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Parametric Control to A Type of Quasi-linear Second-order Systems via Output Feedback

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ABSTRACT
This paper considers the design of output feedback control for a type of quasi-linear second-order systems with the time-varying coefficient matrices containing the state variables and a time-varying parameter vector. Based on the solution to a type of second-order generalized Sylvester matrix equations, general complete parameterization of a quasi-linear output feedback controller is established with respect to the state variables, the time-varying parameter vector, the constant closed-loop system and another two groups of arbitrary parameters, and also for the left and right closed-loop eigenvectors matrices. With the proposed parametric output feedback control, the closed-loop system can be transformed into a constant linear system with desired eigenstructure. Finally, simulation results are provided to illustrate the convenience and effectiveness of application in the general spacecraft rendezvous problem.

KEYWORDS
quasi-linear second-order systems; parametric control; eigenstructure assignment; output feedback; spacecraft rendezvous

1. Introduction

Second-order systems represent the dynamic process of many phenomena in nature, and also have a lot of applications, such as vibration control for engineering structures (Omidi & Mahmoodi (2016); Sun, Gao, & Kaynak (2015)), orbital and attitude control for spacecraft (Luo, Zhang, & Tang (2014); Pukdeboon (2016); Zhu, Wang, Shen, & Poh (2017)), manipulator control for robot and mechanical control (Asada & Slotine (1986); Jayakody, Shi, Katupitiya, & Kinkaid (2016); Mattila, Koivumäki, Caldwell, & Semini (2017)), etc. Note that most of these practical systems are in fact quasi-linear, that is, the dynamical models are really originally nonlinear but can be written in a linear format. For example, the dynamical model of robotic systems is given by

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = u, \tag{1}$$

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where q is the generalized coordinate vector; u is the control torque vector; $M(q) > 0$ is the inertia matrix and $C(q, \dot{q})$ comprises the Coriolis and centrifugal effects, which are dependent on the coordinate vector. Therefore, most of presented results (e.g., Chu & Datta (1996); Datta & Rincon (1993); Duan (2004); Duan & Liu (2002)) for the constant second-order system are not applicable to the above quasi-linear systems.

In fact, there have been many approaches to nonlinear control systems, such as, model predictive control, H_∞ control, feedback linearization, etc (See Goodwin, Dona, Rojas, & Perrier (2001); Grüne & Pannek (2017); Jiffri, Paoletti, & Mottershead (2016); Lee (2011); Slotine & Li (1991) and the references therein). Each approach has its scope of application, because of the nonlinear nature, it is unrealistic to propose one approach for applying all kinds of nonlinear systems. Furthermore, the closed-loop system is generally a nonlinear one after applying a nonlinear control approach. Although, it can be shown that the closed-loop system is stable, one is not very familiar with its dynamic performance. By constrast, linear system is simple and well known that the stability and response characteristics are totally determined by the closed-loop eigenstructure.

Regarding applications, the first step is to derive an equivalent first-order system, without thinking the advantage that the original second-order model may offer. In general, the original models of the practical system are in the second-order models, retaining the models in second-order format has many advantages. For example, the physical meanings of variables as well as the system coefficients would have been lost in first-order form. In addition, additional computation load is given in the first-order form and the advantage in the design of controller is no longer exist. In this paper, we propose a parametric control approach for output feedback control of a type of nonlinear systems, namely, quasi-linear systems. The proposed approach is based on the solution to a type of generalized Sylvester matrix (Duan (2015)). A complete explicit parameterization for output feedback controller is given, under which the closed-loop system can be transformed into an arbitrary linear constant matrix, also for the left and right closed-loop eigenvector matrices.

The proposed approach is fundamentally different from the well-known feedback linearization for nonlinear control systems, mainly in the following aspects. Firstly, the central idea of feedback linearization is to algebraically transform a nonlinear system dynamics into a (fully or partly) linear one, so that linear control techniques can be applied. Feedback linearization techniques can be viewed as ways of transforming original system models into equivalent models of a simpler form. While the proposed approach tries to give a linear closed-loop system by fully utilizing the design degrees of freedom. Secondly, analytically solving the partial differential equations defining input-state linearizing transformations is generally not systematic, while the proposed approach provides a systematic way. Thirdly, feedback linearization is hard to apply when the system is non-minimum phase or weakly non-minimum phase, while the proposed approach does not necessarily require the system to be minimum phase.

The remainder of this article is organised as follows. The problem formulation is presented, and some notations and assumptions are given in Section 2. In Section 3, the parametric forms of output feedback controller are proposed for two cases, the constant matrix is arbitrary and diagonal. In Section 4, an example of spacecraft rendezvous problem can be solved by the proposed approach, and results show the effectiveness and simplicity.

2. Problem Formulation

In this paper we propose a type of second-order systems in the following form

$$\begin{cases} A_2(\theta,q,\dot q)\ddot q + A_1(\theta,q,\dot q)\dot q + A_0(\theta,q,\dot q)q = B(\theta,q,\dot q)u \\ y_0 = C_0(\theta,q,\dot q)q, \quad y_1 = C_1(\theta,q,\dot q)\dot q \end{cases} \tag{2}$$

where $q \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, $y_0 \in \mathbb{R}^{m_0}$ and $y_1 \in \mathbb{R}^{m_1}$, are the state vector, the control vector, the measured output and derivative output vectors, respectively; The matrices $A_2(\theta,q,\dot q)$, $A_1(\theta,q,\dot q)$, $A_0(\theta,q,\dot q) \in \mathbb{R}^{n\times n}$, $B(\theta,q,\dot q) \in \mathbb{R}^{n\times r}$ and $C_0(\theta,q,\dot q) \in \mathbb{R}^{m_0\times n}$, $C_1(\theta,q,\dot q) \in \mathbb{R}^{m_1\times n}, m_0 + m_1 = m$ are the system coefficient matrices which are piecewise continuous functions of q, $\dot q$ and θ, where θ is a time-varying parameter vector which satisfies

$$\theta(t) = \begin{bmatrix} \theta_1(t) & \cdots & \theta_l(t) \end{bmatrix}^{\mathrm{T}} \in \Omega \subset \mathbb{R}^l, t \geq 0. \tag{3}$$

where Ω is a compact set.

Assumption 1. $B(\theta,q,\dot q)$, $C_0(\theta,q,\dot q)$ and $C_1(\theta,q,\dot q)$ are uniformly bounded with respect to q, $\dot q$ and $\theta(t) \in \Omega$.

Assumption 2. rank $A_2(\theta,q,\dot q) = n$.

For the above system (2), we choose the following output feedback control

$$\begin{aligned} u &= K_0(\theta,q,\dot q)y_0 + K_1(\theta,q,\dot q)y_1 + v \\ &= K_0(\theta,q,\dot q)C_0(\theta,q,\dot q)q + K_1(\theta,q,\dot q)C_1(\theta,q,\dot q)\dot q \\ &= \begin{bmatrix} K_0(\theta,q,\dot q)C_0(\theta,q,\dot q) & K_1(\theta,q,\dot q)C_1(\theta,q,\dot q) \end{bmatrix} \begin{bmatrix} q \\ \dot q \end{bmatrix}, \end{aligned} \tag{4}$$

where $K_0(\theta,q,\dot q) \in \mathbb{R}^{r\times m_0}$ and $K_1(\theta,q,\dot q) \in \mathbb{R}^{r\times m_1}$ are the feedback gain matrices to be designed, which are also piecewise continuous functions of q, $\dot q$ and θ, and v is the external input.

