Newton's Laws as an Interpretive Tool in System Dynamics

Abstract

This paper proposes an understanding of system dynamics models using concepts from Newtonian mechanics. By considering the second derivative form of a model, and extending the concept of feedback loop impact, it is shown that Newton's three laws of motion have their equivalent in system dynamics, and that the impacts of the forces on the stocks are the measure of the force that determines stock behaviour. The concepts of mass, inertia, momentum, and friction are explored as to their usefulness in understanding model behaviour. The Newtonian understanding is applied to two standard system dynamics models, inventory-workforce and economic long-wave, where their behaviour is analysed using force dominance on the stocks and the laws of motion. The method, and conceptual understanding, is commended for further exploration.

Key Words: Loop dominance, force, Newton's laws of motion, loop impact, model analysis, structural methods.

1 Introduction

A fundamental principle of system dynamics is that the structure of a system is responsible for its behaviour (Sterman, 2000, p. 28). Such structure is expressed in the stock/flow relationships and the feedback loops. The latter are particularly important as they represent endogeneity in the system, and help explain complex behaviour in the variables through changes in dominance of different types of feedback.

In order to quantify feedback loop dominance, numerous methods have been developed (Kampmann & Oliva, 2009; Duggan & Oliva, 2013; Sato, 2016). Broadly, the methods can be divided into two categories: those that relate loop gains to system behaviour, expressed in eigenvalues and eigenvectors, through elasticity analysis (e.g. Forrester, 1982; Goncalves, 2009; Kampmann, 2012); and those that relate pathway connections to variable behaviour expressed in their change and graphical curvature over time (e.g. Ford, 1999; Mojtahedzadeh et.al., 2004; Hayward & Boswell, 2014). In each method the main structural element under consideration is the feedback loop, whose changes in dominance become the chief way of explaining variable behaviour in an intuitive way that cuts through model complexity.

The two categories of loop dominance methods are related: the loop gain is the product of the link, or pathway, gains between adjacent elements in the loop (Kampmann, 2012); also the loop gain is the product of the impacts, or pathway participations, between adjacent stocks in the loop (Hayward & Boswell, 2014). Whatever method is used, a loop with n stocks has n degrees of freedom, and thus requires n numbers to fully describe its effect, whether they are metrics, eigenvalues, or impacts. For $n > 1$, a single number, such as loop gain, is insufficient to fully capture the dynamics of the loop. As such loop dominance analysis becomes harder to interpret the greater the number and order of feedback loops in a model. The question is thus posed as to whether concepts from other modelling methodologies could assist loop dominance analysis as an explanatory framework in system dynamics models.

In order identify an alternative methodology it is noted that the link between structure and behaviour in the pathway participation metric (PPM) method (Mojtahedzadeh et.al., 2004), and the loop impact method (Hayward & Boswell, 2014), is an equation for the second derivative of stock variables in terms of other variables. In each method the model equations are differentiated, placing them in second derivative form, thus focusing on the curvature in variable behaviour, whilst retaining the causal structure of the model in each term that contributes to that curvature. This understanding is analogous to that of Newtonian mechanics where acceleration is determined by various forces, here identified with different loops. As will be shown, the impact of such forces can be extended to exogenous causes.

The purpose of this paper is to use concepts from Newtonian mechanics to understand system dynamics models. Firstly, the concept of loop impact, as introduced by Hayward & Boswell (2014), is investigated further by discussing the notion of the "impact" of a cause on motion generally. The concept is reinterpreted as the impact of a force on a stock. Secondly, laws of system dynamics are proposed by analogy with Newton's laws of motion, with the concepts of force, momentum, mass, inertia and friction explored as to their usefulness in understanding the behaviour of any system dynamics model. Thirdly, a notation is introduced to enable the analytical computation of impact. Finally, the ideas presented are applied to two existing system dynamics models to evaluate their use. Put informally this paper addresses the question: how would Sir Isaac Newton have understood behaviour in a system dynamics model in order to assist the insights of Professor Jay Forrester?

2 The Concept of "Impact"

Impact of Force on Motion

The proposal is to use Hayward & Boswell's (2014) definition of loop impact as a ratio measure of the "force" of one stock on the motion of another stock. In order to help understand the concept of impact on motion generally, an example is given of its use in Newtonian mechanics.

The impact of a force $F = m\ddot{x}$ on the changes in a variable x can be defined as the ratio of the force to the momentum $p = m\dot{x}$:

$$\mathrm{I}_{Fx} \triangleq \frac{F}{p} = \frac{\mathrm{d}^2x/\mathrm{d}t^2}{\mathrm{d}x/\mathrm{d}t} \tag{1}$$

where mass is assumed constant The subscripts on I indicate cause and effect. Thus impact is the logarithmic time derivative of velocity and is independent of mass.

To illustrate this definition consider the situation where a ball is thrown upwards with initial velocity u, subject to a constant gravitational force of acceleration $-g$. Air resistance is assumed negligible. Let x be the vertical displacement of the ball from the ground, then $\mathrm{d}^2x/\mathrm{d}t^2 = -g$. Thus $\mathrm{d}x/\mathrm{d}t = u - gt$, where t is time, and $x = ut - \frac{1}{2}gt^2$, assuming the ball starts at $x = 0$. Thus, from (1), the impact $\mathrm{I}_{F_g\,x}$ of the force of gravity F_g on the ball's motion x is given by:

$$\mathrm{I}_{F_g\,x} = \frac{g}{gt - u}$$

Although gravity is constant, its impact on the motion of the ball is not constant. The impact is greatest as the ball is slowing down, near the top of its motion, seen in the greater curvature in the graph of x against time, figure 1. The impact is infinite while the ball is temporarily at rest, $t = u/g$, and then changes polarity as the ball starts falling. The sign of the impact indicates whether the force is reinforcing the motion, positive, or resisting it, negative.

Physically the impact of a force is measuring the extent to which the force can change the motion of the object (indicated by its position variable), given that it is already in motion. For the thrown ball, when it is moving very fast, the constant gravitational force is inducing only a small change in the motion of the ball, whereas when the ball is moving slowly the change in its motion due to gravity is much larger, in percentage terms. For example, if the ball is moving at 50 m/s, then the change of velocity in a tenth of a second is 2%, however for a velocity of 2 m/s the change is 50% .

The impact of a force on motion can be viewed as a measure of curvature, similar to the radius of curvature used in analytical geometry. However, whereas the latter is constant when the curve is circular, the impact is constant when the curve is exponential. Consider a force that induces exponential acceleration, $\mathrm{d}^2x/\mathrm{d}t^2 = e^{at}$, with a constant . Then $\mathrm{d}x/\mathrm{d}t = e^{at}/a$,

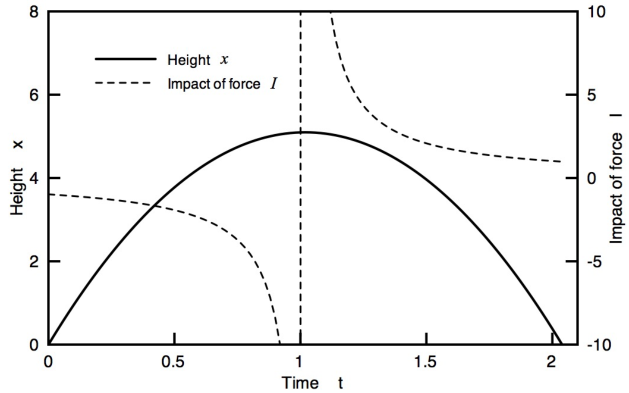

Fig. 1: Impact of force of gravity on vertical motion of a ball.

and the impact of the force on x is $\mathrm{I}_{F_{\mathrm{exp}}\,x} = a$, a constant. The impact is positive if $a > 0$ reinforcing the motion, and negative if $a < 0$, resisting the motion [1].

Impact of a Stock on a Stock

Consider a stock y influencing a stock x, figure 2, equation (2):

$$\mathrm{d}x/\mathrm{d}t = f(y) \tag{2}$$

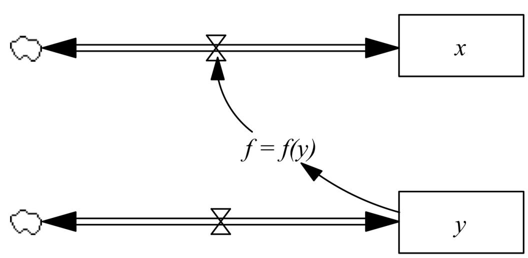

Fig. 2: One stock, x, influenced by another, y.

Following Hayward & Boswell (2014), and (1), the impact of y on x is defined by:

$$\mathrm{I}_{\underline{yx}} \triangleq \frac{\mathrm{d}^2x/\mathrm{d}t^2}{\mathrm{d}x/\mathrm{d}t} = f'(y)\frac{\dot{y}}{\dot{x}} \tag{3}$$

where the underline subscript indicates the causal pathway of the impact. This definition is referred to as the pathway participation between y and x by Mojtahedzadeh et.al. (2004), where $f'(y)$ is the link gain (Kampmann, 2012). Although (3) is called loop impact in Hayward & Boswell (2014), it is clear that the concept of impact is independent of whether the link between the stocks is part of a feedback loop or not. Thus (3) will be called the **Stock Impact** of y on x.

