Stability Analysis of A Class of Hybrid Stochastic Retarded Systems under Asynchronous Switching

Yu Kang*, Di-Hua Zhai, Guo-Ping Liu, Yun-Bo Zhao, Ping Zhao

***Abstract*—The stability of a class of hybrid stochastic retarded systems (HSRSs) with an asynchronous switching controller is investigated. In this model, the controller design relies on the observed jumping parameters, which are however delayed and thus can not be measured in real-time precisely. This delayed time interval, referred to as the "asynchronous switching interval", is Markovian and dependent on the actual switching signal. The sufficient conditions under which the system is either stochastically asymptotic stable or input-to-state stable are obtained by applying the extended Razumikhin-type theorem to the asynchronous switching interval. These results are less conservative as it is only required that the designed Lyapunov function is non-decreasing. It is shown that the stability of the considered system can be guaranteed by a sufficiently small mode transition rate of the underlying Markov process, which is a conclusion similar to that in asynchronous deterministic switched systems. The effectiveness and correctness of the obtained results are finally verified by a numerical example.**

***Index Terms*—Hybrid stochastic retarded systems, Markovian switching, asynchronous switching, time-delay, Razumikhin-type theorem, stochastic stability.**

I. Introduction

A Switched system consists of a family of subsystems (or, the modes) and a switching signal governing the switches between the modes [1]–[5]. It is called a randomly switched system if the mode switches are governed by a stochastic process statistically independent from the system states and, further, a Markovian jump linear system (MJLS) if the stochastic process is Markovian and the system dynamic is linear [6], [7]. MJLSs have received considerable attentions in theory [8]–[17], and are found to be appropriate models for manufacturing systems [18], power systems [19], robots control systems [20], etc.

For a switched systems, mode-dependent controller has received more and more attention, which is believed to be less conservative. The mode-dependent controller design for switched systems are often assumed to be strictly synchronized [21]–[26], which may not generally hold in reality due to unknown and unpredictable issues such as time-delay, disturbance, component and interconnection failures, etc. Specifically, in practical systems, time-delay often appears in switched systems either in input control or in output measurements, due to the distance between the place where control signal is generated and the place where control signal is applied to the plant as well as significant communication distance between the sensor and the controller. On the other hand, for the mode-dependent controller design, the switching information is necessary. However, due to the existence of environmental noises, disturbances, and small modelling uncertainties, considerable time is needed in the mode detection of the plant. It thus presents a great challenge at the boundary of switched systems and time delay systems, and the concept of asynchronous switching is proposed to deal with this phenomenon. Roughly speaking, the so-called "asynchronous switching" is caused by the detection delay of switching signal which results in the mismatched period of designed controller in each subsystem. The subsystems may be unstable between these mismatched periods. Considerable studies have been reported in this area, for example, state feedback stabilization [27], input-to-state stabilization [28], and output feedback stabilization [29], the use of the average dwell time approach [30]–[34], just to name a few.

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In the past two decades, almost all the research on asynchronous switching systems are for deterministic switched systems while the asynchronous randomly switched systems have received little attention, especially for nonlinear systems. The switching signal's stochastic properties of randomly switched systems lead to the following two difficulties in the analysis of the systems stability. Firstly, since the switching signal is a stochastic process, the methods in deterministic switched systems, e.g., dwell time approach or average dwell time approach, are difficult to be used directly. Secondly, the detected switching signal is still a stochastic process. The relationship between the detected switching signal and the origin switching signal further increases the complexity of the problem. Recently, the asynchronous issues of MJLSs have also been studied [35]–[37]. Among them, [35] and [36] investigated the stability and stabilization problem for a class of discrete-time MJLSs via time-delayed controller. In [37], by defining two Markov processes, the stability of the continuous-time MJLSs with detection delays and false alarms in detected switching signal and discrete-time MJLSs with constant time delays or random communication delays in mode signal are developed. Surprisingly, the studies on the stability analysis for asynchronous stochastic nonlinear systems with Markovian switching are scarce. This motivates our present study.

In this paper, we focus on the stability analysis of a class of hybrid stochastic retarded systems (HSRSs) under asynchronous switching. In HSRSs, each subsystem is described by a stochastic functional differential equation, and the switching rule between those subsystems is a continuous-time Markov process. We will consider the asynchronous case with random detection delay and model the detected switching signal as a Markov process conditional on the real Markovian switching signal. The Razumikhin-type sufficient criteria for globally asymptotically stability in probability (GASiP) [38], α-globally asymptotically stability in the mean (α-GASiM) [39], pth moment exponentially stability [40], stochastic input-to-state stability (SISS) [38], α-input-to-state stability in the mean (α-ISSiM) [39] and pth moment input-to-state stability (pth moment ISS) [41], are given. It is shown that, the stability of HSRSs under asynchronous switching can be guaranteed provided that the mode transition rate is sufficiently small, i.e., a larger instability margin can be compensated for by a smaller transition rate.

The remainder of the paper is organized as follows. The problem is formulated and necessary definitions and lemmas are given in Section II. The global asymptotic stability and input-to-state stability are then discussed in Section III and Section IV, respectively. Then, the main results are extended to a class of hybrid stochastic delay systems and the simulation results are given in Section V. Section VI concludes the paper.

Notations: $\mathbb{N}_+$ and $\mathbb{R}_+$ denote the set of all positive integers and nonnegative real numbers, respectively; $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote n-dimensional real space and $n\times m$ dimensional real matrix space, respectively. For vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm $|x| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$. Let $\tau \geq 0$ and $C([-\tau, 0]; \mathbb{R}^n)$ denote the family of all continuous $\mathbb{R}^n$-valued functions φ on $[-\tau, 0]$ with the norm $\|\varphi\| = \sup\{|\varphi(\theta)| : -\tau \leq \theta \leq 0\}$. Let $C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ be the family of all $\mathcal{F}_0$-measurable bounded $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$. For $p > 0$ and $t \geq 0$, let $L^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$ denote the family of all $\mathcal{F}_t$-measurable $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables $\phi = \{\phi(\theta) : -\tau \leq \theta \leq 0\}$ such that $\sup_{-\tau\leq\theta\leq 0} \mathbb{E}\{|\phi(\theta)|^p\} < \infty$. A^C denotes the complementary set of set A. $C^{i,k}$ denotes all the functions with ith continuously differentiable first component and kth continuously differentiable second component. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}$ if α is continuous, strictly increasing and $\alpha(0) = 0$. And if α is also unbounded, then it is of class $\mathcal{K}_\infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$, if $\beta(\cdot, t)$ is of class $\mathcal{K}$ in the first argument for each fixed $t \geq 0$ and $\beta(s, t)$ decreases to 0 as $t \to +\infty$ for each fixed $s \geq 0$. We denote the class $\mathcal{CK}$ ($\mathcal{CK}_\infty$) function and $\mathcal{VK}$ ($\mathcal{VK}_\infty$) function as the subset of class $\mathcal{K}$ ($\mathcal{K}_\infty$) function, which are concave and convex, respectively. Finally, the composition of two functions $\alpha : A \to B$ and $\beta : B \to C$ is denoted by $\alpha \circ \beta : A \to C$.

II. Preliminaries and Problem Formulation

Consider the following asynchronous Markovian switching nonlinear systems:

$$\begin{cases} \mathrm{d}x(t) = f(t, x_t, \nu(t), r(t))\mathrm{d}t \\ \qquad\qquad + g(t, x_t, \nu(t), r(t))\mathrm{d}w(t) \\ \nu(t) = h(t, x_t, u(t), r'(t)) \end{cases} \tag{1}$$

with the initial state $x_0 = \{x(\theta) : -\tau \leq \theta \leq 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ and $r_0 = r(0) = i_0$, where $x_t = \{x(t+\theta) : -\tau \leq \theta \leq 0\}$ is a $C([-\tau, 0]; \mathbb{R}^n)$-valued random variable. $w(t) = (w_1(t), w_2(t), \ldots, w_m(t))^T$ is a m-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, with Ω being the sample space, $\mathcal{F}$ being a sigma-algebra, $\{\mathcal{F}_t\}_{t\geq 0}$ being a filtration and satisfies the usual conditions and $\mathbb{P}$ being a complete probability measure. $r(t)$ is a right-continuous Markov process on the probability space taking values in a finite state space $\mathcal{S} = \{1, 2, \cdots, N\}$

with generator $\Pi = \{\pi_{ij}\}_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta)=j \mid r(t)=i\} = \begin{cases} \pi_{ij}\Delta + o(\Delta), & i \neq j \\ 1+\pi_{ii}\Delta + o(\Delta), & i = j \end{cases} \tag{2}$$

where $\Delta > 0$ is a sufficiently small positive number, and $\lim_{\Delta\to 0}\frac{o(\Delta)}{\Delta} = 0$. $\pi_{ij} \geq 0$ is the transition rate from i to j ($j \neq i$), and $\pi_{ii} = -\sum_{j=1,j\neq i}^{N}\pi_{ij}$. Let $\bar{\pi} \triangleq \max_{i\in\mathcal{S}}\{|\pi_{ii}|\}$ and $\tilde{\pi} \triangleq \max_{i,j\in\mathcal{S}}\{\pi_{ij}\}$ and assume the Markov process $r(t)$ is independent of the Brownian motion $w(t)$.

In addition, in system (1), $\nu(t) \in \mathcal{L}_\infty^l$ is the asynchronous control input, which relies on the detected switching signal $r'(t)$. $\mathcal{L}_\infty^l$ denotes the set of all the measurable and locally essentially bounded input $\nu(t) \in \mathbb{R}^l$ on $[0,\infty)$ with the norm

$$\begin{cases} \|\nu(s)\| = \inf\limits_{\mathcal{A}\subset\Omega,\mathbb{P}(\mathcal{A})=0} \sup\{|u(\omega,s)| : \omega \in \Omega\backslash\mathcal{A}\} \\ \|\nu(s)\|_{[t_0,\infty)} = \sup\limits_{s\in[t_0,\infty)} \|u(s)\| \end{cases} \tag{3}$$

where $u(t) \in \mathcal{L}_\infty^k$ is the reference input. Moreover, $f : \mathbb{R}_+ \times C([-\tau,0];\mathbb{R}^n) \times \mathcal{S} \times \mathbb{R}^l \to \mathbb{R}^n$, $g : \mathbb{R}_+ \times C([-\tau,0];\mathbb{R}^n) \times \mathcal{S} \times \mathbb{R}^l \to \mathbb{R}^{n\times m}$ and $h : \mathbb{R}_+ \times C([-\tau,0];\mathbb{R}^n) \times \mathbb{R}^k \times \mathcal{S} \to \mathbb{R}^l$ are measurable functions with $f(t,0,0,i) \equiv 0$, $g(t,0,0,i) \equiv 0$ and $h(t,0,0,i) \equiv 0$ for any $i \in \mathcal{S}$. Let

$$\begin{aligned} \bar{f}(t,x_t,u,\bar{r}(t)) &= \bar{f}(t,x_t,u,r(t),r'(t)) \\ &= f(t,x_t,h(t,x_t,u,r'(t)),r(t)) \\ \bar{g}(t,x_t,u,\bar{r}(t)) &= \bar{g}(t,x_t,u,r(t),r'(t)) \\ &= g(t,x_t,h(t,x_t,u,r'(t)),r(t)) \end{aligned}$$

For convenience, let $\bar{f}_{ij}(t,x_t,u(t))$ and $\bar{g}_{ij}(t,x_t,u(t))$ denote $\bar{f}(t,x_t,u(t),i,j)$ and $\bar{g}(t,x_t,u(t),i,j)$, respectively, for any $i,j \in \mathcal{S}$. Specifically, when $i = j$, the mode-dependent controller and the system operate synchronously, while when $i \neq j$, they operate asynchronously. Due to $\nu(t)$ relies not on $r(t)$ but on $r'(t)$, when $r'(t) \neq r(t)$, i.e, on the asynchronous time interval, the designed controller is an mismatched one for the controlled system, which may cause the degradation of control loop performance and even make it unstable. The stability of the control system with the existence of asynchronous switching will be our main concern.

In the paper, it is also assumed that $\bar{f}$, $\bar{g}$ satisfy the local Lipschitz condition and the linear growth condition, hence for the closed-loop system

$$\begin{aligned} \mathrm{d}x(t) &= \bar{f}(t,x_t,u(t),r(t),r'(t))\mathrm{d}t \\ &\quad + \bar{g}(t,x_t,u(t),r(t),r'(t))\mathrm{d}w(t) \end{aligned} \tag{4}$$

there exists an unique solution on $t \geq -\tau$.

