# Tradeoffs between transmission intervals and delays for decentralized networked control systems based on a gain assignment approach

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***Abstract*—This paper investigates the stability problem of interconnected networked control systems (NCSs) in which there are many local networks and each network is independent of the others. Several transmission imperfections including variable transmission intervals, variable transmission delays and communication constraints, are permitted by each local network. A gain assignment criterion is proposed to assign the gain closely related to the maximum allowable transmission interval (MATI) and maximum allowable delay (MAD) of each local network. Moreover, the tradeoffs between the MATI and MAD of each local network can be computed by solving a series of differential equations with the acquired gains. Finally, a common used example is given to show the improvement of the obtained results over the existing literature.**

***Index Terms*—Networked control systems, interconnected systems, communication constraints, transmission delays.**

## I. Introduction

Studies on NCSs have attracted a lot of attention in recent years due to the great benefit of NCSs over the traditional control systems, such as the low cost and easy installation and maintenance [1]–[5]. However, the existence of network may degrade the system performance and even destabilize the NCSs due to the existence of network imperfections including variable transmission intervals, variable delays, communication constraints, quantization effects and packet dropouts [6], [3]. Quantitative analysis for these influences has attracted a lot of investigation (see [1]–[3], [6]–[8], and references cited therein).

Many existing literature focusing on stability analysis of NCSs with transmission protocols consider only one global network (see e.g. [1], [2], [6], [7]). The communication in the network is hence synchronized according to a global clock. While there are also many situations such as large-scale systems in which the NCSs consist of a number of local networks and each network is independent of the others. Each local network has its own clock and thus the whole NCSs communicate asynchronously. This problem has not received much consideration in the existing literature. The authors in [9] consider the stability of interconnected networked control systems in which each local network is affected by variable transmission intervals, variable small delays and disturbances by small gain theorem, and connect the input-to-state gains to the network parameters such as MATI and MAD specifically. The paper [10] models the interconnected NCS into a hybrid system consisting of multiple subsystems and combines the emulation-based stability analysis for NCSs and technique from large scaled systems. Explicit MATI of each local network is given in [10]. However, both of the results in [9] and [10] may be conservative since the Lyapunov function is chosen for each local network randomly and no effective gain assignment approaches are given to assign the gain closely related to the MATI and MAD. How to improve the MATI and MAD motivates the research of this paper.

In this paper, we consider the stability of interconnected NCSs in which each local network is affected by variable transmission intervals, variable small delays and communication constraints. The considered system is linear and a hybrid technique is adopted to analyze the stability of such systems inspired from [6]. The main contribution is concluded as the following points. A common Lyapunov function is used for the $x$ subsystem and a centralized gain assignment approach is proposed to assign the gains with respect to different $e_i$ subsystems. Such gain assignment approach can effectively optimize the gain closely related to the MATI and MAD. The tradeoffs between MATI and MAD of each local network can be given by solving a series of differential equations. Finally, a common used example shows that the MATI and MAD acquired by our approaches for each local network are much less conservative compared with those in [9]. Even in free of delays, the MATI is also much less conservative than that acquired in [6].

$|.|$ and $\langle .,. \rangle$ denote respectively the Euclidean norm and the usual inner product of real vectors. Given a matrix $Q$, $|Q|$ denotes the matrix 2-norm. For $N \in \mathbb{N}$, $\bar{N} = \{1, 2, \cdots, N\}$. For $l$ vectors $x_i \in \mathbb{R}^{n_i}, i = 1, 2, \cdots, l$, denote $(x_1, x_2, \cdots, x_l) = [x_1^T, x_2^T, \cdots, x_l^T]^T$. $1_N \in \mathbb{R}^N$ consists of element 1 and $0_N \in \mathbb{R}^N$ consists of element 0. $\wedge$ and $\vee$ denote logical 'and' and 'or', respectively.

## II. Interconnected networked control systems

Consider the following interconnected continuous-time linear systems with subsystems $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_N$

$$\mathcal{P}_i : \begin{cases} \dot{x}_{p,i} = A_{p,i,i}x_{p,i} + \sum\limits_{j \neq i} A_{p,j,i}x_{p,j} + B_{p,i}\hat{u}_i \\ y_i = C_{p,i}x_{p,i} \end{cases} i \in \bar{N} \tag{1}$$

where $x_{p,i}$ is the state of $\mathcal{P}_i$ subsystem, $\hat{u}_i$ is the actual local control input, and $y_i$ is the output of subsystem $\mathcal{P}_i, i \in \bar{N}$. $A_{p,i,i}$, $B_{p,i}$ and $C_{p,i}$ are respectively the state matrix, input matrix and output matrix of the subsystem $\mathcal{P}_i$.