Impact in Feedback Loops

If a system has feedback loops then stock impact will represent the impact of the loop on the stocks. Consider a linear system with two stocks, figure 3, equations (4–5):

$$\dot{x} = ax + by \tag{4}$$
$$\dot{y} = cx + dy \tag{5}$$

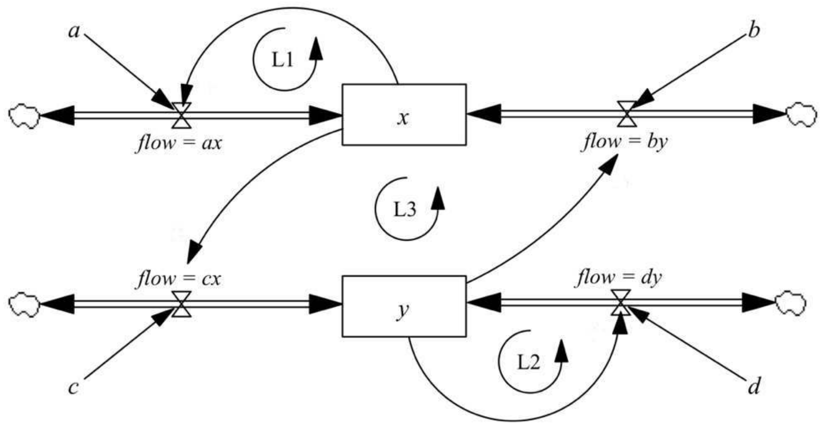

Fig. 3: Generic linear 2 stock system.

with constants a, b, c, d. There are two first order feedback loops, L_1, L_2 with gains $G_1 = a$ and $G_2 = d$; and one second order loop, L_3 with gain $G_3 = bc$. Using definition (3), and identifying the terms with the links in figure 3, the stock impacts are given by:

$$\mathrm{I}_{\underline{xx}}(L_1) = a \tag{6}$$
$$\mathrm{I}_{\underline{yy}}(L_2) = d \tag{7}$$
$$\mathrm{I}_{\underline{yx}}(L_3) = \frac{b(cx + dy)}{ax + by} \tag{8}$$
$$\mathrm{I}_{\underline{xy}}(L_3) = \frac{c(ax + by)}{cx + dy} \tag{9}$$

where the notation includes the name of the loop in which the impact is embedded.

The stock impacts $\mathrm{I}_{\underline{xx}}(L_1)$ and $\mathrm{I}_{\underline{yy}}(L_2)$ are the direct impact of a stock on itself, via first order feedback loops, L_1, L_2, and are equal to the loop gains. Thus it is natural to refer to them as **loop impacts**. The stock impacts $\mathrm{I}_{\underline{xy}}(L_3)$ and $\mathrm{I}_{\underline{yx}}(L_3)$ are part of the same loop, L_3, thus they could also be called loop impacts, as they represent the effect of the loop on each stock. Though in general not equal, their product equals the loop gain, $\mathrm{I}_{\underline{xy}}(L_3)\mathrm{I}_{\underline{yx}}(L_3) = G_3 = bc$; a special case of the loop impact product result (Hayward & Boswell, 2014, appendix C). Note the polarity of the impacts (8–9) may change sign, but do so together such that the polarity of the loop gain is preserved.

However the concept of impact is more general than that of feedback loops. Let $c = 0$ in (4–5), figure 3, which breaks the loop L_3 with $\mathrm{I}_{\underline{xy}} = 0$. (The loop name, $L3$ must be dropped.) y is now an exogenous influence on x, with impact $\mathrm{I}_{\underline{yx}} = bdy/(ax + by)$. This can no longer be referred to as "loop" impact as there is no loop, thus the term stock impact is preferred in this case.

Force of a Stock on a Stock

It is possible to view the causal connection between stocks as a force in the Newtonian sense. Consider the model in figure 2. Differentiating (2) gives the acceleration of x: $\mathrm{d}^2x/\mathrm{d}t^2 = f'(y)\dot{y}$. The value of the force that y exerts on x depends on the time derivative of y, rather than y itself. Nevertheless this constitutes the force of y on x as the stock impact (3) is of same form as the impact of a force introduced earlier (1). As will be shown later, the value of y is related to the momentum of x.

Thus in the second order linear model, figure 3, x can be seen to be exerting forces on itself and y, with y exerting forces on itself and x, where the impacts of the forces are given by (6–9). Unlike Newtonian mechanics there is no requirement for the two stocks to have the same units as they are not position coordinates. Thus the forces exerted by x may not have the same units as those exerted by y. However the *impacts* of the forces do have the same units: "per unit time", independent of the stock units. Thus stock impact enables forces from different stocks to be compared regardless of units.

3 Newton's Laws of Stock Dynamics

In order investigate a Newtonian interpretation of system dynamics, some conventions will be assumed. Firstly, if a variable is given as a stock then it is assumed to have at least one flow, even if the flow does not explicitly appear in a model diagram due to a zero value. Secondly, a flow with no connecting element is assumed to be constant in time. If the intention is for a flow to vary over time then a connection from a time-dependent converter is used to indicate this. The three laws which Newton developed for mechanics are now widened to apply to any type of stock as three laws of stock dynamics, regardless of the variable the stock represents.

Law 1 – Uniform Motion

A stock will remain level or change uniformly unless acted upon by a force.

This law is the equivalent of Newton's first law of motion, applied to a single stock x, and represents the system in figure 4, with equation $\dot{x} = k$, where k is the net flow. The law applies regardless of the number of flows. The stock either stays at "rest" or in motion at the same "speed", unaffected by any force to change its net flow. The graph of x against time will be linear, for example figure 5; the lack of curvature indicating no force. At this stage no concept of mass or momentum is required.

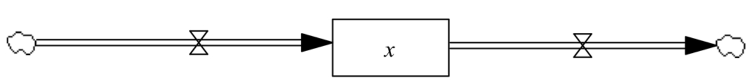

Fig. 4: Model of stock with no applied forces.

For the model in figure 4 to give unique behaviour, both its initial value, x_0, and its initial net flow, that is its initial "velocity" $\dot{x}_0$, are required, as in Newtonian mechanics.

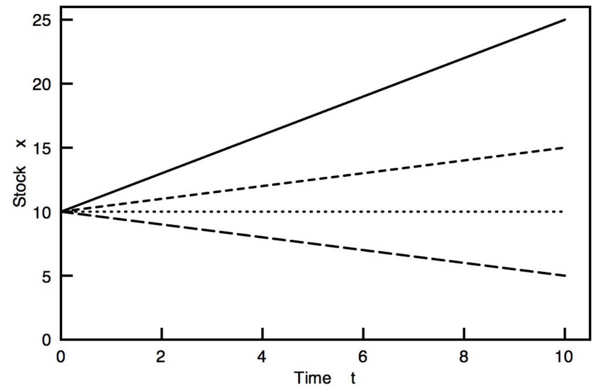

Fig. 5: Behaviour of stock with no applied forces for four different net flows.

Law 2 – Change of Motion Due to Force

The acceleration of a stock produced by a net force is in proportion to that force and the inverse "mass" of the stock.

Consider the model in figure 6, where $\dot{x} = ay$ for a stock x influenced by y, controlled by a constant converter a. Differentiating gives the acceleration of x: $\ddot{x} = a\dot{y} = \dot{y}/m$, where $a \triangleq 1/m$. Thus acceleration is determined by the measure of the force on x due to y, that is $\mathrm{d}y/\mathrm{d}t$ as noted in section 2, and the inverse of the mass. This is one expression of Newton's second law of motion, where force is converted into acceleration by the reciprocal of the mass. Thus the reciprocal of the converter a represents the mass of the stock x with respect to the influence y.

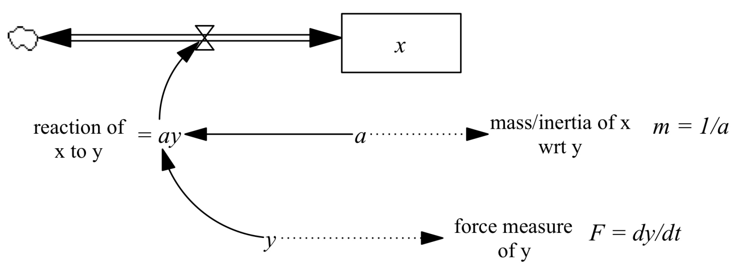

Fig. 6: Stock with applied force.

The force F in the Newtonian sense is given as the time derivative of the converter y that is influencing the stock x, which reproduces the classic form of Newton's second law:

$$F = \frac{\mathrm{d}y}{\mathrm{d}t} = \left(\frac{1}{a}\right)\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \tag{10}$$

Thus y is the equivalent of the momentum of x implying that the rate of change of stock momentum is proportional to the applied force.

To illustrate these Newtonian concepts consider the effect of a step change in y on x, at $t = 2$, for three different values of sensitivity a. The lower values of a means that y has less effect on the stock x, figure 7 (a). Thus a high mass system, $a = 0.1$ has less response to the force induced by y than a low mass system, $a = 0.3$, due to its greater inertial resistance.

The force, $\dot{y}$, is a pulse, figure 7 (b). Thus a converter with a step change represents an impulsive force[2].

