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# An approximation method for the solution of nonlinear integral equations

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**Abstract**

A Chebyshev collocation method has been presented to solve nonlinear integral equations in terms of Chebyshev polynomials. This method transforms the integral equation to a matrix equation which corresponds to a system of nonlinear algebraic equations with unknown Chebyshev coefficients. Finally, some examples are presented to illustrate the method and results discussed.


*Keywords:* Chebyshev polynomials and series; Collocation method; Nonlinear integral equation

## 1. Introduction

A Chebyshev-matrix method for solving nonlinear integral equations have been presented by Sezer and Doğan [7].

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In this study, Chebyshev collocation method, which is given by Akyüz and Sezer [1], is developed for nonlinear integral equation of Fredholm and Volterra types in the forms

$$y(x) = f(x) + \lambda \int_{-1}^{1} K(x,t)[y(t)]^2 \, dt \tag{1.1}$$

and

$$y(x) = f(x) + \lambda \int_{-1}^{x} K(x,t)[y(t)]^2 \, dt, \tag{1.2}$$

where $\lambda$ is a real parameter. We assume that these equations have solution as truncated Chebyshev series defined by

$$y(x) = \sum_{j=0}^{N}{}' a_j T_j(x), \quad -1 \leqslant x \leqslant 1, \tag{1.3}$$

where $T_j(x)$ denote the Chebyshev polynomials of the first kind, $a_j$ are unknown Chebyshev coefficients, $N$ is chosen any positive integer and $\sum'$ is a sum whose first term is halved. To obtain the Chebyshev polynomial solution of (1.1) and (1.2) it is assumed that $f(x)$ and $K(x,t)$ are defined on $[-1,1]$.

If the integrals are bounded in the range $[0,1]$, then solution can be obtained by means of the shifted Chebyshev polynomials $T_j^*(t)$.

## 2. Fundamental relations

We suppose that kernel functions and solutions of equations (1.1) and (1.2) can be expressed as a truncated Chebyshev series. Then (1.3) can be written in the matrix form

$$y(x) = T(x)A, \tag{2.1}$$

where

$$T(x) = [\, T_0(x) \quad T_1(x) \quad \cdots \quad T_N(x) \,], \quad A = \left[\, \tfrac{a_0}{2} \quad a_1 \quad \cdots \quad a_N \,\right]^{\mathrm{T}}.$$

Besides, $[y(t)]^2$ function can be written in the matrix form [7]

$$[y(t)]^2 = \overline{T(t)}B \tag{2.2}$$

in which

$$\overline{T(t)} = [\, T_0(t) \quad T_1(t) \quad \cdots \quad T_{2N}(t) \,], \quad B = \left[\, \tfrac{b_0}{2} \quad b_1 \quad \cdots \quad b_{2N} \,\right]^{\mathrm{T}}$$

and the elements $b_i$ of the column matrix $B$ consist of $a_i$ and $a_{-i} = a_i$ as follows:

$$b_i = \begin{cases} \dfrac{\left(a_{\frac{i}{2}}\right)^2}{2} + \sum\limits_{r=1}^{N-\frac{i}{2}} \left(a_{\frac{i}{2}-r}\right)\left(a_{\frac{i}{2}+r}\right) & \text{for even } i, \\ \sum\limits_{r=1}^{N-\frac{i-1}{2}} \left(a_{\frac{i+1}{2}-r}\right)\left(a_{\frac{i-1}{2}+r}\right) & \text{for odd } i. \end{cases}$$

