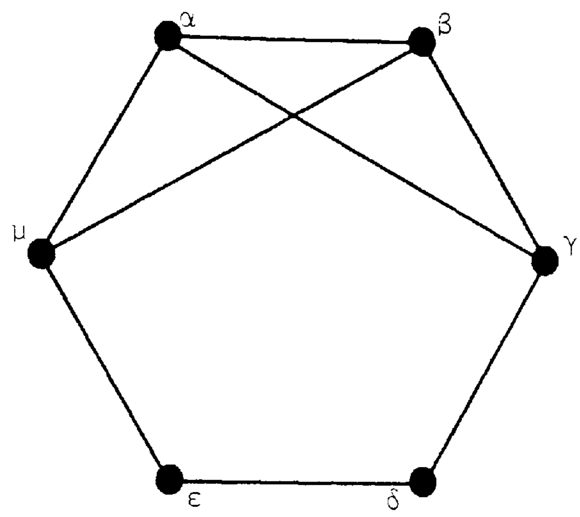


FIG. 4.6

Let the vertices of the graph be labelled α, β, γ, δ, ε and μ.

We form an equation for each vertex by considering the vertices adjacent to that vertex. The following equations are obtained,

$$\beta + \gamma + \mu = k \qquad \text{------(1)}$$

$$\alpha + \gamma + \mu = k \qquad \text{------(2)}$$

$$\alpha + \beta + \delta = k \qquad \text{------(3)}$$

$$\varepsilon + \gamma = k \qquad \text{------(4)}$$

$$\mu + \delta = k \qquad \text{------(5)}$$

$$\alpha + \beta + \varepsilon = k \qquad \text{------(6)}$$

Solving the equations, from (1), (2), (3) and (6)

$$\underline{\alpha = \beta, \quad \varepsilon = \delta}$$

We can therefore write,

$$\alpha + \gamma + \mu = k$$
$$2\alpha + \delta = k$$
$$\gamma + \delta = k$$
$$\mu + \delta = k$$

Thus $\underline{\mu = \gamma}$

giving $\alpha + 2\gamma = k$

and since $2\alpha + \delta = k$

and $\gamma + \delta = k$

We have $\underline{\gamma = 2\alpha}$

giving $\alpha + 4\alpha = k$

i.e. $\underline{\alpha = \frac{k}{5}}$

From equations (2) and (5)

$$3\alpha + \mu = k$$
$$\mu + \delta = k$$

Therefore $\underline{\delta = 3\alpha = \frac{3k}{5}}$

Thus $\alpha = \frac{k}{5}$, $\gamma = \frac{2k}{5}$, $\mu = \frac{2k}{5}$, $\beta = \frac{k}{5}$, $\delta = \frac{3k}{5}$, $\varepsilon = \frac{3k}{5}$

Applying the properties (a) to (e) to this example we find that property (b) cannot be satisfied because the cut set $\{\mu, \gamma\}$ gives,

$\gamma + \mu = \frac{4k}{5}$ which is not a cut set greater than or equal to k. Hence this graph is not considered because there is a cut set which is too small.

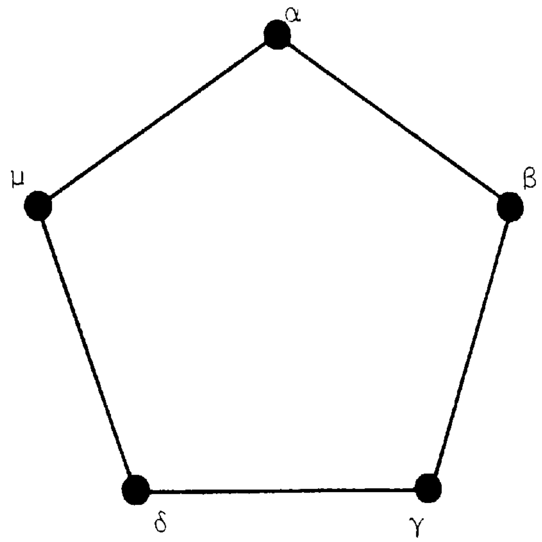


FIG. 4.7

Let the vertices of the graph be labelled α, β, γ, δ and μ.

The following equations are obtained,

$$\mu + \beta = k \qquad \text{------(1)}$$

$$\alpha + \gamma = k \qquad \text{------(2)}$$

$$\delta + \beta = k \qquad \text{------(3)}$$

$$\mu + \gamma = k \qquad \text{------(4)}$$

$$\delta + \alpha = k \qquad \text{------(5)}$$

From (1) and (4) $\mu + \beta = \mu + \gamma$

$$\underline{\beta = \gamma}$$

Similarly $\alpha = \mu$, $\mu = \delta$, $\beta = \alpha$.

Thus by substituting in equations (1) to (5), we have,

$$\underline{\alpha = \beta = \gamma = \delta = \mu = \frac{k}{2}}$$

Applying the properties (a) to (e) to this example we find that all properties are satisfied and the graph is constructed.

We now give a complete list of the infinite families of graphs generated. For $R_k \leqq 7$ there is a complete list of the families with the smallest number of cut sets with $|V|$ vertices, some infinite families are included for $R_k = 8$ but the list is not complete. The families of graphs generated are shown in FIG. 4.8 to FIG. 4.22.

<u>Case A</u> $\frac{k}{|V|} = \frac{1}{2}$, Number of vertex cut sets $R_k = 2$.

$G = K_{kk}$ Number of vertex cut sets which are vertex neighbour sets = 2.

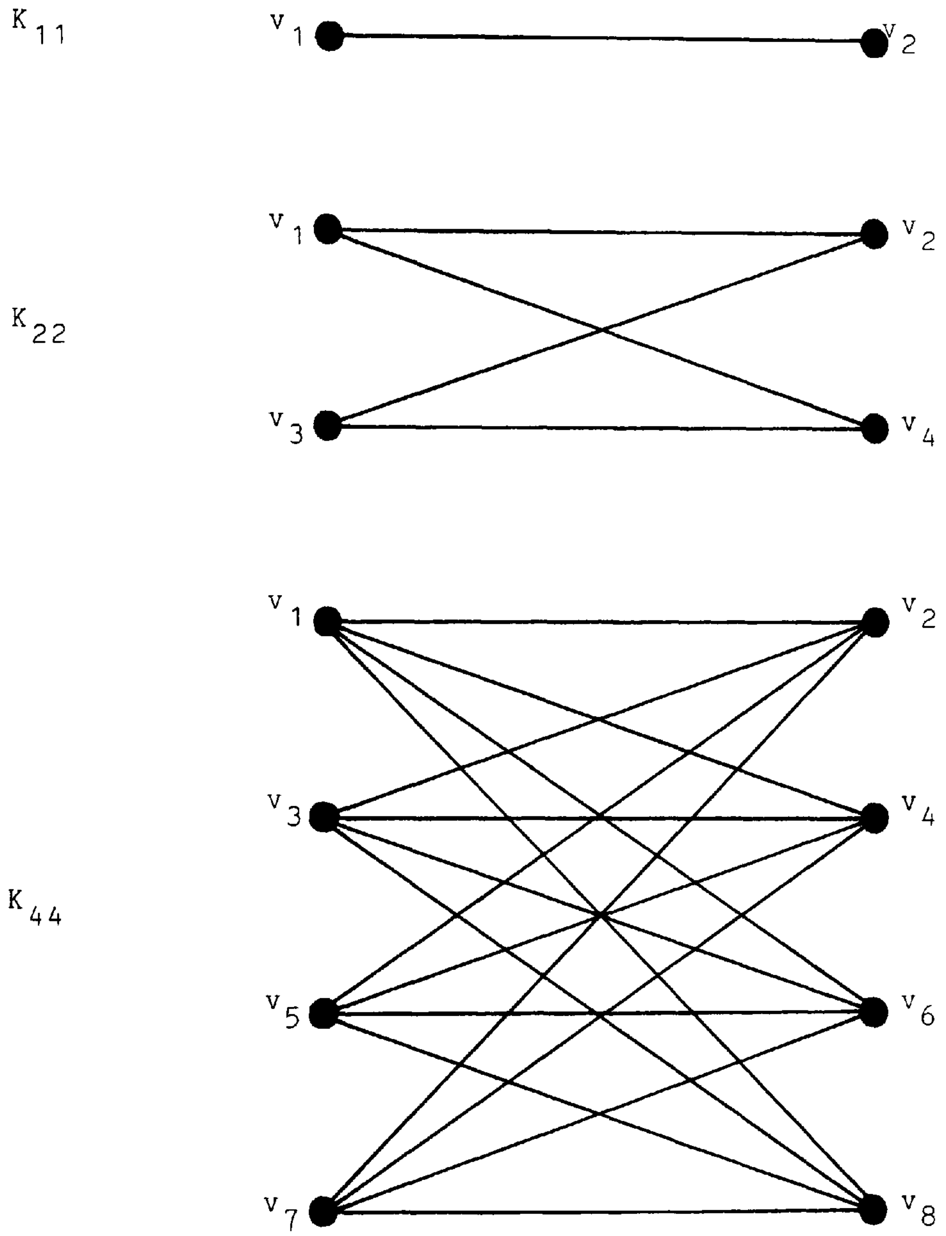


FIG. 4.8

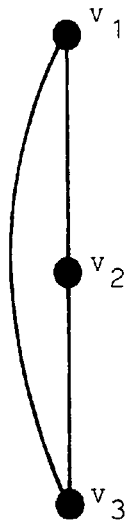
Case B $\frac{k}{|V|} = \frac{2}{3}$,

Number of vertex cut sets $R_k = 3$

$G = K_{\frac{k}{2}, \frac{k}{2}, \frac{k}{2}}$

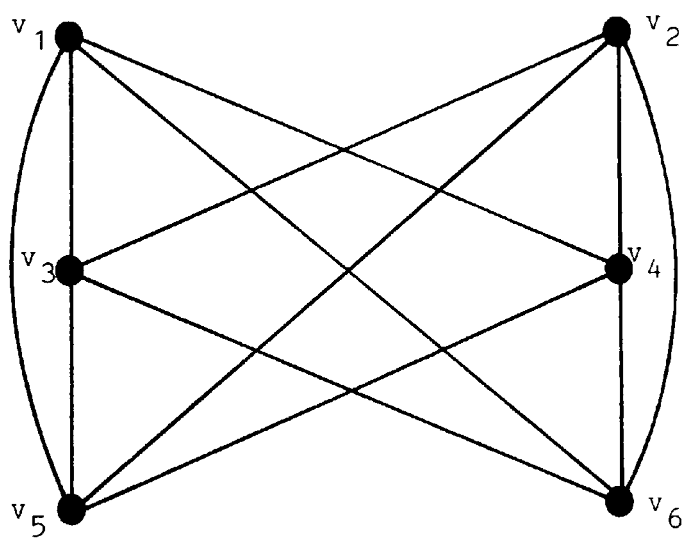
Number of vertex cut sets which are vertex neighbour sets = 3.

K_{111}



k = 2

K_{222}



k = 4

FIG. 4.9

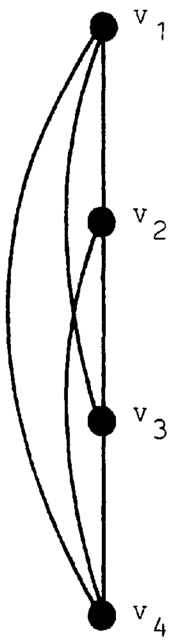
Case C $\frac{k}{|V|} = \frac{3}{4}$,

$G = K_{\frac{k}{3}, \frac{k}{3}, \frac{k}{3}, \frac{k}{3}}$

Number of vertex cut sets $R_k = 4$

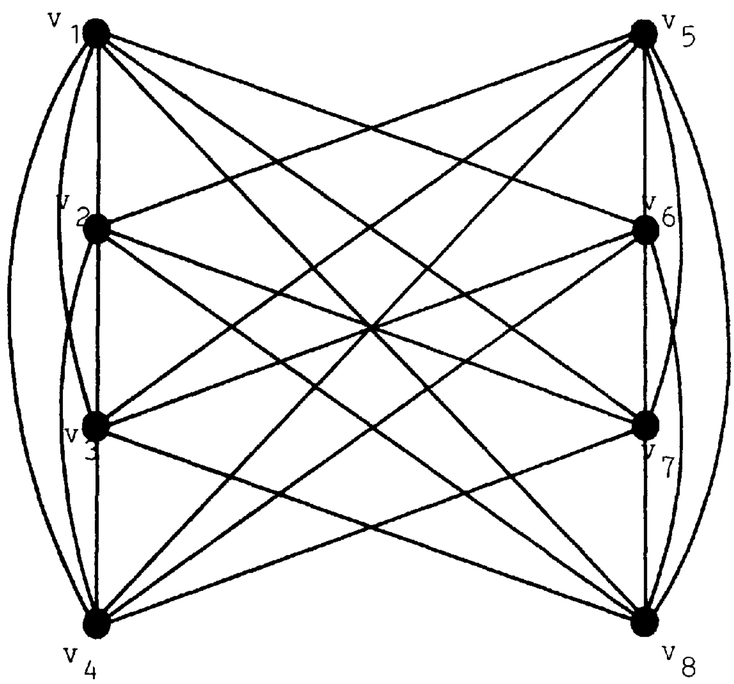
Number of vertex cut sets which are vertex neighbour sets = 4.

K_{1111}



k = 3

K_{2222}



k = 6

FIG. 4.10

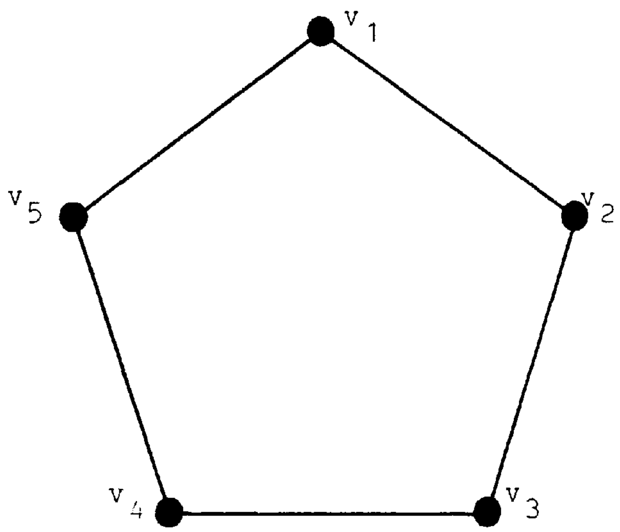
Case D $\frac{k}{|V|} = \frac{2}{5}$

Number of vertex cut sets $R_k = 5$

G = Construction A applied to the pentagon with $m = \frac{k}{2}$

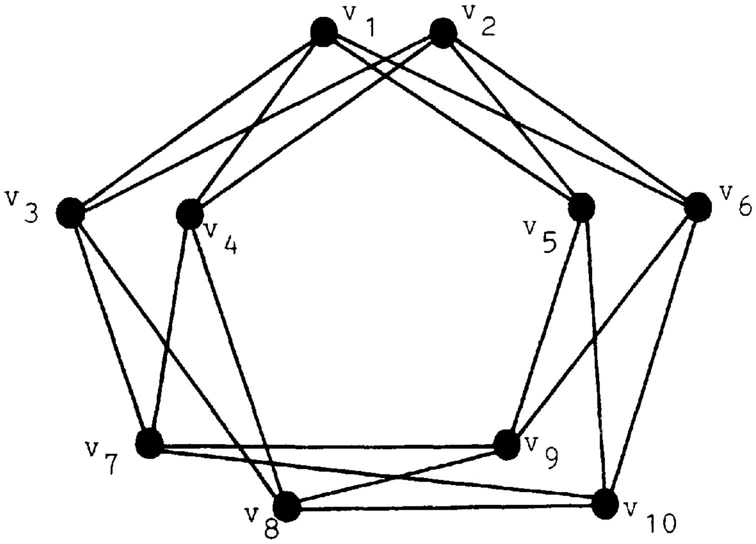
Number of vertex cut sets which are vertex neighbour sets = 5.

m = 1



k = 2

m = 2



k = 4

FIG. 4.11

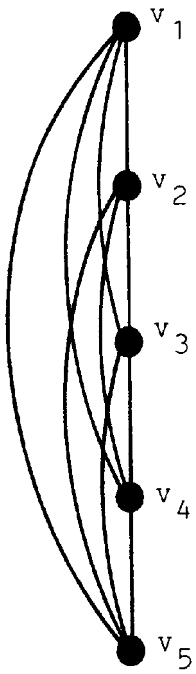
Case E $\frac{k}{|V|} = \frac{4}{5}$,

Number of vertex cut sets $R_k = 5$

$G = K_{\frac{k}{4}, \frac{k}{4}, \frac{k}{4}, \frac{k}{4}, \frac{k}{4}}$

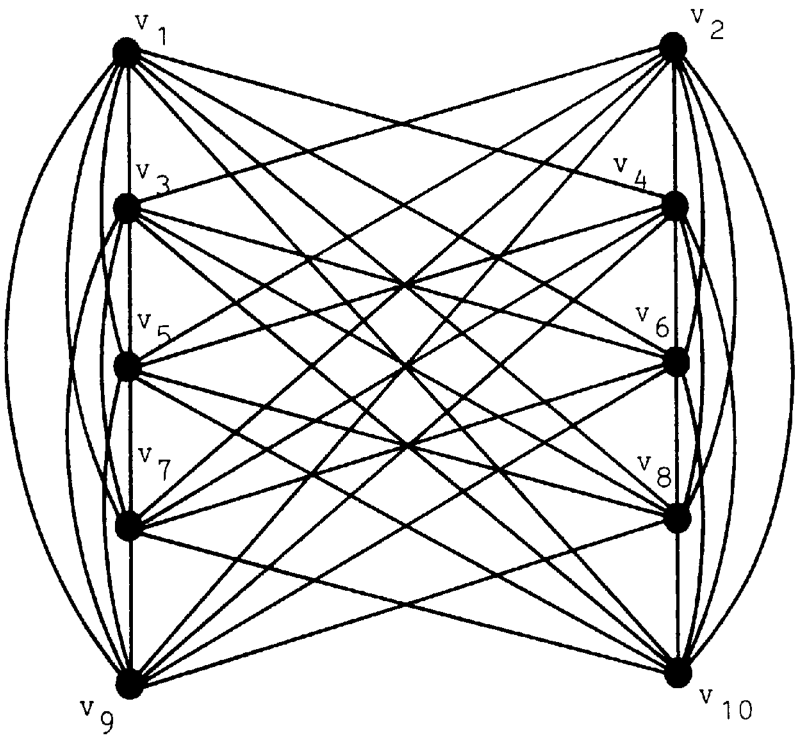
Number of vertex cut sets which are vertex neighbour sets = 5.

