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THE NATURAL OPERATORS LIFTING k–PROJECTABLE VECTOR FIELDS TO PRODUCT-PRESERVING BUNDLE FUNCTORS ON k–FIBERED MANIFOLDS

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Abstract. For any product-preserving bundle functor F defined on the category $k-\mathcal{FM}$ of k–fibered manifolds, we determine all natural operators transforming k–projectable vector fields on $Y \in Ob(k-\mathcal{FM})$ to vector fields on FY. We also determine all natural affinors on FY. We prove a composition property analogous to that concerning Weil bundles.

0. Preliminaries. The classical results by Kainz and Michor [**6**], Luciano [**11**] and Eck [**3**] read that the product-preserving bundle functors on the category $\mathcal{M}f$ of manifolds are just Weil bundles, [**17**]. Let us remind Kolář's result [**7**].

For a bundle functor F on $\mathcal{M}f$, denote by $\mathcal{F}$ the flow operator lifting vector fields to F. Further, consider an element c of a Weil algebra A and let $L(c)_M : TT^AM \to TT^AM$ denote the natural affinor by Koszul ([**7**], [**8**]). Then we have a natural operator $L(c)_M \circ \mathcal{T}^A : TM \rightsquigarrow TT^AM$ lifting vector fields on a manifold M to a Weil bundle T^AM.

The Lie algebra associated to the Lie group $Aut(A)$ of all algebra automorphisms of A is identified with the algebra of derivations $Der(A)$ of A. For any $D \in Der(A)$ consider its one-parameter subgroup $\delta(t) \in Aut(A)$. It determines the vector field $D_M = \frac{d}{dt}{}_0\delta(t)_M$ on T^AM, where we identify Weil algebra homomorphisms with the corresponding natural transformations. Finally, we

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obtain a natural operator $\Lambda_{D,M} : TM \rightsquigarrow TT^AM$ defined by $\Lambda_{D,M}(X) = D_M$ for any vector field X on M. Then Kolář's result reads as follows.

All natural operators $TM \rightsquigarrow TT^AM$ are of the form $L(c)_M \circ \mathcal{T}^A + \Lambda_{D,M}$ for some $c \in A$ and $D \in Der(A)$.

Let us remind some results concerning product-preserving bundle functors on the category $\mathcal{FM}$ of fibered manifolds, [**12**], [**2**], [**16**]. They are just of the form T^μ for a homomorphism $\mu : A \to B$ of Weil algebras. Bundle functors T^μ are defined as follows. Let $i, j : \mathcal{M}f \to \mathcal{FM}$ be functors defined by $i(M) = id_M : M \to M$ and $j(M) = (M \to pt)$ for a manifold M and the single-point manifold pt. If $F : \mathcal{FM} \to \mathcal{FM}$ preserves the product then so do $G^F = F \circ i$ and $H^F = F \circ j$ and so there are Weil algebras A and B such that $G^F = T^A$ and $H^F = T^B$. Further, there is an obvious natural identity transformation $\tau_M : i(M) \to j(M)$ and thus we have a natural transformation $\mu_M = F\tau_M : T^AM \to T^BM$ corresponding to a Weil algebra homomorphism $\mu : A \to B$. Then the functor T^μ can be defined as the pull-back $T^AM \times_{T^BM} T^BY$ with respect to μ and T^Bp for a fibered manifold $p : Y \to M$. Then $F = T^\mu$ modulo a natural equivalence.

Let $\overline{F}$ be another product-preserving bundle functor on $\mathcal{FM}$. Then the result of [**12**] also yields natural transformations $\eta : F \to \overline{F}$ in the form of couples of $(\mu, \overline{\mu})$–related natural transformations $\nu = \eta \circ i : T^A \to T^{\overline{A}}$ and $\rho : \eta \circ j : T^B \to T^{\overline{B}}$ for a Weil algebra homomorphisms $\nu : A \to \overline{A}$ and $\sigma : B \to \overline{B}$.