In the next, we make some definitions for the Markov process $r(t)$ and the detected switching signal $r'(t)$. Firstly, $r(t)$ is assumed to be a regular Markov process with standard transition probability matrix. Let the sequence $\{t_l\}_{l\geq 0}$ denote the switching instants sequence of $r(t)$, and $r(t_l) = i_l$, $t_0 = 0$. When $i_l = i$, $t_{l+1} - t_l$ is called the sojourn-time of Markov process in mode i. As usual, the sojourn-time sequence $\{t_{l+1} - t_l\}_{l\geq 0}$ belongs to an exponential distribution with rate parameter $\lambda(i)$, where $0 \leq \lambda(i) < \infty$ is the transition rate of r(t) in mode i. Further, for all $i,j \in \mathcal{S}$ and $i \neq j$, $\mathbb{E}\{t_{l+1} - t_l | i_l = i, i_{l+1} = j\} = \frac{1}{\lambda(i)}$, where $\lambda(i)$ denotes the reciprocal of the average sojourn-time of Markov process $r(t)$ in mode i. According to (2), we also have $\lambda(i) = -\pi_{ii}$. On the other hand, the detected switching $r'(t)$ is considered as $r'(t) = r(t - d(t))$, and it is the only switching signal which can be obtained and used by the controller. Let $\{t'_l\}_{l\geq 0}$ denote the switching instants sequence of $r'(t)$. As in [7], the following statements are assumed to describe the characteristic of $r'(t)$. When $r(t)$ jumps from i to j, $r'(t)$ follows $r(t)$ with a delay and this delay is also an independent exponentially distributed random variable with the mean $\frac{1}{\pi_{ij}^0}$, and

$$\mathbb{P}\{r'(t+\Delta)=j \mid \begin{matrix} r'(s)=i, s\in[t^*,t] \\ r(t^*)=j, r(t^{*-})=i \end{matrix}\} = \begin{cases} \pi_{ij}^0\Delta + o(\Delta), & i \neq j \\ 1+\pi_{ii}^0\Delta + o(\Delta), & i = j \end{cases} \tag{5}$$

Clearly, when letting $\pi_{ij}^0 \to \infty$, the detection is instantaneous. It is assumed that π_{ij}^0 is sufficiently large and $0 \leq d(t) \leq d \leq \inf\{t_{l+1} - t_l\}$. Further, $r'(t)$ is causal, meaning that the ordering of the switching instants of $r'(t)$ is the same as the ordering of the corresponding switching instants of $r(t)$. Thus, it follows that $0 = t_0 = t'_0 < t_1 \leq t'_1 < t_2 \leq t'_2 < \cdots < t_l \leq t'_l < t_{l+1} < \cdots$, where $t'_l = t_l + d(t_l)$ for any $l \geq 1$. Define a virtual switching signal $\bar{r}(t)$, from $[0,\infty)$ to $\mathcal{S}\times\mathcal{S}$, by $\bar{r}(t) = (r(t), r'(t))$. Let $\{\bar{t}_l\}_{l\geq 0}$ denote the switching instants of $\bar{r}(t)$. Then, for any $l \geq 1$, $\bar{t}_0 = t'_0 = t_0$, $\bar{t}_{2l-1} = t_l$ and $\bar{t}_{2l} = t'_l$.

Remark 2.1: (i) In Ref. [7], the detection process is described by both non-zero detection delay and false alarms due to environmental noises, disturbances, and small modelling uncertainties, etc. The false alarm is assumed to be an independent exponential distribution with rate π_{ij}^1. However, the existence of false alarms increases the difficulty and complexity of the closed-loop systems, and therefore this issue is out of the scope of this paper. However, this issue together with non-zero detection delay are our ongoing work.

(ii) Various algorithms exist for the detection of

Markovian switching signal. In this paper, we choose the method discussed in [7], referred to as the optimal minimum probability of error bayesian detector. As in [7], $r'(t)$ is assumed to have the similar characteristics as $r(t)$, and hence, $r'(t)$ is regarded as a conditional Markov process. For $r'(t)$, non-Markovian conditional switches can also be our future work.

To prove the main results, the following lemma is required.

Lemma 2.1: For any given $V(x(t),t,\bar{r}(t)) = V(x(t),t,r(t),r'(t)) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S} \times \mathcal{S};\mathbb{R}_+)$, associated with system (4), the diffusion operator $\mathfrak{L}V$, from $C([-\tau,0];\mathbb{R}^n)\times\mathbb{R}_+\times\mathcal{S}\times\mathcal{S}$ to $\mathbb{R}$, can be described as follows.

Case 1. When $r'(t)=r(t)=i$, then

$$\begin{aligned}&\mathfrak{L}V(x_t,t,i,i)\\ =\;& V_t(x(t),t,i,i)+V_x(x(t),t,i,i)\bar{f}_{ii}(t,x_t,u)\\ &+\frac{1}{2}trace[\bar{g}_{ii}^T(t,x_t,u)V_{xx}(x(t),t,i,i)\bar{g}_{ii}(t,x_t,u)]\\ &+\sum_{k=1}^{N}\pi_{ik}V(x(t),t,k,i)\end{aligned} \tag{6}$$

Case 2. When $r'(t)=i$, $r(t)=j$ and $j\neq i$, then

$$\begin{aligned}&\mathfrak{L}V(x_t,t,j,i)\\ =\;& V_t(x(t),t,j,i)+V_x(x(t),t,j,i)\bar{f}_{ji}(t,x_t,u)\\ &+\frac{1}{2}trace[\bar{g}_{ji}^T(t,x_t,u)V_{xx}(x(t),t,j,i)\bar{g}_{ji}(t,x_t,u)]\\ &+\pi_{ij}^0V(x(t),t,j,j)-\pi_{ij}^0V(x(t),t,j,i)\end{aligned} \tag{7}$$

Remark 2.2: Lemma 2.1 is from (2) in [41] and Lemma 3 in [37]. When $r'(t)\equiv r(t)$ for all $t\geq 0$, (6) is the same as (2) in [41]. Otherwise (6) and (7) are similar to the ones in Lemma 3 in [37]. Lemma 3 in [37] considers also false alarms of $r'(t)$. In this paper, the causality of $r'(t)$ means $\Pi^1=\{\pi_{ij}^1\}_{N\times N}=0$ and (6) follows.

III. Global Asymptotic Stability

From the definition of ISS, an ISS system is GAS if the input $u\equiv 0$. Therefore, the GAS property is useful for ISS. In this section, GAS in probability and in pth moment are considered.

To begin with, a useful lemma is stated as follows.

Lemma 3.1: Let $V(t) = e^{\lambda t}V(x(t),t,\bar{r}(t)) = e^{\lambda t}V(x(t),t,r(t),r'(t))$ for all $t\geq 0$ and $\lambda\geq 0$, then

$$\begin{aligned}&\mathrm{D}^+\mathbb{E}\{V(t)\}=\mathbb{E}\{\mathfrak{L}V(t)\}\\ =\;&\lambda\mathbb{E}\{V(t)\}+e^{\lambda t}\mathbb{E}\{\mathfrak{L}V(x_t,t,r(t),r'(t))\}\end{aligned} \tag{8}$$

where $\mathrm{D}^+\mathbb{E}\{V(t)\}=\limsup_{dt\to 0^+}\frac{\mathbb{E}\{V(t+dt)\}-\mathbb{E}\{V(t)\}}{dt}$.

Proof: Firstly, for any $k_1,k_2\in\mathcal{S}$, it follows

$$\begin{aligned}&\mathbb{E}\{V(t+dt)|x(t),r(t)=k_1,r'(t)=k_2,t\}\\ =\;&\mathbb{E}\{V(t)+\lambda V(t)dt|x(t),r(t)=k_1,r'(t)=k_2,t\}\\ &+\mathbb{E}\{e^{\lambda t}V_t(x(t),t,\bar{r}(t))dt|x(t),r(t)=k_1,r'(t)=k_2,t\}\\ &+\mathbb{E}\{e^{\lambda t}V_x(x(t),t,\bar{r}(t))\bar{f}(t,x_t,u,\bar{r}(t))dt\\ &+\frac{1}{2}e^{\lambda t}trace[\bar{g}^T(t,x_t,u,\bar{r}(t))V_{xx}(x(t),t,\bar{r}(t))\\ &\times\bar{g}(t,x_t,u,\bar{r}(t))]dt|x(t),r(t)=k_1,r'(t)=k_2,t\}\\ &+\mathbb{E}\{e^{\lambda t}V(x(t),t,r(t+dt),r'(t))\\ &+e^{\lambda t}V(x(t),t,r(t),r'(t+dt))|x(t),r(t)=k_1,\\ &r'(t)=k_2,t\}+o(dt)\end{aligned} \tag{9}$$

which is in accordance with Lemma 2.1. We complete the proof by considering the following two cases: $r(t)=r'(t)=i$ and $r'(t)=i$, $r(t)=j$, respectively, where $i,j\in\mathcal{S}$ and $j\neq i$.

Case 1. $r'(t)=r(t)=i$. In this case, only the true mode switches may occur. Using the conclusion in [7], it follows

$$\begin{aligned}&\mathbb{E}\{e^{\lambda t}V(x(t),t,r(t+dt),r'(t))|x(t),r(t)=r'(t)=i,t\}\\ &=\sum_{j=1}^{N}\pi_{ij}[e^{\lambda t}V(x(t),t,j,i)-e^{\lambda t}V(x(t),t,i,i)]dt\\ &=\sum_{j=1}^{N}\pi_{ij}e^{\lambda t}V(x(t),t,j,i)dt\\ &\mathbb{E}\{e^{\lambda t}V(x(t),t,r(t),r'(t+dt))|x(t),r(t)=r'(t)=i,t\}\\ &=\pi_{ii}^1[e^{\lambda t}V(x(t),t,i,i)-e^{\lambda t}V(x(t),t,i,i)]dt=0\end{aligned}$$

Then,

$$\begin{aligned}&\mathbb{E}\{V(t+dt)|x(t),r'(t)=r(t)=i,t\}\\ &=\mathbb{E}\{V(t)|x(t),r'(t)=r(t)=i,t\}\\ &+[\lambda e^{\lambda t}V(x(t),t,i,i)+e^{\lambda t}\mathfrak{L}V(x_t,t,i,i)]dt+o(dt)\end{aligned} \tag{10}$$

where $\mathfrak{L}V(x_t,t,i,i)$ is defined in (6). Taking the expectation on the both sides of (10),

$$\begin{aligned}&\mathrm{D}^+\mathbb{E}\{e^{\lambda t}V(x(t),t,i,i)\}=\mathbb{E}\{\lambda e^{\lambda t}V(x(t),t,i,i)\\ &+e^{\lambda t}\mathfrak{L}V(x_t,t,i,i)\}\end{aligned} \tag{11}$$

Case 2. $r'(t)=i$, $r(t)=j$. This situation corresponds to the detection delay, and it is assumed that the true mode $r(t)$ doesn't switch during this short time lapse. The only possible switch is that $r'(t)$ switches from i to j, corresponding to the end of the transient, and this switch occurs on the average after $\frac{1}{\pi_{ij}^0}$ seconds. Then,

$$\mathbb{E}\{e^{\lambda t}V(x(t),t,r(t+dt),r'(t))|\begin{matrix}x(t),r(t)=j\\ r'(t)=i,t\end{matrix}\}$$

$$= \pi_{jj}[e^{\lambda t}V(x(t),t,j,i) - e^{\lambda t}V(x(t),t,j,i)]dt = 0$$

$$\mathbb{E}\{e^{\lambda t}V(x(t),t,r(t),r'(t+dt))| \begin{array}{c} x(t),r(t)=j \\ r'(t)=i,t \end{array} \}$$

$$= \pi^0_{ij}[e^{\lambda t}V(x(t),t,j,j) - e^{\lambda t}V(x(t),t,j,i)]dt$$

Thus, similar to (10), it holds that

$$\mathrm{D}^+\mathbb{E}\{e^{\lambda t}V(x(t),t,j,i)\} = \mathbb{E}\{\lambda e^{\lambda t}V(x(t),t,j,i) + e^{\lambda t}\mathfrak{L}V(x_t,t,j,i)\} \tag{12}$$

where $\mathfrak{L}V(x_t,t,j,i)$ in this case is defined in (7).

Combining (11) and (12), and considering the arbitrary of i, j, it follows (8), for all $t \geq 0$. Thus, we complete the proof. ■

Using Lemma 3.1, the criteria of GASiP for system (4) is obtained.

