WIENER AMALGAM SPACES IN GENERALIZED HARMONIC ANALYSIS AND WAVELET THEORY

by

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ABSTRACT

Title of Dissertation: WIENER AMALGAM SPACES IN GENERALIZED HARMONIC ANALYSIS AND WAVELET THEORY

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This thesis is divided into four parts. Part I, Introduction and Notation, describes the results contained in the thesis and their background. Part II, Wiener Amalgam Spaces, is an expository introduction to Feichtinger's general amalgam space theory, which is used in the remainder of the thesis to formulate and prove results. Part III, Generalized Harmonic Analysis, presents new results in that area. Part IV, Wavelet Theory, contains exposition and miscellaneous results on Gabor (also known as Weyl–Heisenberg) wavelets.

Amalgam, or mixed-norm, spaces are Banach spaces of functions determined by a norm which distinguishes between local and global properties of functions. Specific cases were introduced by Wiener. Feichtinger has developed a far-reaching generalization of amalgam spaces, which allows general function spaces norms as local or global components. We use Feichtinger's amalgam theory, on d-dimensional Euclidean space under componentwise multiplication, to prove that the Wiener transform (introduced by Wiener to analyze the spectra of infinite-energy signals) is an invertible mapping of

the amalgam space with local L^2 and global L^q components onto an appropriate space defined in terms of the variation of functions, for each q between one and infinity. As corollaries, we obtain results of Beurling on the Fourier transform and results of Lau and Chen on the Wiener transform. Moreover, our results are carried out in higher dimensions. In addition, we prove that the higher-dimensional variation spaces are complete by using Masani's helices; this generalizes a one-dimensional result of Lau and Chen.

In wavelet theory, we present a survey of frames in Hilbert and Banach spaces and the use of the Zak transform in analyzing Gabor wavelets. Frames are an alternative to unconditional bases in these spaces; like bases, they provide representations of each element of the space in terms of the frame elements, and do so in a way in which the scalars in the representation are explicitly known. However, unlike bases, the representations need not be unique. We then discuss the specific case of Gabor frames in the space of square-integrable functions, concentrating on the role of the Zak transform in the analysis of such frames.

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This work would have been impossible without the mathematical framework of the Wiener amalgam spaces, provided by Hans Feichtinger of the University of Vienna. I thank Dr. Feichtinger for numerous preprints, discussions, suggestions, and encouragement.

Also critical was the work of Benedetto with George Benke and Ward Evans of The MITRE Corporation on the higher-dimensional Wiener–Plancherel formula. I thank Dr. Benke, my group leader at MITRE, for suggesting and supporting my non-thesis work on wavelets, and Dr. Evans for numerous mathematical discussions and advice.

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TABLE OF CONTENTS

PART I

INTRODUCTION AND NOTATION

CHAPTER 0

INTRODUCTION

This thesis falls naturally into several parts.

Part I, Introduction and Notation, describes the results contained in this thesis and their background, and lays out the notational scheme used throughout. Part I consists of Chapters 0 and 1.

Part II, Wiener Amalgam Spaces, is an expository introduction to Feichtinger's general amalgam space theory, which is used in the remainder of the thesis to formulate and prove results. Part II consists of a single chapter, Chapter 2.

Part III, Generalized Harmonic Analysis, contains new results in that area. The results depend heavily on the use of amalgam spaces. Our major result links and extends results of Wiener, Beurling, Lau and Chen, and Benedetto, Benke, and Evans into a single isomorphism theorem. Part III consists of Chapters 3 through 5.

Finally, Part IV, Wavelet Theory, contains exposition and miscellaneous new results in that area. Part IV consists of Chapters 6 and 7.

We introduce each of Parts II, III, and IV below, in Sections 0.1, 0.2, and 0.3, respectively.

Section 0.1. Amalgam spaces.

The classical L^p spaces on the real line $\mathbf{R}$ consist of those functions f for which the norm

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(t)|^p \, dt\right)^{1/p}$$

is finite. These spaces play a prominent role in modern analysis, yet often are difficult to use in applications because the L^p norm does not distinguish between local and global properties. For example, all rearrangements of a given function have identical L^p norms. Thus, it is not possible to recognize from the norm of a function whether it is, say, the characteristic function of an interval or the sum of many characteristic functions of small intervals spread widely over $\mathbf{R}$. As another example, "local" and "global" inclusions in L^p behave differently, with the result that there are no inclusion relations for L^p as a whole. To illustrate this, let $K \subset \mathbf{R}$ be a compact set, and let ℓ^p be the space of sequences $\{c_k\}$ which are p-summable, i.e., $\sum |c_k|^p < \infty$. Define the following subspaces of $L^p(\mathbf{R})$:

$$L^p(K) = \{f \in L^p(\mathbf{R}) : \text{supp}(f) \subset K\},$$

$$G^p = \{f = \sum c_k \, \chi_{[k,k+1]} : \{c_k\} \in \ell^p\},$$

where $\chi_{[k,k+1]}$ is the characteristic function of the interval $[k, k+1]$. Functions in $L^p(K)$ have only "local" behavior, while functions in G^p have only "global" behavior, in some sense. "Local" inclusions behave as follows:

$$p_1 \geq p_2 \quad \Rightarrow \quad L^{p_1}(K) \subset L^{p_2}(K),$$

while "global" inclusions behave as:

$$q_1 \le q_2 \quad \Rightarrow \quad \ell^{q_1} \subset \ell^{q_2},\ G^{q_1} \subset G^{q_2}.$$

No $L^p(\mathbf{R})$ is contained in any other $L^q(\mathbf{R})$.

Amalgam spaces decouple the connection between local and global properties which is inherent in the definition of L^p spaces. Their first use was by Norbert Wiener, in the formulation of his generalized harmonic analysis. In the notation of this thesis, he defined the spaces $W(L^1, L^2)$ and $W(L^2, L^1)$ in [**W4**], and $W(L^1, L^\infty)$ and $W(L^\infty, L^1)$ in [**W1**; **W2**], where $W(L^p, L^q)$ is the *standard amalgam space* defined by the norm

$$\|f\|_{W(L^p,L^q)} = \left(\sum_{n\in\mathbf{Z}} \left(\int_n^{n+1} |f(t)|^p\,dt\right)^{q/p}\right)^{1/q}, \tag{0.1.1}$$

the usual adjustments being made if p or q is infinity. Amalgams have been reinvented many times in the literature; the first systematic study appears to have been undertaken by Holland in [**Ho**]; an excellent review article is [**FS**].

The amalgams $W(L^p, L^q)$ distinguish between local L^p and global L^q properties of functions in the ways we expect. For example, rearrangements do not have identical norms in general, and inclusions behave correctly:

$$p_1 \ge p_2,\ q_1 \le q_2 \quad \Rightarrow \quad W(L^{p_1}, L^{q_1}) \subset W(L^{p_2}, L^{q_2}).$$

The dual space of $W(L^p, L^q)$ is $W(L^{p'}, L^{q'})$, where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. For $1 \le p,\ q \le 2$ we have a Hausdorff–Young property for the Fourier transform:

$$W(L^p, L^q)^\wedge \subset W(L^{q'}, L^{p'});$$

note that local and global properties are interchanged on the Fourier transform side.

H. Feichtinger recently proposed a far-reaching generalization of amalgam spaces to general topological groups and general local/global function spaces, e.g., [**F2**; **F8**], cf., Chapter 2. Given Banach spaces B, C of functions on a locally compact group G, he defines spaces $W(B,C)$ of functions or distributions which are "locally in B" and "globally in C". Moreover, his generalization is powerful and natural. Some properties which follow immediately from his theory are the following.

Inclusions. If $B_1 \subset B_2$ and $C_1 \subset C_2$ then $W(B_1,C_1) \subset W(B_2,C_2)$.

Duality. If a space of test functions (e.g., the Schwartz space $\mathcal{S}(\mathbf{R})$ of smooth, rapidly decreasing functions) is dense in B and C then $W(B,C)' = W(B',C')$.

Complex interpolation. Complex interpolation can be carried out in each component of $W(B,C)$ separately.

Pointwise multiplications. If $B_1 \cdot B_2 \subset B_3$ and $C_1 \cdot C_2 \subset C_3$ then $W(B_1,C_1) \cdot W(B_2,C_2) \subset W(B_3,C_3)$.

Convolutions. If $B_1 * B_2 \subset B_3$ and $C_1 * C_2 \subset C_3$ then $W(B_1,C_1) * W(B_2,C_2) \subset W(B_3,C_3)$.

Many other specific results follow immediately from Feichtinger's theory by choosing Sobolev spaces, Besov spaces, weighted L^p spaces, the Fourier algebra A, etc., as the local or global components, with various choices of topological groups.

Feichtinger refers to his spaces $W(B,C)$ as *Wiener-type spaces*; following a suggestion of J. Benedetto, and in order to promote the link between Feichtinger's generalization and the amalgams previously defined in the literature, we call them *Wiener amalgam spaces.* Taking G to be the group $\mathbf{R}$ under addition with Haar measure dt, the local component B to be $L^p(\mathbf{R})$, and the global component C to be $L^q(\mathbf{R})$, results in a Wiener amalgam space coinciding precisely with the standard amalgam space defined by (0.1.1).

In this thesis we obtain new results, and new proofs and generalizations of previously known results, in generalized harmonic analysis (Part III) and in wavelet theory (Part IV), by using amalgam spaces. Except for the amalgam space connection, the results in the two parts are unrelated, although we believe that the application of wavelets to generalized harmonic analysis could produce new results in the future.

For the benefit of the reader, we present in Part II a self-contained introduction to Feichtinger's theory. Since his theory is not needed in the later parts in its full generality, we present a simplified theory in which we allow only weighted L^p spaces as local or global components. This results in a considerable technical simplification of the proofs without destroying their essential flavor. Thus, Part II can be considered an elementary introduction to the general theory as presented in [**F8**]. In addition, we prove only those results directly related to our needs in this thesis, e.g., completeness, translation invariance, equivalence of discrete norms, inclusions, and duality. Part II is purely expository, and it is not necessary to read Part II in order to appreciate the results in Parts III and IV.

While Part II is written in terms of general topological groups, the results in Parts III and IV use the Wiener amalgam spaces on two specific topological groups. Part III uses the *multiplicative* group $\mathbf{R}^d_* = \{x \in \mathbf{R}^d : x_j \neq 0, \text{ all } j\}$, under componentwise multiplication, with Haar measure $dt/|t_1 \cdots t_d|$. Part IV uses the *additive* group $\mathbf{R}^d$, under componentwise addition, with Haar measure dt. To clearly distinguish between amalgam spaces on these two groups, we use the following notation in Parts III and IV (and in Sections 0.2 and 0.3 of this chapter):

$$W(L^p, L^q) = W(L^p(\mathbf{R}^d), L^q(\mathbf{R}^d)) \tag{0.1.2}$$

and

$$W_*(L^p, L^q) = W(L^p(\mathbf{R}^d_*), L^q(\mathbf{R}^d_*)). \tag{0.1.3}$$

The amalgam space $W(L^p, L^q)$ on the group $\mathbf{R}^d$ is precisely the higher-dimensional analogue of the standard amalgam space defined in (0.1.1); the intervals $[n, n+1]$ are simply replaced by cubes $[n_1, n_1+1] \times \cdots \times [n_d, n_d+1]$ for $n \in \mathbf{Z}^d$. We point out, however, that this norm is only equivalent to the fundamental norms used by Feichtinger as the basic definition for $W(L^p, L^q)$. We refer to a norm such as (0.1.1) as a *discrete-type* norm for $W(L^p, L^q)$; the fundamental defining norm is instead a *continuous-type* norm (Definition

2.2.2). Such norms more clearly illustrate the local L^p/global L^q features of $W(L^p, L^q)$.

For the one-dimensional case ($d = 1$), the discrete-type norm for $W_*(L^p, L^q)$ on the group $\mathbf{R}_*$ is

$$\|f\|_{W_*(L^p,L^q)} = \left(\sum_{n \in \mathbf{Z}, \pm} \left(\int_{\pm[2^n, 2^{n+1}]} |f(t)| \frac{dt}{|t|} \right)^{q/p} \right)^{1/q}. \tag{0.1.4}$$

The higher-dimensional version of this norm is obtained by replacing the intervals $\pm[2^n, 2^{n+1}]$ by rectangles $\pm[2^{n_1}, 2^{n_1+1}] \times \cdots \times \pm[2^{n_d}, 2^{n_d+1}]$, and by using the Haar measure $dt/|t_1 \cdots t_d|$.

Special Acknowledgement. We thank Dr. Feichtinger for permission to use several of his unpublished lecture notes in this section.

Section 0.2. Generalized harmonic analysis.

In this section we summarize and present background for results obtained in Part III of this thesis. Items a-d below discuss the background of our problems in generalized harmonic analysis, e-f discuss our results, g discusses future research possibilities, and h outlines Part III by chapters.

a. *The Wiener–Plancherel formula.* The Fourier transform provides the basic definition of spectrum for finite-energy functions on the real line. Central to its definition is the *Plancherel formula*

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt \;=\; \int_{-\infty}^{\infty} |\hat{f}(\gamma)|^2\, d\gamma,$$

where the Fourier transform is defined by

$$\hat{f}(\gamma) \;=\; \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i\gamma t}\, dt,$$

cf., Section 1.8. In order to deal with infinite-energy but finite-power functions, Wiener introduced what we now call the *Wiener transform*, and proved the *Wiener–Plancherel formula*, e.g., [**W1**]. These are defined as follows. Given a function f on the real line $\mathbf{R}$, its Wiener transform is (formally)

$$(0.2.1) \qquad Wf(\gamma) \;=\; \int_{-\infty}^{\infty} f(t)\, \frac{e^{-2\pi i\gamma t} - \chi_{[-1,1]}(t)}{-2\pi i t}\, dt.$$

If f has *bounded quadratic means*, i.e., if

$$(0.2.2) \qquad \sup_{T>0} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2\, dt < \infty,$$

then Wf is well-defined (Theorem 4.1.7). The Wiener–Plancherel formula states that for such f,

$$(0.2.3) \qquad \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 \, dt \; = \; \lim_{\lambda\to 0} \frac{2}{\lambda} \int_{-\infty}^{\infty} |\Delta_\lambda Wf(\gamma)|^2 \, d\gamma,$$

meaning that if one limit exists then the other does also and they are equal, and where Δ_λ is the *symmetric difference operator*

$$\Delta_\lambda F(\gamma) \; = \; F(\gamma+\lambda) \; - \; F(\gamma-\lambda).$$

Note that if f has finite energy, i.e., if $f \in L^2(\mathbf{R})$, then the left-hand side of (0.2.3) is zero.

Wiener called the theory associated with (0.2.1) and (0.2.3) *generalized harmonic analysis* as it generalizes the usual finite-energy harmonic analysis. For background, perspective, and proof of (0.2.3) and associated subjects, see [**B7**].

The Wiener–Plancherel formula has been extended to higher dimensions in [**BBE**], [**B1**], and [**Ben**]. The paper [**BBE**] adopted a "rectangular" approach to higher dimensions, while [**B1**] and [**Ben**] adopted a "spherical" approach. We prove our results in generalized harmonic analysis in higher dimensions following the rectangular approach of [**BBE**]. For clarity, we concentrate in this introduction on one-dimensional statements, and summarize higher-dimensional results in item f below.

b. *Lau's extension of the Wiener–Plancherel formula.* K.-S. Lau and J. K. Lee observed in [**LL**] that the space of functions f for which the limit on the left-hand side of (0.2.3) exists is nonlinear, and, more generally, that

$$(0.2.4) \qquad B(p,\lim) = \{f \in L^p_{\mathrm{loc}}(\mathbf{R}) : \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(t)|^p \, dt \text{ exists}\}$$

is nonlinear (the case $p = 2$ had originally been proved in [**HW**]). Therefore $B(p,\lim)$ cannot be dealt with using the methods of ordinary functional analysis. However, the Wiener transform W is defined for all f with bounded quadratic means, hence for all $f \in B(2,\limsup)$, where $B(p,\limsup)$ is the space of functions f for which the norm

$$(0.2.5) \qquad \|f\|_{B(p,\limsup)} = \limsup_{T\to\infty} \left(\frac{1}{2T}\int_{-T}^{T} |f(t)|^p \, dt\right)^{1/p}$$

is finite. Marcinkiewicz, in [**Mar**], proved that $B(p,\limsup)$ is a Banach space once functions f, $g \in B(p,\limsup)$ with $\|f - g\|_{B(p,\limsup)} = 0$ are identified. Lau and Lee proved that the Wiener transform W is a topological isomorphism of $B(2,\limsup)$ onto the space $V(2,\limsup)$, where

$$(0.2.6) \qquad \|F\|_{V(p,\limsup)} = \limsup_{\lambda\to 0} \left(\frac{2}{\lambda}\int_{-\infty}^{\infty} |\Delta_\lambda F(\gamma)|^p \, d\gamma\right)^{1/p}.$$

Since $V(p,\limsup)$ is not *solid*, i.e., $|F| \le |G|$ does not necessarily imply $\|F\|_{V(p,\limsup)} \le \|G\|_{V(p,\limsup)}$, the completeness of $V(p,\limsup)$ is a difficult question. Using the helix techniques of Masani, Lau and Lee were able to prove that $V(p,\limsup)$ is a Banach space (once functions F, G with $\|F - G\|_{V(2,\limsup)} = 0$ are identified), cf., [**LL**; **M1**; **M3**].

Following Lau and Lee's work on $B(p,\limsup)$, Lau and Chen proved in [**CL1**] that the Wiener transform W extends to a topological isomorphism of the space $B(2,\infty)$, where

$$\|f\|_{B(p,\infty)} = \sup_{T>0} \left(\frac{1}{2T} \int_{-T}^{T} |f(t)|^p \, dt \right)^{1/p}, \tag{0.2.7}$$

onto $V(2,\infty)$, where

$$\|F\|_{V(p,\infty)} = \sup_{\lambda>0} \left(\frac{2}{\lambda} \int_{-\infty}^{\infty} |\Delta_\lambda W f(\gamma)|^p \, d\gamma \right)^{1/p}. \tag{0.2.8}$$

We reproduce the proof of this result in Section 4.4–4.5. Our results include and generalize this result, both to a larger class of spaces and to higher dimensions.

It is clear that $B(p,\infty)$ is a Banach space, without the need to form equivalence classes other than the usual a.e. ones. Lau and Chen proved that $V(p,\infty)$ is also a Banach space (after the formation of equivalence classes), by using Masani's helix techniques.

c. *Beurling's A^p and B^p spaces.* In one of his deep investigations into spectral synthesis, Beurling introduced the following spaces, e.g., [**Be1**]:

$$B^p = \bigcap_{w \in \Lambda} L^p_w(\mathbf{R}) \tag{0.2.9}$$

and

$$A^{p'} = \bigcup_{w \in \Lambda} L^{p'}_{w'}(\mathbf{R}), \tag{0.2.10}$$

where Λ is the class of even, positive, integrable weights which are decreasing on $(0,\infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$,

$$w' = w^{1-p'},$$

and L^p_w is defined by the norm

$$\|f\|_{L^p_w} = \left(\int_{-\infty}^{\infty} |f(t)|^p \, w(t)\, dt\right)^{1/p}.$$

Beurling proved the following facts.

A^p and B^p are Banach spaces.

$A^p \subset L^1(\mathbf{R})$ and is a convolution algebra.

$B^p \supset L^\infty(\mathbf{R})$.

$(A^p)' = B^{p'}$, under the duality

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\, dt. \tag{0.2.11}$$

$B^p = B(p, \infty)$.

In addition, he proved that the *Fourier* transform on A^2 satisfies an isomorphism property similar to the one proved by Lau and Chen for the *Wiener* transform on $B(2,\infty) = B^2$. Recasting his result into our terminology, he essentially proved that the Fourier transform is a topological isomorphism of A^2 onto a space $V(2,1)$ defined by the norm

$$\|F\|_{V(2,1)} = \int_0^{\infty} \left(\frac{2}{\lambda}\int_{-\infty}^{\infty} |\Delta_\lambda F(\gamma)|^2 \, d\gamma\right)^{1/2} \frac{d\lambda}{\lambda}. \tag{0.2.12}$$

The proof required tricky estimates involving the weights w; we reproduce it in Section 4.4.

Many of Beurling's results in [**Be1**] (with the exception of the Fourier transform isomorphism theorem) were actually proved in higher dimensions, but with a spherical approach, rather than the rectangular approach of this thesis.

d. *Feichtinger's contribution.* As discussed in Section 0.1, Feichtinger has produced a general theory of amalgam spaces on topological groups. In **[F4]**, he characterized $B(p,\infty)$ as an amalgam space by proving that

$$(0.2.13) \qquad B(p,\infty) = W_*(L^p, L^\infty) = W(L^p(\mathbf{R}_*), L^\infty(\mathbf{R}_*)),$$

under equivalent norms. This insight provided us with a framework to link Beurling's and Lau's isomorphism results, and to prove our own results. The characterization as an amalgam space provided us with equivalent discrete-type norms, which are the basic machinery we use to prove our major theorems.

e. *Our results.* For clarity, we discuss one-dimensional versions of our results first, and make remarks on the higher-dimensional formulations in item f.

We generalize Feichtinger's characterization of $B(p,\infty)$ as the amalgam space $W_*(L^p, L^\infty)$ as follows. Define $B(p,q)$ to be the space of functions f for which the norm

$$(0.2.14) \qquad \|f\|_{B(p,q)} = \left(\int_0^\infty \left(\frac{1}{2T}\int_{-T}^{T} |f(t)|^p\, dt\right)^{q/p} \frac{dT}{T}\right)^{1/q}$$

is finite, with the standard adjustments if p or q is infinity. In Theorem 3.2.4 we prove that

$$(0.2.15) \qquad B(p,q) = W_*(L^p, L^q),$$

with equivalent norms. This provides us with discrete-type norms for all $B(p,q)$, cf., (0.1.4).

Recall now that $(A^p)' = B^{p'}$, with duality defined by (0.2.11). From (0.2.13), we have $B^{p'} = B(p',\infty) = W_*(L^{p'},L^\infty)$. It follows immediately from Feichtinger's amalgam theory that

$$W_*(L^p,L^1)' = W_*(L^{p'},L^\infty).$$

However, these amalgam spaces are on the multiplicative group $\mathbf{R}_*$, so the duality is with respect to the Haar measure on $\mathbf{R}_*$, i.e., with

$$\langle f,g\rangle = \int_{\mathbf{R}_*} f(t)\,\overline{g(t)}\,\frac{dt}{|t|}.$$

It therefore follows that

$$A^p = t\,W_*(L^p,L^1),$$

i.e.,

$$f \in A^p \quad \Leftrightarrow \quad tf(t) \in W_*(L^p,L^1) = B(p,1).$$

Except for the convergence factor $\chi_{[-1,1]}(t)$, the Fourier transform of $f \in A^2$ therefore corresponds to the Wiener transform of $tf(t) \in B(2,1)$, i.e.,

$$\begin{aligned}\hat{f}(\gamma) &= \int_{-\infty}^{\infty} f(t)\,e^{-2\pi i\gamma t}\,dt \\ &= -2\pi i \int_{-\infty}^{\infty} tf(t)\,\frac{e^{-2\pi i\gamma t}}{-2\pi i t}\,dt \\ &\approx -2\pi i\,W(tf)(\gamma).\end{aligned}$$

Since the convergence factor is not needed to make the integral defining Wg converge for $g \in B(2,1)$, and is irrelevant once we compute $\Delta_\lambda Wg$, the Beurling isomorphism theorem for the Fourier transform on A^2 therefore implies that the Wiener transform is a topological isomorphism of $B(2,1)$ onto $V(2,1)$. Comparing this to the Lau result, that W is a topological isomorphism of $B(2,\infty)$ onto $V(2,\infty)$, we anticipate the major result of Part III, namely, that W is a topological isomorphism of $B(2,q)$ onto $V(2,q)$ for each $1 \leq q \leq \infty$ (Theorem 4.5.5), where $V(p,q)$ is defined by the norm

$$(0.2.16) \qquad \|F\|_{V(p,q)} = \left(\int_0^\infty \left(\frac{2}{\lambda}\int_{-\infty}^\infty |\Delta_\lambda F(\gamma)|^p \, d\gamma\right)^{q/p} \frac{d\lambda}{\lambda}\right)^{1/q}.$$

We prove our isomorphism theorem directly, without interpolation. This avoids lengthly technical details establishing the interpolation properties of the non-solid spaces $V(p,q)$. Moreover, our use of Wiener amalgam spaces to prove this result gives new proofs of the Beurling and Lau results using a single technique, rather than the very different techniques used by the original authors.

Although not needed to prove our isomorphism theorem, we show in Section 3.4 that $B(p,q)$ can be written as a union or intersection of weighted L^p spaces, similar to the Beurling characterizations of A^p, B^p given in (0.2.9) and (0.2.10), cf., Proposition 3.4.6. This characterization allows us to relate the spaces $B(p,q)$ to other spaces which have appeared in harmonic analysis, cf., Remark 3.4.7.

f. *Higher dimensions.* Benedetto, Benke, and Evans, in [BBE], extended the Wiener–Plancherel formula (0.2.3) to higher dimensions, in a "rectangular" way. This nontrivial task included the determination of correct higher-dimensional analogues of limits, the Wiener transform W, and the symmetric difference operators Δ_λ, as well as the formulation and proof of new Tauberian theorems. The term "rectangular" stems from the fact that the intervals $[-T,T]$ in (0.2.3) are replaced by rectangular boxes $R_T = \prod_{j=1}^{d}[-T_j,T_j]$ for $T = (T_1,\ldots,T_d) \in \mathbf{R}_+^d$. For example, the space $B(p,\infty)$ is defined in higher dimensions in a rectangular way by the norm

$$(0.2.17) \qquad \|f\|_{B(p,\infty)} = \sup_{T\in\mathbf{R}_+^d} \left(\frac{1}{|R_T|}\int_{R_T} |f(t)|^p\,dt\right)^{1/p}.$$

The rectangular higher-dimensional definitions of limits are given in Section 3.1, of the Wiener transform in Section 4.1, and of the difference operator in Section 4.2. Using those definitions, the Wiener–Plancherel formula becomes the following: for $f \in B(2,\infty)$,

$$(0.2.18) \qquad \lim_{T\to\infty} \frac{1}{|R_T|}\int_{R_T} |f(t)|^2\,dt = \lim_{\lambda\to 0} \frac{2^d}{|\lambda_1\cdots\lambda_d|}\int_{\hat{\mathbf{R}}^d} |\Delta_\lambda Wf(\gamma)|^2\,d\gamma.$$

We prove all our results in higher dimensions using the higher-dimensional rectangular definitions. This includes the characterization of $B(p,q)$ as an amalgam space, the convergence of the Wiener transform on $B(2,q)$, the isomorphic nature of the Wiener transform as a mapping of $B(2,q)$ onto $V(2,q)$, and the proof of the completeness of the higher-dimensional variation spaces $V(p,q)$.

The completeness of $V(p,q)$ is proved in the final chapter of Part III. For one dimension, the completeness follows as a corollary of results of Lau and Chen based on Masani's helix techniques. In higher dimensions, the proof requires an iteration of those techniques (Theorem 5.2.3). We review the definitions and basic properties of helices in that chapter, and, while not appropriate for proving the completeness of $V(p,q)$, we also indicate how to extend helices directly to $\mathbf{R}^d$.

g. *Future results.* Benedetto has completed, and Benke is completing, work on spherical higher-dimensional analogues of the Wiener–Plancherel formula, cf., [**B1**] and [**Ben**], spherical in the sense that the intervals $[-T,T]$ in (0.2.3) are replaced by spheres of radius T. The resulting spherical formulas appear to be even more interesting than their rectangular counterparts. A major goal for future research is therefore to determine the spherical analogues of our isomorphism theorems. Another goal is to investigate higher-dimensional analogues of the Lau and Lee isomorphism theorem on $B(2, \limsup)$, both in rectangular and spherical settings.

A related area in which we expect our amalgam space methods to be of use is the following. In [**CL1**], Lau and Chen proved modified Wiener–Plancherel isomorphism theorems, obtained by replacing the factors $1/2T$ by $1/(2T)^\alpha$. Such results have applications to fractals, Hausdorff measures, etc., cf., [**E2; St1; St2**]. A goal for future research is therefore to prove our isomorphism theorem in such a setting. As a step in this direction, we prove in Section 3.5 that the spaces $B_\rho(p,q)$, obtained by replacing the factors $1/|R_T|$ in the definition of the higher-dimensional $B(p,q)$ by general functions $\rho(T)$, can be written as weighted Wiener amalgam spaces on the multiplicative group.

h. *Outline.* We outline Part III by chapters.

In Chapter 3 we present the definitions and fundamental characterizations of the Besicovitch spaces $B(p,q)$. We prove that $B(p,q)$ coincides with the Wiener amalgam space $W_*(L^p, L^q)$ and prove bounds for the norm equivalence. We discuss the relationship of $B(p,q)$ to unions or intersections of weighted L^p spaces. We discuss the effect of replacing the factor $1/2T$ in the definition of $B(p,q)$ (or $1/|R_T|$ in higher dimensions) by a general function $\rho(T)$, and show that the resulting spaces are again Wiener amalgam spaces, with weighted L^p components.

In Chapter 4 we discuss the Wiener transform. We prove that it is defined on each space $B(2,q)$ for $1 \leq q \leq \infty$, and determine the basic properties of $\Delta_\lambda Wf$. We reproduce the Beurling and Lau proofs of the isomorphic nature of W on $B(2,1)$ and $B(2,\infty)$, respectively, and then prove, directly, the continuity and invertibility of W on each of the spaces $B(2,q)$ by using the Wiener amalgam norms derived in Chapter 3.

In Chapter 5 we prove that the variation spaces $V(p,q)$ are Banach spaces by using an adaptation of Masani's helix techniques. We review the basic definitions and properties of helices and give Lau and Chen's proof that $V(p,\infty)$ is complete when $d = 1$, then extend this proof to higher dimensions by using an iterated helix technique.