For a bundle functor F on $\mathcal{FM}$, denote by $\mathcal{F}$ the flow operator lifting projectable vector fields to F. Further, consider an element c of A and let $L(c)_Y : TT^\mu Y \to TT^\mu Y$, $L(c)_Y(y_1, y_2) = (L(c)_M(y_1), L(\mu(c))_Y(y_2))$, $(y_1, y_2) \in TT^\mu Y = TT^AM \times_{TT^BM} TT^BY$ be the modification of the Koszul affinor. Then we have a natural operator $L(c)_Y \circ \mathcal{T}^\mu : T_{proj}Y \rightsquigarrow TT^\mu Y$ lifting projectable vector fields on a fibered manifold Y to $T^\mu Y$ for a Weil algebra homomorphism $\mu : A \to B$.

The Lie algebra associated to the Lie group $Aut(\mu) = \{(\nu, \rho) \in Aut(A) \times Aut(B) \mid \rho \circ \mu = \mu \circ \nu\}$ of all automorphisms of μ is identified with the algebra of derivations $Der(\mu) = \{D = (D_1, D_2) \in Der(A) \times Der(B) \mid D_2 \circ \mu = \mu \circ D_1\}$ of μ. For any $D \in Der(\mu)$ consider its one-parameter subgroup $\delta(t) \in Aut(\mu)$. It determines the vector field $D_Y = \frac{d}{dt}_0 \delta(t)_Y$ on $T^\mu Y$, where we identify homomorphisms of μ with the corresponding natural transformations. Finally, we obtain a natural operator $\Lambda_{D,Y} : T_{proj}Y \rightsquigarrow TT^\mu Y$ defined by $\Lambda_{D,Y}(X) = D_Y$ for any projectable vector field X on Y. Then a result of Tomáš [**16**] reads

All natural operators $T_{proj}Y \rightsquigarrow TT^{\mu}Y$ are of the form $L(c)_Y \circ \mathcal{T}^{\mu} + \Lambda_{D,Y}$ for some $c \in A$ and $D \in Der(\mu)$.

Let us recall the concept of k–fibered manifolds. It is a sequence of surjective submersions

$$Y = Y_k \xrightarrow{p_k} Y_{k-1} \xrightarrow{p_{k-1}} \dots \xrightarrow{p_1} Y_0 \tag{1}$$

between manifolds. Given another k–fibered manifold $\overline{Y} = \overline{Y}_k \xrightarrow{\overline{p}_k} \overline{Y}_{k-1} \xrightarrow{\overline{p}_{k-1}} \dots \xrightarrow{\overline{p}_1} \overline{Y}_0$, a map $f : Y \to \overline{Y}$ is called a morphism of k–fibered manifolds if there are the so-called underline maps $f_j : X_j \to \overline{X}_j$ for $j = 0, \dots, k-1$ such that $f_{j-1} \circ p_j = \overline{p}_j \circ f_j$ for $j = 1, \dots, k$, where $f_k = f$. Thus we have the category $k - \mathcal{FM}$ of k–fibered manifolds which is local and admissible in the sense of [**8**]. Clearly, the category $1 - \mathcal{FM}$ of 1–fibered manifolds coincides with the category $\mathcal{FM}$ of fibered manifolds.

Let us remind some results concerning product-preserving bundle functors on the category $k - \mathcal{FM}$ of k–fibered manifolds, [**13**]. They are just of the form T^{μ} for a sequence

$$\mu = (A_k \xrightarrow{\mu^k} A_{k-1} \xrightarrow{\mu^{k-1}} \dots \xrightarrow{\mu^1} A_0) \tag{2}$$

