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Original article

Representations of functionals on absolute weighted spaces and adjoint operators

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Abstract

In the present paper, we establish general representations of continuous linear functionals, which play important roles in Functional Analysis, of the absolute weighted spaces which have recently been introduced in Sarıgöl (2016, 2011), and also determine their norms. Further making use of this we give adjoint operators of matrix mappings defined on these spaces.

Keywords: Sequence spaces; BK spaces; Absolute summability; Continuous linear functional

1. Introduction

Any vector subspace of w, the space of all (real- or) complex valued sequences, is called *a sequence space*. A sequence space X is a BK-space if it is a Banach space provided that each of the maps $p_n : X \to \mathbb{C}$ defined by $p_n(x) = x_n$ is continuous for all $n \geq 0$. A BK-space X is said to have AK property if $\phi \subset X$ and $\{e^{(n)}\}$ is a basis for X, where $e^{(n)}$ is a sequence whose only non-zero term is 1 in kth place for each $n \geq 0$ and $\phi = \text{span}\{e^{(n)}\}$, the set of all finitely non-zero sequences. For example, ℓ_k, the space of all k-absolutely convergent series, is AK-space for $k \geq 1$.

Let X, Y be sequence spaces and $A = (a_{nv})$ be an infinite matrix of complex numbers. If $Ax = (A_n(x)) \in Y$ for every $x \in X$, then we say that A defines a matrix transformation from X into Y, and denote it by $A \in (X, Y)$, where $A_n(x) = \sum_{v=0}^{\infty} a_{nv} x_v$, provided that the series converges for $n \geq 0$.

Now, let Σa_v be a given infinite series with nth partial sums (s_n) and (θ_n) be a sequence of nonnegative terms. Then the series Σa_v is said to be summable $|A, \theta_n|_k$, $k \geq 1$, if

$$\sum_{n=0}^{\infty} \theta_n^{k-1} |\Delta A_n(s)|^k < \infty, \; A_{-1}(s) = 0,$$

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$\Delta A_n(s) = A_n(s) - A_{n-1}(s)$ [1]. If A is the matrix of weighted mean $(\overline{N}, p_n)$ $(resp.\ \theta_n = P_n/p_n)$, then summability $|A, \theta_n|_k$ reduces to summability $|\overline{N}, p_n, \theta_n|_k$ $(resp.\ |\overline{N}, p_n|_k$, [2]), [3]. Further, if $\theta_n = n$ for $n \geq 1$ and A is the matrix of Cesàro mean (C, α), then it is the same as summability $|C, \alpha|_k$ in Flett's notation [4]. By a weighted mean matrix we state

$$a_{nv} = \begin{cases} p_v/P_n, & 0 \leq v \leq n \\ 0, & v > n \end{cases} \tag{1.1}$$

where (p_n) is a sequence of positive numbers with $P_n = p_0 + p_1 + \cdots + p_n \to \infty$ as $n \to \infty$. In [5], the space $\left|\overline{N}_p^\theta\right|_k$, $k \geq 1$, was defined as the set of all series summable by $|\overline{N}, p_n, \theta_n|_k$, *i.e.*,

$$\left|\overline{N}_p^\theta\right|_k = \left\{ a = (a_v) : \sum_{n=1}^{\infty} \left| \gamma_n \sum_{v=1}^{n} P_{v-1} a_v \right|^k < \infty \right\},$$

which is a BK-space with respect to the norm (*see* [6])

$$\|a\|_{\left|\overline{N}_p^\theta\right|_k} = \left\{ |a_0|^k + \sum_{n=1}^{\infty} \left| \gamma_n(p, \theta) \sum_{v=1}^{n} P_{v-1} a_v \right|^k \right\}^{\frac{1}{k}}, \tag{1.2}$$

where

$$\gamma_0(p, \theta) = \theta_0^{1/k'},\ \gamma_n(p, \theta) = \frac{\theta_n^{1/k'} p_n}{P_n P_{n-1}},\ n \geq 1. \tag{1.3}$$

Hence it is clear that $a \in \left|\overline{N}_p^\theta\right|_k$ if and only if $T(a) \in l_k$, the set of all k -absolutely convergent series, where ,

$$T_0(a) = \gamma_0(p, \theta) a_0,\ T_n(a) = \gamma_n(p, \theta) \sum_{v=1}^{n} P_{v-1} a_v, \tag{1.4}$$

$1/k + 1/\widetilde{k} = 1$ for $k > 1$, and $1/\widetilde{k} = 0$ for $k = 1$.