Section 0.3. Wavelet theory.

Part IV of this thesis is a survey of results in wavelet theory, especially frames, Gabor systems, and the Zak transform. Part IV is largely expository; results of many authors have been combined with examples, remarks, and minor results of our own into a survey of one portion of wavelet theory. Most of the work on Part IV was completed prior to 1988, when we were hired by The MITRE Corporation to pursue work in wavelets. After that point we concentrated our thesis work on generalized harmonic analysis. Our work on wavelets for MITRE has appeared under separate cover, e.g., [**BHW; H1; H2; HW1; HW2**]. The paper [**HW2**] is a comprehensive introduction to wavelet theory from the point of view of frames.

In item a below we discuss the basic problem of wavelet theory. Item b discusses frames, which are an alternative to orthonormal or unconditional bases. Items c and d discuss Gabor and affine wavelets, respectively, and item e discusses the general wavelet theory of Feichtinger and Gröchenig. Item f outlines Part IV by chapters.

a. *Wavelet theory.* The basic problem of wavelet theory is to find good bases, or good substitutes for bases, for Banach function spaces, especially $L^2(\mathbf{R}^d)$, the Hilbert space of square-integrable functions on d-dimensional Euclidean space. The term "good" has, of course, many interpretations, including, but not limited to, the following. The basis elements should be easily generated from a single (or finitely many) functions through a combination of the fundamental operations of translation, modulation (translation in frequency, i.e., multiplication by $e^{2\pi i\gamma\cdot t}$), and dilation. The basis elements should be well localized in time and frequency, i.e., both the basis elements and their Fourier transforms should have good decay. Both the basis elements and their Fourier transforms should be smooth, preferably infinitely differentiable.

Two basic approaches to constructing such systems have developed. These are the Gabor (or Weyl–Heisenberg) wavelet systems and the affine wavelet systems, discussed below in items c and d. We point out that it has recently become unfashionable to refer to Gabor systems as wavelets, the term wavelet instead being reserved for affine systems.

b. *Frames.* Frames were invented by Duffin and Schaeffer, in the course of an investigation into nonharmonic Fourier series, as an alternative to orthonormal bases in Hilbert spaces [**DS**].

A sequence $\{e_n\}$ of vectors in a Hilbert space H is an orthonormal basis if the sequence is orthonormal, i.e., $\langle e_m, e_n\rangle = 0$ if $m \neq n$ and $\langle e_n, e_n\rangle = 1$, and the Plancherel formula holds, i.e., $\sum |\langle x, e_n\rangle|^2 = \|x\|^2$ for all $x \in H$. It follows that if $x \in H$ then there exist unique scalars $\{c_n\}$ such that $x = \sum c_n e_n$. A sequence $\{x_n\}$ in H is a frame if there exist numbers A, $B > 0$ such that $A\,\|x\|^2 \leq \sum |\langle x, e_n\rangle|^2 \leq B\,\|x\|^2$ for $x \in H$. The vectors $\{x_n\}$ need not be orthogonal, yet it follows that given $x \in H$ there exist scalars $\{c_n\}$ such that $x = \sum c_n x_n$. Unlike orthonormal bases, these scalars need not be unique. However, they are given explicitly, and the series $x = \sum c_n x_n$ converges unconditionally, i.e., all rearrangements converge (and converge to x), cf., Proposition 6.2.8. Frames which are *exact*, i.e., for which the representations $x = \sum c_n x_n$ are unique, are bounded unconditional bases for the Hilbert space, and vice versa (Proposition 6.3.3).

Frames thus provide representations of elements of a Hilbert space in terms of the frame elements, like orthonormal bases. Since the definition of frame is less restrictive than the definition of orthonormal bases, frames are usually easier to construct in applications.

c. *Gabor systems.* A Gabor system is generated from a single function (the *mother wavelet*) by translations and modulations; in particular, a Gabor system for $L^2(\mathbf{R})$ has the form $\{g_{mn}\}_{m,n\in\mathbf{Z}}$, where

$$g_{mn}(t) = e^{2\pi imbt}\, g(t - na),$$

and $g \in L^2(\mathbf{R})$ and a, $b > 0$ are fixed. Gabor systems have a long history and are closely related to several well-known signal processing tools, e.g., the short-time Fourier transform, the Wigner distribution, and the radar ambiguity function, cf., [**DeJ**]. They have applications to many areas, e.g., quantum mechanics [**BZ**; **BZZ**; **Z1**; **Z2**; **Z3**] and holography and optical computing [**Sch1**; **Sch2**; **Sch3**]. We restrict our discussion here to one dimension; the extension to higher dimensions is essentially trivial.

We concentrate in this thesis on the case of Gabor systems satisfying $ab = 1$. This case is especially amenable to analysis through the use of the *Zak transform*, a tool which has been reinvented many times in the literature. According to Schempp, a discrete form of the Zak transform was used by Gauss. Janssen, in [**J1**], lists some of the other occurances of the Zak transform. Zak used the transform in quantum mechanics to study the Gabor system generated by the Gaussian function $g(t) = e^{-\pi t^2}$. Some of the earliest results on the Zak transform were obtained by Auslander and Tolimieri by topological methods, e.g., [**AT2**], cf., [**AT1; AGT; AGTE**]. Important new results on the Zak transform have been obtained analytically by Janssen, e.g., [**J2; J3; J4**].

The Zak transform is a unitary map of $L^2(\mathbf{R})$ onto $L^2(Q)$, where $Q = [0,1] \times [0,1]$ is the unit cube in $\mathbf{R} \times \hat{\mathbf{R}}$. The Zak transform of g_{mn} has a particularly simple form, namely, $Zg_{mn}(t,\omega) = e^{2\pi imt}\, e^{2\pi in\omega}\, Zg(t,\omega)$. It follows immediately from this formula that a Gabor system with $ab = 1$ is complete if and only if $Zg \neq 0$ a.e., is an orthonormal basis if and only if $|Zg| = 1$ a.e., and is a frame if and only if $|Zg|$ is essentially constant, cf., Propsition 7.3.3.

The value $ab = 1$ has been shown to be a critical value for Gabor systems, cf., [**D1; Ri**]. In particular, any Gabor system with $ab > 1$ must be incomplete, and any Gabor system with $ab < 1$ which is a frame must be inexact. We prove in Proposition 7.3.3 that any Gabor system with $ab = 1$ which is a frame must be exact, whence $\{g_{mn}\}$ is a bounded unconditional basis for $L^2(\mathbf{R})$. It has been shown that if a Gabor system with $ab = 1$ is a frame then the mother wavelet g cannot be well localized both in time and frequency, in particular, $\|tg(t)\|_2 \, \|\gamma\hat{g}(\gamma)\|_2 = \infty$. This is the *Balian–Low theorem*, cf., [**Bal; Bat; BHW; D1; DJ; Low**]. In this thesis we present a simple proof of a related phenomenon, namely, that if g is the mother wavelet for a Gabor frame with $ab = 1$ then either g is discontinuous or has poor decay at infinity, precisely, $g \notin W(C_0, L^1)$, the Wiener amalgam space on the real line with local C_0 and global L^1 components, cf., Corollary 7.5.3.

In summary, Gabor frames with $ab = 1$ are easily analyzed using the Zak transform, but exhibit poor localization properties. It has been shown in [**DGM**] (where the idea of considering Gabor or affine systems which are frames instead of orthonormal bases was introduced) that the Balian–Low phenomenon does not occur if $ab < 1$, i.e., Gabor systems which are inexact frames can be generated by mother wavelets which are smooth (even infinitely differentiable) and have good decay (even compact support). We mention also that the Balian–Low phenomena is essentially nonexistent in a discrete setting, i.e., when considering Gabor frames for discrete signals in $\ell^2(\mathbf{Z})$, cf., [**H1**].

d. *Affine systems.* An affine system has the form $\{\varphi_{mn}\}_{m,n\in\mathbf{Z}}$, where

$$\varphi_{mn}(t) \;=\; a^{-n/2}\,\varphi(a^{-n}t - mb),$$

and the function φ and numbers $a > 1$, $b > 0$ are fixed. Although affine systems will not be discussed in the main part of the thesis, we include them here for completeness and comparison. A classical example is the *Haar system*, formed by taking $\varphi = \chi_{[0,1/2]} - \chi_{[1/2,1]}$, $a = 2$, and $b = 1$. The Haar system forms an orthonormal basis for $L^2(\mathbf{R})$.

In [**FJ**], Frazier and Jawerth introduced affine systems which are not bases, but have properties similar to frames, i.e., any element in the space can be written in terms of the affine system elements. They proved that such affine frames can be constructed in a wide range of function spaces, including the Besov and Triebel–Lizorkin spaces. Moreover, the space which the function belongs to is characterized by the behavior of the coefficients needed to write the function in terms of the affine frame elements.

Later, Daubechies, Grossmann, and Meyer used Hilbert space methods to construct affine frames in $L^2(\mathbf{R})$, cf., [**DGM**]. Daubechies, Mallat, and Meyer have recently shown that it is possible to find affine systems in $L^2(\mathbf{R})$ which are orthonormal bases, and which are generated by functions which are smooth and localized (unlike the Haar system). For example, it is possible to construct a mother wavelet φ which generates an affine orthonormal basis and which is compactly supported and k ($< \infty$) times differentiable, or is infinitely supported, infinitely differentiable, and exponentially decaying both in time and frequency, or is infinitely differentiable and has a compactly supported Fourier transform, cf., [**D2; Mal; Me1**]. Thus affine systems do not display the Balian–Low phenomenon. The existence of affine orthonormal bases has led to the introduction of fast (order N) algorithms for signal analysis, cf., [**D2; Mal**]. These algorithms have applications in signal processing, image processing, edge detection, etc., e.g., [**Gr; KMG**]. The algorithms are fast and easy to implement; we have used them at The MITRE Corporation for signal analysis.

e. *Feichtinger and Gröchenig's unified theory.* A Gabor system $\{g_{mn}\}$ can be viewed as the orbit of the function g under the Schroedinger representation of the Heisenberg group on a function space (see [**HW**] for details). An affine system $\{\varphi_{mn}\}$ can similarly be viewed as the orbit of φ under the translation/dilation representation of the $ax + b$ group on a function space. Thus Gabor and affine systems are structurally similar from the group representation point of view. Feichtinger and Gröchenig have developed a general wavelet theory from this group representation viewpoint, e.g., [**F3; F5; F6; FG2; FG3; FG4**]. Roughly stated, given a general representation on a general function space (satisfying certain conditions), they have shown that for a large class of mother wavelets g, any orbit $\{g_{mn}\}$ which is "dense enough" will induce representations of the functions in the function space in terms of the $\{g_{mn}\}$. Moreover, the function space is characterized by the coefficients needed to represent functions in terms of the $\{g_{mn}\}$. The techniques they developed to prove this general theory have also been applicable to other areas, in particular, to the problem of reconstructing a band-limited signal from irregularly sampled data, e.g., [**FG1**].

f. *Outline.* In Chapter 6 we present a survey of frames (and a dual concept known as sets of atoms) in Hilbert spaces, with some remarks on the extension of these concepts to Banach spaces. We discuss the representations of elements in the space provided by frames, and characterize when the representations will be unique, i.e, when the frame is exact. We determine the exact relationship between frames and sets of atoms, showing that while atoms are more general, in practice the two concepts will be equivalent. We prove a general stability theorem for atoms in Banach spaces, showing that the elements of a set of atoms may be perturbed by a small amount without destroying the atomic properties.

In Chapter 7 we discuss Gabor systems and the Zak transform. We show that Gabor systems with $ab = 1$ can be analyzed through the use of the Zak transform. We analyze the structure of the Zak transform, and prove that it is a continuous mapping of the Wiener amalgam space $W(L^p, L^1)$ into the Lebesgue space $L^p(Q)$. We use this to prove a variant of the Balian–Low theorem, that a mother wavelet for a Gabor frame with $ab = 1$ cannot be continuous and have good decay at infinity, in particular, $g \notin W(C_0, L^1)$. We conclude by discussing some questions similar to ones which arise from the application of the Zak transform to Gabor frames. In particular, we generalize slightly a result of Boas and Pollard which shows that if finitely many elements are removed from an orthonormal basis for $L^2(X)$ then it is always possible to find a single function to multiply the remaining elements by so that the resulting sequence is complete. We show this need not be true if infinitely many elements are deleted, and discuss some related results by other authors.

CHAPTER 1

NOTATION AND DEFINITIONS

Section 1.1. Basic symbols.

a. $\mathbf{C}$ is the set of complex numbers. The modulus or absolute value of $z \in \mathbf{C}$ is denoted by $|z|$, the complex conjugate by $\bar{z}$.

$\mathbf{R}$ is the real line thought of as the time axis, and $\hat{\mathbf{R}}$ is its dual group, the real line as the frequency axis. $\mathbf{R}^d$ is d-dimensional Euclidean space, the set of d-tuples of real numbers, and $\hat{\mathbf{R}}^d$ is its dual group.

$\mathbf{Z}$ is the set of integers, and $\mathbf{Z}^d$ the set of d-tuples of integers.

b. An element $x \in \mathbf{R}^d$ is written in terms of its components as $x = (x_1, \dots, x_d)$. Given a, $b \in \mathbf{R}^d$ we define

$$\begin{aligned} a \cdot b &= a_1 b_1 + \cdots + a_d b_d, \\ |a| &= \left(a_1{}^2 + \cdots + a_d{}^2\right)^{1/2}, \\ \Pi(a) &= a_1 \cdots a_d. \end{aligned}$$

All other operations on elements of $\mathbf{R}^d$ are to be interpreted componentwise, including logical operations. For example, if a, $b \in \mathbf{R}^d$ then

$$\begin{aligned} a + b &= (a_1 + b_1, \dots, a_d + b_d), \\ ab &= (a_1 b_1, \dots, a_d b_d), \\ a/b &= (a_1/b_1, \dots, a_d/b_d), \\ a^b &= (a_1{}^{b_1}, \dots, a_d{}^{b_d}), \end{aligned}$$

$$\cos a = (\cos a_1, \ldots, \cos a_d),$$

$$a > b \Leftrightarrow a_j > b_j \text{ for } j = 1, \ldots, d.$$

An operation between $a \in \mathbf{R}^d$ and $c \in \mathbf{R}$ is treated by identifying $c \in \mathbf{R}$ with $(c, \ldots, c) \in \mathbf{R}^d$, e.g.,

$$a + c = (a_1 + c, \ldots, a_d + c),$$

$$c/a = (c/a_1, \ldots, c/a_d),$$

$$c^a = (c^{a_1}, \ldots, c^{a_d}),$$

$$a > c \Leftrightarrow a_j > c \text{ for } j = 1, \ldots, d.$$

c. The concatenation of $a \in \mathbf{R}^d$, $b \in \mathbf{R}^k$ is $(a, b) = (a_1, \ldots, a_d, b_1, \ldots, b_k) \in \mathbf{R}^{d+k}$.

Section 1.2. Special sets.

a. The **coordinate axes**, or more precisely, the **coordinate hyperplanes**, in $\mathbf{R}^d$ are

$$\mathbf{A}_d = \{x \in \mathbf{R}^d : \Pi(x) = 0\}.$$

b. The *d*-**dimensional multiplicative group** is

$$\mathbf{R}^d_* = \mathbf{R}^d \backslash \mathbf{A}_d,$$

under componentwise multiplication. The identity element of $\mathbf{R}^d_*$ is $(1, \dots, 1)$.

c. The **unit sphere** in $\mathbf{R}^d$ is

$$\mathbf{S}_{d-1} = \{x \in \mathbf{R}^d : |x| = 1\}.$$

d. The **set of signs** in $\mathbf{R}^d$ is

$$\Omega^d = \{-1, 1\}^d = \{\sigma \in \mathbf{R}^d : \sigma_j = \pm 1 \text{ for } j = 1, \dots, d\}.$$

e. If $E \subset \mathbf{R}^d$ then

$$E_+ = E^+ = \{x \in E : x > 0\}.$$

f. A **rectangle** in $\mathbf{R}^d$ is a rectangular box whose sides are parallel to the coordinate axes. Given a, $b \in \mathbf{R}^d$ with $a \le b$, the **open rectangle determined by** a, b is

$$(a, b) = \prod_{j=1}^{d} (a_j, b_j) = \{x \in \mathbf{R}^d : a < x < b\}.$$

We similarly define the closed rectangle $[a, b]$ and the **half-open rectangles** $[a, b)$ and $(a, b]$. The **side lengths** of any such rectangle are the components of the d-tuple $b - a$. We allow a or b to be scalars, identifying $a \in \mathbf{R}$ with $(a, \ldots, a) \in \mathbf{R}^d$. For example, $[0, b)$ is the rectangle with one vertex at the origin and the other at b. If both a and b are scalars then some dimensional confusion could result; however, the dimension should always be clear from context. For example, $[0, 1]$ is a cube in $\mathbf{R}^d$ for any d.

Given $T \in \mathbf{R}^d_+$ we define

$$R_T = [-T, T].$$

Section 1.3. Functions.

a. The **characteristic function** of a set E is $\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$

The **Kronecker delta** is $\delta_{xy} = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases}$

b. A real-valued function f on a set E is **positive** if $f(t) > 0$ for $t \in E$. It is **nonnegative** if $f(t) \geq 0$ for $t \in E$.

c. A function $f\colon \mathbf{R}^d \to \mathbf{C}$ is **P-periodic**, where $P \in \mathbf{R}^d$, if $f(t+P) = f(t)$ for $t \in \mathbf{R}^d$.

d. A function $f\colon \mathbf{R}^d \to \mathbf{C}$ is **symmetric** if $f(t) = f(-t)$ for $t \in \mathbf{R}^d$, **radial** if $f(s) = f(t)$ whenever $|s| = |t|$, and **even** if $f(\sigma t) = f(t)$ for $t \in \mathbf{R}^d$ and $\sigma \in \Omega^d$. These three notions are equivalent if $d = 1$ but not if $d > 1$. Every radial function is even, and every even function is symmetric. If $d > 1$ then the function $f(t) = |\Pi(t)|$ is even but not radial, and $f(t) = \operatorname{sign}(t_1) \cdot \operatorname{sign}(t_2)$ is symmetric but not even.

e. A function $f\colon \mathbf{R}^d \to \mathbf{C}$ is **rectangular** if there exist functions $f_j\colon \mathbf{R} \to \mathbf{C}$ such that

$$f(t) \;=\; \prod_{j=1}^{d} f_j(t_j)$$

for all $t \in \mathbf{R}^d$.

f. A real-valued function f is (rectangularly) **decreasing** on a set $E \subset \mathbf{R}^d$ if given $s,\, t \in E$,

$$s < t \;\;\Rightarrow\;\; f(s) \geq f(t).$$

In other words, f is decreasing in each component. f is **strictly decreasing** if $f(s) > f(t)$ when $s < t$. We similarly define **increasing** and **strictly**

increasing.

g. Given a real-valued function f on $\mathbf{R}_+^d$, its **least decreasing majorant** $f^\star$ is

$$f^\star(t) = \sup_{s \geq t} f(s).$$

Its **greatest decreasing minorant** $f_\star$ is

$$f_\star(t) = \inf_{s \leq t} f(s).$$

Clearly $f_\star \leq f \leq f^\star$, and f is decreasing if and only if $f = f^\star = f_\star$. If f is rectangular then $f^\star(t) = \prod_1^d f_j{}^\star(t_j)$ and $f_\star(t) = \prod_1^d f_{j\star}(t_j)$.

h. The following function spaces are defined specifically for functions on $\mathbf{R}^d$; other function spaces are defined in Section 1.7. Given $k \in \mathbf{Z}^d$ with $k \geq 0$ we define

$$\begin{aligned}
C(\mathbf{R}^d) &= \{f : f \text{ is continuous}\}, \\
C_c(\mathbf{R}^d) &= \{f \in C(\mathbf{R}^d) : \operatorname{supp}(f) \text{ is compact}\}, \\
C^k(\mathbf{R}^d) &= \{f : \partial^\alpha f \in C(\mathbf{R}^d) \text{ for } \alpha \in \mathbf{Z}^d, 0 \leq \alpha \leq k\}, \\
C_c^k(\mathbf{R}^d) &= C^k(\mathbf{R}^d) \cap C_c(\mathbf{R}^d),
\end{aligned}$$

where

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{(\partial_1)^{\alpha_1} \cdots (\partial_d)^{\alpha_d}}.$$

$C^\infty(\mathbf{R}^d)$ and $C_c^\infty(\mathbf{R}^d)$ are defined analogously. The **Schwartz space** of rapidly decreasing functions is

$$\mathcal{S}(\mathbf{R}^d) = \Big\{f \in C^\infty(\mathbf{R}^d) : \sup_{t \in \mathbf{R}^d} |\Pi(t^k)\, \partial_\alpha f(t)| < \infty \text{ for } k, \alpha \in \mathbf{Z}^d, k, \alpha \geq 0\Big\}.$$

The space of **tempered distributions**, denoted $\mathcal{S}'(\mathbf{R}^d)$, is the topological dual of **S**.

Section 1.4. Convergence.

Given a normed linear space X and a sequence $\{x_n\}_{n\in\mathbf{Z}_+}$ of elements of X, we say that the series $\sum x_n$ **converges** to $x \in X$, and write $\sum x_n = x$, if $s_N \to x$, where $s_N = \sum_{n=1}^{N} x_n$. The series converges **unconditionally** if $\sum x_{\beta(n)}$ converges for every permutation β of $\mathbf{Z}_+$. It converges **absolutely** if $\sum \|x_n\| < \infty$. Absolute convergence implies unconditional convergence. If X is finite-dimensional, the converse is also true.

LEMMA 1.4.1. *Given a normed linear space X, the following statements are equivalent.*

a. *X is complete.*

b. *If $\{x_n\}_{n\in\mathbf{Z}_+} \subset X$ and $\sum \|x_n\| < \infty$ then $\sum x_n$ converges in X.*

LEMMA 1.4.2 [S]. *Given a sequence $\{x_n\}_{n\in\mathbf{Z}_+}$ in a Banach space X, the following statements are equivalent.*

a. *$\sum x_n$ converges unconditionally.*

b. *$x = \lim_F \sum_{n\in F} x_n$ exists, where the limit is with respect to the net of finite subsets of $\mathbf{Z}_+$ ordered by inclusion. In other words, for every $\varepsilon > 0$ there is a finite set $G \subset \mathbf{Z}_+$ such that $\|x - \sum_{n\in F} x_n\| < \varepsilon$ for every finite $F \subset \mathbf{Z}_+$ with $F \supset G$.*

c. *For each $\varepsilon > 0$ there is an $N \in \mathbf{Z}_+$ such that for each finite $F \subset \mathbf{Z}_+$ with $\min(F) > N$ we have $\|\sum_{n\in F} x_n\| < \varepsilon$.*

d. *$\sum x_{n_j}$ converges for every increasing sequence $0 < n_1 < n_2 < \ldots$.*

e. $\sum \sigma_n x_n$ *converges for every choice of signs* $\sigma_n = \pm 1$.

f. $\sum c_n x_n$ *converges for every bounded sequence of scalars* $\{c_n\}$.

In case these hold, $\sum x_{\beta(n)} = \sum x_n$ *for every permutation* β *of* $\mathbf{Z}_+$.

LEMMA 1.4.3. *Given a Banach space* X *and a sequence* $\{x_n\}_{n \in \mathbf{Z}_+} \subset X$.

a. *If* $x = \sum x_n$ *converges then* $\|x\| \le \sum \|x_n\| \le \infty$.

b. *If* $\sum \|x_n\| < \infty$ *then* $x = \sum x_n$ *converges unconditionally.*

PROOF: a. Given $\varepsilon > 0$, there exists by definition an $N > 0$ such that $\|x - \sum_1^N x_n\| \le \varepsilon$. Therefore,

$$\|x\| \le \left\|x - \sum_1^N x_n\right\| + \left\|\sum_1^N x_n\right\| \le \varepsilon + \sum_1^N \|x_n\| \le \varepsilon + \sum_1^\infty \|x_n\|.$$

Letting $\varepsilon \to 0$ gives the result.

b. Follows immediately from the triangle inequality and Lemma 1.4.2. ∎

Section 1.5. Operators.

a. Assume X and Y are Banach spaces, and that $S: X \to Y$.

S is **linear** if $S(ax + by) = aSx + bSy$ for $x, y \in X$ and $a, b \in \mathbf{C}$.

S is **injective** if $Sx \neq Sy$ whenever $x \neq y$.

The **range** of S is $\text{Range}(S) = \{Sx : x \in X\}$.

S is **surjective** if $\text{Range}(S) = Y$.

S is **bijective** if it is both injective and surjective.

The **norm** of S is $\|S\| = \sup\{\|Sx\|_Y : x \in X, \|x\|_X = 1\}$.

S is **bounded** if $\|S\| < \infty$. A linear operator is bounded if and only if it is **continuous**, i.e., if $x_n \to x$ implies $Sx_n \to Sx$.

The **adjoint** of S is the unique operator $S': Y' \to X'$ such that $\langle Sx, y' \rangle = \langle x, S'y' \rangle$ for all $x \in X$ and $y' \in Y'$, where X', Y' are the Banach space duals of X, Y, respectively.

S is **invertible**, or a **topological isomorphism**, if S is linear, bijective, continuous, and $S^{-1}: Y \to X$ is continuous.

S is an **isometry** if $\|Sx\|_Y = \|x\|_X$ for all $x \in X$.

S is **unitary** if it is a linear bijective isometry.

$L(X, Y) = \{S: X \to Y : S \text{ is linear and continuous}\}$.

$L(X) = L(X, X)$.

b. Assume H is a Hilbert space and $S, T: H \to H$.

S is **self-adjoint** if $\langle Sx, y \rangle = \langle x, Sy \rangle$ for $x, y \in H$.

S is **positive**, denoted $S \geq 0$, if $\langle Sx, x \rangle \geq 0$ for $x \in H$. All positive operators are self-adjoint.

$S \geq T$ if $S - T \geq 0$.

c. For functions f on $\mathbf{R}^d$ we define the following operators.

$$\begin{array}{llll} \text{Translation:} & T_a f(t) = f(t-a), & \text{for} & a \in \mathbf{R}^d, \\ \text{Modulation:} & E_a f(t) = e^{2\pi i a \cdot t} f(t), & \text{for} & a \in \mathbf{R}^d, \\ \text{Dilation:} & D_a f(t) = f(t/a), & \text{for} & a \in \mathbf{R}^d_*. \end{array}$$

We also use the symbol E_a to refer to the **exponential function** $E_a(t) = e^{2\pi i a \cdot t}$, where $a,\ t \in \mathbf{R}^d$.

Section 1.6. Topological groups.

Although some sections of this thesis are written in terms of abstract topological groups, in practice we use only the additive and multiplicative groups on $\mathbf{R}^d$.

a. The set $\mathbf{R}^d$ is a topological group under componentwise addition, with Haar measure equaling Lebesgue measure dt. The set $\mathbf{R}^d$ will always be assumed to have this operation and measure. The group translation operator is ordinary translation: $T_a f(t) = f(t-a)$. The measure of a set $E \subset \mathbf{R}^d$ with respect to Lebesgue measure is denoted by $|E|$.

The sets $\mathbf{R}^d_*$ and $\mathbf{R}^d_+$ are topological groups under componentwise multiplication, with Haar measure $dt/|\Pi(t)|$. The sets $\mathbf{R}^d_*$ and $\mathbf{R}^d_+$ will always be assumed to have this operation and measure. The group translation operator for these groups is dilation: $D_a f(t) = f(t/a)$. The measure of a set $E \subset \mathbf{R}^d_*$ with respect to this Haar measure is denoted by $|E|$.

Integrals with unspecified limits are assumed to be over $\mathbf{R}^d$ with respect to Lebesgue measure dt.

b. We point out the following facts about the multiplicative group $\mathbf{R}^d_*$.

Compact sets in $\mathbf{R}^d_*$ are bounded away from both ∞ and the coordinate axes. A connected compact set is entirely contained in one quadrant of $\mathbf{R}^d_*$.

Haar measure $dt/|\Pi(t)|$ is dilation invariant.

Given $E \subset \mathbf{R}^d_*$, $|E|_* = 0$ if and only if $|E| = 0$. Therefore the term *almost everywhere* (a.e.) has the same meaning in the additive and multiplicative groups. To see this, assume $E \subset \mathbf{R}_*^d$ with $|E| = 0$ is compact and contained in one quadrant of $\mathbf{R}^d$, say $\mathbf{R}_+^d$. Then $E \subset [a,b] \subset \mathbf{R}_+^d$, so

$$|E|_* \;=\; \int_E \frac{dt}{|\Pi(t)|} \;\leq\; \frac{1}{|\Pi(a)|} \int_E dt \;=\; 0.$$

The general case follows since $\mathbf{R}_*^d$ is σ-finite, and the converse is similar.

Section 1.7. Function spaces.

Let G be a σ-finite, locally compact group with left Haar measure dx. A positive function w on G, i.e., $w: G \to \mathbf{R}_+$, is a **weight** on G. In this thesis, all functions defined on topological groups or measure spaces are assumed to be measurable.

a. Given $1 \le p \le \infty$ and a weight w on G, we define the **weighted L^p-space**

$$L^p_w(G) = \{f: G \to \mathbf{C} : \|f\|_{L^p_w(G)} < \infty\},$$

where

$$\|f\|_{L^p_w(G)} = \begin{cases} \left(\int_G |f(x)|^p \, w(x) \, dx\right)^{1/p}, & \text{if } 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{x \in G} |f(x)| \, w(x), & \text{if } p = \infty. \end{cases}$$

If $w \equiv 1$ then we write $L^p(G) = L^p_w(G)$. When G is understood we write L^p_w or L^p. We let $\|\cdot\|_p = \|\cdot\|_{L^p}$. When G is countable and dx is counting measure we write $\ell^p_w(G)$ instead of $L^p_w(G)$.