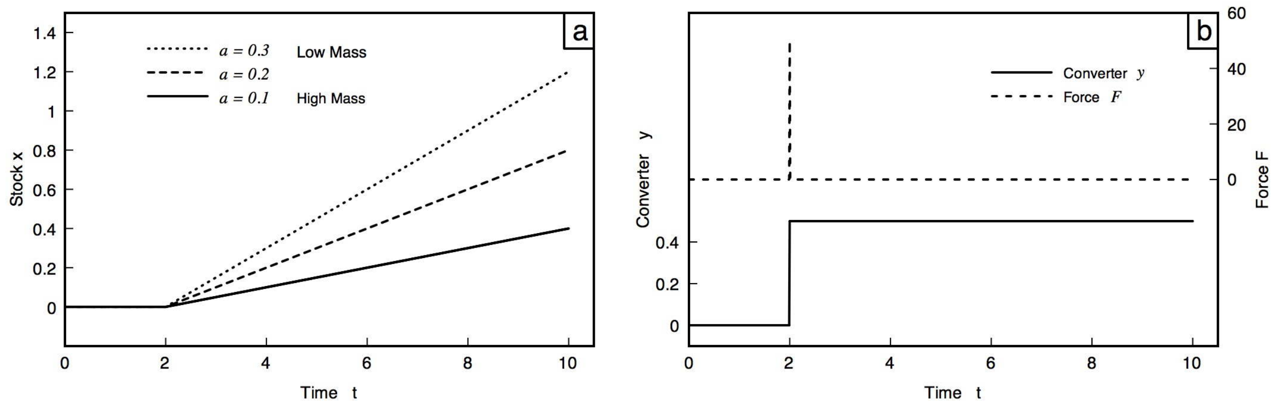

Fig. 7: (a) Effect of pulse force on stock x showing the inertial effect of high mass, low sensitivity a. (b) Step value in converter y is equivalent to pulse force $F = \mathrm{d}y/\mathrm{d}t$

Forces Due to Stocks

The second law of stock dynamics governs the influence of stocks on each other in a way analogous to a force. For example let a stock $y \geq 0$ influence two other stocks x_1, x_2 with different sensitivities to y, figure 8, equations $\dot{x}_i = a_i y$ for $i = 1, 2$, and $\dot{y} = k$. Changes in y induce deviations from uniform motion on the x_i, thus y's rate of change, k, quantifies the force of y on the x_i, whose masses with respect to y are a_i^{-1}. The stock y is the momentum of both x_i.

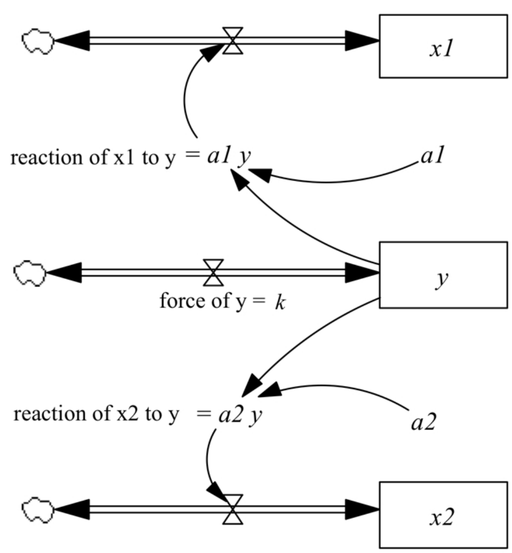

Fig. 8: Stocks with a common applied force.

To illustrate, consider the effect of a constant force $k = -4$. Let $a_1 = 0.2$ and $a_2 = 0.4$, making x_1 the least sensitive of the two stocks to the common force. For $y_0 = 20$, both stocks x_i are brought to rest by $t = 5$, with x_2 reaching the higher value, figure 9. The stock

x_2 is affected more by the force from y than x_1 as it has the least inertia, thus less resistance to change. With respect to y stock x_2 is "lighter" than x_1.

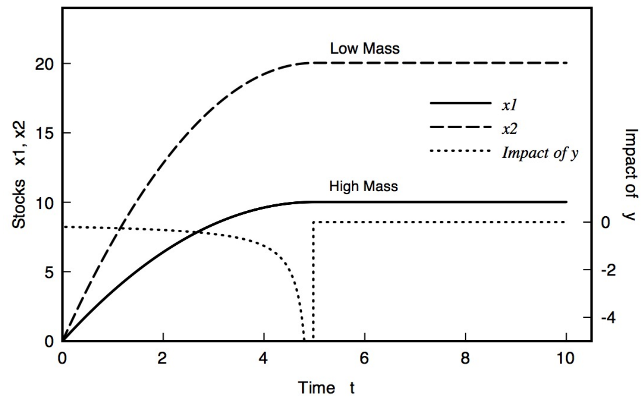

Fig. 9: Effect of stock y, representing a constant force, on other stocks x_1, x_2

The impact of the force from y on each of the stocks is the same, figure 9, as it is a ratio measure, $\ddot{x}_i/\dot{x}_i = \dot{y}/y = -k/(y_0 - kt)$ for $t < 5$. The impact of y is independent of the sensitivities of x_1, x_2 as y induces the same deviation from uniform motion on both stocks. In this case the deviation increases until both stocks are at rest, $t > 5$, becoming uniform with y having zero impact. Impact is a scale-free measure of force, with units $[T^{-1}]$, enabling forces on stocks with different units to be compared. This interpretation of impact is independent of the stock connections being part of a loop, thus independent of the concept of feedback.

Forces Due to Feedback Loops

When feedback is involved then a stock in a loop will be affected by a force, and exert a force on another. To illustrate the Newtonian interpretation of a second order loop consider the second order linear system (4–5), figure 3 with $a = d = 0$ so that only loop L_3 remains. Following (10) parameters b, c are the inverse of the masses of each stock with respect to the other, thus the loop gain, $G_3 = bc$, is the inverse of the product of the masses and thus represents the inertial resistance of the loop. If $G_3 < 0$ the system oscillates indefinitely with angular frequency equal to $\sqrt{|G_3|}$. Thus a higher "mass" second order loop will be more sluggish and oscillate more slowly.

For a Newtonian interpretation of a first order loop set $b = 0$ in (4) to decouple the x stock from y, $\dot{x} = ax$, governed by loop L_1. For a first order balancing loop then $a < 0$, a draining process. This is a form of frictional resistance $\ddot{x} = a\dot{x}$, a force from a stock on itself acting as an energy sink. By contrast, for a reinforcing loop, $a > 0$, the compounding process is an energy source where the stock accelerates itself. Likewise if the second order loop L_3 is reinforcing, each stock is accelerating the other, another form of energy source to the system governed by its loop gain. Thus only a first order balancing loop acts as a dissipative force in a linear system.

These Newtonian analogies can be used to interpret the behaviour of the full system (4–5) in terms of the balance of forces associated with the loops. The condition for system stability depends on the loop gains: $G_1 + G_2 < 0$ and $G_1G_2 > G_3$ (see appendix). Consider a system where L_2 is reinforcing and the other two loops are balancing, such that the system is stable.

The second order loop balancing, L_3, could be seen as an attempt by x to stabilise the behaviour of y. The loop impacts on each stock (6–9) are computed and their dominance determined following the method of Hayward & Boswell (2014), figure 10(a).

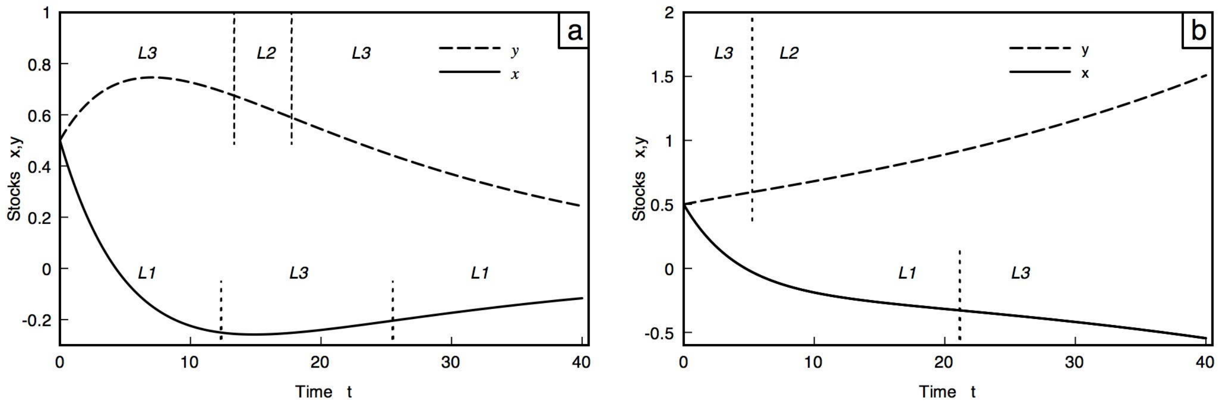

Fig. 10: Second Order Linear System (4–5) with $a = -0.25, b = -0.1, d = 0.03$, showing the change of force/loop dominance on each stock for the same loop structure: (a) Stable, with $c = 0.15$; (b) unstable, with $c = 0.01$.

Stock x goes through three phases where the stabilisation of the frictional force L_1 is temporarily replaced by destabilisation due to L_3 whose impact polarity on x is positive in the middle period, figure 10(a). Stability is achieved as L_1 eventually dominates L_3. By contrast stock y has L_3 dominating initially where, with a change of polarity at y's maximum, $t = 7.2$, it accelerates y downwards. This destabilisation continues while reinforcing loop L_2 is briefly dominant, but is eventually stabilised by L_3, which now has negative impact polarity, the opposite of its polarity on x in this period. The negative polarity of L_3's loop gain is preserved throughout. Thus when its impact on x is accelerating, its impact on y is stabilising.