The plant (1) is controlled by $N$ local controllers $\mathcal{C}_i, i \in \bar{N}$. Each controller communicates with sensors and actuators of the corresponding plant via the communication network $\mathcal{N}_i, i \in \bar{N}$. The $i$-th controller $\mathcal{C}_i$ to $\mathcal{P}_i$ is given as follows

$$\mathcal{C}_i : \begin{cases} \dot{x}_{c,i} = A_{c,i}x_{c,i} + B_{c,i}\hat{y}_i \\ u_i = C_{c,i}x_{c,i} \end{cases} i \in \bar{N} \tag{2}$$

where $x_{c,i}$ is the state of controller $\mathcal{C}_i$, $\hat{y}_i$ is the most recently received information on the output $y_i$ of subsystem $\mathcal{P}_i$, and $u_i$ is the output of controller $\mathcal{C}_i$. $A_{c,i}$, $B_{c,i}$ and $C_{c,i}$ are respectively the state matrix, input matrix and output matrix of the controller $\mathcal{C}_i$.

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The communication networks $\mathcal{N}_1, \mathcal{N}_2, \cdots, \mathcal{N}_N$ are independent with each other. Each local network $\mathcal{N}_i, i \in \bar{N}$ has its own transmission/sampling clock and consists of $n_i \in \mathbb{N}$ nodes. Let $\{t_j^i\}_{j=1}^{\infty}$ be a sequence of sampling times corresponding to the local network $\mathcal{N}_i, i \in \bar{N}$ and there exists a $\delta > 0$ such that $\delta \le t_{j+1}^i - t_j^i$ for all $i \in \bar{N}$ and $j \in \mathbb{N}$. In each network $\mathcal{N}_i$, the sensors sample the output $(y_i, u_i)$ of the plant and the controller at time $t_j^i$, while only the node authorized by the $i$-th transmission protocol can transmit the values via the network $\mathcal{N}_i$. Due to the existence of transmission delays, the transmitted values will arrive at the destination after delay $\tau_j^i$. Next, a standard assumption related to the transmission intervals and delays is introduced ( [6], [3]).

*Assumption 1:* The transmission times satisfy $\delta \le t_{j+1}^i - t_j^i \le \tau_{MATI}^i$ for all $i \in \bar{N}$ and $j \in \mathbb{N}$ and the delays satisfy $0 \le \tau_j^i \le \min\{\tau_{MAD}^i, t_{j+1}^i - t_j^i\}$ for all $i \in \bar{N}$ and $j \in \mathbb{N}$, where $\delta \in (0, \tau_{MATI}^i]$ for all $i \in \bar{N}$ is arbitrary.

The updates of the entries in $\hat{y}_i$ or $\hat{u}_i$ at $t_j^i + \tau_j^i$ satisfy

$$\hat{y}_i((t_j^i + \tau_j^i)^+) = y_i(t_j^i) + h_{y,i}(j, e_i(t_j^i)) \tag{3}$$

$$\hat{u}_i((t_j^i + \tau_j^i)^+) = u_i(t_j^i) + h_{u,i}(j, e_i(t_j^i)) \tag{4}$$

where the functions $h_i = (h_{y,i}, h_{u,i})$ is called the $i-th$ network protocol that can, for instance, be the Round Robin protocol (RR) or the Try-Once-Discard protocol (TOD), $e_i \in \mathbb{R}^{n_{e_i}}$ denotes the vector $(e_i^y, e_i^u)$ with $e_i^y = \hat{y}_i - y_i$ and $e_i^u = \hat{u}_i - u_i$. Between the updates of $\hat{y}_i$ and $\hat{u}_i$, assume

$$\begin{cases} \dot{\hat{y}}_i(t) = 0 \\ \dot{\hat{u}}_i(t) = 0 \end{cases} \qquad t \in [t_j^i + \tau_j^i, t_{j+1}^i + \tau_{j+1}^i]$$

for all $i \in \bar{N}$.

## III. Hybrid model

In this section, the modeling technique is inspired from [6].

Let $x_i = (x_{p,i}, x_{c,i}) \in \mathbb{R}^{\bar{n}_i}$ and $x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^{n_x}$. From [6], the $i$-th interconnection can be modeled into a hybrid system where the flow dynamic is given as

$$\begin{cases} \dot{x}_i = A_i x + B_i e_i \\ \dot{e}_i = M_i x + J_i e_i \\ \dot{l}_i = 0 & (l_i = 1 \wedge \tau_i \in [0, \tau_{MAD}^i]) \\ \dot{\tau}_i = 1 & \vee (l_i = 0 \wedge \tau_i \in [0, \tau_{MATI}^i]) \\ \dot{s}_i = 0 \\ \dot{\kappa}_i = 0 \end{cases} \tag{5}$$

where $A_i = \begin{bmatrix} A_i(1) & A_i(2) & \cdots & A_i(N) \end{bmatrix}$ and

$$A_i(1) = \begin{bmatrix} A_{p,1,i} & B_{p,i}C_{c,i} \\ B_{c,i}C_{p,i} & A_{c,i} \end{bmatrix} \in \mathbb{R}^{\bar{n}_i \times \bar{n}_i}, \tag{6}$$

$$A_i(k) = \begin{bmatrix} A_{p,k,i} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{\bar{n}_i \times \bar{n}_k}, k = 2, \cdots, N \tag{7}$$

$$B_i = \begin{bmatrix} 0 & B_{p,i} \\ B_{c,i} & 0 \end{bmatrix}, \tag{8}$$

$$M_i = \begin{bmatrix} -C_{p,i} \\ -C_{c,i} \end{bmatrix} A_i, J_i = \begin{bmatrix} -C_{p,i} \\ -C_{c,i} \end{bmatrix} B_i. \tag{9}$$