of k Weil algebra homomorphisms. Bundle functors T^{μ} are defined as follows. Let $i^{[l]} : \mathcal{M}f \to k - \mathcal{FM}$ for $l = 0, \dots, k$ be a sequence of functors defined by $i^{[l]}(M) = pt_M^{[l+1]} = (M \xrightarrow{id_M} M \xrightarrow{id_M} \dots \xrightarrow{id_M} M \to pt \to \dots \to pt)$, $k - l$ times of the single-point manifold pt, and $i^{[l]}(f) = f$. If $F : k - \mathcal{FM} \to \mathcal{FM}$ preserves the product then so do $G^{l,F} = F \circ i^{[l]}$ and so there are Weil algebras A_l such that $G^{l,F} = T^{A_l}$ for $l = 0, \dots, k$. Further, there are obvious identity natural transformations $\tau^l_M : i^{[l]}(M) \to i^{[l-1]}(M)$ and thus we have a sequence of natural transformations $\mu^l_M = F\tau^l_M$ corresponding to a sequence $\mu = (A_k \xrightarrow{\mu^k} A_{k-1} \xrightarrow{\mu^{k-1}} \dots \xrightarrow{\mu^1} A_0)$ of Weil algebra homomorphisms. For any k–fibered manifold Y of the form (1) we have

$$\begin{aligned} T^{\mu}Y =& \{y = (y_k, y_{k-1}, \dots, y_0) \in T^{A_k}Y_0 \times T^{A_{k-1}}Y_1 \times \dots . \times T^{A_0}Y_k \mid \\ & \mu^{k-l}_{Y_l}(y_{k-l}) = T^{A_{k-l-1}}p_{l+1}(y_{k-l-1}),\ l = 0, \dots, k-1\}. \end{aligned} \tag{3}$$

For a $k - \mathcal{FM}$–map $f : Y \to \overline{Y}$, $T^{\mu}f : T^{\mu}Y \to T^{\mu}\overline{Y}$ is the restriction and correstriction of $T^{A_k}f_0 \times T^{A_{k-1}}f_1 \times \dots \times T^{A_0}f_k$. Then $F = T^{\mu}$ modulo a natural equivalence.

Let $\overline{F}$ be another product-preserving bundle functor on $k - \mathcal{FM}$. Then the results of [**13**] also yield natural transformations $\eta : F \to \overline{F}$ in the form of sequences $\nu = (\nu^k, \dots, \nu^0)$ of $(\mu, \overline{\mu})$–related natural transformations $\nu^l = \eta \circ i^{[l]} : T^{A_l} \to T^{\overline{A}_l}$ for Weil algebra homomorphisms $\nu^l : A_l \to \overline{A}_l$.

We shall investigate k–projectable vector fields. A vector field X on a k–fibered manifold Y of the form (1) is called k–projectable if there are vector fields X_l on Y_l for $l = 0, \dots, k-1$ which are related to X by the respective compositions of projections of Y. The flow of X is formed by local $k - \mathcal{FM}$–isomorphisms. The space of all k–projectable vector fields on Y will be denoted by $\mathcal{X}_{k-proj}(Y)$.

Natural operators lifting vector fields are used in practically each paper in which the problem of prolongations of geometric structures was studied. For example A. Morimoto [**15**] used liftings of functions and vector fields has been to define the complete lifting of connections. That is why such natural operators are classified in [**4**], [**7**], [**16**] and other papers (over 100 references). For example, in the case of the tangent bundle TM of a manifold M (in our notation, $k = 0$), any natural operator lifting vector fields from M to TM is a linear combination of the complete lifting, the vertical lifting and the Liouville (dilatation) vector field.

A torsion of a connection Γ on TM is the Nijenhuis bracket $[\Gamma, J]$ of Γ with the almost tangent structure J on TM. This fact has been generalized in [**9**] in such a way that a torsion of a connection Γ with respect to a natural affinor A is $[\Gamma, A]$. Thus natural affinors can be used to study torsions of connections. That is why they have been classified in [**1**], [**5**], [**10**] and other papers (over 20 references). For example, any natural affinor on TM is a linear combination of the identity affinor and the almost tangent structure on TM.