By using the above controller, the closed-loop system can be obtained as follows

$$A_2(\theta,q,\dot q)\ddot q + A_1^c(\theta,q,\dot q)\dot q + A_0^c(\theta,q,\dot q)q = B(\theta,q,\dot q)v \tag{5}$$

where

$$\begin{cases} A_0^c(\theta,q,\dot q) = A_0(\theta,q,\dot q) - B(\theta,q,\dot q)K_0(\theta,q,\dot q)C_0(\theta,q,\dot q) \\ A_1^c(\theta,q,\dot q) = A_1(\theta,q,\dot q) - B(\theta,q,\dot q)K_1(\theta,q,\dot q)C_1(\theta,q,\dot q) \end{cases} \tag{6}$$

Let

$$x = \begin{bmatrix} q \\ \dot q \end{bmatrix} \tag{7}$$

the closed-loop system (5)-(6) can be written into the first-order form

$$E_c(\theta,x)\dot x = A_c(\theta,x)x + B_c(\theta,x)v \tag{8}$$

with

$$\begin{aligned} E_c(\theta,x) &= \mathrm{diag}(I_n, A_2(\theta,x)), \\ A_c(\theta,x) &= \begin{bmatrix} 0 & I_n \\ -A_0^c(\theta,x) & -A_1^c(\theta,x) \end{bmatrix}, \\ B_c(\theta,x) &= \begin{bmatrix} 0 \\ B(\theta,x) \end{bmatrix}. \end{aligned}$$

This paper considers the design of output feedback control in the form of (4) such that the closed-loop system matrix pair $(E_c(\theta,x), A_c(\theta,x))$ possesses a constant linear system with a desired eigenstructure. That is, the design of the purpose is to let $(E_c(\theta,x), A_c(\theta,x))$ be similar to an arbitrary given constant matrix $F \in \mathbb{R}^{2n\times 2n}$.

2.1. *The Closed-loop Eigenvector Matrices*

Let the pair of matrices $T_c(\theta,x)$ and $V_c(\theta,x)$ be the left and right closed-loop eigenvector matrices of the closed-loop system matrix pair $(E_c(\theta,x), A_c(\theta,x))$. For the closed-loop eigenvector matrices we introduce the follow lemmas.

Lemma 2.1. *The closed-loop system matrix pair $(E_c(\theta,x), A_c(\theta,x))$ is given by (8). There exists the following left closed-loop eigenvector matrice*

$$T_c(\theta,x) = \begin{bmatrix} T_0^{\mathrm{T}}(\theta,x) & T_1^{\mathrm{T}}(\theta,x) \end{bmatrix}^{\mathrm{T}}, T_i^{\mathrm{T}}(\theta,x) \in \mathbb{R}^{n\times 2n}, i = 0,1, \tag{9}$$

satisfying

$$\begin{bmatrix} T_0^{\mathrm{T}}(\theta,x) & T_1^{\mathrm{T}}(\theta,x) \end{bmatrix} A_c(\theta,x) = F \begin{bmatrix} T_0^{\mathrm{T}}(\theta,x) & T_1^{\mathrm{T}}(\theta,x) \end{bmatrix} E_c(\theta,x), \tag{10}$$

if and only if

$$F^2 T^{\mathrm{T}}(\theta,x) A_2(\theta,x) + F T^{\mathrm{T}}(\theta,x) A_1^c(\theta,x) + T^{\mathrm{T}}(\theta,x) A_0^c(\theta,x) = 0, \tag{11}$$

and

$$\begin{cases} T_0^{\mathrm{T}}(\theta,x) = F T^{\mathrm{T}}(\theta,x) A_2(\theta,x) + T^{\mathrm{T}}(\theta,x) A_1^c(\theta,x) \\ T_1^{\mathrm{T}}(\theta,x) = T^{\mathrm{T}}(\theta,x) \end{cases}. \tag{12}$$

Lemma 2.2. *The closed-loop system matrix pair $(E_c(\theta,x), A_c(\theta,x))$ is given by (8). There exists the following right closed-loop eigenvector matrice*

$$V_c(\theta,x) = \begin{bmatrix} V_0^{\mathrm{T}}(\theta,x) & V_1^{\mathrm{T}}(\theta,x) \end{bmatrix}^{\mathrm{T}}, V_i^{\mathrm{T}}(\theta,x) \in \mathbb{R}^{n\times 2n}, i = 0,1, \tag{13}$$

satisfying

$$A_c(\theta,x) \begin{bmatrix} V_0(\theta,x) \\ V_1(\theta,x) \end{bmatrix} = E_c(\theta,x) \begin{bmatrix} V_0(\theta,x) \\ V_1(\theta,x) \end{bmatrix} F, \tag{14}$$

if and only if

$$A_2(\theta,x)V(\theta,x)F^2 + A_1^c(\theta,x)V(\theta,x)F + A_0^c(\theta,x)V(\theta,x) = 0, \tag{15}$$

and

$$\begin{cases} V_0(\theta,x) = V(\theta,x) \\ V_1(\theta,x) = V(\theta,x)F \end{cases}. \tag{16}$$

2.2. *Problem Statement*

Based on the above discussion, the parametric control of second-order systems with time-varying coffecients (2) via output feedback control (4) can be stated as follows.

Problem 2.3 (ESAO)**.** *Given the system (2) satisfying Assumptions 1 and 2, and an arbitrary constant matrix $F \in \mathbb{R}^{2n\times 2n}$, find a pair of eigenvector matrices $T_c(\theta,x), V_c(\theta,x) \in \mathbb{R}^{2n\times 2n}$, and the gain matrices of output feedback $K_0(\theta,q,\dot{q}) \in \mathbb{R}^{r\times m_0}$ and $K_1(\theta,q,\dot{q}) \in \mathbb{R}^{r\times m_1}$ such that*

$$T_c^{\mathrm{T}}(\theta,x)E_c(\theta,x)V_c(\theta,x) = I_{2n}, \tag{17}$$

and

$$T_c^{\mathrm{T}}(\theta,x)A_c(\theta,x)V_c(\theta,x) = F. \tag{18}$$

3. Solution to Problem ESAO

There exists the following time-varying right coprime factorization (RCF)

$$\left[s^2A_2(\theta,x) + sA_1(\theta,x) + A_0(\theta,x)\right]^{-1} B(\theta,x) = N(\theta,x,s)D^{-1}(\theta,x,s), \tag{19}$$

where $N(\theta,x,s) \in \mathbb{R}^{n\times r}[s]$ and $D(\theta,x,s) \in \mathbb{R}^{r\times r}[s]$ are a pair of polynomial matrices. Denote $D(\theta,x,s) = [d_{ij}(\theta,x,s)]_{r\times r}$ and

$$\omega = \max\left\{\deg(d_{ij}(\theta,x,s)), i = 1,2,\ldots,r, j = 1,2,\ldots,r\right\},$$

then $N(\theta,x,s)$ and $D(\theta,x,s)$ can be represented in the following form

$$\begin{cases} N(\theta,x,s) = \sum\limits_{i=0}^{\omega} N_i(\theta,x)s^i \\ D(\theta,x,s) = \sum\limits_{i=0}^{\omega} D_i(\theta,x)s^i. \end{cases} \tag{20}$$