That the loop L_3 is balancing and of constant gain is not a good indication of stock behaviour as the impact of its force on each of the stocks is variable and changes polarity. Although there is a brief period, $17.7 < t < 25.5$, where L_3 is dominant on both stocks, and could be said to dominate the system, it is the balance of forces on each stock that determines their behaviour. Stability is achieved because the final dominant forces on each stock have negative impact, that is sufficient friction has been applied to x to control y's behaviour.

The system can be destabilized by reducing x's control over y, reducing c to 0.01. The loop structure remains identical, but now $G_1G_2 < G_3$, thus $x \to -\infty$, $y \to \infty$. The force on y due to x, via L_3, is now unable to regain dominance over that of L_2, the destabilizing force, figure 10(b). For x, the loop L_1 is unable to regain dominance over L_3, as friction is insufficient to counteract the force from y. L_3 now has a smaller gain, that is more inertial mass, and is therefore less effective in controlling the destabilizing influence of L_2. If instead the gain of L_3 were increased, the system remains stable, with damped oscillations as $(G_1 - G_2)^2 < -4G_3$ (see appendix). This less massive system is over responsive to the corrective effects of L_3 as the frictional dissipation L_1 is relatively less effective.

Thus the loop impact method of Hayward & Boswell (2014) can also be understood in terms of force dominance on each stock, with the use of Newtonian terms such as mass, inertia and friction providing an alternative explanation of behaviour.

Law 3 – Equal and Opposite Forces

The force on a stock through a flow has an equal and opposite force on a stock at the other end of the flow.

Consider a stock x with a draining process flowing into stock y, figure 11, equations $\dot{x} = -ax$, $\dot{y} = ax$. Not only is the flow conserved, but the forces are equal and opposite in the Newtonian sense, $\ddot{x} = -\ddot{y}$. Therefore the loop impacts are identical $\mathrm{I}_{\underline{xx}}(B) = \mathrm{I}_{\underline{xy}}(B) = -a$, thus the behaviour of x and y mirror each other, both decelerating to stability, assuming $x, y \geq 0$, figure 12.

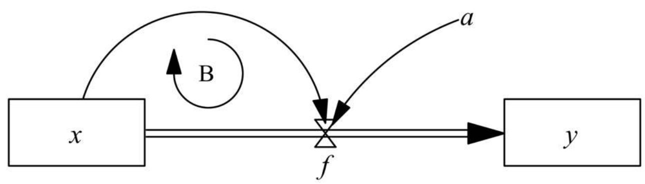

Fig. 11: The force due to loop B has an equal and opposite effect on each stock.

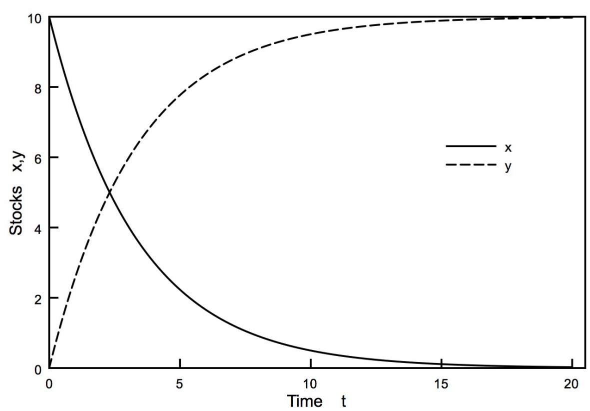

Fig. 12: Equal and opposite forces on stocks in a conserved flow, with same impact on behaviour.

4 Notational Refinement

A Newtonian interpretation of system behaviour requires the computation of stock impacts. Though this could be achieved numerically following Hayward & Boswell (2014), for transparency an analytical approach is preferred to emphasise that Newtonian concepts are not dependent on a particular loop dominance method. The symbolic notation of section 3 is developed to enable differentiation along a causal pathway even when there are multiple pathways between stocks.

To illustrate the notation consider a first order limits-to-growth model with harvesting, figure 13, which has three causal pathways between the stock x and itself. The differential equation for the system, $\dot{x} = a(1-x)x - bx$, does not have sufficient information to compute loop impacts without reference to the system dynamics diagram as the causal pathways have been eliminated in its derivation.

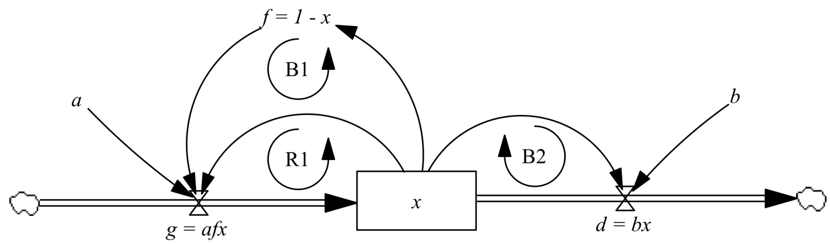

Fig. 13: Limits-to-growth with harvesting model.

To retain the causal structure, the left hand side variable of each model equation is added as a subscript on the right hand side variables when the equation is discharged. Using the model equations in figure 13: $\dot{x} = g - d = af_{\underline{g}}x_{\underline{g}} - bx_{\underline{d}} = a(1 - x_{\underline{fg}})x_{\underline{g}} - bx_{\underline{d}}$

Thus the differential equation for the system dynamics model is written so that each pathway from x to itself is labelled by the underlined subscripts, (11):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(1 - x_{\underline{fg}})x_{\underline{g}} - bx_{\underline{d}} \tag{11}$$

It is thus clear which x in the differential equation is connected with each causal pathway, that is with which loop. This labelling method is easily extended to a many-stock system. The general form of this process is given in appendix B. The underline distinguishes causal subscripts from other subscript use.

Computing the loop impacts requires differentiation along a pathway (Hayward & Boswell, 2014), which is achieved by partial differentiation of the right hand side of (11) with respect to x annotated with the appropriate pathway index. For example the impact of loop $B1$ on x is defined as the partial derivative by $x_{\underline{fg}}$ (compare equation (3)):

$$\mathrm{I}_{\underline{xfgx}} \triangleq \left.\frac{\partial \dot{x}}{\partial x}\right\|_{\underline{fg}} \triangleq \frac{\partial \dot{x}}{\partial x_{\underline{fg}}} \tag{12}$$

The double vertical line in (12) indicates the causal pathway derivative. Thus the three loop impacts are:

$$\begin{aligned}
\mathrm{I}_{\underline{xgx}}(R) &= a(1 - x_{\underline{fg}}) = a(1 - x) \\
\mathrm{I}_{\underline{xfgx}}(B1) &= -ax_{\underline{g}} = -ax \\
\mathrm{I}_{\underline{xdx}}(B2) &= -b
\end{aligned}$$

where the subscripts on x are finally dropped to allow for computation of loop dominance. These are the impacts of the three forces on x. As will be shown in the next section the pathway derivative method can be extended to many-stock systems. A general definition is given in appendix B.

5 Application of the Newtonian Interpretation

Inventory-Workforce Model

For a simple application of Newtonian ideas in system dynamics consider a two state inventory-workforce model subject to an exogenous demand on sales, figure 14. There are two forces on the inventory, one from demand, and one directly from workforce connected with loop $B2$. There are three forces on the workforce: the frictional self force $B1$; one from the inventory stock via the inventory adjustment control, part of $B2$; and an external force from demand via sales which is not connected with a feedback loop.

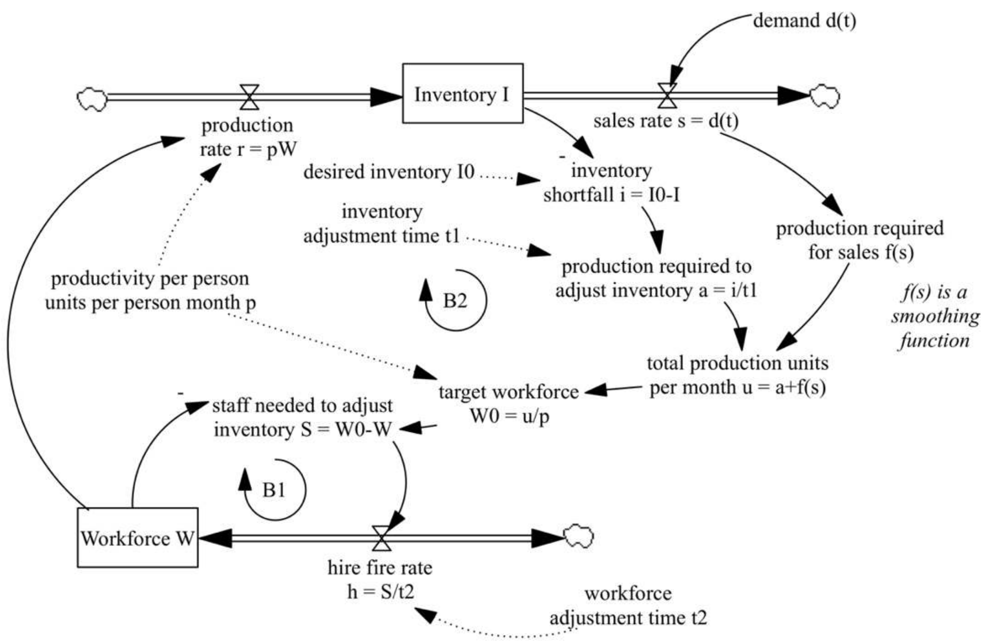

Fig. 14: Two state inventory-workforce model. (Ventana, 2011)