Theorem 3.1: If there exist functions $\alpha_1 \in \mathcal{K}_\infty$, $\alpha_2 \in \mathcal{CK}_\infty$, $\mu \geq 1$, $q > 1$, λ_2, $0 < \varsigma < 1$, and $V(x(t),t,\bar{r}(t)) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S} \times \mathcal{S};\mathbb{R}_+)$, such that

$$\alpha_1(|x(t)|) \leq V(x(t),t,\bar{r}(t)) \leq \alpha_2(|x(t)|) \tag{13}$$

For any $l \in \mathbb{N}_+$, there exists $\bar{\lambda}_1 \in (0,\lambda_1)$ such that

$$\mathbb{E}\{\mathfrak{L}V(\varphi(\theta),t,\bar{r}(t))\} \leq \begin{cases} -\lambda_1\mathbb{E}\{V(\varphi(0),t,\bar{r}(t))\}, t \in [\bar{t}_{2l-2},\bar{t}_{2l-1}) \\ \lambda_2\mathbb{E}\{V(\varphi(0),t,\bar{r}(t))\}, \quad t \in [\bar{t}_{2l-1},\bar{t}_{2l}) \end{cases} \tag{14}$$

provided those $\varphi \in L^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n)$ satisfying that

$$\min_{i,j\in\mathcal{S}} \mathbb{E}\{V(\varphi(\theta),t+\theta,i,j)\} < q\mathbb{E}\{V(\varphi(0),t,\bar{r}(t))\} \tag{15}$$

where

$$e^{\bar{\lambda}_1\tau} < q \tag{16}$$

Moreover,

$$\mathbb{E}\{V(x(\bar{t}_l),\bar{t}_l,\bar{r}(\bar{t}_l))\} \leq \mu\mathbb{E}\{V(x(\bar{t}_l),\bar{t}_l,\bar{r}(\bar{t}_{l-1}))\} \tag{17}$$

and there exists some $\bar{\lambda}_2 \in (\lambda_2,\infty)$ such that

$$\mu^2 e^{(\bar{\lambda}_1+\bar{\lambda}_2)d}\bar{\pi} - \tilde{\pi} \leq \varsigma\bar{\lambda}_1 \tag{18}$$

Then, system (4) with $u \equiv 0$ is GASiP.

Proof: According to (8) in Lemma 3.1, we have

$$\mathrm{D}^+\mathbb{E}\{V(x(t),t,\bar{r}(t))\} = \mathbb{E}\{\mathfrak{L}V(x_t,t,\bar{r}(t))\} \tag{19}$$

for any $t \in [\bar{t}_{2l-2},\bar{t}_{2l-1}) \cup [\bar{t}_{2l-1},\bar{t}_{2l})$, $l \in \mathbb{N}_+$, with $\bar{t}_0 = t_0 = t'_0 = 0$.

On the one hand, from (13), using Jensen's inequality, one can obtain

$$\mathbb{E}\{V(x(t),t,i_0,i_0)\} = \mathbb{E}\{V(x(t),t,\bar{r}(t))\} \leq \mathbb{E}\{\alpha_2(|x(t)|)\} \leq \alpha_2(\mathbb{E}\{\|\xi\|\})$$

for any $t \in [t_0-\tau,t_0]$. In the following, we shall prove that

$$\mathbb{E}\{V(x(t),t,i_0,i_0)\} \leq \alpha_2(\mathbb{E}\{\|\xi\|\})e^{-\bar{\lambda}_1(t-t_0)} \tag{20}$$

for $t \in [\bar{t}_0,\bar{t}_1) = [t_0,t_1)$. Suppose (20) is not true, i.e., there exists some $t \in (t_0,t_1)$ such that

$$\mathbb{E}\{V(x(t),t,i_0,i_0)\} > \alpha_2(\mathbb{E}\{\|\xi\|\})e^{-\bar{\lambda}_1(t-t_0)} \tag{21}$$

Let

$$t^* = \inf\{t \in (t_0,t_1) : \mathbb{E}\{V(x(t),t,i_0,i_0)\} > \alpha_2(\mathbb{E}\{\|\xi\|\})e^{-\bar{\lambda}_1(t-t_0)}\}$$

Then $t^* \in (t_0,t_1)$ and $\mathbb{E}\{V(x(t^*),t^*,i_0,i_0)\} = \alpha_2(\mathbb{E}\{\|\xi\|\})e^{-\bar{\lambda}_1(t^*-t_0)}$. Further, there exists a sequence $\{\tilde{t}_n\}$ ($\tilde{t}_n \in (t^*,t_1)$, for any $n \in \mathbb{N}_+$) with $\lim_{n\to\infty}\tilde{t}_n = t^*$, such that

$$\mathbb{E}\{V(x(\tilde{t}_n),\tilde{t}_n),i_0,i_0\} > \alpha_2(\mathbb{E}\{\|\xi\|\})e^{-\bar{\lambda}_1(\tilde{t}_n-t_0)} \tag{22}$$

From the definition of t^*, for any $\theta \in [-\tau,0]$, it follows

$$\mathbb{E}\{V(x(t^*+\theta),t^*+\theta,i_0,i_0)\} \leq e^{-\bar{\lambda}_1\theta}\mathbb{E}\{V(x(t^*),t^*,i_0,i_0)\} \leq e^{\bar{\lambda}_1\tau}\mathbb{E}\{V(x(t^*),t^*,i_0,i_0)\}$$

and further,

$$\min_{i,j\in\mathcal{S}} \mathbb{E}\{V(x(t^*+\theta),t^*+\theta,i,j)\} < q\mathbb{E}\{V(x(t^*),t^*,i_0,i_0)\},\ \theta \in [-\tau,0]$$

Thus, from (14) and (19), we obtain

$$\mathrm{D}^+\mathbb{E}\{V(x(t^*),t^*,i_0,i_0)\} \leq -\lambda_1\mathbb{E}\{V(x(t^*),t^*,i_0,i_0)\} < -\bar{\lambda}_1\mathbb{E}\{V(x(t^*),t^*,i_0,i_0)\}$$

Thus, for $h > 0$ which is sufficient small, it holds

$$\mathrm{D}^+\mathbb{E}\{V(x(t^*),t^*,i_0,i_0)\} \leq -\bar{\lambda}_1\mathbb{E}\{V(x(t^*),t^*,i_0,i_0)\}$$

for $t \in [t^*,t^*+h]$.

Hence,

$$\mathbb{E}\{V(x(t^*+h),t^*+h,i_0,i_0)\} \leq \mathbb{E}\{V(x(t^*),t^*,i_0,i_0)\}e^{-\bar{\lambda}_1 h}$$

which is a contradiction to (22). Therefore, (20) holds. Combining the continuity of function $V(x(t),t,i_0,i_0)$ and (17), we have

$$\mathbb{E}\{V(x(\bar{t}_1),\bar{t}_1,\bar{r}(\bar{t}_1))\} \leq \mu\mathbb{E}\{V(x(\bar{t}_1),\bar{t}_1,\bar{r}(\bar{t}_0))\} \leq \mu\alpha_2(\mathbb{E}\{\|\xi\|\})e^{-\bar{\lambda}_1(t_1-t_0)} \tag{23}$$

Let $W(t,\bar{r}(t)) = e^{\bar{\lambda}_1 t}V(x(t),t,\bar{r}(t))$. In the sequel, we will show that for any $t \in [\bar{t}_{2l-1},\bar{t}_{2l+1})$

$$\mathbb{E}\{W(t,\bar{r}(t))\} \le \mu\mathbb{E}\{W(\bar{t}_{2l-1},\bar{r}(\bar{t}_{2l-1}))\}e^{(\bar{\lambda}_1+\bar{\lambda}_2)d} \tag{24}$$

The following three cases are considered: $t \in [\bar{t}_{2l-1},\bar{t}_{2l})$, $t=\bar{t}_{2l}$ and $t \in (\bar{t}_{2l},\bar{t}_{2l+1})$.

First, when $t \in [\bar{t}_{2l-1},\bar{t}_{2l})$, we claim that

$$\begin{aligned} & \mathbb{E}\{W(t,\bar{r}(t))\} \\ \le\ & \mu\mathbb{E}\{W(\bar{t}_{2l-1},\bar{r}(\bar{t}_{2l-1}))\}e^{(\bar{\lambda}_1+\bar{\lambda}_2)(t-\bar{t}_{2l-1})} \end{aligned} \tag{25}$$

Suppose (25) is not true. Then, there exists some $t \in [\bar{t}_{2l-1},\bar{t}_{2l})$ such that

$$\begin{aligned} & \mathbb{E}\{W(t,\bar{r}(t))\} \\ >\ & \mu\mathbb{E}\{W(\bar{t}_{2l-1},\bar{r}(\bar{t}_{2l-1}))\}e^{(\bar{\lambda}_1+\bar{\lambda}_2)(t-\bar{t}_{2l-1})} \end{aligned}$$

Let $t^* = \inf\{t \in [\bar{t}_{2l-1},\bar{t}_{2l}) : \mathbb{E}\{W(t,\bar{r}(\bar{t}_{2l}))\} > \mu\mathbb{E}\{W(\bar{t}_{2l-1},\bar{r}(\bar{t}_{2l-1}))\}e^{(\bar{\lambda}_1+\bar{\lambda}_2)(t-\bar{t}_{2l-1})}\}$, thus

$$\begin{aligned} & \mathbb{E}\{W(t^*,\bar{r}(\bar{t}_{2l}))\} \\ =\ & \mu\mathbb{E}\{W(\bar{t}_{2l-1},\bar{r}(\bar{t}_{2l-1}))\}e^{(\bar{\lambda}_1+\bar{\lambda}_2)(t^*-\bar{t}_{2l-1})} \end{aligned}$$

Considering the continuity, there exists a list of sequence $\{\tilde{t}_n\}_{n\in\mathbb{N}_+} \in (t^*,\bar{t}_{2l})$ with $\lim_{n\to\infty}\tilde{t}_n = t^*$ such that

$$\begin{aligned} & \mathbb{E}\{W(\tilde{t}_n,\bar{r}(\bar{t}_{2l}))\} \\ >\ & \mu\mathbb{E}\{W(\bar{t}_{2l-1},\bar{r}(\bar{t}_{2l-1}))\}e^{(\bar{\lambda}_1+\bar{\lambda}_2)(\tilde{t}_n-\bar{t}_{2l-1})} \end{aligned} \tag{26}$$

Define $U(t) = e^{-(\bar{\lambda}_1+\bar{\lambda}_2)t}\mathbb{E}\{W(t,\bar{r}(t))\}$, then

$$\begin{aligned} \mathrm{D}^+U(t) \ =\ & -\bar{\lambda}_2 e^{-\bar{\lambda}_2 t}\mathbb{E}\{V(x(t),t,\bar{r}(t))\} \\ & +e^{-\bar{\lambda}_2 t}\mathrm{D}^+\mathbb{E}\{V(x(t),t,\bar{r}(t))\} \end{aligned}$$

From the definition of t^*, for any $\theta \in [-\tau,0]$, it follows

$$\begin{aligned} & \mu\mathbb{E}\{W(\bar{t}_{2l-1},\bar{r}(\bar{t}_{2l-1}))\}e^{(\bar{\lambda}_1+\bar{\lambda}_2)(t^*+\theta-\bar{t}_{2l-1})} \\ & = \mathbb{E}\{W(t^*,\bar{r}(\bar{t}_{2l-1}))\}e^{(\bar{\lambda}_1+\bar{\lambda}_2)\theta} \\ & \ge \mathbb{E}\{W(t^*+\theta,\bar{r}(\bar{t}_{2l-1}))\} \end{aligned}$$

which means

$$\begin{aligned} & \mathbb{E}\{V(x(t^*+\theta),t^*+\theta,\bar{r}(\bar{t}_{2l-1}))\} \\ & \le \mathbb{E}\{V(x(t^*),t^*,\bar{r}(\bar{t}_{2l-1}))\}e^{\bar{\lambda}_2\theta} \\ & \le \mathbb{E}\{V(x(t^*),t^*,\bar{r}(\bar{t}_{2l-1}))\} \end{aligned} \tag{27}$$

Hence,

$$\begin{aligned} & \min_{i,j\in\mathcal{S}}\mathbb{E}\{V(x(t^*+\theta),t^*+\theta,i,j)\} \\ & < q\mathbb{E}\{V(x(t^*),t^*,\bar{r}(\bar{t}_{2l-1}))\} \end{aligned}$$

Then,

$$\begin{aligned} \mathrm{D}^+U(t^*) = & -\bar{\lambda}_2 e^{-\bar{\lambda}_2 t^*}\mathbb{E}\{V(x(t^*),t^*,\bar{r}(\bar{t}_{2l-1}))\} \\ & +e^{-\bar{\lambda}_2 t^*}\mathrm{D}^+\mathbb{E}\{V(x(t^*),t^*,\bar{r}(\bar{t}_{2l-1}))\} \\ \le & -(\bar{\lambda}_2-\lambda_2)e^{-\bar{\lambda}_2 t^*}\mathbb{E}\{V(x(t^*),t^*,\bar{r}(\bar{t}_{2l-1}))\} \end{aligned}$$

Note that either $\mathbb{E}\{V(x(t^*),t^*,\bar{r}(\bar{t}_{2l-1}))\} = 0$ or $\mathbb{E}\{V(x(t^*),t^*,\bar{r}(\bar{t}_{2l-1}))\} > 0$. In the case $\mathbb{E}\{V(x(t^*),t^*,\bar{r}(\bar{t}_{2l-1}))\} = 0$, we have $x(t^*) = 0$ a.s. From (27) and (13), we have $x(t^*+\theta) = 0$ a.s. for any $\theta \in [-\tau,0]$. Recalling that $h(t^*,0,0,r'(\bar{t}_{2l-1})) = 0$, $f(t^*,0,0,r(\bar{t}_{2l-1})) = 0$ and $g(t^*,0,0,r(\bar{t}_{2l-1})) = 0$, hence $\bar{f}(t^*,0,0,\bar{r}(\bar{t}_{2l-1})) = 0$ and $g(t^*,0,0,\bar{r}(\bar{t}_{2l-1})) = 0$. Thus, one sees that $x(t^*+h) = 0$ a.s., for all $h > 0$, i.e., $\mathbb{E}\{W(t^*+h,\bar{r}(\bar{t}_{2l-1}))\} = 0$, which is a contradiction of (26). On the other hand, in the case $\mathbb{E}\{V(x(t^*),t^*,\bar{r}(\bar{t}_{2l-1}))\} > 0$, there exists a positive number h which is sufficient small such that

$$\mathrm{D}^+U(t) \le 0,\ t \in [t^*,t^*+h]$$

which means

$$\mathbb{E}\{W(t^*+h,\bar{r}(\bar{t}_{2l-1}))\} \le e^{(\bar{\lambda}_1+\bar{\lambda}_2)h}\mathbb{E}\{W(t^*,\bar{r}(\bar{t}_{2l-1}))\}$$

and it is a contradiction to (26). Therefore (25) holds. Further, (24) holds on $t \in [\bar{t}_{2l-1},\bar{t}_{2l})$.