$L^p_w(G)$ is a Banach space for $1 \le p \le \infty$. The **dual index** to p is $p' = p/(p-1)$, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. The **dual weight** to w is $w' = w^{1-p'}$. We have $(L^p_w)' = L^{p'}_{w'}$ for $1 \le p < \infty$, where the prime denotes the Banach space dual and the duality is defined by

$$\langle f, g \rangle = \int_G f(x) \, \overline{g(x)} \, dx$$

for $f \in L^p_w(G)$, $g \in L^{p'}_{w'}(G)$. Note that $L^2(G)$ is a Hilbert space under this inner product.

b. We define the following additional spaces of functions on G.

$$L^p_{\text{loc}}(G) = \{f: G \to \mathbf{C} : f \cdot \chi_K \in L^p(G), \text{ all compact } K \subset G\},$$

$$C(G) = \{f: G \to \mathbf{C} : f \text{ is continuous}\},$$

$$C_c(G) = \{f \in C(G) : \text{supp}(f) \text{ is compact}\},$$

$$C_b(G) = \{f \in C(G) : f \text{ is bounded}\},$$

$$C_0(G) = \{f \in C(G) : f \text{ vanishes at infinity}\},$$

where ***vanishing at infinity*** means that for each $\varepsilon > 0$ there exists a compact $K \subset G$ such that $|f(x)| < \varepsilon$ for all $x \notin K$.

$(C_b(G), \|\cdot\|_\infty)$ and $(C_0(G), \|\cdot\|_\infty)$ are Banach spaces; $C_c(G)$ is dense in $(L^p(G), \|\cdot\|_p)$ for $1 \le p < \infty$, and in $(C_0(G), \|\cdot\|_\infty)$.

c. A **Banach function space**, or BF-space, on G is a Banach space B continuously embedded into $L^1_{\text{loc}}(G)$, i.e., for each compact $K \subset G$ there is a $C_K > 0$ such that $\|f \cdot \chi_K\|_{L^1(G)} \le C_K \|f\|_B$ for each $f \in B$.

A BF-space B is **solid** if given f, $g \in B$ with $|f| \le |g|$ a.e. we have $\|f\|_B \le \|g\|_B$. The spaces $L^p_w(G)$ and $C_0(G)$ are solid. $L^p_w(G)$ possesses the stronger property that if $f \in L^1_{\text{loc}}(G)$ and $g \in B = L^p_w(G)$ with $|f| \le |g|$ a.e. then $f \in B$ and $\|f\|_B \le \|g\|_B$. $C_0(G)$ need not satisfy this, e.g., take $G = \mathbf{R}^d$.

d. Given $a \in G$, the **left** and **right group translation operators** are

$$L_a f(x) = f(a^{-1}x) \qquad \text{and} \qquad R_a f(x) = f(xa^{-1}).$$

e. Let B be a Banach function space on G.

B is **closed under left translations** if $L_a(B) \subset B$ for each $a \in G$.

B is **left translation invariant** if it is closed under left translations and $L_a: B \to B$ is continuous for each $a \in G$. If each L_a is an isometry then B is **left translation isometric.**

Translation is **strongly continuous** in B if $\lim_{a \to b} \|L_a f - L_b f\|_B = 0$ for all $f \in B$ and $b \in G$, where the limit is taken in the group topology sense, i.e., for each $\varepsilon > 0$ there is a neighborhood U of b such that $\|L_a f - L_b f\|_B < \varepsilon$ for $a \in U$, cf., Section 1.9a.

B is **left homogeneous** if it is left translation isometric and translation is strongly continuous in B.

B is a **left Segal algebra** if it is left homogeneous and is dense in $L^1(G)$ in the L^1-norm.

Similar definitions are made with *right* in place of *left.* If the term left or right is omitted, it is assumed that both hold, for example, if G is abelian.

f. The following inclusions hold for ℓ^p. If $0 < p \leq q \leq \infty$ then $\ell^p \subset \ell^q$, with $\|\cdot\|_{\ell^p} \geq \|\cdot\|_{\ell^q}$.

For $0 < p < 1$, ℓ^p is not a Banach space, but is a complete metric space with distance defined by $d(f,g) = \|f - g\|_p^p$. The triangle inequality for this distance is equivalent to the estimate

$$\|f+g\|_p^p \leq \|f\|_p^p + \|g\|_p^p.$$

g. If $E \subset G$ has finite measure and $1 \leq p \leq q < \infty$ then

$$\left(\frac{1}{|E|}\int_E |f(t)|^p\,dt\right)^{1/p} \leq \left(\frac{1}{|E|}\int_E |f(t)|^q\,dt\right)^{1/q}.$$

This also holds for $1 \leq p < q = \infty$ if the right-hand side is replaced by $\operatorname{ess\,sup}_{t\in E}|f(t)|$. Equivalently,

$$\|f\cdot\chi_E\|_{L^p} \leq |E|^{\frac{1}{p}-\frac{1}{q}}\,\|f\cdot\chi_E\|_{L^q}$$

for all $1 \leq p \leq q \leq \infty$, with the interpretation $1/\infty = 0$.

Section 1.8. The Fourier transform.

a. The **Fourier transform** of a function $f \in L^1(\mathbf{R}^d)$ is

$$\hat{f}(\gamma) = \int f(t)\, e^{-2\pi i \gamma \cdot t}\, dt,$$

defined for $\gamma \in \hat{\mathbf{R}}^d$. The **inverse Fourier transform** is

$$\check{f}(\gamma) = \hat{f}(-\gamma) = \int f(t)\, e^{2\pi i \gamma \cdot t}\, dt.$$

The Fourier transform of $f \in L^2(\mathbf{R}^d)$ is $\hat{f} = \lim_{n\to\infty} (f \cdot \chi_{R_n})^\wedge$, where the limit is in the L^2-norm.

b. The **Plancherel formula** is $\|f\|_{L^2(\mathbf{R}^d)} = \|\hat{f}\|_{L^2(\hat{\mathbf{R}}^d)} = \|\check{f}\|_{L^2(\hat{\mathbf{R}}^d)}$.

The **Parseval formula** is $\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle = \langle \check{f}, \check{g}\rangle$, where $\langle \cdot, \cdot \rangle$ is the $L^2(\mathbf{R}^d)$ inner product.

The **inversion formula** is $f = f^{\wedge\vee} = f^{\vee\wedge}$ for $f \in L^2(\mathbf{R}^d)$.

If $f \in \mathcal{S}(\mathbf{R}^d)$ then we have the **Poisson summation formula**

$$\sum_{k\in\mathbf{Z}^d} f(k) = \sum_{k\in\mathbf{Z}^d} \hat{f}(k).$$

c. We have the formulas

$$(T_a f)^\wedge = E_{-a}\hat{f} \qquad \text{and} \qquad (E_a f)^\wedge = T_a \hat{f}.$$

Section 1.9. Group representations.

Let G be a locally compact group and X a Banach space.

a. A **representation** of G on X is a homomorphism of G into $L(X)$, i.e., a mapping $U: G \rightarrow L(X)$ such that

$$U_{xy} = U_x U_y$$

for $x, y \in G$.

U is **unitary** if each $U_x: X \rightarrow X$ is a unitary operator.

U is **strongly continuous** if $\lim_{x \to y} U_x = U_y$, where the limit is taken in the strong operator topology. That is,

$$\lim_{x \to y} \|U_x f - U_y f\| = 0$$

for all $f \in X$, where this limit is in the group topology.

b. If $X = H$, a Hilbert space, then we make the following additional definitions.

A element $g \in H$ is **admissible** if $\int_G |\langle U_x g, g\rangle|^2 \, dx < \infty$.

g is **cyclic** if $\text{span}\{U_x g\}_{x \in G}$ is dense in H.

U is **square-integrable** if there exists an admissible $g \in H \backslash \{0\}$.

U is **irreducible** if every $g \in H \backslash \{0\}$ is cyclic.

c. The following result is well-known, e.g., [**GMP**].

PROPOSITION 1.9.1. *If U is a square-integrable and irreducible representation of a locally compact group G on a Hilbert space H then there exists a* *unique self-adjoint positive operator* $C\colon \text{Domain}(C) \to H$ *such that*

a. $\text{Domain}(C) = \{g \in H : g$ *is admissible*$\}$,

b. *given any* $f_1, f_2 \in H$ *and any admissible* $g_1, g_2 \in H$,

$$\int_G \langle f_1, U_x g_1\rangle \langle U_x g_2, f_2\rangle \, dx \;=\; \langle f_1, f_2\rangle \langle C g_1, C g_2\rangle.$$

Moreover, if G is unimodular then C is a multiple of the identity.

Setting $f_1 = f_2 = g_1 = g_2 = g$ in Proposition 1.9.1, we obtain

$$\int_G |\langle g, U_x g\rangle|^2 \, dx \;=\; \|g\|^2 \, \|Cg\|^2.$$

Setting $f_1 = f_2 = f$ and $g_1 = g_2 = g$, we obtain

$$\int_G |\langle f, U_x g\rangle|^2 \, dx \;=\; \|f\|^2 \, \|Cg\|^2 \;=\; \frac{\|f\|^2}{\|g\|^2} \int_G |\langle g, U_x g\rangle|^2 \, dx.$$

PART II

WIENER AMALGAM SPACES

CHAPTER 2

WIENER AMALGAM SPACES

In this chapter we discuss the far-reaching generalization of amalgam spaces derived by Feichtinger, e.g., [**F2**; **F8**]. Given Banach function spaces B, C on a locally compact group G, satisfying certain conditions, he defined spaces $W(B,C)$ of distributions which are, roughly speaking, locally in B in globally in C. The space $W(L^p(\mathbf{R}), L^q(\mathbf{R}))$ coincides with the standard amalgam space defined in (0.1.1). While each $W(B,C)$ can be described in terms of a discrete-type norm like (0.1.1), the fundamental norm describing the local/global properties is a continuous-type norm (cf., Sections 2.2 and 2.4). These equivalent continuous and discrete norms provide flexibility in using the $W(B,C)$ in applications.

Feichtinger calls the spaces $W(B,C)$ *Wiener-type spaces*; following a suggestion of J. Benedetto, and in order to promote the link between Feichtinger's generalization and amalgams occuring previously in the literature, we call them *Wiener amalgam spaces*.

Wiener amalgam spaces lie at the heart of many of the main results of this thesis, especially those in Part III (Generalized Harmonic Analysis). In those chapters, we use the Wiener amalgam spaces $W(L^p(\mathbf{R}^d_*), L^q(\mathbf{R}^d_*))$, on the *multiplicative* d-dimensional group $\mathbf{R}^d_*$. It is the discrete-type norms on this space which provide the machinery for our major results. Amalgams play a smaller, but still important role in Part IV (Wavelet Theory). There we use the standard amalgam spaces $W(L^p(\mathbf{R}^d), L^q(\mathbf{R}^d))$ on the *additive* group $\mathbf{R}^d$.

The purpose of this chapter is to review fundamental facts about the Wiener amalgams $W(B, C)$. As noted above, the main results in this thesis use only the cases $B = L^p(G)$, $C = L^q(G)$; some minor results use $B = L^p_v(G)$ or $C = C_0(G)$. We therefore present the Feichtinger theory only for the spaces $W(L^p_v(G), L^q_w(G))$. This results in a considerable technical simplification of the general $W(B, C)$ theory. This chapter can therefore be regarded as an elementary introduction to the general theory presented in [**F8**].

The results in this chapter are known; we have collected results and proofs from many sources, including [**F1–F8; FG; Ho; FS; Wa**] and others. The credit for this chapter therefore belongs primarily to Feichtinger and secondarily to others; we have synthesized their results into a single expository chapter.

We now outline this chapter by sections.

In Section 2.1 we characterize those weights w for which the weighted L^p space $L^p_w(G)$ is translation invariant.

Section 2.2 contains the basic definitions of the Wiener amalgam spaces in terms of continuous-type norms, and proofs of fundamental properties such as completeness and translation invariance.

In Section 2.3 we determine various inclusion relations between the spaces $W(L^p_v(G), L^q_w(G))$.

In Section 2.4 we derive equivalent discrete norms for Wiener amalgam spaces. These norms are the ones which will be used in the proofs of the major results in Parts III and IV.

Finally, in Section 2.5 we prove duality relationships between the amalgam spaces.

We assume throughout this chapter that G is a σ-finite, locally compact group. Since in later chapters we use only $G = \mathbf{R}^d$ or $G = \mathbf{R}^d_*$, we assume for simplicity that G is unimodular, i.e., left and right Haar measure coincide. We denote this Haar measure by dx, the identity element by e, the left group translation operator by $L_a f(x) = f(a^{-1}x)$, and the right group translation operator by $R_a f(x) = f(xa^{-1})$, cf., Section 1.7. The measure of a set $E \subset G$ with respect to Haar measure is denoted by $|E|$. A positive function $w: G \to \mathbf{R}_+$ is called a **weight**.

Section 2.1. Moderate weights.

In this section we characterize the class of weights w for which the Banach function space $L^p_w(G)$ is translation invariant. The results in this section are known. In particular, the proofs given here are extensions to locally compact groups of Walnut's proofs on the additive group $\mathbf{R}^d$ [**Wa**]. See also the original results in [**Ed; Gau**].

DEFINITION 2.1.1. a. A weight $m: G \to \mathbf{R}_+$ is **submultiplicative** if $m(e) = 1$ and $m(xy) \le m(x)\,m(y)$ for $x, y \in G$.

b. A weight $w: G \to \mathbf{R}_+$ is **right moderate** if there exists a submultiplicative function m such that $w(xy) \le w(x)\,m(y)$ for $x, y \in G$.

Corresponding definitions and theorems for *left moderate* weights are assumed throughout this section. If the term left or right is omitted, it is assumed that both hold.

PROPOSITION 2.1.2. *If w is right moderate with associated submultiplicative function m then $w(e)/m(x^{-1}) \le w(x) \le w(e)\,m(x)$ for all $x \in G$.*

PROOF: We compute $w(x) = w(ex) \le w(e)\,m(x)$ and $w(e) = w(xx^{-1}) \le w(x)\,m(x^{-1})$. ∎

PROPOSITION 2.1.3. *Given w right moderate with associated submultiplicative function m, and given $r \in \mathbf{R}$.*

a. *If $r \ge 0$ then w^r is right moderate with associated submultiplicative function m^r.*

b. *If* $r \le 0$ *then* w^r *is right moderate with associated submultiplicative function* $\tilde{m}^r$*, where* $\tilde{m}(x) = m(x^{-1})$.

PROOF: Part a is clear, and therefore for part b we need only consider $r = -1$. That $\tilde{m}$ is submultiplicative is also clear, and that w^{-1} is right moderate follows immediately from the computation

$$w(xy)\,m(y^{-1}) \ge w(xyy^{-1}) = w(x). \quad \blacksquare$$

THEOREM 2.1.4. *Submultiplicative functions are locally bounded.*

PROOF: Assume m is submultiplicative. We claim first that if m is bounded on any open neighborhood of the identity then it is bounded on every compact set. To see this, assume m is bounded on some open U containing e, and let K be a compact set. Then $K \subset \bigcup_1^N x_k U$ for some $x_1, \dots, x_N \in G$. Let $R = \max\{m(x_k)\}$. If $x \in K$ then $x = x_k y$ for some k and some $y \in U$, so $m(x) = m(x_k y) \le m(x_k)\,m(y) \le R\,\|m \cdot \chi_U\|_\infty$. Therefore m is bounded on K, as claimed.

Now suppose that m was unbounded on every open neighborhood of e. Let U be an open neighborhood of e with compact closure, such that $U = U^{-1}$. Since

$$|aU \,\Delta\, U| \;=\; \int_G |\chi_{aU} - \chi_U| \;=\; \|L_a\chi_U - \chi_U\|_{L^1(G)}$$

and left translation is strongly continuous in $L^1(G)$, there exists a neighborhood V of e such that

$$|aU \,\Delta\, U| \;<\; \tfrac{1}{2}|U| \tag{2.1.1}$$

for $a \in V$.

Now, for each $N \in \mathbf{Z}_+$ there exists by assumption an $x_N \in V$ such that $m(x_N) \geq N^2$. Therefore, given $x \in G$ we have

$$N^2 \leq m(x_N) = m(x_N x^{-1} x) \leq m(x_N x^{-1}) m(x),$$

so either $m(x_N x^{-1}) \geq N$ or $m(x) \geq N$. Defining

$$A_N = \{x \in U : m(x) \geq N\},$$

we therefore have

$$A_N \supset x_N (U \backslash A_N)^{-1} \cap U, \tag{2.1.2}$$

since if $y \in U$ and $y = x_N x^{-1}$ for some $x \in U \backslash A_N$ then $m(x) < N$, so $m(y) = m(x_N x^{-1}) \geq N$, whence $y \in A_N$. Since

$$X \backslash Y \supset X \backslash Z \quad \Rightarrow \quad X \cap Z \supset X \backslash Y$$

and

$$x_N U^{-1} \backslash U \supset x_N (U \backslash A_N)^{-1} \backslash U,$$

it follows that

$$\begin{aligned} &x_N (U \backslash A_N)^{-1} \cap U \\ &\qquad \supset x_N (U \backslash A_N)^{-1} \backslash (x_N U^{-1} \backslash U) \\ &\qquad = x_N (U \backslash A_N)^{-1} \backslash (x_N U \backslash U) \\ &\qquad \supset x_N (U \backslash A_N)^{-1} \backslash (x_N U \, \Delta \, U). \end{aligned} \tag{2.1.3}$$

Since $x_N \in V$ we therefore have from (2.1.1), (2.1.2), and (2.1.3) that

$$\begin{aligned} |A_N| &\geq |x_N(U\backslash A_N)^{-1} \cap U| \\ &\geq |x_N(U\backslash A_N)^{-1} \setminus (x_N U \,\Delta\, U)| \\ &\geq |x_N(U\backslash A_N)^{-1}| - |x_N U \,\Delta\, U| \\ &= |U\backslash A_N| - |x_N U \,\Delta\, U| \\ &\geq |U| - |A_N| - \tfrac{1}{2}|U|, \end{aligned}$$

whence

$$|A_N| \geq \tfrac{1}{4}|U|.$$

Since the sets A_N are nested in U, this implies $|\cap A_N| \geq \frac{1}{4}|U| > 0$. However, m is finite-valued, so $\cap A_N = \emptyset$, a contradiction. ▮

COROLLARY 2.1.5. *Every right moderate function is locally bounded.*

PROOF: Assume w was right moderate but unbounded on some compact set K, and fix any $x \in G$. Let m be the submultiplicative function associated with w. By Theorem 2.1.4, m is locally bounded, so $M = \|m \cdot \chi_{x^{-1}K}\|_\infty < \infty$. Now, given $R > 0$ there exists $y \in K$ such that $w(y) > RM$. Therefore,

$$RM \;<\; w(y) \;\leq\; w(x)\,m(x^{-1}y) \;\leq\; w(x)\,M.$$

As x and R are arbitrary, this is a contradiction. ▮

THEOREM 2.1.6. *Given a positive $w \in L^1_{\mathrm{loc}}(G)$, the following statements are equivalent.*

a. *w is right moderate.*

b. *$L^p_w(G)$ is closed under right translations for some (and therefore every) $1 \le p \le \infty$.*

c. *$L^p_w(G)$ is right translation invariant for some (and therefore every) $1 \le p \le \infty$.*

d. *For each compact $K \subset G$,*

$$A(K) = \sup_{x \in G,\, y \in K} \frac{w(xy)}{w(x)} < \infty.$$

e. *Given any compact set $K \subset G$ there exists a constant $B = B(K)$ such that*

$$\sup_{xK} w \le B \inf_{xK} w$$

for every $x \in G$.

f. *Given any compact set $K \subset G$ there exist constants $C = C(K)$, $D = D(K)$ such that*

$$C\, w(y) \le \int_{xK} w(t)\, dt \le D\, w(y)$$

for all $y \in xK$.

g. *Given any compact set $K \subset G$ and given $k \in K$ there exist constants $E = E(K, k)$, $F = F(K, k)$ such that*

$$E\, w(xk) \le \int_{xK} w(t)\, dt \le F\, w(xk)$$

for all $x \in G$.

PROOF: a $\Rightarrow$ c. Assume w is right moderate with associated submultiplicative function m. Given $1 \le p < \infty$, $f \in L^p_w$, and $a \in G$, we then have

$$\begin{aligned}\|R_a f\|^p_{L^p_w} &= \int_G |f(xa^{-1})|^p\, w(x)\, dx \\ &= \int_G |f(x)|^p\, w(xa)\, dx \\ &\le m(a) \int_G |f(x)|^p\, w(x)\, dx \\ &\le m(a)\, \|f\|^p_{L^p_w}.\end{aligned}$$

Thus R_a maps L^p_w into itself, and does so continuously, with $\|R_a\| \le m(a)^{1/p}$. The case $p = \infty$ is similar, with the result $\|R_a\| \le m(a)$.

c $\Rightarrow$ a. Assume that c holds, and fix $1 \le p < \infty$. For $a \in G$ define $m(a) = \|R_a\|^p$. Note that $m(e) = \|I\|^p = 1$ and

$$m(ab) = \|R_{ab}\|^p = \|R_b\, R_a\|^p \le \|R_b\|^p\, \|R_a\|^p = m(b)\, m(a),$$

so m is submultiplicative.

We show now that w is right moderate with m as associated submultiplicative function. Fix any $a \in G$ and $f \in L^p_w$. Then

$$\begin{aligned}\int_G |f(x)|^p\, w(xa)\, dx &= \int_G |f(xa^{-1})|^p\, w(x)\, dx \\ &= \|R_a f\|^p_{L^p_w} \\ &\le m(a)\, \|f\|^p_{L^p_w} \\ &= m(a) \int_G |f(x)|^p\, w(x)\, dx.\end{aligned}$$

Since this is true for every $f \in L^p_w$, we have $w(xa) \le w(x)\,m(a)$ for a.e. $x \in G$, so w is moderate. The case $p = \infty$ is similar (set $m(a) = \|R_a\|$).

b $\Rightarrow$ c. Assume L^p_w is closed under right translations for some $1 \le p \le \infty$. Given $a \in G$, assume $f_n \in L^p_w$ are such that $f_n \to f \in L^p_w$ and $R_a f_n \to g \in L^p_w$ as $n \to \infty$. Then we can find a subsequence $\{f_{n_k}\}$ where both convergences are pointwise a.e. Then $R_a f_{n_k} \to R_a f$, g pointwise a.e., whence $R_a f = g$ a.e. R_a is therefore continuous by the closed graph theorem.

a $\Rightarrow$ d. Assume w is right moderate with associated submultiplicative function m. By Theorem 2.1.4 we have $m \in L^\infty_{\mathrm{loc}}$, so if $y \in K$, a compact set in G, and $x \in G$ then

$$w(xy) \;\le\; w(x)\,m(y) \;\le\; w(x)\,\|m \cdot \chi_K\|_\infty.$$

Thus $A(K) \le \|m \cdot \chi_K\|_\infty < \infty$.

d $\Rightarrow$ c. Assume d holds, and let $K \subset G$ be compact. Given $a \in K$ and $f \in L^p_w(G)$, where $1 \le p < \infty$, we have

$$\begin{aligned}
\|R_a f\|^p_{L^p_w} &= \int_G |f(xa^{-1})|^p\, w(x)\,dx \\
&= \int_G |f(x)|^p\, \frac{w(xa)}{w(x)}\, w(x)\,dx \\
&\le A(K) \int_G |f(x)|^p\, w(x)\,dx \\
&= A(K)\,\|f\|^p_{L^p_w}.
\end{aligned}$$

Therefore R_a maps L^p_w into itself, and does so continuously, with $\|R_a\| \le A(K)^{1/p}$. The case $p = \infty$ is similar.

d $\Rightarrow$ e. Assume d holds, and let $K \subset G$ be a compact set and x any element of G. Set $L = K \cup K^{-1}$, and note that L is both compact and symmetric (i.e., $L^{-1} = L$). If $y \in xK$ then $x^{-1}y \in K \subset L$, so $y^{-1}x \in L$. Hence,

$$w(y) \;=\; \frac{w(xx^{-1}y)}{w(x)}\,w(x) \;\le\; A(L)\,w(x)$$

and

$$w(x) \;=\; \frac{w(yy^{-1}x)}{w(y)}\,w(y) \;\le\; A(L)\,w(y),$$

so

$$\sup_{xK} w \;\le\; A(L)\,w(x) \;\le\; A(L)^2 \inf_{xK} w.$$

e $\Rightarrow$ d. Assume e holds, and let $K \subset G$ be compact. Let $L \supset K$ be compact, with $e \in L$. Given $x \in G$ and $y \in K$ we then have

$$w(xy) \;\le\; \sup_{xK} w \;\le\; \sup_{xL} w \;\le\; B(L) \inf_{xL} w \;\le\; B(L)\,w(xe) \;=\; B(L)\,w(x)$$

since $e \in L$. Therefore, $A(K) \le B(L) < \infty$.

e $\Rightarrow$ f. Assume e holds, let $K \subset G$ be compact, and let x be any element of G. Then

$$\frac{|K|}{B(K)} \sup_{xK} w \;\le\; |xK| \inf_{xK} w \;\le\; \int_{xK} w \;\le\; |xK| \sup_{xK} w \;\le\; |K|\,B(K) \inf_{xK} w.$$

g $\Rightarrow$ e. Assume g holds, and let $K \subset G$ be compact and x any element of G. Then $L = K \cup K^{-1} \cup \{e\}$, $L' = LL$, and $L'' = L'L'$ are all compact symmetric sets containing e. The symmetry implies that $xL \subset yL' \subset xL''$

and $yL \subset xL' \subset yL''$ for $y \in xL$. Therefore,

$$\begin{aligned}\frac{1}{w(y)}\int_{xL'} w &\le \frac{1}{w(y)}\int_{yL''} w \\ &\le F(L'',e) \\ &= \frac{F(L'',e)}{E(L',e)}\,E(L',e) \\ &\le \frac{F(L'',e)}{E(L',e)}\,\frac{1}{w(x)}\int_{xL'} w,\end{aligned}$$

so $w(x) \le \frac{F(L'',e)}{E(L',e)}\,w(y)$ for $y \in xL$. Similarly,

$$\begin{aligned}\frac{1}{w(y)}\int_{xL'} w &\ge \frac{1}{w(y)}\int_{yL} w \\ &\ge \frac{E(L,e)}{F(L',e)}\,F(L',e) \\ &\ge \frac{E(L,e)}{F(L',e)}\,\frac{1}{w(x)}\int_{xL'} w.\end{aligned}$$

Thus $w(x) \ge \frac{E(L,e)}{F(L',e)}\,w(y)$ for $y \in xL$, so

$$\begin{aligned}\sup_{xK} w &\le \sup_{xL} w \\ &\le \frac{F(L',e)}{E(L,e)}\,w(x) \\ &\le \frac{F(L',e)\,F(L'',e)}{E(L,e)\,E(L',e)}\,\inf_{xL} w \\ &\le \frac{F(L',e)\,F(L'',e)}{E(L,e)\,E(L',e)}\,\inf_{xK} w. \ \blacksquare\end{aligned}$$

PROPOSITION 2.1.7. *Every right moderate weight is equivalent to a continuous right moderate weight in the sense that if w is right moderate then there exists a continuous right moderate v and constants A, $B > 0$ such that*

$$A\,v(x) \le w(x) \le B\,v(x)$$

for all $x \in G$.

Proof: Assume w is right moderate, and let $k \in C_c(G)$ be any function such that $k \geq 0$ and $\int_G k = 1$. Let $K \supset \operatorname{supp}(k)$ be a compact symmetric neighborhood of e. Since w is locally bounded, we can define

$$v(x) \;=\; (w * k)(x) \;=\; \int_G w(t)\, k(xt^{-1})\, dt \;=\; \int_G w(tx)\, k(t^{-1})\, dt.$$

Clearly v is positive and continuous, and

$$v(xy) \;=\; \int_G w(txy)\, k(t^{-1})\, dt \;\leq\; m(y) \int_G w(tx)\, k(t^{-1})\, dt \;=\; v(x)\, m(y),$$

so v is right moderate. Also,

$$\begin{aligned} v(x) &= \int_G w(tx)\, k(t^{-1})\, dt \\ &\leq \sup_{xK} w \cdot \int_K k(t^{-1})\, dt \\ &\leq B(K) \inf_{xK} w \\ &\leq B(K)\, w(x), \end{aligned}$$

where $B(K)$ is as in Theorem 2.1.6e. Similarly, $v(x) \geq B(K)^{-1}\, w(x)$, so we are done. ∎

Section 2.2. Definition and basic properties.

In this section we define and derive basic properties of the spaces $W(B,C)$. Our proofs will hold when B, C are weighted L^p spaces $L^p_w(G)$, where $1 \leq p \leq \infty$ and $w: G \to \mathbf{R}_+$. For these spaces, integrability (local and global) is the only defining factor. This simplifies the proofs from the general abstract case; we attempt to indicate what technical modifications are necessary to cover the general case. Note that $L^p_w(G)$ is solid in the sense of Section 1.7c, and is right translation invariant if and only if w is right moderate (Theorem 2.1.6c). The primary space we are interested in other than the weighted L^p spaces is $C_0(G)$, the continuous functions on G vanishing at infinity.

With the weighted L^p spaces as a model, we define

$$B_{\text{loc}} = \{f: G \to \mathbf{C} : f \cdot \chi_K \in B \text{ for every compact } K \subset G\}.$$

Remark 2.2.1. This definition is not the proper one to make if B has properties other than integrability, e.g., smoothness. For $B = C_0(G)$ it would be appropriate to take

$$B_{\text{loc}} = \{f \in M(G) = C_c(G)' : f\varphi \in B \text{ for every } \varphi \in C_c(G)\},$$

with corresponding technical difficulties added to the proofs. For the general case we would assume that there is a homogeneous Banach space A such that:

a. A is continuously embedded into $(C_b(G), \|\cdot\|_\infty)$.

b. A is a regular Banach algebra under pointwise multiplication.

c. A is closed under complex conjugation.

d. B is continuously contained in A_c', where

$$A_c = \{f \in A : \text{supp}(f) \text{ is compact}\}.$$

e. A is a Banach module over B with respect to pointwise multiplication, i.e., if $f \in A$ and $g \in B$ then $fg \in B$ with $\|fg\|_B \leq \|f\|_A \|g\|_B$.