Using the equations in figure 14 the differential equations of the inventory-workforce model with causal connections are:

$$\dot{I} = pW_{\underline{r}} - d_{\underline{s}}(t) \tag{13}$$

$$\dot{W} = \frac{1}{t_2}\left[\frac{1}{p}\left(\frac{I_0 - I_{\underline{iauW_0Sh}}}{t_1} + f(d_{\underline{sfuW_0Sh}}(t))\right) - W_{\underline{Sh}}\right] \tag{14}$$

The impacts of the stocks, and exogenous demand, on a target stock, are derived by pathway differentiation. For example, following (3), the impact of I on W though the inventory shortfall pathway is defined as the partial derivative connected with the W of that network.

$$\mathrm{I}_{\underline{IiauW_0ShW}} \triangleq \left.\frac{\partial \dot{W}}{\partial I}\right\|_{\underline{iauW_0Sh}} \times \frac{\dot{I}}{\dot{W}} \triangleq \frac{\partial \dot{W}}{\partial I_{\underline{iauW_0Sh}}}\frac{\dot{I}}{\dot{W}} \tag{15}$$

Thus the three impacts on stock W are:

$$\mathrm{I}_{\underline{IiauW_0ShW}}(B2) = \frac{pW - d(t)}{I_0 - I + t_1 f(d) - t_1 pW} \tag{16}$$

$$\mathrm{I}_{\underline{dsfuW_0ShW}} = \frac{t_1 f'(d)\dot{d}(t)}{I_0 - I + t_1 f(d) - t_1 pW} \tag{17}$$

$$\mathrm{I}_{\underline{WShW}}(B1)) = -\frac{1}{t_2} \tag{18}$$

The impact subscripts can be omitted in this case because there is a unique identification between loops and pathways: $\mathrm{I}(B2), \mathrm{I}_d, \mathrm{I}(B1)$.

To investigate the impact of the forces on the stocks, let the system start in equilibrium with the inventory at 100, the workforce at 20, and a demand of 10. Let demand rise steadily from 10 to 12 for $t = 5 \ldots 10$. The force exerted by the demand only applies during this period, and is the slope of the line, figure 15(a). Thus demand has an impact on the inventory from $t = 5$, however at $t = 9.1$ the impact of $B2$ exceeds demand impact and dominates the rest of the inventory's return to equilibrium.

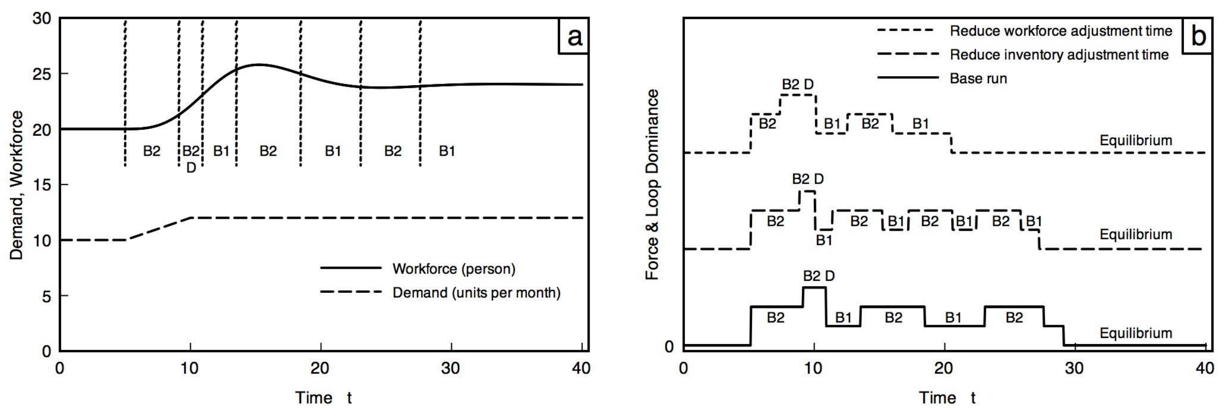

Fig. 15: Results of inventory-workforce model. D indicates dominance of exogenous demand $d(t)$. (a) Base run for *workforce* with transitions of force/loop impacts, $t_1 = t_2 = 2.5$, $p = 0.5$, $I_0 = 100$. (b) Force/loop dominance on *workforce*, base run compared with reducing inventory and workforce adjustment times, $t_1 = 1$ and $t_2 = 1$ respectively.

There are three forces on *Workforce* whose impacts (16–18) are compared using Hayward & Boswell's (2014) definition of dominance[3]. Initially, from $t = 5$, $B2$ dominates as the workforce increases to match the production required by the inventory. By $t = 9.2$ it takes both $B2$ and demand, indicated by D, to dominate the behaviour of *workforce*, thus showing the corrective action from sales is helping to adjust the desired workforce. At $t = 10.9$ the frictional force of the workforce, due to the stock adjustment loop $B1$, starts to slow the growth[4].

At $t = 13.5$ the corrective force $B2$ again dominates causing the workforce to peak, and accelerate downwards. The impacts of $B2$ on the two stocks has changed polarity with the change in direction. The remainder of the motion is repeated change between $B2$ and $B1$, the latter dissipative force damping the oscillations caused by $B2$.

Let the scenario in figure 15(a) be the base run, and compare the differing effects of reducing the inventory adjustment time t_1 and workforce adjustment time t_2. Using the Newtonian analogy, t_2 controls the friction, and t_1 affects the inertia of the second order loop B_2. Figure 15(b) compares their effects on the force/loop impacts on *Workforce*.

When inventory adjustment time is reduced $B1$'s first dominance is earlier compared with the base run, but its appearance is too brief to control the impact of $B2$. The reduced inventory adjustment time t_1 has lowered the "mass" associated with $B2$. It now has less inertia, is harder to control and thus exhibits more oscillations compared with the base run 15(b) middle run[5].

Reducing the workforce adjustment time, figure 15(b) top run, also allows $B1$ to dominate earlier, but this has been achieved by reducing the duration of $B2$'s dominance, as $B1$ is now stronger compared with $B2$. The weakness of $B2$ can be seen in the first period of its dominance, which needs a longer period of assistance from the demand, $B2\,D$, to overcome $B1$. This results in fewer oscillations and thus equilibrium is reached faster. In Newtonian terms increasing friction has achieved more control than reducing the inertia of the system.

From (16–17) it is seen that both $\mathrm{I}(B2)$ and $\mathrm{I}_{\underline{d}}$ are independent of the workforce adjustment time, t_2, thus it is possible to increase the frictional force without any direct effect on these impacts, just the indirect effect through W; hence the success of that policy on reducing oscillations. Reducing t_1 increases $\mathrm{I}(B2)$, the reduction in inertia in this loop referred to earlier. $\mathrm{I}_d/\mathrm{I}(B2) \propto t_1$ showing that reducing inventory adjustment time, weakens the effect of the target of sales compared with production. The resulting higher relative impact of $B2$ on W increases the number of oscillations.

Economic Long-Wave Model

For a more challenging application of the Newtonian view of system dynamics consider Sterman's (1985) economic long-wave model, which has become one of the benchmarks for analytical methods. The stock/flow diagram of Kampmann (2012) is given in figure 16, where the equations of his table 2 are embedded in the diagram[6]. Although the model has only 3 stock variables, the connections are complex with 36 loops, of which 16 are independent (Kampmann, 2012). More than one independent loop set can be chosen because many of the loops share edges in parts of their structure.