By considering the continuity of $W(t,\bar{r}(\bar{t}_{2l-1}))$ at time $t=\bar{t}_{2l}$, it follows

$$\mathbb{E}\{W(\bar{t}_{2l},\bar{r}(\bar{t}_{2l}))\} \le \mu\mathbb{E}\{W(\bar{t}_{2l-1},\bar{r}(\bar{t}_{2l-1}))\}e^{(\bar{\lambda}_1+\bar{\lambda}_2)d}$$

Following the similar analysis on interval $(\bar{t}_{2l-1},\bar{t}_{2l})$, one can prove that (24) holds on $(\bar{t}_{2l},\bar{t}_{2l+1})$, and then it holds on $[\bar{t}_{2l-1},\bar{t}_{2l+1})$. Thus,

$$\begin{aligned} & \mathbb{E}\{V(x(t),t,\bar{r}(t))\} \\ \le\ & \mu\mathbb{E}\{V(x(\bar{t}_{2l-1}),\bar{t}_{2l-1},\bar{r}(\bar{t}_{2l-1}))\}e^{-\bar{\lambda}_1(t-\bar{t}_{2l-1})} \\ & \times e^{(\bar{\lambda}_1+\bar{\lambda}_2)d},\ t \in [\bar{t}_{2l-1},\bar{t}_{2l+1}) \end{aligned} \tag{28}$$

By considering the continuity of $V(x(t),t,\bar{r}(\bar{t}_{2l}))$, one can see that (28) holds at time $\bar{t}_{2l+1}$, and then,

$$\begin{aligned} & \mathbb{E}\{V(x(t_{l+1}),t_{l+1},\bar{r}(t_{l+1}))\} \\ & \le \mu^2\mathbb{E}\{V(x(t_l),t_l,\bar{r}(t_l))\}e^{-\bar{\lambda}_1(t_{l+1}-t_l)}e^{(\bar{\lambda}_1+\bar{\lambda}_2)d} \end{aligned} \tag{29}$$

For any $t \ge \bar{t}_1 = t_1$, iterating (28) from $l=1$ to $l = N_r(t,t_1)+1$, one can obtain

$$\begin{aligned} & \mathbb{E}\{V(x(t),t,\bar{r}(t))\} \\ & \le \mu^2\mathbb{E}\{V(x(t_{N_r(t,t_1)+1}),t_{N_r(t,t_1)+1},\bar{r}(t_{N_r(t,t_1)+1}))\} \\ & \quad \times e^{(\bar{\lambda}_1+\bar{\lambda}_2)d}e^{-\bar{\lambda}_1(t-t_{N_r(t,t_1)+1})} \\ & = \mathbb{E}\{\mu^{2(N_r(t,t_1)+1-N_r(t,t_1))}e^{(N_r(t,t_1)+1-N_r(t,t_1))(\bar{\lambda}_1+\bar{\lambda}_2)d}\} \\ & \quad \times \mathbb{E}\{V(x(t_{N_r(t,t_1)+1}),t_{N_r(t,t_1)+1},\bar{r}(t_{N_r(t,t_1)+1}))\} \\ & \quad \times e^{-\bar{\lambda}_1(t-t_{N_r(t,t_1)+1})} \\ & \le \mathbb{E}\{\mu^{2(N_r(t,t_1)+1-N_r(t,t_1))}e^{(N_r(t,t_1)+1-N_r(t,t_1))(\bar{\lambda}_1+\bar{\lambda}_2)d}\} \\ & \quad \times \mu^2 e^{(\bar{\lambda}_1+\bar{\lambda}_2)d}\mathbb{E}\{V(x(t_{N_r(t,t_1)}),t_{N_r(t,t_1)},\bar{r}(t_{N_r(t,t_1)}))\} \\ & \quad \times e^{-\bar{\lambda}_1(t-t_{N_r(t,t_1)})} \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\{\mu^{2(N_r(t,t_1)+2-N_r(t,t_1))}e^{(N_r(t,t_1)+2-N_r(t,t_1))(\bar{\lambda}_1+\bar{\lambda}_2)d}\} \\
&\quad \times \mathbb{E}\{V(x(t_{N_r(t,t_1)}), t_{N_r(t,t_1)}, \bar{r}(t_{N_r(t,t_1)}))\} \\
&\quad \times e^{-\bar{\lambda}_1(t-t_{N_r(t,t_1)})} \\
&\leq \cdots \\
&\leq \mathbb{E}\{\mu^{2(N_r(t,t_1)-2)}e^{(N_r(t,t_1)-2)(\bar{\lambda}_1+\bar{\lambda}_2)d}\}\mu^2 e^{(\bar{\lambda}_1+\bar{\lambda}_2)d} \\
&\quad \times \mathbb{E}\{V(x(t_2), t_2, \bar{r}(t_2))\}e^{-\bar{\lambda}_1(t-t_2)} \\
&= \mathbb{E}\{\mu^{2(N_r(t,t_1)-1)}e^{(N_r(t,t_1)-1)(\bar{\lambda}_1+\bar{\lambda}_2)d}\} \\
&\quad \times \mathbb{E}\{V(x(t_2), t_2, \bar{r}(t_2))\}e^{-\bar{\lambda}_1(t-t_2)} \\
&\leq \mathbb{E}\{\mu^{2(N_r(t,t_1)-1)}e^{(N_r(t,t_1)-1)(\bar{\lambda}_1+\bar{\lambda}_2)d}\}\mu^2 e^{(\bar{\lambda}_1+\bar{\lambda}_2)d} \\
&\quad \times \mathbb{E}\{V(x(t_1), t_1, \bar{r}(t_1))\}e^{-\bar{\lambda}_1(t-t_1)} \\
&= \mathbb{E}\{\mu^{2N_r(t,t_1)}e^{N_r(t,t_1)(\bar{\lambda}_1+\bar{\lambda}_2)d}\} \\
&\quad \times \mathbb{E}\{V(x(t_1), t_1, \bar{r}(t_1))\}e^{-\bar{\lambda}_1(t-t_1)} \qquad (30)
\end{aligned}
$$

Combining (23) with (30), we arrive at

$$
\begin{aligned}
&\mathbb{E}\{V(x(t), t, \bar{r}(t))\} \leq \mathbb{E}\{\mu^{2N_r(t,0)}e^{(\bar{\lambda}_1+\bar{\lambda}_2)N_r(t,0)d}\} \\
&\quad \times \alpha_2(\mathbb{E}\{\|\xi\|\})e^{-\bar{\lambda}_1 t} \qquad (31)
\end{aligned}
$$

for any $t \geq t_0 - \tau$. According to Lemma 6 in [42], let $s = 2\ln(\mu) + (\bar{\lambda}_1 + \bar{\lambda}_2)d$, there exists a positive number $M > 0$ such that

$$
\begin{aligned}
&e^{-\varsigma\bar{\lambda}_1 t}\mathbb{E}\{\mu^{2N_r(t,0)}e^{(\bar{\lambda}_1+\bar{\lambda}_2)N_r(t,0)d}\} \\
&\quad \leq Me^{-\varsigma\bar{\lambda}_1 t} + e^{[\mu^2 e^{(\bar{\lambda}_1+\bar{\lambda}_2)d}\bar{\pi} - \tilde{\pi} - \varsigma\bar{\lambda}_1]t}
\end{aligned}
$$

When $\varsigma\bar{\lambda}_1 \geq \mu^2 e^{(\bar{\lambda}_1+\bar{\lambda}_2)d}\bar{\pi} - \tilde{\pi}$, we have

$$
e^{-\varsigma\bar{\lambda}_1 t}\mathbb{E}\{\mu^{2N_r(t,0)}e^{(\bar{\lambda}_1+\bar{\lambda}_2)N_r(t,0)d}\} \leq M + 1 < \infty
$$

Then,

$$
\begin{aligned}
\mathbb{E}\{V(x(t), t, \bar{r}(t))\} &\leq \bar{M}e^{-(1-\varsigma)\bar{\lambda}_1 t}\alpha_2(\mathbb{E}\{\|\xi\|\}) \\
&\triangleq \bar{\beta}(\mathbb{E}\{\|\xi\|\}, t) \qquad (32)
\end{aligned}
$$

for any $M + 1 \leq \bar{M} < \infty$. It's no difficulty to verify $\bar{\beta}(\cdot,\cdot) \in \mathcal{KL}$ when $0 < \varsigma < 1$.

Then, for any $\varepsilon > 0$, take $\tilde{\beta} = \frac{\bar{\beta}}{\varepsilon}$. Obviously, $\tilde{\beta}(\cdot,\cdot) \in \mathcal{KL}$. Using Chebyshev's inequality, we have

$$
\begin{aligned}
&\mathbb{P}\{V(x(t), t, \bar{r}(t)) \geq \tilde{\beta}(\mathbb{E}\{\|\xi\|\}, t)\} \\
&\quad \leq \frac{\mathbb{E}\{V(x(t), t, \bar{r}(t))\}}{\tilde{\beta}(\mathbb{E}\{\|\xi\|\}, t)} < \varepsilon
\end{aligned}
$$

i.e.,

$$
\mathbb{P}\{|x(t)| < \beta(\mathbb{E}\{\|\xi\|\}, t)\} \geq 1 - \varepsilon
$$

where $\beta(r, s) = \alpha_1^{-1} \circ \tilde{\beta}(r, s) \in \mathcal{KL}$. Thus, we complete the proof. ■

Remark 3.1: (i). Assumption (14) is widely used in Razumikhin-type stability criterion and imposes less restrictions on the functions $\bar{f}(t, \varphi(\theta), u(t), \bar{r}(t))$ and $\bar{g}(t, \varphi(\theta), u(t), \bar{r}(t))$, as described in [40]. When $t \in [\bar{t}_{2l-1}, \bar{t}_{2l})$, condition (14) corresponds to the asynchronous case and λ_2 may or may not be positive. In what follows, λ_2 is assumed to positive, and λ_1 and λ_2 denote the minimal stability margin and maximal instability margin, respectively.

(ii). In Theorem 3.1, condition (18) is given to guarantee the stability. Indeed, for any $i \in \mathcal{S}$, there may exist a mismatched period. Those mismatched period are usually bounded with $d < \infty$. In this case, a larger mode sojourn-time is more appropriate. Based on (18), for fixed λ_1, μ and ς, a larger instability margin λ_2 or a larger upper bound on detection delay d can be compensated by a smaller $\bar{\pi}$. By considering $\bar{\pi} = \max_{i\in\mathcal{S}}\{|\pi_{ii}|\}$, one can obtain a smaller $\bar{\pi}$ by decreasing $|\pi_{ii}|$. Then the sojourn-time of $r(t)$ in mode i, $\mathbb{E}\{t_{l+1} - t_l | i_l = i, i_{l+1} = j\} = \frac{1}{|\pi_{ii}|}$. Furthermore, one can claim that the average value of the sojourn-time of $r(t)$ is less than or equal to $\frac{1}{\bar{\pi}}$, and, the smaller $\bar{\pi}$ is the larger the sojourn-time is. Thus, the stability of the hybrid stochastic retarded systems under asynchronous switching can be guaranteed by a sufficient small detection delay and a sufficient small mode transition rate $\bar{\pi}$. This result has a similar spirit as for asynchronous deterministic switched systems based on average dwell time approach where the closed-loop stability can be guaranteed by a sufficient large average dwell time.

The following two corollaries can be get directly from Theorem 3.1 and its proof. Their proofs are omitted.

Corollary 3.1: If $\mu < \frac{\lambda_1+\tilde{\pi}}{\bar{\pi}}$, and the conditions (13)-(17) hold, system (4) under a strictly synchronous controller $\nu(t)$ with $u \equiv 0$ is GASiP.

Remark 3.2: The similar conclusion can be seen in Corollary 12 in [42], which considers the GAS a.s. of a class of Markovian switching nonlinear systems. Corollary 3.1 provides a sufficient criterion in stochastic case with retarded delays.