Then we would define

$$B_{\text{loc}} = \{f \in A_c' : f\varphi \in B \text{ for } \varphi \in A_c\}.$$

This can be shown to be independent of the choice of A.

DEFINITION 2.2.2. Fix a compact set $Q \subset G$ with nonempty interior. For $f \in B_{\text{loc}}$ and $x \in G$ define

$$F_f(x) = F_f^Q(x) = \|f \cdot \chi_{xQ}\|_B.$$

The **Wiener amalgam space** $W(B,C)$ is

$$W(B,C) = \{f \in B_{\text{loc}} : F_f \in C\},$$

with norm

$$\|f\|_{W(B,C)} = \|F_f\|_C = \big\| \, \|f \cdot \chi_{xQ}\|_B \, \big\|_C.$$

We refer to B as the **local component** and C as the **global component** of $W(B,C)$.

REMARK 2.2.3. For the general case, we would define

$$F_f(x) = \inf\{\|g\|_B : g \in B \text{ and } g\varphi = f\varphi \text{ for } \varphi \in A_c \text{ with } \text{supp}(\varphi) \subset xQ\},$$

and again set $\|f\|_{W(B,C)} = \|F_f\|_C$.

EXAMPLE 2.2.4. We compare $W(L^\infty, L^1)$ to $W(C_0, L^1)$.

Since both L^∞ and C_0 are equipped with the L^∞-norm, $W(L^\infty, L^1)$ and $W(C_0, L^1)$ have the same norm. However, the definitions of $(L^\infty)_{\text{loc}}$ and $(C_0)_{\text{loc}}$ differ, so they are distinct spaces. In fact,

$$W(C_0, L^1) = \{f \in W(L^\infty, L^1) : f \text{ is continuous}\}.$$

THEOREM 2.2.5. a. *$W(B,C)$ is a Banach space.*

b. *If C is solid and right translation invariant then $W(B,C)$ is independent of the choice of Q, i.e., different choices of Q define the same space with equivalent norms.*

PROOF: a. That $\|\cdot\|_{W(B,C)}$ is a norm is clear, so we prove that $W(B,C)$ is complete in this norm. Assume $\{f_n\}_{n\in\mathbf{Z}_+} \subset W(B,C)$ with $\sum \|f_n\|_{W(B,C)} < \infty$. By Lemma 1.4.1 it suffices to prove that $\sum f_n$ converges to an element of $W(B,C)$. Now, $\sum \|f_n\|_{W(B,C)} = \sum \|F_{f_n}\|_C$ and C is complete, so $\sum F_{f_n}$ must converge to an element of C. Therefore,

$$\sum F_{f_n}(x) = \sum \|f_n \cdot \chi_{xQ}\|_B < \infty$$

for a.e. $x \in G$. Since B is also complete, $\sum f_n \cdot \chi_{xQ}$ must converge to an element $g_x \in B$. Clearly $g_x = g_y$ a.e. on $xQ \cap yQ$, so we can define a function g a.e. by $g(t) = g_x(t)$ for $t \in xQ$, i.e., $g \cdot \chi_{xQ} = g_x$. Applying Lemma 1.4.3 twice, we have

$$\begin{aligned} \|g\|_{W(B,C)} &= \big\| \|g \cdot \chi_{xQ}\|_B \big\|_C \\ &= \Big\| \big\|\sum f_n \cdot \chi_{xQ}\big\|_B \Big\|_C \end{aligned}$$

$$\leq \sum \| \, \|f_n \cdot \chi_{xQ}\|_B \, \|_C$$

$$= \sum \|f_n\|_{W(B,C)}$$

$$< \infty,$$

so $g \in W(B,C)$. A similar computation shows

$$\Big\|g - \sum_1^N f_n\Big\|_{W(B,C)} \to 0 \qquad \text{as } N \to \infty,$$

so $\sum f_n$ converges in $W(B,C)$ to g. Therefore $W(B,C)$ is complete.

b. Assume that C is solid and right translation invariant, and let Q_1, Q_2 be two compact subsets of G with nonempty interiors. Then we can find points $x_1, \dots, x_N \in G$ such that

$$Q_2 \subset \bigcup_1^N x_k Q_1.$$

For $x \in G$ we therefore have

$$\begin{aligned}
F_f^{Q_2}(x) &= \|f \cdot \chi_{xQ_2}\|_B \\
&\leq \|f \cdot \chi_{\cup x x_k Q_1}\|_B \\
&\leq \Big\|f \cdot \sum_1^N \chi_{x x_k Q_1}\Big\|_B \\
&\leq \sum_1^N \|f \cdot \chi_{x x_k Q_1}\|_B \\
&= \sum_1^N F_f^{Q_1}(x x_k) \\
&= \sum_1^N (R_{x_k^{-1}} F_f^{Q_1})(x).
\end{aligned}$$

Since both $F_f^{Q_1}$ and $\sum R_{x_k^{-1}} F_f^{Q_1}$ are elements of C and C is solid, we have

$$\begin{aligned}
\|F_f^{Q_2}\|_C &\leq \Big\| \sum_1^N R_{x_k^{-1}} F_f^{Q_1} \Big\|_C \\
&\leq \sum_1^N \|R_{x_k^{-1}} F_f^{Q_1}\|_C \\
&\leq \sum_1^N M \,\|F_f^{Q_1}\|_C \\
&= MN \,\|F_f^{Q_1}\|_C,
\end{aligned}$$

where $M = \max\{\|R_{x_k^{-1}}\|_C\} < \infty$ by the translation invariance of C. A symmetric argument gives the reverse inequality and completes the proof. ∎

REMARK 2.2.6. General amalgams $W(B,C)$ can be shown to be also independent of the choice of Banach algebra A (cf., Remark 2.2.1).

We assume from now onwards that C is solid and right translation invariant.

LEMMA 2.2.7. $L_a \chi_E = \chi_{aE}$, $R_a \chi_E = \chi_{Ea}$.

PROOF:

$$L_a \chi_E(x) = 1 \Leftrightarrow \chi_E(a^{-1}x) = 1 \Leftrightarrow a^{-1}x \in E \Leftrightarrow x \in aE \Leftrightarrow \chi_{aE}(x) = 1.$$

The second statement is similar. ∎

PROPOSITION 2.2.8. *If B and C are left translation invariant then so is $W(B,C)$, with*

$$\|L_a\|_{W(B,C)} \leq \|L_a\|_B \,\|L_a\|_C.$$

If B, C are left translation isometric then so is $W(B,C)$.

PROOF: Assume $f \in W(B,C)$. As B is left translation invariant we have $L_a f \in B$, so $F_{L_a f} \in C$. Now,

$$\begin{aligned} F_{L_a f}(x) &= \|L_a f \cdot \chi_{xQ}\|_B \\ &= \|L_a(f \cdot \chi_{a^{-1}xQ})\|_B \\ &\leq \|L_a\|_B \; \|f \cdot \chi_{a^{-1}xQ}\|_B \\ &= \|L_a\|_B \; F_f(a^{-1}x) \\ &= \|L_a\|_B \; (L_a F_f)(x). \end{aligned} \tag{2.2.1}$$

Since C is left translation invariant, $L_a F_f \in C$. Therefore, since C is solid,

$$\begin{aligned} \|L_a f\|_{W(B,C)} &= \|F_{L_a f}\|_C \\ &\leq \|L_a\|_B \; \|L_a F_f\|_C \\ &\leq \|L_a\|_B \; \|L_a\|_C \; \|f\|_{W(B,C)}. \end{aligned}$$

The translation isometric case is similar. ∎

PROPOSITION 2.2.9. *If left translation is strongly continuous in B, C is translation invariant, and $C_c(G)$ is dense in C, then left translation is strongly continuous in $W(B,C)$.*

PROOF: Fix $f \in W(B,C)$ and $\varepsilon > 0$. Then there exists a $k \in C_c(G)$ such that $\|F_f \cdot (1-k)\|_C < \varepsilon$. Let $K = \text{supp}(k)$. For $a \in G$ we then have

(2.2.2)

$$\begin{aligned} &\|L_a F_f \cdot (1-k)\|_C \\ &\quad \leq \|L_a(F_f \cdot (1-k))\|_C \\ &\qquad + \|(L_a F_f) \cdot (1-k) - L_a(F_f \cdot (1-k))\|_C \end{aligned}$$

$$\leq \|L_a\|_C \, \|F_f \cdot (1-k)\|_C$$

$$+ \, \|L_a\|_C \, \|F_f \cdot L_{a^{-1}}(1-k) - F_f \cdot (1-k)\|_C$$

$$\leq \varepsilon \|L_a\|_C + \|L_a\|_C \, \|F_f \cdot (L_{a^{-1}}k - k)\|_C$$

$$\leq \varepsilon \|L_a\|_C + \|L_a\|_C \, \|F_f\|_C \, \|L_{a^{-1}}k - k\|_\infty,$$

where the last inequality follows from the fact that C is solid and both F_f and $F_f \cdot (L_{a^{-1}}k - k) \in C$. Since $\|L_{a^{-1}}k - k\|_\infty \to 0$ as $a \to e$, we can find a neighborhood U of e (with compact closure) such that

$$\text{(2.2.3)} \qquad \|L_{a^{-1}}k - k\|_\infty \leq \varepsilon$$

for all $a \in U$. Now, $\|L_a\|_B$ and $\|L_a\|_C$ are locally bounded as functions of a since they are submultiplicative functions on G (Theorem 2.1.4). Therefore, $M = \sup_{a \in U} \|L_a\|_B, \|L_a\|_C < \infty$, which, combined with (2.2.2) and (2.2.3), gives

$$\text{(2.2.4)} \qquad \|L_aF_f \cdot (1-k)\|_C \leq M + \varepsilon M \|F_f\|_C \, \varepsilon$$

for $a \in U$. Combining (2.2.4) with (2.2.1), we have

$$\|F_{L_af} \cdot (1-k)\|_C \leq \|L_a\|_B \, \big\|L_aF_f \cdot (1-k)\big\|_C$$

$$\leq \varepsilon M^2 \big(1 + \|F_f\|_C\big)$$

$$= R\varepsilon$$

for $a \in U$. Since $F_{L_af-f} \leq F_{L_af} + F_f$ and $\text{supp}(k) = K$, we therefore have for $a \in U$ that

$$
\begin{aligned}
&\|L_a f - f\|_{W(B,C)} \\
&\qquad = \|F_{L_a f - f}\|_C \\
&\qquad \le \|F_{L_a f - f} \cdot (1-k)\|_C + \|F_{L_a f - f} \cdot k\|_C \\
&\qquad \le \|F_{L_a f} \cdot (1-k)\|_C + \|F_f \cdot (1-k)\|_C + \|k\|_C \sup_{x \in K} F_{L_a f - f}(x) \\
&\qquad \le R\varepsilon + \varepsilon + \|k\|_C \sup_{x \in K} \|(L_a f - f) \cdot \chi_{xQ}\|_B.
\end{aligned}
$$

The result now follows from the fact that K is compact and left translation is strongly continuous in B. ∎

COROLLARY 2.2.10. *If B is left homogeneous, C is translation isometric, and $C_c(G)$ is dense in C, then $W(B,C)$ is left homogeneous.*

EXAMPLE 2.2.11. $W(B, L^p(G))$ is left homogeneous for $1 \le p < \infty$.

Section 2.3. Inclusion relations.

In this section we derive some inclusion relations between the amalgams $W(L^p_v, L^q_w)$. We assume throughout the remainder of this chapter that v, w are weights on G with w right moderate (so, in particular, L^q_w is right translation invariant by Theorem 2.1.6).

Note that

$$\|f\|_{L^p_{w^p}} = \left(\int_G |f(t)w(t)|^p\,dt\right)^{1/p} = \|fw\|_{L^p}.$$

This expression is sometimes used as the defining norm for L^p_w, rather than $L^p_{w^p}$, as we use it. Some of the results in this section would be easier to state if we adopted this alternate definition of L^p_w, but it will be convenient in the main part of the thesis to keep the w's on the "outside". Recall from Proposition 2.1.3 that w^p is right moderate if and only if w is.

For simplicity and consistency in dealing with the case $p = \infty$ we let $w^\infty = w$, so $\|f\|_{L^\infty_w} = \|f\|_{L^\infty_{w^\infty}}$.

A moderate weight can be transferred between the local and global components, as follows.

PROPOSITION 2.3.1. *Given* $1 \le p,\ q \le \infty$,

$$W(L^p_{v^p}, L^q_{w^q}) = W(L^p_{(vw)^p}, L^q).$$

PROOF: Assume $1 \le p,\ q < \infty$. Since w is right moderate, there exists by Theorem 2.1.6e a constant B such that $B^{-1}w(t) \le w(x) \le Bw(t)$ for all

$x \in G$ and $t \in xQ$. Therefore,

$$\begin{aligned}
\|f\|_{W(L^p_{v^p}, L^q_{w^q})} &= \| \, \|f \cdot \chi_{xQ}\|_{L^p_{v^p}} \, \|_{L^q_{w^q}} \\
&= \left(\int_G \left(\int_{xQ} |f(t)\, v(t)|^p \, dt \right)^{q/p} w(x)^q \, dx \right)^{1/q} \\
&\leq \left(\int_G \left(\int_{xQ} |f(t)\, v(t)\, B\, w(t)|^p \, dt \right)^{q/p} dx \right)^{1/q} \\
&= B \, \|f\|_{W(L^p_{(vw)^p}, L^q)}.
\end{aligned}$$

The opposite inequality, and the remaining cases, are similar. ∎

When the local and global components are comparable, we have the following.

PROPOSITION 2.3.2. *Given* $1 \leq p \leq \infty$,

$$W(L^p_v, L^p_w) = L^p_{vw},$$

with equivalence of norms. If $e \in Q$ *and* $|Q| = 1$ *then*

$$\| \cdot \|_{W(L^p_v, L^p)} = \| \cdot \|_{L^p_v}.$$

PROOF: Without loss of generality assume $e \in Q$. For $1 \leq p < \infty$ we have

$$\begin{aligned}
\|f\|^p_{W(L^p_v, L^p_w)} &= \int_G \int_{xQ} |f(t)|^p \, v(t) \, dt \; w(x) \, dx \\
&= \int_G \int_G |f(t)|^p \, v(t) \, \chi_{xQ}(t) \, w(x) \, dt \, dx \\
&= \int_G |f(t)|^p \, v(t) \int_G w(x) \, \chi_{tQ^{-1}}(x) \, dx \, dt.
\end{aligned}$$

Since w is right moderate, there exist by Theorem 2.1.6f constants C, $D > 0$ such that

$$C\,w(t) \;\le\; \int_{tQ^{-1}} w(x)\,dx \;\le\; D\,w(t)$$

for $t \in G$ (note that if $w \equiv 1$ then $C = D = |Q^{-1}| = |Q|$). Thus,

$$\|f\|^p_{W(L^p_v,L^p_w)} \;\le\; D \int_G |f(t)|^p\, v(t)\, w(t)\, dt \;=\; D\,\|f\|^p_{L^p_{vw}}.$$

The opposite inequality, and the case $p = \infty$, are similar. ∎

One simple inclusion relation is the following.

PROPOSITION 2.3.3. *Assume B is solid. If $1 \le p < \infty$ and $w \in L^1(G)$, or if $p = \infty$ and $w \in L^\infty(G)$, then $W(B, L^p_w) \supset B$.*

PROOF: Assume $1 \le p < \infty$ and $w \in L^1$. If $f \in B$ then

$$\begin{aligned}
\|f\|_{W(B,L^p_w)} &= \big\|\, \|f \cdot \chi_{xQ}\|_B \,\big\|_{L^p_w} \\
&= \left(\int_G \|f \cdot \chi_{xQ}\|^p_B\, w(x)\, dx\right)^{1/p} \\
&\le \left(\int_G \|f\|^p_B\, w(x)\, dx\right)^{1/p} \\
&= \|f\|_B\, \|w\|_{L^1}.
\end{aligned}$$

The remaining case is similar. ∎

The standard inclusion relations for L^p spaces on compact sets imply inclusion relations with respect to the local components of Wiener spaces.

PROPOSITION 2.3.4. *Given $1 \le p \le q \le \infty$,*

$$W(L^p_{v^p}, C) \;\supset\; W(L^q_{v^q}, C)$$

with

$$\|\cdot\|_{W(L^p_{v^p},C)} \leq |Q|^{\frac{1}{p}-\frac{1}{q}}\, \|\cdot\|_{W(L^q_{v^q},C)}.$$

PROOF: From Section 1.7g,

$$\begin{aligned} \|f\cdot \chi_{xQ}\|_{L^p_{v^p}} &= \|fv\cdot \chi_{xQ}\|_{L^p} \\ &\leq |xQ|^{\frac{1}{p}-\frac{1}{q}}\, \|fv\cdot \chi_{xQ}\|_{L^q} \\ &= |Q|^{\frac{1}{p}-\frac{1}{q}}\, \|f\cdot \chi_{xQ}\|_{L^q_{v^q}}. \end{aligned}$$

The result now follows from the solidity of C. ∎

The following lemma is an integral version of Minkowski's inequality, e.g., [WZ, p. 143].

LEMMA 2.3.5. *Given measure spaces (X,μ) and (Y,ν), and given $1 \leq p < \infty$. If F is measurable and nonnegative on $X \times Y$ then*

$$\left(\int_Y \left(\int_X F(x,y)\,d\mu(x)\right)^p d\nu(y)\right)^{1/p} \leq \int_X \left(\int_Y F(x,y)^p\,d\nu(y)\right)^{1/p} d\mu(x).$$

PROPOSITION 2.3.6.

a. *If $1 \leq p \leq q \leq \infty$ then $W(L^p_{v^p}, L^q_{w^q}) \supset L^p_{(vw)^p} \cup L^q_{(vw)^q}$.*

b. *If $1 \leq q \leq p \leq \infty$ then $W(L^p_{v^p}, L^q_{w^q}) \subset L^p_{(vw)^p} \cap L^q_{(vw)^q}$.*

PROOF: Assume $1 \leq p \leq q < \infty$. By Propositions 2.3.2 and 2.3.4,

$$\|f\|_{W(L^p_{v^q},L^q_{w^q})} \leq |Q|^{\frac{1}{p}-\frac{1}{q}}\, \|f\|_{W(L^q_{v^q},L^q_{w^q})} \sim |Q|^{\frac{1}{p}-\frac{1}{q}}\, \|f\|_{L^q_{(vw)^q}}.$$

Thus $W(L^p_{v^p}, L^q_{w^q}) \supset L^q_{(vw)^q}$.

For the second containment we use the Minkowski integral inequality. First, write

(2.3.1)

$$\|f\|_{W(L^p_{v^p},L^q_{w^q})} = \left(\int_G \left(\int_G |f(t)|^p \chi_{xQ}(t)\, v(t)^p\, dt\right)^{q/p} w(x)^q\, dx\right)^{1/q}$$
$$= \left(\int_G \left(\int_G |F(x,t)|^p\, v(t)^p\, dt\right)^{q/p} w(x)^q\, dx\right)^{1/q},$$

where

$$F(x,t) = f(t)\cdot \chi_{xQ}(t).$$

As $q/p \geq 1$ we can apply Lemma 2.3.5, using the measures $\mu = v(t)^p\, dt$ and $\nu = w(x)^q\, dx$, to obtain

$$\|f\|^p_{W(L^p_v,L^q_w)} = \left(\int_G \left(\int_G |F(x,t)|^p\, v(t)^p\, dt\right)^{q/p} w(x)^q\, dx\right)^{p/q}$$
$$\leq \int_G \left(\int_G |F(x,t)|^{p\cdot q/p}\, w(x)^q\, dx\right)^{p/q} v(t)^p\, dt$$
$$= \int_G \left(\int_G |f(t)|^q \chi_{xQ}(t)\, w(x)^q\, dx\right)^{p/q} v(t)^p\, dt$$
$$= \int_G |f(t)|^p \left(\int_G \chi_{tQ^{-1}}(x)\, w(x)^q\, dx\right)^{p/q} v(t)^p\, dt.$$

Without loss of generality, assume $e \in Q$. Then since w^q is right moderate, there exist by Theorem 2.1.6g constants C, $D > 0$ such that

$$C\, w(t)^q \leq \int_{tQ^{-1}} w(x)^q\, dx \leq D\, w(t)^q$$

for all $t \in G$. Thus,

$$\|f\|^p_{W(L^p_{v^p},L^q_{w^q})} \leq \int_G |f(t)|^p \left(D\, w(t)^q\right)^{p/q} v(t)^p\, dt = D^{p/q}\, \|f\|_{L^p_{(vw)^p}},$$

as desired. The remaining cases are similar, with $1 \leq q \leq p < \infty$ following by applying the Minkowski integral inequality to (2.3.1), but in the opposite direction. ∎

The following theorem, a Hölder's inequality for Wiener amalgams, can be extended to a duality theorem. However, we delay consideration of duality until after we develop equivalent discrete norms, cf., Theorem 2.5.1.

PROPOSITION 2.3.7. *Given* $1 \le p,\ q \le \infty$,

$$\|fg\|_{W(L^1,L^1)} \le \|f\|_{W(L^p_v,L^q_w)}\, \|g\|_{W(L^{p'}_{v'},L^{q'}_{w'})},$$

where $\frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}=1$, $v'=v^{1-p'}$, *and* $w'=w^{1-q'}$.

PROOF: Since $(L^p_v)'=L^{p'}_{v'}$ and $(L^q_w)'=L^{q'}_{w'}$, we have

$$\begin{aligned}
\|f\|_{W(L^p_v,L^q_w)}\, \|g\|_{W(L^{p'}_{v'},L^{q'}_{w'})} &= \big\| \|f\cdot\chi_{xQ}\|_{L^p_v} \big\|_{L^q_w} \big\| \|g\cdot\chi_{xQ}\|_{L^{p'}_{v'}} \big\|_{L^{q'}_{w'}} \\
&\ge \big\| \|f\cdot\chi_{xQ}\|_{L^p_v} \|g\cdot\chi_{xQ}\|_{L^{p'}_{v'}} \big\|_{L^1} \\
&\ge \big\| \|fg\cdot\chi_{xQ}\|_{L^1} \big\|_{L^1} \\
&= \|fg\|_{W(L^1,L^1)}. \ \blacksquare
\end{aligned}$$

REMARK 2.3.8. From Proposition 2.3.2, $\|fg\|_{W(L^1,L^1)} \sim \|fg\|_{L^1}$, and is equality if we chose Q so that $e\in Q$ and $|Q|=1$.

Section 2.4. Discrete norms.

In this section we derive equivalent discrete-type norms for the Wiener amalgam spaces, analogous to the equivalent norms (0.1.1) for the amalgams $W(L^p(\mathbf{R}), L^q(\mathbf{R}))$.

We continue to assume throughout this section that C is solid and right translation invariant.

DEFINITION 2.4.1. A set of functions $\{\psi_i\}_{i\in J}$ on G is a **bounded uniform partition of unity**, or **BUPU**, if

a. $\sum \psi_i \equiv 1$,

b. $\sup \|\psi_i\|_\infty < \infty$,

c. there exists a compact set $U \subset G$ (with nonempty interior) and points $y_i \in G$ such that $\mathrm{supp}(\psi_i) \subset y_i U$ for all i,

d. for each compact $K \subset G$,

$$\sup_{x\in G} \#\{i : x \in y_i K\} \;=\; \sup_i \#\{j : y_i K \cap y_j K \neq \emptyset\} \;<\; \infty.$$

We say that the BUPU has **size** U, and call $\{y_i\}$ the **associated points.**

It has been shown in [**F7**] that it is possible to find BUPU's of any prescribed size in any homogeneous Banach space.

THEOREM 2.4.2. *If $\{\psi_i\}$ is a BUPU of size U with associated points $\{y_i\}$, then*

$$\|f\|_{W(B,C)} \sim \Big\|\sum \|f\psi_i\|_B \, \chi_{y_i V}\Big\|_C \qquad (2.4.1)$$

for every compact set $V \supset U$.

PROOF: For simplicity, denote the right-side norm of (2.4.1) by $\|\cdot\|_V$. This clearly is a norm, so we first show that it is independent of V in the sense that different choices of V give equivalent norms. Fix f, and let V_1, $V_2 \supset U$ be compact sets with nonempty interiors. Then we can find $x_1, \ldots, x_N$ such that

$$V_2 \subset \bigcup_1^N V_1 x_k.$$

Defining $G_V = \sum \|f\psi_i\|_B \, \chi_{y_i V}$, we therefore have for $x \in G$ that

$$\begin{aligned}
G_{V_2}(x) &= \sum_i \|f\psi_i\|_B \, \chi_{y_i V_2}(x) \\
&\leq \sum_i \|f\psi_i\|_B \, \chi_{\cup y_i V_1 x_k}(x) \\
&\leq \sum_i \|f\psi_i\|_B \sum_{k=1}^N \chi_{y_i V_1 x_k}(x) \\
&= \sum_{k=1}^N \sum_i \|f\psi_i\|_B \, \chi_{y_i V_1}(x x_k^{-1}) \\
&= \sum_{k=1}^N G_{V_1}(x x_k^{-1}) \\
&= \sum_{k=1}^N (R_{x_k^{-1}} G_{V_1})(x).
\end{aligned}$$

Since C is solid and right translation invariant, this implies

$$\begin{aligned}
\|f\|_{V_2} &= \|G_{V_2}\|_C \\
&\leq \Big\| \sum_{k=1}^N R_{x_k^{-1}} G_{V_1} \Big\|_C \\
&\leq \sum_{k=1}^N \|R_{x_k^{-1}}\|_C \, \|G_{V_1}\|_C
\end{aligned}$$

$$\leq MN \, \|G_{V_1}\|_C$$

$$= MN \, \|f\|_{V_1},$$

where $M = \max\{\|R_{x_k^{-1}}\|_C\} < \infty$. A symmetric argument gives the reverse inequality, so $\|\cdot\|$ is in fact independent of V.

Now we show that the left- and right-hand sides of (2.4.1) are equivalent. Fix Q large enough that $U^{-1}U \subset Q$. If $x \in y_iU$ then $y_i \in xU^{-1}$, so $y_iU \subset xU^{-1}U \subset xQ$. Therefore $f\psi_i = f\psi_i\chi_{xQ}$ since $\mathrm{supp}(\psi_i) \subset y_iU \subset xQ$, so

$$\|f\psi_i\|_B \;=\; \|f\psi_i\chi_{xQ}\|_B \;\leq\; \|\psi_i\|_\infty \, \|f\cdot\chi_{xQ}\|_B \;\leq\; M \, \|f\cdot\chi_{xQ}\|_B,$$

where $M = \sup \|\psi_i\|_\infty < \infty$. Hence,

$$\begin{aligned} G_U(x) &= \sum_i \|f\psi_i\|_B \, \chi_{y_iU}(x) \\ &= \sum_{\{i : x \in y_iU\}} \|f\psi_i\|_B \\ &\leq \#\{i : x \in y_iU\} \, M \, \|f\cdot\chi_{xQ}\|_B \\ &\leq C_U \, M \, \|f\cdot\chi_{xQ}\|_B, \end{aligned}$$

where $C_U = \sup_{x\in G} \#\{i : x \in y_iU\} < \infty$. Since C is solid, this implies

$$(2.4.2) \quad \|f\|_U \;=\; \|G_U\|_C \;\leq\; C_U \, M \, \big\| \, \|f\cdot\chi_{xQ}\|_B \, \big\|_C \;=\; C_U \, M \, \|f\|_{W(B,C)}.$$

To prove the opposite inequality, let $V \supset U$ be such that $V \supset UQ^{-1}$. Given $x \in G$, define

$$M_x \;=\; \{i : y_iU \cap xQ \neq \emptyset\}.$$

If $i \in M_x$ then $y_i u = xq$ for some $u \in U$ and $q \in Q$, so $x = y_i u q^{-1} \in y_i V$. Therefore,

$$\begin{aligned}
\|f \cdot \chi_{xQ}\|_B &= \Big\| \sum_i (f \cdot \chi_{xQ})\, \psi_i \Big\|_B \\
&\le \sum_{i \in M_x} \|f \psi_i \chi_{xQ}\|_B \\
&\le \sum_{i \in M_x} \|f \psi_i\|_B \\
&= \sum_{i \in M_x} \|f \psi_i\|_B \, \chi_{y_i V}(x) \\
&\le \sum_i \|f \psi_i\|_B \, \chi_{y_i V}(x) \\
&= G_V(x).
\end{aligned}$$

Since C is solid, we therefore have

$$\|f\|_{W(B,C)} = \big\| \|f \cdot \chi_{xQ}\|_B \big\|_C \le \|G_V\|_C = \|f\|_V. \tag{2.4.3}$$

From (2.4.2), (2.4.3), and the fact that $\|\cdot\|_V$ is independent of V, we conclude that $\|\cdot\|_V \sim \|\cdot\|_{W(B,C)}$. ∎

Example 2.4.3. Assume $\{y_i\}$ and U are such that $\{y_i U\}$ is a partition of G. Then $\{\chi_{y_i U}\}$ is a BUPU of size U, so

$$\begin{aligned}
\|f\|_{W(L^p_v, L^q_w)} &\sim \Big\| \sum \|f \cdot \chi_{y_i U}\|_{L^p_v} \, \chi_{y_i U} \Big\|_{L^q_w} \\
&= \left(\int_G \Big| \sum \|f \cdot \chi_{y_i U}\|_{L^p_v} \, \chi_{y_i U}(x) \Big|^q \, w(x)\, dx \right)^{1/q} \\
&= \left(\sum \int_{y_i U} \|f \cdot \chi_{y_i U}\|^q_{L^p_v} \, w(x)\, dx \right)^{1/q} \\
&= \left(\sum \|f \cdot \chi_{y_i U}\|^q_{L^p_v} \int_{y_i U} w(x)\, dx \right)^{1/q},
\end{aligned}$$

where the interchange of summation and integration is justified by the fact that $\{y_iU\}$ is a partition of G. Since w is right moderate we have by Theorem 2.1.6e, f that the values $\int_{y_iU} w$ are uniformly equivalent to the values of w at any point in y_iU or to its supremum or infimum on y_iU. Thus, for example,

$$\|f\|_{W(L^p_v,L^q_w)} \sim \left(\sum \|f\cdot\chi_{y_iU}\|^q_{L^p_v}\, w(z_i)\right)^{1/q} = \|\{\|f\cdot\chi_{y_iU}\|_{L^p_v}\}\|_{\ell^q_\omega},$$

where $z_i \in y_iU$ is any set of fixed vectors, and ω is the weight on the index set J defined by $\omega(i) = w(z_i)$.