The jump dynamic is described as

$$\begin{aligned} &G_i(x_i^+, e_i^+, l_i^+, \tau_i^+, s_i^+, \kappa_i^+) \\ &= \begin{cases} (x_i, e_i, 1, 0, h_i(\kappa_i, e_i) - e_i, \kappa_i + 1), \\ l_i = 0 \wedge \tau_i \in [\delta, \tau_{MATI}^i]; \\ (x_i, s_i + e_i, 0, \tau_i, -s_i - e_i, \kappa_i), \\ l_i = 1 \wedge \tau_i \in [0, \tau_{MAD}^i]. \end{cases} \end{aligned} \tag{10}$$

Note that in the hybrid system (5)-(10), the variables $\tau_i$, $\kappa_i$, $s_i$ and $l_i$ are respectively the timer variable, counter variable tracking the transmission, storing variable and Boolean identifying the transmission event and the update event [6].

Denote $e = (e_1, e_2, \cdots, e_N) \in \mathbb{R}^{n_e}$, $\tau = (\tau_1, \tau_2, \cdots, \tau_N) \in \mathbb{R}^N$, $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_N) \in \mathbb{R}^N$, $l = (l_1, l_2, \cdots, l_N) \in \mathbb{R}^N$, $s = (s_1, s_2, \cdots, s_N) \in \mathbb{R}^{n_e}$ and $\xi = (x, e, l, \tau, s, \kappa) \in \mathbb{R}^{n_\xi}$ with $n_\xi = n_x + 2n_e + 3N$. The entire interconnected NCSs can be described by the following hybrid system $\mathcal{H}_{NCS}$

$$\mathcal{H}_{NCS} \quad \begin{cases} \dot{\xi} \in F_{NCS}(\xi), \xi \in C_{NCS} \\ \xi^+ \in G_{NCS}(\xi), \xi \in D_{NCS}. \end{cases} \tag{11}$$

The flow dynamic is given as

$$F_{NCS}(\xi) = (Ax + Be, Mx + Je, 0_N, 1_N, 0_{n_e}, 0_N) \tag{12}$$

where

$$A = \begin{bmatrix} A_1 \\ \vdots \\ A_N \end{bmatrix}, B = \begin{bmatrix} B_1 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & B_N \end{bmatrix} \tag{13}$$

$$M = \begin{bmatrix} M_1 \\ \vdots \\ M_N \end{bmatrix}, J = \begin{bmatrix} J_1 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & J_N \end{bmatrix}. \tag{14}$$

The flow set is given as

$$\begin{aligned} C_{NCS} = \{\xi \in \mathbb{R}^{n_\xi} | (x, e) \in \mathbb{R}^{n_x + n_e}, (s, \kappa) \in \mathbb{R}^{n_e} \times \mathbb{N}^N, \\ (l_i, \tau_i) \in (\{0\} \times [0, \tau_{MATI}^i]) \bigcup (\{1\} \times [0, \tau_{MAD}^i]), \\ \forall i \in \bar{N}\}. \end{aligned} \tag{15}$$

The jump dynamic is described as

$$G_{NCS}(\xi) = \{G_i(\xi) : \xi \in D_{NCS,i}\}. \tag{16}$$

$G_i$ is given as

$$\begin{aligned} &G_i(x, e, l, \tau, s, \kappa) = \\ &\begin{cases} (x, e, \bar{l}, \bar{\tau}, \bar{s}, \bar{\kappa}) & (x, e, l, \tau, s, \kappa) \in D_{NCS,i} \wedge l_i = 0 \\ (x, e^*, l^*, \tau, s^*, \kappa) & (x, e, l, \tau, s, \kappa) \in D_{NCS,i} \wedge l_i = 1 \end{cases} \end{aligned} \tag{17}$$

with

$$\begin{aligned} \bar{l} &= (l_1, \cdots, l_{i-1}, 1, l_{i+1}, \cdots, l_N) \\ \bar{\tau} &= (\tau_1, \cdots, \tau_{i-1}, 0, \tau_{i+1}, \cdots, \tau_N) \\ \bar{s} &= (s_1, \cdots, s_{i-1}, h_i(\kappa_i, e_i) - e_i, s_{i+1}, \cdots, s_N) \\ \bar{\kappa} &= (\kappa_1, \cdots, \kappa_{i-1}, \kappa_i + 1, \kappa_{i+1}, \cdots, \kappa_N) \\ e^* &= (e_1, \cdots, e_{i-1}, e_i + s_i, e_{i+1}, \cdots, e_N) \\ l^* &= (l_1, \cdots, l_{i-1}, 0, l_{i+1}, \cdots, l_N) \\ s^* &= (s_1, \cdots, s_{i-1}, -s_i - e_i, s_{i+1}, \cdots, s_N). \end{aligned} \tag{18}$$