Corollary 3.2: Under the assumptions in Theorem 3.1, system (4) with $u \equiv 0$ is also α_1-GASiM. Specially, if $\alpha_1 \in \mathcal{VK}_\infty$, system (4) with $u \equiv 0$ is GASiM. Furthermore, if $\alpha_1(s) = c_1 s^p$, $\alpha_2(s) = c_2 s^p$, where c_1 and c_2 are positive numbers, system (4) with $u \equiv 0$ is pth moment exponentially stable.

IV. Input-to-State Stability

In this section, based on the conclusions in Theorem 3.1, we will provide the sufficient conditions of SISS and pth moment ISS for system (4).

Theorem 4.1: If there exist functions $\alpha_1 \in \mathcal{K}_\infty$, $\alpha_2 \in \mathcal{CK}_\infty$, $\chi \in \mathcal{K}$, scalers $\mu \geq 1$, $q > 1$, $\lambda_1 > 0$, λ_2, $0 < \varsigma < 1$ and $V(x(t), t, \bar{r}(t)) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times$

$\mathcal{S}\times\mathcal{S};\mathbb{R}_+)$, such that for any $l\in\mathbb{N}_+$,

$$\begin{aligned}&|\varphi(0)|\geq\chi(\|u\|_{[0,\infty)})\Rightarrow\mathbb{E}\{\mathfrak{L}V(\varphi(\theta),t,\bar{r}(t))\}\\&\leq\begin{cases}-\lambda_1\mathbb{E}\{V(\varphi(0),t,\bar{r}(t))\}, & t\in[\bar{t}_{2l-2},\bar{t}_{2l-1})\\ \lambda_2\mathbb{E}\{V(\varphi(0),t,\bar{r}(t))\}, & t\in[\bar{t}_{2l-1},\bar{t}_{2l})\end{cases}\end{aligned}$$

provided those $\varphi\in L^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n)$ satisfying that (15) and (16). Moreover, (13), (17) and (18) hold, Then, system (4) is SISS.

Proof: Let the time sequences $\{\underline{t}_i\}_{i\geq1}$ and $\{\tilde{t}_i\}_{i\geq1}$ denote the time that the trajectory enters and leaves the set $\mathfrak{B}=\{\varphi\in L^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n):|\varphi(0)|<\chi(\|u\|_{[t_0,\infty)})\}$, respectively. In the following, we will complete the proof by considering the following two cases: $\xi\in\mathfrak{B}^C$ and $\xi\in\mathfrak{B}\setminus\{0\}$, respectively.

Case 1. $\xi\in\mathfrak{B}^C$. In this case, for any $t\in[0,\underline{t}_1)$, $|x(t)|\geq\chi(\|u\|_{[0,\infty)})$. According to Theorem 3.1, for any $\varepsilon'>0$, there exists a $\mathcal{KL}$ function β such that

$$\mathbb{P}\{|x(t)|<\beta(\mathbb{E}\{\|\xi\|\},t)\}\geq1-\varepsilon',\ \forall t\in[0,\underline{t}_1)\quad(33)$$

Now consider the interval $t\in[\underline{t}_1,\infty)$. Define $\tilde{t}_1=\inf\{t>\underline{t}_1:|x(t)|\geq\chi(\|u\|_{[t_0,\infty)})\}$, and $\inf\emptyset=\infty$. Clearly, for any $t\in[\underline{t}_1,\tilde{t}_1)$, we have

$$\mathbb{P}\{|x(t)|<\chi(\|u\|_{[0,\infty)})\}=1\geq1-\varepsilon'',\ \forall\varepsilon''>0\quad(34)$$

Define $\underline{t}_2=\min\{t\geq\tilde{t}_1:|x(t)|<\chi(\|u\|_{[t_0,\infty)})\}$. According to Theorem 3.1, we also have

$$\mathbb{P}\{|x(t)|<\beta(x(\tilde{t}_1),t-\tilde{t}_1)\}\geq1-\varepsilon',\ \forall t\in[\tilde{t}_1,\underline{t}_2).$$

Similarly, for any $i\geq2$, we define

$$\begin{cases}\underline{t}_i=\min\{t\geq\tilde{t}_{i-1}:|x(t)|<\chi(\|u\|_{[t_0,\infty)})\}\\ \tilde{t}_i=\inf\{t>\underline{t}_i:|x(t)|\geq\chi(\|u\|_{[t_0,\infty)})\}\end{cases}$$

By repeating the above induction, for any $i\geq1$, when $t\in[\underline{t}_i,\tilde{t}_i)$, we can obtain

$$\mathbb{P}\{|x(t)|<\chi(\|u\|_{[t_0,\infty)})\}=1\geq1-\varepsilon''$$

and when $t\in[\tilde{t}_i,\underline{t}_{i+1})$,

$$\mathbb{P}\{|x(t)|<\beta(\mathbb{E}\{|x(\tilde{t}_i)|\},t-\tilde{t}_i)\}\geq1-\varepsilon'$$

From the proof of Theorem 3.1, the $\mathcal{KL}$ function $\beta(r,s)$ satisfies

$$\beta(r,s)\leq\alpha_1^{-1}(\bar{M}e^{-\lambda_3 s}\alpha_2(r))$$

for some $\bar{M}\geq0$, where $\lambda_3\in(0,(1-\varsigma)\bar{\lambda}_1)$. Since $\alpha_1\in\mathcal{K}_\infty$, further, we can get

$$\beta(r,s)\leq\alpha_1^{-1}(\bar{M}\alpha_2(r))$$

Thus, for any $i\geq1$, when $t\in[\underline{t}_i,\tilde{t}_i)$,

$$\mathbb{P}\{|x(t)|<\chi(\|u\|_{[t_0,\infty)})\}=1\geq1-\varepsilon''\quad(35)$$

and when $t\in[\tilde{t}_i,\underline{t}_{i+1})$,

$$\begin{aligned}&\mathbb{P}\{|x(t)|<\alpha_1^{-1}(\bar{M}\alpha_2(\mathbb{E}\{|x(\tilde{t}_i)|\}))\}\\&\geq\mathbb{P}\{|x|<\beta(\mathbb{E}\{|x(\tilde{t}_i)|\},t-\bar{t}_i)\}\geq1-\varepsilon'\end{aligned}\quad(36)$$

Considering the continuity of $x(t)$, we have

$$\mathbb{E}\{|x(\tilde{t}_i)|\}<\chi(\|u\|_{[t_0,\infty)}),\ a.s.\quad(37)$$

Substituting (37) into (35) and (36), we obtain

$$\mathbb{P}\{|x(t)|<\gamma(\|u\|_{[0,\infty)})\}\geq1-\varepsilon''',\ \forall t\geq\underline{t}_1\quad(38)$$

where $\varepsilon'''=\max\{\varepsilon',\varepsilon''\}$, $\gamma(s)=\max\{\chi(s),\alpha_1^{-1}(\bar{S}\alpha_2(s))\}$. It's easy to verify that $\gamma\in\mathcal{K}$. Then, combining (33) and (38), we have

$$\mathbb{P}\{|x(t)|<\beta(\mathbb{E}\{\|\xi\|\},t)+\gamma(\|u\|_{[0,\infty)})\}\geq1-\varepsilon\quad(39)$$

for any $\xi\in\mathfrak{B}^C$, $t\geq0$, where $\varepsilon=\max\{\varepsilon',\varepsilon'''\}$.

Case 2. $\xi\in\mathfrak{B}\setminus\{0\}$. In this case, $\underline{t}_1=0$ a.s. When $t>0$, we have $\mathbb{P}\{t\in(\underline{t}_1,\infty)\}=\mathbb{P}\{t\in(t_0,\infty)\}=1$. Following the proof of **Case 1.**, the inequality (38) still holds. Then

$$\begin{aligned}&\mathbb{P}\{|x(t)|<\beta(\mathbb{E}\{\|\xi\|\},t)+\gamma(\|u\|_{[0,\infty)})\}\\&\geq\mathbb{P}\{|x(t)|<\gamma(\|u\|_{[0,\infty)})\}\geq1-\varepsilon'''\end{aligned}\quad(40)$$

for any $t\in(0,\infty)$. When $t=0$, by the definition of the set $\mathfrak{B}$ and the definition of γ, we can obtain

$$\begin{aligned}&\mathbb{P}\{|x(0)|<\beta(\mathbb{E}\{\|\xi\|\},0)+\gamma(\|u\|_{[0,\infty)})\}\\&\geq\mathbb{P}\{|x(0)|<\chi(\|u\|_{[0,\infty)})\}=1\end{aligned}$$

which implies, for any $\varepsilon_1>0$,

$$\mathbb{P}\{|x(0)|<\beta(\mathbb{E}\{\|\xi\|\},0)+\gamma(\|u\|_{[0,\infty)})\}\geq1-\varepsilon_1\quad(41)$$

Combining (40) and (41), we have

$$\mathbb{P}\{|x(t)|<\beta(\mathbb{E}\{\|\xi\|\},t)+\gamma(\|u\|_{[0,\infty)})\}\geq1-\varepsilon\quad(42)$$

for all $t\geq0$, $\xi\in\mathfrak{B}\setminus\{0\}$, where $\varepsilon=\max\{\varepsilon''',\varepsilon_1\}$.

Combining the proof of **Case 1.** and the proof of **Case 2.**, for any $\varepsilon>0$, $t\geq0$ and $\xi\in C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$, we have

$$\mathbb{P}\{|x(t)|<\beta(\mathbb{E}\{\|\xi\|\},t)+\gamma(\|u\|_{[0,\infty)})\}\geq1-\varepsilon$$

By causality we get

$$\mathbb{P}\{|x(t)|<\beta(\mathbb{E}\{\|\xi\|\},t)+\gamma(\|u\|_{[0,t)})\}\geq1-\varepsilon$$

Thus, we complete the proof. ■

Remark 4.1: Since the existence of asynchronous period, if $x(t^*)\in\mathfrak{B}$ for some $t^*\geq0$, we cannot guarantee that $|x(t)|<\chi(\|u\|_{[0,\infty)})$ a.s., for any $t>t^*$. But, from (38), it will be upper bounded by $\|u\|_{[0,\infty)}$ in probability.

Similar to Corollary 3.2, we have the following results.

Corollary 4.1: Under the hypotheses of Theorem 4.1, system (4) is also α_1-ISSiM. Specially, if $\alpha_1(s) = c_1 s^p$, $\alpha_2(s) = c_2 s^p$, where c_1 and c_2 are positive numbers, system (4) is pth moment ISS.

V. Application and Example

Hybrid stochastic delay system (HSDS), described by stochastic differential delay equations with Markovian switching, is an important class of hybrid stochastic retarded systems and is frequently used in engineering. In this section, the conclusions established in previous section are applied to the stability analysis of a class of HSDSs under asynchronous switching.

Let us consider the following hybrid system which has been discussed in [41] and the reference therein.

$$\begin{cases} \mathrm{d}x(t) = F(t, x(t), x(t-d_1(t,r(t))), \nu(t), r(t))\mathrm{d}t \\ \qquad + G(t, x(t), x(t-d_1(t,r(t))), \nu(t), r(t))\mathrm{d}w(t) \\ \nu(t) = H(t, x(t), u(t), r'(t)) \end{cases} \tag{43}$$

on $t \geq 0$, where $d_1 : \mathbb{R}_+ \times \mathcal{S} \to [0, \tau]$ is Borel measurable while F, G and H are measurable functions with $F(t,0,0,0,i) \equiv 0$, $G(t,0,0,0,i) \equiv 0$ and $H(t,0,0,i) \equiv 0$, for all $t \geq 0$ and $i \in \mathcal{S}$. Let $\bar{F}(t, x(t), x(t-d_1(t,r(t))), u(t), \bar{r}(t)) = F(t, x(t), x(t-d_1(t,r(t))), H(t, x(t), u(t), r'(t)), r(t))$, $\bar{G}(t, x(t), x(t-d_1(t,r(t))), u(t), \bar{r}(t)) = G(t, x(t), x(t-d_1(t,r(t))), H(t, x(t), u(t), r'(t)), r(t))$, and $d_{1r(t)}(t) = d_1(t, r(t))$. We assume $\bar{F}$ and $\bar{G}$ satisfy the local Lipschitz condition and the linear growth condition. Then, the closed-loop system,

$$\begin{aligned} \mathrm{d}x(t) &= \bar{F}_{ij}(t, x(t), x(t-d_{1i}(t)), u(t))\mathrm{d}t \\ &+ \bar{G}_{ij}(t, x(t), x(t-d_{1i}(t)), u(t))\mathrm{d}w(t) \end{aligned} \tag{44}$$

has unique solution on $t \geq -\tau$.

In fact, system (44) is a special case of (4) when $\bar{f}_{ij}(t, \varphi(0), \varphi, u) = \bar{F}_{ij}(t, \varphi(0), \varphi(-d_{1i}(t)), u)$ and $\bar{g}_{ij}(t, \varphi(0), \varphi, u) = \bar{G}_{ij}(t, \varphi(0), \varphi(-d_{1i}(t)), u)$ for $(\varphi, t, i, j) \in C([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+ \times \mathcal{S} \times \mathcal{S}$.

