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Bishop Independence

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Research Article

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Abstract

Bishop Independence concerns determining the maximum number of Bishops that can be placed on a board such that no Bishop can attack any other Bishop. This paper presents the solution to the Bishop Independence problem, determining the Bishop Independence number, for all sizes of boards on the following topologies: the cylinder, the Möbius strip, the torus, the Klein bottle and the surface of a cube.

Keywords: Graph, Bishop Graph, Independence, Bishop Independence, Chess, Chessboard
Mathematics Subject Classification: 05C69, 00A08

1 Introduction

Puzzles on the chessboard are of interest to the mathematical community. Their relation to graph problems is of special interest to combinatorialists. For the graph G with set of vertices V, the set $S \subset V$ is independent if no two vertices in S are adjacent in G. The independence number $\beta_0(G)$ is the maximum cardinality of an independent set of the graph G. This directly relates to the problem of finding the independence number of a specified chess piece. The board can be represented by a graph in which each square corresponds to a vertex and an edge exists between two vertices A and B if a chess piece on square A can reach square B in a single move. An arrangement of pieces on the board is independent if the pieces are placed such that no piece can move to another's position in a single move; such an arrangement of pieces is equivalent to a subset of V. Hence it is also independent in the graph. Thus the independence problem in chess, that of finding the maximum cardinality of an independent arrangement for a specific type of piece, is equivalent to finding $\beta_0(G)$ for a graph G that represents the board for the given chess piece. Since the relationships of cells and pieces are more easily interpreted on a board, the independence problem is here formulated and interpreted on a board rather than on a graph.

Initial work in combinatorial chess problems focused on the Knights Tour problem, which is the task of taking a Knight placed on a chessboard and moving it according to the rules of chess such that it

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visits each square of the board exactly once, ending on a square a single move away from its original position. Work on this problem concerns determining if such a tour exists for given board sizes, and Schwenk [1] presented the first full proof for the existence of Knights tours on the rectangular board. Following Schwenk's work Watkins provided proofs for the existence of Knights tours on the Torus [2], Cylinder, Möbius strip and Klein bottle [3]. Further, DeMaio extended Knights Tours to the cube [4] and rectangular prism [5] and solved the Bishop Tour problem on the rectangle [6]. Recently Erde et. al [7] introduced chess problems to n-dimensions in their paper on Knights Tours.

In this paper the independence problem is explored for the Bishop chess piece. Yaglom and Yaglom determine the independence number of Bishops for the case of $n \times n$ square boards [8], and DeMaio and Faust consider the domination and independence problems for Bishops on the torus [9]. Succinctly expressed summaries of these results and of other similar results can be found in [10]. Recently Berghammer presented a proof for the domination and independence numbers for Bishops on the rectangular board [11]. This paper gives an alternative proof for Bishop independence on the torus, and provides the Bishop independence number for all sizes of board on: the cylinder, the Möbius strip, the torus, the Klein bottle and the surface of a cube.

A Bishop can, in a single move, travel along a diagonal of a given board any number of squares in one direction. A Bishop is said to *cover* a square if it can reach that square in a single move. We define $B^A_{m,n}$ to represent permissible Bishop moves on an $m \times n$ board (that is a board with m rows and n columns) oriented on the topology A, where A is null for the flat surface and takes the value C for the cylinder, T for the torus, S for the Möbius strip, K for the Klein bottle. Similarly we denote B^3_n to represent permissible Bishop moves on the surface of an $n \times n \times n$ cube with faces acting as connected boards. Since Bishops can move to any square on a diagonal on which they are currently placed it is impossible for two Bishops to be placed on the same diagonal and be independent. There can never be more independent Bishops than there are diagonals in one direction, and thus counting these on a given board gives a trivial upper bound on the independence number. Let the diagonal that runs from bottom left to top right and all diagonals that run parallel to it be termed positive diagonals, and the diagonal that runs from top left to bottom right and all diagonals that run parallel to it be termed negative diagonals.