There also exists the following time-varying right coprime factorization (RCF)

$$\left[s^2A_2^{\mathrm{T}}\theta,x) + sA_1^{\mathrm{T}}(\theta,x) + A_0^{\mathrm{T}}(\theta,x)\right]^{-1} \begin{bmatrix} C_0^{\mathrm{T}}(\theta,x) & sC_1^{\mathrm{T}}(\theta,x) \end{bmatrix} = H(\theta,x,s)L^{-1}(\theta,x,s), \tag{21}$$

where $H(\theta, x, s) \in \mathbb{R}^{n\times m}[s]$ and $L(\theta, x, s) \in \mathbb{R}^{m\times m}[s]$, $m = m_0 + m_1$ are a pair of polynomial matrices. Denote $L(\theta, x, s) = [l_{ij}(\theta, x, s)]_{m\times m}$ and

$$\tau = \max\left\{\deg(l_{ij}(\theta, x, s)), i = 1, 2, \ldots, m, j = 1, 2, \ldots, m\right\},$$

then $H(\theta, x, s)$ and $L(\theta, x, s)$ can be represented in the following form

$$\begin{cases} H(\theta, x, s) = \sum_{i=0}^{\tau} H_i(\theta, x)s^i \\ L(\theta, x, s) = \begin{bmatrix} L_0(\theta, x, s) \\ L_1(\theta, x, s) \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{\tau} L_{0i}(\theta, x)s^i \\ \sum_{i=0}^{\tau} L_{1i}(\theta, x)s^i \end{bmatrix}. \end{cases} \tag{22}$$

where $L_0(\theta, x, s) \in \mathbb{R}^{m_0\times m}[s]$ and $L_1(\theta, x, s) \in \mathbb{R}^{m_1\times m}[s]$.

3.1. *Case of F Arbitrary*

With the above deduction, we give the following theorem regarding to the Problem 2.3 (EASO).

Theorem 3.1. *Let $N(\theta, x, s), D(\theta, x, s)$ and $H(\theta, x, s), L(\theta, x, s)$ be two pair of polynomial matrices satisfying RCF (19) and (21), respectively.*

1. Problem EASO has a solution if and only if there exist two matrices $Z_b \in \mathbb{R}^{m\times 2n}$ and $Z_c \in \mathbb{R}^{r\times 2n}$ satisfying

$$\begin{bmatrix} I & F & \cdots & F^{\tau} \end{bmatrix} \Phi(Z_b, Z_c) \begin{bmatrix} I \\ F \\ \vdots \\ F^{\omega} \end{bmatrix} = I_{2n}, \tag{23}$$

where

$$\Phi(Z_b, Z_c) = \left[\Phi_{ij}(Z_b, Z_c)\right]_{\tau n\times\omega n}, \tag{24}$$

with

$$\begin{aligned} \Phi_{ij}(Z_b, Z_c) = {} & F Z_b^{\mathrm{T}} H_{i-1}^{\mathrm{T}}(\theta, x) A_2 N_{j-1}(\theta, x) Z_c + Z_b^{\mathrm{T}} H_{i-1}^{\mathrm{T}}(\theta, x) A_1 N_{j-1}(\theta, x) Z_c \\ & - Z_b^{\mathrm{T}} L_{1(i-1)}^{\mathrm{T}}(\theta, x) C_1 N_{j-1}(\theta, x) Z_c + Z_b^{\mathrm{T}} H_{i-1}(\theta, x)^{\mathrm{T}} A_2 N_{j-1}(\theta, x) Z_c F, \\ & i = 1, 2, \ldots, \omega + 1, j = 1, 2, \ldots, \tau + 1. \end{aligned} \tag{25}$$

2. When the above condition is met, the left and right eigenvector matrices $T_c(\theta, x)$ and $V_c(\theta, x)$ can be obtained as

$$T_c(Z_b, F) = \begin{bmatrix} A_2(\theta, x)^{\mathrm{T}} T(Z_b, F) F^{\mathrm{T}} + A_1^{\mathrm{T}}(\theta, x) T(Z_b, F) - C_1^{\mathrm{T}}(\theta, x) W_b(Z_b, F) \\ T(Z_b, F) \end{bmatrix}, \tag{26}$$

and

$$V_c(Z_c, F) = \begin{bmatrix} V(Z_c, F) \\ V(Z_c, F)F \end{bmatrix}, \tag{27}$$

where

$$\begin{cases} T(Z_b, F) = H_0(\theta, x)Z_b + H_1(\theta, x)Z_bF^{\mathrm{T}} + \cdots + H_\tau(\theta, x)Z_b(F^{\mathrm{T}})^\tau \\ V(Z_c, F) = N_0(\theta, x)Z_c + N_1(\theta, x)Z_cF + \cdots + N_\omega(\theta, x)Z_cF^\omega \end{cases}. \tag{28}$$

while all the general parametric solutions for the output feedback matrices $K_0(\theta, x)$ and $K_1(\theta, x)$ can be obtained as either

$$\begin{aligned} &\begin{bmatrix} K_1(\theta, x) & K_2(\theta, x) \end{bmatrix} \\ =&W_c(Z_c, F)\left(C(\theta, x)V_c(Z_c, F)\right)^{\mathrm{T}}\left(C(\theta, x)V_c(Z_c, F)(C(\theta, x)V_c(Z_c, F))^{\mathrm{T}}\right)^{-1}, \end{aligned} \tag{29}$$

or

$$\begin{aligned} &\begin{bmatrix} K_1(\theta, x) & K_2(\theta, x) \end{bmatrix} \\ =&\left(B^{\mathrm{T}}(\theta, x)T(Z_b, F)T^{\mathrm{T}}(Z_b, F)B(\theta, x)\right)^{-1}B^{\mathrm{T}}(\theta, x)T(Z_b, F)W_b^{\mathrm{T}}(Z_b, F), \end{aligned} \tag{30}$$

where

$$C(\theta, x) = \mathrm{diag}(C_0(\theta, x), C_1(\theta, x)), \tag{31}$$

and

$$W_b^{\mathrm{T}}(Z_b, F) = \begin{bmatrix} W_{b0}^{\mathrm{T}}(Z_b, F) & W_{b1}^{\mathrm{T}}(Z_b, F) \end{bmatrix}, \tag{32}$$

$$\begin{cases} W_{b0}(Z_b, F) = L_{00}(\theta, x)Z_b + L_{01}(\theta, x)Z_bF^{\mathrm{T}} + \cdots + L_{0\tau}(\theta, x)Z_b(F^{\mathrm{T}})^\tau \\ W_{b1}(Z_b, F) = L_{10}(\theta, x)Z_b + L_{11}(\theta, x)Z_bF^{\mathrm{T}} + \cdots + L_{1\tau}(\theta, x)Z_b(F^{\mathrm{T}})^\tau \\ W_c(Z_c, F) = D_0(\theta, x)Z_c + D_1(\theta, x)Z_cF + \cdots + D_\omega(\theta, x)Z_cF^\omega \end{cases}. \tag{33}$$

where $Z_b \in \mathbb{R}^{m\times 2n}$ and $Z_c \in \mathbb{R}^{r\times 2n}$ are two groups of arbitrary parameter vectors that represent the degrees of freedom in the solutions.

Proof. The process of proof is divided into three steps.