2. Representations of functionals on the space $\left|\overline{N}_p^\theta\right|_k$

It is known that the continuous dual of a normed space X, denoted by X^*, is defined by the set of all bounded linear functionals on U, and also it is a fundamental principle of functional analysis that investigations of spaces are often combined with those of the dual spaces. In this connection duals of many spaces have been considered [7]. For example, $c^* \cong l_1$, $l_1^* \cong l_\infty$, $l_k^* \cong l_{k'}$ for $1 < k < \infty$, where c, l_∞ and $l_{k'}$ are the sets of all convergent, bounded sequences and k'-absolutely convergent series, respectively. Also their representations and norms are as follows:

$$f(x) = a \lim_n x_n + \sum_{n=0}^{\infty} a_n x_n,\quad \|f\|_{c^*} = |a| + \|a\|_{l_1}$$

$$f(x) = \sum_{n=0}^{\infty} a_n x_n,\quad \|f\|_{l_k^*} = \|a\|_\infty \quad (0 < k \leq 1)$$

$$f(x) = \sum_{n=0}^{\infty} a_n x_n,\quad \|f\|_{l_k^*} = \|a\|_{l_k} \ (1 < k < \infty).$$

In this section showing that $\left|\overline{N}_p^\theta\right|_k^*$ and $\left|\overline{N}_p^\theta\right|_1^*$ are isometrically isomorphic to $l_{k'}$ and l_∞, respectively, we give general representations of linear functionals on them and determine their norms.

First we characterize the property AK of the space $\left|\overline{N}_p^\theta\right|_k$, which plays important role to prove the theorems.

Theorem 2.1. *Let $1 \leq k < \infty$ and $\theta = (\theta_n)$ be a sequence of nonnegative numbers. Then, in order that $\left|\overline{N}_p^{\theta}\right|_k$ is a BK-space with property AK, it is necessary and sufficient*

$$\sup_m \sum_{n=m}^{\infty} \left| \frac{\gamma_n(p,\theta)}{\gamma_m(p,\theta)} \right|^k < \infty. \tag{2.1}$$

Proof. $\left|\overline{N}_p^{\theta}\right|_k$ is a BK-space (see [6]). Now, if $x \in \phi$, then there exists a positive integer m such that $x = (x_0, x_1, x_m, 0, \ldots)$, and so $\phi \subset \left|\overline{N}_p^{\theta}\right|_k$ iff

$$\sum_{m=n}^{\infty} \left| \gamma_m(p,\theta) \sum_{v=1}^{n} P_{v-1} x_v \right|^k = \left| \sum_{v=1}^{n} P_{v-1} x_v \right|^k \sum_{m=n}^{\infty} |\gamma_m(p,\theta)|^k < \infty,$$

and $\left(e^{(n)}\right)$ is a base of $\left|\overline{N}_p^{\theta}\right|_k$ iff

$$\left\| x - \sum_{n=0}^{m} x_n e^{(n)} \right\|_{\left|\overline{N}_p^{\theta}\right|_k} \to 0 \text{ as } m \to \infty.$$

On the other hand, if $T(x) \in l_k$ for any $x \in \left|\overline{N}_p^{\theta}\right|_k$, then it follows from Minkowski's inequality that

$$\left\| x - \sum_{n=0}^{m} x_n e^{(n)} \right\|_{\left|\overline{N}_p^{\theta}\right|_k} = \left\{ \sum_{n=m+1}^{\infty} \left| T_n(x) - \gamma_n(p,\theta) \frac{T_m(x)}{\gamma_m(p,\theta)} \right|^k \right\}^{\frac{1}{k}} \to 0 \text{ as } m \to \infty$$

if and only if

$$\sum_{n=m+1}^{\infty} \left| \gamma_n(p,\theta) \frac{T_m(x)}{\gamma_m(p,\theta)} \right|^k = |T_m(x)|^k \sum_{n=m+1}^{\infty} \left| \frac{\gamma_n(p,\theta)}{\gamma_m(p,\theta)} \right|^k \to 0 \text{ as } m \to \infty,$$

or, equivalently, (2.1) holds, which states $x = \sum_{n=0}^{\infty} x_n e^n$. Further, by triangle inequality, it has a unique expression. This proves the result.