The jump set is as follows

$$D_{NCS} = \bigcup_{i=1}^{N} D_{NCS,i} \tag{19}$$

where

$$\begin{aligned} D_{NCS,i} = \{\xi | (x, e) \in \mathbb{R}^{n_x + n_e}, (s, \kappa) \in \mathbb{R}^{n_e} \times \mathbb{N}^N, \\ (l_j, \tau_j) \in (\{1\} \times [0, \tau_{MAD}^j]) \bigcup (\{0\} \times [0, \tau_{MATI}^j]), \\ \forall j \in \{1, 2, \cdots, i-1\}, \\ (l_i, \tau_i) \in (\{1\} \times [0, \tau_{MAD}^i]) \bigcup (\{0\} \times [\delta, \tau_{MATI}^i]), \\ (l_j, \tau_j) \in (\{1\} \times [0, \tau_{MAD}^j]) \bigcup (\{0\} \times [0, \tau_{MATI}^j]), \\ \forall j \in \{i+1, i+2, \cdots, N\}\}. \end{aligned} \tag{20}$$

## IV. Stability analysis

The following conditions require that the transmission protocols of each local network be UGES (uniformly globally exponentially stable) protocols which are proposed in [2].

*Condition 1:* Each local protocol given by $h_i, i \in \bar{N}$ is UGES in the sense that there exists a function $W_I : \mathbb{N} \times \mathbb{R}^{n_{e_i}} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument such that, for all $e_i \in \mathbb{R}^{n_{e_i}}$ and all $\kappa_i \in \mathbb{N}$, it holds that

$$\begin{aligned} \underline{\alpha}_{W,i}|e_i| \leq W_i(\kappa_i, e_i) \leq \overline{\alpha}_{W,i}|e_i| \\ W_i(\kappa_i+1, h_i(\kappa_i, e_i)) \leq \lambda_i W_i(\kappa_i, e_i) \end{aligned} \tag{21}$$

for some positive numbers $\underline{\alpha}_{W,i}$, $\overline{\alpha}_{W,i}$ and $0 < \lambda_i < 1$, $i \in \bar{N}$.

*Remark 1:* For each network $\mathcal{N}_i$, $\lambda_i = \sqrt{\frac{n_i-1}{n_i}}$, $\underline{\alpha}_{W,i} = 1$ and $\overline{\alpha}_{W,i} = \sqrt{n_i}$ if the transmission protocol is RR protocol and $\lambda_i = \sqrt{\frac{n_i-1}{n_i}}$, $\underline{\alpha}_{W,i} = 1$ and $\overline{\alpha}_{W,i} = 1$ if TOD protocol is adopted [2].

*Condition 2:* For all $i \in \bar{N}$, the function $W_i$ given in Condition 1 satisfies that

$$W_i(\kappa_i+1, e_i) \leq \lambda_{W,i} W_i(\kappa_i, e_i) \tag{22}$$

for some constant $\lambda_{W,i}, i \in \bar{N}$ and that for almost all $e_i \in \mathbb{R}^{n_{e_i}}$ and all $\kappa_i \in \mathbb{N}$, $|\frac{\partial W_i}{\partial e_i}| \leq \eta_i$ for some constant $\eta_i > 0, i \in \bar{N}$.

*Remark 2:* For the network $\mathcal{N}_i$, $\lambda_{W,i} = \sqrt{n_i}$ and $\eta_i = \sqrt{n_i}$ if RR protocol is used and $\lambda_{W,i} = \eta_i = 1$ if TOD protocol is used [6]. The "almost everywhere" is used in Condition 2 since $\frac{\partial W_i}{\partial e_i}$ may exist for all $e_i \in \mathbb{R}^{n_{e_i}}$ except a set with measure of zero. For example, $W_i = |e_i|$ if TOD protocol is adopted by the network $\mathcal{N}_i$ and $W_i$ is differentiable almost everywhere.

*Condition 3:* For some $\epsilon > 0$, there exists a positive definite matrix $P$, constants $\gamma_i > 0, i = 1, 2, \cdots, N$ such that the following linear matrix inequality (LMI) holds

$$\begin{bmatrix} Q + \sum_{i=1}^N \eta_i^2 M_i^T M_i + \epsilon I_{n_x} & PB \\ * & \epsilon I_{n_e} - \sum_{i=1}^N \underline{\alpha}_{W,i}^2 \gamma_i^2 \Gamma_i \end{bmatrix} < 0$$

where $Q = A^T P + PA$ and $\Gamma_i$ is a $n_e$-dimensional block diagonal matrix in which the $i$-th diagonal block is the $n_{e_i}$-dimensional identity matrix and the $k$-th diagonal block is the $n_{e_k}$-dimensional zero matrix for each $k \in \bar{N}$ with $k \neq i$.