K_{11111}



k = 4

K_{22222}



k = 8

FIG. 4.12

Case F $\frac{k}{|V|} = \frac{5}{8}$

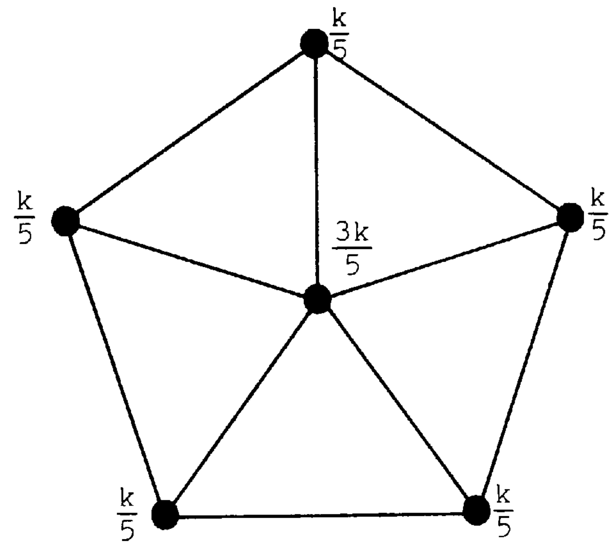
G consists of a pentagon with each vertex labelled $\frac{k}{5}$ together with one vertex labelled $\frac{3k}{5}$ adjacent to every vertex of the pentagon

Number of vertex cut sets

$R_k = 6$

Number of vertex cut sets which are vertex neighbour sets = 6.

G



k = 5

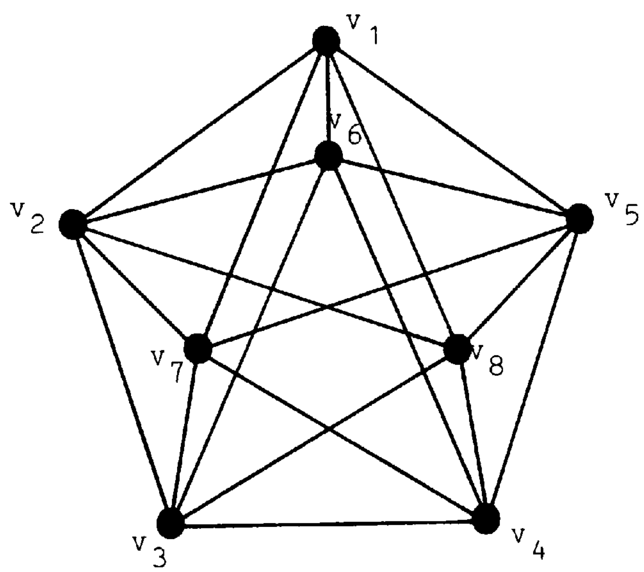


FIG. 4.13

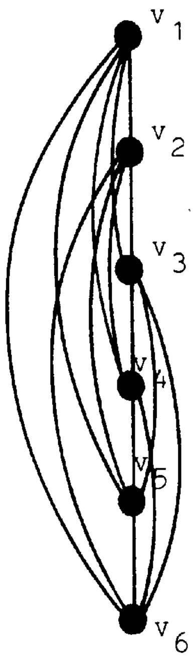
Case G $\frac{k}{|V|} = \frac{5}{6}$

Number of vertex cut sets $R_k = 6$

$G = K_{\frac{k}{5}, \frac{k}{5}, \frac{k}{5}, \frac{k}{5}, \frac{k}{5}, \frac{k}{5}}$

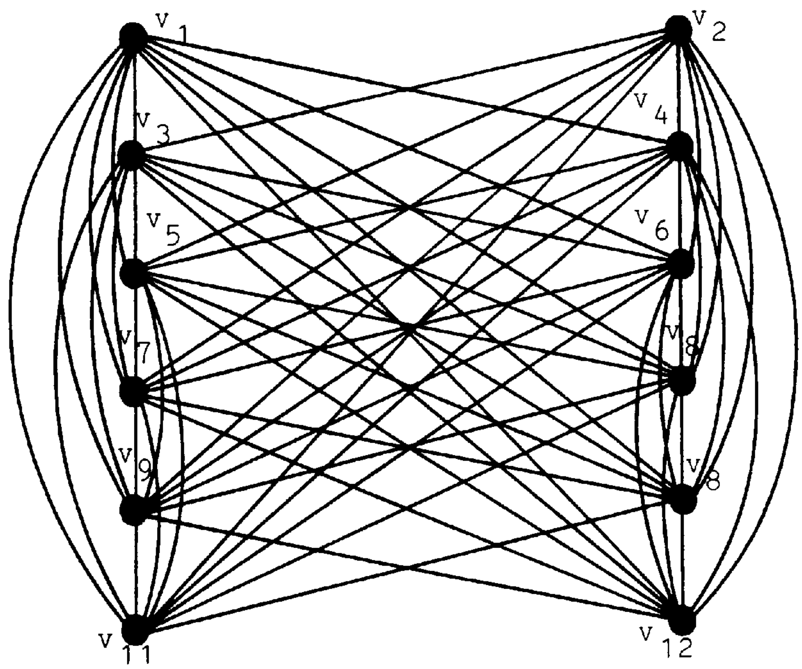
Number of vertex cut sets which are vertex neighbour sets = 6.

K_{111111}



k = 5

K_{222222}



k = 10

FIG. 4.14

Case H $\frac{k}{|V|} = \frac{4}{7}$

G has vertices $v_0, v_1, \dots v_6$, $v_i v_j$ are adjacent if $|i-j| \equiv 1$ or 3 (mod 7); then apply Construction A with $m = \frac{k}{4}$

Number of vertex cut sets $R_k = 7$

Number of vertex cut sets which are vertex neighbour sets = 7.

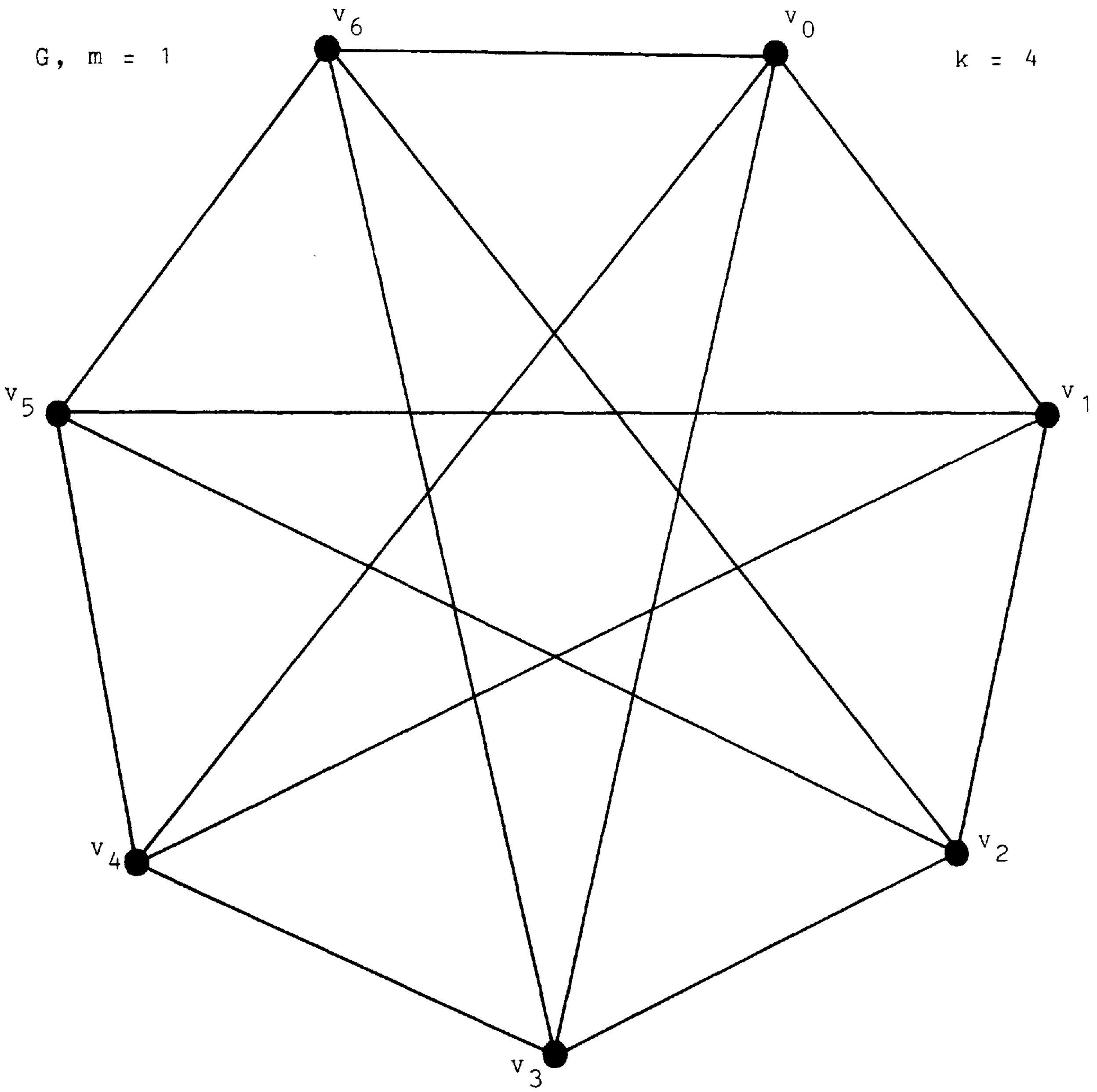


FIG. 4.15

Case I $\frac{k}{|V|} = \frac{8}{11}$

G has vertices v_0, v_1, ... v_4, w_1, w_2 with $v_i v_j$ adjacent if $|i-j| \equiv 1 \pmod 5$; w_1, w_2 adjacent; and $v_i w_j$ adjacent for each i, j (i=0, 1, ... 4; j=1, 2). Each v_i is labelled $\frac{k}{8}$ and each w_j is labelled $\frac{3k}{8}$.

Number of vertex cut sets

$R_k = 7$

Number of vertex cut sets which are vertex neighbour sets = 7.

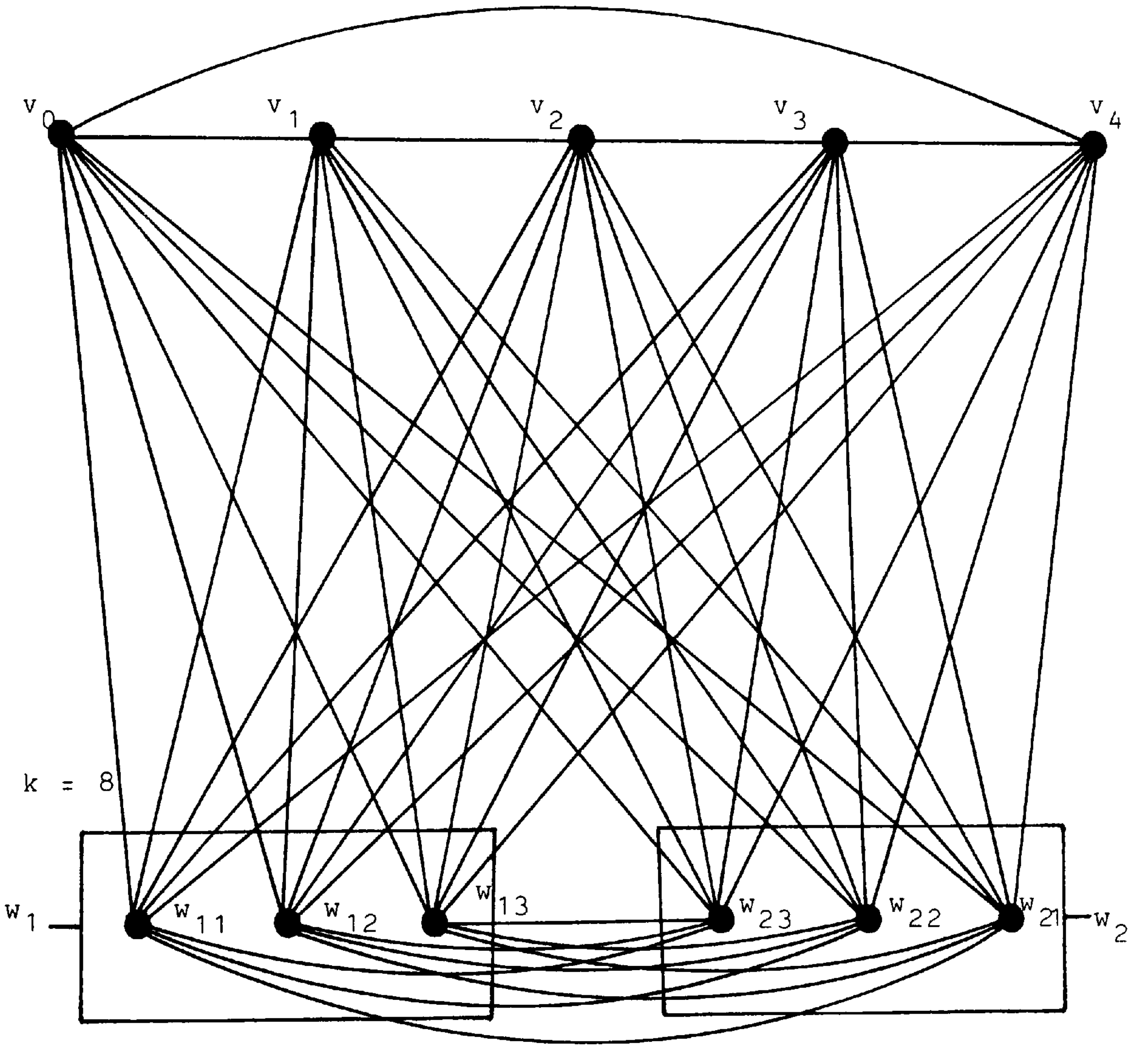


FIG. 4.16

Case J $\frac{k}{|V|} = \frac{6}{7}$

$G = K_{\frac{k}{6}, \frac{k}{6}, \frac{k}{6}, \frac{k}{6}, \frac{k}{6}, \frac{k}{6}, \frac{k}{6}}$

Number of vertex cut sets $R_k = 7$

Number of vertex cut sets which are vertex neighbour sets = 7.