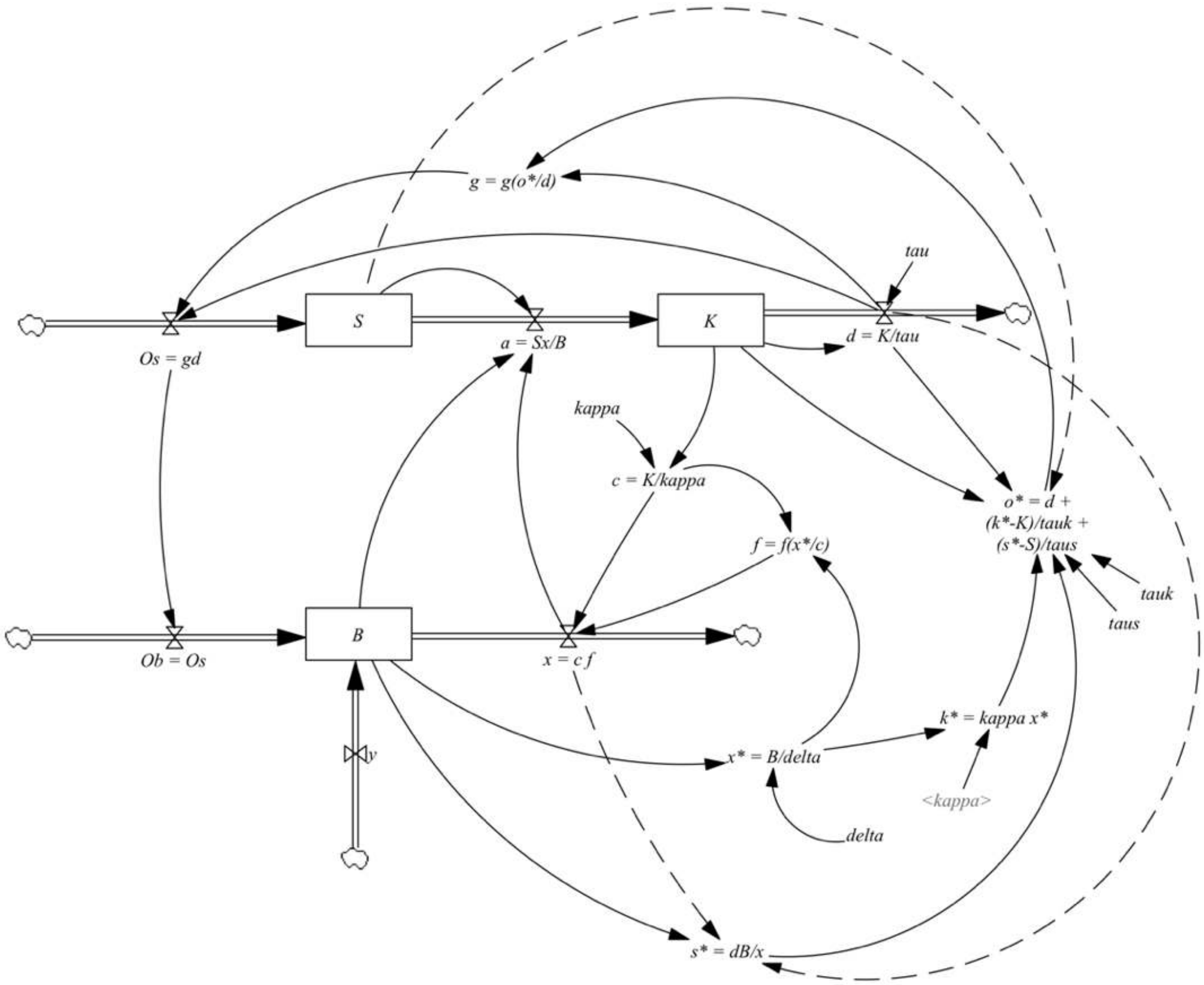

Fig. 16: Economic long-wave model, with model equations given on each stock, flow and auxiliary. f and g are graphical converters, expressed in functional form. Some connectors are dashed for readability.

Following the method of the previous sections the model is written as three causally connected differential equations, using figure 16 and (Kampmann, 2012, table 2):

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \frac{S_{\underline{a}} K_{\underline{cxa}}\, f\left[\frac{\kappa B_{x^*\underline{\underline{fxa}}}}{\delta K_{c\underline{fxa}}}\right]}{\kappa B_{\underline{a}}} - \frac{K_{\underline{d}}}{\tau} \tag{19}$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{K_{\underline{do}}}{\tau} g\left[\frac{K_{\underline{do^*go}} + \frac{\kappa\tau B_{\underline{\underline{x^*k^*o^*go}}} - \delta\tau K_{\underline{\underline{o^*go}}}}{\delta\tau_k} + \frac{\frac{\kappa K_{\underline{ds^*o^*go}} B_{\underline{s^*o^*go}}}{K_{\underline{cxs^*o^*go}} f\left[\frac{\kappa B_{x^*\underline{\underline{fxs^*o^*go}}}}{\delta K_{c\underline{fxs^*o^*go}}}\right]} - \tau S_{\underline{o^*go}}}{\tau_s}}{K_{\underline{dgo}}}\right] - \frac{S_{\underline{a}} K_{\underline{cxa}}\, f\left[\frac{\kappa B_{x^*\underline{\underline{fxa}}}}{\delta K_{c\underline{fxa}}}\right]}{\kappa B_{\underline{a}}} \tag{20}$$

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{K_{\underline{do}}}{\tau} g\left[\frac{K_{\underline{\underline{do^*go}}} + \frac{\kappa\tau B_{\underline{\underline{x^*k^*o^*go}}} - \delta\tau K_{\underline{\underline{o^*go}}}}{\delta\tau_k} + \frac{\frac{\kappa K_{\underline{ds^*o^*go}} B_{\underline{s^*o^*go}}}{K_{\underline{cxs^*o^*go}} f\left[\frac{\kappa B_{x^*\underline{\underline{fxs^*o^*go}}}}{\delta K_{c\underline{fxs^*o^*go}}}\right]} - \tau S_{\underline{o^*go}}}{\tau_s}}{K_{\underline{dgo}}}\right] + y - \frac{K_{\underline{cx}}}{\kappa} f\left[\frac{\kappa B_{x^*\underline{fx}}}{\delta K_{\underline{cfx}}}\right] \tag{21}$$

where the use of subscripted variables has prevented any algebraic reduction taking place through factoring or cancellation of stocks, unless they have come via the same pathway. Thus equations (19–21) preserve the network topology of the stock/flow diagram.

From a Newtonian viewpoint, the number of forces on each stock due to other stocks is unique. For example equation (19) shows that K has 6 forces, indicated by the 6 different pathway subscripted variables. 3 are self forces: via the flow d, ($K_{\underline{d}}$); via x directly, ($K_{\underline{cxa}}$); and via c through the graphical function f, ($K_{c\underline{fxa}}$). These are first order loops. There is one force from S via the flow a, which is the equal and opposite reaction (Newton's third law) from the draining loop on S. Finally, there are 2 forces from B, one directly via a and one through the function f. Forces on S and B can be likewise enumerated. The impacts of all these forces are computed by pathway differentiation, similar to (15), and incorporated in the simulation model[7].

The results are examined from $t = 128$, once the limit cycle behaviour is established. The change of dominance of stock impacts on K are given in figure 17(a). Growth is dominated by $\mathrm{I}_{\underline{SaK}}$, that is the reaction of K to the outflow of S. The only exception is a brief period from K itself via the $KcxaK$ pathway, enhancing the accelerated growth. Thus it can be said that the growth in K is largely a reaction to the outflow of S, that is a result of Newton's third law of motion. The same reaction force governs the change from growth to decline of K and the acceleration that immediately follows. This cause of behaviour is standard in chain models (Hayward & Boswell, 2014). The remainder of the decline is caused by the frictional dissipation force, impact $\mathrm{I}_{\underline{KdK}}$.

Comparing the impacts of the forces, figure 17(b), shows the frictional force $\mathrm{I}_{\underline{KdK}} = -1/\tau$, is a small constant, but that all the other forces collapse to near zero throughout K's decline allowing $\mathrm{I}_{\underline{KdK}}$ to dominate. The two forces via the graphical function f are very small,

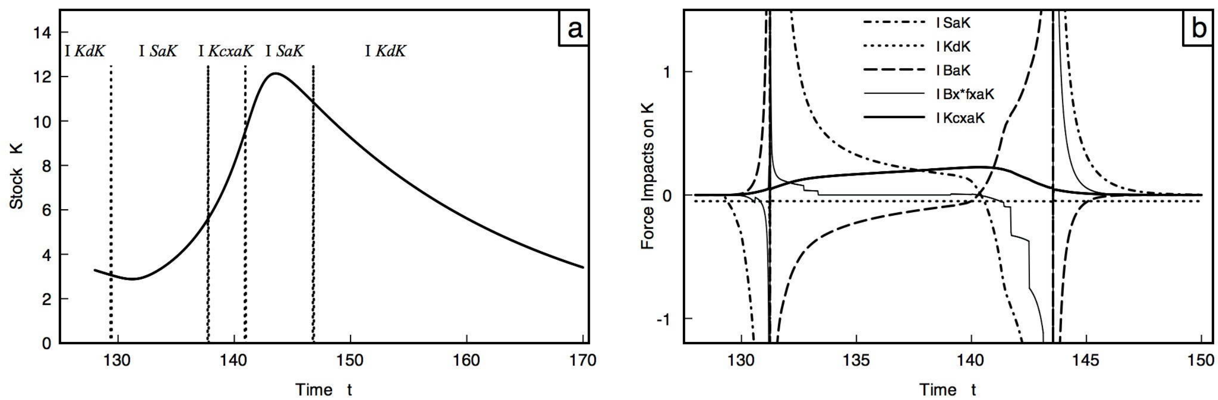

Fig. 17: Economic long-wave model with $\tau_k = \tau_s = \delta = 1.5$, $\tau = 20$, $\kappa = 3$ and $y = 1$. (a) Stock K with regions of force (stock impact) dominance. (b) Stock impacts for 5 of the 6 forces on K. $\text{I}_{\underline{KcfxaK}}$ is too small to show; all bar $\text{I}_{\underline{KaK}}$ are zero during K's decline.

except for $\text{I}_{\underline{Bx^*fxaK}}$ near the turning points only[8], even then it is swamped by the other forces. Other forces, such as $\text{I}_{\underline{BaK}}$, are non-zero and do affect the motion, but they are not strong enough to explain the type of curvature, the acceleration and deceleration; they only influence its extent. Thus K's behaviour can generally be explained by the relative effects of a reaction to a force on S (Newton's third law) and friction on K.

For the stock S there are 16 forces: 4 direct forces; 2 via f; and 10 via g. To simplify the understanding, the 2 forces via f are ignored as these pathways have already been shown to be small, figure 17(b), and the ones via g are treated as a single combined force. Thus 5 impacts on S can be compared, figure 18(a). Essentially there are four phases: a short period of growth, due to a short impulse from g, about $t = 130$; a period of steady growth, where a number of forces dominate in combination (given by the arrow on the figure); a short impulse from g that causes S to decelerate, after $t = 140$; and its subsequent decline through $\text{I}_{\underline{SaS}}$, the frictional force on S. The force dominance in the second period, the steady growth in S, is largely spurious as the forces through g during this period are zero, and the remaining four forces, $\text{I}_{\underline{KdoS}}$, $\text{I}_{\underline{SaS}}$, $\text{I}_{\underline{KcxaS}}$, $\text{I}_{\underline{BaS}}$, though not zero, nevertheless balance to almost zero, figure 18(b) (labelled *Sum of other forces* on the figure). Instead this second phase of steady growth is best explained by Newton's first law of motion, with S increasing under its own momentum after the impulse caused through g in the first phase. g only has effect in two short phases because all forces through it have their impacts proportional to $\dot{g}$, and most of the time g is either zero, or at saturation, thus horizontal with no gradient, figure 18(b).