*Remark 3:* The LMI condition is very natural and reasonable and essentially requires that the whole $x$ subsystem is input-to-state stable (ISS). In order to make this point clear, we write out the equivalent condition of the LMI condition as follows

$$\begin{aligned} x^T(A^TP + PA)x + 2x^T PBe \leq -\sum_{i=1}^N \eta_i^2 x^T M_i^T M_i x \\ -\epsilon x^T x - \epsilon e^T e + \sum_{i=1}^N \underline{\alpha}_{W,i}^2 \gamma_i^2 e_i^T e_i. \end{aligned} \tag{23}$$

Note that $A$ is a Hurwitz matrix since feedback controller has been used. Such a matrix inequality is linear since $Q$, $M_i$, $\epsilon$, $\underline{\alpha}_{W,i}$ and $\eta_i$ are all known and the unknown matrix variables are $\gamma_i$ and $P$. Therefore, the LMI is solvable and can be optimized. A benchmark example is also utilized to illustrate this point in Section V.

Consider the following function for each $i \in \bar{N}$

$$\widetilde{W}_i(\kappa_i, l_i, e_i, s_i) = \begin{cases} \max\{W_i(\kappa_i, e_i), W_i(\kappa_i, e_i + s_i)\}, l_i = 0 \\ \max\{\frac{\lambda_i}{\lambda_{W,i}} W_i(\kappa_i, e_i), W_i(\kappa_i, e_i + s_i)\}, l_i = 1. \end{cases} \tag{24}$$

*Lemma 1:* Consider the system $\mathcal{H}_{NCS}$ such that Condition 1, 2 hold. For each $i \in \bar{N}$, the function $\widetilde{W}_i$ defined in (24) satisfies the following conditions for all $\kappa_i \in \mathbb{N}$, $l_i \in \{0, 1\}$ and $s_i, e_i \in \mathbb{R}^{n_{e_i}}$

$$\begin{aligned} \underline{\beta}_{W,i}|(e_i, s_i)| \leq \widetilde{W}_i(\kappa_i, l_i, e_i, s_i) \leq \overline{\beta}_{W,i}|(e_i, s_i)| \\ \widetilde{W}_i(\kappa_i+1, 1, e_i, h_i(\kappa_i, e_i) - e_i) \leq \lambda_i \widetilde{W}_i(\kappa_i, l_i, e_i, s_i) \\ \widetilde{W}_i(\kappa_i, 0, e_i + s_i, -s_i - e_i) \leq \widetilde{W}_i(\kappa_i, 1, e_i, s_i) \end{aligned} \tag{25}$$

where $\underline{\beta}_{W,i}$ and $\overline{\beta}_{W,i}$ are some positive constants. Moreover it holds that for all $\kappa_i \in \mathbb{N}$, $l_i \in \{0, 1\}$, $s_i \in \mathbb{R}^{n_{e_i}}$ and almost all $e_i \in \mathbb{R}^{n_{e_i}}$

$$\begin{aligned} \left\langle \frac{\partial \widetilde{W}_i(\kappa_i, l_i, e_i, s_i)}{\partial e_i}, M_i x + J_i e_i \right\rangle \quad &\leq \quad L_{i,l_i} \widetilde{W}_i(\kappa_i, l_i, e_i, s_i) \\ &+ \quad \eta_i |M_i x| \end{aligned} \tag{26}$$

where $L_{i,0} = \frac{\eta_i |J_i|}{\underline{\alpha}_{W,i}}; L_{i,1} = \frac{\eta_i |J_i| \lambda_{W,i}}{\lambda_i \underline{\alpha}_{W,i}}$.

Lemma 1 is a direct result of Theorem V.3 in [6].

Inspired by [6], we introduce the following differential equations

$$\dot{\phi}_{i,0} \quad = \quad -L_{i,0}\phi_{i,0} - \gamma_{i,0}(\phi_{i,0}^2 + 1) \tag{27}$$

$$\dot{\phi}_{i,1} \quad = \quad -L_{i,1}\phi_{i,1} - \gamma_{i,0}(\phi_{i,1}^2 + \frac{\gamma_{i,1}^2}{\gamma_{i,0}^2}) \tag{28}$$

where $L_{i,0} = \frac{\eta_i |J_i|}{\underline{\alpha}_{W,i}}$, $L_{i,1} = \frac{\eta_i |J_i| \lambda_{W,i}}{\lambda_i \underline{\alpha}_{W,i}}$, $\gamma_{i,0} = \gamma_i$, $\gamma_{i,1} = \frac{\gamma_i \lambda_{W,i}}{\lambda_i}$.

Next, we will give sufficient conditions to ensure that $\mathcal{H}_{NCS}$ is uniformly globally asymptotically stable (UGAS).

*Theorem 1:* Consider the system $\mathcal{H}_{NCS}$ and Condition 1, 2, 3 hold. Suppose that for each $i \in \bar{N}$, $\tau_{MATI}^i \geq \tau_{MAD}^i \geq 0$ satisfy

$$\phi_{i,0}(\tau_i) \geq \lambda_i^2 \phi_{i,1}(0) \text{ for all } 0 \leq \tau_i \leq \tau_{MATI}^i \tag{29}$$

$$\phi_{i,1}(\tau_i) \geq \phi_{i,0}(\tau_i) \quad \text{for all } 0 \leq \tau_i \leq \tau_{MAD}^i \tag{30}$$

for solutions $\phi_{i,0}$ and $\phi_{i,1}$ of differential equations (27) and (28) corresponding to certain chosen initial condition $\phi_{i,\ell}(0) > 0, \ell = 0, 1$, with $\phi_{i,1}(0) \geq \phi_{i,0}(0) \geq \lambda_i^2 \phi_{i,1}(0) \geq 0$, $\phi_{i,0}(\tau_{MATI}^i) > 0$. Then $\mathcal{H}_{NCS}$ is UGAS.