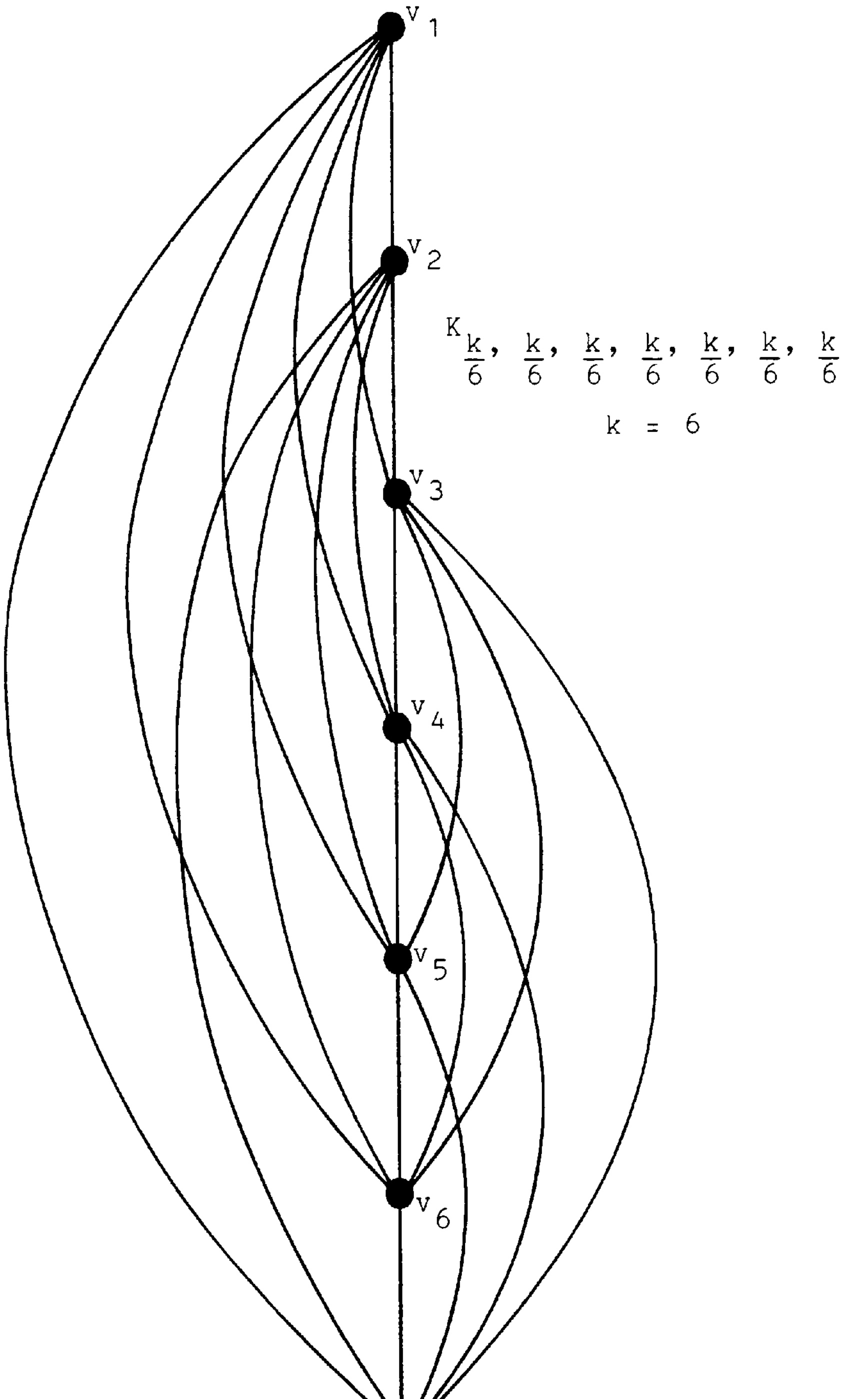


FIG. 4.17

Case K $\frac{k}{|V|} = \frac{4}{9}$

G has vertices v_0, v_1, v_2, v_3, v_4 each labelled $\frac{k}{4}$ and w_1, w_2 labelled $\frac{k}{2}$; v_0 is adjacent to v_1, v_2, v_3, v_4 and the following edges are also present:

(v_1v_4), (v_2v_3), (w_1v_2), (w_1v_1), (w_2v_3), (w_2v_4), (w_1w_2)

Number of vertex cut sets R_k = 7

Number of vertex cut sets which are vertex neighbour sets = 7.

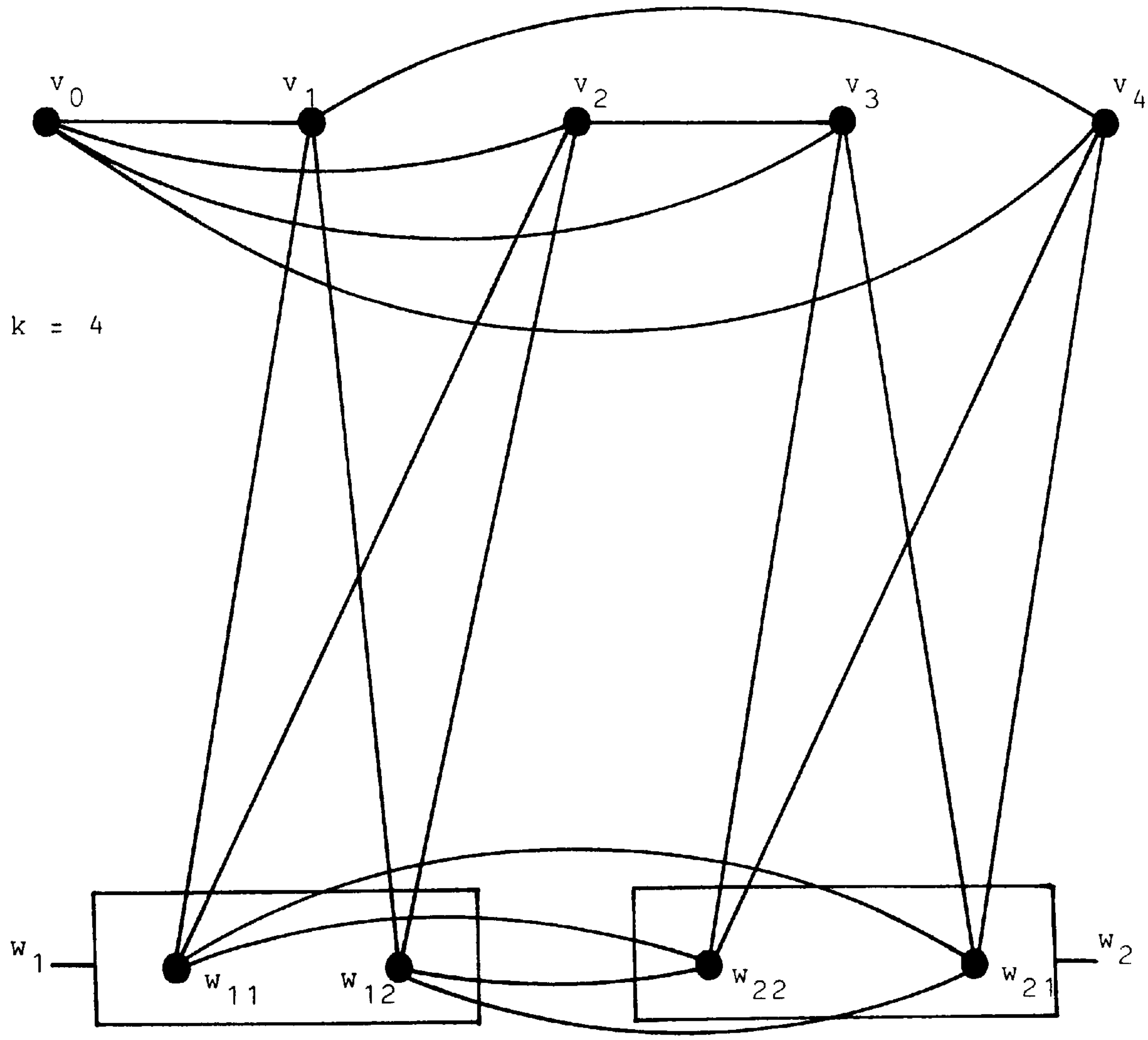


FIG. 4.18

Case L $\frac{k}{|V|} = \frac{9}{14}$

Number of vertex cut sets

$R_k = 8$

Number of vertex cut sets which are neighbour sets of a vertex = 8.

G has vertices u labelled $\frac{5k}{9}$;

$v_1, v_2 \dots v_5$ labelled $\frac{k}{9}$;

w_1, w_2 labelled $\frac{2k}{9}$;

u is adjacent to $v_1, v_2 \dots v_5, w_1, w_2$ and the following edges are also present:

(v_1v_2), (v_3v_4), (v_5w_1), (w_1w_2), (v_1v_3), (v_1v_4), (v_1v_5), (v_2v_5), (v_2w_2), (v_3w_2), (v_4w_1)

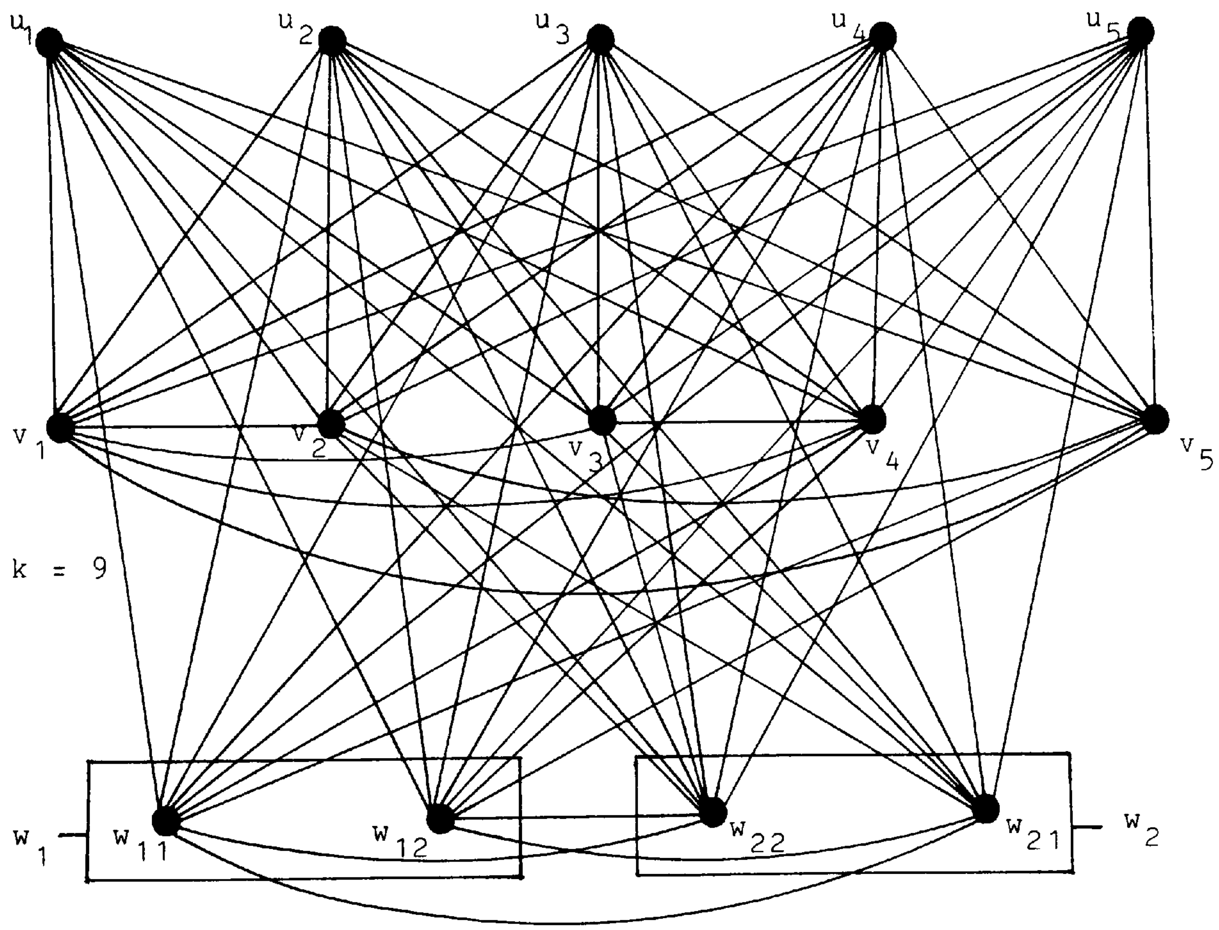


FIG. 4.19

Case M $\frac{k}{|V|} = \frac{3}{8}$

G has vertices v_0, v_1 ... v_7 with v_i, v_j adjacent if $|i-j| \equiv 1$ or 4 (mod 8). Each v_i is labelled $\frac{k}{3}$.

Number of vertex cut sets

$R_k = 8$

Number of vertex cut sets which are neighbour sets of a vertex = 8.

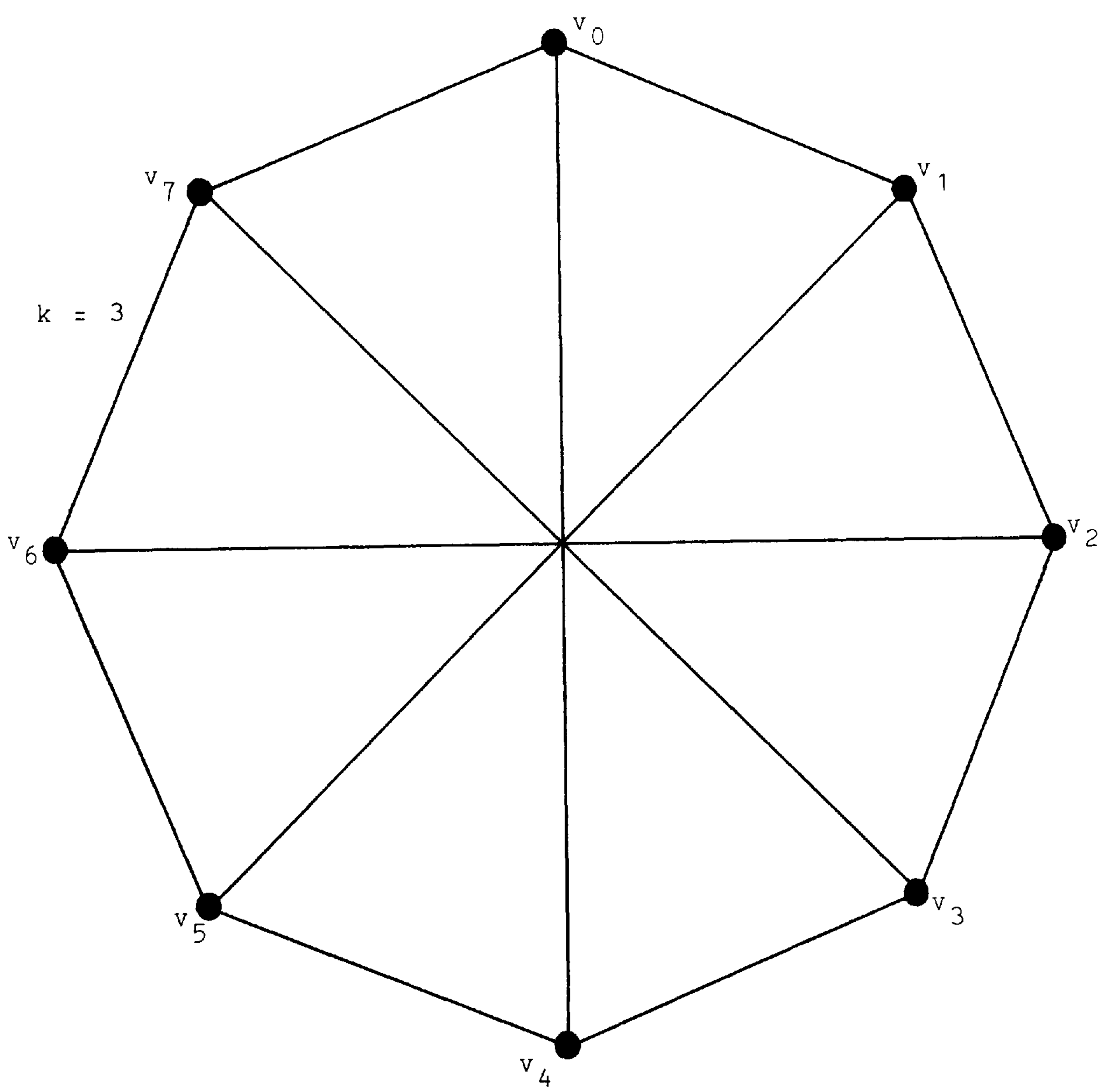


FIG. 4.20

Case N $\frac{k}{|V|} = \frac{7}{8}$

Number of vertex cut sets $R_k = 8$

$G = K_{\frac{k}{7}, \frac{k}{7}, \frac{k}{7}, \frac{k}{7}, \frac{k}{7}, \frac{k}{7}, \frac{k}{7}, \frac{k}{7}}$