1. Some properties of product preserving bundle functors on $k - \mathcal{FM}$. According to the Weil theory [**6**], for Weil algebras A and B there is the canonical identification $T^A \circ T^B M = T^{B \otimes A} M$. We generalize this fact on $k - \mathcal{FM}$. This extends the respective result of Tomáš's [**16**].

Consider $T^\mu Y$ in the form (3), where μ is of the form (2) and Y is of the form (1). It is easy to see that $T^\mu Y$ is a k–fibered manifold if we consider it in the form

$$T^\mu Y = T^{\mu^{[k]}} Y_{[k]} \to T^{\mu^{[k-1]}} Y_{[k-1]} \to \dots \to T^{\mu^{[0]}} Y_{[0]} \ , \tag{4}$$

where $\mu^{[l]} = (A_k \xrightarrow{\mu^k} A_{k-1} \xrightarrow{\mu^{k-1}} \dots \xrightarrow{\mu^{k-l+1}} A_{k-l})$ is the truncation of μ (it is a sequence of l Weil algebra homomorphisms) and $Y_{[l]} = Y_l \xrightarrow{p_l} Y_{l-1} \xrightarrow{p_{l-1}} \dots \xrightarrow{p_1} Y_0$ is the truncation of Y (it is an $l - \mathcal{FM}$–object) and where $T^{\mu^{[l]}} Y_{[l]}$ is defined as in (3) (in particular, $T^{\mu^{[0]}} Y_{[0]} = T^{A_0} Y_0$). Here the arrows in (4) are the restrictions and correstrictions of the obvious projections $T^{A_k} Y_0 \times \dots \times$

$T^{A_{k-l}}Y_l \to T^{A_k}Y_0 \times \cdots \times T^{A_{k-l+1}}Y_{l-1}$. Then $T^\mu : k-\mathcal{FM} \to \mathcal{FM}$ is a functor $k-\mathcal{FM} \to k-\mathcal{FM}$. Thus we can compose product-preserving bundle functors on $k-\mathcal{FM}$.

Proposition 1. *Let $T^\mu, T^{\overline{\mu}} : k-\mathcal{FM} \to \mathcal{FM}$ be product-preserving bundle functors corresponding to sequences μ and $\overline{\mu}$ of the form* (2). *Then $T^\mu \circ T^{\overline{\mu}} = T^{\overline{\mu}\otimes\mu}$, where (of course) $\overline{\mu}\otimes\mu = (\overline{A}_k \otimes A_k \xrightarrow{\overline{\mu}^k\otimes\mu^k} \overline{A}_{k-1}\otimes A_{k-1} \xrightarrow{\overline{\mu}^{k-1}\otimes\mu^{k-1}} \dots \xrightarrow{\overline{\mu}^1\otimes\mu^1} \overline{A}_0 \otimes A_0)$.*

Proof. Let $\tilde{\mu} = (\tilde{A}_k \xrightarrow{\tilde{\mu}^k} \tilde{A}_{k-1} \xrightarrow{\tilde{\mu}^{k-1}} \dots \xrightarrow{\tilde{\mu}^1} \tilde{A}_0)$ be the sequence of the form (2) corresponding to the composition $T^\mu \circ T^{\overline{\mu}}$. It can be computed as described in Section 0. Thus by the mentioned Weil theory [**6**], there is $\tilde{A}_l = \overline{A}_l \otimes A_l$ (as there is the identification $\tilde{A}_l = T^{A_l} \circ T^{\overline{A}_l}(\mathbf{R}) = T^{\overline{A}_l\otimes A_l}(\mathbf{R}) = \overline{A}_l \otimes A_l$). This identification is $(\tilde{\mu}, \overline{\mu}\otimes\mu)$–related. □

We describe some special case of T^μ. Let μ be of the form (2), where $A_k = A_{k-1} = \dots = A_0 = A$ and $\mu^l = id_A$ for $l = 1, \dots, k$. We will write id^A for such μ. Then $T^{id^A}Y = T^AY$. In particular, $T^{id}Y = TY$, where $id = id^{\mathbf{D}}$ and $\mathbf{D}$ is the Weil algebra of dual numbers.