On more complex topologies the representation of the board as a rectangular $m \times n$ grid can still be maintained. The standard rectangular board has boundaries on its sides which inhibit movement of pieces. As will be described for each topology explored, the removal of boundaries and the introduction of identities that grant movement from one edge to another creates a flat representation of the topologies with direct analogue to the rectangular board. Diagonals that appear distinct may join across an identity, and consequently a distinct diagonal is one which starts at one boundary and ends at another, or which joins itself at a square creating a cycle. A sub-diagonal is then defined as a part of a distinct diagonal that is contained between edges within the rectangular representation. Although the traditional chequered black and white colouring of the chess board is not necessarily maintained across the identities of boards on different topologies, the rectangular representations can still be considered to be coloured in this way. In this case a sub-diagonal will only have squares of the same colour and we define a collection of sub-diagonals or a distinct diagonal to be monochromatic if all the squares are of the same colour. Further any pair of diagonals that intersect one another will do so on a square if they are both the same colour.

2 Rectangular Boards

$B_{m,n}$ represents permissible Bishop moves on a standard $m \times n$ rectangular board. This is the flat topology, upon which there are four boundaries to Bishop movement. All diagonals are therefore distinct diagonals. An $m \times n$ board is equivalent to an $n \times m$ board, and thus it is assumed without loss of generality that $m \leq n$.

Theorem 2.1. [8] $\beta_0(B_{n,n}) = 2n - 2$.

Theorem 2.2. [11] *$\beta_0(B_{m,n}) = n + m - 2$ for m and n both even; otherwise $\beta_0(B_{m,n}) = n + m - 1$ for $m \neq n$.*

3 Cylindrical Boards

The standard rectangular chess board can be transformed into a cylinder by joining together one pair of opposing sides. The flat cylinder has two boundaries on the sides that are not joined and an identity between the remaining two sides. $B^C_{m,n}$ represents permissible Bishop moves on an $m \times n$ cylindrical board for which the right and left sides of the flat cylinder are identified. Boundaries exist on the top and bottom row.

Theorem 3.1. $\beta_0(B^C_{m,n}) = n$.

Proof. Without loss of generality, all distinct positive diagonals start in the bottom row. Hence $\beta_0(B^C_{m,n}) \leq n$, and placing Bishops on all the squares of this row creates an arrangement of n independent Bishops. ☐

4 The Möbius Strip

The standard rectangular chess board can be transformed into a Möbius strip by joining together one pair of opposing sides such that the non-adjacent corners meet. The Möbius strip has a reverse identity (indicated by the arrows in Fig. 1), and as a result its top and bottom rows form a single boundary and the two faces of the board are joined. $B^S_{m,n}$ represents permissible Bishop moves on an $m \times n$ Möbius strip for which the right and left sides of the board are reverse identified.

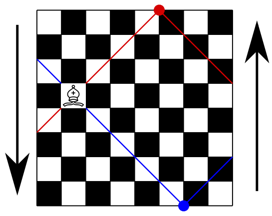

Fig. 1: Identities required to transform a rectangular chess board into a Möbius strip

Theorem 4.1. *For $B^S_{m,n}$:*

$$\beta_0(B^S_{m,n}) = \begin{cases} n-1 & m \leq n,\ m \text{ even and } n \text{ odd}, \\ n & m \leq n,\ m \text{ odd or } n \text{ even}, \\ n & m > n,\ m \text{ and } n \text{ both odd or both even}, \\ 2n & m > n, \text{ and exactly one of } m, n \text{ odd}. \end{cases}$$

Proof. Consider the case $m \leq n$. There exists a set of diagonals that do not cross the identity and it is impossible for any distinct diagonal to cross the identity twice. There are a total of $2n + 2m - 2$ sub-diagonals. Since $2m-2$ of those diagonals cross the identity and are joined, there are $2n+2m-2-(2m-2) = 2n$ distinct diagonals. Since a Bishop placed on any square will cover two distinct diagonals, $\beta_0(B^S_{m,n}) \leq n$. However for m even and n odd, distinct diagonals are monochromatic, and the number of these is odd. As each Bishop covers two distinct diagonals, $\beta_0(B^S_{m,n}) \leq n-1$.