EXAMPLE 2.4.4. Set $G = \mathbf{R}^d$, $U = [0,1]$ (the unit cube in $\mathbf{R}^d$), and $y_n = n$ for $n \in \mathbf{Z}^d$. Then, by Example 2.4.3, the norm for $W(L^p(\mathbf{R}^d), L^q(\mathbf{R}^d))$ is equivalent to the discrete norm

$$\|f\|_{W(L^p,L^q)} \sim \left(\sum_{n\in\mathbf{Z}^d} \|f\cdot\chi_{[n,n+1]}\|^q_{L^p(\mathbf{R}^d)}\right)^{1/q}.$$

Thus $W(L^p(\mathbf{R}), L^q(\mathbf{R}))$ is identical with the standard amalgam spaces defined in (0.1.1).

EXAMPLE 2.4.5. "Dyadic amalgams" dyad(L^p, ℓ^q), considered by some authors, are defined by the norms

$$\|f\|_{\mathrm{dyad}(L^p,\ell^q)} = \left(\sum_{n\in\mathbf{Z},\pm} \|f\cdot\chi_{\pm[2^n,2^{n+1}]}\|^q_{L^p(\mathbf{R})}\right)^{1/q},$$

e.g., [FS]. The sets $\{\pm[2^n, 2^{n+1}]\}$ form a dyadic partition of $\mathbf{R}_*$, and are group translates in $\mathbf{R}_*$ of the compact set $[1,2]$ since $\pm[2^n,2^{n+1}] = \pm 2^n\cdot[1,2]$. However, the dyadic amalgam spaces are not Wiener amalgam spaces on the multiplicative group $\mathbf{R}_*$ because of the use of Lebesgue measure dt rather than the Haar measure $dt/|t|$ for $\mathbf{R}_*$. For example, by Example 2.4.3, a discrete norm for $W_*(L^p, L^q) = W(L^p(\mathbf{R}_*), L^q(\mathbf{R}_*))$ based on the BUPU $\{\chi_{\pm[2^n,2^{n+1}]}\}$ is

$$\|f\|_{W_*(L^p,L^q)} \sim \left(\sum_{n\in\mathbf{Z},\pm} \|f\cdot\chi_{\pm[2^n,2^{n+1}]}\|^q_{L^p(\mathbf{R}_*)}\right)^{1/q},$$

where we recall that when dealing with the groups $\mathbf{R}^d$ and $\mathbf{R}^d_*$ we use the notations

$$W(L^p, L^q) = W(L^p(\mathbf{R}^d), L^q(\mathbf{R}^d))$$

and

$$W_*(L^p, L^q) = W(L^p(\mathbf{R}^d_*), L^q(\mathbf{R}^d_*)),$$

cf., (0.1.2) and (0.1.3). Since

$$\big\| |t|^{1/p} g(t)\big\|_{L^p(\mathbf{R}_*)} = \|g\|_{L^p(\mathbf{R})},$$

it follows that

$$\mathrm{dyad}(L^p, \ell^q) = |t|^{1/p}\, W_*(L^p, L^q),$$

i.e., $f \in \mathrm{dyad}(L^p, \ell^q)$ if and only if $|t|^{1/p} f \in W_*(L^p, L^q)$.

EXAMPLE 2.4.6. A d-dimensional discrete norm for the amalgam spaces $W_*(L^p, L^q) = W(L^p(\mathbf{R}^d_*), L^q(\mathbf{R}^d_*))$ on the multiplicative group $\mathbf{R}^d_*$ would be the following. Let $G = \mathbf{R}^d_*$ and $U = [1,2] \subset \mathbf{R}^d_*$. Recall the defintion $\Omega^d = \{-1,1\}^d$, i.e., Ω^d is the set of d-tuples of ±1's. Then $\{\sigma 2^n[1,2]\}_{n\in\mathbf{Z}^d,\sigma\in\Omega^d}$ is a partition of $\mathbf{R}^d_*$, where $2^n = (2^{n_1},\dots,2^{n_d})$ as usual. Therefore,

$$\|f\|_{W_*(L^p,L^q)} \sim \left(\sum_{n\in\mathbf{Z}^d,\sigma\in\Omega^d} \|f\cdot\chi_{\sigma[2^n,2^{n+1}]}\|^q_{L^p(\mathbf{R}^d_*)}\right)^{1/q}.$$

REMARK 2.4.7. In Example 2.4.3 we assumed that we could find a BUPU $\{\psi_i\}$ such that the supports of the ψ_i were disjoint. This may not be possible, or even desirable, in general. However, we have by definition that the supports of any BUPU $\{\psi_i\}$ do not overlap "too much", i.e.,

$$\sup_i \#\{j : |\operatorname{supp}(\psi_i) \cap \operatorname{supp}(\psi_i)| > 0\} < \infty.$$

This allows us to prove that $W(B, L^q_w)$ has an equivalent discrete-type norm based on any BUPU (Theorem 2.4.11).

DEFINTION 2.4.8. A family $\{E_i\}_{i \in J}$ of subsets of a measure space (X, μ) has **a bounded number of overlaps** if

$$K = \sup_i \#\{j : \mu(E_i \cap E_j) > 0\} < \infty.$$

We call K the **maximum number of overlaps** since no E_i can intersect more than K of the E_j. Note that $K = \|\sum \chi_{E_i}\|_\infty$.

LEMMA 2.4.9. *Given a measure space (X, μ) and a family $\{E_i\}_{n \in J}$ with maximum number of overlaps K. Then there is a finite partition $\{J_r\}_{r=1}^K$ of J such that*

$$i \neq j \in J_r \quad \Rightarrow \quad \mu(E_i \cap E_j) = 0. \tag{2.4.4}$$

PROOF: Let J_1 be a maximal subset of J with respect to (2.4.4) for $r = 1$. Inductively define J_r for $r \geq 2$ as a maximal subset of $J \setminus \bigcup_1^{r-1} J_s$ having property (2.4.4). Suppose $i \in J \setminus \bigcup_1^K J_r$. Then given $1 \leq r \leq K$, we have

$i \in J \setminus \bigcup_1^{r-1} J_s$ and $i \notin J_r$. Since J_r is maximal in $J \setminus \bigcup_1^{r-1} J_s$ with respect to (2.4.4), $J_r \cup \{i\}$ cannot satisfy (2.4.4). That means there is a $j_r \in J_r$ such that $\mu(E_i \cap E_{j_r}) > 0$. Hence, for each $l \in \{i, j_1, \dots, j_K\}$ we have $\mu(E_i \cap E_l) > 0$. However, the J_r are disjoint, so $i, j_1, \dots, j_K$ are distinct, which contradicts the definition of K. Therefore $J = \bigcup_1^K J_r$. ∎

PROPOSITION 2.4.10. *Given a measure space (X, μ) and given $1 \le p \le \infty$. Assume $\{f_n\}_{n \in J} \subset L^p(X, d\mu)$ are nonnegative functions such that $\{\mathrm{supp}(f_n)\}$ has a maximum of K overlaps.*

a. *If $1 \le p < \infty$ then for each finite set $F \subset J$ we have*

$$\left(\sum_{n \in F} \|f_n\|_p^p\right)^{1/p} \le \left\|\sum_{n \in F} f_n\right\|_p \le K^{1/p'} \left(\sum_{n \in F} \|f_n\|_p^p\right)^{1/p}. \tag{2.4.5}$$

Therefore, $\sum \|f_n\|_p^p < \infty$ if and only if $\sum f_n$ converges in $L^p(X, d\mu)$. In this case the convergence is unconditional, and we can replace F by J in (2.4.5).

b. *If $p = \infty$ then for each finite set $F \subset J$ we have*

$$\sup_{n \in F} \|f_n\|_\infty \le \left\|\sum_{n \in F} f_n\right\|_\infty \le K \sup_{n \in F} \|f_n\|_\infty. \tag{2.4.6}$$

Therefore, $\sup \|f_n\|_\infty < \infty$ if and only if $\sum f_n$ converges in $L^\infty(X, d\mu)$. In this case the convergence is unconditional, and we can replace F by J in (2.4.6).

PROOF: We prove only a as b is similar. By Lemma 2.4.9, we can partitition J as $J = \bigcup_1^K J_r$, with the property that $\mu(\mathrm{supp}(f_m) \cap \mathrm{supp}(f_n)) = 0$ whenever $m \ne n \in J_r$. Recall now that for any $c_n \ge 0$ we have

$$\left(\sum_1^K c_n{}^p\right)^{1/p} \le \sum_1^K c_n \le K^{1/p'} \left(\sum_1^K c_n{}^p\right)^{1/p}.$$

Therefore,

$$\begin{aligned}
\Big\|\sum_{n\in F} f_n\Big\|_p^p &= \int_X \Big|\sum_{n\in F} f_n\Big|^p \, d\mu \\
&= \int_X \Big|\sum_{r=1}^{K} \sum_{n\in F\cap J_r} f_n\Big|^p \, d\mu \\
&\le K^{p/p'} \sum_{r=1}^{N} \int_X \Big|\sum_{n\in F\cap J_r} f_n\Big|^p \, d\mu \\
&= K^{p/p'} \sum_{r=1}^{N} \sum_{n\in F\cap J_r} \int_X |f_n|^p \, d\mu \\
&= K^{p/p'} \sum_{n\in F} \|f_n\|_p^p,
\end{aligned}$$

where the next-to-last equality follows from the fact that the supports of the f_n for $n \in F \cap J_r$ are all disjoint. The opposite inequality is similar, and the statements about convergence follow directly from Lemma 1.4.2. ∎

THEOREM 2.4.11. *Given $1 \le q \le \infty$. Let $\{\psi_i\}$ be a BUPU of size U with associated points $\{y_i\}$, and fix any $z_i \in y_iU$. Then*

$$\|f\|_{W(B,L^q_w)} \sim \Big(\sum \|f\psi_i\|_B^q \, w(z_i)\Big)^{1/q} = \big\|\{\|f\psi_i\|_B\}\big\|_{\ell^q_\omega},$$

where $\omega(i) = w(z_i)$.

PROOF: Assume $1 \le q < \infty$; the case $q = \infty$ is similar. By Theorem 2.4.2 we have

$$\|f\|_{W(B,L^q_w)} \sim \Big\|\sum \|f\psi_i\|_B \, \chi_{y_iU}\Big\|_{L^q_w}.$$

Since $\{\chi_{y_iU} \cdot w^{1/q}\}$ satisfies the hypotheses of Proposition 2.4.10,

$$\left\| \sum \|f\psi_i\|_B \, \chi_{y_iU} \right\|_{L^q_w} \sim \left(\sum \left\| \, \|f\psi_i\|_B \, \chi_{y_iU} \right\|^q_{L^q_w} \right)^{1/q}$$

$$= \left(\sum \|f\psi_i\|^q_B \, \|\chi_{y_iU}\|^q_{L^q_w} \right)^{1/q}.$$

Finally, by Theorem 2.1.6f,

$$\|\chi_{y_iU}\|^q_{L^q_w} = \int_{y_iU} w \sim w(z_i),$$

which completes the proof. ∎

COROLLARY 2.4.12. *Given* $1 \le p \le q \le \infty$,

$$W(B, L^p_{w^p}) \subset W(B, L^q_{w^q}).$$

PROOF: Fixing any BUPU $\{\psi_i\}$, we have by Theorem 2.4.11 that

$$\|f\|_{W(B,L^p_{w^p})} \sim \left\| \{ \|f\psi_i\|_B \} \right\|_{\ell^p_{\omega^p}}$$

$$= \left\| \{ \|f\psi_i\|_B \, \omega(i) \} \right\|_{\ell^p}$$

$$\ge \left\| \{ \|f\psi_i\|_B \, \omega(i) \} \right\|_{\ell^q}$$

$$= \left\| \{ \|f\psi_i\|_B \} \right\|_{\ell^q_{\omega^q}}$$

$$\sim \|f\|_{W(B,L^q_{w^q})}. \quad ∎$$

From Proposition 2.3.4 and Corollary 2.4.12 we obtain the following.

COROLLARY 2.4.13. *Given* $1 \le p_2 \le p_1 \le \infty$ *and* $1 \le q_1 \le q_2 \le \infty$,

$$W(L^{p_1}, L^{q_1}) \subset W(L^{p_2}, L^{q_2}).$$

REMARK 2.4.14. Propositions 2.3.1 and 2.3.4 and Corollary 2.4.12 combine to give a simple proof of Proposition 2.3.6. For, if $p \leq q$ then, by Propositions 2.3.1 and 2.3.4,

$$W(L^p_{v^p}, L^q_{w^q}) \supset W(L^q_{v^q}, L^q_{w^q}) = L^q_{(vw)^q},$$

and, by Corollary 2.4.12 and Proposition 2.3.1,

$$W(L^p_{v^p}, L^q_{w^q}) \supset W(L^p_{v^p}, L^p_{w^p}) = L^p_{(vw)^p}.$$

Section 2.5. Duality.

In this section we prove, using the discrete norms obtained in Section 2.4, that $W(B,C)' = W(B',C')$.

THEOREM 2.5.1. *Given* $1 \le p,\ q < \infty$,

$$W(L^p_v, L^q_w)' \;=\; W(L^{p'}_{v'}, L^{q'}_{w'}),$$

where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$, $v' = v^{1-p'}$, $w' = w^{1-q'}$, *and the duality is given by*

$$\langle f,g\rangle \;=\; \int_G f(t)\overline{g(t)}\,dt$$

for $f \in W(L^p_v, L^q_w)$ *and* $g \in W(L^{p'}_{v'}, L^{q'}_{w'})$.

PROOF: We assume for simplicity that $\{\chi_{y_iU}\}$ is a BUPU for G, so by Theorem 2.4.11,

$$\|f\|_{W(L^p_v,L^q_w)} \;=\; \left(\sum \|f\cdot\chi_{y_iU}\|^q_{L^p_v}\,\omega(i)\right)^{1/q} \tag{2.5.1}$$

and

$$\|g\|_{W(L^{p'}_{v'},L^{q'}_{w'})} \;=\; \left(\sum \|g\cdot\chi_{y_iU}\|^{q'}_{L^{p'}_{v'}}\,\omega'(i)\right)^{1/q'}, \tag{2.5.2}$$

where $z_i \in y_iU$ is any fixed choice of vectors and $\omega(i) = w(z_i)$, $\omega'(i) = w'(z_i)$. The case of a general BUPU $\{\psi_i\}$ is similar, with some added technical complications.

a. Given $f \in W(L^p_v, L^q_w)$ and $g \in W(L^{p'}_{v'}, L^{q'}_{w'})$ we have

$$\begin{aligned}\int_G |f(t)\,g(t)|\,dt \;&=\; \sum \int_{y_iU} |f(t)\,g(t)|\,dt\\ &\le\; \sum \|f\cdot\chi_{y_iU}\|_{L^p_v}\ \|g\cdot\chi_{y_iU}\|_{L^{p'}_{v'}}\end{aligned}$$

$$\leq \left(\sum \|f\cdot\chi_{y_iU}\|_{L^p_v}^q\,\omega(i)\right)^{1/q}\left(\sum \|g\cdot\chi_{y_iU}\|_{L^{p'}_{v'}}^{q'}\,\omega'(i)\right)^{1/q'}$$

$$= \|f\|_{W(L^p_v,L^q_w)}\,\|g\|_{W(L^{p'}_{v'},L^{q'}_{w'})}.$$

Therefore $\int_G f\bar{g}$ is well-defined, and

$$|\langle f,g\rangle| \leq \|f\|_{W(L^p_v,L^q_w)}\,\|g\|_{W(L^{p'}_{v'},L^{q'}_{w'})},$$

so g determines a continuous linear functional on $W(L^p_v,L^q_w)$.

b. We show now that

$$\|g\|_{W(L^{p'}_{v'},L^{q'}_{w'})} = \sup\{|\langle f,g\rangle| : \|f\|_{W(L^p_v,L^q_w)} = 1\}.$$

To see this, assume for simplicity that $1 < p,q < \infty$, fix $g \in W(L^{p'}_{v'},L^{q'}_{w'})$, and define $g_i = g\cdot\chi_{y_iU}$. Let

$$f_i(t) = \begin{cases} |g_i(t)|^{p'}\,v'(t)\,/\,\overline{g_i(t)}, & g_i(t) \neq 0,\\ 0, & g_i(t) = 0.\end{cases}$$

Then $\text{supp}(f_i) \subset y_iU$, and

$$|f_i(t)|^p\,v(t) = |g_i(t)|^{p(p'-1)}\,v(t)^{p(1-p')}\,v(t) = |g_i(t)|^{p'}\,v'(t),$$

so $\|f_i\|_{L^p_v}^p = \|g_i\|_{L^{p'}_{v'}}^{p'} < \infty$. Moreover,

(2.5.3)

$$\begin{aligned}\langle f_i,g_i\rangle &= \int_G f_i(t)\,\overline{g_i(t)}\,dt\\ &= \int_G |g_i(t)|^{p'}\,v'(t)\,dt\\ &= \left(\int_G |g_i(t)|^{p'}\,v'(t)\,dt\right)^{1/p}\left(\int_G |g_i(t)|^{p'}\,v'(t)\,dt\right)^{1/p'}\end{aligned}$$

$$= \left(\int_G |f_i(t)|^p\, v(t)\, dt\right)^{1/p} \left(\int_G |g_i(t)|^{p'}\, v'(t)\, dt\right)^{1/p'}$$

$$= \|f_i\|_{L^p_v}\, \|g_i\|_{L^{p'}_{v'}}$$

$$= a_i\, b_i.$$

Define

$$c_i = \begin{cases} b_i^{p'}\, \omega'(i) \,/\, (a_i\, b_i), & a_i b_i \neq 0, \\ 0, & a_i b_i = 0. \end{cases}$$

Then

$$(c_i\, a_i)^q\, \omega(i) = b_i^{q(q'-1)}\, \omega(i)^{q(1-q')}\, \omega(i) = b_i^{q'}\, \omega'(i),$$

so $\|\{c_i\, a_i\}\|^q_{\ell^q_\omega} = \|\{b_i\}\|^{q'}_{\ell^{q'}_{\omega'}}$. Moreover,

(2.5.4)

$$\begin{aligned} \sum c_i\, a_i\, b_i &= \sum b_i^{q'}\, \omega'(i) \\ &= \left(\sum b_i^{q'}\, \omega'(i)\right)^{1/q} \left(\sum b_i^{q'}\, \omega'(i)\right)^{1/q'} \\ &= \left(\sum (c_i\, a_i)^q\, \omega(i)\right)^{1/q} \left(\sum b_i^{q'}\, \omega'(i)\right)^{1/q'} \\ &= \|\{c_i\, a_i\}\|_{\ell^q_\omega}\, \|\{b_i\}\|_{\ell^{q'}_{\omega'}}. \end{aligned}$$

Note that

(2.5.5) $$\|g\|_{W(L^{p'}_{v'}, L^{q'}_{\omega'})} = \|\{b_i\}\|_{\ell^{q'}_{\omega'}},$$

and define $f = \sum c_i\, f_i$; this is possible as $\{\chi_{y_i U}\}$ is a BUPU. We have

(2.5.6) $$\|f\|^q_{W(L^p_v, L^q_\omega)} = \|\{c_i\, a_i\}\|^q_{\ell^q_\omega} = \|\{b_i\}\|^{q'}_{\ell^{q'}_{\omega'}} < \infty,$$

so $f \in W(L^p_v, L^q_w)$. And, from (2.5.3), (2.5.4), (2.5.5), and (2.5.6) we have

$$\langle f, g\rangle = \sum c_i \langle f_i, g_i\rangle = \sum c_i\, a_i\, b_i = \|f\|_{W(L^p_v, L^q_w)} \|g\|_{W(L^{p'}_{v'}, L^{q'}_{w'})},$$

which completes the claim.

c. Finally, assume that $\mu \in W(L^p_v, L^q_w)'$ is given. Fix i, and note that $L^p_v(y_iU) \subset W(L^p_v, L^q_w)$, where

$$L^p_v(y_iU) = \{h \in L^p_v(G) : \text{supp}(h) \subset y_iU\},$$

since, by (2.5.1),

$$h \in L^p_v(y_iU) \Rightarrow \|h\|_{W(L^p_v, L^q_w)} = \|h \cdot \chi_{y_iU}\|_{L^p_v}\, \omega(i).$$

Therefore μ, restricted to $L^p_v(y_iU)$, defines a continuous linear functional on $L^p_v(y_iU)$, so there exists a $g_i \in L^p_v(y_iU)' = L^{p'}_{v'}(y_iU)$ such that $\langle h, \mu\rangle = \langle h, g_i\rangle$ for $h \in L^p_v(y_iU)$. Since $\text{supp}(g_i) \subset y_iU$ and $\{y_iU\}$ is a partition of G, we can define $g = \sum g_i$.

To show that $g \in W(L^{p'}_{v'}, L^{q'}_{w'})$, we first claim that $\{\|g_i\|_{L^{p'}_{v'}}\} \in \ell^{q'}_{w'}$. Given $\{c_i\} \in \ell^q_w$ and $\varepsilon > 0$, choose $f_i \in L^p_v(y_iU)$ such that $\|f_i\|_{L^p_v} \le 1$ and

$$|\langle f_i, g_i\rangle| \ge \|g_i\|_{L^{p'}_{v'}} - \frac{\varepsilon}{2^i |c_i|}.$$

Note that $f = \sum c_i f_i \in W(L^p_v, L^q_w)$ since

$$\|f\|_{W(L^p_v, L^q_w)} = \left(\sum \|c_i f_i\|^q_{L^p_v}\, \omega(i)\right)^{1/q} \le \left(\sum |c_i|^q\, \omega(i)\right)^{1/q} < \infty.$$

Hence,

$$\left|\sum c_i \langle f_i, g_i\rangle\right| = \left|\sum c_i \langle f_i, \mu\rangle\right| \tag{2.5.6}$$

$$= \left|\left\langle \sum c_i f_i,\ \mu\right\rangle\right|$$

$$= |\langle f, \mu\rangle|$$

$$\leq \|f\|_{W(L^p_v, L^q_w)} \|\mu\|$$

$$\leq \left\|\{c_i\}\right\|_{\ell^q_\omega} \|\mu\|.$$

Without loss of generality, fix the phase of c_i so that $c_i \langle f_i, g_i\rangle \geq 0$. Then, using (2.5.6),

$$\sum |c_i| \, \|g_i\|_{L^{p'}_{v'}} \leq \sum |c_i| \left(|\langle f_i, g_i\rangle| + \frac{\varepsilon}{2^i |c_i|}\right)$$

$$= \left|\sum c_i \langle f_i, g_i\rangle\right| + \varepsilon$$

$$\leq \left\|\{c_i\}\right\|_{\ell^q_\omega} \|\mu\| + \varepsilon.$$

Thus $\left\{\|g_i\|_{L^{p'}_{v'}}\right\} \in (\ell^q_\omega)' = \ell^{q'}_{\omega'}$, as claimed. Hence,

$$\|g\|_{W(L^{p'}_{v'}, L^{q'}_{w'})} = \left(\sum \|g \cdot \chi_{y_i U}\|^{q'}_{L^{p'}_{v'}} \, \omega'(i)\right)^{1/q'}$$

$$= \left\|\left\{\|g_i\|_{L^{p'}_{v'}}\right\}\right\|_{\ell^{q'}_{\omega'}}$$

$$< \infty,$$

so $g \in W(L^{p'}_{v'}, L^{q'}_{w'})$. Clearly $\langle f, g\rangle = \langle f, \mu\rangle$ for all $f \in W(L^p_v, L^q_w)$, so we are done. ∎

PART III

GENERALIZED HARMONIC ANALYSIS

CHAPTER 3

BESICOVITCH SPACES

In this chapter we establish the basic properties of the Besicovitch spaces $B(p,q)$. These space were defined, for the one-dimensional case, in (0.2.14); the general definition is given in Definition 3.2.1. Our main result, Theorem 3.2.4, is that $B(p,q)$ coincides with the Wiener amalgam space $W_*(L^p, L^q)$, where we recall that for notational simplicity, and to avoid confusion between amalgams on the additive and multiplicative groups, we adopted the notations

$$W(L^p, L^q) \;=\; W(L^p(\mathbf{R}^d), L^q(\mathbf{R}^d))$$

and

$$W_*(L^p, L^q) \;=\; W(L^p(\mathbf{R}^d_*), L^q(\mathbf{R}^d_*)),$$

cf., (0.1.2) and (0.1.3).

Our identification of $B(p,q)$ as $W_*(L^p, L^q)$ immediately provides us with equivalent discrete norms for $B(p,q)$, and implies duality and inclusion relations. These basic properties provide the machinery for our results on the Wiener transform in Chapter 4. Although not pursued in this thesis, the Wiener space identification implies other properties as well, e.g., convolution relations on the multiplicative group.

We begin in Section 3.1 by considering higher-dimensional analogues of the nonlinear spaces $B(p, \lim)$ defined, for one-dimension, by (0.2.4). We review the definitions of higher-dimensional limits from [**BBE**] (needed to define $B(p, \lim)$) and prove the nonlinearity of $B(p, \lim)$ in higher dimensions.

In Section 3.2 we prove the fundamental equality $B(p,q) = W_*(L^p, L^q)$ and establish bounds for the norm equivalence. We do this in terms of the discrete norm for $W_*(L^p, L^q)$, as it is this norm that we use to prove our results in later chapters. We also discuss the inclusion and duality relations that follow from this identification.

In Section 3.3 we prove a higher-dimensional analogue of a theorem due to Beurling, which characterizes $B(p, \infty)$ as an intersection of weighted L^p-spaces. We give Beurling's proof, for $d = 1$, and two new proofs for $d \geq 1$. One proof uses the Wiener amalgam norms, and is generalized in the following section to a larger class of spaces, while the other proof is valid only for $B(p, \infty)$.

In Section 3.4 we attempt to characterize $B(p,q)$ as an appropriate union or intersection of weighted L^p-spaces. This reveals links between $B(p,q)$ and other function spaces which have arisen in harmonic analysis.

Finally, in Section 3.5 we examine the effect of replacing the factors $1/|R_T|$ in the definition of $B(p,q)$ by general functions $\rho(T)$. We show that the resulting spaces $B_\rho(p,q)$ are weighted Wiener amalgam spaces on the multiplicative group.

Section 3.1. Rectangular limits.

The paper [**BBE**] extended the Wiener–Plancherel formula (0.2.3) to higher dimensions. The higher-dimensional version, (0.2.18), requires the use of special d-dimensional rectangular limits. It is the purpose of this section to define these rectangular limits, and to show that the spaces $B(p, \lim)$, consisting of functions for which the left-hand limit of (0.2.18) exists, are nonlinear, and therefore not conducive to the usual methods of functional analysis.

DEFINITION 3.1.1 [**BBE**]. Given a function $f: \mathbf{R}^d \to \mathbf{C}$ and given $z \in \mathbf{C}$.

a. We write $\lim_{t\to\infty} f(t) = z$ if $\lim_{r\in\mathbf{R}, r\to\infty} f(rc) = z$ for every $c \in \mathbf{S}_{d-1}\backslash\mathbf{A}_d$. That is, $f(t)$ converges to z along every ray from the origin to infinity except for those rays which lie in the coordinate hyperplanes.

b. We write $\mathrm{Glim}_{t\to\infty} f(t) = z$ if for each $\varepsilon > 0$ there exists a $T \in \mathbf{R}^d_+$ such that $|z - f(t)| < \varepsilon$ for all $t \notin R_T$. This is the natural definition of convergence for $\mathbf{R}^d$ considered as a locally compact group, and indicates convergence to z along every path whose points are eventually arbitrarily far from the origin.

c. We write $\mathrm{Ulim}_{t\to\infty} f(t) = z$ if for each $\varepsilon > 0$ there exists a $T \in \mathbf{R}^d_+$ such that $|z - f(t)| < \varepsilon$ for all $t \in \mathbf{R}^d$ such that $|t_j| > T_j$ for each j. The letter U stands for "unrestricted"; this notion plays a role in multi-dimensional Fourier series, cf., [**A**; **Zy**].

We make corresponding definitions for the limits as $t \to 0$, and make the obvious adjustments for f defined only on $\mathbf{R}^d_+$. For real-valued f we allow

$z = \pm\infty$.

It is clear that if $\text{Glim}\, f(t)$ exists then $\text{Ulim}\, f(t)$ will exist also, and if $\text{Ulim}\, f(t)$ exists then $\lim f(t)$ exists also. In one dimension, the three limits are identical. The following example shows they are distinct for $d \geq 2$.

EXAMPLE 3.1.2. Parts a and b are from [**BBE**].

a. Set $d = 2$ and $f = \chi_E$, where

$$E = \{(u,v) \in \mathbf{R}^2_+ : 0 < v < u^{1/2}\}.$$

Given $c \in \mathbf{S}_1$ we have $\lim_{r \in \mathbf{R}, r \to \infty} f(rc) = 0$, so $\lim_{t \to \infty} f(t)$ exists and is zero. However, $\text{Ulim}_{t \to \infty} f(t)$ does not exist. In fact, given $T \in \mathbf{R}^2_+$ we can find $s, t \in \mathbf{R}^2$ with $s, t > T$ such that $f(s) = 0$ while $f(t) = 1$.

b. Set $f = \chi_E$ where

$$E = \{t \in \mathbf{R}^d : |t_j| > 1 \text{ for all } j\}.$$

Then $\text{Ulim}_{t \to \infty} f(t) = 1$ although $\text{Glim}_{t \to \infty} f(t)$ does not exist.

c. Set $f = \chi_E$, where

$$E = \{t \in \mathbf{R}^d : 0 < t_1 < 1\}.$$

Then $\text{Ulim}_{t \to \infty} f(t) = 0$ although $\text{Glim}_{t \to \infty} f(t)$ does not exist.