Step 1. Obtain the parametric expressions of (28) and (33).

Substituting the expressions of $A_0^c(\theta, x)$ and $A_1^c(\theta, x)$ in (6) into (11) and (15), respectively, yields

$$\begin{aligned} &F^2T^{\mathrm{T}}(\theta, x)A_2(\theta, x) + FT^{\mathrm{T}}(\theta, x)A_1(\theta, x) + T^{\mathrm{T}}(\theta, x)A_0(\theta, x) \\ &\quad = FT^{\mathrm{T}}(\theta, x)B(\theta, x)K_1(\theta, x)C_1(\theta, x) + T^{\mathrm{T}}(\theta, x)B(\theta, x)K_0(\theta, x)C_0(\theta, x), \end{aligned} \tag{34}$$

and

$$\begin{aligned} &A_2(\theta,x)V(\theta,x)F^2 + A_1(\theta,x)V(\theta,x)F + A_0(\theta,x)V(\theta,x) \\ &\qquad = B(\theta,x)K_1(\theta,x)C_1(\theta,x)V(\theta,x)F + B(\theta,x)K_0(\theta,x)C_0(\theta,x)V(\theta,x). \end{aligned} \tag{35}$$

Let

$$W_{b0}^{\mathrm{T}}(\theta,x) = T^{\mathrm{T}}(\theta,x)B(\theta,x)K_0(\theta,x), W_{b1}^{\mathrm{T}}(\theta,x) = T^{\mathrm{T}}(\theta,x)B(\theta,x)K_1(\theta,x), \tag{36}$$

and

$$\begin{aligned} W_c(\theta,x) &= K_1(\theta,x)C_1(\theta,x)V(\theta,x)F + K_0(\theta,x)C_0(\theta,x)V(\theta,x) \\ &= K(\theta,x)C(\theta,x)V_c(\theta,x), \end{aligned} \tag{37}$$

where

$$K(\theta,x) = \left[\begin{array}{cc} K_0(\theta,x) & K_1(\theta,x) \end{array}\right],$$

then (34) and (35) become the second-order generalized Sylvester equations

$$\begin{aligned} F^2T^{\mathrm{T}}(\theta,x)A_2(\theta,x) + &FT^{\mathrm{T}}(\theta,x)A_1(\theta,x) + T^{\mathrm{T}}(\theta,x)A_0(\theta,x) \\ &= FW_{b1}^{\mathrm{T}}(\theta,x)C_1(\theta,x) + W_{b0}^{\mathrm{T}}(\theta,x)C_0(\theta,x), \end{aligned} \tag{38}$$

and

$$A_2(\theta,x)V(\theta,x)F^2 + A_1(\theta,x)V(\theta,x)F + A_0(\theta,x)V(\theta,x) = B(\theta,x)W_c(\theta,x). \tag{39}$$

Therefore, using the general solution to the second-order generalized Sylvester equation (Duan (2015)), we can obtain the parametric solutions as given in (28) and (33).

Step 2. Derive the equation (23).

Consider equation (12), the definitions of $A_1^c(\theta,x)$ and $W_{b1}^{\mathrm{T}}(\theta,x)$, we have

$$\begin{cases} T_1 = T(Z_b,F) \\ T_0 = A_2^{\mathrm{T}}(\theta,x)T(Z_b,F)F^{\mathrm{T}} + A_1^{\mathrm{T}}(\theta,x)T(Z_b,F) - C_1^{\mathrm{T}}(\theta,x)W_{b1}(Z_b,F) \end{cases}, \tag{40}$$

and from (16), we have

$$\begin{cases} V_0 = V(Z_c,F) \\ V_1 = V(Z_c,F)F \end{cases}. \tag{41}$$

Combine equations (17), (8), (40) and (41), we can obtain

$$\begin{aligned} &FT^{\mathrm{T}}(Z_b,F)A_2V(Z_c,F) + T^{\mathrm{T}}(Z_b,F)A_1V(Z_c,F) \\ &\qquad - W_{b1}^{\mathrm{T}}(Z_b,F)C_1V(Z_c,F) + T^{\mathrm{T}}(Z_b,F)A_2V(Z_c,F)F = I_{2n}. \end{aligned} \tag{42}$$

Substitute the parametric solutions in (28) and (33) into the above equation, yields

$$\begin{aligned}
&F\sum_{i=0}^{\tau}F^{i}Z_b^{\mathrm{T}}H_i^{\mathrm{T}}(\theta,x)A_2\sum_{j=0}^{\omega}N_j(\theta,x)Z_cF^{j}+\sum_{i=0}^{\tau}F^{i}Z_b^{\mathrm{T}}H_i^{\mathrm{T}}(\theta,x)A_1\sum_{j=0}^{\omega}N_j(\theta,x)Z_cF^{j}\\
&\quad-\sum_{i=0}^{\tau}F^{i}Z_b^{\mathrm{T}}L_{1i}^{\mathrm{T}}(\theta,x)C_1\sum_{j=0}^{\omega}N_j(\theta,x)Z_cF^{j}+\sum_{i=0}^{\tau}F^{i}Z_b^{\mathrm{T}}H_i^{\mathrm{T}}(\theta,x)A_2\sum_{j=0}^{\omega}N_j(\theta,x)Z_cF^{j}F=I_{2n}.
\end{aligned} \tag{43}$$

Therefore, the above equation is equivalently written as in (23).

Step 3. Derive the parametric solutions of output feedback matrices $K_0(\theta,x)$ and $K_1(\theta,x)$ in (29) or (30).

When the second-order generalized Sylvester equations (38) and (39) have solutions as given in (28) and (33), the gain matrix $K_0(\theta,x)$ and $K_1(\theta,x)$ can be solved either from (37) as in (29) or from (36) as in (30) directly. Now we need to prove (36) and (37) have a common solution $[K_0(\theta,x)\quad K_1(\theta,x)]$ if and only if the following equation holds

$$T^{\mathrm{T}}(\theta,x)B(\theta,x)W_c(\theta,x)=W_b^{\mathrm{T}}(\theta,x)C(\theta,x)V_c(\theta,x). \tag{44}$$

Noting (12), we obtain

$$\begin{aligned}
T^{\mathrm{T}}A_1&=W_{b1}^{\mathrm{T}}C_1+T_0^{\mathrm{T}}-FT^{\mathrm{T}}A_2,\\
T^{\mathrm{T}}A_0&=W_{b0}^{\mathrm{T}}C_0-FT_0^{\mathrm{T}}.
\end{aligned}$$

Further, consider the equation (17), we have

$$\begin{aligned}
T^{\mathrm{T}}BW_c&=T^{\mathrm{T}}A_2VF^2+T^{\mathrm{T}}A_1VF+T^{\mathrm{T}}A_0V\\
&=T_1^{\mathrm{T}}A_2VF^2+W_{b1}^{\mathrm{T}}C_1VF+T_0^{\mathrm{T}}VF-FT_1^{\mathrm{T}}A_2VF+W_{b0}^{\mathrm{T}}C_0V-FT_0^{\mathrm{T}}V\\
&=(T_1^{\mathrm{T}}A_2VF+T_0^{\mathrm{T}}V)F-F(T_1^{\mathrm{T}}A_2VF+T_0^{\mathrm{T}}V)+\begin{bmatrix}W_{b0}^{\mathrm{T}} & W_{b1}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}C_0 & 0\\ 0 & C_1\end{bmatrix}\begin{bmatrix}V\\ VF\end{bmatrix}\\
&=F-F+W_b^{\mathrm{T}}CV_c\\
&=W_b^{\mathrm{T}}CV_c
\end{aligned}$$