Examining the impacts of the 10 forces through g shows that most are negligible. The rapid acceleration of S about $t = 130$ is dominated by $\text{I}_{\underline{Bk^*o^*gos}}$, the target setting for K. The change from growth to decline of S about $t = 140$ is dominated by $\text{I}_{\underline{Ko^*goS}}$ (initially assisted by $\text{I}_{\underline{KcxaS}}$), with the following short period of deceleration again dominated by $\text{I}_{\underline{Bk^*o^*gos}}$. Both $\text{I}_{\underline{Bk^*o^*gos}}$ and $\text{I}_{\underline{Ko^*goS}}$ are connected with the capital, K, adjustment process. Thus the dramatic changes in S are caused by two brief, but intense, periods of acceleration and deceleration caused by capital adjustment, with the remainder of the behaviour either following Newton's first law, uniform growth, or frictional dissipation, giving exponential decline. That g is effectively acting like a switch, goes some way to explaining the severe non-linearity of the limit cycle.

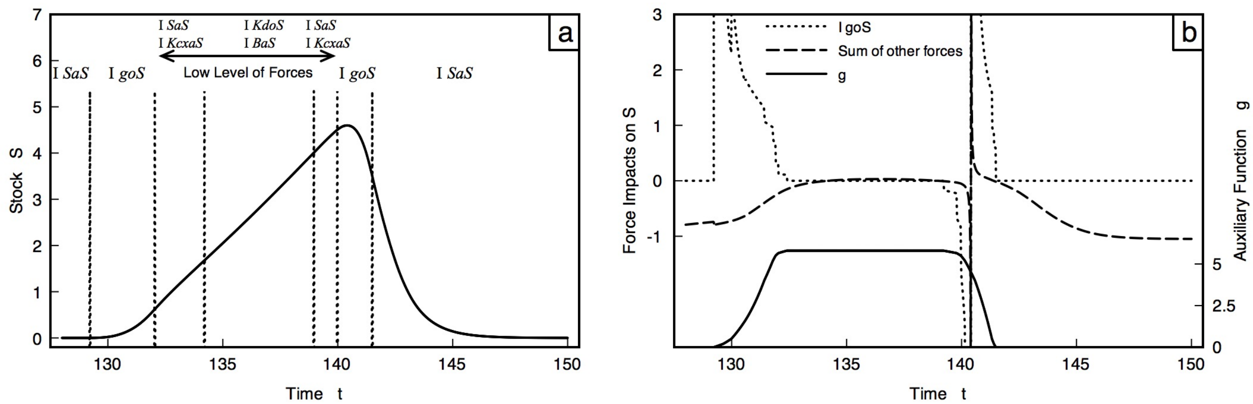

Fig. 18: Economic long-wave model. (a) Stock S with regions of force dominance, negligible force on region marked by arrow. (b) Force impacts for those through g, ($\text{I}_{\underline{goS}}$), compared with the sum of the other 4 forces on S, ($\text{I}_{\underline{KdoS}} + \text{I}_{\underline{SaS}} + \text{I}_{\underline{KcxaS}} + \text{I}_{\underline{BaS}}$), showing a period of negligible force, $t = 132..139$. Graph g against time, showing it has no force (gradient) except at critical turning points of S.

The impacts on the backlog B follow a similar pattern to those on S. Those through g dominate in two brief periods causing B to change direction. $I_{\underline{KcxB}}$ and $I_{\underline{KcfxB}}$ self cancel except when the function f is at its highest value, as can be seen in (21) by replacing f by a straight line and cancelling the Ks. Thus, like S the main rise in B is through Newton's first law, with its decline governed by friction, loop $I_{\underline{Bx*fxB}}$, the overtime loop (Kampmann, 2012). It is therefore possible to use Newtonian concepts, particularly that of force, to explain the behaviour of this complex, many-loop, system in a relatively simple and intuitive way.

6 Conclusion

This paper set out to show that the behaviour of system dynamics models can be interpreted using concepts borrowed from Newtonian mechanics. By developing the concept of loop impact, proposed by Hayward & Boswell (2014), it was shown that the causal link between adjacent stocks represented a force, measured by the net rate of change of the stock corresponding to that force. Key to the analogy is the concept of *impact* as a ratio measure of acceleration; a concept applicable to both mechanical forces and causal links between stocks. This concept corresponds with the definition of loop impact, but is also applicable to exogenous influences. Newton's three laws of motion have their analogy in system dynamics and, together with the concept of mass, inertia, momentum and friction, have a natural interpretation which can assist with understanding model behaviour.

Newtonian concepts were applied to an inventory-workforce model, and the economic long-wave model, helping to simplify the connection between model structure and behaviour. For the inventory-workforce model the inertial mass of the second order loop was key to understanding the effect of interventions. For the economic long-wave model, the combination of Newtons first and third laws, and frictional damping as a response to a sudden impulse, explained much of the behaviour.

A subscript notation was developed for the model differential equations that allowed the network topology of the system dynamics model to remain intact, enabling easier analytical computation of the loop and exogenous impacts on stocks. Such an analytical notation preserves the causal structure of the system dynamics model even when presented in differential equation form. The concept of a pathway derivative was defined in order to measure change along a specific causal pathway, something partial derivative notation is unable to achieve if there are multiple pathways between elements. Although the notation could be refined, for example by merging the symbols for adjacent elements where there is no branching, it is hoped that having a symbolic representation for system dynamics models, and measures of causal impact, could act as a bridge between system dynamics methodology and the more analytical mathematical modelling approach.

The Newtonian approach presented here is seen as an enhancement to the feedback understanding of a model rather than a replacement. In many models the stock impacts are a measure of the effect of feedback on a stock, and give an indication of loop dominance where loops can be easily be identified, as in the inventory-workforce model. In a model as complex as the economic long-wave, there may be a number of loops that can be associated with a given stock impact. Thus, provided a suitable independent loop set can found, the impacts can be applied to the force effects of feedback loops on stocks.

The proposed method, and its Newtonian interpretation, needs to be applied to a wide range of models in order to investigate the extent to which the analogy can help understand model behaviour. It is hoped that the work presented in this paper gives encouragement for further research into a Newtonian understanding of model behaviour and its connection with system structure.

Notes

1. The impact of a force, in the sense defined here, is the impact on the *motion of an object*, units $[T^{-1}]$. It should not be confused with the impact an object makes when it collides with another object, which is a force applied over a very short period of time, units $[MLT^{-2}]$, also called a shock.

2. The force is a delta function but appears as a finite spike in figure 7 (b) due to the fixed step length used in the numerical integration.

3. In Hayward & Boswell's (2014) loop impact method, dominance is defined as the smallest number of loops that explain the majority of the curvature in stock behaviour, using the loops with the largest impacts, and whose polarities match the curvature. Sato (2016) describes this causal explanation as sufficient but not necessary. The authors are indebted to Jeremy Sato for his insightful work and personal conversations in this matter.

4. Demand continues to have an impact on the workforce after it has stop changing at $t = 10$ because of the smoothing function delay in *production required for sales*.

5. A threshold is required to stop registering very small impacts, defining the numerical approximation to equilibrium.

6. A typographical error in the depreciation equation of Kampmann's (2012) paper is corrected using his accompanying Vensim model, $d = K/\tau$; denominator is τ rather than δ.

7. The formulae omitted for brevity. The simulation was also checked using Hayward & Boswell's (2014) numerical approximation to impact.

8. The impact is not smooth due to f being based on a look-up table. Replacing f with a smoothed function would eliminate the sudden changes in impacts.

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Appendix A: Second Order Linear System

For a second order linear system the criteria for stability of the equilibrium can be expressed in terms of loop gains. The Jacobian J of the linear system, (4–5) figure 3, can be written as $J = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, which has eigenvalues $\lambda = [a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}]/2$. The system is stable if both eigenvalues are negative, that is trace $p = a + d < 0$ and determinant $q = ad - bc > 0$ (Drazin, 1992, ch.6). Using the gains of the loops $G_1 = a$, $G_2 = d$ and $G_3 = bc$, then stability can be determined by the sum of the first order loop gains being negative, $G_1 + G_2 < 0$, and their product being bigger than the second order loop gain, $G_1 G_2 > G_3$. Thus for stability at least one the first order loops must be balancing. That is, there is sufficient dissipation in the system to counteract the effects of the second order loop. One corollary of these conditions is that the system is stable only if: either L_3 is reinforcing and both L_1 and L_2 are balancing; or L_3 is balancing and at least one of the first order loops is also balancing.

The system oscillates $(a + d)^2 < 4(ad - bc)$. This can be expressed in gains as $(G_1 - G_2)^2 + 4G_3 < 0$. Thus $G_3 < 0$ is a necessary condition for oscillation, that is the second order loop L_3 must be balancing.

Classification criteria for stability, saddle behaviour, oscillation etc. can be expressed on the the standard $p - q$ plane (Drazin, 1992, p.176) using the loop gains, figure 19.