*Proof:* For each $i \in \bar{N}$, let $\phi_{i,0} = \tilde{\phi}_{i,0}$ and $\phi_{i,1} = \frac{\gamma_{i,1}}{\gamma_{i,0}} \tilde{\phi}_{i,1}$ in differential equations (27) and (28), then (27) and (28) and conditions (29) and (30) can be transformed into

$$\dot{\tilde{\phi}}_{i,l_i} = -2L_{i,l_i}\tilde{\phi}_{i,l_i} - \gamma_{i,l_i}(\tilde{\phi}_{i,l_i}^2 + 1), l_i = 0, 1 \tag{31}$$

and

$$\gamma_{i,0}\tilde{\phi}_{i,0}(\tau_i) \geq \lambda_i^2 \gamma_{i,1} \tilde{\phi}_{i,1}(0) \text{ for all } \tau_i \in [0, \tau_{MATI}^i] \tag{32}$$

$$\gamma_{i,1}\tilde{\phi}_{i,1}(\tau_i) \geq \gamma_{i,0}\tilde{\phi}_{i,0}(\tau_i) \text{ for all } \tau_i \in [0, \tau_{MAD}^i]. \tag{33}$$

Consider the following function

$$U(\xi) = x^T P x + \sum_{i=1}^N \gamma_{i,l_i} \tilde{\phi}_{i,l_i}(\tau_i) \widetilde{W}_i^2(\kappa_i, l_i, e_i, s_i). \tag{34}$$

When $\xi \in D_{NCS,i}$ and $l_i = 0$, we have that $\tau_i \in [\delta, \tau_{MATI}^i]$ and obtain, using (17), that

$$\begin{aligned} U(\xi^+) \leq x^T P x + \sum_{j \neq i} \gamma_{j,l_j} \tilde{\phi}_{j,l_j}(\tau_j) \widetilde{W}_j^2(\kappa_j, l_j, e_j, s_j) \\ + \gamma_{i,0} \tilde{\phi}_{i,0}(\tau_i) \widetilde{W}_i^2(\kappa_i, 0, e_i, s_i). \end{aligned} \tag{35}$$

When $\xi \in D_{NCS,i}$ and $l_i = 1$, similarly, by using (17), we have

$$\begin{aligned} U(\xi^+) \leq x^T P x + \sum_{j \neq i} \gamma_{j,l_j} \tilde{\phi}_{j,l_j}(\tau_j) \widetilde{W}_j^2(\kappa_j, l_j, e_j, s_j) \\ + \gamma_{i,1} \tilde{\phi}_{i,1}(\tau_i) \widetilde{W}_i^2(\kappa_i, 1, e_i, s_i). \end{aligned} \tag{36}$$

Since from Condition 1, $W_i(\kappa_i, e_i)$ is locally Lipschitz with respect to $e_i$ for each $i \in \bar{N}$, $\widetilde{W}_i(\kappa_i, l_i, e_i, s_i)$ is locally Lipschitz with respect to $(e_i, s_i)$ from the construction in (24). Based on the property that locally Lipschitz function is differentiable almost everywhere, for all $(x,\ell,s,\kappa)$ and almost all $(e,s)$, it holds that

$$
\begin{aligned}
&\langle\nabla U(\xi),F(\xi)\rangle \le x^T(A^TP+PA)x+2x^TPBe\\
&-\sum_{i=1}^{N}\gamma_{i,l_i}(2L_{i,l_i}\tilde{\phi}_{i,l_i}(\tau_i)+\gamma_{i,l_i}(\tilde{\phi}_{i,l_i}^2+1))\widetilde{W}_i^2(\kappa_i,l_i,e_i,s_i)\\
&+\sum_{i=1}^{N}2\gamma_{i,l_i}\tilde{\phi}_{i,l_i}(\tau_i)L_{i,l_i}\widetilde{W}_i^2(\kappa_i,l_i,e_i,s_i)\\
&+\sum_{i=1}^{N}2\gamma_{i,l_i}\tilde{\phi}_{i,l_i}(\tau_i)\widetilde{W}_i(\kappa_i,l_i,e_i,s_i)\eta_i\sqrt{x^TM_i^TM_ix}\\
&\le -\sum_{i=1}^{N}\eta_i^2x^TM_i^TM_ix-\epsilon(|x|^2+|e|^2)\\
&+\sum_{i=1}^{N}\gamma_{i,l_i}^2\widetilde{W}_i^2(\kappa_i,l_i,e_i,s_i)\\
&-\sum_{i=1}^{N}\gamma_{i,l_i}(2L_{i,l_i}\tilde{\phi}_{i,l_i}(\tau_i)+\gamma_{i,l_i}(\tilde{\phi}_{i,l_i}^2+1))\widetilde{W}_i^2(\kappa_i,l_i,e_i,s_i)\\
&+\sum_{i=1}^{N}2\gamma_{i,l_i}\tilde{\phi}_{i,l_i}(\tau_i)L_{i,l_i}\widetilde{W}_i^2(\kappa_i,l_i,e_i,s_i)\\
&+\sum_{i=1}^{N}2\gamma_{i,l_i}\tilde{\phi}_{i,l_i}(\tau_i)\widetilde{W}_i(\kappa_i,l_i,e_i,s_i)\eta_i\sqrt{x^TM_i^TM_ix}\\
&\le -\epsilon(|e|^2+|x|^2).
\end{aligned}
$$

