On the Number of Sudoku Grids

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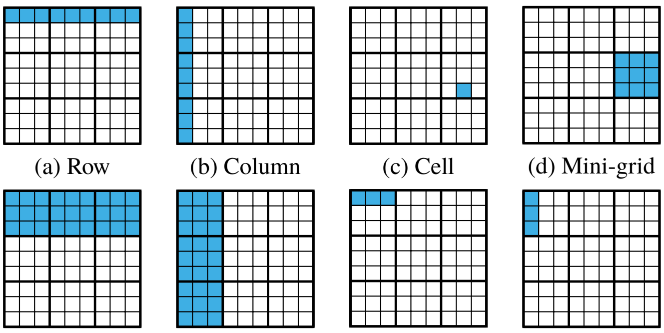
The subject of the number of sudoku grids has sparked much recent attention in *Mathematics Today* (R. Franklin (October 2012), Charles Evans (April 2013) and Ian Stewart (June 2013)). This feature addresses known results for numbers of sudoku grids of different sizes, and explains how a well-formed symmetry group can be used to simplify the enumeration of sudoku grids.

Sudoku is usually thought of as a 9×9 puzzle which is further subdivided into mini-grids of size 3×3, with each of the 81 cells of the grid to be filled with the digits 1 to 9 such that each digit appears exactly once in each row, column and mini-grid. In fact, although published sudoku puzzles are generally 9×9 in size, other dimensions can be used, and for every non-prime dimension n there is an $n\times n$ sudoku grid. However, for some sizes of sudoku more than one size of mini-grid can be chosen. As examples, a 6×6 sudoku (known as rudoku) can have mini-grids of size 3×2 or 2×3 (although these are essentially a rotation from one to the other), and a 12×12 sudoku can have mini-grids of size either 3×4 or 6×2 (leading to very different puzzles). Published puzzles show incomplete grids, with a number of cells pre-filled with fixed, or given digits, chosen to ensure that a solution is unique.



(a) Row (b) Column (c) Cell (d) Mini-grid

(e) Band (f) Stack (g) Tier (h) Pillar

Figure 1: Illustration of terminology used for sudoku grids.

Formally a sudoku grid, $S^{x,y}$, is a $n\times n$ array subdivided into n mini-grids of size $x\times y$ (where $n=xy$); the values $1,\ldots,n$ are contained within the array in such a way that each value occurs exactly once in every *row*, *column* and *mini-grid*. We denote a sudoku grid of size $n\times n$ with mini-grids of size $x\times y$, where $n=xy$ as $S^{x,y}$, and the number of ways of arranging the values in $S^{x,y}$ as $\mathrm{S}^{x,y}(n)$. ${S^{x,y}}_{a,b}$ is a specific mini-grid in band a and stack b of $S^{x,y}$ and $[{S^{x,y}}_{a,b}]_{i,j}$ a specific cell in tier i and pillar j of the mini-grid ${S^{x,y}}_{a,b}$ (see Figure 1).

Counting the number of sudoku grids is known to be a difficult problem, similar in nature to that of counting the number of Latin squares. It is proposed in [1] that adaptations of methods used for counting the number of Latin squares could be used to count the number of sudoku grids. The number of sudoku grids has been calculated (and in most cases verified) for sizes up to 12×12 (with mini-grids of size 3×4). An historical account of this work, which mostly comprises computational calculations rather than mathematical proof, is given in [2]. Mathematical proofs for the numbers of sudoku grids of small sizes have been achieved, with proofs for larger sizes still being elusive. Counting the number of 4×4 sudoku grids is trivial (and a full mathematical enumeration is given as an example in [3]), and the only known non-trivial mathematical enumeration is for 6×6 sudoku grids [4]. The method of counting employed requires categorisation of the arrangements of numbers within partially filled grids. The properties of these partially filled grids are analysed and the number of ways of completing them calculated. A similar counting method is performed to enumerate the first band of a 9×9 sudoku grid in [5], but the number of ways of completing the grid is then calculated computationally.