Kernel function $K(x,t)$ can be expanded to univariate Chebyshev series for each $x_i$ in the form

$$K(x_i,t) = \sum_{r=0}^{N}{}'' k_r(x_i)T_r(t),$$

where a summation symbol with double primes denotes a sum with first and last terms halved, $x_i$ are the Chebyshev collocation points defined by

$$x_i = \cos\left(\frac{i\pi}{N}\right), \quad i = 0,1,\ldots,N \tag{2.3}$$

and Chebyshev coefficients $k_r(x_i)$ are determined by means of the relation

$$k_r(x_i) = \frac{2}{N}\sum_{j=0}^{N}{}'' K(x_i,t_j)T_r(t_j), \quad t_j = \cos\left(\frac{j\pi}{N}\right),$$

which is given in [2]. Then the matrix representation of $K(x_i,t)$ can be given by

$$K(x_i,t) = K(x_i)T(t)^{\mathrm{T}}, \tag{2.4}$$

where

$$K(x_i) = \left[\frac{k_0(x_i)}{2} \quad k_1(x_i) \quad \cdots \quad k_{N-1}(x_i) \quad \frac{k_N(x_i)}{2}\right].$$

## 3. The method for solution of nonlinear Fredholm integral equations

In this section, we consider Fredholm equation in (1.1) and approximate to solution by means of finite Chebyshev series defined in (1.3). The aim is to find Chebyshev coefficients, that is, the matrix $A$. For this reason, firstly, the Chebyshev collocation points defined by (2.3) are substituted into Eq. (1.1) and then it is obtained a matrix equation of the form

$$Y = F + \lambda I \tag{3.1}$$

in which $I(x)$ denotes the integral part of Eq. (1.1) and

$$Y = \begin{bmatrix} y(x_0) \\ y(x_1) \\ \vdots \\ y(x_N) \end{bmatrix}, \quad F = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix}, \quad I = \begin{bmatrix} I(x_0) \\ I(x_1) \\ \vdots \\ I(x_N) \end{bmatrix}, \quad T = \begin{bmatrix} T(x_0) \\ T(x_1) \\ \vdots \\ T(x_N) \end{bmatrix}.$$

When Chebyshev collocation points are put in relation (2.1), the matrix $Y$ becomes

$$Y = TA, \tag{3.2}$$

where the blocked matrix $T$ is defined above. In similar way, substituting the relations (2.2) and (2.4) in $I(x_i)$ and for $i = 0, 1, \ldots, N$, $j = 0, 1, \ldots, 2N$ using the relation

$$Z = \int_{-1}^{1} T(t)^{\mathrm{T}} \overline{T(t)}\, \mathrm{d}t = \left[ \int_{-1}^{1} T_i(t) T_j(t)\, \mathrm{d}t \right] = [z_{ij}],$$

whose entries are given in [3] as

$$z_{ij} = \begin{cases} \frac{1}{1-(i+j)^2} + \frac{1}{1-(i-j)^2} & \text{for even } i+j, \\ 0 & \text{for odd } i+j, \end{cases}$$

we have

$$I(x_i) = K(x_i) ZB. \tag{3.3}$$

Therefore, we get the matrix $I$ in terms of Chebyshev coefficients matrix in the form

$$I = KZB, \tag{3.4}$$

where

$$K = [\, K(x_0) \quad K(x_1) \quad \cdots \quad K(x_N) \,]^{\mathrm{T}}.$$

Finally using the relation (3.2) and (3.4), we have the fundamental matrix equation

$$TA - \lambda KZB = F, \tag{3.5}$$

which corresponds to a system of $(N+1)$ nonlinear algebraic equations with the $(N+1)$ unknown Chebyshev coefficients. Thus the unknown coefficients $a_j$ can be computed from this equation and consequently the solution of Fredholm integral equation is found in the form of truncated Chebyshev series.

If the integral is bounded by the range $[0, 1]$, the solution of integral equation is defined as

$$y(x) = \sum_{j=0}^{N}{}' a_j^* T_j^*(x), \quad 0 \leqslant x \leqslant 1, \tag{3.6}$$

where $T_j^*(x)$ denote the shifted Chebyshev polynomials and the Chebyshev collocation points in $[0,1]$ are

$$x_i = \frac{1}{2}\left[1 + \cos\left(\frac{i\pi}{N}\right)\right], \quad i = 0, 1, \ldots, N. \tag{3.7}$$

If the previous procedure is used, the fundamental matrix equation becomes

$$T^*A^* - \lambda K^* Z^* B^* = F, \tag{3.8}$$

where $T = T^*$ and $Z = 2Z^*$.