The above derivations show that $U(\xi)$ is a Lyapunov function. UGAS follows from the standard arguments in [11]. ■

*Remark 4:* In Theorem 1, the MATI and MAD for the local network $\mathcal{N}_i$ are closely related to the gain parameters $\gamma_{i,l_i}, l_i=0,1$ and can be computed by solving the differential equation pair (27)-(28).

*Remark 5:* The aim of Theorem 1 is to give algorithm to compute the allowable maximum transmission interval and allowable maximum delay for NCS with multiple local networks. Each pair of MATI and MAD of each network can be computed explicitly by utilizing computation software such as MATLAB. Based on the algorithm provided in Theorem 1, in actually, a tradeoff curve of MATI and MAD can be explicitly given (please see Section V for detailed procedure).

*Corollary 1:* Suppose that all the conditions in Theorem 1 hold. If for each $i\in\bar{N}$, $\phi_{i,0}(0)=\phi_{i,1}(0)=\lambda_i^{-1}$ in case $\lambda_i\neq 0$, then $\tau_{MATI}^i$ can be given as follows

$$
\tau_{MATI}^i=
\begin{cases}
\dfrac{1}{L_{i,0}r_i}\arctan\left(\dfrac{r_i(1-\lambda_i)}{2\frac{\lambda_i}{1+\lambda_i}(\frac{\gamma_{i,0}}{L_{i,0}})+1+\lambda_i}\right),\gamma_{i,0}>L_{i,0}\\
\dfrac{1-\lambda_i}{L_{i,0}(1+\lambda_i)},\gamma_{i,0}=L_{i,0}\\
\dfrac{1}{L_{i,0}r_i}\text{arctanh}\left(\dfrac{r_i(1-\lambda_i)}{2\frac{\lambda_i}{1+\lambda_i}(\frac{\gamma_{i,0}}{L_{i,0}})+1+\lambda_i}\right),\gamma_{i,0}>L_{i,0}
\end{cases}
\quad (37)
$$

where $r_i=\sqrt{|(\gamma_{i,0}/L_{i,0})^2-1|}$.

The proof idea of Corollary 1 is similar to that of [11].

*Proof:* In the absence of delays, $\tau_{MATI}^i$ for $i$-th network will be such that the differential equation (29) with initial value $\phi_{i,0}(0)=\lambda_i^{-1}$ satisfy $\phi_{i,0}(\tau_{MATI}^i)=\lambda_i$ by Theorem 1. The solution of (29) can be explicitly given from [11]. Therefore, Corollary 1 holds. ■

*Remark 6:* From Lemma 1, $\gamma_{i,0}=\gamma_i$ holds and $\gamma_i$ can be obtained by solving LMI in Condition 3.

## V. Case Studies

The common used example in [10], [9] is adopted.

Consider the following linearized system of the pendula

$$
\dot{x}=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}x+\begin{bmatrix}B_{11} & 0\\ 0 & B_{22}\end{bmatrix}\hat{u} \quad (38)
$$

where $x=(x_1,x_2)$ with $x_i=(p_i,\dot{p}_i,\theta_i,\dot{\theta}_i)$ is the state of the subsystem $\mathcal{P}_i, i=1,2$. The matrices $A_{11}$, $A_{12}$, $A_{21}$, $A_{22}$, $B_{11}$ and $B_{22}$ are the same as that in [10], [9].

Each subsystem employs its own local network over which the state values $y_i=x_i$ are transmitted to the controller at transmission times. Static controller $u=Kx$ is adopted with

$$
\begin{aligned}
K_1 &= \begin{bmatrix} 11396 & 7196.2 & 573.96 & 1199.0 \end{bmatrix} \quad (39)\\
K_2 &= \begin{bmatrix} 29241 & 18135 & 2875.3 & 3693.9 \end{bmatrix}. \quad (40)
\end{aligned}
$$

The resulting closed-loop system can be written as

$$
\begin{aligned}
\dot{x}_1 &= (A_{11}+B_{11}K_1)x_1+A_{12}x_2+B_{11}K_1e_1 \quad (41)\\
\dot{x}_2 &= (A_{22}+B_{22}K_2)x_2+A_{21}x_1+B_{22}K_2e_2 \quad (42)\\
\dot{e}_1 &= -\dot{x}_1 \quad (43)\\
\dot{e}_2 &= -\dot{x}_2. \quad (44)
\end{aligned}
$$

Under TOD protocol for the network $\mathcal{N}_i$, $W_i=|e_i|$, $\underline{\alpha}_{W,i}=\overline{\alpha}_{W,i}=\eta_i=\lambda_{W,i}=1$, $\lambda_i=\sqrt{\frac{n_i-1}{n_i}}$ where $n_i$ is the number of nodes in the network $\mathcal{N}_i$.