2. Natural vector fields on bundle functors T^μ. Consider a sequence μ of the form (2). The group

$$\begin{aligned} Aut(\mu) =& \{\nu = (\nu^k, \nu^{k-1}, \dots, \nu^0) \in Aut(A_k) \times Aut(A_{k-1}) \times \cdots \times Aut(A_0) \mid \\ & \nu^{l-1} \circ \mu^l = \mu^l \circ \nu^l \ , l = 1, \dots, k\} \end{aligned}$$

of all automorphisms of μ is a closed subgroup in $Aut(A_k) \times Aut(A_{k-1}) \times \cdots \times Aut(A_0)$. Thus $Aut(\mu)$ is a Lie group. Let

$$\begin{aligned} Der(\mu) =& \{D = (D^k, D^{k-1}, \dots, D^0) \in Der(A_k) \times Der(A_{k-1}) \times \dots \times Der(A_0) \mid \\ & D^{l-1} \circ \mu^l = \mu^l \circ D^l \ , l = 1, \dots, k\} \end{aligned}$$

be the Lie algebra of all derivations of μ.

Proposition 2. *Let $Lie(Aut(\mu))$ be the Lie algebra of the Lie group $Aut(\mu)$ of all automorphisms of μ of the form* (2). *Then $Lie(Aut(\mu)) = Der(\mu)$.*

Proof. We know that $Lie(Aut(A)) = Der(A)$ for any Weil algebra A ([**7**]). Consequently, the proposition follows dirrectly from the application of exponential mapping concept. □

Let us recall that a natural operator $\Lambda_Y : T_{k-proj}Y \rightsquigarrow TT^\mu Y$ is a system of regular $k-\mathcal{FM}$–invariant operators

$$\Lambda_Y : \mathcal{X}_{k-proj}(Y) \to \mathcal{X}(T^\mu Y)$$

for any $k-\mathcal{FM}$–object Y. The $k-\mathcal{FM}$–invariance means that for any $k-\mathcal{FM}$–objects $Y,\overline{Y}$, any k–projectable vector fields $X\in\mathcal{X}_{k-proj}(Y)$ and $\overline{X}\in\mathcal{X}_{k-proj}(\overline{Y})$ and any $k-\mathcal{FM}$–map $f:Y\to\overline{Y}$, if X and $\overline{X}$ are f–related (i.e. $Tf\circ X=\overline{X}\circ f$) then $\Lambda_Y(X)$ and $\Lambda_{\overline{Y}}(\overline{X})$ are $T^\mu f$–related. The regularity means that Λ_Y transforms smoothly parametrized families of k–projectable vector fields into smoothly parametrized families of vector fields.

A natural operator $\Lambda_Y : T_{k-proj}Y \rightsquigarrow TT^\mu Y$ is called absolute (or a natural vector field on T^μ) if Λ_Y is a constant function for any $Y\in Obj(k-\mathcal{FM})$.

Proposition 2 enables us to modify the definition of an absolute operator $\Lambda_Y : T_{k-proj}Y \rightsquigarrow TT^\mu Y$ as follows. Let $D\in Der(\mu)=Lie(Aut(\mu))$ and let $\delta(t)\in Aut(\mu)$ be a one-parameter subgroup corresponding to D. It determines the vector field $D_Y=\frac{d}{dt}{}_0\delta(t)_Y$ on $T^\mu Y$, where we identify homomorphisms of μ with the corresponding natural transformations. Finally, we obtain a natural operator $\Lambda_{D,Y} : T_{k-proj}Y \rightsquigarrow TT^\mu Y$ defined by $\Lambda_{D,Y}(X)=D_Y$ for any k–projectable vector field X on $Y\in Ob(k-\mathcal{FM})$.