DEFINITION 3.1.3. Given $f \in L^1_{\text{loc}}(\mathbf{R}^d)$ and a set $E \subset \mathbf{R}^d$ with finite measure, the **mean of** f **on** E is

$$M_E(f) = \frac{1}{|E|} \int_E f(t)\, dt.$$

If it exists, the (rectangular) **mean of** f is

$$M(f) = \lim_{T \to \infty} M_{R_T}(f).$$

EXAMPLE 3.1.4. We give examples of functions which do or do not possess means. One-dimensional versions of parts d and e appeared in [**Ba1**], of part f in [**HW**], and of part g in [**LL**].

a. If $f \in L^1(\mathbf{R}^d)$ then $M(f)$ exists and is zero. For, given $T \in \mathbf{R}^d_+$ we have

$$|M_{R_T}(f)| \leq \frac{1}{|R_T|} \int_{R_T} |f| \leq \frac{\|f\|_1}{|R_T|},$$

whence $\mathrm{Glim}_{T \to \infty} M_{R_T}(f) = 0$.

b. If $f \in L^1_{\mathrm{loc}}(\mathbf{R}^d)$ is P-periodic, where $P \in \mathbf{R}^d_+$, then

$$M(f) = \frac{1}{|I|} \int_I f(t)\, dt,$$

where $I \subset \mathbf{R}^d$ is any rectangle with side lengths P.

To see this, fix $T \in \mathbf{R}^d_+$, and let $N = N(T) \in \mathbf{Z}^d_+$ be the unique vector such that $NP \leq T < (N+1)P$. Note that

$$\frac{1}{|R_{NP}|} \int_{R_{NP}} f = \frac{1}{|I|} \int_I f$$

since f is P-periodic. Therefore,

$$\left| \frac{1}{|R_T|} \int_{R_T} f - \frac{1}{|I|} \int_I f \right|$$
$$= \left| \frac{1}{|R_T|} \int_{R_T} f - \frac{1}{|R_{NP}|} \int_{R_{NP}} f \right|$$

$$
\begin{aligned}
&\leq \left| \frac{1}{|R_T|}\int_{R_T} f - \frac{1}{|R_{NP}|}\int_{R_T} f \right| + \left| \frac{1}{|R_{NP}|}\int_{R_T} f - \frac{1}{|R_{NP}|}\int_{R_{NP}} f \right| \\
&\leq \frac{|R_T| - |R_{NP}|}{|R_T|\,|R_{NP}|}\int_{R_T} |f| + \frac{1}{|R_{NP}|}\int_{R_T\setminus R_{NP}} |f| \\
&\leq \frac{|R_{(N+1)P}| - |R_{NP}|}{|R_{NP}|\,|R_{NP}|}\int_{R_{(N+1)P}} |f| + \frac{1}{|R_{NP}|}\int_{R_{(N+1)P}\setminus R_{NP}} |f| \\
&= \frac{|R_{(N+1)P}| - |R_{NP}|}{|R_{NP}|}\,\frac{|R_{(N+1)P}|}{|R_{NP}|}\,\frac{1}{|R_{(N+1)P}|}\int_{R_{(N+1)P}} |f| \\
&\qquad + \frac{|R_{(N+1)P}|}{|R_{NP}|}\,\frac{1}{|R_{(N+1)P}|}\int_{R_{(N+1)P}} |f| - \frac{1}{|R_{NP}|}\int_{R_{NP}} |f| \\
&= C\,\frac{|R_{(N+1)P}| - |R_{NP}|}{|R_{NP}|}\,\frac{|R_{(N+1)P}|}{|R_{NP}|} + C\,\frac{|R_{(N+1)P}|}{|R_{NP}|} - C \\
&= C\,\frac{\Pi(N+1) - \Pi(N)}{\Pi(N)}\,\frac{\Pi(N+1)}{\Pi(N)} + C\,\frac{\Pi(N+1) - \Pi(N)}{\Pi(N)},
\end{aligned}
$$

where $C = \frac{1}{|I|}\int_I |f|$. Since

$$
\operatorname*{Glim}_{N\to\infty} \frac{\Pi(N+1) - \Pi(N)}{\Pi(N)} = 0 \quad \text{and} \quad \operatorname*{Glim}_{N\to\infty} \frac{\Pi(N+1)}{\Pi(N)} = 1,
$$

the result follows.

c. From part b and the fact that $E_b(t) = e^{2\pi i b\cdot t}$ is $1/b$-periodic, we have

$$
M(E_b) = |\Pi(b)| \int_{[0,1/b]} E_b = \delta_{0b}.
$$

d. The function $f(t) = |\Pi(t)|$ does not have a mean, since $\frac{1}{|R_T|}\int_{R_T} f = |\Pi(T/2)|$.

e. The function $f(t) = |\Pi(t)|^i$ is bounded, yet does not possess a mean, since $\frac{1}{|R_T|}\int_{R_T} f = |\Pi(T)|^i/(i+1)^d$. Note, however, that $M(|f|^p)$ does exist for all $p > 0$ since $|f| \equiv 1$.

f. By part b, any $f \in L^1_{\mathrm{loc}}(\mathbf{R}^d)$ which is periodic possesses a mean, even though it need not be bounded. All bounded periodic functions possess means.

g. Let $\{t_n\}_{n \in \mathbf{Z}}$ be any sequence of positive real numbers strictly increasing to infinity which satisfies

$$\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = 0.$$

Set $t_0 = 0$ and let

$$E = \{x \in \mathbf{R}^d_+ : t_{2n} \leq x_1 < t_{2n+1} \text{ for some } n \geq 0\}.$$

Then the function $f = \chi_E$ does not possess a mean, despite the fact that it is bounded and takes only the values 0 and 1.

To see this, fix $\mathbf{c} \in \mathbf{S}^+_{d-1}$ and define $T_n = t_n \mathbf{c}$. Note that $R_{T_1} \subset R_{T_2} \subset \cdots$, and $|R_{T_n}| = (2t_n)^d \, |\Pi(\mathbf{c})|$. Therefore,

$$\begin{aligned}
\frac{1}{|R_{T_{2n}}|} \int_{R_{T_{2n}}} f &\leq \frac{1}{|R_{T_{2n}}|} \int_{R_{T_{2n-1}}} 1 \\
&= \frac{|R_{T_{2n-1}}|}{|R_{T_{2n}}|} \\
&= \left(\frac{t_{2n-1}}{t_{2n}}\right)^d \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

However,

$$\begin{aligned}
\frac{1}{|R_{T_{2n+1}}|} \int_{R_{T_{2n+1}}} f &\geq \frac{1}{|R_{T_{2n+1}}|} \int_{R_{T_{2n+1}} \setminus R_{T_{2n}}} 1 \\
&= \frac{|R_{T_{2n+1}}| - |R_{T_{2n}}|}{|R_{T_{2n+1}}|}
\end{aligned}$$

$$= 1 - \left(\frac{t_{2n}}{t_{2n+1}}\right)^d$$

$$\to 1 \quad \text{as } n \to \infty.$$

Therefore $M(f)$ does not exist. Also, $M(|f|^p)$ does not exist for any p since $|f| \equiv f$.

h. Let f be as in part g, and set $h = 1/2$ and $g = f - h$. Since M is linear and $M(h)$ exists while $M(f)$ does not, we conclude that $M(g)$ does not exist. However, $|g| \equiv 1/2$, so $M(|g|^p)$ exists for all p, even though $M(|f|^p)$ does not.

EXAMPLE 3.1.5. We show that

$$B(p, \lim) = \{f \in L^p_{\text{loc}}(\mathbf{R}^d) : \lim_{T\to\infty} \frac{1}{|R_T|} \int_{R_T} |f(t)|^p \, dt \text{ exists}\}$$

is nonlinear.

a. Let f, g, and h be as in Example 3.1.4g. Since $|g| = |h| \equiv 1/2$ we have $g, h \in B(p, \lim)$. However, $M(f) = M(|f|^p)$ does not exist, so $f = g + h \notin B(p, \lim)$.

b. For $p = 2$ we give another example, whose one-dimensional version appeared in [**Ba1**] and [**HW**].

Let $f \in B(2, \lim)$ be any function such that $M(f)$ does not exist (see Example 3.1.4e for a complex-valued example, or Example 3.1.4g for a real-valued example). Given any $T \in \mathbf{R}^d_+$ we have

$$M_{R_T}(|1 + f|^2) = M_{R_T}(1) + 2\,\text{Re}(M_{R_T}(f)) + M_{R_T}(|f|^2).$$

Now, $M(1)$ and $M(|f|^2)$ both exist while $M(f)$ does not. Therefore $1 + f \notin B(2, \lim)$ even though $1, f \in B(2, \lim)$.

Although nonlinear, $B(p, \lim)$ is a large space. For example, it contains $L^p(\mathbf{R}^d)$ and all periodic functions which are integrable over their periods, including all constant functions. Examples 3.1.4e and f show that $B(p, \lim) \setminus L^\infty(\mathbf{R}^d) \neq \emptyset$ and $L^\infty(\mathbf{R}^d) \setminus B(p, \lim) \neq \emptyset$.

REMARK 3.1.6. The original Wiener–Plancherel formula, (0.2.3), was proved by Wiener for functions in $B(p, \lim)$, in one dimension. Because $B(p, \lim)$ is nonlinear, Lau and his colleagues extended the Wiener transform to larger spaces. In [**LL**], where they proved that $B(p, \lim)$ is nonlinear, Lau and Lee proved (also for $d = 1$) that the Wiener transform W is a topological isomorphism of the *Marcinkiewicz space* $B(2, \limsup)$ onto the variation space $V(2, \limsup)$, where $B(p, \limsup)$ and $V(p, \limsup)$ are as defined in (0.2.5) and (0.2.6), respectively. Of course, $B(2, \limsup) \supset B(2, \lim)$, and, by the Wiener–Plancherel formula, the Wiener transform is an isometry when restricted to $B(2, \lim)$. However, Lau and Lee proved that W is not an isometry on all of $B(2, \limsup)$, not even on the linear span of $B(2, \lim)$ in $B(2, \limsup)$.

Following the Lau and Lee results in [**LL**], Lau and Chen proved in [**CL1**] that it is also possible to extend the Wiener transform W from $B(2, \lim)$ to $B(2, \infty)$, and that W is a topological isomorphism of $B(2, \infty)$ onto $V(2, \infty)$, where $B(p, \infty)$ and $V(p, \infty)$ are as defined in (0.2.7) and (0.2.8), respectively. This result forms one cornerstone for our results in Chapter 4, for we prove there that W is in fact a topological isomorphism of a whole range of spaces $B(2, q)$ onto $V(2, q)$ for $1 \leq q \leq \infty$. Moreover, we do this in higher dimensions.

A goal for future research is to extend the Lau and Lee results for $B(2, \limsup)$ to higher dimensions as well. As a step in this direction, we make a few remarks on the definition of d-dimensional rectangular limsups.

DEFINITION 3.1.7. Given a real-valued function $f: \mathbf{R}^d \to \mathbf{R}$.

a. $\limsup_{t\to\infty} f(t) = \sup_{c\in \mathbf{S}_{d-1}} \limsup_{r\in\mathbf{R}, r\to\infty} f(rc)$.

b. $\text{Glimsup}_{t\to\infty} f(t) = \inf_{T\in\mathbf{R}^d_+} \sup_{t\notin R_T} f(t)$.

c. $\text{Ulimsup}_{t\to\infty} f(t) = \inf_{T\in\mathbf{R}^d_+} \sup_{t\in\mathbf{R}^d, |t_j|>T_j} f(t)$.

We make corresponding definitions for liminfs, for $t \to 0$, for $f: \mathbf{R}^d_+ \to \mathbf{R}$, etc.

Note that the numbers defined above always exist in the extended real sense, i.e., $-\infty \le \limsup f \le \infty$. Given $f: \mathbf{R}^d \to \mathbf{R}$ we have

$$\text{Gliminf } f \le \text{Uliminf } f \le \liminf f$$
$$\le \limsup f \le \text{Ulimsup } f \le \text{Glimsup } f.$$

However, these are not equalities in general, cf., Example 3.1.8. Also, it is clear that

$$\lim_{t\to\infty} f(t) \text{ exists} \Leftrightarrow \liminf_{t\to\infty} f(t) = \limsup_{t\to\infty} f(t),$$
$$\operatorname*{Glim}_{t\to\infty} f(t) \text{ exists} \Leftrightarrow \operatorname*{Gliminf}_{t\to\infty} f(t) = \operatorname*{Glimsup}_{t\to\infty} f(t),$$
$$\operatorname*{Ulim}_{t\to\infty} f(t) \text{ exists} \Leftrightarrow \operatorname*{Uliminf}_{t\to\infty} f(t) = \operatorname*{Ulimsup}_{t\to\infty} f(t).$$

EXAMPLE 3.1.8. a. Let f be as in Example 3.1.2a. Then $\liminf f = \limsup f = 0$, $\text{Uliminf} f = 0$, $\text{Ulimsup} f = 1$, $\text{Gliminf} f = 0$, $\text{Glimsup} f = 1$.

b. Let f be as in Example 3.1.2b. Then $\liminf f = \limsup f = 1$, $\text{Uliminf} f = \text{Ulimsup} f = 1$, $\text{Gliminf} f = 0$, $\text{Glimsup} f = 1$.

c. Let f be as in Example 3.1.2c. Then $\liminf f = \limsup f = 0$, $\text{Uliminf} f = \text{Ulimsup} f = 0$, $\text{Gliminf} f = 0$, $\text{Glimsup} f = 1$.

Section 3.2. Equivalence with Wiener amalgam spaces.

In [**F4**], Feichtinger derived an equivalent norm for $B(p,\infty)$ based on dyadic decompositions of $\mathbf{R}$ (in fact, this was done in higher dimensions, but with a spherical approach, rather than the rectangular approach of this thesis). Essentially, he proved that $B(p,\infty) = W_*(L^p, L^\infty)$, under equivalent norms. We prove and extend this equality in this section, namely, we show that $B(p,q) = W_*(L^p, L^q)$ for all p, q, and do this in higher dimensions with a rectangular approach.

We adopt the discrete norm for $W_*(L^p, L^q)$ defined in Example 2.4.6 as standard, i.e., we take $\{\chi_{\sigma[2^n,2^{n+1}]}\}_{n\in\mathbf{Z}^d,\sigma\in\Omega^d}$ as a standard BUPU, with the result that

$$(3.2.1) \qquad \|f\|_{W_*(L^p,L^q)} = \left(\sum_{n\in\mathbf{Z}^d,\sigma\in\Omega^d}\left(\int_{\sigma[2^n,2^{n+1}]}|f(t)|^p\,\frac{dt}{|\Pi(t)|}\right)^{q/p}\right)^{1/q},$$

the standard adjustments being made if p or q is infinity.

DEFINITION 3.2.1. Given $1 \le p,\ q < \infty$, the **Besicovitch space** $B(p,q)$ is the space of functions $f\colon \mathbf{R}^d \to \mathbf{C}$ for which the norm

$$\|f\|_{B(p,q)} = \left(\int_{\mathbf{R}^d_+}\left(\frac{1}{|R_T|}\int_{R_T}|f(t)|^p\,dt\right)^{q/p}\frac{dT}{\Pi(T)}\right)^{1/q}$$

is finite. The standard adjustments are made if p or q is infinity, namely,

$$\|f\|_{B(p,\infty)} = \operatorname*{ess\,sup}_{T\in\mathbf{R}^d_+}\left(\frac{1}{|R_T|}\int_{R_T}|f(t)|^p\,dt\right)^{1/p},$$

$$\|f\|_{B(\infty,q)} = \left(\int_{\mathbf{R}^d_+}\left(\operatorname*{ess\,sup}_{t\in R_T}|f(t)|\right)^q\frac{dT}{\Pi(T)}\right)^{1/q},$$

$$\|f\|_{B(\infty,\infty)} = \operatorname*{ess\ sup}_{T\in\mathbf{R}^d_+}\left(\operatorname*{ess\ sup}_{t\in R_T}|f(t)|\right).$$

That $\|\cdot\|_{B(p,q)}$ is a norm is evident. It follows from Theorem **3.2.4** that $B(p,q)$ is a Banach space.

Our characterization of $B(p,q)$ as a Wiener amalgam space begins with the easiest case, namely, $p = q$.

PROPOSITION 3.2.2. *Given* $1 \le p \le \infty$,

$$B(p,p) = L^p(\mathbf{R}^d_*) = W_*(L^p, L^p),$$

with

$$\|\cdot\|_{B(p,p)} = 2^{-d/p}\,\|\cdot\|_{L^p(\mathbf{R}^d_*)} = 2^{-d/p}\,\|\cdot\|_{W_*(L^p,L^p)}.$$

PROOF: The case $p = \infty$ is clear, so assume $1 \le p < \infty$. The second equality is trivial, since

$$\begin{aligned}\|f\|_{W_*(L^p,L^p)} &= \left(\sum_{n,\sigma}\left(\int_{\sigma[2^n,2^{n+1}]}|f(t)|^p\,\frac{dt}{|\Pi(t)|}\right)^{p/p}\right)^{1/p}\\ &= \left(\int_{\mathbf{R}^d_*}|f(t)|^p\,\frac{dt}{|\Pi(t)|}\right)^{1/p}\\ &= \|f\|_{L^p}.\end{aligned}$$

For the first equality, compute

$$\begin{aligned}\|f\|^p_{B(p,p)} &= \int_{\mathbf{R}^d_+}\left(\frac{1}{|R_T|}\int_{R_T}|f(t)|^p\,dt\right)^{p/p}\frac{dT}{\Pi(T)}\\ &= \int_{\mathbf{R}^d_+}\frac{1}{2^d\,\Pi(T)}\int_{R_T}|f(t)|^p\,dt\,\frac{dT}{\Pi(T)}\\ &= 2^{-d}\int_0^\infty\cdots\int_0^\infty\int_{-T_d}^{T_d}\cdots\int_{-T_1}^{T_1}|f(t)|^p\,dt_1\cdots dt_d\,\frac{dT_1}{{T_1}^2}\cdots\frac{dT_d}{{T_d}^2}\end{aligned}$$

$$= 2^{-d} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |f(t)|^p \int_{|t_d|}^{\infty} \cdots \int_{|t_1|}^{\infty} \frac{dT_1}{T_1{}^2} \cdots \frac{dT_d}{T_d{}^2}\, dt_1 \cdots dt_d$$

$$= 2^{-d} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |f(t)|^p \frac{1}{|t_1|} \cdots \frac{1}{|t_d|}\, dt_1 \cdots dt_d$$

$$= 2^{-d} \int_{\mathbf{R}_*^d} |f(t)|^p \frac{dt}{|\Pi(t)|}$$

$$= 2^{-d}\, \|f\|_{L^p}^p. \blacksquare$$

LEMMA 3.2.3. *Given* $\alpha \in \mathbf{R}_+^d$.

a. $\displaystyle\sum_{k \in \mathbf{Z}_+^d} \Pi(2^{-k\alpha}) = \Pi\Big(\frac{1}{2^\alpha - 1}\Big).$

b. $\displaystyle\sum_{k \in \mathbf{Z}_+^d} \Pi(2^{-(k-1)\alpha}) = \Pi\Big(\frac{2^\alpha}{2^\alpha - 1}\Big).$

PROOF: We compute

$$\sum_{k \in \mathbf{Z}_+^d} \Pi(2^{-k\alpha}) = \sum_{k_d \in \mathbf{Z}_+} \cdots \sum_{k_1 \in \mathbf{Z}_+} 2^{-k_1\alpha_1} \cdots 2^{-k_d\alpha_d}$$

$$= \prod_{j=1}^{d} \sum_{k_j \in \mathbf{Z}_+} 2^{-k_j\alpha_j}$$

$$= \prod_{j=1}^{d} \frac{1}{2^{\alpha_j} - 1}.$$

The second statement is similar. $\blacksquare$

The following is the main result of this chapter, in which we characterize $B(p,q)$ as a Wiener amalgam space. The bounds given for the norm equivalence in Theorem 3.2.4 are not sharp, cf., Remark 3.2.5.

THEOREM 3.2.4. *Given* $1 \le p < \infty$ *and* $1 \le q \le \infty$,

$$B(p,q) = W_*(L^p, L^q),$$

with equivalence of norms given by

$$C\,\|\cdot\|_{W_*(L^p,L^q)} \;\leq\; \|\cdot\|_{B(p,q)} \;\leq\; D\,\|\cdot\|_{W_*(L^p,L^q)}, \tag{3.2.2}$$

where

$$C \;=\; (\log 2)^{d/q}\, 2^{-(\frac{1}{q}+\frac{3}{p})d},$$
$$D \;=\; (\log 2)^{d/q} \begin{cases} 2^{2d/p}, & p<q,\\ \left(\frac{2^{q/p}}{2^{q/p}-1}\right)^{d/q}, & p\geq q.\end{cases}$$

PROOF: Assume for simplicity that $1 \leq p,\, q < \infty$ (the $q = \infty$ case is similar).

a. Fix any $\sigma \in \Omega^d$. Then we compute

$$\begin{aligned}
\|f\|^q_{B(p,q)} &= \int_{\mathbf{R}^d_+} \left(\frac{1}{|R_T|}\int_{R_T} |f(t)|^p\,dt\right)^{q/p} \frac{dT}{\Pi(T)}\\
&= \sum_n \int_{[2^n,2^{n+1}]} \left(\frac{1}{2^d\,\Pi(T)}\int_{R_T} |f(t)|^p\,dt\right)^{q/p} \frac{dT}{\Pi(T)}\\
&\geq \sum_n \int_{[2^n,2^{n+1}]} \left(\frac{1}{2^d\,\Pi(2^{n+1})}\int_{R_{2^n}} |f(t)|^p\,dt\right)^{q/p} \frac{dT}{\Pi(T)}\\
&= \sum_n (\log 2)^d \left(\frac{1}{2^{3d}\,\Pi(2^{n-1})}\int_{R_{2^n}} |f(t)|^p\,dt\right)^{q/p}\\
&\geq (\log 2)^d\, 2^{-3dq/p} \sum_n \left(\int_{\sigma[2^{n-1},2^n]} |f(t)|^p\,\frac{dt}{\Pi(2^{n-1})}\right)^{q/p}\\
&\geq (\log 2)^d\, 2^{-3dq/p} \sum_n \left(\int_{\sigma[2^{n-1},2^n]} |f(t)|^p\,\frac{dt}{|\Pi(t)|}\right)^{q/p},
\end{aligned}$$

where the summations in n run over $\mathbf{Z}^d$. Therefore,

$$\begin{aligned}
2^d\,\|f\|^q_{B(p,q)} &= \sum_{\sigma\in\Omega^d} \|f\|^q_{B(p,q)}\\
&\geq (\log 2)^d\, 2^{-3dq/p} \sum_{n,\sigma} \left(\int_{\sigma[2^{n-1},2^n]} |f(t)|^p\,\frac{dt}{|\Pi(t)|}\right)^{q/p}
\end{aligned}$$

$$= (\log 2)^d\, 2^{-3dq/p}\, \|f\|^q_{W_*(L^p,L^q)},$$

from which the first inequality in (3.2.2) follows.

b. Note that

$$R_{2^{n+1}} = \bigcup_{m\in\mathbf{Z}^d_+,\tau\in\Omega^d} \tau[2^{n-m+1}, 2^{n-m+2}].$$

Therefore,

$$\begin{aligned}
&\|f\|^q_{B(p,q)} \\
&= \int_{\mathbf{R}^d_+} \left(\frac{1}{|R_T|}\int_{R_T} |f(t)|^p\,dt\right)^{q/p} \frac{dT}{\Pi(T)} \\
&= \sum_n \int_{[2^n,2^{n+1}]} \left(\frac{1}{2^d\,\Pi(T)}\int_{R_T} |f(t)|^p\,dt\right)^{q/p} \frac{dT}{\Pi(T)} \\
&\le \sum_n \int_{[2^n,2^{n+1}]} \left(\frac{1}{2^d\,\Pi(2^n)}\int_{R_{2^{n+1}}} |f(t)|^p\,dt\right)^{q/p} \frac{dT}{\Pi(T)} \\
&= \sum_n (\log 2)^d \left(\frac{1}{\Pi(2^{n+1})}\int_{R_{2^{n+1}}} |f(t)|^p\,dt\right)^{q/p} \\
&= (\log 2)^d \sum_n \left(\frac{1}{\Pi(2^{n+1})}\sum_{m,\tau}\int_{\tau[2^{n-m+1},2^{n-m+2}]} |f(t)|^p\,dt\right)^{q/p} \\
&\le (\log 2)^d \sum_n \left(\sum_{m,\tau}\frac{\Pi(2^{n-m+2})}{\Pi(2^{n+1})}\int_{\tau[2^{n-m+1},2^{n-m+2}]} |f(t)|^p\,\frac{dt}{|\Pi(t)|}\right)^{q/p} \\
&= (\log 2)^d \sum_n \left(\sum_{m,\tau}\frac{1}{\Pi(2^{m-1})}\int_{\tau[2^{n-m+1},2^{n-m+2}]} |f(t)|^p\,\frac{dt}{|\Pi(t)|}\right)^{q/p} \\
&= (\log 2)^d \sum_n \left|\sum_{m,\tau} F_{m,\tau}(n)\right|^{q/p} \\
&= (\log 2)^d \left\|\sum_{m,\tau} F_{m,\tau}\right\|^{q/p}_{\ell^{q/p}},
\end{aligned}$$

where $F_{m,\tau}$ is the sequence

$$F_{m,\tau}(n) = \frac{1}{\Pi(2^{m-1})} \int_{\tau[2^{n-m+1}, 2^{n-m+2}]} |f(t)|^p \frac{dt}{|\Pi(t)|}.$$

c. Assume $p \leq q$, i.e., $q/p \geq 1$. Then we may apply Minkowski's inequality in the Banach space $\ell^{q/p}$ to the calculation in part b. The summations in the following calculation are over $m \in \mathbf{Z}_+^d$, $n \in \mathbf{Z}^d$, and $\sigma, \tau \in \Omega^d$.

$$\begin{aligned}
&\Big\|\sum_{m,\tau} F_{m,\tau}\Big\|_{\ell^{q/p}} \\
&\quad \leq \sum_{m,\tau} \|F_{m,\tau}\|_{\ell^{q/p}} \\
&\quad = \sum_{m,\tau} \Big(\sum_{n} |F_{m,\tau}(n)|^{q/p}\Big)^{p/q} \\
&\quad = \sum_{m,\tau} \bigg(\sum_{n} \bigg(\frac{1}{\Pi(2^{m-1})} \int_{\tau[2^{n-m+1}, 2^{n-m+2}]} |f(t)|^p \frac{dt}{|\Pi(t)|}\bigg)^{q/p}\bigg)^{p/q} \\
&\quad = \sum_{m,\tau} \frac{1}{\Pi(2^{m-1})} \bigg(\sum_{n} \bigg(\int_{\tau[2^n, 2^{n+1}]} |f(t)|^p \frac{dt}{|\Pi(t)|}\bigg)^{q/p}\bigg)^{p/q} \\
&\quad \leq \sum_{m,\tau} \frac{1}{\Pi(2^{m-1})} \bigg(\sum_{n,\sigma} \bigg(\int_{\sigma[2^n, 2^{n+1}]} |f(t)|^p \frac{dt}{|\Pi(t)|}\bigg)^{q/p}\bigg)^{p/q} \\
&\quad = \sum_{m,\tau} \frac{1}{\Pi(2^{m-1})} \|f\|^p_{W_*(L^p, L^q)} \\
&\quad = 2^{2d} \|f\|^p_{W_*(L^p, L^q)},
\end{aligned}$$

since the summation in m is over $\mathbf{Z}_+^d$. The second inequality in (3.2.2) therefore follows for this case.

d. Finally, assume $q \leq p$. Since $0 < q/p \leq 1$, we may apply the triangle inequality in the metric space $\ell^{q/p}$ to the calculation in part b (cf., Section 1.7f).

The summations in the following calculation are over $m \in \mathbf{Z}_+^d$, $n \in \mathbf{Z}^d$, and $\sigma, \tau \in \Omega^d$.

$$
\begin{aligned}
\Big\|\sum_{m,\tau} F_{m,\tau}\Big\|_{\ell^{q/p}}^{q/p}, \\
&\leq \sum_{m,\tau} \|F_{m,\tau}\|_{\ell^{q/p}}^{q/p} \\
&= \sum_{m,\tau} \sum_{n} |F_{m,\tau}(n)|^{q/p} \\
&= \sum_{m,\tau} \sum_{n} \left(\frac{1}{\Pi(2^{m-1})} \int_{\tau[2^{n-m+1}, 2^{n-m+2}]} |f(t)|^p \, \frac{dt}{|\Pi(t)|} \right)^{q/p} \\
&= \sum_{m} \sum_{n,\tau} \left(\frac{1}{\Pi(2^{m-1})} \int_{\tau[2^{n-m+1}, 2^{n-m+2}]} |f(t)|^p \, \frac{dt}{|\Pi(t)|} \right)^{q/p} \\
&= \sum_{m} \frac{1}{\Pi(2^{m-1})^{q/p}} \sum_{n,\tau} \left(\int_{\tau[2^{n-m+1}, 2^{n-m+2}]} |f(t)|^p \, \frac{dt}{|\Pi(t)|} \right)^{q/p} \\
&= \sum_{m} \frac{1}{\Pi(2^{m-1})^{q/p}} \sum_{n,\tau} \left(\int_{\tau[2^{n}, 2^{n+1}]} |f(t)|^p \, \frac{dt}{|\Pi(t)|} \right)^{q/p} \\
&= \sum_{m} \frac{1}{\Pi(2^{m-1})^{q/p}} \|f\|_{W_*(L^p, L^q)}^q \\
&= \left(\frac{2^{q/p}}{2^{q/p} - 1} \right)^d \|f\|_{W_*(L^p, L^q)}^q,
\end{aligned}
$$

where the last equality follows from Lemma 3.2.3 and the fact that the summation in m is over $\mathbf{Z}_+^d$. The second inequality in (3.2.2) therefore follows for this case. ∎

REMARK 3.2.5. The bounds for the norm equivalence given in Theorem 3.2.4 are, in general, not the best possible.