## 4. The method for solution of nonlinear Volterra integral equations

We now consider the nonlinear Volterra integral equations in (1.2). To obtain the solution of this equation in terms of Chebyshev polynomials we first define the integral part of (1.2) by $J(x)$ and then following the previous procedure we obtain

$$TA = F + \lambda J. \tag{4.1}$$

Using the relations (2.2) and (2.4) in $J(x_i)$, and then for $i = 0, 1, \ldots, N$, $j = 0, 1, \ldots, 2N$

$$Z(x_i) = \int_{-1}^{x_i} T(t)^{\mathrm{T}}\overline{T(t)}\,\mathrm{d}t = \left[\int_{-1}^{x_i} T_i(t)T_j(t)\,\mathrm{d}t\right] = [z_{ij}(x_i)],$$

where

$$z_{ij}(x) = \frac{1}{4}\begin{cases} 2x^2 - 2 \quad \text{for } i+j=1, \\ \frac{T_{i+j+1}(x)}{i+j+1} - \frac{T_{i+j-1}}{i+j-1} - \frac{1}{i+j+1} + \frac{1}{i+j-1} + x^2 - 1 \quad \text{for } |i-j| = 1, \\ \frac{T_{i+j+1}(x)}{i+j+1} + \frac{T_{1-i-j}(x)}{1-i-j} + \frac{T_{1+i-j}(x)}{1+i-j} + \frac{T_{1-i+j}(x)}{1-i+j} + 2\left[\frac{1}{1-(i+j)^2} + \frac{1}{1-(i-j)^2}\right] \\ \quad \text{for even } i+j, \\ \frac{T_{i+j+1}(x)}{i+j+1} + \frac{T_{1-i-j}(x)}{1-i-j} + \frac{T_{1+i-j}(x)}{1+i-j} + \frac{T_{1-i+j}(x)}{1-i+j} - 2\left[\frac{1}{1-(i+j)^2} + \frac{1}{1-(i-j)^2}\right] \\ \quad \text{for odd } i+j, \end{cases}$$

we have

$$J(x_i) = K(x_i)Z(x_i)B, \quad i = 0, 1, \ldots, N$$

or compact notation

$$J = \overline{\mathrm{K}}\mathscr{Z}B, \tag{4.2}$$

where $\overline{\mathrm{K}}$ and $\mathscr{Z}$ are $(N+1)$-by-$(N+1)^2$ and $(N+1)^2$-by-$(2N+1)$ matrices respectively and can be written by the blocked matrices as follows:

$$\overline{\mathrm{K}} = \begin{bmatrix} K(x_0) & 0 & \cdots & 0 \\ 0 & K(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K(x_N) \end{bmatrix}, \quad \mathscr{Z} = \begin{bmatrix} Z(x_0) \\ Z(x_1) \\ \vdots \\ Z(x_N) \end{bmatrix}, \quad J = \begin{bmatrix} J(x_0) \\ J(x_1) \\ \vdots \\ J(x_N) \end{bmatrix}.$$

Inserting the relation (4.2) in (4.1), the fundamental matrix equation of Volterra type is obtained

$$TA - \lambda\overline{\mathrm{K}}\mathscr{Z}B = F. \tag{4.3}$$

Unknown Chebyshev coefficients are computed from this nonlinear algebraic system and thereby we get Chebyshev series approach.

In addition, when the range is taken as $[0, 1]$, it is followed the above procedure using the Chebyshev collocation points in (3.7). Therefore the fundamental matrix equation is obtained as

$$T^*A^* - \lambda\overline{\mathrm{K}^*}\mathscr{Z}^*B^* = F, \tag{4.4}$$

where $T = T^*$ and $\mathscr{Z} = 2\mathscr{Z}^*$. Solving this nonlinear system, unknown coefficients $a_j^*$ are found.