With the above deduction, the proof is completed. □

3.2. *Case of F Diagonal*

In fact, the matrix F takes to be the diagonal form in many applications

$$F=\operatorname{diag}(s_1,s_2,\cdots,s_{2n}), \tag{45}$$

where $s_i\in\mathbb{C}^{-}$, $i=1,2,\dots,2n$ are a group of self-conjugate complex poles. In this situation, the general solutions to the second-order generalized Sylvester equation (38) can be given as

$$\begin{cases}T=\begin{bmatrix}t_1 & t_2 & \cdots & t_{2n}\end{bmatrix}\\ t_i=H(\theta,x,s_i)z_i^b, i=1,2,\dots,2n\end{cases}, \tag{46}$$

and

$$\begin{cases}W_b=\begin{bmatrix}w_1^b & w_2^b & \cdots & w_{2n}^b\end{bmatrix}\\ w_i^b=L(\theta,x,s_i)z_i^b, i=1,2,\dots,2n\end{cases}, \tag{47}$$

and also the second-order generalized Sylvester equation (39) can be given as

$$\begin{cases} V = \left[\begin{array}{cccc} v_1 & v_2 & \cdots & v_{2n} \end{array} \right] \\ v_i = N(\theta, x, s_i) z_i^c, i = 1, 2, \ldots, 2n \end{cases}, \tag{48}$$

and

$$\begin{cases} W_c = \left[\begin{array}{cccc} w_1^c & w_2^c & \cdots & w_{2n}^c \end{array} \right] \\ w_i^c = D(\theta, x, s_i) z_i^c, i = 1, 2, \ldots, 2n \end{cases}, \tag{49}$$

where $z_i^b \in \mathbb{R}^m$ and $z_i^c \in \mathbb{R}^r$, $i = 1, 2, \ldots, 2n$, are two groups of arbitrary parameter vectors that represent the degrees of freedom in the solutions.

Following the above deduction, in the case of the matrix F is diagonal, we propose the following theorem regarding to the Problem 2.3 (EASO).

Theorem 3.2. *Let $N(\theta, x, s), D(\theta, x, s)$ and $H(\theta, x, s), L(\theta, x, s)$ be two pair of polynomial matrices satisfying RCF (19) and (21), respectively, and the matrix F takes the form of (45).*

1. Problem EASO has a solution if and only if there exist two groups of parameter vectors $z_i^b \in \mathbb{R}^m$ and $z_i^c \in \mathbb{R}^r$, $i = 1, 2, \ldots, 2n$, satisfying

$$\begin{aligned} s_i {z_i^b}^{\mathrm{T}} H^{\mathrm{T}}(s_i) A_2 N(s_j) z_j^c + {z_i^b}^{\mathrm{T}} H^{\mathrm{T}}(s_i) A_1 N(s_j) z_j^c - {z_i^b}^{\mathrm{T}} L_1^{\mathrm{T}}(s_i) C_1 N(s_j) z_j^c \\ + {z_i^b}^{\mathrm{T}} H^{\mathrm{T}}(s_i) A_2 N(s_j) z_j^c s_j = \delta_{ij}, i, j = 1, 2, \ldots, 2n, \end{aligned} \tag{50}$$

where δ_{ij} are the elements of the identity matrix, that is, for $i = j, \delta_{ij} = 1$; otherwise $\delta_{ij} = 0$.

2. When the above condition is met, the left and right eigenvector matrices $T_c(\theta, x)$ and $V_c(\theta, x)$ can be parametrized by columns as

$$\begin{cases} T_{ci}(\{s_k, z_k^b\}) = \left[\begin{array}{c} A_2^{\mathrm{T}} H(\theta, x, s_i) z_i^b s_i + A_1^{\mathrm{T}} H(\theta, x, s_i) z_i^b - C_1^{\mathrm{T}} L_1(\theta, x, s_i) z_i^b \\ H(\theta, x, s_i) z_i^b \end{array} \right] \\ V_{ci}(\{s_k, z_k^c\}) = \left[\begin{array}{c} N(\theta, x, s_i) z_i^c \\ N(\theta, x, s_i) z_i^c s_i \end{array} \right], i = 1, 2, \ldots, 2n \end{cases}, \tag{51}$$

while all the general parametric solutions for the output feedback matrices $K_0(\theta, x)$ and $K_1(\theta, x)$ can be obtained as either (29) or (30), and the matrices W_b and W_c can be parametrized by columns as

$$\begin{cases} W_{bi}(\{s_k, z_k^b\}) = L(\theta, x, s_i) z_i^b \\ W_{ci}(\{s_k, z_k^c\}) = D(\theta, x, s_i) z_i^c, i = 1, 2, \ldots, 2n \end{cases}, \tag{52}$$

where $z_i^b \in \mathbb{R}^m$ and $z_i^c \in \mathbb{R}^r, i = 1, 2, \ldots, 2n$, are two groups of arbitrary parameter vectors satisfying condition (50).

Proof. According to Theorem 3.1, when the matrix F takes the diagonal form of (45), the matrices T_c, V_c, W_b and W_c can be the form of columns given by (51) and (52). It is easy to prove the results of Theorem 3.2. □

3.3. *General Procedure*

Based on Theorem 3.1 and 3.2, we can propose a general procedure for solving the parametric forms of output feedback control of the second-order system with time-varying coefficients of (2).

Step 1 Design the structure of the constant matrix F.

We choose the matrix F in a Jordan form or a diagonal form. It is required that matrix F is Hurwitz, that is, the eigenvalues of the matrix F lie in the left-half of the $s-$ plane

$$\lambda_i(F) \in \mathbb{C}^-, i = 1, 2, \ldots, 2n. \tag{53}$$

Step 2 Obtain two pairs of right coprime factorizations (RCF) $\{N(\theta, x, s), D(\theta, x, s)\}$ and $\{H(\theta, x, s), L(\theta, x, s)\}$.

From the right coprime factorizations (19) and (21), two pairs of particular solutions can be given by

$$\begin{cases} N(s) = \mathrm{adj}(s^2 A_2(\theta, x) + sA_1(\theta, x) + A_0(\theta, x))B(\theta, x) \\ D(s) = \det(s^2 A_2(\theta, x) + sA_1(\theta, x) + A_0(\theta, x))I_n \end{cases}, \tag{54}$$

and

$$\begin{cases} H(s) = \mathrm{adj}(s^2 A_2^{\mathrm{T}}(\theta, x) + sA_1^{\mathrm{T}}(\theta, x) + A_0^{\mathrm{T}}(\theta, x)) \begin{bmatrix} C_0^{\mathrm{T}}(\theta, x) & sC_1^{\mathrm{T}}(\theta, x) \end{bmatrix} \\ L(s) = \det(s^2 A_2^{\mathrm{T}}(\theta, x) + sA_1^{\mathrm{T}}(\theta, x) + A_0^{\mathrm{T}}(\theta, x))I_n \end{cases}. \tag{55}$$

Step 2 Compute the output feeback gain matrices $[K_0(\theta, x) \quad K_1(\theta, x)]$.