For example, for the case $p = q$ we can compare the exact bounds determined in Proposition 3.2.2, namely,

$$\|\cdot\|_{B(p,p)} = 2^{-d/p}\|\cdot\|_{W_*(L^p,L^p)},$$

to the approximate bounds given in Theorem 3.2.4, i.e.,

$$(2^{-4}\log 2)^{d/p}\|\cdot\|_{W_*(L^p,L^p)} \leq \|\cdot\|_{B(p,p)} \leq (2\log 2)^{d/p}\|\cdot\|_{W_*(L^p,L^p)}.$$

Since

$$(2^{-4}\log 2)^{d/p} < 2^{-d/p} < (2\log 2)^{d/p},$$

we conclude that the bounds in Theorem 3.2.4 are not best possible.

REMARK 3.2.6. Our recognition of $B(p,q)$ as the Wiener space $W_*(L^p, L^q)$ immediately provides us with inclusion and duality relations.

a. *Inclusions.* From Corollary 2.4.13,

$$p_1 \geq p_2,\ q_1 \leq q_2 \quad \Rightarrow \quad B(p_1,q_1) \subset B(p_2,q_2).$$

From Proposition 2.3.2 (cf., Proposition 3.2.2),

$$B(p,p) = W_*(L^p,L^p) = L^p(\mathbf{R}^d_*).$$

Therefore,

$$p \leq q \quad \Rightarrow \quad B(p,q) \supset L^p(\mathbf{R}^d_*) \cap L^q(\mathbf{R}^d_*)$$

and

$$p \geq q \quad \Rightarrow \quad B(p,q) \subset L^p(\mathbf{R}^d_*) \cup L^q(\mathbf{R}^d_*),$$

cf., Proposition 2.3.6.

b. *Dilation invariance.* By Proposition 2.2.8, $W_*(L^p, L^q)$ is dilation invariant, i.e., $\|D_\lambda f\|_{W_*(L^p,L^q)} \sim \|f\|_{W_*(L^p,L^q)}$ for each $\lambda \in \mathbf{R}^d_*$, where D_λ is the dilation operator $D_\lambda f(t) = f(t/\lambda)$. In fact, $B(p,q)$ is dilation isometric, since

$$\begin{aligned}
\|D_\lambda f\|^q_{B(p,q)} &= \int_{\mathbf{R}^d_+} \left(\frac{1}{|R_T|} \int_{R_T} |f(t/\lambda)|^p \, dt \right)^{q/p} \frac{dT}{\Pi(T)} \\
&= \int_{\mathbf{R}^d_+} \left(\frac{1}{|R_T|} \int_{R_{\lambda T}} |f(t)|^p \, \frac{dt}{|\Pi(\lambda)|} \right)^{q/p} \frac{dT}{\Pi(T)} \\
&= \int_{\mathbf{R}^d_+} \left(\frac{1}{|R_{\lambda T}|} \int_{R_{\lambda T}} |f(t)|^p \, dt \right)^{q/p} \frac{dT}{\Pi(T)} \\
&= \int_{\mathbf{R}^d_+} \left(\frac{1}{|R_T|} \int_{R_T} |f(t)|^p \, dt \right)^{q/p} \frac{dT}{\Pi(T)} \\
&= \|f\|^q_{B(p,q)}.
\end{aligned}$$

c. *Duality.* From Theorem 2.5.1, if $1 \le p,\ q < \infty$ then

$$B(p,q)' = B(p',q'),$$

with duality given by

$$\langle f, g \rangle = \int_{\mathbf{R}^d_*} f(t)\,\overline{g(t)}\,\frac{dt}{|\Pi(t)|}.$$

Since the norm in $B(p,q)$ is only equivalent to the norm in $W_*(L^p, L^q)$, we can conclude only that the norm in $B(p,q)'$ is equivalent to the norm in $B(p',q')$. The following computation shows that the canonical norm for $B(p,q)'$ is a constant multiple of the norm for $B(p',q')$. Given $f \in B(p,q)$ and $g \in B(p',q')$,

$$
\begin{aligned}
&\|f\|_{B(p,q)}\,\|g\|_{B(p',q')} \\
&\quad = \left(\int_{\mathbf{R}^d_+}\left(\frac{1}{|R_T|}\int_{R_T}|f(t)|^p\,dt\right)^{q/p}\frac{dT}{\Pi(T)}\right)^{1/q} \\
&\qquad\quad \times \left(\int_{\mathbf{R}^d_+}\left(\frac{1}{|R_T|}\int_{R_T}|g(t)|^{p'}\,dt\right)^{q'/p'}\frac{dT}{\Pi(T)}\right)^{1/q'} \\
&\quad \geq \int_{\mathbf{R}^d_+}\left(\frac{1}{|R_T|}\int_{R_T}|f(t)|^p\,dt\right)^{1/p}\left(\frac{1}{|R_T|}\int_{R_T}|g(t)|^{p'}\,dt\right)^{1/p'}\frac{dT}{\Pi(T)} \\
&\quad \geq \int_{\mathbf{R}^d_+}\frac{1}{|R_T|}\int_{R_T}|f(t)\,g(t)|\,dt\,\frac{dT}{\Pi(T)} \\
&\quad = \|fg\|_{B(1,1)} \\
&\quad = 2^{-d}\,\|fg\|_{L^1(\mathbf{R}^d_*)} \\
&\quad \geq 2^{-d}\left|\int_{\mathbf{R}^d_*} f(t)\,\overline{g(t)}\,\frac{dt}{|\Pi(t)|}\right| \\
&\quad = 2^{-d}\,|\langle f,g\rangle|.
\end{aligned}
$$

The norm for $B(p,q)'$ would be equal to the norm for $B(p',q')$ if we defined the duality by

$$
\langle f,g\rangle = \int_{\mathbf{R}^d_+}\frac{1}{|R_T|}\int_{R_T} f(t)\,\overline{g(t)}\,dt\,\frac{dT}{\Pi(T)},
$$

i.e., duality according to the norm for $B(1,1)$.

REMARK 3.2.7. Although the sets $W_*(L^p,L^q)$ and $B(p,q)$ coincide by Theorem 3.2.4, they have distinct, albeit equivalent, norms. We retain this distinction in the remainder of this thesis, stating results in terms of $W_*(L^p,L^q)$ when we intend to use the discrete norm, or in terms of $B(p,q)$ when we intend to use the norm for that space. If the norm is not important, we refer to the space as $B(p,q)$.

We close this section with a few remarks about the space $B(p,\infty)$.

LEMMA 3.2.8. *Given $0 < p < \infty$ and $b \in \mathbf{R}^d_*$, we have $E_b - 1 \in B(p,\lim) \subset B(p,\infty)$, with*

$$\|E_b - 1\|_{B(p,\infty)} \geq 2^{d(\frac{1}{2}-\frac{1}{p})}.$$

PROOF: Without loss of generality assume $b \in \mathbf{R}^d_+$. Since $E_b - 1$ is $1/b$-periodic and bounded it is an element of $B(p,\lim)$ by Example 3.1.4b. Moreover, that example also implies that

$$\begin{aligned}
\|E_b - 1\|^p_{B(p,\infty)} &\geq \lim_{T\to\infty} \frac{1}{|R_T|} \int_{R_T} |E_b(t) - 1|^p \, dt \\
&= \frac{1}{|[0,1/b]|} \int_{[0,1/b]} |E_b(t) - 1|^p \, dt \\
&= \int_{[0,1]} |E_1(t) - 1|^p \, dt \\
&= \prod_{j=1}^{d} \int_0^1 |e^{2\pi i t_j} - 1|^p \, dt_j \\
&\geq \prod_{j=1}^{d} \int_{1/4}^{3/4} |e^{2\pi i t_j} - 1|^p \, dt_j \\
&\geq \prod_{j=1}^{d} \int_{1/4}^{3/4} (\sqrt{2})^p \, dt_j \\
&= 2^{(\frac{p}{2}-1)d}. \ \blacksquare
\end{aligned}$$

PROPOSITION 3.2.9. *$B(p,\infty)$ is not separable for $1 \leq p \leq \infty$.*

PROOF: The case $p = \infty$ follows from the fact $B(\infty,\infty) = L^\infty$. For $p < \infty$, we have by Lemma 3.2.8 that

$$\|E_a - E_b\|_{B(p,\infty)} = \|E_{a-b} - 1\|_{B(p,\infty)} \geq 2^{d(\frac{1}{2}-\frac{1}{p})} > 0,$$

if $a \neq b$. Thus $\{E_b\}_{b\in\mathbf{R}^d}$ is an uncountable separated set in $B(p,\infty)$. ∎

The statement and proof of the following result is adapted from the one-dimensional version presented in [**Ba1**].

Proposition 3.2.10. *Given $1 \leq p < \infty$, $B(p,\lim)$ is a proper, closed, nonlinear subset of $B(p,\infty)$. Moreover, if $\{f_n\}_{n\in\mathbf{Z}_+} \subset B(p,\lim)$, $f \in B(p,\infty)$, and $f_n \to f$ in $B(p,\infty)$, then $f \in B(p,\lim)$ and*

$$M(|f|^p) = \lim_{n\to\infty} M(|f_n|^p),$$

where M is the mean value operator of Definition 3.1.3.

Proof: Clearly $B(p,\lim) \subset B(p,\infty)$, and is nonlinear by Example 3.1.5. In Example 3.1.4 we showed that $B(p,\lim) \setminus L^\infty(\mathbf{R}^d) \neq \emptyset$ and $L^\infty(\mathbf{R}^d) \setminus B(p,\lim) \neq \emptyset$. Since both $B(p,\lim)$ and $L^\infty(\mathbf{R}^d)$ are contained in $B(p,\infty)$, $B(p,\lim)$ must be a proper subset of $B(p,\infty)$.

Now assume that $f_n \in B(p,\lim)$ and $f_n \to f \in B(p,\infty)$. Set $M_n = M(|f_n|^p)$, and note that

$$\begin{aligned} |M_m - M_n| &\leq \lim_{T\to\infty} M_{R_T}(|f_m - f_n|^p) \\ &\leq \sup_{T\in\mathbf{R}^d_+} M_{R_T}(|f_m - f_n|^p) \\ &= \|f_m - f_n\|^p_{B(p,\infty)} \\ &\to 0 \quad \text{as } m,n\to\infty, \end{aligned}$$

where M_{R_T} is the mean value on R_T. M_n must therefore converge to some number M as $n \to \infty$.

By hypothesis,

$$\varepsilon_n = \sup_{T \in \mathbf{R}_+^d} M_{R_T}(|f - f_n|^p) = \|f - f_n\|_{B(p,\infty)}^p \to 0 \quad \text{as } n \to \infty.$$

Therefore, for $T \in \mathbf{R}_+^d$,

$$M_{R_T}(|f_n|^p) - \varepsilon_n \leq M_{R_T}(|f|^p) \leq M_{R_T}(|f_n|^p) + \varepsilon_n.$$

Thus,

$$\begin{aligned} M_n - \varepsilon_n &= M(|f_n|^p) - \varepsilon_n \\ &= \liminf_{T \to \infty} M_{R_T}(|f_n|^p) - \varepsilon_n \\ &\leq \liminf_{T \to \infty} M_{R_T}(|f|^p) \\ &\leq \limsup_{T \to \infty} M_{R_T}(|f|^p) \\ &\leq \limsup_{T \to \infty} M_{R_T}(|f_n|^p) + \varepsilon_n \\ &= M(|f_n|^p) + \varepsilon_n \\ &= M_n + \varepsilon_n, \end{aligned}$$

where the liminfs and limsups are the d-dimensional versions defined in Section 3.1. Letting $n \to \infty$, it follows that $M(|f|^p) = \lim_{T\to\infty} M_{R_T}(|f|^p)$ exists and equals $M = \lim_{n\to\infty} M_n = \lim_{n\to\infty} M(|f_n|^p)$. ∎

EXAMPLE 3.2.11. Hartman and Wintner **[HW]** gave the following example (for $d = 1$) of functions $\{f_n\}_{n\in\mathbf{Z}_+} \subset B(2, \lim) \cap L^\infty(\mathbf{R})$ and $f \in B(2, \lim)$ such that $M(|f - f_n|^2) \to 0$ as $n \to \infty$ but $f \notin L^\infty(\mathbf{R})$.

Fix any $f \in L^2[0,1)\backslash L^\infty[0,1)$, and extend f periodically to $\mathbf{R}$. Then $f \in B(2,\lim)$ by Example 3.1.4b. Let S_N be the N^{th} partial sum of the Fourier series of f, i.e.,

$$S_N(t) = \sum_{n=-N}^{N} c_n\, e^{2\pi int},$$

where

$$c_n = \int_0^1 f(t)\, e^{-2\pi int}\, dt.$$

Clearly $S_N \in B(2,\lim) \cap L^\infty(\mathbf{R})$, and, by Example 3.1.4b,

$$\begin{aligned} M(|f - S_N|^2) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t) - S_N(t)|^2\, dt \\ &= \int_0^1 |f(t) - S_N(t)|^2\, dt \\ &\to 0 \quad \text{as } N \to \infty. \end{aligned}$$

This example extends trivially to higher dimensions as follows. Let f, S_N be as above, and define $g(t) = f(t_1)$ and $T_N(t) = S_N(t_1)$ for $t \in \mathbf{R}^d$. Then $g \in B(2,\lim) \setminus L^\infty(\mathbf{R}^d)$, $T_N \in B(2,\lim) \cap L^\infty(\mathbf{R}^d)$, and $M(|g - T_N|^2) \to 0$ as $N \to \infty$.

Section 3.3. Beurling's characterization of $B(p,\infty)$.

Wiener, in [**W1**], proved that $B(p,\infty)$ is contained in a certain weighted L^p space (for $d = 1$). This result has been generalized by Beurling, Lau, Benedetto, and others, and we generalize it in Section 3.4 to the $B(p,q)$ spaces. In this section, we discuss the $B(p,\infty)$ case.

We begin by proving and extending Wiener's result, which in its original form is the following theorem with $d = 1$ and $a = 2$. Lau and Lee generalized this to $d = 1$, $a > 1$ in [**LL**]. Benedetto, Benke, and Evans proved a $d \geq 1$, $a = 2$ result in [**BBE**]. Our proof is a combination of the [**LL**] and [**BBE**] results. The proof is essentially Wiener's, i.e., integration by parts.

THEOREM 3.3.1. *Given $1 \leq p < \infty$ and $a \in \mathbf{R}^d_+$ with $a > 1$,*

$$B(p,\infty) \subset L^p_v(\mathbf{R}^d)$$

where

$$v(t) = \prod_{j=1}^{d} \frac{1}{1 + |t_j|^{a_j}}.$$

Moreover, the containment is proper.

PROOF: a. For clarity in proving the containment we restrict ourselves to $d = 2$ (the general case being similar). Fix a, $b > 0$ and $f \in B(p,\infty)$. Define

$$\varphi(x,y) = |f(x,y)|^p + |f(x,-y)|^p + |f(-x,y)|^p + |f(-x,-y)|^p$$

and

$$\psi(S,T) = \int_0^T \int_0^S \varphi(x,y)\,dx\,dy = \int_{-T}^{T} \int_{-S}^{S} |f(x,y)|^p\,dx\,dy.$$

Then

$$M \;=\; \sup_{S,T>0} \frac{1}{ST}\,\psi(S,T) \;=\; 4\,\|f\|^p_{B(p,\infty)} \;<\; \infty.$$

We compute:

(3.3.1)

$$\begin{aligned}
&\int_0^T\int_0^S \frac{\varphi(x,y)}{(1+x^a)(1+y^b)}\,dx\,dy \\
&= \int_0^T\int_0^S \frac{1}{1+x^a}\,\frac{1}{1+y^b}\,\partial_x\,\partial_y\psi(x,y)\,dx\,dy \\
&= \int_0^T \frac{1}{1+y^b}\left(\int_0^S \frac{1}{1+x^a}\,\partial_x\left(\partial_y\psi(x,y)\right)dx\right)dy \\
&= \int_0^T \frac{1}{1+y^b}\left(\frac{\partial_y\psi(S,y)}{1+S^a} \;+\; a\int_0^S \frac{x^{a-1}}{(1+x^a)^2}\,\partial_y\psi(x,y)\,dx\right)dy \\
&= I_1(S,T) \;+\; I_2(S,T).
\end{aligned}$$

Before estimating I_1 and I_2, note that

$$\int_0^T \frac{1}{1+y^b}\,\partial_y\psi(x,y)\,dy \quad=\quad \frac{\psi(x,T)}{1+T^b} \;+\; b\int_0^T \frac{y^{b-1}}{(1+y^b)^2}\,\psi(x,y)\,dy$$

and that

$$\frac{x^{a-1}}{1+x^a} \;\le\; \frac{1}{x} \qquad\text{and}\qquad \frac{y^{b-1}}{1+y^b} \;\le\; \frac{1}{y}$$

for all x, $y > 0$. Therefore,

(3.3.2)

$$\begin{aligned}
I_1(S,T) &= \frac{1}{1+S^a}\int_0^T \frac{1}{1+y^b}\,\partial_y\psi(S,y)\,dy \\
&= \frac{1}{1+S^a}\left(\frac{\psi(S,T)}{1+T^b} \;+\; b\int_0^T \frac{y^{b-1}}{(1+y^b)^2}\,\psi(S,y)\,dy\right)
\end{aligned}$$

$$\leq \frac{S}{1+S^a}\frac{T}{1+T^b}\frac{\psi(S,T)}{ST} + \frac{bS}{1+S^a}\int_0^T \frac{1}{1+y^b}\frac{\psi(S,y)}{Sy}\,dy$$

$$\leq \frac{S}{1+S^a}\frac{T}{1+T^b}M + \frac{S}{1+S^a}bM\int_0^T \frac{1}{1+y^b}\,dy.$$

Similarly,

(3.3.3)

$$\begin{aligned}
I_2(S,T) &= a\int_0^S \frac{x^{a-1}}{(1+x^a)^2}\left(\int_0^T \frac{1}{1+y^b}\,\partial_y\psi(x,y)\,dy\right)dx \\
&= a\int_0^S \frac{x^{a-1}}{(1+x^a)^2}\left(\frac{\psi(x,T)}{1+T^b} + b\int_0^T \frac{y^{b-1}}{(1+y^b)^2}\psi(x,y)\,dy\right)dx \\
&\leq a\frac{T}{1+T^b}\int_0^S \frac{1}{1+x^a}\frac{\psi(x,T)}{xT}\,dx \\
&\quad + ab\int_0^S\int_0^T \frac{1}{1+x^a}\frac{1}{1+y^b}\frac{\psi(x,y)}{xy}\,dy\,dx \\
&\leq \frac{T}{1+T^b}aM\int_0^S \frac{1}{1+x^a}\,dx \\
&\quad + abM\int_0^S \frac{1}{1+x^a}\,dx\int_0^T \frac{1}{1+y^b}\,dy.
\end{aligned}$$

Combining (3.3.1), (3.3.2), and (3.3.3), letting S, $T \to \infty$, and noting that $\lim_{S\to\infty} S/(1+S^a) = \lim_{T\to\infty} T/(1+T^b) = 0$, we obtain

(3.3.4)

$$\begin{aligned}
\|f\|_{L_v^p}^p &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|^p\,v(x,y)\,dx\,dy \\
&= \int_0^{\infty}\int_0^{\infty} \frac{\varphi(x,y)}{(1+x^a)(1+y^b)}\,dx\,dy \\
&\leq a\,b\,C(a)\,C(b)\,M \\
&= 4\,a\,b\,C(a)\,C(b)\,\|f\|_{B(p,\infty)}^p,
\end{aligned}$$

where

$$C(r) = \int_0^\infty \frac{1}{1+x^r}\,dx \le \int_0^1 dx + \int_1^\infty x^{-r}\,dx = \frac{r}{r-1} < \infty.$$

This completes proof of the containment.

b. To see that the containment is proper, first assume that $d = 1$ and define

$$f(t) = \begin{cases} t^{(a-1)/(2p)}, & t \ge 1, \\ 0, & t < 1. \end{cases}$$

Then

$$\int_{\mathbf{R}} \frac{|f(t)|^p}{1+t^a}\,dt = \int_1^\infty \frac{t^{(a-1)/2}}{1+t^a}\,dt \le \int_1^\infty \frac{t^{(a-1)/2}}{t^a}\,dt = \frac{2}{a-1} < \infty,$$

so $f \in L^p_v(\mathbf{R})$. However,

$$\frac{1}{2T}\int_{-T}^{T} |f(t)|^p\,dt = \frac{1}{2T}\int_1^T t^{(a-1)/2}\,dt = \frac{T^{(a-1)/2} - T^{-1}}{a+1} \to \infty$$

as $T \to \infty$, so $f \notin B(p,\infty)$. A higher-dimensional example follows immediately by defining $g(t) = \prod_{j=1}^d f(t_j)$ for $t \in \mathbf{R}^d$.

Another example is furnished by

$$f(t) = \begin{cases} (\log t)^{1/p}, & t \ge 1, \\ 0, & t < 1, \end{cases}$$

for $t \in \mathbf{R}$. Since

$$\frac{1}{2T}\int_{-T}^{T} |f(t)|^p\,dt = \frac{1}{2T}\int_1^T \log t\,dt = \frac{1}{2}(T^{-1} - 1 + \log T) \to \infty$$

as $T \to \infty$, we have $f \notin B(p,\infty)$. However,

$$\int_{\mathbf{R}} \frac{|f(t)|^p}{1+t^a}\,dt \le \int_1^\infty \frac{\log t}{t^a}\,dt = \frac{1}{(a-1)^2} < \infty,$$

so $f \in L^p_v(\mathbf{R})$. The higher-dimensional case follows as before. ∎

By Theorem 3.3.1, $B(p, \infty)$ is contained in a weighted L^p space. Beurling proved in **[Be1]** that $B(p, \infty)$ equals the intersection of all weighted L^p spaces $L^p_w(\mathbf{R})$ over the class of weights w which are positive, even, integrable, and decreasing on $\mathbf{R}_+$. We reproduce his proof in Proposition 3.3.8 and Theorem 3.3.9, as well as giving new proofs of our own. To be precise, Beurling actually proved this characterization in higher dimensions, but in a spherical setting, rather than the rectangular setting of this thesis. We prove our characterization in higher dimensions, but in a rectangular setting.

The higher-dimensional, rectangular analogue of the Beurling class is the following.

DEFINITION 3.3.2. a. $\Lambda = \Lambda(\mathbf{R})$ denotes the class of all positive, even, integrable (with respect to Lebesgue measure) weights w on $\mathbf{R}$ which are decreasing on $\mathbf{R}_+$.

b. $\Lambda = \Lambda(\mathbf{R}^d)$ denotes the class of weights w on $\mathbf{R}^d$ for which there exist $w_j \in \Lambda(\mathbf{R})$ such that

$$w(t) = \prod_{j=1}^{d} w_j(t_j) \tag{3.3.5}$$

for $t \in \mathbf{R}^d$.

REMARK 3.3.3. Given $w \in \Lambda(\mathbf{R}^d)$.

a. w is rectangular, positive, integrable, and even, and is decreasing on $\mathbf{R}^d_+$, cf., Section 1.3d-f. Rectangular refers to the fact that w has the form (3.3.5).

Even means that $w(\sigma t) = w(t)$ for all $t \in \mathbf{R}^d$ and $\sigma \in \Omega^d$. Decreasing means that if $s \geq t \in \mathbf{R}_+^d$ then $w(s) \leq w(t)$, i.e., w is decreasing in each component.

b. As each w_j is decreasing on $\mathbf{R}_+$ it must be continuous except at countably many points. Therefore w is continuous a.e.

EXAMPLE 3.3.4. a. The weight v appearing in Theorem 3.3.1 is an element of $\Lambda(\mathbf{R}^d)$.

b. Set $d = 1$, and define

$$k(t) = \left|\frac{\sin 2\pi t}{\pi t}\right|^2$$

for $t \in \mathbf{R}$. Note that k is even and integrable, though not positive and not decreasing on $\mathbf{R}_+$, so $k \notin \Lambda(\mathbf{R})$.

Let $k^\star$ be the least decreasing majorant of k on $\mathbf{R}_+$ and $k_\star$ the greatest decreasing minorant of k on $\mathbf{R}_+$, cf., Section 1.3g. That is, for $t \in \mathbf{R}_+$,

$$k^\star(t) = \sup_{s \geq t} k(s) \qquad \text{and} \qquad k_\star(t) = \inf_{0 \leq s \leq t} k(s).$$

Extend $k^\star$ and $k_\star$ evenly to $\mathbf{R}$. We clearly have

$$\int_0^\infty k^\star(t)\,dt \leq \int_0^1 4\,dt + \int_1^\infty (\pi t)^{-2}\,dt < \infty.$$

Thus $k^\star$ is even and integrable, and is decreasing on $\mathbf{R}_+$. Since $k^\star$ is positive it is therefore an element of $\Lambda(\mathbf{R})$. Numerically,

$$\int_0^\infty k^\star(t)\,dt \approx 1.068 > 1 = \int_0^\infty k(t)\,dt.$$

$k_\star$ is also even and integrable, and is decreasing on $\mathbf{R}_+$. Since $k_\star$ is nonnegative, but not positive, $k_\star$ is not an element of $\Lambda(\mathbf{R})$. Note that $k_\star = k \cdot \chi_{[0,1/2]}$, so

$$\int_0^\infty k_\star(t)\,dt \;=\; \int_0^{1/2} k(t)\,dt \;\approx\; 0.903 \;<\; 1 \;=\; \int_0^\infty k(t)\,dt.$$

c. For arbitrary d, define

$$K(t) \;=\; \Pi\Big(\frac{\sin 2\pi t}{\pi t}\Big)^2 \;=\; \prod_{j=1}^{d} \Big|\frac{\sin 2\pi t_j}{\pi t_j}\Big|^2 \;=\; \prod_{j=1}^{d} k(t_j)$$

for $t \in \mathbf{R}^d$. Let $K^\star$ be the least decreasing majorant of K on $\mathbf{R}^d_+$ and $K_\star$ the greatest decreasing minorant of k on $\mathbf{R}^d_+$. That is, for $t \in \mathbf{R}^d_+$,

$$K^\star(t) \;=\; \sup_{s\in[t,\infty)} K(s) \;=\; \prod_{j=1}^{d} k^\star(t_j)$$

and

$$K_\star(t) \;=\; \inf_{s\in[0,t]} K(s) \;=\; \prod_{j=1}^{d} k_\star(t_j).$$

Extend $K^\star$ and $K_\star$ evenly to $\mathbf{R}^d$. It follows from part b that $K^\star \in \Lambda(\mathbf{R}^d)$ while K, $K_\star \notin \Lambda(\mathbf{R}^d)$.

The functions k and K play an important role in Chapter 4.

LEMMA 3.3.5. *Given a nonnegative, even function w on $\mathbf{R}^d$ which is decreasing on $\mathbf{R}^d_+$.*

a. $\sup_{T\in\mathbf{R}^d_+} \Pi(T)\,w(T) \;\le\; \int_{\mathbf{R}^d_+} w(t)\,dt.$

b. $\sup_{T\in\mathbf{R}^d_+} \Pi(T)\,w(T) \;\le\; 2^d \sup_{n\in\mathbf{Z}^d} \Pi(2^n)\,w(2^n).$

c. $\int_{\mathbf{R}^d_+} w(t)\,dt \;\le\; \sum_{n\in\mathbf{Z}^d} \Pi(2^n)\,w(2^n) \;\le\; 2^d \int_{\mathbf{R}^d_+} w(t)\,dt.$

d. $\lim_{T\to 0,\infty} \Pi(T)\, w(T) = 0.$

PROOF: a. As w is decreasing on $\mathbf{R}^d_+$,

$$\int_{\mathbf{R}^d_+} w(t)\,dt \ \geq \ \int_{[0,T]} w(t)\,dt \ \geq \ \int_{[0,T]} w(T)\,dt \ = \ \Pi(T)\,w(T).$$

b. If $T \in [2^n, 2^{n+1}]$ then $\Pi(T)\,w(T) \leq \Pi(2^{n+1})\,w(2^n)$ since w is decreasing.

c. Since w is decreasing on $\mathbf{R}^d_+$,

$$\begin{aligned}
\int_{\mathbf{R}^d_+} w(t)\,dt &= \sum \int_{[2^n,2^{n+1}]} w(t)\,dt \\
&\leq \sum \int_{[2^n,2^{n+1}]} w(2^n)\,dt \\
&= \sum \Pi(2^n)\,w(2^n) \\
&= \sum 2^d \int_{[2^{n-1},2^n]} w(2^n)\,dt \\
&\leq 2^d \sum \int_{[2^{n-1},2^n]} w(t)\,dt \\
&= 2^d \int_{\mathbf{R}^d_+} w(t)\,dt.
\end{aligned}$$

d. If $T \in [2^n, 2^{n+1}]$ then $\Pi(2^n)\,w(2^{n+1}) \leq \Pi(T)\,w(T) \leq \Pi(2^{n+1})\,w(2^n)$ since w is decreasing on $\mathbf{R}_+$. This, combined with part c, gives the result. ∎

The spaces A^p, B^p defined below are rectangular analogues of the spherical spaces defined in [**Be1**].