## 5. Accuracy of solution

We can easily check the accuracy of the solutions obtained in the forms (1.3) and (3.6) as follows.

The solution (1.3) or the corresponding polynomial expansion must satisfy approximately the (1.1) or (1.2) for $-1 \leqslant x_i \leqslant 1$, $i = 0, 1, \ldots, N$, that is

$$D(x_i) = y(x_i) - f(x_i) - \lambda I(x_i) \cong 0$$

or

$$|D(x_i)| \cong 10^{-k_i}, \tag{5.1}$$

where $k_i$ are positive integers.

If $\max 10^{-k_i} = 10^{-k}$ ($k$ any positive integer) is prescribed, then the truncation limit $N$ is increased until the difference $|D(x_i)|$ becomes smaller than the prescribed $10^{-k}$ at each of the points $x_i$. Thus, we can get better the solution (1.3) by choosing $k$ appropriately so that $10^{-k}$ is very close to zero.

In the similar way, accuracy of the solution (3.6) for nonlinear Fredholm and Volterra integral equations in the range $0 \leqslant x \leqslant 1$ can be checked.

## 6. Illustrations

In this section, we consider five problems. All results were computed using Mathcad 2000 professional.

**Example 1.** Let us first consider the nonlinear Fredholm integral equation

$$y(x) = x^2 - \frac{8}{15}x - \frac{7}{6} + \int_0^1 (x+t)[y(t)]^2\,dt$$

and seek the solution $y(x)$ as a truncated Chebyshev series

$$y(x) = \sum_{j=0}^{2}{}' a_j^* T_j^*(x) \quad 0 \leqslant x \leqslant 1, \tag{6.1}$$

so that

$$f(x) = x^2 - \frac{8}{15}x - \frac{7}{6}, \quad K(x,t) = x + t, \quad \lambda = 1, \quad N = 2.$$

For $N = 2$, the Chebyshev collocation points in $[0, 1]$ are found from (3.7) as

$$x_0 = 1, \quad x_1 = 0.5, \quad x_2 = 0$$

and the fundamental matrix of the problem is defined by

$$T^*A^* - K^*Z^*B^* = F, \tag{6.2}$$

where

$$T^* = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \quad A^* = \begin{bmatrix} a_0^*/2 \\ a_1^* \\ a_2^* \end{bmatrix}, \quad F = \begin{bmatrix} -\frac{7}{10} \\ -\frac{71}{60} \\ -\frac{7}{6} \end{bmatrix}, \quad K^* = \begin{bmatrix} 1.5 & 0.5 & 0 \\ 1 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix},$$

$$Z^* = \begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 & -\frac{1}{15} \\ 0 & \frac{1}{3} & 0 & -\frac{1}{5} & 0 \\ -\frac{1}{3} & 0 & \frac{7}{15} & 0 & -\frac{19}{105} \end{bmatrix}, \quad B^* = \begin{bmatrix} \frac{1}{2}\left(\frac{a_0^{*2}}{2} + a_1^{*2} + a_2^{*2}\right) \\ a_0^*a_1^* + a_1^*a_2^* \\ \frac{a_1^{*2}}{2} + a_0^*a_2^* \\ a_1^*a_2^* \\ \frac{a_2^{*2}}{2} \end{bmatrix}.$$

If these matrices are substituted in (6.2), it is obtained nonlinear algebraic system. This system yields the solution

$$a_0^* = -1.25, \quad a_1^* = 0.5, \quad a_2^* = 0.125.$$

Substituting these values in (6.1) we have

$$y(x) = x^2 - 1,$$

which is the exact solution.