Compute the output feedback gain matrices $[K_0(\theta, x) \quad K_1(\theta, x)]$ through the formulas (29) or (30), by using the matrice T_c, V_c, W_b and W_c solved by (26)-(33) or (51)-(52).

4. An Example - Spacecraft Rendezvous Problem

4.1. *Solution to Spacecraft Rendezvous Problem*

Consider the spacecraft rendezvous problem as shown in the Figure 1, when the chaser and the target spacecraft are relatively close to each other, we can get simple linear equations for the chasers relative motion, linking the chasers relative position, velocity and acceleration as follows

$$\begin{bmatrix} \ddot{x}_r \\ \ddot{y}_r \\ \ddot{z}_r \end{bmatrix} - \begin{bmatrix} 2k\dot{\theta}^{\frac{3}{2}} x_r + 2\dot{\theta}\dot{y}_r + \dot{\theta}^2 x_r + \ddot{\theta} y_r \\ -k\dot{\theta}^{\frac{3}{2}} y_r - 2\dot{\theta}\dot{x}_r + \dot{\theta}^2 y_r - \ddot{\theta} x_r \\ -k\dot{\theta}^{\frac{3}{2}} z_r \end{bmatrix} = u, \tag{56}$$

where $k = \frac{\mu}{\mathcal{C}^{\frac{3}{2}}} =$ constant, $\mathcal{C}$ is the orbital angular momentum of target, μ is a gravitational parameter; θ is the true anomaly; x_r, y_r and z_r indicate the radial, along-track and out of plane components of the position vector of the chaser satellite in the target satellites local-vertical-local-horizontal (LVLH) frame, respectively. The

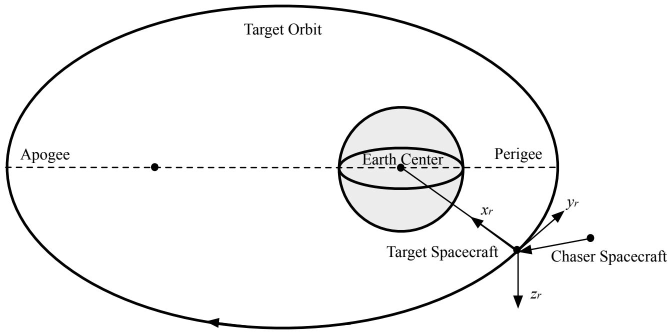

Figure 1. Spacecraft rendezvous.

above equation system described in (56) is known as Lawden's equations (Lawden (1963)) or Tschauner-Hempel equations (Tschauner & Hempel (1965)).

The relative motion equation (56) can be rewritten as the (2) form with

$$q = \begin{bmatrix} x_r & y_r & z_r \end{bmatrix}^{\mathrm{T}},$$

$$\begin{aligned} A_2(\theta, q, \dot{q}) &= B(\theta, q, \dot{q}) = I_3, \\ A_1(\theta, q, \dot{q}) &= \begin{bmatrix} 0 & -2\dot{\theta} & 0 \\ 2\dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ A_0(\theta, q, \dot{q}) &= \begin{bmatrix} -2k\dot{\theta}^{\frac{3}{2}} - \dot{\theta}^2 & -\ddot{\theta} & 0 \\ \ddot{\theta} & k\dot{\theta}^{\frac{3}{2}} - \dot{\theta}^2 & 0 \\ 0 & 0 & k\dot{\theta}^{\frac{3}{2}} \end{bmatrix}, \\ C_0(\theta, q, \dot{q}) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C_1(\theta, q, \dot{q}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \end{aligned} \tag{57}$$

We can easily deduce two right coprime factorizations (RCF) (19) and (21) obviously hold for (57) and

$$\begin{cases} N(\theta, s) = I_3 \\ D(\theta, s) = \begin{bmatrix} s^2 - 2k\dot{\theta}^{\frac{3}{2}} - \dot{\theta}^2 & -2\dot{\theta}s - \ddot{\theta} & 0 \\ 2\dot{\theta}s + \ddot{\theta} & s^2 + k\dot{\theta}^{\frac{3}{2}} - \dot{\theta}^2 & 0 \\ 0 & 0 & s^2 + k\dot{\theta}^{\frac{3}{2}} \end{bmatrix} \end{cases}, \tag{58}$$

and

$$\begin{cases} H(\theta, s) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \\ L(\theta, s) = \begin{bmatrix} L_0(\theta, s) \\ L_1(\theta, s) \end{bmatrix} \\ \quad = \begin{bmatrix} s^2 - 2k\dot{\theta}^{\frac{3}{2}} - \dot{\theta}^2 & 2\dot{\theta}s + \ddot{\theta} & 0 & -s & 0 & 0 \\ -2\dot{\theta}s - \ddot{\theta} & s^2 + k\dot{\theta}^{\frac{3}{2}} - \dot{\theta}^2 & 0 & 0 & 0 & -s \\ 0 & 0 & s^2 + k\dot{\theta}^{\frac{3}{2}} & 0 & -s & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{cases}. \tag{59}$$

Let

$$\begin{cases} F = \operatorname{diag}(s_1, s_2, s_3, s_4, s_5, s_6) = \operatorname{diag}(-0.01, -0.02, -0.03, -0.04, -0.05, -0.06) \\ Z_c = \begin{bmatrix} z_1^c & z_2^c & \cdots & z_{2n}^c \end{bmatrix} = \begin{bmatrix} I_3 & I_3 \end{bmatrix} \end{cases}, \tag{60}$$

then the condition (17) can be transformed into the form of columns

$$V_c^{\mathrm{T}}(Z_c, F) E_c^{\mathrm{T}} T_{ci} = e_i, \tag{61}$$

where T_{ci} is given by (51), and $e_i, i = 1, 2, \ldots, 6$, are the i-th column of identity matrix.

If

$$\operatorname{rank}\left(V_c^{\mathrm{T}}(Z_c, F) E_c^{\mathrm{T}} \begin{bmatrix} A_2^{\mathrm{T}} H(\theta, x, s_i) s_i + A_1^{\mathrm{T}} H(\theta, x, s_i) - C_1^{\mathrm{T}} L_1(\theta, x, s_i) \\ H(\theta, x, s_i) \end{bmatrix} \right) = 6,$$

we can easily obtain the z_i^b as follows

$$z_i^b = \left(V_c^{\mathrm{T}}(Z_c, F) E_c^{\mathrm{T}} \begin{bmatrix} A_2^{\mathrm{T}} H(\theta, x, s_i) s_i + A_1^{\mathrm{T}} H(\theta, x, s_i) - C_1^{\mathrm{T}} L_1(\theta, x, s_i) \\ H(\theta, x, s_i) \end{bmatrix} \right)^{-1} e_i,$$
$$i = 1, 2, \ldots, 6.$$

With the above sets of parameters, we can get

$$\begin{aligned} Z_b &= \begin{bmatrix} z_1^b & z_2^b & \cdots & z_{2n}^b \end{bmatrix} \\ &= \begin{bmatrix} 100/3 & 0 & 0 & -100/3 & 0 & 0 \\ 0 & 100/3 & 0 & 0 & -100/3 & 0 \\ 0 & 0 & 100/3 & 0 & 0 & -100/3 \\ -5/3 & 200\dot{\theta}/3 & 0 & 5/3 & -200\dot{\theta}/3 & 0 \\ 0 & 0 & -3 & 0 & 0 & 3 \\ -200\dot{\theta}/3 & -7/3 & 0 & 200\dot{\theta}/3 & 7/3 & 0 \end{bmatrix}. \end{aligned} \tag{62}$$