For m even, placing Bishops in all squares of the first column provides an initial arrangement of m independent Bishops. These Bishops cover all squares in the first $\frac{m}{2}+1$ columns, including the column in which they are placed, and the last $\frac{m}{2}$ columns of the board. Thus of the n columns, $n-(\frac{m}{2}+1)-\frac{m}{2} = n-m-1$ columns remain that are not completely covered by the initial arrangement and such columns occur for $n > m+1$. For $n = m$ the initial arrangement meets the bound, hence $\beta_0(B^S_{m,n}) = n$. For $n = m+1$ the initial arrangement meets the bound, specifically for m even and n odd, hence $\beta_0(B^S_{m,n}) = n-1$. For $n > m+1$ the initial arrangement does not cover the middle two rows of the board for the $n-m-1$ non-completely covered columns. Thus there is a $2 \times (n-m-1)$ rectangular sub-board on which Bishops can be placed on any square and be independent of the initial arrangement. Placing Bishops in both squares of the odd columns for such a rectangular sub-board gives an independent arrangement of $n-m-1$ Bishops for $n-m-1$ even, and an independent arrangement of $n-m$ Bishops for $n-m-1$ odd. This meets the Bishop independence number for a $2(n-m-1)$ rectangular sub-board as given in Theorem 2.2. On the Möbius strip a total arrangement of $m+(n-m) = n$ independent Bishops can be placed for n even, as the expression $n-m-1$ is odd, and so this meets the upper bound of $\beta_0(B^S_{m,n}) \leq n$. For n odd, the expression $n-m-1$ is even and this gives a total arrangement on the Möbius strip of $m+(n-m-1) = n-1$ independent Bishops, which meets the upper bound of $\beta_0(B^S_{m,n}) \leq n-1$ for m even and n odd.

For m odd, placing Bishops along all squares of the middle row creates an arrangement of n Bishops, each with separate positive and negative diagonals and thus independent and hence $\beta_0(B^S_{m,n}) = n$.

Consider the case $m > n$. Every distinct diagonal will cross the identity at least once and some may cross the identity multiple times. Hence all distinct diagonals are comprised of both positive and negative sub-diagonals, and every distinct diagonal will intersect itself (not necessarily on a square). If the diagonals intersect on a square (which occurs for m even and n odd, or for m odd and n even), $\beta_0(B^S_{m,n})$ is equal to the number of distinct diagonals. If m and n are both odd, or they are both even, distinct diagonals change colour on crossing the reverse identity, and so do not intersect on a square. Hence $\beta_0(B^S_{m,n})$ is equal to half the number of distinct diagonals. There are $4n$ diagonals for an $m \times n$ board with $m > n$ that only cross the identity at one end and thus there are $2n$ distinct diagonals. □

5 Toroidal Boards

The standard rectangular chess board can be transformed into a torus by joining together one pair of opposing sides to form a cylinder (indicated by arrows marked 'A' in Fig. 2) and then combining the remaining pair of opposing sides (the arrows marked 'B' in Fig.2) by closing the cylinder. The torus has an identity between each pair of opposite edges, and no boundaries. Thus there is no restriction on movement and all distinct diagonals will close on themselves. $B^T_{m,n}$ represents permissible Bishop moves on an $m \times n$ toroidal board for which opposing sides of the torus are identified. As for the rectangular board, an $m \times n$ toroidal board is equivalent to an $n \times m$ toroidal board, thus it is assumed in the following proof that $m \leq n$ without loss of generality. We define a pass as a set of sub-diagonals which are all part of the same distinct diagonal, and which together cover a single square in each column. A distinct diagonal is then made up of a number of passes, the final pass ending such that it connects to the beginning of the first pass.

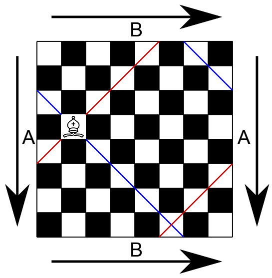

Fig. 2: Identities required to transform a rectangular chess board into a torus

Theorem 5.1. $\beta_0(B^T_{m,n})$ = gcd(m,n).