Under RR protocol for the network $\mathcal{N}_i$, let $e_i=(e_i(1),e_i(2),\cdots,e_i(n_i))$ where $e_i(k)$ corresponds to the error of the $k$-th node of the network $\mathcal{N}_i$. Note that $n_i$ denotes the number of the nodes in the network $\mathcal{N}_i$. Then $W(\kappa_i,e_i)$ can be represented by $W(\kappa_i,e_i)=\sqrt{\sum_{k=1}^{n_i}a_k^2(\kappa_i)|e_i(k)|^2}$ where $a_k(\kappa_i)$ is the time-varying coefficient satisfying $|a_k(\kappa_i)|\le\sqrt{n_i}$ for all $k\in\{1,2,\cdots,n_i\}$ and all $\kappa_i\in\mathbb{N}$ (see Lemma 2 of [12]). Besides, $\underline{\alpha}_{W,i}=1$, $\overline{\alpha}_{W,i}=\eta_i=\lambda_{W,i}=\sqrt{n_i}$, $\lambda_i=\sqrt{\frac{n_i-1}{n_i}}$. To preclude zeno behavior, $\delta$ is $1e-6$ which implies that the networks have a lower bound of sampling periods.

In free of delays, we firstly compare our results with those in [10] under TOD protocol for each local network. Set $\gamma_1=\gamma_2$ in Condition 3 and use the LMI tool box of Matlab to minimize $\gamma$, then we have $\gamma_1=\gamma_2=820.6963$. When each local network has only one node, $\tau_{MATI}^1=0.0016$ and $\tau_{MATI}^2=0.0013$. When each local network has two nodes, $\tau_{MATI}^1=2.6767e-04$ and $\tau_{MATI}^2=1.7169e-04$. See the acquired $\tau_{MATI}^i, i=\{1,2\}$ in Section V of [10] under the same constraints. It can be seen that when each local network has one node, the order of magnitude for $\tau_{MATI}^i, i\in\{1,2\}$ is $1e-6$ in [10], which is caused partly by the difficulties in choosing the appropriate Lyapunov function for each local network. Next we compare our result with that in [9] by admitting delays. Fig. 1 is an illustration of how to solve the MATI and MAD for each local network by Theorem 1. The intersection of blue line with dot and red line with dot is the MAD of the network $\mathcal{N}_1$ and the intersection of the red line with dot and the horizon line with dot is the MATI of the Network $\mathcal{N}_1$. Tradeoffs between MATI and MAD for each local network under TOD and RR protocols and different numbers of nodes for each local network are illustrated in Fig. 2. The order of magnitude for the MATI and MAD under RR and TOD protocols with the number of nodes $n_1=n_2=2,3$ is $1e-4$ while the order of magnitude for the MATI and MAD in [9] is $1e-7$. This shows that our results are much less conservative than that in [9].

## VI. Conclusion

This paper has addressed the stability problem of linear interconnected NCSs. A gain assignment criterion has been given to choose

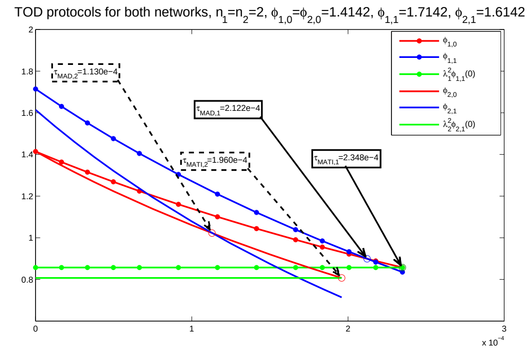


Fig. 1. $\phi_{i,\ell}, i = 1, 2, \ell = 1, 2, n_1 = n_2 = 2$ with TOD protocols for both networks

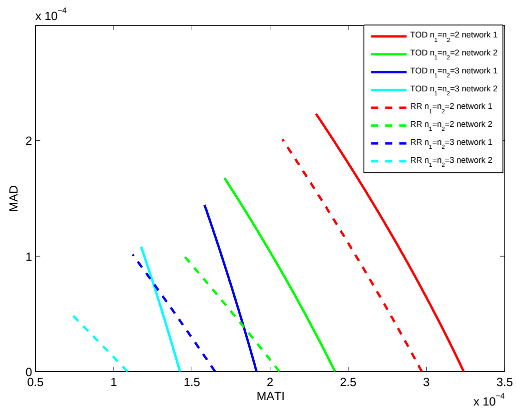


Fig. 2. Tradeoffs between MATI and MAD under different protocols and numbers of nodes

approximate gain closely related to the MATI and MAD, and the acquired MATI has been less conservative compared with the existing literature. An example has been given to show the effectiveness and improvement of the proposed approach.

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