Number of vertex cut sets which are neighbour sets of a vertex = 8.

k = 7

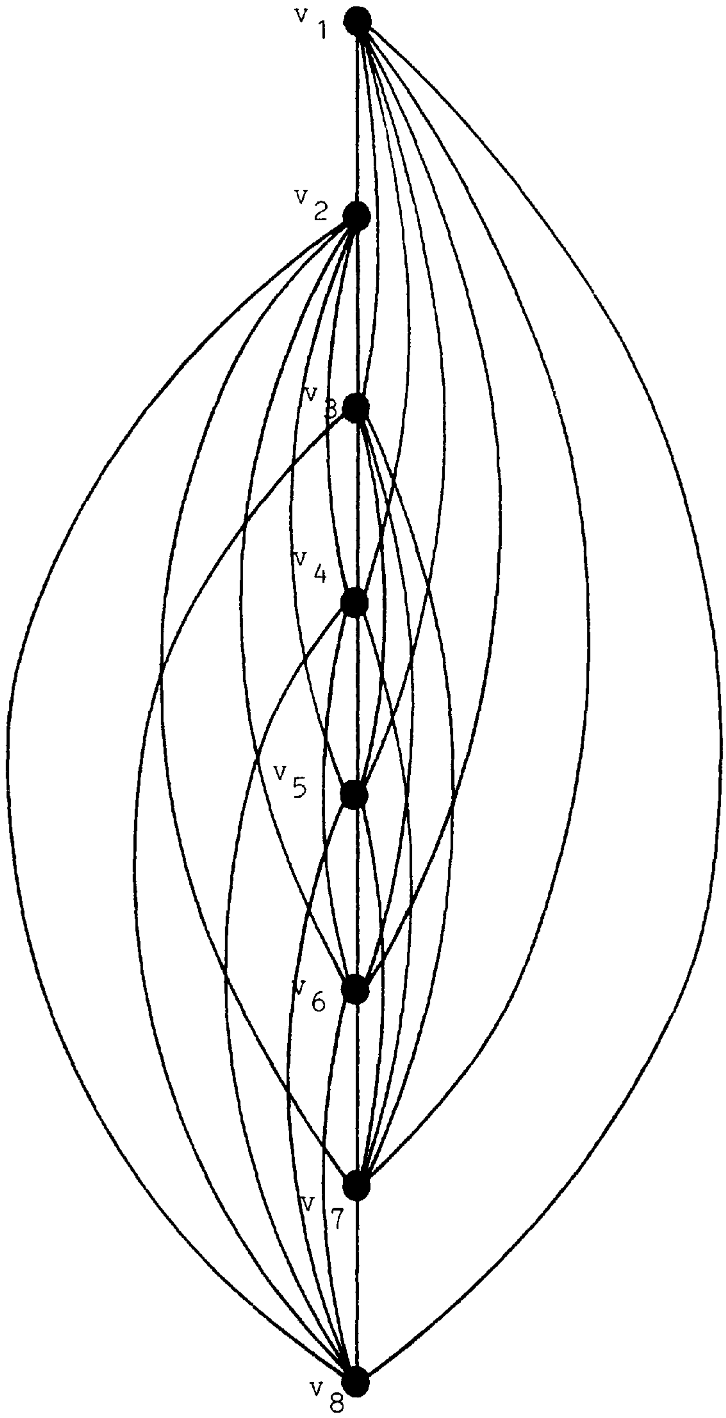


FIG. 4.21

Case 0 $\frac{k}{|V|} = \frac{3}{5}$

G is obtained by applying Construction A to the graph of FIG. 4.22

Number of vertex cut sets $R_k = 8$

Number of vertex cut sets which are neighbour sets of a vertex = 8.

k = 6

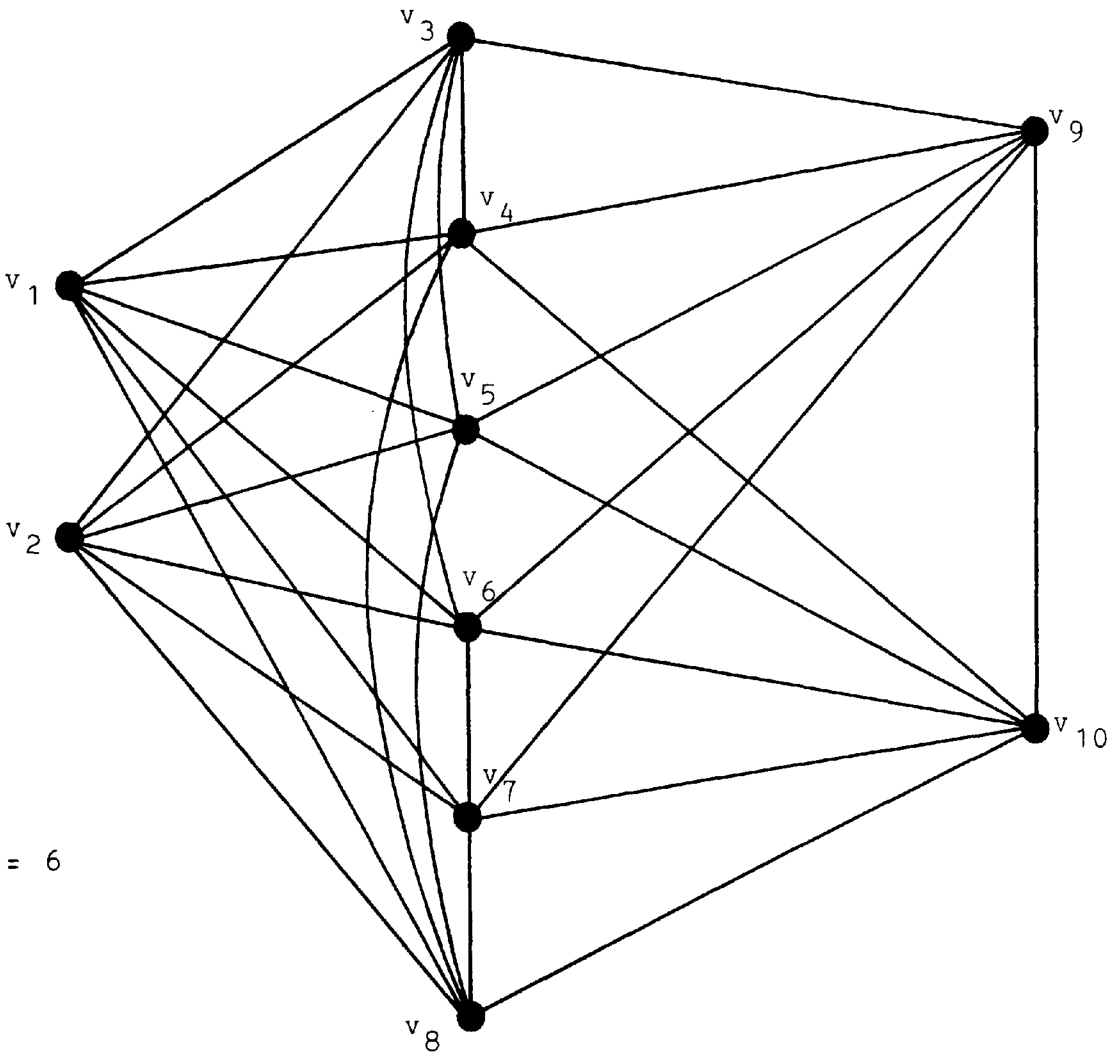


FIG. 4.22

CHAPTER 5

CHAPTER 5

Comparison Of The Probability Of Connection Of Graphs With The Smallest Number Of Minimum Vertex Cut Sets When The Probability Of Failure Of A Vertex Is Close To 1

In this Chapter, we consider graphs in which each vertex has a probability q of failure. The edges are assumed to be perfectly reliable. We examine the class of graphs exhibited in the last Chapter, and compare their reliability when the probability of failure is close to 1.

H. Frank [23] has described the problem of finding graphs with the minimum probability of disconnection if the probability of failure of any vertex is sufficiently small and shows that several families of complete multipartite graphs satisfy this criterion. Dealing with the same problem the graphs of D. H. Smith [45] described in Chapter four show that in many cases it is possible to construct a larger class of optimal graphs with the minimum number of vertex cut sets with k vertices.

Given a graph G with $|V|$ vertices and $|E|$ edges and probability of vertex failure q, then the probability of connection $P_c(G)$ is given by,

$$\underset{\text{(Probability of connection)}}{P_c(G)} = \sum_{i=0}^{|V|} D_i \, q^i (i-q)^{|V|-i}$$

Where D_i = the number of sets of i vertices whose removal from the graph G leaves connected subgraphs with $|V| - i$ vertices.

It follows that,

$$
\begin{aligned}
P_c(G) \quad &= D_0 q^0 (1-q)^{|V|-0} + D_1 q^1 (1-q)^{|V|-1} + \ldots \\
\text{(Probability of) connection} \quad &+ D_{|V|-3} q^{|V|-3} (1-q)^3 + D_{|V|-2} q^{|V|-2} (1-q)^2 \\
&+ D_{|V|-1} q^{|V|-1} (1-q) + D_{|V|} q^{|V|}
\end{aligned}
$$

It can be seen that for q close to 1, if we consider the coefficients $D_{|V|}$, $D_{|V|-1}$, $D_{|V|-2}$, $D_{|V|-3}$, ... , then maximising these values is an important step in finding graphs which have a maximum probability of connection and in this sense are more reliable.

Consider the term $D_{|V|}q^{|V|}$, for any graph G it follows that $D_{|V|} = 1$.

For $D_{|V|-1}$ we have $D_{|V|-1} = |V|$ the number of vertices in the graph.

For $D_{|V|-2}$ we require two vertices to be left connected, it follows therefore that $D_{|V|-2} = |E|$, the number of edges in the graph i.e. $|E| = \frac{|V|\rho}{2}$ where $\rho = k$.

In the case of the coefficient $D_{|V|-3}$ we need the connected subgraph with three vertices. The subgraphs left are illustrated in FIG. 5.1.

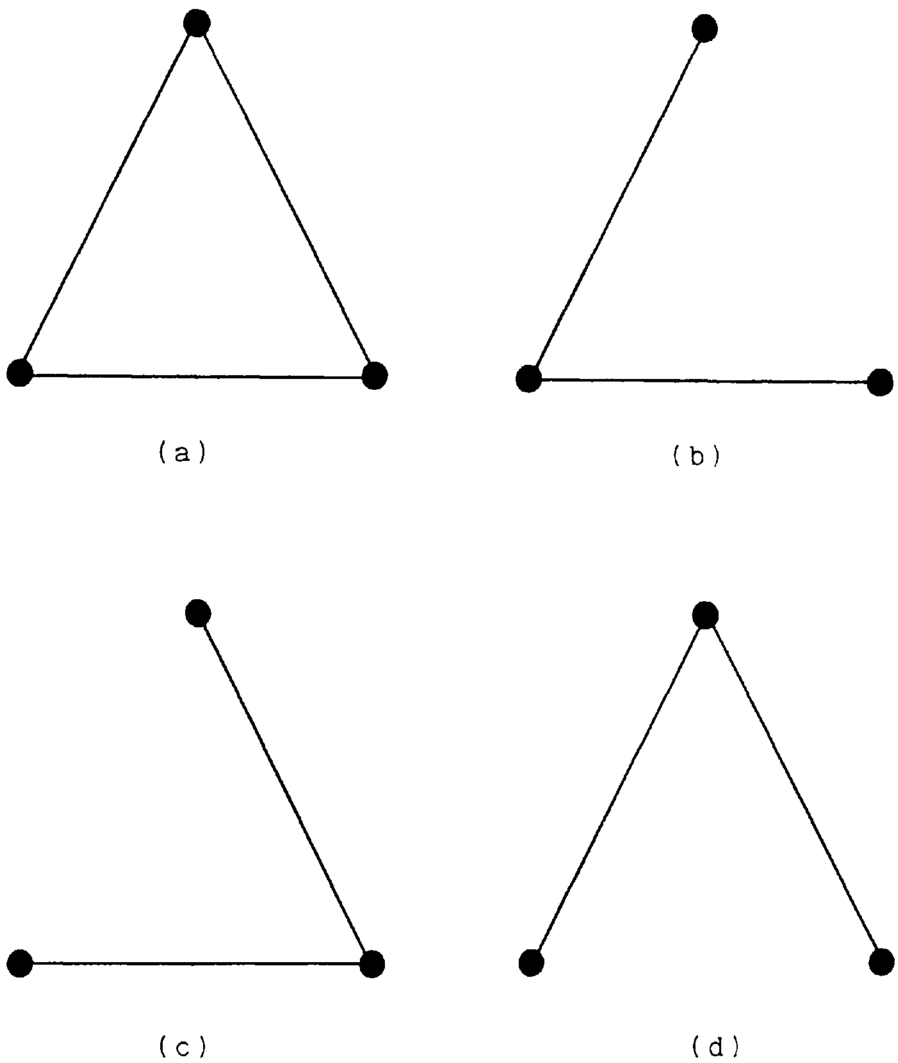


FIG. 5.1 Illustration of the connected subgraphs containing three vertices.

Using the illustrations given in FIG. 5.1 we proceed as follows,

$D_{|V|-3}$ = Number of triangles + $|V|\binom{k}{2}$ - 3 x number of triangles
(as each triangle is counted four times by the first two terms)

i.e. $D_{|V|-3} = |V|\binom{k}{2}$ - 2 x number of triangles.

To maximise $D_{|V|-3}$ we therefore require graphs that contain no triangles.

The graph shown in FIG. 5.2 contains no triangles and we use this example to show the calculation of the coefficient $D_{|V|-3}$ for a non-regular graph.

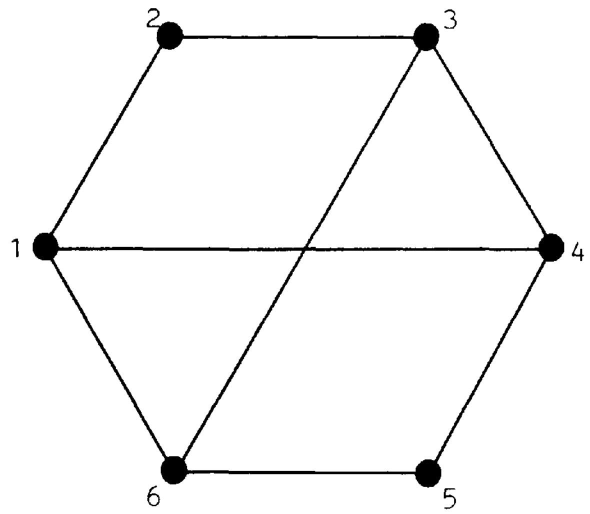


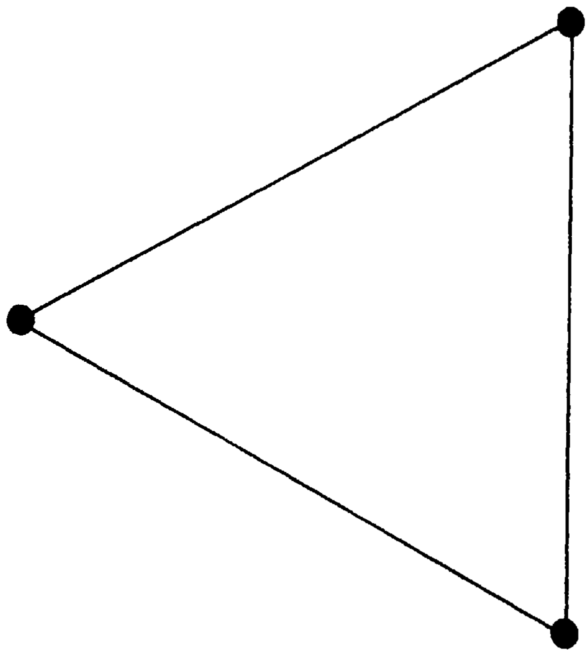
FIG. 5.2

$$D_{|V|-3} = 4\binom{3}{2} + 2\binom{2}{2} - 0 = \underline{14}$$

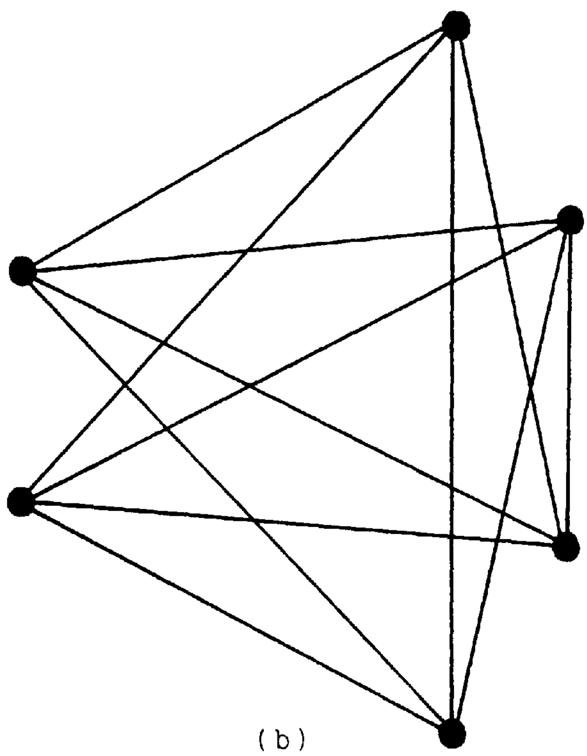
i.e. the 14 connected subgraphs with 3 vertices are,

{1, 2, 3}; {1, 2, 4}; {1, 3, 4}; {1, 2, 6};{1, 4, 5};
{1, 5, 6}; {2, 3, 4}; {1, 3, 6}; {2, 3, 6};{3, 4, 6};
{1, 4, 6}; {3, 4, 5}; {3, 5, 6}; {4, 5, 6}.