Proof. For an $m \times mk_1$ board, $k_1 \in \mathbb{N}$, a distinct diagonal closes in one pass. For an $m \times (mk_2 + r)$ board, $k_2 \in \mathbb{N}_0$ and $r \in \mathbb{N}$ such that $r < m$, a distinct diagonal closes in $\frac{1}{r}$lcm(r, m) passes. In each pass, another sub-diagonal is combined with the starting diagonal. Hence there are $\frac{m \times r}{\text{lcm}(m,r)} =$ gcd(m, r) $=$ gcd(m, n) distinct positive diagonals. Hence $\beta_0(B^T_{m,n}) \leq$ gcd(m,n) and letting p = gcd(m, n), placing Bishops on the top p squares of the first column of the board creates an arrangement of p Bishops, each with separate positive and negative diagonals and which are thus independent. □

6 The Klein Bottle

The standard rectangular chess board can be transformed into a Klein bottle by first joining together one pair of opposing sides to form a cylinder (indicated by arrows marked 'B' in Fig.3) and then combining the remaining pair of opposing sides (the arrows marked 'A' in Fig.3) such that opposing faces are joined. The Klein bottle has one identity, one reverse identity, and no boundaries. Thus there is no restriction on movement and all distinct diagonals will close on themselves. $B^K_{m,n}$ represents permissible Bishop moves on an $m \times n$ Klein Bottle for which the top and bottom sides are identified, and the left and right sides are reverse identified. Once a distinct diagonal has made one pass of the board as a set of positive sub-diagonals it will become a pass as a set of negative sub-diagonals. Similarly, a pass as a set of negative sub-diagonals will become a pass as a set of positive sub-diagonals, and therefore for a diagonal to close it must be made up of an even number of passes. Hence diagonals are only defined as negative or positive during passes.

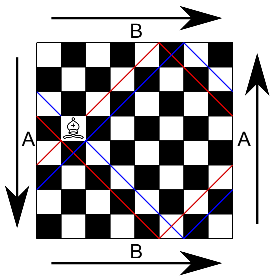

Fig. 3: Identities required to transform a rectangular chess board into a Klein Bottle

Lemma 6.1. $\beta_0(B^K_{m,mk+r})$ = $\beta_0(B^K_{m,r})$, *$k \in \mathbb{N}_0$ and $r \in \mathbb{N}$*

Proof. An $m \times r$ board on a Klein bottle can be extended by inserting an $m \times m$ cylinder between any of its columns. As the inserted cylinder has the same number of rows as columns, the number of distinct diagonals is not affected by the addition of the cylinder, nor is the pattern of the diagonals disrupted. Hence the independence number for the board is not changed. □

Theorem 6.2. *For* $B^K_{m,n}$*:*

$$\beta_0(B^K_{m,n}) = \begin{cases} 2\gcd(m,n) & \text{if } m \text{ is even and } n \text{ is odd}, \\ \gcd(m,n) & \text{if } m \text{ is odd}, \\ \gcd(m,n) & \text{if } m \text{ and } n \text{ are even and } \frac{m}{\gcd(m,n)} \text{ is even}, \\ \frac{1}{2}\gcd(m,n) & \text{if } m \text{ and } n \text{ are even and } \frac{m}{\gcd(m,n)} \text{ is odd}. \end{cases}$$

Proof. For an $m \times mk_1$ board, $k_1 \in \mathbb{N}$, on the Klein bottle, two passes are required for a diagonal to close thus there are $D = 2m/2 = m$ distinct diagonals. For an $m \times (mk_2 + r)$ board on the Klein bottle, $k_2 \in \mathbb{N}_0$ and $r \in \mathbb{N}$ such that $r < m$, a distinct diagonal closes in $L = \frac{1}{r}\text{lcm}(r,m)$ passes for L even. For L odd a distinct diagonal closes in $2L$ passes. When L is even there are $D = 2m/L = 2\text{gcd}(m,r) = 2\text{gcd}(m,n)$ distinct diagonals, and when L is odd there are $D = 2m/2L = \text{gcd}(m,r) = \text{gcd}(m,n)$ distinct diagonals.