Further, we have

$$T = 100 \begin{bmatrix} 1/3 & 0 & 0 & -1/3 & 0 & 0 \\ 0 & 1/3 & 0 & 0 & -1/3 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & -1/3 \end{bmatrix}, V = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

$$W_b = \left[\begin{matrix} -200k\dot{\theta}^{\frac{3}{2}}/3 - 100\dot{\theta}^2/3 - 1/75 & 100\ddot{\theta}/3 & 0 \\ -100\ddot{\theta}/3 & 100k\dot{\theta}^{\frac{3}{2}}/3 - 100\dot{\theta}^2/3 - 1/30 & 0 \\ 0 & 0 & 100k\dot{\theta}^{\frac{3}{2}}/3 - 3/50 \\ -5/3 & 200\dot{\theta}/3 & 0 \\ 0 & 0 & -3 \\ -200\dot{\theta}/3 & -7/3 & 0 \end{matrix} \right.$$
$$\left. \begin{matrix} 200k\dot{\theta}^{\frac{3}{2}}/3 + 100\dot{\theta}^2/3 + 1/75 & -100\ddot{\theta}/3 & 0 \\ 100\ddot{\theta}/3 & -100k\dot{\theta}^{\frac{3}{2}}/3 + 100\dot{\theta}^2/3 + 1/30 & 0 \\ 0 & 0 & -100k\dot{\theta}^{\frac{3}{2}}/3 + 3/50 \\ 5/3 & -200\dot{\theta}/3 & 0 \\ 0 & 0 & 3 \\ 200\dot{\theta}/3 & 7/3 & 0 \end{matrix} \right],$$

$$W_c = \left[\begin{matrix} 1/10000 - \dot{\theta}^2 - 2k\dot{\theta}^{\frac{3}{2}} & \dot{\theta}/25 - \ddot{\theta} & 0 \\ \ddot{\theta} - \dot{\theta}/50 & k\dot{\theta}^{\frac{3}{2}} - \dot{\theta}^2 + 1/2500 & 0 \\ 0 & 0 & k\dot{\theta}^{\frac{3}{2}} + 9/10000 \end{matrix} \right.$$
$$\left. \begin{matrix} 1/625 - \dot{\theta}^2 - 2k\dot{\theta}^{\frac{3}{2}} & \dot{\theta}/10 - \ddot{\theta} & 0 \\ \ddot{\theta} - 2\dot{\theta}/25 & k\dot{\theta}^{\frac{3}{2}} - \dot{\theta}^2 + 1/400 & 0 \\ 0 & 0 & k\dot{\theta}^{\frac{3}{2}} + 9/2500 \end{matrix} \right].$$

Thus, we can lead to the output feedback gain matrices $K_0(\theta, x)$ and $K_1(\theta, x)$ based on (29) or (30), respectively

$$\begin{aligned} K_0(\theta, x) &= \begin{bmatrix} -2k\dot{\theta}^{\frac{3}{2}} - \dot{\theta}^2 - 1/2500 & -\ddot{\theta} & 0 \\ \ddot{\theta} & k\dot{\theta}^{\frac{3}{2}} - \dot{\theta}^2 - 1/1000 & 0 \\ 0 & 0 & k\dot{\theta}^{\frac{3}{2}} - 9/5000 \end{bmatrix}, \\ K_1(\theta, x) &= \begin{bmatrix} -1/20 & 0 & -2\dot{\theta} \\ 2\dot{\theta} & 0 & -7/100 \\ 0 & -9/100 & 0 \end{bmatrix}. \end{aligned} \tag{63}$$

and after applying the above output feedback control, the closed-loop system can be obtained

$$\ddot{q} + \begin{bmatrix} 1/20 & 0 & 0 \\ 0 & 7/100 & 0 \\ 0 & 0 & 9/100 \end{bmatrix} \dot{q} + \begin{bmatrix} 1/2500 & 0 & 0 \\ 0 & 1/1000 & 0 \\ 0 & 0 & 9/5000 \end{bmatrix} q = 0, \tag{64}$$

which can be tested to possess the desired eigenvalues (60).

4.2. *Numerical Simulation*

In this subsection, the results of numerical simulations are presented to compare the performance of the proposed output feedback controller (63) with that in Gao, Teo, & Duan (2015). Suppose the orbital parameters of the target spacecraft are as follows: the semimajor axis $a = 24616$ km, the eccentricity $e = 0.73074$, and the period T is 38436s. The main orbital parameters of the target spacecraft are summarized in Table 1.

Table 1. The orbital parameters of the target spacecraft

Parameters	Symbol	Values
Target orbit		
Semimajor axis	a	2.4616×10^7 m
Eccentricity	e	0.73074
Angular momentum	$\mathcal{C}$	6.762×10^{10} m^2/s
Constant k	k	2.267×10^{-2}/s$^{1/2}$
Period	T	$38,436$ s
Gravitational parameter	μ	3.986×10^{14} m^3/s^2

Suppose that at the initial moment, the target spacecraft is located at the perigee, that is, the initial value of the true anomaly is $\theta_0 = 0$. Based on Kepler's Law, the orbital height of target spacecraft is given by

$$r_t = \frac{a(1-e^2)}{1+e\cos\theta}, \tag{65}$$

where r_t is the vector from the center of gravity to the target spacecraft. Further, $\dot{\theta}$ and $\ddot{\theta}$ are the angular velocity and acceleration of target spacecraft can be obtained by, respectively

$$\begin{cases} \dot{\theta} = \sqrt{\dfrac{\mu(1+e\cos\theta)}{r_t^3}}, \\ \ddot{\theta} = \dfrac{-2\mu e\sin\theta}{r_t^3}. \end{cases} \tag{66}$$

From the above equations (66) are periodic with period T, the period of the target orbit. The results are recorded in Fig 2. It follows that both $\dot{\theta}$ and $\ddot{\theta}$ are T periodic.

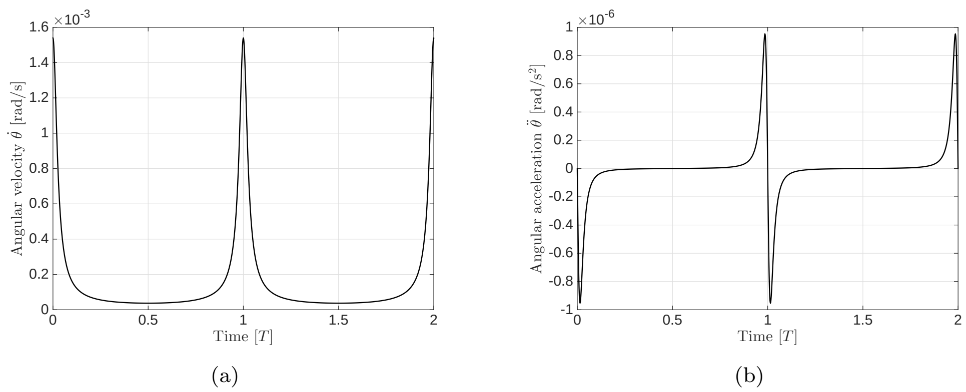

Figure 2. Angular velocity $\dot{\theta}$ and acceleration $\ddot{\theta}$ of target spacecraft.