Proposition 3. *Let F be a product-preserving bundle functor on $k-\mathcal{FM}$. Then every absolute operator $\Lambda_Y : T_{k-proj}Y \rightsquigarrow TFY$ is of the form $\Lambda_{D,Y}$ for some $D\in Der(\mu)$, where μ is the sequence of the form* (2) *corresponding to F.*

Proof. The flow $Fl_t^{\Lambda_Y}$ of $\Lambda_Y\in\mathcal{X}(FY)$ is $k-\mathcal{FM}$–invariant and (thus) global, because FY is a $k-\mathcal{FM}$–orbit of any open neighbourhood of $0\in A_k^{m_k}\times\cdots\times A_0^{m_0}=F((i^{[k]}(\mathbf{R})^{m_k}\times\cdots\times(i^{[0]}(\mathbf{R}))^{m_0})$ for some $m_k,\dots,m_0$. Thus $Fl_t^{\Lambda_Y}: FY\to FY$ is a natural transformation. Let $\eta_t\in Aut(\mu)$ correspond to $Fl_t^{\Lambda_Y}$. Then $D=\frac{d}{dt}{}_0\eta_t\in Der(\mu)$ and $\Lambda_{D,Y}=\Lambda_Y$. □

3. Natural affinors on T^μ and natural operators $T_{k-proj}Y \rightsquigarrow TT^\mu$. Let μ be a sequence of the form (2) and let Y be a k–fibered manifold of the form (1).

Let us recall that a natural affinor on $T^\mu Y$ is a system of $k-\mathcal{FM}$–invariant affinors (i.e., tensor fields of type (1,1))

$$L_Y : TT^\mu Y\to TT^\mu Y$$

on $T^\mu Y$ for any $k-\mathcal{FM}$–object Y. The $k-\mathcal{FM}$–invariance means that for any $k-\mathcal{FM}$–map $f:Y\to\overline{Y}$, there is $L_{\overline{Y}}\circ TT^\mu f=TT^\mu f\circ L_Y$.

For $(y_k,y_{k-1},\dots,y_0)\in T(T^{A_k}Y_0\times T^{A_{k-1}}Y_1\times\cdots\times T^{A_0}Y_k)\bigcap TT^\mu Y$ and $c\in A_k$ we put

$$\begin{aligned}&L(c)_Y(y_k,y_{k-1},\dots,y_0)=\\ &\qquad(L(c)_{Y_k}(y_k),L(\mu^k(c))_{Y_{k-1}}(y_{k-1}),\dots,L(\mu^1\circ\ldots\circ\mu^{k-1}\circ\mu^k(c))_{Y_0}(y_0)),\end{aligned}\tag{5}$$

where $L(a)_M : TT^AM \to TT^AM$ is the Koszul affinor, [**7**]. We call $L(c)_Y$ the modified Koszul affinor on $T^\mu Y$.

The following theorem characterizes all natural affinors on $T^\mu Y$.

Theorem 1. *Let μ be a sequence of the form* (2) *and $Y \in Ob(k - \mathcal{FM})$ be of the form* (1). *Then every natural affinor on $T^\mu Y$ is of the form $L(c)_Y$ for some $c \in A_k$.*

Theorem 1 generalizes the result of [**1**] for Weil functors on $\mathcal{M}f$ and the result of Tomáš's [**16**] for product-preserving bundle functors on $\mathcal{FM}$ to all product-preserving bundle functors on $k - \mathcal{FM}$. A proof of Theorem 1 will follow a proof of Theorem 2.

For a k–projectable vector field $X \in \mathcal{X}_{k-proj}(Y)$, one can define its flow prolongation $\mathcal{F}X = \frac{d}{dt}{}_0 F(Fl_t^X) \in \mathcal{X}(FY)$ to a product-preserving bundle functor $F = T^\mu$ on $k - \mathcal{FM}$. (We know that the flow of X is formed by local $k - \mathcal{FM}$–isomorphisms, and then we can apply $F = T^\mu$ and obtain a flow on FY.) One can verify the Kolář formula

$$\mathcal{F}X = \eta_Y \circ FX \ , \tag{6}$$

where $\eta_Y : FTY = T^{id\otimes\mu}Y \tilde{=} T^{\mu\otimes id}Y = TFY$ is the exchange isomorphism and X is considered as $k - \mathcal{FM}$–map $X : Y \to TY = T^{id}Y$. We will not use this formula.