For m and n even, passes of a distinct diagonal are monochromatic, with negative passes being the opposite colour to positive passes. Thus negative and positive passes of the same distinct diagonal will not intersect on a square and all Bishops will cover two distinct diagonals. Therefore $\beta_0(B^K_{m,n}) \leq D/2$, and the gcd$(m,n)$ is even. Hence there are $D/2$ distinct diagonals with white positive passes and $D/2$ distinct diagonals with black positive passes, forming two groups for which each diagonal in one group intersects every diagonal in the other group on a square. Selecting pairs, one from each group in any combination, a Bishop is placed on any one of the squares at which the diagonals intersect. This creates an arrangement of $D/2$ Bishops.

For m even and n odd, distinct diagonals are monochromatic and any distinct diagonal will intersect itself on a square.

For m odd, any distinct diagonal can be represented as the main positive diagonal on an $m \times m$ cylinder with top and bottom sides identified. From Lemma 6.1, the board can be expanded such that each distinct diagonal corresponds to the main diagonal on its own distinct cylinder. Considering any such distinct diagonal and corresponding cylinder, the distinct diagonal will make at least one negative pass on the cylinder, either along the main negative diagonal or as two negative sub-diagonals. For the first case, the two diagonals will intersect on the middle square of the cylinder. For the second case, one negative sub-diagonal is black and the other white; they both intersect the main diagonal and therefore one negative sub-diagonal will intersect at a square and the other will not.

Hence for m and/or n odd, $\beta_0(B^K_{m,n}) = D$ since each distinct diagonal intersects itself on a square, and so that square lies only on the distinct diagonal. For $m \neq kn$, $k \in \mathbb{N}$; when m is odd, L is also odd hence $\beta_0(B^K_{m,n}) = \text{gcd}(m,n)$. When m is even and n is odd, L is even hence $\beta_0(B^K_{m,n}) = 2\text{gcd}(m,n)$. □

The Klein bottle and torus are the only closed surfaces upon which the square grid structure of the chessboard can be perfectly imposed. This is of mathematical interest since $B^K_{m,n}$ and $B^T_{m,n}$ represent the only regular Bishop graphs. However, this does not prevent the exploration of other closed surfaces.

7 The surface of a cube

When imposing the grid structure of the chessboard onto the surface of a cube (Fig.4 and Fig.5, in which pairs of equivalently labelled arrows indicate identities) non regularity occurs only for the 24 squares at the corners of the faces. These squares have only 7 surrounding squares as opposed to the 8 neighbours possessed by the remaining squares. Thus a Bishop on a corner square is located on only three diagonals rather than four. Hence despite being a closed surface, not all distinct diagonals on the cube are cycles. B_n^3 represents the permissible Bishop moves on the surface of an $n \times n \times n$ cube for which each face is an $n \times n$ board and movement across connected face edges is permitted.

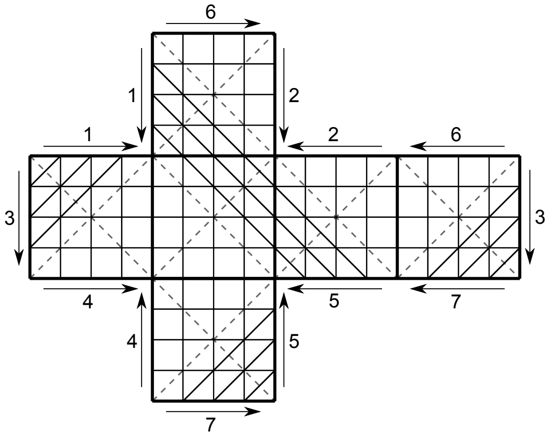

Fig. 4: Identities required to represent a chessboard on a cube as a net, for n even

Theorem 7.1. *For B_n^3:*

$$\beta_0(B_n^3) = \begin{cases} 2n & \text{for n even} \\ 2n+4 & \text{for n odd.} \end{cases}$$

Proof. The only boundaries to Bishop movement on the surface of a cube are the corners of each face. Thus there are only 12 open diagonals on the surface of a cube, and these are the 2 major diagonals (sub diagonals containing most squares) of each face.