In the following, we use Theorem 4.1 to establish a useful criterion for system (44).

Corollary 5.1: If there exist functions $\alpha_1 \in \mathcal{K}_\infty$, $\alpha_2 \in \mathcal{CK}_\infty$, $\chi \in \mathcal{K}$, some numbers $\mu \geq 1$, $q > 1$, $\lambda_k > 0$, $\lambda_{k1} > 0$, $k = 1, 2$, $0 < \varsigma < 1$ and $V(x(t), t, \bar{r}(t)) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S} \times \mathcal{S}; \mathbb{R}_+)$, such that (13) and (17) holds, and moreover, for any $l \in \mathbb{N}_+$,

$$\begin{aligned} &\mathfrak{L}V(x(t), y_1(t), t, \bar{r}(t)) \leq -\lambda_1 V(x(t), t, \bar{r}(t)) \\ &+ \lambda_{11} \min_{m,n \in \mathcal{S}} \{V(y_1(t), t-d_{1i_l}(t), m, n)\} \\ &+ \chi(\|u\|_{[0,\infty)}), \ t \in [\bar{t}_{2l-2}, \bar{t}_{2l-1}) \end{aligned} \tag{45}$$

and

$$\begin{aligned} &\mathfrak{L}V(x(t), y_1(t), t, \bar{r}(t)) \leq \lambda_2 V(x(t), t, \bar{r}(t)) \\ &+ \lambda_{21} \min_{m,n \in \mathcal{S}} \{V(y_1(t), t-d_{1i_l}(t), m, n)\} \\ &+ \chi(\|u\|_{[0,\infty)}), \ t \in [\bar{t}_{2l-1}, \bar{t}_{2l}) \end{aligned} \tag{46}$$

where $y_1(t) = x(t - d_1(t, r(t)))$. Further, if there exists $\lambda_0 > 0$, and $\bar{\lambda}_1 = \lambda_1 - q\lambda_{11} - \lambda_0 > 0$, $\bar{\lambda}_2 = \lambda_2 + q\lambda_{21} + \lambda_0 > 0$, $\hat{\lambda}_1 \in (0, \bar{\lambda}_1)$ and $\hat{\lambda}_2 \in (\bar{\lambda}_2, \infty)$ such that

$$e^{\hat{\lambda}_1 \tau} \leq q \tag{47}$$

and

$$\mu^2 \bar{\pi} e^{(\hat{\lambda}_1 + \hat{\lambda}_2)d} - \tilde{\pi} \leq \varsigma \hat{\lambda}_1 \tag{48}$$

Then, system (44) is SISS.

Proof: From (45) and (46), there exists $0 < \lambda_0 < \lambda_1$ such that

$$\begin{aligned} &|x(t)| \geq \bar{\chi}(\|u\|_{[0,\infty)}) \Rightarrow \mathfrak{L}V(x(t), y_1(t), t, \bar{r}(t)) \\ &\leq \lambda_{11} \min_{m,n \in \mathcal{S}} V(y_1(t), t-d_{1i_l}(t), m, n) \\ &- \tilde{\lambda}_1 V(x(t), t, \bar{r}(t)), \ t \in [\bar{t}_{2l-2}, \bar{t}_{2l-1}) \end{aligned} \tag{49}$$

and

$$\begin{aligned} &|x(t)| \geq \bar{\chi}(\|u\|_{[0,\infty)}) \Rightarrow \mathfrak{L}V(x(t), y_1(t), t, \bar{r}(t)) \\ &\leq \lambda_{21} \min_{m \in \mathcal{S}} V(y_1(t), t-d_{1i_l}(t), m, n) \\ &+ \tilde{\lambda}_2 V(x(t), t, \bar{r}(t)), \ t \in [\bar{t}_{2l-1}, \bar{t}_{2l}) \end{aligned} \tag{50}$$

for any $l \geq 0$, where $\tilde{\lambda}_1 = \lambda_1 - \lambda_0 > 0$, $\tilde{\lambda}_2 = \lambda_2 + \lambda_0$, and $\bar{\chi}(s) = \lambda_0^{-1} \alpha_1^{-1} \circ \chi(s)$. Clearly, $\bar{\chi}(\cdot) \in \mathcal{K}$. By using Fatou's lemma, we have

$$\begin{aligned} &|x(t)| \geq \bar{\chi}(\|u\|_{[0,\infty)}) \Rightarrow \mathbb{E}\{\mathfrak{L}V(x(t), y_1(t), t, \bar{r}(t))\} \\ &\leq -\bar{\lambda}_1 \mathbb{E}\{V(x(t), t, \bar{r}(t))\}, \ t \in [\bar{t}_{2l-2}, \bar{t}_{2l-1}) \end{aligned}$$

and

$$\begin{aligned} &|x(t)| \geq \bar{\chi}(\|u\|_{[0,\infty)}) \Rightarrow \mathbb{E}\{\mathfrak{L}V(x(t), y_1(t), t, \bar{r}(t))\} \\ &\leq \bar{\lambda}_2 \mathbb{E}\{V(x(t), t, \bar{r}(t))\}, \ t \in [\bar{t}_{2l-1}, \bar{t}_{2l}) \end{aligned}$$

whenever (15) holds. Thus, all the conditions in the Theorem 4.1 are satisfied, which means system (44) is SISS. ■

Corollary 5.2: Under the hypotheses of Corollary 5.1, system (44) is also α_1-ISSiM. Specially, if $\alpha_1(s) = c_1 s^p$, $\alpha_2(s) = c_2 s^p$, where c_1 and c_2 are positive numbers, system (44) is pth moment ISS.

From the definitions of SISS and pth moment ISS, we can find that, if the input $u = 0$, a SISS/pth moment ISS system is GASiP/pth moment stable. A pth moment ISS system is also SISS. Therefore, in what follows, for a class of asynchronous HSDSs, only the conditions of the pth moment ISS will be given.

Consider the following system,

$$\begin{aligned}\mathrm{d}x(t) &= [A(r(t))x(t) + B(r(t))\nu(t)\\ &+ f(t, x(t-d_1(t,r(t))), r(t))]\mathrm{d}t + [C(r(t))x(t)\\ &+ g(t, x(t-d_1(t,r(t))), r(t))]\mathrm{d}w(t) \end{aligned} \tag{51}$$

where $x(t) \in \mathbb{R}^n$, $\nu(t) \in \mathcal{L}^l_\infty$. (For such system, the linear case with constant delay has been discussed in [43] and the reference therein.) Assume that $|f(t, x(t-d_1(t,r(t))), r(t))| \leq \|U_1(r(t))\||x(t-d_1(t,r(t)))|$, $|g(t, x(t-d_1(t,r(t))), r(t))| \leq \|U_2(r(t))\| \times|x(t-d_1(t,r(t)))|$.

The mode-dependent controller is designed as

$$\nu(t) = K(r'(t))x(t) + u(t) \tag{52}$$

where $u(t)$ is the reference input. For convenience, when $r(t) = i$, for any operate h, let h_i denote $h(i)$, and $y_1(t) = x(t-d_{1i}(t))$. Then, the closed-loop system is

$$\begin{aligned}\mathrm{d}x(t) &= [A_i x(t) + B_i K_j x(t) + B_i u(t)\\ &+ f_i(t, y_1(t))]\mathrm{d}t + [C_i x(t) + g_i(t, y_1(t))]\mathrm{d}w(t)\end{aligned} \tag{53}$$

Taking $V(x(t), \bar{r}(t)) = x^T(t)P(\bar{r}(t))x(t)$, where $P(\bar{r}(t)) = P^T(\bar{r}(t)) > 0$, if for any $\varepsilon_i > 0$, $i = 1, 2, 3$, such that

$$\begin{bmatrix} \Sigma_{111} & \Sigma_{112} & \Sigma_{113} \\ * & \Sigma_{122} & \Sigma_{123} \\ * & * & \Sigma_{133} \end{bmatrix} < 0 \tag{54}$$

$$\begin{bmatrix} \Sigma_{211} & X_{ii} \\ * & -\lambda_{11}X_{ii} \end{bmatrix} < 0 \tag{55}$$

$$\begin{bmatrix} \Sigma_{311} & \Sigma_{312} & \Sigma_{313} \\ * & \Sigma_{322} & \Sigma_{323} \\ * & * & \Sigma_{333} \end{bmatrix} < 0 \tag{56}$$

$$\begin{bmatrix} \Sigma_{411} & X_{ij} \\ * & -\lambda_{21}X_{ij} \end{bmatrix} < 0 \tag{57}$$

where $X_{ii} = P_{ii}^{-1}$, $X_{ij} = P_{ij}^{-1}$, $P_{ii} < \beta_1 I$ and $P_{ij} < \beta_2 I$, $\Sigma_{111} = \frac{1}{\beta_2\pi_{ii}}I$, $\Sigma_{112} = X_{ii}$, $\Sigma_{113} = 0$, $\Sigma_{122} = -\frac{1}{1+\varepsilon_3}X_{ii}$, $\Sigma_{123} = C_iX_{ii}$, $\Sigma_{133} = X_{ii}A_i^T + A_iX_{ii} + 2B_iY_{ii} + \pi_{ii}X_{ii} + \varepsilon_1 B_iB_i^T + \varepsilon_2 I + \lambda_1 X_{ii}$, $\Sigma_{211} = -(\varepsilon_2^{-1}\|U_{1i}\|^2 I + (1+\varepsilon_3^{-1})\beta_1\|U_{2i}\|^2 I)^{-1}I$, $\Sigma_{311} = -\frac{1}{\pi^0_{ji}}X_{ii}$, $\Sigma_{312} = X_{ij}$, $\Sigma_{313} = 0$, $\Sigma_{322} = -\frac{1}{1+\varepsilon_3}X_{ij}$, $\Sigma_{323} = C_iX_{ij}$, $\Sigma_{333} = X_{ij}A_i^T + A_iX_{ij} + 2B_iK_jX_{ij} - \pi^0_{ji}X_{ij} + \varepsilon_1 B_iB_i^T + \varepsilon_2 I - \lambda_2 X_{ij}$, $\Sigma_{411} = -(\varepsilon_2^{-1}\|U_{1i}\|^2 I + (1+\varepsilon_3^{-1})\beta_2\|U_{2i}\|^2 I)^{-1}I$. Let $\chi(s) = \varepsilon_1^{-1}s^2$, and if there exists $\mu \geq 1$, $q > 1$, $\lambda_0 > 0$, such that (17), (47), (48) hold, where $\bar{\lambda}_1 = \lambda_1 - q\lambda_{11} - \lambda_0 > 0$, $\bar{\lambda}_2 = \lambda_2 + q\lambda_{21} + \lambda_0 > 0$, $\hat{\lambda}_1 \in (0, \bar{\lambda}_1)$ and $\hat{\lambda}_2 \in (\bar{\lambda}_2, \infty)$. Then from Corollary 5.2, system (53) is 2th moment ISS. For more details, see Appendix A.

For given system (51) with asynchronous controller (52), when doing the stability analysis, we first give μ, λ_1, λ_2, λ_{11} and λ_{21}, which meet the conditions of Corollary 5.2. If there exist ε_1, ε_2, ε_3, β_1 and β_2, such that (54) and (55) hold, then we can get P_{ii} and the candidate controllers gains K_i, where $i \in \mathcal{S}$. To verify the effectiveness of the candidate controllers, we need to solve (56), (57) and (17). If a feasible solution exists, then one can obtain P_{ij} and the admissible controllers gains, where $i, j \in \mathcal{S}$, $j \neq i$.

Example 5.1: To demonstrate the effectiveness, we choose the parameters in system (53) as $A_1 = [1.5, 1.5; 0, -3]$, $A_2 = [-0.5, 10; 15, 2.5]$, $B_1 = [-1, 2; 0, -1]$, $B_2 = [-2, 1; 0, 2]$, $C_1 = [0.1, 0; 0, 0.1]$, $C_2 = [0.2, 0; 0.1, 0.2]$, and

$$f_1(t, y_1(t)) = \begin{bmatrix} 0.1\cos(t) & 0.1 \\ 0 & -0.1\sin(t) \end{bmatrix} y_1(t)$$

$$f_2(t, y_1(t)) = \begin{bmatrix} 0.1(\cos(t))^2 & 0 \\ 0 & 0.1\sin(t) \end{bmatrix} y_1(t)$$

$$g_1(t, y_1(t)) = \begin{bmatrix} 0.1\cos(t) & 0 \\ 0 & -0.1\sin(t) \end{bmatrix} y_1(t)$$

$$g_2(t, y_1(t)) = \begin{bmatrix} 0.1\cos(t) & 0 \\ 0.1 & 0.1(\sin(t))^2 \end{bmatrix} y_1(t)$$

Then, we have $|f_1(t, y_1(t))| \leq \|U_{11}\||y_1(t)|$, $|f_2(t, y_1(t))| \leq \|U_{12}\||y_1(t)|$, $|g_1(t, y_1(t))| \leq \|U_{21}\||y_1(t)|$, $|g_2(t, y_1(t))| \leq \|U_{22}\||y_1(t)|$, where $U_{11} = [0.1, 0.1; 0, -0.1]$, $U_{12} = [0.1, 0; 0, 0.1]$, $U_{21} = [0.1, 0; 0, -0.1]$, $U_{22} = [0.1, 0; 0.1, 0.1]$, and $d_{11}(t) = 0.05\cos(2t)$, $d_{12}(t) = 0.07\sin(t)$, $d_{21}(t) = 0.06\sin(t)$, $d_{22}(t) = 0.08\cos(t)$, $\tau = 0.08$. We also assume that $d = 0.2$, and $\Pi = [-0.01, 0.01; 0.01, -0.01]$, $\Pi^0 = [-70, 70; 50, -50]$.