Following on from the example giving the calculation of $D_{|V|-3}$ we now explain how the number of triangles for each case are obtained. The diagrams illustrated in FIG. 5.3 (a) and (b) show a single triangle and a graph obtained from the single triangle by replacing each vertex by m vertices.



(a)



(b)

FIG. 5.3

Referring to the diagrams in FIG. 5.3(a) and (b) we have,

1 triangle $\Rightarrow$ m^3 triangles.

i.e. if m = 2, we have 8 triangles for the graph shown in FIG. 5.3(b).

If we now use the base graph in case F in Chapter four as an example with each vertex labelled $\frac{k}{5}$, then in general the number of triangles generated in any other graph in case F can be written as,

$$\text{Number of triangles} = \left(\frac{k}{5}\right)^3 \times \text{number of triangles in the base graph}$$

The number of triangles for each base graph is obtained by inspection of the base graph and these numbers are then used in the general expression for the number of triangles in the respective cases of Chapter four using the particular vertex label given for each case. The results are given in the table of values for $D_{|V|-3}$ in FIG. 5.4.

Case	$\frac{k}{\lvert V\rvert}$	$D_{\lvert V\rvert}$	$D_{\lvert V\rvert-1}$	$D_{\lvert V\rvert-2}$	$D_{\lvert V\rvert-3}$
A	$\frac{1}{2}$	1	$\lvert V\rvert$	$\frac{\lvert V\rvert k}{2}$	$\lvert V\rvert\binom{k}{2}$
B	$\frac{2}{3}$	1	$\lvert V\rvert$	$\frac{\lvert V\rvert k}{2}$	$\lvert V\rvert\binom{k}{2}-2\left(\frac{k}{2}\right)^3$
C	$\frac{3}{4}$	1	$\lvert V\rvert$	$\frac{\lvert V\rvert k}{2}$	$\lvert V\rvert\binom{k}{2}-8\left(\frac{k}{3}\right)^3$
D	$\frac{2}{5}$	1	$\lvert V\rvert$	$\frac{\lvert V\rvert k}{2}$	$\lvert V\rvert\binom{k}{2}$
E	$\frac{4}{5}$	1	$\lvert V\rvert$	$\frac{\lvert V\rvert k}{2}$	$\lvert V\rvert\binom{k}{2}-20\left(\frac{k}{4}\right)^3$
F	$\frac{5}{8}$	1	$\lvert V\rvert$	$\frac{\lvert V\rvert k}{2}$	$\lvert V\rvert\binom{k}{2}-30\left(\frac{k}{5}\right)^3$
G	$\frac{5}{6}$	1	$\lvert V\rvert$	$\frac{\lvert V\rvert k}{2}$	$\lvert V\rvert\binom{k}{2}-40\left(\frac{k}{5}\right)^3$
H	$\frac{4}{7}$	1	$\lvert V\rvert$	$\frac{\lvert V\rvert k}{2}$	$\lvert V\rvert\binom{k}{2}-14\left(\frac{k}{4}\right)^3$
I	$\frac{8}{11}$	1	$\lvert V\rvert$	$\frac{\lvert V\rvert k}{2}$	$\lvert V\rvert\binom{k}{2}-150\left(\frac{k}{8}\right)^3$
J	$\frac{6}{7}$	1	$\lvert V\rvert$	$\frac{\lvert V\rvert k}{2}$	$\lvert V\rvert\binom{k}{2}-70\left(\frac{k}{6}\right)^3$
K	$\frac{4}{9}$	1	$\lvert V\rvert$	$\frac{\lvert V\rvert k}{2}$	$\lvert V\rvert\binom{k}{2}-4\left(\frac{k}{4}\right)^3$
L	$\frac{9}{14}$	1	$\lvert V\rvert$	$\frac{\lvert V\rvert k}{2}$	$\lvert V\rvert\binom{k}{2}-184\left(\frac{k}{9}\right)^3$
M	$\frac{3}{8}$	1	$\lvert V\rvert$	$\frac{\lvert V\rvert k}{2}$	$\lvert V\rvert\binom{k}{2}$
N	$\frac{7}{8}$	1	$\lvert V\rvert$	$\frac{\lvert V\rvert k}{2}$	$\lvert V\rvert\binom{k}{2}-112\left(\frac{k}{7}\right)^3$
O	$\frac{3}{5}$	1	$\lvert V\rvert$	$\frac{\lvert V\rvert k}{2}$	$\lvert V\rvert\binom{k}{2}-50\left(\frac{k}{6}\right)^3$

FIG. 5.4

We use a result given by B. Bollobas [14] corollary 1.6, page 297 to compare the optimal value for the number of triangles in a graph with the number of triangles in the Smith graphs. This result states that; given a graph G with $|V|$ vertices and $|E|$ edges then the number of triangles ($t(G)$) in that graph is given by,

$$t(G) \geqq \left(\frac{|E|}{3|V|}\right)\left(4|E| - |V|^2\right).$$

If we let $\frac{k}{|V|} = \alpha$, and $|E| = \frac{|V|k}{2}$

Then $t(G) \geqq \frac{k}{6}\left(\frac{2k^2}{\alpha} - \frac{k^2}{\alpha^2}\right) \geqq k^3\left(\frac{1}{3\alpha} - \frac{1}{6\alpha^2}\right)$

i.e. $t(G) \geqq k^3 \times B$ where $B = \left(\frac{1}{3\alpha} - \frac{1}{6\alpha^2}\right)$

Taking the value of α for each of the graphs in FIG. 4.7 to FIG. 4.21 in Chapter four we calculate the value of B and hence obtain a general expression for the number of triangles in each graph i.e.

$t(G) \geqq Bk^3$ the optimal graphs being those with $t(G) = Bk^3$.

A comparison is then made of the number of triangles in each of the graphs listed in the cases A, B, C, ... O, of Chapter four and those graphs giving the optimal value $t(G) = Bk^3$. The tabulated results also give the percentage deviation from the optimal value for each case and are shown in FIG. 5.5. The percentage deviation is defined as,

$$\frac{(t(G)\ \text{Smith-Optimal Value})}{\text{Optimal Value}} \times 100\%$$

The table of results in FIG. 5.6 compares the number of triangles in each of forty-six specific circulants with the number of triangles in each of the graphs of Smith which have the same values of k and $|V|$. The circulants in our comparison are defined as follows:

Let G be a circulant graph with $G = C_{|V|} < a_1, a_2, \dots a_{\frac{k}{2}} >$ where $0 < a_1 < a_2 < \dots < a_{\frac{k}{2}} < \frac{(|V| + 1)}{2}$, has $i \pm a_1$, $i \pm a_2$, ... $i \pm a_{\frac{k}{2}}$ < (Mod $|V|$) adjacent to each point i (i.e. $a_1 = 1$, $a_2 = 2$, etc.) If k is odd we also join vertex i to vertex $i + \frac{|V|}{2}$.

The eigenvalues (λ_i) and the number of triangles t(G) for each circulant are obtained using a computer program, the value of t(G) being given by,

$$t(G) = \frac{1}{6} \sum_{i=1}^{|V|} \lambda_i^3 \qquad \text{(where } \lambda_i = \text{the eigenvalues of the graph)}$$

The formula for t(G) is derived by D. M. Cvetkovic, M. Doob, H. Sachs [19] page 85.

An example of the calculations involved for each graph in compiling this table is now given using case D, FIG. 4.10, Chapter four and the circulant graph shown in FIG. 5.7.

<u>Smith Graph, Case D, k = 4, $|V| = 10$</u> The graph is constructed using Construction A applied to the pentagon with $m = \frac{k}{2}$. For this graph $t(G) = 0$ as shown in the table of values FIG. 5.5, which gives the general result for the number of triangles in the graphs in this case.

Case	$\frac{k}{\lvert V \rvert}$	t(G) Number of triangles Smith graphs	$t(G)=Bk^3$ Optimal Value	Percentage deviation
A	$\frac{1}{2}$	0	0	0
B	$\frac{2}{3}$	$0.125k^3$	$0.125k^3$	0
C	$\frac{3}{4}$	$0.1481481k^3$	$0.1481481k^3$	0
D	$\frac{2}{5}$	0	0	0
E	$\frac{4}{5}$	$0.15625k^3$	$0.15625k^3$	0
F	$\frac{5}{8}$	$0.12k^3$	$0.1066666k^3$	+12.5
G	$\frac{5}{6}$	$0.16k^3$	$0.16k^3$	0
H	$\frac{4}{7}$	$0.109375k^3$	$0.0729166k^3$	+50
I	$\frac{8}{11}$	$0.1464843k^3$	$0.1432291k^3$	+2.27
J	$\frac{6}{7}$	$0.162037k^3$	$0.162037k^3$	0
K	$\frac{4}{9}$	$0.03125k^3$	0	
L	$\frac{9}{14}$	$0.1262002k^3$	$0.1152263k^3$	+9.5
M	$\frac{3}{8}$	0	0	0
N	$\frac{7}{8}$	$0.1632653k^3$	$0.1632653k^3$	0
O	$\frac{3}{5}$	$0.11574074k^3$	$0.0925925k^3$	+25

FIG. 5.5

k	$\|V\|$	$\frac{k}{\|V\|}$	Number of triangles Smith graphs	Number of triangles Circulant graphs
4	8	$\frac{1}{2}$	0	8
5	10	$\frac{1}{2}$	0	10
6	12	$\frac{1}{2}$	0	36
7	14	$\frac{1}{2}$	0	42
8	16	$\frac{1}{2}$	0	96
4	6	$\frac{2}{3}$	8	8
6	9	$\frac{2}{3}$	27	30
8	12	$\frac{2}{3}$	64	76
6	8	$\frac{3}{4}$	32	32
9	12	$\frac{3}{4}$	108	112
12	16	$\frac{3}{4}$	256	272
4	10	$\frac{2}{5}$	0	10
6	15	$\frac{2}{5}$	0	45
8	20	$\frac{2}{5}$	0	120
8	10	$\frac{4}{5}$	80	80
12	15	$\frac{4}{5}$	270	275
16	20	$\frac{4}{5}$	640	660
5	8	$\frac{5}{8}$	15	16
10	16	$\frac{5}{8}$	120	160
15	24	$\frac{5}{8}$	405	575

FIG. 5.6

k	|V|	$\frac{k}{\|V\|}$	Number of triangles Smith graphs	Number of triangles Circulant graphs
5	6	$\frac{5}{6}$	20	20
10	12	$\frac{5}{6}$	160	160
15	18	$\frac{5}{6}$	540	546
4	7	$\frac{4}{7}$	7	7
8	14	$\frac{4}{7}$	56	84
12	21	$\frac{4}{7}$	203	314
8	11	$\frac{8}{11}$	75	77
16	22	$\frac{8}{11}$	600	659
24	33	$\frac{8}{11}$	2025	2284
6	7	$\frac{6}{7}$	35	35
12	14	$\frac{6}{7}$	280	280
18	21	$\frac{6}{7}$	945	952
4	9	$\frac{4}{9}$	2	9
8	18	$\frac{4}{9}$	16	108
12	27	$\frac{4}{9}$	54	403
9	14	$\frac{9}{14}$	92	114
18	28	$\frac{9}{14}$	736	1005
27	42	$\frac{9}{14}$	2484	3520
3	8	$\frac{3}{8}$	0	0
6	16	$\frac{3}{8}$	0	48

FIG. 5.6 (continued)

k	$\|V\|$	$\frac{k}{\|V\|}$	Number of triangles Smith graphs	Number of triangles Circulant graphs
9	24	$\frac{3}{8}$	0	191
7	8	$\frac{7}{8}$	56	56
14	16	$\frac{7}{8}$	448	448
21	24	$\frac{7}{8}$	1512	1520
6	10	$\frac{3}{5}$	25	30
12	20	$\frac{3}{5}$	200	299

FIG. 5.6 (continued)

It is noted from the table of values in FIG. 5.6 that the Smith graphs are better in the sense that the number of triangles in each graph is less than or equal to the number of triangles in the circulant graph for the same values of k and $|V|$.

Circulant Graph $k = 4, \ |V| = 10$

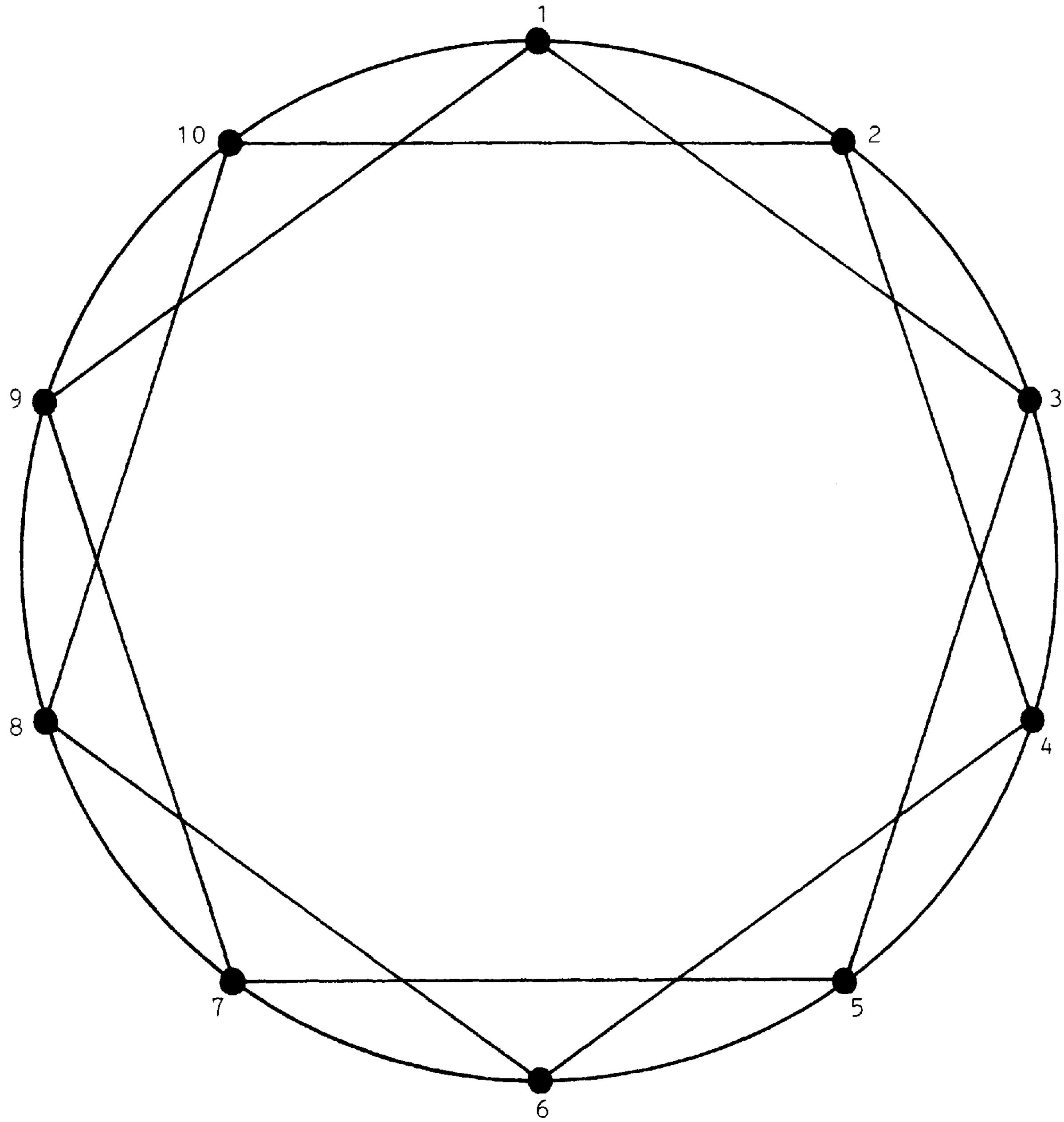


FIG. 5.7

$\lambda_i = 4, -1, -1, -1, -1, \sqrt{5}, \sqrt{5}, -\sqrt{5}, -\sqrt{5}, 0.$

$$t(G) = \frac{1}{6} \sum_{i=1}^{|V|} \lambda_i^3 = 10.$$

CHAPTER 6

CHAPTER 6

Comparison of the Probability of Disconnection of Graphs When the Probability of Edge Failure p is close to 0

We consider finite, simple, undirected regular graphs in which each edge has a probability p of failure. Failures of edges are assumed independent. We recall from Chapter two, section 2.5 that the probability of disconnection of the graph can be written as,

$$\underset{\text{(Probability of) disconnection}}{P_d(G)} = \sum_{i=\lambda}^{|E|} R_i p^i (1-p)^{|E|-i}$$

Where R_i is the number of edge cut sets with i edges.