Theorem 2.2. *(i-) Let $1 < k < \infty$ and $\theta = (\theta_n)$ be a sequence of nonnegative numbers. If (p_n) is a sequence of nonnegative numbers satisfying (2.1), then, $\left|\overline{N}_p^{\theta}\right|_k^*$ is isometrically isomorphic to $l_{k'}$, i.e., $\left|\overline{N}_p^{\theta}\right|_k^* \cong l_{k'}$. Moreover if $f \in \left|\overline{N}_p^{\theta}\right|_k^*$*

$$f(x) = \lambda_0 x_0 + \sum_{v=1}^{\infty} \left(\sum_{n=v}^{\infty} \lambda_n \gamma_n(p,\theta) \right) P_{v-1} x_v; \; x \in \left|\overline{N}_p^{\theta}\right|_k \tag{2.2}$$

and

$$\|f\|_{\left|\overline{N}_p^{\theta}\right|_k^*} = \|\lambda\|_{l_{k'}} \tag{2.3}$$

where $\lambda \in l_{k'}$.

$(ii$-) Let $k = 1$ and $sup_n P_n/p_n < \infty$. Then, $\left|\overline{N}_p\right|_1^*$ is isometrically isomorphic to l_∞, *i.e.*, $\left|\overline{N}_p^{\theta}\right|_1^* \cong l_\infty$, and if $f \in \left|\overline{N}_p\right|_1^*$, then it is defined by (2.2) and

$$\|f\|_{\left|\overline{N}_p^{\theta}\right|_1'} = \|\lambda\|_\infty \tag{2.4}$$

where $\lambda \in l_\infty$.

Proof. (i) Define $T : l_{k'} \to \left|\overline{N}_p^\theta\right|_k^*$ by $T(\lambda) = f$, where f is as in (2.2). Trivially, T is well defined by (2.1), linear and injective. Also, T is surjective. In fact, take $f \in \left|\overline{N}_p^\theta\right|_k^*$. By Lemma 1.6 in [1] we see that (1.4) defines an isometry between $\left|\overline{N}_p^\theta\right|_k$ and l_k with respect to the norms (1.2) and $\|x\|_{l_k} = \left\{\sum_{n=0}^\infty |x_n|^k\right\}^{1/k}$. This means that $x \in \left|\overline{N}_p^\theta\right|_k$ if and only if $T(x) \in l_k$, and $\|x\|_{\left|\overline{N}_p^\theta\right|_k} = \|T(x)\|_{l_k}$. Further, $f \in \left|\overline{N}_p^\theta\right|_k^*$ if and only if $F \in l_k'$, where

$$f(x) = F(T(x)) = F(T), \text{ for all } x \in \left|\overline{N}_p^\theta\right|_k,$$

and also

$$\|f\| = \sup_{\|x\|_{\left|\overline{N}_p^\theta\right|_k}=1} |f(x)| = \sup_{\|T\|_{l_k}=1} |F(T)| = \|F\|.$$

It is well known from [6] that $l_k' \cong l_{k'}$, which shows that $F \in l_k^*$ if and only if there exists $\lambda \in l_{k'}$ such that

$$F(T) = \sum_{n=0}^\infty \lambda_n T_n(x), \text{ for all } T(x) \in l_k,$$
$$\|F\| = \|\lambda\|_{l_{k'}}. \tag{2.5}$$

So it follows that for every $x \in \left|\overline{N}_p^\theta\right|_k$

$$f(x) = \lambda_0 x_0 + \sum_{n=1}^\infty \lambda_n \gamma_n(p,\theta) \sum_{v=1}^n P_{v-1} x_v. \tag{2.6}$$