Sudoku grids possess a great deal of structure and therefore there are a number of symmetry operations applicable. However, as stated in [6] and [7] some of these symmetry operations result in grids which are 'fixed'. A sudoku grid is fixed if the application of some symmetry operation on it results in an identical grid (examples of this are given in [8]). However, there are a number of operations for which no fixed grid can be produced, and these allow us to construct reduced sudoku grids, which are comparable to reduced Latin squares. A reduced Latin square is one in which the values in the first row and first column are in numerical order. The concept of reduction is not as straightforward for sudoku grids, but a similar approach can be adopted [3].

A *reduced sudoku grid*, $s^{x,y}$, is a sudoku grid, $S^{x,y}$, having the following properties:

- the values in ${S^{x,y}}_{1,1}$ are in canonical form, $[{S^{x,y}}_{1,1}]_{i,j} = (i-1)y+j$;
- for each mini-grid ${S^{x,y}}_{1,b}$, for $b=2,\ldots,x$, the values in $[{S^{x,y}}_{1,b}]_{1,j}$ for $j=1,\ldots,y$ are increasing;
- for each mini-grid ${S^{x,y}}_{a,1}$, for $a=2,\ldots,y$, the values in $[{S^{x,y}}_{a,1}]_{i,1}$ for $i=1,\ldots,x$ are increasing;
- $[{S^{x,y}}_{1,b}]_{1,1} < [{S^{x,y}}_{1,b+1}]_{1,1}$ for $b=2,\ldots,x-1$;
- $[{S^{x,y}}_{a,1}]_{1,1} < [{S^{x,y}}_{a+1,1}]_{1,1}$ for $a=2,\ldots,y-1$.

An example of a reduced sudoku grid is given in Figure 2(a), and an isomorphic sudoku grid that can be formed from it (by permuting the values $(1,9,2,5,8)(3,7,6)(4)$ and permuting the rows and columns) is given in Figure 2(b).

Every sudoku grid is isomorphic to exactly one reduced sudoku grid, and hence the number of sudoku grids is a multiple of the number of reduced sudoku grids.

1	2	3	4	5	9	6	7	8
4	5	6	7	1	8	3	9	2
7	8	9	3	6	2	1	5	4
2	9	4	5	7	6	8	1	3
6	7	1	8	4	3	5	2	9
8	3	5	9	2	1	4	6	7
3	6	7	1	9	4	2	8	5
5	4	2	6	8	7	9	3	1
9	1	8	2	3	5	7	4	6

(a) Reduced sudoku grid

9	5	7	6	1	3	2	8	4
4	8	3	2	5	7	1	9	6
6	1	2	8	4	9	5	3	7
1	7	8	3	6	4	9	5	2
5	2	4	9	7	1	3	6	8
3	6	9	5	2	8	7	4	1
8	4	5	7	9	2	6	1	3
2	9	1	4	3	6	8	7	5
7	3	6	1	8	5	4	2	9

(b) Sudoku grid isomorphic to (a)

Figure 2: Example sudoku grid in reduced form and a sudoku grid isomorphic to it.

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Theorem 1 [3] *If* $\mathrm{s}^{x,y}(n)$ *is the number of reduced sudoku grids of size* $n \times n$ *with mini-grids of size* $x \times y$ (*where* $n = xy$) *then* $\mathrm{S}^{x,y}(n)$ *is given by*

$$\mathrm{S}^{x,y}(n) = (n-1)!x!^y y!^x \mathrm{s}^{x,y}(n). \tag{1}$$

For a 9×9 sudoku grid the size of the symmetry group is $1{,}881{,}169{,}920$. In [5], the number of 9×9 sudoku grids was determined to be $6{,}670{,}903{,}752{,}021{,}072{,}936{,}960$. This value is divisible by the size of the symmetry group, suggesting that there are $3{,}546{,}146{,}300{,}288$ reduced sudoku grids.

For many years researchers have attempted to find a general equation to determine the number of Latin squares of any given size. This problem remains open, and this is also the case for sudoku grids. Any progress in either domain will provide insight for the other.

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