We remarked in Chapter one, section 1.3 that when link failures do occur in a network and the network is disconnected it is less damaging for one node to be isolated from the rest of the network, than for half the nodes to be isolated from the other half. This is reflected in the following definition:

Definition

A graph is (k, k+j)-edge-connected if it has edge connectivity $\lambda = k$, has an edge cut set with k+j edges and all edge cut sets E with $|E| < k+j$ have the property that G-E has at most one component which is not an isolated vertex.

We recall from Chapter three, section 3.1, that the number of edge cut sets R_{k+j} of size k+j in a regular graph is given by,

$$R_{k+j} \geq |V| \binom{|E|-k}{j}, \text{ where } j < k - 1.$$

the lower bound counting only those edge cut sets which are obtained from edge cut sets incident at vertices. Equality is obtained when the graph is (k, k+j)-edge-connected and so the first j coefficients R_k, R_{k+1}, R_{k+2}, R_{k+3}, ... R_{k+j-1} are minimised in the expression for $P_d(G)$, the probability of disconnection.

The reason for considering the values of R_{k+j} the number of edge cut sets with k+j edges, is that the probability of disconnection for a network having equal and independent edge failures can be reduced to finding all the R_{k+j} values of the corresponding graph. To minimise $P_d(G)$, one must first maximise λ and then minimise all the R_{k+j}.

We examine the list of Smith graphs given in Chapter four and give the general value of j for which each case is (k, k+j)-edge-connected. The following definition and theorem indicates how these values of j were obtained.

Definition

A graph is (k, k+j)-connected if it has connectivity k, has a vertex cut set with k+j vertices and all vertex cut sets X with $|X| < k+j$ have the property that G-X has at most one component which is not an isolated vertex.

Before giving a theorem which shows the connection between the (k, k+j)-connected and (k, k+j)-edge-connected definitions we note from Chapter two, section 2.5 that the probability of disconnection $P_d(G)$ for graphs with the probability of vertex failure close to 0 is minimised if S_k the number of vertex cut sets with k vertices ($k=\rho=$degree) is minimised.

The Smith graphs give various infinite families of graphs with the smallest number of minimum vertex cut sets.

Theorem

Let $j \leqq k-2$, $k \geqq 4$, and G be a graph with $|E|$ edges ($|E| \geqq k+j$) and with minimum degree k. Then if G is $(k, k+j)$-connected it is $(k, k+j)$-edge-connected.

We note that the graph of the triangular prism with $k=3$, $j=1$ illustrated in Chapter two, section 2.7, FIG. 2.25, is a counter example to the corresponding result when $k = 3$.

T. Evans and D. H. Smith [21] give a proof of the above theorem.

We now consider the Smith graphs and examine each case A, B, C, ... O to determine the value of j for which each case is $(k, k+j)$-connected but not $(k, k+j+1)$-connected. An example of how the general value of j is obtained for the various cases is illustrated as follows:

Case K $\quad \frac{k}{|V|} = \frac{4}{9}$

The vertex cut set $V = \{v_0\ w_{11}\ w_{12}\ w_{22}\ w_{21}\}$ in the Smith graph in Chapter four, FIG. 4.17 gives the disconnected graph shown in FIG. 6.1.

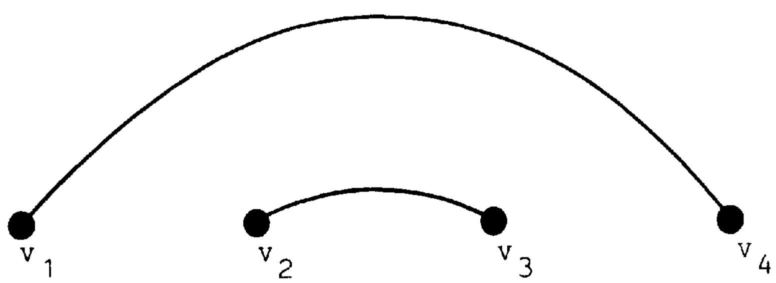


FIG. 6.1

This disconnected graph shows that the original graph fails to be (k, k+j)-connected. The number of vertices in the cut set which gives the disconnected graph of FIG. 6.1 is $\frac{5k}{4}$, and no smaller cut sets fail to satisfy the (k, k+j)-connected definition. In general the graph in case K is (k, k+j)-connected for $j \leq \frac{k}{4}$.

The values of j for which each graph is (k, k+j)-connected are tabulated in FIG. 6.2. The theorem then guarantees that if $k \geqq 4$ the graph is (k, k+j)-edge-connected for the same values of j except in cases A and D which are (k, 2k-2)-edge-connected. The results for the value of j for which each case is (k, k+j)-edge-connected are shown in FIG. 6.3. These values of j show that the number of edge cut sets R_i in the Smith graphs is minimised, where $i = k+j-1$, and hence minimise the first j coefficients in the expression for $P_d(G)$ the probability of disconnection of the graph (edge failure).

F. T. Boesch and J. F. Wang [12] give results for a special class of circulants $G = C_{|V|}(1, 2, \ldots S)$, $2 \leqq S \leqq \frac{|V|}{2}$ (having degree $\rho = \lambda = k$) as defined in Chapter three, section 3.1, their work showing that this special class of circulants not only minimise R_λ but all R_i for $\lambda \leqq i \leqq 2k - 3$ (where i = the number of edges in an edge cut set). We note from the results in FIG. 6.3 that cases A and D of the Smith graphs satisfying the (k, k+j)-edge-connected definition with $j \leqq k-2$ share with the special class of circulants the property that the coefficients $R_i (\lambda \leqq i \leqq 2k - 3)$ are minimised. For small p the graphs in cases B, C and E to O are at least near optimal with respect to edge failures.

Case	$\frac{k}{\lvert V\rvert}$	Values of j for which graph is (k, k+j)-connected
A	$\frac{1}{2}$	$j \leqq k-2$
B	$\frac{2}{3}$	$j \leqq \frac{k}{2}-2$
C	$\frac{3}{4}$	$j \leqq \frac{k}{3}-2$
D	$\frac{2}{5}$	$j \leqq \frac{3k}{2}-2$
E	$\frac{4}{5}$	$j \leqq \frac{k}{4}-2$
F	$\frac{5}{8}$	$j \leqq \frac{3k}{5}-2$
G	$\frac{5}{6}$	$j \leqq \frac{k}{5}-2$
H	$\frac{4}{7}$	$j \leqq \frac{3k}{4}-2$
I	$\frac{8}{11}$	$j \leqq \frac{3k}{8}-2$
J	$\frac{6}{7}$	$j \leqq \frac{k}{6}-2$
K	$\frac{4}{9}$	$j \leqq \frac{k}{4}$
L	$\frac{9}{14}$	$j \leqq \frac{k}{9}$
M	$\frac{3}{8}$	$j \leqq \frac{k}{3}$
N	$\frac{7}{8}$	$j \leqq \frac{k}{7}-2$
O	$\frac{3}{5}$	$j \leqq \frac{2k}{3}-2$

FIG. 6.2

Case	$\frac{k}{\lceil V \rceil}$	Values of j for which graph is (k, k+j)-edge-connected
A	$\frac{1}{2}$	$j \leq k-2$
B	$\frac{2}{3}$	$j \leq \frac{k}{2}-2$
C	$\frac{3}{4}$	$j \leq \frac{k}{3}-2$
D	$\frac{2}{5}$	$j \leq k-2$
E	$\frac{4}{5}$	$j \leq \frac{k}{4}-2$
F	$\frac{5}{8}$	$j \leq \frac{3k}{5}-2$
G	$\frac{5}{6}$	$j \leq \frac{k}{5}-2$
H	$\frac{4}{7}$	$j \leq \frac{3k}{4}-2$
I	$\frac{8}{11}$	$j \leq \frac{3k}{8}-2$
J	$\frac{6}{7}$	$j \leq \frac{k}{6}-2$
K	$\frac{4}{9}$	$j \leq \frac{k}{4}$
L	$\frac{9}{14}$	$j \leq \frac{k}{9}$
M	$\frac{3}{8}$	$j \leq \frac{k}{3}$
N	$\frac{7}{8}$	$j \leq \frac{k}{7}-2$
O	$\frac{3}{5}$	$j \leq \frac{2k}{3}-2$

FIG. 6.3

CHAPTER 7

CHAPTER 7

Comparison of the probability of disconnection of graphs with the smallest number of minimum vertex cut sets when the probability of failure of an edge is close to 1

In this Chapter consideration is given to finite, simple, undirected graphs in which each edge has a probability p of failure. Failures of edges are assumed to be independent.

The complexity of a graph as stated in Chapter three is of interest because it enables us to compare graphs when the probability p of edge failure is close to 1. We minimise the probability of disconnection $P_d(G)$ of the graph by maximising the value of the complexity $\left(T_{|V|-1}\right)$, as given in Chapter three, section 3.1.

In general the complexity of a regular graph G with degree $\rho = k$ is given by,

$$T_{|V|-1} = \frac{1}{|V|} \prod_{r=1}^{|V|-1} (k - \lambda_r)$$

where $\lambda_0, \lambda_1, \ldots \lambda_{|V|-1}$ are the eigenvalues of G.

Using the result given in Chapter three, section 3.2 on the comparison of (k, k+1)-connected graphs and (mk, mk+1)-connected graphs we calculate the complexity of the base graph G of each of the graphs of Smith [45] constructed in Chapter four, and give a general result for the complexity of any of the infinite families of such graphs which are spread through the range $\frac{3}{8} \leq \frac{k}{|V|} < 1$.

The complexity of the base graph is given by,

$$T_{|V|-1} = \frac{1}{|V|} \prod_{r=1}^{|V|-1} (k-\lambda_r)$$

It follows from Chapter three, section 3.2 that a graph obtained by applying construction A with $m|V|$ vertices, degree $\rho = mk$ will have complexity given by,

$$T_{m|V|-1} = \frac{1}{m|V|} \prod_{r=1}^{m|V|-1} (mk-m\lambda_r)$$

where $\lambda_r = 0$ for $r = |V|, |V|+1, \ldots m|V|-1$

giving,

$$T_{m|V|-1} = \frac{1}{m|V|} \left((mk-m\lambda_1)(mk-m\lambda_2) \ldots (mk-m\lambda_{m|V|-1}) \right)$$

Thus $$T_{m|V|-1} = \frac{m^{m|V|-1}}{m|V|} \left((k-\lambda_1)(k-\lambda_2) \ldots (k-\lambda_{m|V|-1}) \right)$$

i.e. $$T_{m|V|-1} = m^{|V|m-2} k^{|V|(m-1)} T_{|V|-1}$$

The eigenvalues for each of the base graphs G constructed in Chapter four are obtained using a computer program and are tabulated in FIG. 7.1. These values are then used to calculate the complexity of each base graph and also the complexity of a number of graphs in each of the cases A, B, ... O, using the general formula,

$$T_{m|V|-1} = m^{|V|m-2} k^{|V|(m-1)} T_{|V|-1}$$

An example of the calculation of the complexity for the graphs in case A for m = 1 to 8 is given below. Similar calculations are used for some of the graphs in the other cases and these are given with the results for case A in FIG. 7.2.

Case A $\qquad k=1, \quad |V|=2, \quad \frac{k}{|V|} = \frac{1}{2}$.

m	1	2	3	4	5	6	7	8
k	1	2	3	4	5	6	7	8
|V|	2	4	6	8	10	12	14	16

For the base graph $\lambda_r = 1, -1$.

Thus $T_{|V|-1} = \frac{1}{|V|} \prod_{r=0}^{|V|-1} (k-\lambda_r) = 1$.

and using $T_{m|V|-1} = m^{|V|m-2} k^{|V|(m-1)} T_{|V|-1}$,

we obtain the following:

when

$m = 2, \quad T_{m|V|-1} = 2^2 = 4$.

$m = 3, \quad T_{m|V|-1} = 3^4 = 81$.

$m = 4, \quad T_{m|V|-1} = 4^6 = 4.096 \times 10^3$.

In general, $T_{m|V|-1} = m^{2m-2}$.

In FIG. 7.3 the values of the complexity for some of the Smith graphs are compared with the complexity of circulant graphs with the same values of k and |V|. The circulant graphs have degree $\rho=k$ and are defined in Chapter three, section 3.6. The values of k considered are 4, 5, 6, 7 and 8 with $|V|$ taking the values 5, 6, 7, ... 30 for each value of k.

The values of the complexity for each circulant graph are obtained from a computer program using the eigenvalue and complexity equations given in Chapter Three, Section 3.6. The maximum value of complexity is of interest and it is this value which is noted for each value of k and $|V|$, other values of complexity are obtained depending on the construction of the circulant graph. A general illustration of the construction of the circulant graphs used and the values of the complexity obtained are given in FIG. 7.4 together with examples of the various circulant constructions for $k=4$, $|V|=9$.

We recall from Chapter Two, Section 2.6 that the expression $\frac{1}{|V|}\left(\frac{|V|\rho}{|V|-1}\right)^{|V|-1}$ is an upper bound for the complexity of a regular graph. Using this equation an upper bound for the complexity of a regualr graph is calculated and compared with the values of complexity obtained for the Smith graphs having the same values of k and $|V|$. The results are given in FIG. 7.3. Also given in FIG. 7.3 is the percentage deviation from the upper bound which we define as follows:

$$\text{Percentage deviation} = \frac{\text{Smith value of complexity} - \text{upper bound}}{\text{upper bound}} \times 100$$

A further comparison of the complexity of the Smith graph is made if we briefly consider bipartite distance regular graphs of diameter 3. A distance regular graph of diameter three with $|V|$ vertices and degree $\rho = k$ corresponds to a symmetric block design with parameters v, v, k, k, ℓ where $|V|=2v$, $k=k$. These graphs are defined in Chapter Three, Section 3.3, together with the equation and proof for the calculation of the complexity of such graphs. We now give the necessary conditions for a symmetric block design to exist and then show by means of examples when such graphs exist and also how the complexity of such graphs can be calculated.

Symmetric block designs

$|V| = 2v$, $k = k$.

For such a design to exist we must have,

1) $(v-1)\ell = k(k-1)$.