To get (2.2), it is sufficient to show that the order of summation in (2.6) can be interchanged. Now, by (2.1), since the series $\sum_{n=v}^\infty \lambda_n \gamma_n(p,\theta)$ is convergent, we write this sum as

$$\begin{aligned} f(x) &= \lambda_0 x_0 + \lim_{K\to\infty} \sum_{n=1}^K \lambda_n \gamma_n(p,\theta) \sum_{v=1}^n P_{v-1} x_v \\ &= \lambda_0 x_0 + \lim_{K\to\infty} \sum_{v=1}^K P_{v-1} x_v \sum_{n=v}^K \lambda_n \gamma_n(p,\theta). \\ &= \lambda_0 x_0 + \lim_{K\to\infty} \sum_{v=1}^K P_{v-1} x_v \left\{\sum_{n=v}^\infty - \sum_{n=K+1}^\infty\right\} \lambda_n \gamma_n(p,\theta). \end{aligned}$$

Thus it remains to show that

$$\left|\sum_{v=1}^K P_{v-1} x_v \sum_{n=K+1}^\infty \lambda_n \gamma_n\right| \to 0 \text{ as } K \to \infty.$$

But, it is easily seen from Hölder's inequality and (2.1) that

$$\begin{aligned} \left|\sum_{v=1}^K P_{v-1} x_v \sum_{n=K+1}^\infty \lambda_n \gamma_n\right| &\le |T_K(x)| \sum_{n=K+1}^\infty \left|\frac{\gamma_n}{\gamma_K} \lambda_n\right| \\ &\le M \left(\sum_{n=K+1}^\infty |\lambda_n|^{k'}\right)^{\frac{1}{k'}} \to 0 \text{ as } K \to \infty \end{aligned}$$

where

$$M = \sup_K |T_K(x)| \left(\sum_{n=K+1}^\infty \left|\frac{\gamma_n(p,\theta)}{\gamma_K(p,\theta)}\right|^k\right)^{\frac{1}{k}}.$$

Thus (2.2) holds, and also $\|L(\lambda)\|_{\left|\overline{N}_p^\theta\right|_k^*} = \|f\|_{\left|\overline{N}_p^\theta\right|_k^*} = \|\lambda\|_{l_{k'}}$ by (2.5), which completes the proof.

The proof of part (ii) follows from lines in part (i) considering that $l_1^* \cong l_\infty$ and (2.1) reduces to $\sup_n (P_n/p_n) < \infty$.

Also, using Theorem 2.2, we give a general representation of adjoint operator. We first recall related concepts. Let X, Y be normed spaces and $A : X \to Y$ be a bounded linear operator. Then, adjoint operator of A, denoted by A^*, is defined $A^* : Y^* \to X^*$ such that $A^*(f) = f o A$.

Making use of Theorem 2.2 we can prove the following theorem which establishes representation of adjoint operator of matrix operator on $\left|\overline{N}_p^\theta\right|_k$ for $k \geq 1$.

Theorem 2.3. *Let* (p_n) *and* (q_n) *be sequences of nonnegative numbers satisfying* $\sup_n P_n/p_n < \infty$ *and* (2.1), *respectively. If* $A \in \left(\left|\overline{N}_p\right|, \left|\overline{N}_q^\theta\right|_k\right)$, $k \geq 1$, *then the adjoint operator* $A^* : \left|\overline{N}_q^\theta\right|_k^* \to \left|\overline{N}_p\right|^*$ *is defined by*

$$g(x) = A^*(f)(x) = \sum_{j=0}^{\infty} \mu_j x_j;\ x \in \left|\overline{N}_q\right|$$

where $\lambda \in l_{k'}$, $\mu \in \ell_\infty$ *and*

$$\mu_0 = \varepsilon_0 a_{00},\, \mu_j = \frac{\theta_j^{-1/k'}}{p_j} \sum_{v=1}^{\infty} Q_{v-1} \left(P_j a_{vj} - P_{j-1} a_{v,j+1}\right) \sum_{n=v}^{\infty} \frac{\lambda_n q_n}{Q_n Q_{n-1}},\ j \geq 1.$$