Consider a single face. Of the $2n-1$ positive diagonals, $n-1$ of these are above the major positive diagonal. These diagonals are parallel to each other, crossing each face once and only once, never intersecting each other on a square or otherwise (see Fig.4). By symmetry this property holds for the $n-1$ positive sub diagonals below the major positive diagonal and equivalently for the negative sub diagonals on the same face. Hence there are $4(n-1)$ distinct closed diagonals comprised of 4 equally sized sets, denoted here *closed diagonal sets*. This gives a total of $4n+8$ distinct diagonals on the surface of an $n \times n \times n$ cube. Since none of these diagonals intersect themselves a Bishop will always cover 2 diagonals and hence a maximum of $2n+4$ Bishops can be placed independently on the surface of an $n \times n \times n$ cube.

For n even the bound cannot be met. Consider one of the four closed diagonal sets. Indexing the members of the set $1, ..., n-1$, by position on a given face with respect to the major diagonal, the set can be further divided into two subsets: the set $\mathfrak{P}$ of diagonals having an odd index and the set $\mathfrak{Q}$ of diagonals having an even index, for which $|\mathfrak{P}| = \frac{n}{2}$ and $|\mathfrak{Q}| = \frac{n-2}{2}$. Let the set P denote the union of the subsets $\mathfrak{P}$ of all four closed diagonal sets, and similarly let Q denote the union of the subsets $\mathfrak{Q}$ of all four closed diagonal sets; as such $|P| = 2n$ and $|Q| = 2n-4$. Since n is even, only diagonals of P intersect the 12 open (major) diagonals on a square. Further, for any closed diagonal set, only the diagonals of subset $\mathfrak{P}$ intersect all of the diagonals of $(Q - \mathfrak{Q})$ on a square. Thus, placed anywhere, a Bishop covers a diagonal belonging to the set $\mathfrak{P}$ and hence the maximum number of Bishops that can be placed independently on the board is equal to the number of diagonals in P which is $2n$. The cardinality of P is less than the number of remaining diagonals, $2n + 8$, and hence every diagonal can be paired with a diagonal in P to meet the bound.

For n odd, the bound can be met. Placing Bishops on the middle square of each face will cover all 12 distinct open diagonals. Taking any one of these Bishops and placing additional Bishops on the $n - 1$ squares to its left, continuing onto the neighbouring face, will cover the diagonals of the closed diagonal sets containing the upper positive diagonals and the lower negative diagonals on that face. Further, placing an additional $n-1$ Bishops on the squares to the right of this initial Bishop, continuing onto the neighbouring face, will cover the diagonals of the remaining two closed diagonal sets containing the lower positive sub diagonals and upper positive negative diagonals of the face on which the initial Bishop is placed. This produces a maximal independent covering of $2n + 4$ Bishops (see Fig.5). $\square$

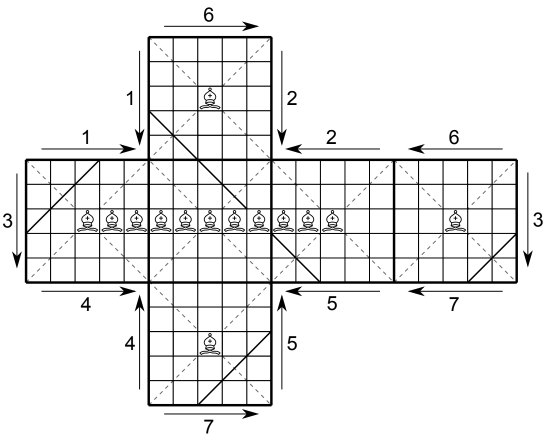

Fig. 5: Identities required to represent a chessboard on a cube as a net, for n odd, with example construction of maximal independent placement of Bishops

8 Conclusion

Bishop Independence numbers for the rectangle, cylinder, Möbius strip, torus, Klein bottle and surface of cube are now known. However the scope of combinatorial chess problems includes other pieces such as the King [8, 12] and Queen. Since Queen movement includes Rook and Bishop movement, results from this paper provide a step towards proving the remaining independence number results for the Queen (the results for the square board are given in [13]). Further, the domination problem, that of placing the minimum number of pieces such that every square is either occupied or can be reached by a piece in a single move, is one of numerous other problems similar to that of independence. Thus results in this paper are expected to be useful in determining the domination number for Bishops on the same surfaces investigated here.

Competing Interests

The authors declare that no competing interests exist.

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