2) if v is even then $n = k-\ell$ is a perfect square.

if v is odd and $n = k-\ell$ then the equation

$$z^2 = nx^2 + (-1)^{(v-1)/2}\ell y^2$$

has a solution in integers x, y, z (not all zero).

Example 1 $k = 3$, $(v-1)\ell = 6$.

If $\ell=1$, $v=7$, $|V|=2v=14$.

If $\ell=2$, $v=4$, $|V|=2v=8$.

Graphs with $k=3$ are $(8,3)$, $(14,3)$

N.B. If v, v, k, k, ℓ exists so does $v, v, v-k, v-k, v-2k+\ell$.

Example 2 $k = 4$, $(v-1)\ell = 12$.

If $\ell=1$, $v=13$, $|V|=2v=26$.

If $\ell=2$, $v=7$, $|V|=2v=14$.

If $\ell=3$, $v=5$, $|V|=2v=10$.

Graphs with $k = 4$ are $(10,4)$, $(14,4)$, $(26,4)$.

Example 3 $\quad k=5, \quad (v-1)\ell=20$

If $\ell=1$, $v=21$, $|V|=42$

If $\ell=2$, $v=11$, $|V|=22$

If $\ell=4$, $v=6$, $|V|=12$

Graphs with k=5 are (12, 5), (22, 5), (42, 5)

The complexity is given by,

$$T_{|V|-1} = \frac{2k}{|V|}\left(\frac{(S+1)k^2-|E|}{S}\right)^S \quad \text{where } |E|=\frac{|V|k}{2} \text{ and } S=\frac{|V|-2}{2}$$

hence using two values of $|V|$ and k as examples we have,

$$|V|=10,\ k=4,\quad T_{|V|-1} = \frac{8}{10}\left(\frac{5 \times 16 - 20}{4}\right)^4 = 4.05 \times 10^4$$

$$|V|=22,\ k=5,\quad T_{|V|-1} = \frac{10}{22}\left(\frac{11 \times 25 - 55}{10}\right)^{10} \approx 1.20726 \times 10^{13}$$

FIG. 7.5 compares the values of complexity of the symmetric block designs with the values obtained for the Smith graphs which have corresponding values of k and $|V|$. FIG. 7.6 compares the complexity of symmetric block designs or bipartite distance regular graphs of diameter three with the maximum and minimum values of complexity for the circulant graphs discussed earlier in this Chapter.

A study of the results given in the various tables in this Chapter show that the values of complexity for many of the Smith graphs are equal to or greater than the maximum values of the complexity for the corresponding circulant graphs, and hence more reliable in the event of failure of edges with p close to 1. Case A of the Smith graphs give graphs with values of complexity equal to the complexity of optimal bipartite distance regular graphs of diameter three where they were found to exist with the same values of k and $|V|$.

The limited number of graphs compared in these cases show the Smith graphs to be as reliable or nearly as reliable as the optimal bipartite distance regular graphs of diameter three.

Comparing the upper bound values of complexity we find that the complexity of a number of the base graphs in the Smith cases equal the upper bound value and are therefore highly reliable. In many other cases the values of complexity were close to the upper bound value.

Case	Degree (k)	Number of Vertices (|V|)	Eigenvalues (λ_r)
A	1	2	1
			-1
B	2	3	2
			-1
			-1
C	3	4	3
			-1
			-1
			-1
D	2	5	2
			$\frac{(\sqrt{5} - 1)}{2}$ (twice)
			$\frac{(-\sqrt{5} - 1)}{2}$ (twice)
E	4	5	4
			-1
			-1
			-1
			-1
F	5	8	5
			$\frac{(\sqrt{5} - 1)}{2}$ (twice)
			0
			0
			$\frac{(-\sqrt{5} - 1)}{2}$ (twice)
			-3

FIG. 7.1

Case	Degree (k)	Number of Vertices (|V|)	Eigenvalues (λ_r)
G	5	6	5
			-1
			-1
			-1
			-1
			-1
H	4	7	4
			0.80194
			0.80194
			-0.55496
			-0.55496
			-2.24698
			-2.24698
I	8	11	8
			$\frac{(\sqrt{5}-1)}{2}$ (twice)
			0
			0
			0
			0
			$\frac{(-\sqrt{5}-1)}{2}$ (twice)
			-3
			-3
J	6	7	6
			-1
			-1
			-1
			-1
			-1
			-1

FIG. 7.1 (continued)

Case	Degree (k)	Number of Vertices (|V|)	Eigenvalues (λ_r)
K	4	9	4
			1
			1
			0.56156
			0
			0
			-1
			-2
			-3.56156
L	9	14	9
			1
			1
			0.56156
			0
			0
			0
			0
			0
			0
			-1
			-2
			-3.56156
			-5
M	3	8	3
			1
			1
			$\sqrt{2} - 1$
			$\sqrt{2} - 1$
			-1
			$-\sqrt{2} - 1$
			$-\sqrt{2} - 1$

FIG. 7.1 (continued)

Case	Degree (k)	Number of Vertices (|V|)	Eigenvalues (λ_r)
N	7	8	7
			-1
			-1
			-1
			-1
			-1
			-1
			-1
O	6	10	6
			1
			1
			0.56156
			0
			0
			-1
			-2
			-2
			-3.56156

FIG. 7.1 (continued)

Degree (k)	Number of Vertices (|V|)	Complexity (Smith graphs)
1	2	1
2	4	4
3	6	4.096×10^{3}
5	10	3.90625×10^{5}
6	12	6.0466176×10^{7}
7	14	1.38412×10^{10}
8	16	4.39804×10^{12}
2	3	3
4	6	3.84×10^{2}
6	9	4.19904×10^{5}
8	12	1.6106×10^{9}
3	4	16
6	8	8.2944×10^{4}
2	5	5
4	10	4.096×10^{4}
6	15	8.16293×10^{9}
8	20	1.12589×10^{16}
4	5	1.25×10^{2}
8	10	3.2768×10^{7}
5	8	2.1025×10^{4}
5	6	1.296×10^{3}
4	7	1.183×10^{3}
8	14	7.93897×10^{10}
8	11	2.2713×10^{8}
6	7	1.6807×10^{4}
4	9	1.248×10^{4}
8	18	2.14404×10^{14}
9	14	3.96582×10^{11}
3	8	3.92×10^{2}
6	16	4.2138×10^{10}
7	8	2.62144×10^{5}
6	10	2.096641×10^{6}

FIG. 7.2

k	$\lvert V \rvert$	Case	$T_{\lvert V \rvert -1}$ Smith graphs	$T_{\lvert V \rvert -1}$ Circulants	$T_{\lvert V \rvert -1}$ Upper Bound	Percentage deviation from the Upper Bound
4	8	A	4.096×10^{3}	4.096×10^{3}	5.215×10^{3}	-21.46
5	10	A	3.90625×10^{5}	3.90342×10^{5}	5.04136×10^{5}	-22.52
6	12	A	6.0466176×10^{7}	6.32964×10^{7}	7.8732978×10^{7}	-23.20
7	14	A	1.38412×10^{10}	1.44976×10^{10}	1.81362×10^{10}	-23.69
8	16	A	4.39804×10^{12}	4.83908×10^{12}	5.78976×10^{12}	-24.04
4	6	B	3.84×10^{2}	3.84×10^{2}	4.25×10^{2}	- 9.65
6	9	B	4.19904×10^{5}	2.66320×10^{5}	4.78837×10^{5}	-12.31
8	12	B	1.6106×10^{9}	1.54974×10^{9}	1.86415×10^{9}	-13.60
3	4	C	16	16	16	0
6	8	C	8.2944×10^{4}	8.2944×10^{4}	8.9107×10^{4}	- 6.92
4	10	D	4.096×10^{4}	4.0962×10^{4}	6.7663×10^{4}	-39.47
6	15	D	8.16293×10^{9}	4.90516×10^{9}	1.37249×10^{10}	-40.52
8	20	D	1.12589×10^{16}	1.31523×10^{16}	1.90955×10^{16}	-41.04
4	5	E	1.25×10^{2}	80	1.25×10^{2}	0
8	10	E	3.2768×10^{7}	3.2769×10^{7}	3.4643942×10^{7}	- 5.41
5	8	F	2.1025×10^{4}	2.0809×10^{4}	2.2487×10^{4}	- 6.5
5	6	G	1.296×10^{3}	1.296×10^{3}	1.296×10^{3}	0
4	7	H	1.183×10^{3}	1.183×10^{3}	1.476×10^{3}	-19.85

FIG. 7.3

k	$\|V\|$	Case	$T_{\|V\|-1}$ Smith graphs	$T_{\|V\|-1}$ Circulants	$T_{\|V\|-1}$ Upper Bound	Percentage deviation from the Upper Bound
8	14	H	7.93897×10^{10}	8.50954×10^{10}	1.02906×10^{11}	-22.85
8	11	I	2.2713×10^{8}	1.72627×10^{8}	2.53182×10^{8}	-10.29
6	7	J	1.6807×10^{4}	1.07×10^{4}	1.6807×10^{4}	0
4	9	K	1.248×10^{4}	1.2322×10^{4}	1.8683×10^{4}	-33.20
8	18	K	2.14404×10^{14}	2.51654×10^{14}	3.30566×10^{14}	-35.14
9	14	L	3.96582×10^{11}	2.5495×10^{11}	4.75801×10^{11}	-16.65
3	8	M	3.92×10^{2}	3.92×10^{2}	6.96×10^{2}	-43.68
6	16	M	4.2138×10^{10}	4.6115×10^{10}	7.73712×10^{10}	-45.54
7	8	N	2.62144×10^{5}	2.46980×10^{5}	2.62144×10^{5}	0
6	10	O	2.096641×10^{6}	1.87551×10^{6}	2.601229×10^{6}	-19.40

FIG. 7.3 (Cont'd)

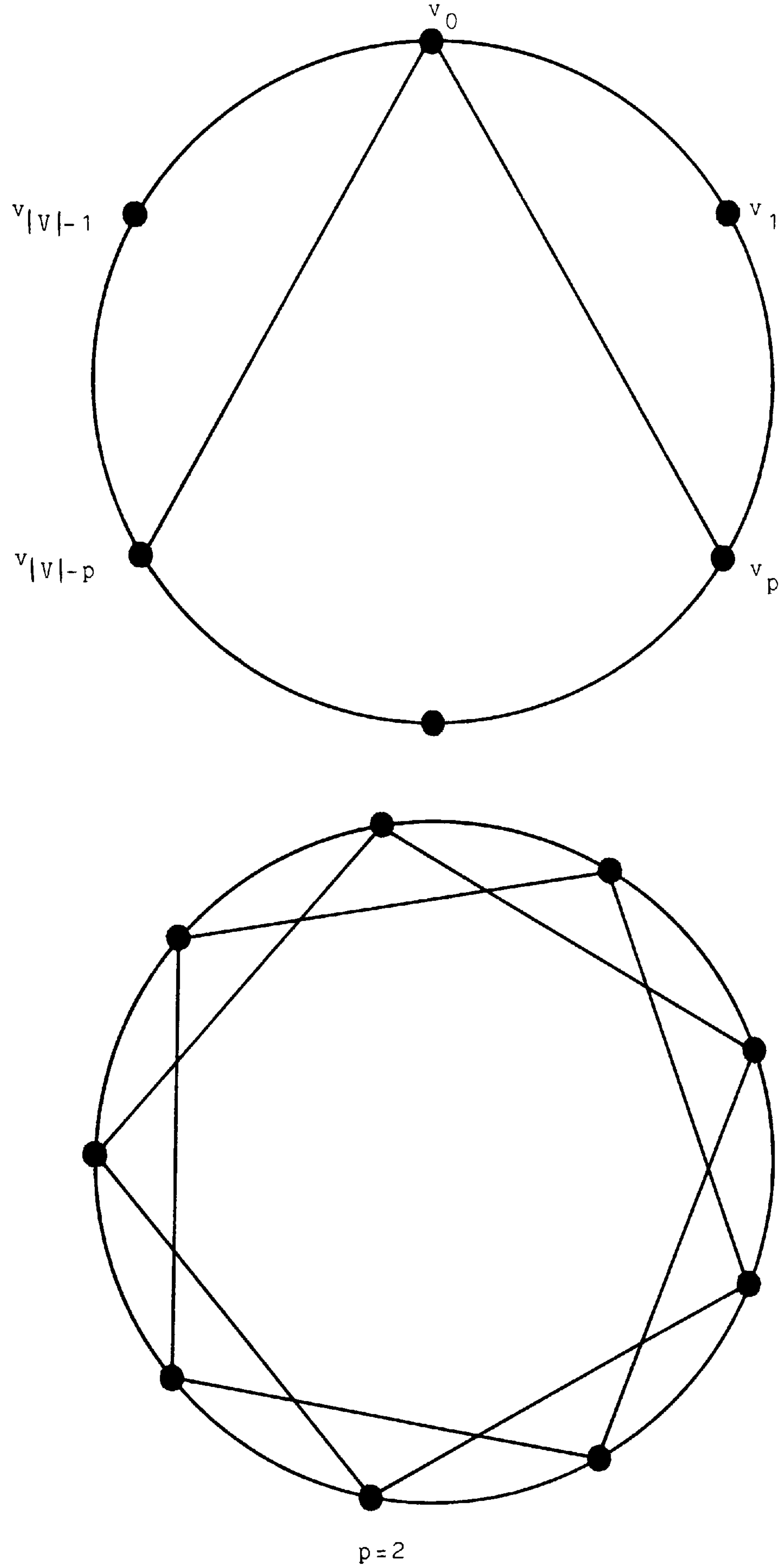
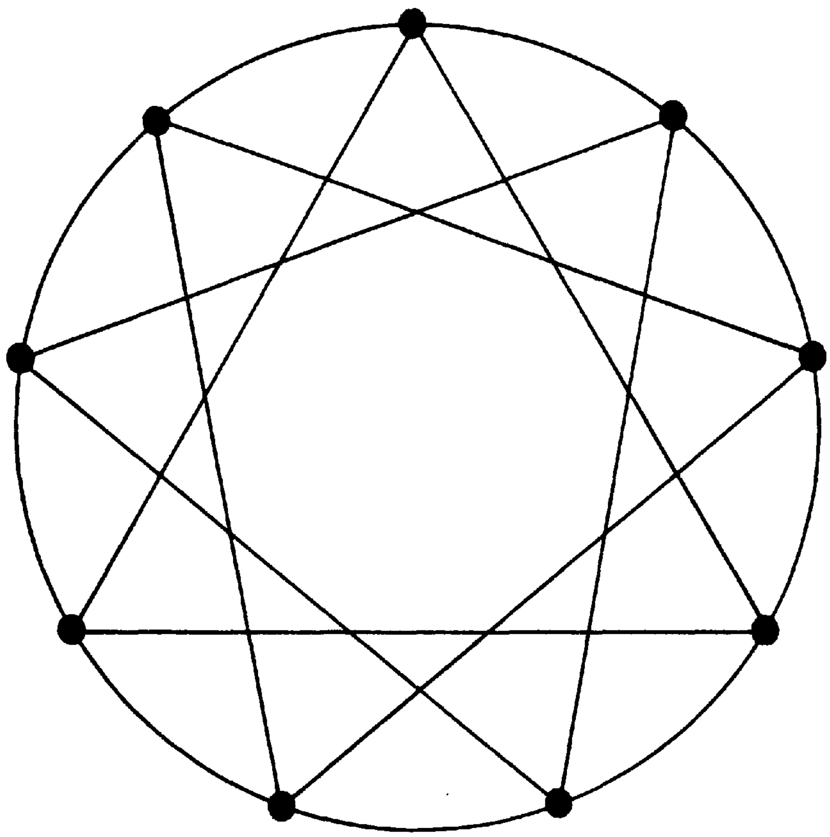
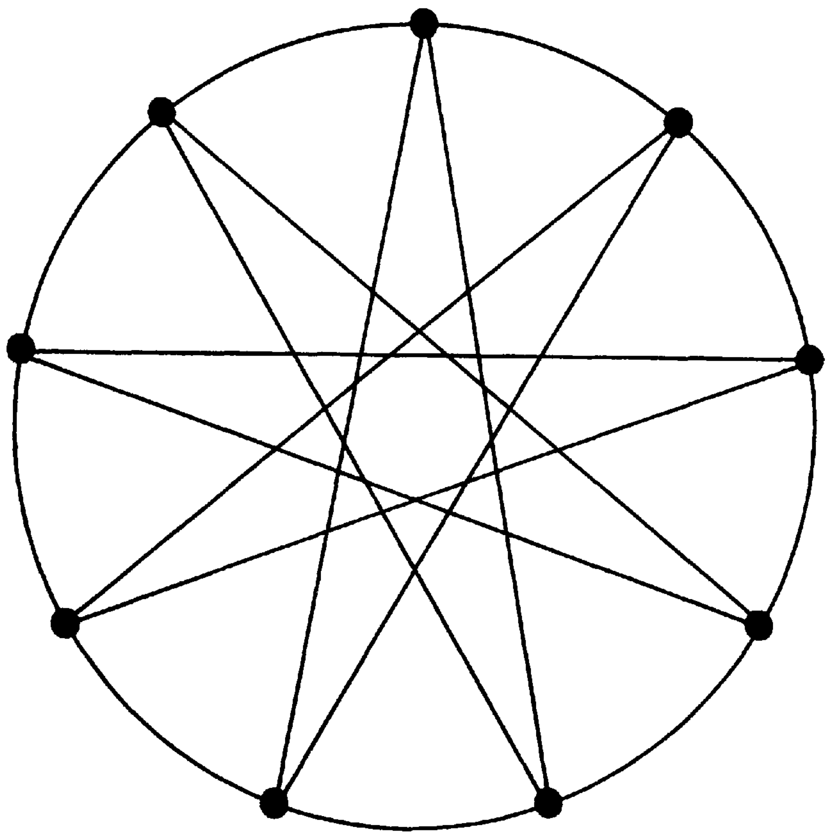