According to above analysis, we choose $\varepsilon_1 = 0.1$, $\varepsilon_2 = 0.6$, $\varepsilon_3 = 1.8$, $\lambda_1 = 20$, $\lambda_2 = 18$, $\lambda_{11} = 1.5$, $\lambda_{21} = 1.5$, $\beta_1 = 8$, $\beta_2 = 3$ and $\mu = 1.5$. There exists $\lambda_0 = 0.01$, $q = 2$, such that $\bar{\lambda}_1 = 16.99$, $\bar{\lambda}_2 = 21.01$. Further, there exists $\hat{\lambda}_1 = 5.097 \in (0, 16.99)$ and $\hat{\lambda}_2 = 21.031 \in (21.01, \infty)$, such that $2 = q > e^{\hat{\lambda}_1\tau} = 1.5034$. It's not difficult to verify that (48) holds with those parameters and $\varsigma = 0.99$, $\bar{\pi} = \tilde{\pi} = 0.01$. By solving (17), (54)-(57), one can obtain that $P_{11} = [0.1854,\ 0;\ 0,\ 0.1854]$, $P_{12} = [0.2670,\ -0.0011;\ -0.0011,\ 0.2703]$, $P_{21} = [0.1943,\ 0.0446;\ 0.0446,\ 0.5675]$, $P_{22} = [0.3826,\ 0.0004;\ 0.0004,\ 0.3823]$, $K_1 = [14.0981,\ 20.9377;\ -0.0706,\ 9.7021]$, $K_2 = [8.3710,\ 9.5497;\ 1.4964,\ -9.0793]$.

The simulation results are shown in Fig.1-Fig.5. Among them, Fig.1 shows the Markovian switching signal which includes the real switching signal and the detected switching signal with non-zero detection delay. The detected switching signal also includes both the case

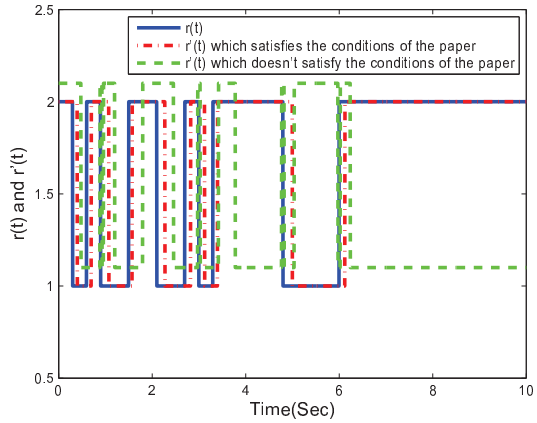

Fig. 1. The switching signal $r(t)$ and the detected $r'(t)$.

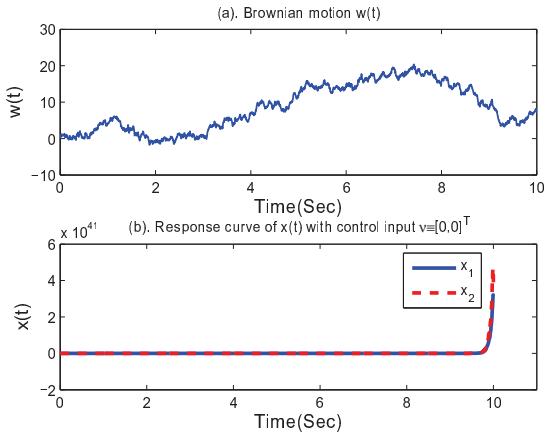

Fig. 2. Response curve of $w(t)$ and $x(t)$.

which $r'(t)$ satisfies the conditions of the Corollary 5.2 and the case which $r'(t)$ doesn't satisfy the conditions of the Corollary 5.2. In the later case, the maximum detection delay is larger than 0.3, then $\mu^2\bar{\pi}e^{(\hat{\lambda}_1+\hat{\lambda}_2)\times 0.3} - \tilde{\pi} = 57.0534 > \hat{\lambda}_1$. Moreover, in order to distinguish the $r'(t)$, we let value 1.1 and 2.1 to express the mode 1 and mode 2 of $r'(t)$ which doesn't satisfy the conditions. Fig.2(a) shows the curve of Brownian motion $w(t)$; Fig.2(b) shows the state trajectories under control input $\nu(t) \equiv 0$, with initial data $x_0 = [3, -1.5]$. Obviously, system (51) under $\nu(t) \equiv 0$ is unstable, i.e., the open-loop system is unstable. Fig.3-Fig.5 show the stability of the closed-loop system, also with initial data $x_0 = [3, -1.5]$. Among them, Fig.3(a), Fig.4(a) and Fig.5(a) show the stability under the strictly synchronous controller, where the reference input $u(t)$, respectively, equals to $[0, 0]^T$, $[3, 3]^T$ and $[3e^{-0.4t}, 5e^{-0.7t}]^T$. The so-called strictly synchronous controller means that the controller in (52) relies not on the the detected switching signal $r'(t)$ but on actual $r(t)$. It can be inferred from them that the system under synchronous switching is

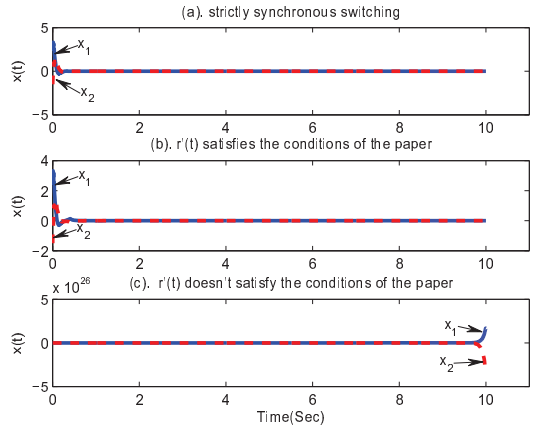

Fig. 3. Response curve of $x(t)$ with reference input $u \equiv [0, 0]^T$.

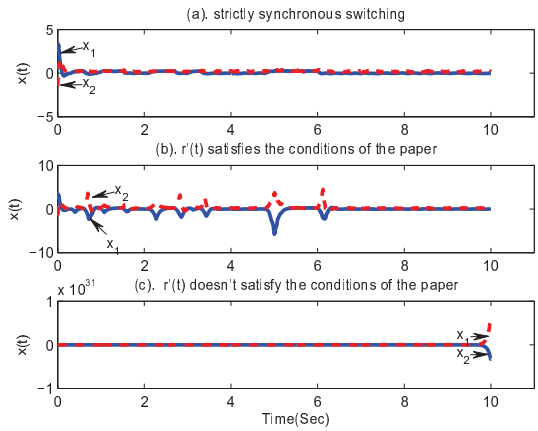

Fig. 4. Response curve of $x(t)$ with reference input $u \equiv [3, 3]^T$.

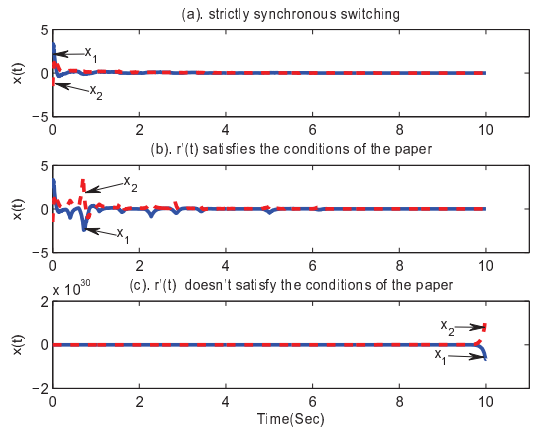

Fig. 5. Response curve of $x(t)$ with reference input $u = [3e^{-0.4t}, 5e^{-0.7t}]^T$.

stable. On the other hand, Fig.3(b), Fig.4(b) and Fig.5(b) show the stability under $r'(t)$ which satisfies the conditions of Corollary 5.2. Obviously, the asymptotic stability and the input-to-state stability under $r'(t)$ which satisfies the conditions can be guaranteed. But compared with Fig.3(a), Fig.4(a) and Fig.5(a), one can see that the mismatched controller which caused by the non-zero detection delay has a great influence on the performance of the system. And moreover, when $r'(t)$ doesn't satisfy the conditions of Corollary 5.2, the system is unstable, as shown in Fig.3(c), Fig.4(c) and Fig.5(c), which corresponds to Fig.3(b), Fig.4(b) and Fig.5(b), respectively. In addition, from Fig.3(a) and Fig.3(b), we can see that the closed-loop system (53) is asymptotically stable, which is in accordance with the assertion that an ISS system is necessarily asymptotically stable. In Fig.4(a) and Fig.4(b), due to the effect of reference input u, the state $x(t)$ will not converge to zero. But, it still remains bounded. In Fig.5(a) and Fig.5(b), since $|u(t)| \to 0$ as $t \to \infty$, system (53) is asymptotically stable, which is also in accordance with Remark 3.1 in [41].

VI. Conclusion

We have examined the stability of a class of hybrid stochastic retarded systems under asynchronous switching, where the detection delay is modeled as a Markovian process. The Razumikhin-type conditions are extended to the interval of asynchronous switching before the matched controller is applied, which allows the Lyapunov functionals to increase during the running time of subsystems. Motivated by asynchronous deterministic switched systems, i.e., the stability of closed-loop systems can be guaranteed by a sufficient large average-dwell time, by considering the properties of Markov process, the conditions of the existence of the admissible asynchronous controller for global asymptotic stability and input-to-state stability are derived. It is shown that the stability of the closed-loop systems can be guaranteed by a sufficient small mode transition rate. The main results have also been applied to a class of hybrid stochastic delay systems, and a numerical example has been provided to demonstrate the effectiveness.

This study ignores the error of the detector (or, false alarms), which makes the analysis more difficult. The asynchronous stability problems with both detection delay and the false alarms, robust stabilization of general nonlinear systems under asynchronous switching, etc., will be our future objective.

Appendix A

Let $V(x(t),\bar{r}(t)) = x^T(t)P(\bar{r}(t))x(t)$, where $P(\bar{r}(t)) = P^T(\bar{r}(t)) > 0$. For any $i,j \in \mathcal{S}$, there exist $\beta_1 > 0$ and $\beta_2 > 0$ such that $P_{ii} < \beta_1 I$ and $P_{ij} < \beta_2 I$, where I is an identity matrix with an appropriate dimension. Since $P_{ij} = P_{ij}^T > 0$, there exists a low-triangular matrix L_{ij} such that $P_{ij} = L_{ij}L_{ij}^T$. From [44], $HFE + E^TF^TH^T \le \varepsilon HH^T + \varepsilon^{-1}E^TE$, $\forall \varepsilon > 0$, when $FF^T \le I$. Then, for any time-interval $[\bar{t}_{2l-1}, \bar{t}_{2l})$, if there exists $\lambda_2 > 0$, $\lambda_{21} > 0$,

$$\begin{aligned}
& \mathfrak{L}V(x(t),y_1(t),i,j) \\
\le\ & x^T(t)[A_i^TP_{ij} + P_{ij}A_i + C_i^TP_{ij}C_i + 2P_{ij}B_iK_j \\
& +\pi_{ji}^0P_{ii} - \pi_{ji}^0P_{ij}]x(t) + 2x^T(t)P_{ij}B_iu(t) \\
& +2x^T(t)P_{ij}f_i(t,y_1(t)) + 2x^T(t)C_i^TP_{ij}g_i(t,y_1(t)) \\
& +g_i^T(t,y_1(t))P_{ij}g_i(t,y_1(t)) \\
\le\ & x^T(t)[A_i^TP_{ij} + P_{ij}A_i + (1+\varepsilon_3)C_i^TP_{ij}C_i \\
& +2P_{ij}B_iK_j + \pi_{ji}^0P_{ii} - \pi_{ji}^0P_{ij} + \varepsilon_1P_{ij}B_iB_i^TP_{ij} \\
& +\varepsilon_2P_{ij}P_{ij}]x(t) + \varepsilon_1^{-1}u^T(t)u(t) \\
& +[\varepsilon_2^{-1}\|U_{1i}\|^2 + (1+\varepsilon_3^{-1})\beta_2\|U_{2i}\|^2]y_1^T(t)y_1(t) \\
\le\ & \lambda_2x^T(t)P_{ij}x(t) + \lambda_{21}y_1^T(t)P_{ij}y_1(t) + \varepsilon_1^{-1}|u(t)|^2
\end{aligned}$$

for any $\varepsilon_i > 0$, $i = 1,2,3,4$. Similarly, when $t \in [\bar{t}_{2l}, \bar{t}_{2l+1})$, if there also exists $\lambda_1 > 0$, $\lambda_{11} > 0$, such that

$$\begin{aligned}
& \mathfrak{L}V(x(t),y_1(t),y_2(t),i,i) \\
\le\ & x^T(t)[A_i^TP_{ii} + P_{ii}A_i + (1+\varepsilon_3)C_i^TP_{ii}C_i \\
& +2P_{ii}B_iK_i + \pi_{ii}P_{ii} - \pi_{ii}\beta_2I + \varepsilon_1P_{ii}B_iB_i^TP_{ii} \\
& +\varepsilon_2P_{ii}P_{ii}]x(t) + \varepsilon_1^{-1}u^T(t)u(t) \\
& +[\varepsilon_2^{-1}\|U_{1i}\|^2 + (1+\varepsilon_3^{-1})\beta_1\|U_{2i}\|^2]y_1^T(t)y_1(t) \\
\le\ & -\lambda_1x^T(t)P_{ii}x(t) + \lambda_{11}y_1^T(t)P_{ii}y_1(t) + \varepsilon_1^{-1}|u(t)|^2
\end{aligned}$$