Choose the initial conditions in the LVLH frame as

$$\begin{cases} q(0) = \left[\begin{array}{ccc} 3000 & -3000 & 3000 \end{array} \right]^{\mathrm{T}} \ \mathrm{m}, \\ \dot{q}(0) = \left[\begin{array}{ccc} 3 & -3 & 3 \end{array} \right]^{\mathrm{T}} \ \mathrm{m/s}. \end{cases} \tag{67}$$

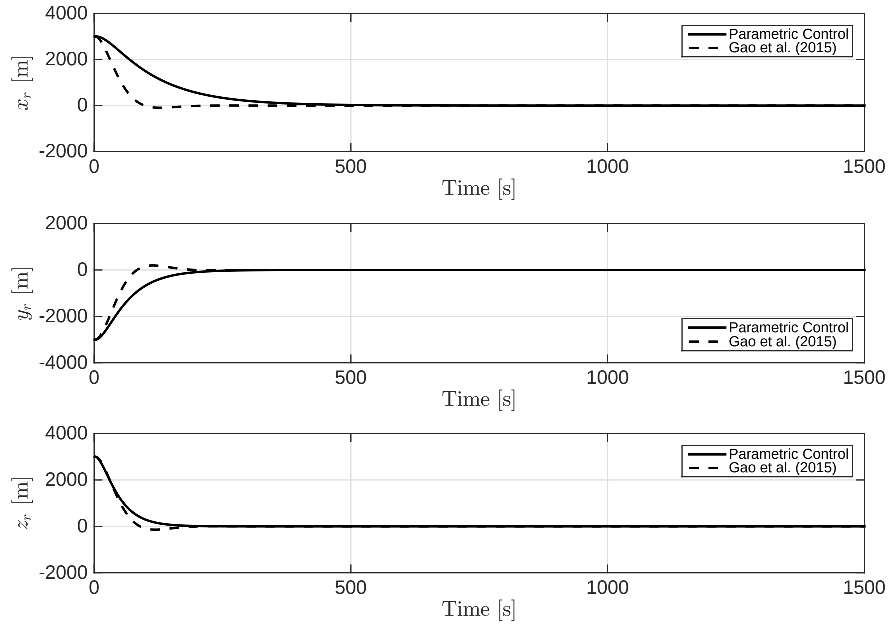

Figure 3. Relative position variables.

The time responses of the relative positions and velocities between the chaser and target spacecraft are shown in Figure 3 and 4. From Figure 3, we can see that the relative positions of the two spacecraft arrive at the coordinate origin by the proposed approach with no overshoot than the cases in Gao et al. (2015). From Figure 4, it can be seen that the maximum of the relative velocities is less than the cases in Gao et al. (2015). From these figures we clearly see that the closed-loop system is asymptotically stable, and, particularly, the parametric control leads to a better transient performance of the closed-loop system than Gao et al. (2015). The control signals of the closed-loop system are shown in Figure 5. It confirms that the proposed approach leads to better transient performances at the cost of less control energy and magnitude of the control signals.

5. Conclusion

This paper has studied the design problem of the output feedback controller for a type of quasi-linear second-order systems with the time-varying coefficient matrices. A general type of quasi-linear second-order systems is proposed, and it is shown that a novel parametric control approach exists, which provides a simple controller parameterization. With this controller, the closed-loop system is a linear constant one with designed eigenstructure. An example of the general spacecraft rendezvous is developed

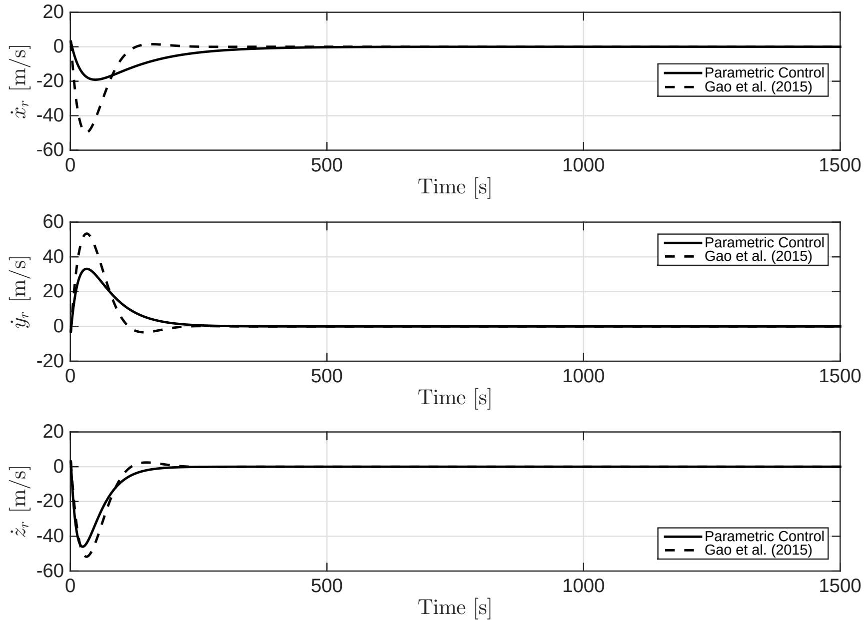

Figure 4. Relative velocity variables.

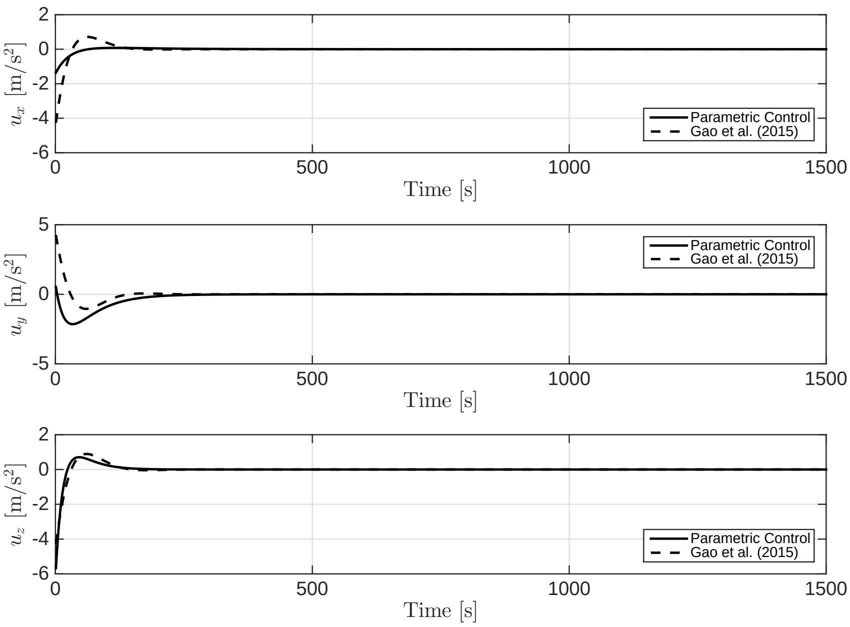

Figure 5. Control signals u_x, u_y and u_z.

to show that the proposed approach can effectually solve the elliptical orbit rendezvous problem. Noted that the proposed method usually needs to obtain accurate mathematical models, which is inevitably conservative, and the next major work is to find robust stability conditions for the uncertain second-order systems.

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