FIG. 7.4



p=3



p=4

FIG. 7.4 (Contd)

k	$\|V\|$	p	$T_{\|V\|-1}$
4	9	2	2.304×10^3
4	9	3	1.0405×10^4
4	9	4	1.2322×10^4

FIG. 7.4 (continued)

$\|V\|$	k	$T_{\|V\|-1}$ Smith graphs	$T_{\|V\|-1}$ Symmetric block design
8	3	3.92×10^2	3.84×10^2
10	4	4.096×10^4	4.05×10^4
18	8	2.14404×10^{14}	2.20582×10^{14}

FIG. 7.5

$\|V\|$	k	$T_{\|V\|-1}$ Symmetric block design	$T_{\|V\|-1}$ Circulant graphs: Maximum	$T_{\|V\|-1}$ Circulant graphs: Minimum
8	3	3.84×10^{2}	3.92×10^{2}	
14	3	5.0421×10^{4}	3.5309×10^{4}	
10	4	4.05×10^{4}	4.0962×10^{4}	5.121×10^{3}
14	4	4.302592×10^{6}	4.37533×10^{6}	1.14754×10^{5}
26	4	7.16864×10^{12}	5.21897×10^{12}	1.81777×10^{9}
12	5	6.635520×10^{6}	7.50523×10^{6}	3.49544×10^{6}
14	5	1.15274×10^{8}	1.30382×10^{8}	9.89921×10^{7}
22	5	1.20726×10^{13}	1.5792×10^{13}	2.48504×10^{12}
26	5	4.11653×10^{13}	5.61268×10^{15}	4.71894×10^{14}
42	5	6.62424×10^{25}		
14	6	1.57565×10^{9}	1.73535×10^{9}	4.20087×10^{8}
22	6	8.55456×10^{17}	1.08133×10^{15}	4.06061×10^{13}
32	6	1.4167×10^{22}		
62	6	1.06571×10^{44}		
16	7	5.13684×10^{11}	5.60237×10^{11}	1.02519×10^{11}
30	7	6.51601×10^{22}	7.89423×10^{22}	5.49057×10^{20}
18	8	6.41959×10^{14}	2.51654×10^{14}	2.6954×10^{13}

FIG. 7.6

CHAPTER 8

CHAPTER 8

The expected number of vertices disconnected from the largest component of a graph

It has been shown in Chapter 4 that it is possible to construct a graph with the minimum number of vertex cut sets with k vertices and thus give a reliable network with the minimum probability of failure if the probability of node failure is close to 0. Although this is desirable from a practical point of view this solution is open to the criticism that although the probability of failure is minimised, when failures do occur a rather large number of nodes may be isolated.

It may be preferable to require that the expected number of vertices M(G) disconnected from the largest remaining component of the graph (or isolated if all components are isolated vertices) be minimised. The expected number of vertices disconnected is given by,

$$M(G) = \sum_{i=k}^{|V|-2} \left(\sum_{v=1}^{S_i} n_{iv} \right) q^i (1-q)^{|V|-i}$$

where S_i = the number of vertex cut sets with i vertices.

n_{iv} = the number of vertices disconnected from the largest component of $G-X_{iv}$ {where X_{iv} is a vertex cut set with i vertices $(i=1,2,...S_i)$} or left isolated if all components are isolated vertices.

k = degree = connectivity.

Since each vertex can be disconnected by at least one vertex cut set with k vertices the minimum value of the coefficient of $q^i(1-q)^{|V|-i}$ is $|V|$ but to attain this minimum we require not just that all vertex cut sets with k vertices be vertex neighbour sets but also the stronger condition that if $|X|$ is a vertex cut set with k vertices then $G-|X|$ has at most one component that is not an isolated vertex. The significance of this stronger condition is demonstrated by the graph in FIG. 8.1. For clarity the bottom diagram in FIG. 8.1 shows how the vertices a, b, c, d, e, f, g are connected, we note also that,

$$\begin{pmatrix} a, \ A \text{ adjacent} \iff a^1, \ A \text{ adjacent etc.} \\ a, \ b \text{ adjacent} \iff a^1, \ b^1 \text{ adjacent etc.} \end{pmatrix}$$

The graph illustrated in FIG. 8.1 is regular with degree ρ = connectivity = 7, $|V| = 22$ and the only vertex cut sets with 7 vertices are vertex neighbour sets. If X_{iv} = A, B, C, D, E, F, G then $n_{iv} = 8$ and it follows that if the other vertex cut sets in the graph with 7 vertices are counted and n_{iv} found then the coefficient of $q^7(1-q)^{15}$ is greater than $|V|$.

i.e.
$$\sum_{v=1}^{S_i} n_{iv} = 8 + 21(1) = 29 > |V|$$

This example shows that the (k, k+j)-connected definition stated gives a stronger condition than that given by S. L. Hakimi and A. T. Amin [26] (see Chapter 2, Section 2.2).

However, the graphs constructed by these authors happen to satisfy the stronger condition also (for j=1), but they are

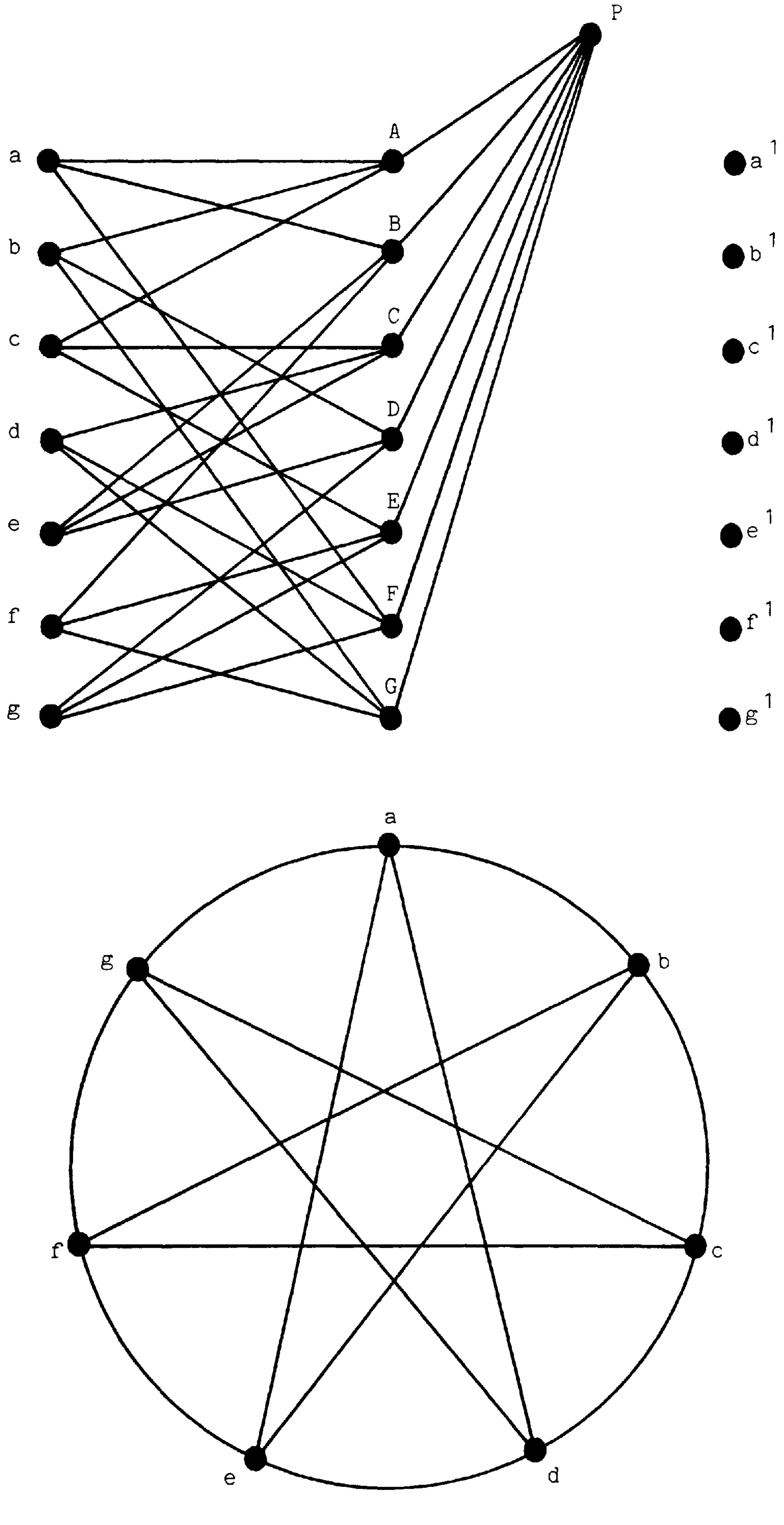


FIG. 8.1

not (k, 2k-2)-connected. For practical values of q it may be better to try to minimise the first few coefficients

$$\sum_{v=1}^{S_i} n_{iv}, \quad \text{say for } i = k,\ k+1,\ \ldots\ k+j-1.$$

A proof that (k, k+j)-connected graphs minimise these coefficients is given as follows:

$$M(G) = \sum_{i=k}^{|V|-2} \left(\sum_{v=1}^{S_i} n_{iv} \right) q^i (1-q)^{|V|-i}$$

If G is (k, k+j)-connected, then

$$\sum_{v=1}^{S_i} n_{iv} = |V| \binom{|V|-k-1}{i-k} \qquad i=k,\ k+1,\ \ldots\ k+j-1.$$

and in general $\displaystyle\sum_{v=1}^{S_i} n_{iv} \geqq |V| \binom{|V|-k-1}{i-k}$,

thus each coefficient of $q^i(1-q)^{|V|-i}$ is a minimum for $i=k,\ k+1,\ \ldots\ k+j-1$. ■

The graphs constructed by S. L. Hakimi and A. T. Amin [26] with $|V|$ vertices and $|E|$ edges and having no more than $|V|$ minimum vertex cut sets each of which is a neighbour set of a vertex have the smallest value of M(G) for some probability of vertex failure close to 0.

The definition of a (k, 2k-2)-connected graph requires $k \leqq \frac{|V|}{2}$ to give a value of $j \leqq k-2$. D. H. Smith [44] constructs (k, 2k-2)-connected graphs with $k \leqq \frac{|V|}{4}$ and gives two constructions and a proof of a theorem for constructing such graphs. He also states that cases with $\frac{|V|}{4} < k \leq \frac{|V|}{2}$ appear to be more complicated to deal with. The constructions and a statement of the theorem are now given,

<u>Case 1</u> $k = 2r$ is even.

<u>Construction A</u> Label vertices 0, 1, 2, ... $|V| - 1$. Vertices i and j are adjacent if $|i-j| \equiv 1, 3, 5, \ldots$ or $2r-1 \pmod{|V|}$.

<u>Case 2</u> $k = 2r+1$ is odd (so $|V|$ must be even).

<u>Construction B</u> Label vertices 0, 1, 2, ... $|V| - 1$. Vertices i and j are adjacent if $|i-j| \equiv 1, 3, 5, \ldots$ or $2r-1 \pmod{|V|}$. If i is odd vertex i is also adjacent to $i+2r+1 \pmod{|V|}$ so that if i is even vertex i is also adjacent to vertex $i-2r-1 \pmod{|V|}$.

<u>Theorem</u> Construction A and Construction B yield graphs that are $(k, 2k-2)$-connected if $k \leq \frac{|V|}{4}$.

Chapter 6 gives the values of j for which the various Smith graphs (Cases A, B, ... O) are $(k, k+j)$-connected and the number of coefficients of M(G) minimised by those graphs.

CHAPTER 9

CHAPTER 9

Conclusions

There has been a considerable amount of work in the area of probabilistic analysis of network reliability. In this thesis we have analysed the design and reliability of a communication network with particular reference to the various infinite families of graphs shown in Chapter 4. If random factors influence the existence of various vertices and edges an important area of investigation is to consider which graphs are optimal in the sense that the probability of disconnection is minimised or the probability of connection is maximised.

In this work we have used the following models for reliability,

1) The edges of a graph are assumed to be reliable, and each vertex is assumed to fail with probability q close to 0.
2) The edges of a graph are assumed to be reliable, and each vertex is assumed to fail with probability q close to 1.
3) The vertices of a graph are assumed to be reliable, and each edge is assumed to fail with probability p close to 0.
4) The vertices of a graph are assumed to be reliable, and each edge is assumed to fail with probability p close to 1.

The graphs constructed in Chapter 4 have a minimum number of vertex cut sets. These various infinite families of graphs contain the same number of minimum vertex cut sets irrespective of the value of k and $|V|$, providing the ratio $\frac{k}{|V|}$ is the value given for each case and lies in the range $\frac{3}{8} \leq \frac{k}{|V|} < 1$.

These graphs are much better than the graphs of S. L. Hakimi and A. T. Amin [26] because the minimum number of incident vertex cut sets is constant for each particular case and is very much less than $|V|$ irrespective of the value of $|V|$, whereas the Hakimi and Amin graphs give the number of incident vertex cut sets in a graph to be no more than $|V|$.

As explained in Chapter 5 finding graphs which have a minimum number of triangles is important in the sense that such graphs have the largest probability of connection for q close to 1. The analysis of the Smith graphs using our second model of reliability shows that in all of the Smith graphs compared the number of triangles was less than or equal to the number of triangles in certain circulant graphs. Thus in most cases the Smith graphs have a larger probability of connection. Compared with a theorectical lower bound for the number of triangles, Cases A, B, C, D, E, G, J, M and N achieved this minimum value for their respective values of k and $|V|$.

In the comparison of graphs with edge failures p close to 0 the Cases A and D of the Smith graphs which satisfy the (k, k+j)-edge-connected definition given in Chapter 6 give a value of $j \leqq k-2$ which equals the value of j given for a special class of circulants, thus minimising the first j coefficients R_i (where $i=k+j-1$) in the expression for the probability of disconnection. In the remaining cases of the Smith graphs i.e. E to O and Cases C and D although the number of coefficients minimised is less the graphs are at least near optimal.

The fourth model of reliability compares the complexity of the Smith graphs and circulants using the eigenvalues of the various graphs. It is found that in 75% of the graphs compared the Smith graphs give the highest value of complexity and would therefore be more reliable in the event of edge failures when p is close to 1.

Comparison with the optimal bipartite regular graph, i.e. the bipartite distance-regular graph of diameter 3 which we explain in Chapter 3, Section 3.3 is equivalent to a balanced incomplete block design, the results show that some of the Smith graphs in Cases A, D and K give values of complexity greater than or nearly equal to the values of the balanced incomplete block design where they exist.

Compared with the upper bound for the complexity of a regular graph, over 70% of the Smith graphs give a value of complexity greater than 75% of the upper bound value and in a number of graphs in Cases C, E, G and N equality is obtained.

Although we have been dealing with small areas of probability i.e. q close to 0, q close to 1, p close to 0, p close to 1 and realizing that the remaining much larger area of probability requires future research, we finally remark that this investigation has produced optimal families of graphs which are highly reliable when q is close to 0. With respect to the other values of probability mentioned above the families of graph were in many instances at least as reliable and in many other instances more reliable when compared with certain standard graphs.

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