Appendix B: Pathway Notation for System Dynamics Models

Causal Pathway Notation

In system dynamics a causal chain is represented by the model equations. For example if x causes y, which in turn causes z, then the model equations are $y = f(x)$ and $z = g(y)$, written in functional form. When the equations are combined the intermediate variable y can be retained as a subscript to the initial cause x in order to indicate the pathway from x to z. Writing $y = f(x_{\underline{y}})$ then the causal link from x to z is $z = g(y) = g(f(x_{\underline{y}}))$.

If z in turn is a cause of w, $w = h(z)$, then w can now be written as a function of x with the pathway through y and z retained. Using $z = g(y_{\underline{z}})$:

$$w = h(z) = h(g(y_{\underline{z}}) = h(g(f(x_{\underline{y}})_{\underline{z}})) = f(g(f(x_{\underline{yz}})))$$

where the definition $f(x_{\underline{y}})_{\underline{z}} \triangleq f(x_{\underline{yz}})$ has been used. The notation can be extended to functions with many arguments.

For example let $y(x) = 3x^2$, $z(x, y) = 2x - 4y$ and $w(x, z) = \sqrt{xz}$. Then

$$w = \sqrt{xz} = \sqrt{x\,(2x_{\underline{z}} - 4y_{\underline{z}})} = \sqrt{x\left(2x_{\underline{z}} - 12x^2_{\underline{zy}}\right)}$$

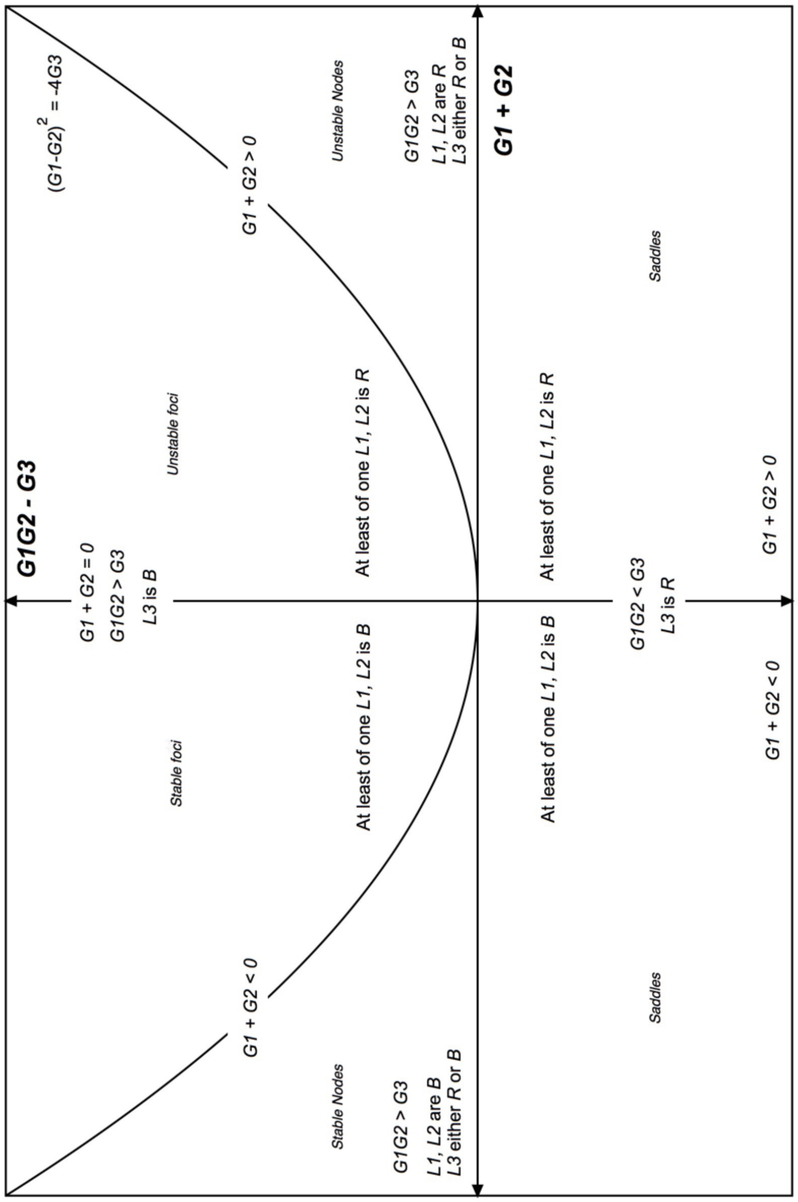

Fig. 19: Stability of second order linear system and loop gain.

where the three different causal pathways from x are indicated by the different subscripts, via z, via zy and directly the x with no subscript. For analytical and numerical investigation the subscripts can be dropped.

Stock Impact Notation

Consider a first order system dynamics model with stock x, with net flows f, and with π causal pathways to itself, i.e. π first order loops. Let a_μ be the name for the collection of intermediary auxiliary variables in pathway μ. The system dynamics model in networked equation form is:

$$\dot{x} = f(x) = f(x_{\underline{a_1}}, x_{\underline{a_2}}, \dots x_{\underline{a_\mu}}, \dots x_{\underline{a_\pi}}) \tag{22}$$

$x_{\underline{a_\mu}}$ is the variable x along pathway a_μ. Equation (22) can be written in a more concise form as:

$$\dot{x} = f(x_{\underline{a_\mu}}) \quad \mu = 1 \dots \pi \tag{23}$$

The stock impacts on x are derived by differentiating (23) by time:

$$\ddot{x} = \frac{\mathrm{d}f}{\mathrm{d}x}\dot{x} = \sum_{\mu=1\dots\pi} \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right\|_{\underline{a_\mu}} \dot{x}$$

where the pathway derivative is defined by:

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right\|_{\underline{a_\mu}} \triangleq \frac{\partial f}{\partial x_{\underline{a_\mu}}}$$

the derivative along one pathway in the first order model. The stock impacts on x are:

$$\mathrm{I}_{\underline{x a_\mu x}} \triangleq \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right\|_{\underline{a_\mu}}$$

By definition all these stock impacts are first order loop impacts.

Consider an nth order system dynamics model with stocks x_i, $i = 1 \dots n$, with net flows f_i, and with π_{ij} causal pathways from x_i to x_j. Let a_{ij} be the names of the list of causal pathways from x_i to x_j. a_{ij} is a matrix of lists of possibly differing lengths π_{ij}. An individual causal pathway in the list is indexed by μ_{ij} drawn from the range $1 \dots \pi_{ij}$, thus giving the matrix of lists $a_{ij\,\mu_{ij}}$, which can be abbreviated to $a_{ij\,\mu}$ without confusion. Each element of each list is a collection of intermediary auxiliary variables in pathway μ_{ij}. The nth order system dynamics model in a concise networked equation form is:

$$\dot{x}_i \;=\; f_i(x_{j\underline{a_{ji\,\mu}}}) \quad i, j = 1 \dots n; \;\; \mu_{ji} = 1 \dots \pi_{jj} \tag{24}$$

$x_{j\underline{a_{ji\,\mu}}}$ is the variable x_j along pathway $a_{ji\,\mu} \equiv a_{ji\,\mu_{ji}}$ connected to x_i. There are π_{ji} pathways connecting these variables.

The stock impacts on x_i are derived by differentiating (24) by time:

$$\ddot{x}_i = \sum_{j=1\dots n} \frac{\partial f_i}{\partial x_j}\frac{\dot{x}_j}{\dot{x}_i}\dot{x}_i = \sum_{j=1\dots n}\; \sum_{\mu_{ji}=1\dots\pi_{ji}} \left.\frac{\partial f_i}{\partial x_j}\right\|_{\underline{a_{ji\,\mu}}} \frac{\dot{x}_j}{\dot{x}_i}\dot{x}_i \tag{25}$$

where the pathway derivative is defined by:

$$\left.\frac{\partial f_i}{\partial x_j}\right\|_{\underline{a_{ji\,\mu}}} \triangleq \frac{\partial f}{\partial x_{j\,\underline{a_{ji\,\mu}}}} \tag{26}$$

the derivative along one pathway $a_{ji\,\mu}$ in the nth order model. Note in $x_{j\,\underline{a_{ji\,\mu}}}$ the underlined subscripts $\underline{a_{ji\,\mu}}$ themselves subscripted, define the pathway, whereas the non-underlined subscript j indicates the variable name. The use of the underline subscript notation should avoid confusion for the many other uses of subscripts in dynamical models, e.g. the system variable identifier i.

The stock impacts of x_j on x_i are:

$$\mathrm{I}_{\underline{x_j a_{ji\,\mu} x_i}} \triangleq \frac{\partial f_i}{\partial x_j}\bigg\|_{\underline{a_{ji\,\mu}}} \frac{\dot{x}_j}{\dot{x}_i}$$

When $i = j$ these are first order loop impacts. When $i \neq j$ these impacts may be part of higher order loops, however this is not guaranteed, as some stocks may not be part of feedback loops. Thus the term stock impact is preferred. If impacts are part of higher order loops they may be referred to as loop impacts, though they may be part of more than one loop. The notation is easily extended to include multiple pathways from any number of exogenous forces.

The notation can be refined further using the comma notation for partial differentiation, $f_{i,j}$. Thus the pathway derivative (26) can be written:

$$f_{i,j}\|_{\underline{a_{ji\,\mu}}} \triangleq f_{i,j\|\underline{a_{ji\,\mu}}}$$

the partial derivative of variable i by variable j along the pathway $\underline{a_{ji\,\mu}}$ from j to i. The double line distinguishes pathway partial differentiation from the conventional form, indicated by the comma.