**Example 2.** Let us find the Chebyshev series solution of Fredholm integral equation

$$y(x) = \frac{1}{2} - \frac{1}{8}x + \int_{-1}^{1} \sin\left[\frac{1}{4}x(t+1)\right][y(t)]^2\,\mathrm{d}t. \tag{6.3}$$

Using the procedure in Section 3 for the interval $[-1,1]$ and taking $N = 3$ and 5, the matrices in Eq. (3.5) are computed. Hence, a nonlinear algebraic system is gained. For $a_0 = 1$, $a_1 = 0$, $a_i = 0$ and $a_0 = 1$, $a_1 = 5$, $a_i = 0$, $2 \leqslant i \leqslant N$, this system is approximately solved using the Mathcad 2000 Professional. Starting from these approximations, that is obtained two different solutions of (6.3) given in Tables 1 and 2, respectively.

The numerical solution of (6.3) in Chebyshev series was given by Shimasaki and Kiyono [8] and Sezer and Dogan [7]. A comparison of these solutions with the our solution is given in Tables 1 and 2.

**Example 3.** Let us find the Chebyshev series solution of nonlinear Volterra integral equation in $[0,1]$

$$y(x) = e^x - 0.5(e^{2x} - 1) + \int_0^x [y(t)]^2\,\mathrm{d}t,$$

with the exact solution $y(x) = e^x$.

Let us suppose that $y(x)$ is approximated by a truncated Chebyshev polynomial of degree six ($N = 6$). Using the procedure in Section 4 for $[0,1]$, we find the approximate solution of this equation.

Taking $h = 0.1$, different variable transformation methods in combination with the Trapezoidal quadrature rule were applied to this equation in [4]. The absolute errors found by presented method are compared with the errors given by variable transforms of Korobov, Sidi and Laurie in Table 3.

Table 1
Comparison of Chebyshev coefficients in the first solution

| $i$ | Shimasaki–Kiyono's $N = 20$, $a_i$ | Sezer–Doğan's $N = 3$, $a_i$ | Presented method $N = 3$, $a_i$ | Presented method $N = 5$, $a_i$ |
|---|---|---|---|---|
| 0 | 0.9999995 | 1 | 1 | 1 |
| 1 | −0.0022401 | −0.002180 | −0.0022381 | −0.0022392 |
| 2 | −0.0000002 | 0 | 0 | 0 |
| 3 | −0.0006414 | −0.000656 | −0.0006416 | −0.0006414 |
| 4 | 0.0 | | | 0 |
| 5 | 0.0000013 | | | 0.0000013 |

$a_i = 0.0$ $(6 \leqslant i \leqslant 20)$.

Table 2
Comparison of Chebyshev coefficients in the second solution

| $i$ | Shimasaki–Kiyono's $N = 20$, $a_i$ | Sezer–Doğan's $N = 3$, $a_i$ | Presented method $N = 3$, $a_i$ | Presented method $N = 5$, $a_i$ |
|---|---|---|---|---|
| 0 | 1.000001 | 1 | 1 | 1 |
| 1 | 5.088390 | 5.088420 | 5.0882234 | 5.088392 |
| 2 | 0.0000004 | 0 | 0 | 0 |
| 3 | −0.0397466 | −0.040228 | −0.0397288 | −0.0397468 |
| 4 | −0.0000001 | | | 0 |
| 5 | 0.0001007 | | | 0.0001007 |
| 6 | 0.0 | | | |
| 7 | −0.0000001 | | | |
| 8 | −0.0000001 | | | |

$a_i = 0.0$ ($9 \leqslant i \leqslant 20$).

Table 3
Error analysis of Example 3

| $x$ | Korobov's | Sidi's | Laurie's | Presented method |
|---|---|---|---|---|
| 0.1 | 0.12E−4 | 0.66E−8 | 0.12E−6 | 0.36E−7 |
| 0.2 | 0.31E−4 | 0.34E−7 | 0.27E−6 | 0.63E−7 |
| 0.3 | 0.60E−4 | 0.11E−6 | 0.47E−6 | 0.35E−7 |
| 0.4 | 0.11E−3 | 0.30E−6 | 0.71E−6 | 0.88E−7 |
| 0.5 | 0.18E−3 | 0.71E−6 | 0.95E−6 | 0.23E−7 |
| 0.6 | 0.29E−3 | 0.15E−5 | 0.12E−5 | 0.70E−7 |
| 0.7 | 0.49E−3 | 0.31E−5 | 0.13E−5 | 0.69E−7 |
| 0.8 | 0.82E−3 | 0.62E−5 | 0.11E−5 | 0.14E−7 |
| 0.9 | 0.14E−2 | 0.12E−4 | 0.27E−6 | 0.12E−7 |
| 1 | 0.25E−2 | 0.23E−4 | 0.20E−5 | 0.86E−7 |