The following theorem modifies Kolář's result [**7**] for Weil functors on $\mathcal{M}f$ and Tomáš's result [**16**] for product-preserving bundle functors on $\mathcal{FM}$ to all product-preserving bundle functors on $k - \mathcal{FM}$.

Theorem 2. *Let F be a product-preserving bundle functor on $k - \mathcal{FM}$. Further, let X be a k–projectable vector field on a k–fibered manifold Y of the form* (1). *Then any natural operator $\Lambda_Y : T_{k-proj}Y \rightsquigarrow TFY$ is of the form*

$$L(c)_Y \circ \mathcal{F}X + \Lambda_{D,Y}$$

for some $c \in A_k$ and $D \in Der(\mu)$, where μ is the sequence of the form (2) *associated to F.*

Proof of Theorem 2. $\Lambda_Y(0)$ is an absolute operator. Thus replacing Λ_Y by $\Lambda_Y - \Lambda_Y(0)$ and appling Proposition 3 we can assume that $\Lambda_Y(0) = 0$.

Since any k–projectable vector field X on $Y \in Ob(k - \mathcal{FM})$ covering non-vanishing vector field on Y_0 is $\frac{\partial}{\partial x}$ on $i^{[k]}(\mathbf{R}) \subset i^{[k]}(\mathbf{R}) \times \dots$ in some $k - \mathcal{FM}$–cordinates (where the dots denote the respective multiproduct of $i^{[l]}(\mathbf{R})$'s), Λ_Y is uniquely determined by $\Lambda_{i^{[k]}(\mathbf{R})\times\dots}(\rho\frac{\partial}{\partial x}) : A_k \times \dots \to A_k \times \dots$, $\rho \in \mathbf{R}$. Using the invariance with respect to the homotheties being $k - \mathcal{FM}$–morphisms $i^{[k]}(\mathbf{R}) \times \dots \to i^{[k]}(\mathbf{R}) \times \dots$ and the homogeneous function theorem and $\Lambda_{i^{[k]}(\mathbf{R})\times\dots}(0) = 0$ we deduce that for any ρ the map $\Lambda_{i^{[k]}(\mathbf{R})\times\dots}(\rho\frac{\partial}{\partial x}) :$

$A_k \times \dots . \to A_k \times \dots$ is constant and linearly dependent on ρ. Then using the invariance with respect to $tid_{i^{[k]}(\mathbf{R})} \times id$ we deduce that the map $\Lambda_{i^{[k]}(\mathbf{R})\times\dots}(\rho\frac{\partial}{\partial x}) : A_k \times \dots \to A_k \times \{0\}$ is constant and linearly dependent on ρ. Then the vector space of all natural operators Λ_Y as above with $\Lambda_Y(0) = 0$ is at most $dim_{\mathbf{R}}A_k$–dimensional. But all natural operators $L(c)_Y \circ \mathcal{F}$ form a $dim_{\mathbf{R}}A_k$–dimensional vector space. Thus the proof is complete. □