Proof. Since $\left|\overline{N}_q^\theta\right|_k$ is a BK-space, by Banach–Steinhaus theorem, $A : \left|\overline{N}_p\right| \to \left|\overline{N}_q^\theta\right|_k$ is a bounded linear operator. Now, given $f \in \left|\overline{N}_q\right|^*$. Then $g \in \left|\overline{N}_p^\theta\right|_k^*$. So, by Theorem 2.2, there exist $\lambda \in l_\infty$ and $\mu \in \ell_{k'}$ such that

$$f(x) = \lambda_0 x_0 + \sum_{v=1}^{\infty} \left(\sum_{n=v}^{\infty} \lambda_n \gamma_n(q,1)\right) Q_{v-1} x_v;\ x \in \left|\overline{N}_q\right|$$

and

$$g(x) = \mu_0 x_0 + \sum_{v=1}^{\infty} \left(\sum_{n=v}^{\infty} \mu_n \gamma_n(p,\theta)\right) P_{v-1} x_v;\ x \in \left|\overline{N}_p^\theta\right|_k.$$

Also, by $g(x) = f(A(x))$,

$$\begin{aligned} g(x) &= \lambda_0 A_0(x) + \sum_{v=1}^{\infty} \left(\sum_{n=v}^{\infty} \lambda_n \gamma_n(q,1)\right) Q_{v-1} A_v(x) \\ &= \sum_{v=0}^{\infty} \sum_{j=0}^{\infty} \varepsilon_v a_{vj} x_j, \end{aligned}$$

where

$$\varepsilon_0 = \lambda_0,\ \varepsilon_v = Q_{v-1} \sum_{n=v}^{\infty} \lambda_n \gamma_n(q,1),\, v \geq 1.$$

Now if we put $x = e^{(j)} \in \left|\overline{N}_p\right|$ for $j = 0, 1, \ldots,$ then we have

$$\mu_0 = \varepsilon_0 a_{00},\ \ P_{j-1} \sum_{n=j}^{\infty} \mu_n \gamma_n(p,\theta) = \sum_{v=0}^{\infty} \varepsilon_v a_{vj} = A_j^T(\varepsilon)$$

where A^T is the transpose of the matrix A. This implies that

$$\begin{aligned}\mu_j &= \frac{1}{\gamma_j(p,\theta)}\left(\frac{A_j^T(\varepsilon)}{P_{j-1}} - \frac{A_{j+1}^T(\varepsilon)}{P_j}\right)\\ &= \frac{\theta_j^{-1/k'}}{p_j}\sum_{v=1}^{\infty} Q_{v-1}\left(P_j a_{vj} - P_{j-1}a_{v,j+1}\right)\sum_{n=v}^{\infty}\lambda_n\gamma_n(q,1)\\ &= \frac{\theta_j^{-1/k'}}{p_j}\sum_{v=1}^{\infty} Q_{v-1}\left(P_j a_{vj} - P_{j-1}a_{v,j+1}\right)\sum_{n=v}^{\infty}\frac{\lambda_n q_n}{Q_n Q_{n-1}}\end{aligned}$$

which completes the proof.

Also, following the lines in Theorem 2.4 we get the following theorem.

Theorem 2.4. *Let* (p_n) *and* (q_n) *be sequences of nonnegative numbers satisfying* $\sup_n P_n/p_n < \infty$ *and* (2.1), *respectively. If* $A \in \left(\left|\overline{N}_q^\theta\right|_k, \left|\overline{N}_p\right|\right)$, $k > 1$, *then the adjoint operator* $A^* : \left|\overline{N}_p\right|^* \to \left|\overline{N}_q^\theta\right|_k^*$ *is defined by*

$$g(x) = A^*(f)(x) = \sum_{j=0}^{\infty}\mu_j x_j;\ x \in \left|\overline{N}_q^\theta\right|_k$$

where $\lambda \in l_{k'}$, $\mu \in \ell_\infty$ *and*

$$\mu_0 = \varepsilon_0 a_{00},\ \mu_j = \frac{1}{p_j}\sum_{v=1}^{\infty} Q_{v-1}\left(P_j a_{vj} - P_{j-1}a_{v,j+1}\right)\sum_{n=v}^{\infty}\frac{\lambda_n\theta_n^{1/k'}q_n}{Q_n Q_{n-1}},\ j \geq 1.$$

Conflict of interest

No conflict of interest was declared by the author.

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