**Example 4.** Let us consider the nonlinear Volterra integral equation in $[-1,1]$

$$y(x) = e^x - (x+1)\sin x + \int_{-1}^{x} e^{-2t} \sin x [y(t)]^2 \, dt.$$

The analytical solution is $y(x) = e^x$. Let us suppose that $y(x)$ is approximated by a truncated Chebyshev series

$$y(x) = \sum_{j=0}^{7}{}' a_j T_j(x), \quad -1 \leqslant x \leqslant 1.$$

Using the procedure in Section 4 for the interval $[-1,1]$, we find the approximate solution of this equation. The same example has been solved by Sezer [6] using Taylor polynomials. Taking $N = 7$, a comparison of these solutions with the exact solution is given in Table 4.

Table 4
Comparison of solutions for Example 4

| $x$ | Taylor solution $y(x)$ | Presented method $y(x)$ | Exact solution $e^x$ |
|---|---|---|---|
| −1.0 | 0.367879 | 0.367879 | 0.367879 |
| −0.8 | 0.449329 | 0.449328 | 0.449329 |
| −0.6 | 0.5488117 | 0.5488142 | 0.5488116 |
| −0.4 | 0.670320 | 0.670324 | 0.670320 |
| −0.2 | 0.8187291 | 0.8187269 | 0.8187308 |
| 0.0 | 0.9999898 | 0.9999877 | 1 |
| 0.2 | 1.221358 | 1.221395 | 1.221403 |
| 0.4 | 1.491666 | 1.491835 | 1.491825 |
| 0.6 | 1.821644 | 1.822138 | 1.822119 |
| 0.8 | 2.224291 | 2.225541 | 2.225541 |
| 1.0 | 2.715301 | 2.718277 | 2.718282 |

Table 5
Comparison of solutions for Example 5

| $x$ | Iteration method $y(x)$ | Haselgrove's solution $y(x)$ | Presented method $y(x)$ |
|---|---|---|---|
| 0.0 | 0.2791588 | 0.2793876 | 0.2791565 |
| 0.1 | 0.3608004 | 0.3609945 | 0.3607984 |
| 0.2 | 0.4437933 | 0.4439571 | 0.4437913 |
| 0.3 | 0.5280324 | 0.5281694 | 0.5280301 |
| 0.4 | 0.6134208 | 0.6135344 | 0.6134181 |
| 0.5 | 0.6998697 | 0.6999627 | 0.6998664 |
| 0.6 | 0.7872971 | 0.7873723 | 0.7872932 |
| 0.7 | 0.8756278 | 0.8756874 | 0.8756232 |
| 0.8 | 0.9647925 | 0.9648387 | 0.9647873 |
| 0.9 | 1.0547276 | 1.0547622 | 1.0547218 |
| 1.0 | 1.1453743 | 1.1453990 | 1.145368 |

**Example 5.** Our last example is nonlinear Fredholm integral equation

$$y(x) = x + 0.5\int_0^1 e^{-xt}[y(t)]^2\,\mathrm{d}t.$$

Using the procedure in section three for interval $[0,1]$, for $N = 6$ approximate solution of this equation are found. Besides, taking $N = 6$, this equation was solved by Chebyshev Iteration method and Haselgrove's method [5]. Obtained results are compared with the results of Iteration method and Haselgrove's results in Table 5. For $N = 6$, whereas an estimated accuracy of order was found $10^{-6}$ by Chebyshev Iteration methods, the accuracy of solution by presented method is found $10^{-9}$ using (5.1).

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