Then,

$$\begin{aligned}
& A_i^TP_{ii} + P_{ii}A_i + (1+\varepsilon_3)C_i^TP_{ii}C_i + 2P_{ii}B_iK_i \\
& +\pi_{ii}P_{ii} - \pi_{ii}\beta_2I + \varepsilon_1P_{ii}B_iB_i^TP_{ii} \\
& +\varepsilon_2P_{ii}P_{ii} + \lambda_1P_{ii} < 0 \qquad (58) \\
& A_i^TP_{ij} + P_{ij}A_i + (1+\varepsilon_3)C_i^TP_{ij}C_i + 2P_{ij}B_iK_j \\
& +\pi_{ji}^0P_{ii} - \pi_{ji}^0P_{ij} + \varepsilon_1P_{ij}B_iB_i^TP_{ij} \\
& +\varepsilon_2P_{ij}P_{ij} - \lambda_2P_{ij} < 0 \qquad (59)
\end{aligned}$$

and

$$\varepsilon_2^{-1}\|U_{1i}\|^2I + (1+\varepsilon_3^{-1})\beta_1\|U_{2i}\|^2I - \lambda_{11}P_{ii} < 0 \quad (60)$$
$$\varepsilon_2^{-1}\|U_{1i}\|^2I + (1+\varepsilon_3^{-1})\beta_2\|U_{2i}\|^2I - \lambda_{21}P_{ij} < 0 \quad (61)$$

Using P_{ii}^{-1} to pre- and post- multiply the left term of equation (58) and (60) respectively yields (54) and (55) hold. Similarly, using P_{ij}^{-1} to pre- and post- multiply the left term of equation (59) and (61) respectively yields (56) and (57) hold.

Thus, when let $\chi(s) = \varepsilon_1^{-1} s^2$, and if there exists $\mu \geq 1$, $q > 1$, $\lambda_0 > 0$, such that (17), (47), (48) and (54)-(57) hold, Then, according to Schurs complement and Corollary 5.2, system (53) is 2th moment ISS.

REFERENCES

[1] D. Liberzon, *Switching in systems and control*, Boston, MA:Birkhauser, 2003.

[2] Z. D. Sun, S. S. Ge, *Stability theory of switched dynamical systems*, London, Springer-Verlag, 2011.

[3] M. S. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Trans. on Automatic Control*, vol.43, no.4, pp. 475-482, 1998.

[4] J. P. Hespanha, A. S. Morse, "Stability of switched systems with average dwell-time," *Proc. of the 38th IEEE Conf. Decision Control*, pp. 2655-2660, 1999.

[5] H. Lin, P. J. Antsaklis, "Stability and stabilizability of switched linear systems: a survey of recent results," *IEEE Trans. on Automatic Control*, vol.54, no.2, pp. 308-322, 2009.

[6] N. N. Krasovskii, E. A. Lidskii, "Analysis design of controller in systems with random attributes-Part 1," *Automatic Remote Control*, vol. 22, pp. 1021-1025, 1961.

[7] M. Mariton, *Jump linear systems in automatic control*, NewYork:Marcel Dekker, 1990.

[8] L. X. Zhang, E. K. Boukas, "Stability and stabilization of Markovian jump linear systems with partly unknown transition probabilities," *Automatica*, vol.45, no.2, pp. 463-468, 2009.

[9] L. G. Wu, P. Shi, H. J. Gao, "State estimation and sliding-mode control of Markovian jump singular systems," *IEEE Trans. on Automatic Control* vol.55, no.5, pp. 1213-1219, 2010.

[10] L. X. Zhang, E. K. Boukas, L. Baron, H. Reza, "Fault detection for discrete-time Markov jump linear systems with partially known transition probabilities," *Int. J. Control*, vol. 83, no. 8, pp. 1564-1572, 2010.

[11] H. J. Gao, Z. Y. Fei, J. Lam, B. Z. Du, "Further results on exponential estimates of Markovian jump systems with mode-dependent time-varying delays," *IEEE Trans. on Automatic Control*, vol.56, no.1, pp. 223-229, 2011.

[12] L. G. Wu, P. Shi, H. J. Gao, C. H. Wang, "H_∞ filtering for 2D Markovian jump systems," *Automatica*, vol.44, no.7, pp. 1849-1858, 2008.

[13] P. Shi, M. Liu, "Discussion 'On the filtering problem for continuous-time Markov jump linear systems with no observation of the Markov chain'," *European J. of Control*, vol.17, no.4, pp. 355-356, 2011.

[14] Z. Y. Fei, H. J. Gao, P. Shi, "New results on stabilization of Markovian jump systems with time delay," *Automatica*, vol.45, no.10, pp. 2300-2306, 2009.

[15] Z. G. Wu, P. Shi, H. Y. Su, J. Chu, "Passivity analysis for discrete-time stochastic Markovian jump neural networks with mixed time-delays," *IEEE Trans. on Neural Networks*, vol.22, no.10, pp. 1566-1575, 2011.

[16] H. L. Dong, Z. L. Wang, H. J. Gao, "Fault detection for Markovian jump systems with sensor saturations and randomly varying nonlinearities," *IEEE Trans. on Circuits and Systems-Part I*, vol.59, no.10, pp. 2354-2362, 2012.

[17] Z. D. Wang, Y. Liu, X. Liu, "Exponential stabilization of a class of stochastic system with Markovian jump parameters and mode-dependent mixed time-delays," *IEEE Trans. on Automatic Control*, vol.55, no.7, pp. 1656-1662, 2010.

[18] M. K. Ghosh, A. Arapostathis, S. I. Marcus, "Optimal control of switching diffusions with application to flexible manufacturing systems," *SIAM J. Control and Optimization*, vol. 31, no. 5, pp. 1183-1204, 1993.

[19] D. D. Sworder, R. O. Rogers, "An LQ-solution to a control problem associated with a solar thermal central receiver," *IEEE Trans. on Automatic Control*, vol.AC-28, no.10, pp. 971-978, 1983.

[20] Y. Kang, Z. J. Li, Y. F. Dong, H. S. Xi, "Markovian-based fault-tolerant control for wheeled mobile manipulators," *IEEE Trans. on Control Systems Technology*, vol.20, no.1, pp. 266-276, 2012.

[21] X. D. Zhao, L. X. Zhang, P. Shi, M. Liu, "Stability and stabilization of switched linear systems with mode-dependent average dwell time," *IEEE Trans. on Automatic Control*, vol.57, no.7, pp. 1809-1815, 2012.

[22] X. D. Zhao, L. X. Zhang, P. Shi, M. Liu, "Stability of switched positive linear systems with average dwell time switching," *Automatica*, vol.48, no.6, pp. 1132-1137, 2012.

[23] J. Lian, F. Zhang, P. Shi, "Sliding mode control of uncertain stochastic hybrid delay systems with average dwell time," *Circuits, Systems, and Signal Processing*, vol.31, no.2, pp. 539-553, 2012.

[24] L. X. Zhang, P. Shi, "H_∞ filtering for a class of switched linear parameter varying systems," *Int. J. Systems Science*, vol.42, no.5, pp. 781-788, 2011.

[25] L. G. Wu, W. C. Daniel, C. W. Li, "H_∞ dynamical output feedback control of switched stochastic systems," *Proc. of the 48th Conf. Decision and Contr.*, pp. 500-505, 2009.

[26] B. Z. Wu, P. Shi, H. Su, J. Chu, "Delay-dependent stability analysis for switched neural networks with time-varying delay," *IEEE Trans. on Syst., Man, Cybern. B, Cybern.*, vol.41, no.6, pp. 1522-1530, 2011.

[27] G. Xie and L. Wang, "Stabilization of switched linear systems with time-delay in detection of switching signal," *Journal of Mathematical Analysis & Applications*, vol.305, no.1, pp. 277-290, 2005.

[28] W. X. Xie, C. Y. Wen, Z. G. Li, "Input-to-state stabilization of switched nonlinear systems," *IEEE Trans. on Automatic Control*, vol.46, no.7, pp. 1111-1116, 2001.

[29] I. Masubuchi and M. Tsutsui, "Advanced performance analysis and robust controller synthesis for time-controlled switched systems with uncertain switchings," *In Proc. IEEE 40th Conf. Decision and Control*, pp. 2466-2471, 2001.

[30] M. S. Mahmoud, P. Shi, "Asynchronous H_∞ filtering of discrete-time switched systems," *Signal Processing*, vol.92, no.10, pp. 2356-2364, 2012.

[31] L. X. Zhang, H. J. Gao, "Asynchronously switched control of swicthed linear systems with average dwell time," *Automatica*, vol.46, pp. 953-958, 2010.

[32] L. X. Zhang, P. Shi, "Stability, l_2-gain and asynchronous H_∞ control of discrete-time switched systems with average dwell time," *IEEE Trans. on Automatic Control*, vol.54, no.9, pp. 2193-2200, 2009.

[33] Z. R. Xiang, R. H. Wang, Q. W. Chen, "Robust stabilization of uncertain stochastic switched non-linear systems under asynchronous switching," *Part I: J. of Systems & Control Engineering*, vol.225, no.1, pp. 8-20, 2011.

[34] Z. R. Xiang, R. H. Wang, Q. W. Chen, "Robust reliable stabilization of stochastic switched nonlinear systems under asynchronous switching," *Applied Mathematics and Computation*, vol.217, no.19, pp. 7725-7736, 2011.

[35] J. L. Xiong, J. Lam, "Stabilization of discrete-time Markovian jump linear systems via time-delayed controllers," *Automatica*, vol.42, no.5, pp. 747-753, 2006.

[36] M. Liu, D. W. C. Ho, Y. G. Niu, "Stabilization of Markovian jump linear system over networks with random communication delay," *Automatica*, vol.45, no.2, pp. 416-421, 2009.

[37] Y. Kang, J. F. Zhang, S. S. Ge, "Robust output feedback H_∞ control of uncertain Markovian jump systems with mode-

dependent time-delays," *Int. J. Control*, vol.81, no.1, pp. 43-61, 2008.

[38] S. J. Liu, J. F. Zhang, Z. P. Jiang, "A notion of stochastic input-to-state stability and its application to stability of cascaded stochastic nonlinear systems," *Acta Math. Applicatae Sinica, English Series*, vol.24, no.1, pp. 141-156, 2008.

[39] P. Zhao, Y. Kang, D. H. Zhai, "On input-to-state stability of stochastic nonlinear systems with Markovian jumping parameters," *Int. J. Control*, vol.85, no.4, pp. 343-349, 2012.

[40] X. R. Mao, "Razumikhin-type theorems on exponential stability of stochastic functional differential equations," *Stochastic Processes and their Applications*, vol. 65, pp. 233-250, 1996.

[41] L. R. Huang, X. R. Mao, "On input-to-state stability of stochastic retarded systems with Markovian switching," *IEEE Trans. on Automatic Control*, vol.54, no.8, pp. 1898-1902, 2009.

[42] D. Chatterjee, D. Liberzon, "On stability of randomly switched nonlinear systems," *IEEE Trans. on Automatic Control*, vol.52, no.12, pp. 2390-2394, 2007.

[43] C. G. Yuan, X. R. Mao, "Robust stability and controllability of stochastic differential delay equations with Markovian switching," *Automatica*, vol.40, no.3, pp. 343-354, 2004.

[44] L. H. Xie, "Output feedback H_∞ control of systems with parameter uncertainty," *Int. J. Control*, vol.63, no.4, pp. 741-750, 1996.