Proof of Theorem 1. The vectors $\mathcal{T}^\mu X_v$ for $X \in \mathcal{X}_{k-proj}(Y)$ and $v \in T^\mu Y$ form a dense subset in $TT^\mu Y$ for sufficiently high fiber-dimensional $Y_k, \dots, Y_0$. (It is a simple consequence the rank theorem imlying that for any Weil algebra A with $width(A) = k$ the vector $\mathcal{T}^A\frac{\partial}{\partial x^1}{}_{j^A(t^1,\dots,t^k,0,\dots,0)} = j^{A\otimes\mathbf{D}}(t^1, \dots, t^k, 0, \dots, 0, t)$ has dense $\mathcal{M}f_m$–orbit in $TT^A\mathbf{R}^m = T^{A\otimes\mathbf{D}}\mathbf{R}^m$ if $m \geq k+1$.) Thus a natural affinor L_Y on $T^\mu Y$ is determined by $L_Y \circ \mathcal{T}^\mu X$ for X as above. But $\Lambda_Y : X \to L_Y \circ \mathcal{T}^\mu X$ is a natural operator with $\Lambda_Y(0) = 0$. Thus by the proof of Theorem 2 there is $\Lambda_Y(X) = L(c)_Y \circ \mathcal{T}^\mu X$ for some $c \in A_k$. Then $L_Y = L(c)_Y$. For arbitrary Y, we locally decompose id_Y by $p \circ j$ for $k - \mathcal{FM}$–maps, where $j : Y \to \overline{Y}$ with sufficiently high fiber-dimensional $\overline{Y}$. Next, we use the equality $L_{\overline{Y}} = L(c)_{\overline{Y}}$ and the invariance of natural affinors with respect to j. □

According to formula (6), it is sufficient to verify it for $X = \frac{\partial}{\partial x}$; see proof of Theorem 2. But then this is simple to verify.

4. Final remarks. Let $m = (m_k, m_{k-1}, \dots, m_0) \in (\mathbf{N} \cup \{0\})^{k+1}$. A k–fibered manifold Y of the form (1) is m–dimensional if $dim(Y_0) = m_0$, $dim(Y_1) = m_0 + m_1$, …, $dim(Y_k) = m_0 + m_1 + \dots + m_k$. All k–fibered manifolds of dimension $m = (m_k, \dots, m_0)$ and their local $k - \mathcal{FM}$–isomorphisms form a category which we will denote by $k - \mathcal{FM}_m$. It is local and admissible in the sense of [**8**].

Let $F = T^\mu : k - \mathcal{FM} \to \mathcal{FM}$ be a product preserving bundle functor and let $\eta : F_{|k-\mathcal{FM}_m} \to F_{|k-\mathcal{FM}_m}$ be a $k - \mathcal{FM}_m$–natural transformation. Assume that $m_k, m_{k-1}, \dots, m_0$ are positive integers. Then by a similar method as for Weil bundles on $\mathcal{M}f$ one can show that there exists one and only one natural transformation $\tilde{\eta} : F \to F$ extending η. Thus by Theorem 1, one can obtain the $k - \mathcal{FM}_m$–version of Theorem 1.

Theorem 1'. *Let μ be a sequence of the form* (2) *and $Y \in Ob(k - \mathcal{FM}_m)$ be of the form* (1)*, $m = (m_k, \dots, m_0)$, $m_k, \dots, m_0$ positive integers. Then every $k - \mathcal{FM}_m$–natural affinor on $T^\mu Y$ is of the form $L(c)_Y$ for some $c \in A_k$.*

By a simple modification of the proof of Theorem 2 one can obtain the $k-\mathcal{FM}_m$–version of Theorem 2.

THEOREM 2'. *Let* μ, Y, m *be as in Theorem 1'. Further, let* X *be a* k–*projectable vector field on a* k–*fibered manifold* Y *of the form* (1) *and dimension* m. *Then any* $k-\mathcal{FM}_m$–*natural operator* $\Lambda_Y : T_{k-proj}Y \rightsquigarrow TT^\mu Y$ *is of the form* $L(c)_Y \circ \mathcal{T}^\mu X + \Lambda_{D,Y}$ *for some* $c \in A_k$ *and* $D \in Der(\mu)$.

The authors would now like to announce that in [**14**] they describe all product preserving bundle functors on the category $\mathcal{F}^2\mathcal{M}$ of fibered-fibered manifolds (i.e. fibered surjective submersions between fibered manifolds) and in a paper being in preparation they extend Kolář's result [**7**] to product-preserving bundle functors on $\mathcal{F}^2\mathcal{M}$.

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