DEFINITION 3.3.6. Given $1 \leq p \leq \infty$.

a. $B^p = \bigcap_{w\in\Lambda} L^p_w(\mathbf{R}^d)$, with norm

$$\|f\|_{B^p} \ = \ \sup_{\substack{w\in\Lambda,\\ \|w\|_1=1}} \|f\|_{L^p_w} \ = \ \sup_{w\in\Lambda} \left(\frac{\int_{\mathbf{R}^d} |f(t)|^p\, w(t)\,dt}{\int_{\mathbf{R}^d} w(t)\,dt} \right)^{1/p}.$$

b. $A^{p'} = \bigcup_{w\in\Lambda} L^{p'}_{w'}(\mathbf{R}^d)$, where $\frac{1}{p}+\frac{1}{p'}=1$ and $w' = w^{1-p'}$, with norm

$$\begin{aligned}\|f\|_{A^{p'}} &= \inf_{\substack{w\in\Lambda,\\ \|w\|_1=1}} \|f\|_{L^{p'}_{w'}} \\ &= \inf_{w\in\Lambda} \left(\int_{\mathbf{R}^d} |f(t)|^{p'}\, w'(t)\,dt\right)^{1/p'} \left(\int_{\mathbf{R}^d} w(t)\,dt\right)^{1/p}.\end{aligned}$$

Note that $B^\infty = L^\infty(\mathbf{R}^d)$ and $A^1 = L^1(\mathbf{R}^d)$.

REMARK 3.3.7. The following facts are proved by Beurling in **[Be1]** (in a spherical setting).

a. A^p, B^p are Banach spaces.

b. $B^p \supset L^\infty(\mathbf{R}^d)$.

c. $A^p \subset L^1(\mathbf{R}^d)$ and is a convolution algebra.

d. $(A^p)' = B^{p'}$, under the duality $\langle f,g\rangle = \int_{\mathbf{R}^d} f(t)\,\overline{g(t)}\,dt$.

e. $B^p = B(p,\infty)$.

Each of these facts is proved in this thesis, some using Beurling's methods, some following from other results. We prove that $B^p = B(p,\infty)$ in Theorem 3.3.9, from which it follows that B^p is a Banach space and contains $L^\infty(\mathbf{R}^d)$. We prove the dualty $(A^p)' = B^{p'}$ in Proposition 3.3.10, and as an immediate corollary obtain that $A^p = \Pi(t)B(p,1)$, from which it follows that A^p is complete and is contained in $L^1(\mathbf{R}^d)$. We prove in Proposition 3.3.13 that A^p is a convolution algebra.

In Theorem 3.3.9 we prove that $B^p = B(p,\infty)$. The key fact is given by the following proposition, for which we give three proofs. First, we give Beurling's proof for the case $d = 1$. This proof is essentially Wiener's integration by parts technique, adapted to cover general $w \in \Lambda(\mathbf{R})$ by using Riemann–Stieltjes integration (the technique can be extended to higher dimensions). Next we present a proof for $d \geq 1$ which uses a method suggested to us by C. Neugebauer, who credited it to R. Bagby [**Bag**]. Finally, we present a simple proof based on discrete norm techniques, which proves the result but with bounds inferior to those obtained by the other techniques.

PROPOSITION 3.3.8. *Given a rectangular, nonnegative, integrable, decreasing function w on $\mathbf{R}_+^d$, i.e., assume there exist $w_j: \mathbf{R}_+ \to [0,\infty)$ which are integrable and decreasing on $\mathbf{R}_+$, such that $w(t) = \prod_1^d w_j(t_j)$ for $t \in \mathbf{R}_+^d$. Then for any nonnegative φ on $\mathbf{R}_+^d$,*

$$\int_{\mathbf{R}_+^d} \varphi(t)\, w(t)\, dt \leq \left(\int_{\mathbf{R}_+^d} w(t)\, dt\right) \sup_{T\in\mathbf{R}_+^d} \frac{1}{\Pi(T)} \int_{[0,T]} \varphi(t)\, dt.$$

PROOF: Set $M = \sup_{T\in\mathbf{R}_+^d} \frac{1}{\Pi(T)} \int_{[0,T]} \varphi(t)\, dt$ and assume without loss of generality that $M < \infty$.

a. We begin with Beurling's proof, for $d = 1$. Define

$$\psi(T) = \int_0^T \varphi(t)\, dt;$$

then

$$M = \sup_{T>0} \frac{1}{T} \psi(T) < \infty. \tag{3.3.6}$$

As $\varphi \in L^1_{\text{loc}}(\mathbf{R})$, ψ is locally absolutely continuous on $\mathbf{R}$. Since w is decreasing, positive, and integrable on $(0,\infty)$ it is of bounded variation on each finite closed interval $[a, b] \subset (0, \infty)$. Therefore the Riemann–Stieltjes integral

$$\int_a^b w(t)\, d\psi(t) \;=\; \int_a^b \varphi(t)\, w(t)\, dt$$

exists. Further, integration by parts, equation (3.3.6), and the fact that w is decreasing gives

(3.3.7)

$$\begin{aligned}
\int_a^b w(t)\, d\psi(t) &= w(b)\,\psi(b) \;-\; w(a)\,\psi(a) \;-\; \int_a^b \psi(t)\, dw(t) \\
&\le w(b)\,\psi(b) \;-\; w(a)\,\psi(a) \;-\; \int_a^b M\, t\, dw(t) \\
&= w(b)\,\psi(b) \;-\; w(a)\,\psi(a) \\
&\qquad - M\Big(b\, w(b) \;-\; a\, w(a) \;-\; \int_a^b w(t)\, dt\Big).
\end{aligned}$$

By Lemma 3.3.5, $\lim_{T\to 0,\infty} T w(T) = 0$. Therefore, by (3.3.6), we have $\lim_{T\to 0,\infty} \psi(T)\, w(T) = 0$ as well. Applying this to (3.3.7),

$$\int_0^\infty \varphi(t)\, w(t)\, dt \;=\; \lim_{\substack{a\to 0,\\ b\to\infty}} \int_a^b w(t)\, d\psi(t) \;\le\; M \int_0^\infty w(t)\, dt.$$

b. We give a second proof for arbitrary $d \ge 1$. Define

$$E \;=\; \big\{(t, u) \in \mathbf{R}_+^d \times \mathbf{R}_+^d : u_j < w_j(t_j), \text{ all } j\big\}.$$

For $u \in \mathbf{R}_+^d$ define

$$\alpha(u) \;=\; \{t \in \mathbf{R}_+^d : (t, u) \in E\} \;=\; \{t \in \mathbf{R}_+^d : w_j(t_j) > u_j, \text{ all } j\}.$$

Note that, since each w_j is decreasing, each $\alpha(u)$ is a (possibly empty) rectangle in $\mathbf{R}_+^d$. Therefore,

$$\int_{\alpha(u)} \varphi(t)\, dt \;\le\; M\, |\alpha(u)|$$

for all $u \in \mathbf{R}_+^d$. Now,

$$\int_{\mathbf{R}_+^d} \chi_E(t,u)\,du \;=\; \prod_{j=1}^{d} \int_0^{w_j(t_j)} du_j \;=\; \prod_{j=1}^{d} w_j(t_j) \;=\; w(t),$$

so

$$|E| \;=\; \int_{\mathbf{R}_+^d}\int_{\mathbf{R}_+^d} \chi_E(t,u)\,du\,dt \;=\; \int_{\mathbf{R}_+^d} w(t)\,dt.$$

Since E can also expressed as

$$E \;=\; \big\{(t,u) \in \mathbf{R}_+^d \times \mathbf{R}_+^d : t \in \alpha(u)\big\},$$

we have

$$\int_{\mathbf{R}_+^d} \chi_E(t,u)\,dt \;=\; \int_{\alpha(u)} dt \;=\; |\alpha(u)|,$$

and therefore,

$$|E| \;=\; \int_{\mathbf{R}_+^d}\int_{\mathbf{R}_+^d} \chi_E(t,u)\,dt\,du \;=\; \int_{\mathbf{R}_+^d} |\alpha(u)|\,du.$$

Hence,

$$\begin{aligned}
\int_{\mathbf{R}_+^d} \varphi(t)\,w(t)\,dt \;&=\; \int_{\mathbf{R}_+^d}\int_{\mathbf{R}_+^d} \varphi(t)\,\chi_E(t,u)\,du\,dt \\
&=\; \int_{\mathbf{R}_+^d}\int_{\mathbf{R}_+^d} \varphi(t)\,\chi_E(t,u)\,dt\,du \\
&=\; \int_{\mathbf{R}_+^d}\int_{\alpha(u)} \varphi(t)\,dt\,du \\
&\le\; \int_{\mathbf{R}_+^d} M\,|\alpha(u)|\,du \\
&=\; M\int_{\mathbf{R}_+^d} w(t)\,dt.
\end{aligned}$$

c. Our third proof has the advantage of simplicity, but results in a constant larger than that obtained in parts a and b. We compute

$$\begin{aligned}\int_{\mathbf{R}^d_+} \varphi(t)\,w(t)\,dt &= \sum_n \int_{[2^n,2^{n+1}]} \varphi(t)\,w(t)\,dt \\ &\le \sum_n w(2^n) \int_{[2^n,2^{n+1}]} \varphi(t)\,dt \\ &\le \sum_n w(2^n) \int_{[0,2^{n+1}]} \varphi(t)\,dt \\ &\le M \sum_n \Pi(2^{n+1})\,w(2^n) \\ &\le 2^{2d} M \int_{\mathbf{R}^d_+} w(t)\,dt,\end{aligned}$$

where the last inequality follows as in Lemma 3.3.5b. Note that this proof does not require that w be rectangular, only that w is decreasing on $\mathbf{R}^d_+$, i.e., decreasing in each component. ∎

THEOREM 3.3.9. *$B(p,\infty) = B^p$, with equality of norms, for $1 \le p \le \infty$.*

PROOF: The case $p = \infty$ follows from the fact that $B(\infty,\infty) = L^\infty(\mathbf{R}^d) = B^\infty$. Therefore, assume $1 \le p < \infty$.

a. Fix $f \in B^p$ and $T \in \mathbf{R}^d_+$, and let $w_n \in \Lambda(\mathbf{R}^d)$ be such that $w_n(t) \searrow \chi_{R_T}(t)$, pointwise, as $n \to \infty$. By the Monotone Convergence Theorem we then have $\int w_n \to \int \chi_{R_T} = |R_T|$. Hence,

$$\int_{R_T} |f(t)|^p\,dt \le \int_{\mathbf{R}^d} |f(t)|^p\,w_n(t)\,dt \le \|f\|^p_{B^p} \int_{\mathbf{R}^d} w_n(t)\,dt \to |R_T|\,\|f\|^p_{B^p}.$$

Thus,

$$\|f\|^p_{B(p,\infty)} = \sup_{T\in\mathbf{R}^d_+} \frac{1}{|R_T|} \int_{R_T} |f(t)|^p\,dt \le \|f\|^p_{B^p}.$$

b. To prove the opposite inequality, fix $f \in B(p,\infty)$ and define

$$\varphi(t) = \sum_{\sigma\in\Omega^d} |f(\sigma t)|^p.$$

Given $w \in \Lambda(\mathbf{R}^d)$, we then have from Proposition 3.3.8 that

$$\int_{\mathbf{R}^d_+} \varphi(t)\,w(t)\,dt \le \left(\int_{\mathbf{R}^d_+} w(t)\,dt\right) \sup_{T\in\mathbf{R}^d_+} \frac{1}{\Pi(T)} \int_{[0,T]} \varphi(t)\,dt. \tag{3.3.8}$$

Now,

$$\frac{1}{\Pi(T)} \int_{[0,T]} \varphi(t)\,dt = \frac{2^d}{|R_T|} \int_{R_T} |f(t)|^p\,dt. \tag{3.3.9}$$

Also, since w is even,

$$\int_{\mathbf{R}^d_+} w(t)\,dt = 2^{-d} \int_{\mathbf{R}^d} w(t)\,dt \tag{3.3.10}$$

and

$$\int_{\mathbf{R}^d_+} \varphi(t)\,w(t)\,dt = \int_{\mathbf{R}^d} |f(t)|^p\,w(t)\,dt. \tag{3.3.11}$$

Substituting (3.3.9), (3.3.10), and (3.3.11) into (3.3.8), we obtain

$$\begin{aligned}
||f||^p_{B^p} &= \sup_{w\in\Lambda} \frac{\int_{\mathbf{R}^d} |f(t)|^p\,w(t)\,dt}{\int_{\mathbf{R}^d} w(t)\,dt} \\
&\le \sup_{T\in\mathbf{R}^d_+} \frac{1}{|R_T|} \int_{R_T} |f(t)|^p\,dt \\
&= ||f||^p_{B(p,\infty)}. \quad \blacksquare
\end{aligned}$$

Theorem 3.3.1 follows as an immediate corollary of Theorem 3.3.9 since v, as defined in Theorem 3.3.1, is an element of $\Lambda(\mathbf{R}^d)$.

We complete this section by proving some facts about the space A^p. First, we give Beurling's proof that $(A^p)' = B^{p'}$. As the completeness of A^p will follow from results in the next section, we assume that A^p is a Banach space.

PROPOSITION 3.3.10. *Given* $1 \le p < \infty$,

$$(A^p)' = B^{p'},$$

under the duality

$$\langle f, g\rangle = \int_{\mathbf{R}^d} f(t)\,\overline{g(t)}\,dt.$$

PROOF: Since $(A^1)' = (L^1)' = L^\infty = B^\infty$, we assume that $1 < p < \infty$. For convenience of notation, we prove the equivalent statement $(A^{p'})' = B^p$.

a. Fix $f \in A^{p'}$, $g \in B^p$, and $w \in \Lambda(\mathbf{R}^d)$ with $\|w\|_1 = 1$. Then since $(L^p_w)' = L^{p'}_{w'}$,

$$|\langle f, g\rangle| \le \|f\|_{L^{p'}_{w'}} \|g\|_{L^p_w} \le \|f\|_{L^{p'}_{w'}} \|g\|_{B^p}.$$

Taking the infimum over all such w we find that $|\langle f, g\rangle| \le \|f\|_{A^{p'}} \|g\|_{B^p}$. Thus each $g \in B^p$ determines a continuous linear functional on $A^{p'}$.

b. We show now that

$$\|g\|_{B^p} = \sup\big\{|\langle f, g\rangle| : \|f\|_{A^{p'}} = 1\big\}.$$

To see this, fix $\varepsilon < 1$. Then there exists a $w \in \Lambda(\mathbf{R}^d)$ with $\|w\|_1 = 1$ such that $\|g\|_{L^p_w} \ge \varepsilon\,\|g\|_{B^p}$. Set

$$f = \begin{cases} |g(t)|^p\, w(t)\,/\,\overline{g(t)}, & g(t) \ne 0, \\ 0, & g(t) = 0. \end{cases}$$

Then

$$\begin{aligned} \int_{\mathbf{R}^d} |f(t)|^{p'}\, w'(t)\,dt &= \int_{\mathbf{R}^d} |g(t)|^{p'(p-1)}\, w(t)^{p'}\, w(t)^{1-p'}\,dt \\ &= \int_{\mathbf{R}^d} |g(t)|^p\, w(t)\,dt. \end{aligned}$$

Thus $f \in L^{p'}_{w'} \subset A^{p'}$. Moreover,

$$
\begin{aligned}
|\langle f, g\rangle| &= \int_{\mathbf{R}^d} f(t)\,\overline{g(t)}\,dt \\
&= \int_{\mathbf{R}^d} |g(t)|^p\, w(t)\,dt \\
&= \left(\int_{\mathbf{R}^d} |g(t)|^p\, w(t)\,dt\right)^{1/p'} \left(\int_{\mathbf{R}^d} |g(t)|^p\, w(t)\,dt\right)^{1/p} \\
&= \left(\int_{\mathbf{R}^d} |f(t)|^{p'}\, w'(t)\,dt\right)^{1/p'} \left(\int_{\mathbf{R}^d} |g(t)|^p\, w(t)\,dt\right)^{1/p} \\
&= \|f\|_{L^{p'}_{w'}}\ \|g\|_{L^p_w} \\
&\geq \|f\|_{A^{p'}}\ \varepsilon\, \|g\|_{B^p}.
\end{aligned}
$$

Since ε is arbitarily close to 1, the claim follows.

c. Finally, assume $\mu \in (A^{p'})'$ is given. As $A^{p'} = \cup L^{p'}_{w'}$, we have $\mu \in (L^{p'}_{w'})' = L^p_w$ for each $w \in \Lambda(\mathbf{R}^d)$. Therefore, for each w there is a function $g_w \in L^p_w$ such that

$$
\langle f, \mu\rangle = \langle f, g_w\rangle = \int_{\mathbf{R}^d} f(t)\,\overline{g_w(t)}\,dt \tag{3.3.12}
$$

for all $f \in L^{p'}_{w'}$. To see that g_w is independent of the choice of w, fix any two weights v, $w \in \Lambda(\mathbf{R}^d)$ with $\|v\|_1 = \|w\|_1 = 1$. Recall that, by definition,

$$
v(t) = \prod_{j=1}^{d} v_j(t_j) \qquad \text{and} \qquad w(t) = \prod_{j=1}^{d} w_j(t_j)
$$

for some v_j, $w_j \in \Lambda(\mathbf{R})$. Define

$$
u(t) = \prod_{j=1}^{d} \frac{v_j(t_j) + w_j(t_j)}{\|v_j + w_j\|_1}.
$$

Clearly $u \in \Lambda(\mathbf{R}^d)$ and $\|u\|_1 = 1$. Also $u \geq C^{-1}v$, $C^{-1}w$, where $C = \prod_1^d \|v_j + w_j\|_1$, so $u' \leq C^{1-p'}v', C^{1-p'}w'$. Therefore $L^{p'}_{u'} \supset L^{p'}_{v'} \cup L^{p'}_{w'}$, whence

$\langle f, g_u\rangle = \langle f, g_v\rangle = \langle f, g_w\rangle$ for all $f \in L^{p'}_{u'}$. Hence $g_u = g_v = g_w$ a.e., so g_w is independent of w, and is denoted hereafter by g. Since $g \in L^p_w$ for all w we have $g \in B^p$, and from (3.3.12), $\langle f, \mu\rangle = \langle f, g\rangle$ for all $f \in A^{p'}$. Thus $\mu = g \in B^p$. ∎

Corollary 3.3.11. *For $1 \le p < \infty$,*

$$A^p = \Pi(t)\, B(p,1),$$

i.e.,

$$f \in A^p \quad \Leftrightarrow \quad \Pi(t)\, f(t) \in B(p,1),$$

and

$$\|f\|_{A^p} = 2^d \, \|\Pi(t)\, f(t)\|_{B(p,1)}.$$

Proof: Given $f \in A^p$, we have

$$\begin{aligned}
\|f\|_{A^p} &= \sup\left\{|\langle f, g\rangle| : \|g\|_{B^{p'}} = 1\right\} \\
&= \sup\left\{\left|\int_{\mathbf{R}^d} f(t)\,\overline{g(t)}\,dt\right| : \|g\|_{B^{p'}} = 1\right\} \\
&= \sup\left\{\left|\int_{\mathbf{R}^d_*} \Pi(t)\, f(t)\,\overline{g(t)}\,\frac{dt}{|\Pi(t)|}\right| : \|g\|_{B(p',\infty)} = 1\right\} \\
&= \sup\left\{|\langle \Pi(t)\, f(t), g(t)\rangle| : \|g\|_{B(p',\infty)} = 1\right\} \\
&= 2^d \left\|\Pi(t)\, f(t)\right\|_{B(p,1)},
\end{aligned}$$

the last equality following by Remark 3.2.6c. ∎

Remark 3.3.12. $A^p = \Pi(t)\, B(p,1) \subset \Pi(t)\, L^1(\mathbf{R}^d_*) = L^1(\mathbf{R}^d)$.

We now give Beurling's proof that A^p is a convolution algebra. It is interesting to note that this does not follow immediately from the identification $A^p = \Pi(t)\, B(p,q) = \Pi(t)\, W_*(L^p, L^1)$. Feichtinger's Wiener amalgam theory does imply that $W_*(L^p, L^1)$ satisfies certain convolution relations; however, those relations are with respect to the group operation in $\mathbf{R}^d_*$, i.e., they are multiplicative convolution relations rather than the ordinary additive ones which appear in the proposition.

PROPOSITION 3.3.13. *A^p is a convolution algebra for all $1 \le p < \infty$.*

PROOF: The case $p = 1$ follows immediately since $A^1 = L^1(\mathbf{R}^d)$, so assume $1 < p < \infty$. For convenience of notation, we will prove the equivalent statement that $A^{p'}$ is a convolution algebra.

Given f, $g \in A^{p'}$ and given $\varepsilon > 1$, let v, $w \in \Lambda(\mathbf{R}^d)$ be such that $\|v\|_1 = \|w\|_1 = 1$ and

$$\|f\|_{L^{p'}_{v'}} \le \varepsilon\, \|f\|_{A^{p'}} \qquad \text{and} \qquad \|g\|_{L^{p'}_{w'}} \le \varepsilon\, \|g\|_{A^{p'}}.$$

As v, $w \in L^1(\mathbf{R}^d)$ we can define $u = v * w$. Note that $\|u\|_1 = \|v\|_1\, \|w\|_1 = 1$ since both v, $w > 0$. Also, $u(t) = \prod_1^d (v_j * w_j)(t_j)$ and $v_j * w_j \in \Lambda(\mathbf{R})$ for all j, so $u \in \Lambda(\mathbf{R}^d)$. Now, $f * g$ exists and is integrable since f, $g \in A^{p'} \subset L^1(\mathbf{R}^d)$. We compute

$$\begin{aligned} |(f*g)(t)| &= \left| \int_{\mathbf{R}^d} f(t-s)\, g(s)\, ds \right| \\ &\le \left(\int_{\mathbf{R}^d} \frac{|f(t-s)\, g(s)|^{p'}}{v(t-s)^{p'/p}\, w(s)^{p'/p}}\, ds \right)^{1/p'} \left(\int_{\mathbf{R}^d} v(t-s)\, w(s)\, ds \right)^{1/p} \end{aligned}$$

$$= \left(\int_{\mathbf{R}^d} |f(t-s)\, g(s)|^{p'}\, v'(t-s)\, w'(s)\, ds \right)^{1/p'} u(t)^{1/p}.$$

Thus,

$$\begin{aligned}
\|f * g\|_{L^{p'}_{u'}}^{p'} &= \int_{\mathbf{R}^d} |(f * g)(t)|^{p'}\, u'(t)\, dt \\
&\leq \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} |f(t-s)\, g(s)|^{p'}\, v'(t-s)\, w'(s)\; u(t)^{p'/p}\, u'(t)\, ds\, dt \\
&= \int_{\mathbf{R}^d} |g(s)|^{p'}\, w'(s) \int_{\mathbf{R}^d} |f(t-s)|^{p'}\, v'(t-s)\, dt\; ds \\
&= \left(\int_{\mathbf{R}^d} |g(s)|^{p'}\, w'(s)\, ds \right) \left(\int_{\mathbf{R}^d} |f(t)|^{p'}\, v'(t)\, dt \right) \\
&= \|g\|_{L^{p'}_{w'}}^{p'}\; \|f\|_{L^{p'}_{v'}}^{p'} \\
&\leq \varepsilon^2\; \|f\|_{A^{p'}}^{p'}\; \|g\|_{A^{p'}}^{p'}.
\end{aligned}$$

Letting $\varepsilon \to 1$ therefore gives the result. ∎

Section 3.4. A characterization of $B(p,q)$.

In this section we attempt to characterize $B(p,q)$ in a manner similar to Beurling's characterization of $B(p,\infty)$ given in Section 3.3, i.e., as a union or intersection of weighted L^p spaces.

DEFINITION 3.4.1. Given $1 \le p < \infty$ and $1 \le q \le \infty$.

a. Given a weight w on $\mathbf{R}^d_*$ we define the weight w_{pq} on $\mathbf{R}^d_*$ by

$$w_{pq}(t) \;=\; |\Pi(t)\, w(t)|^{1-\frac{p}{q}}.$$

b. If $p \le q$ then we define $X(p,q) = \bigcap_{w\in\Lambda} L^p_{w_{pq}}(\mathbf{R}^d_*)$, with norm

$$\begin{aligned}\|f\|_{X(p,q)} &= \sup_{\substack{w\in\Lambda,\\ \|w\|_1=1}} \|f\|_{L^p_{w_{pq}}}\\ &= \sup_{w\in\Lambda} \left(\int_{\mathbf{R}^d_*} |f(t)|^p\, |\Pi(t)\, w(t)|^{1-\frac{p}{q}} \frac{dt}{|\Pi(t)|}\right)^{1/p} \left(\int_{\mathbf{R}^d} w(t)\, dt\right)^{\frac{1}{q}-\frac{1}{p}}.\end{aligned}$$

c. If $q \le p$ then we define $X(p,q) = \bigcup_{w\in\Lambda} L^p_{w_{pq}}(\mathbf{R}^d_*)$, with norm

$$\begin{aligned}\|f\|_{X(p,q)} &= \inf_{\substack{w\in\Lambda,\\ \|w\|_1=1}} \|f\|_{L^p_{w_{pq}}}\\ &= \inf_{w\in\Lambda} \left(\int_{\mathbf{R}^d_*} |f(t)|^p\, |\Pi(t)\, w(t)|^{1-\frac{p}{q}} \frac{dt}{|\Pi(t)|}\right)^{1/p} \left(\int_{\mathbf{R}^d} w(t)\, dt\right)^{\frac{1}{q}-\frac{1}{p}}.\end{aligned}$$

REMARK 3.4.2. a. Since $\int_{\mathbf{R}^d} w(t)\, dt = \int_{\mathbf{R}^d_*} |\Pi(t)\, w(t)|\, \frac{dt}{|\Pi(t)|}$, the normalization of w with respect to Lebesgue measure in Definition 3.4.1 can be considered a normalization of $|\Pi(t)\, w(t)|$ with respect to the Haar measure $dt/|\Pi(t)|$.

b. For $q = \infty$,

$$\begin{aligned}\|f\|_{X(p,\infty)} &= \sup_{w\in\Lambda}\left(\int_{\mathbf{R}^d_*} |f(t)|^p\,|\Pi(t)\,w(t)|\,\frac{dt}{|\Pi(t)|}\right)^{1/p}\left(\int_{\mathbf{R}^d} w(t)\,dt\right)^{-1/p} \\ &= \sup_{w\in\Lambda}\left(\frac{\int_{\mathbf{R}^d} |f(t)|^p\,w(t)\,dt}{\int_{\mathbf{R}^d} w(t)\,dt}\right)^{1/p} \\ &= \|f\|_{B^p}.\end{aligned}$$

Thus $X(p,\infty) = B^p = B(p,\infty)$, cf., Theorem 3.3.9.

c. Since $w_{pp} = 1$, $X(p,p) = L^p(\mathbf{R}^d_*) = B(p,p)$, cf., Proposition 3.2.2.

d. For $q = 1$,

$$\begin{aligned}\|f\|_{X(p,1)} &= \inf_{w\in\Lambda}\left(\int_{\mathbf{R}^d_*} |f(t)|^p\,|\Pi(t)\,w(t)|^{1-p}\,\frac{dt}{|\Pi(t)|}\right)^{1/p}\left(\int_{\mathbf{R}^d} w(t)\,dt\right)^{1-\frac{1}{p}} \\ &= \inf_{w\in\Lambda}\left(\int_{\mathbf{R}^d}\left|\frac{f(t)}{\Pi(t)}\right|^p w(t)^{1-p}\,dt\right)^{1/p}\left(\int_{\mathbf{R}^d} w(t)\,dt\right)^{1/p'} \\ &= \left\|f(t)/\Pi(t)\right\|_{A^p}.\end{aligned}$$

Thus $X(p,1) = \frac{1}{\Pi(t)}\,A^p = B(p,1)$, cf., Corollary 3.3.11.

LEMMA 3.4.3. *If $1 \le p,\, q < \infty$ then* $\dfrac{q-p}{pq} = \dfrac{p'-q'}{p'q'}$.

PROOF: We compute

$$\frac{q-p}{pq} - \frac{p'-q'}{p'q'} = \frac{1}{p} - \frac{1}{q} - \frac{1}{q'} + \frac{1}{p'} = 1 - 1 = 0. \quad \blacksquare$$

LEMMA 3.4.4. *Given a weight w on $\mathbf{R}^d_*$ and given $1 \le p < \infty$ and $1 \le q \le \infty$,*

$$\left(L^p_{w_{pq}}(\mathbf{R}^d_*)\right)' = L^{p'}_{w_{p'q'}}(\mathbf{R}^d_*).$$

PROOF: By Section 1.7a it suffices to show ${w_{pq}}^{1-p'} = w_{p'q'}$. Assume first that $q < \infty$; then, with the help of Lemma 3.4.3, we compute

$$\begin{aligned}\left(\frac{q-p}{q}\right)(1-p') &= \frac{q-p}{q}\,\frac{1}{1-p} \\ &= \frac{q-p}{pq}\,\frac{p}{1-p} \\ &= \frac{p'-q'}{p'q'}\,(-p') \\ &= \frac{q'-p'}{q'}.\end{aligned}$$

Therefore,

$$w_{pq}(t)^{1-p'} = |\Pi(t)\,w(t)|^{(1-p/q)(1-p')} = |\Pi(t)\,w(t)|^{1-p'/q'} = w_{p'q'}(t).$$

If $q = \infty$ then $q' = 1$, so

$$w_{p\infty}(t)^{1-p'} = |\Pi(t)\,w(t)|^{1-p'} = w_{p'1}(t). \quad \blacksquare$$

REMARK 3.4.5. From Lemma 3.4.4 we can prove that $X(p,q)' = X(p',q')$, cf., Proposition 3.3.10. Note that $B(p,q)' = B(p',q')$ by Remark 3.2.6c.

PROPOSITION 3.4.6.

a. *If* $1 \le p \le q \le \infty$ *then* $X(p,q) \supset W_*(L^p, L^q)$.

b. *If* $1 \le q \le p < \infty$ *then* $X(p,q) \subset W_*(L^p, L^q)$.

PROOF: a. For simplicity, assume $q < \infty$ as the case $q = \infty$ is similar.

Fix $f \in W_*(L^p, L^q)$ and $w \in \Lambda(\mathbf{R}^d)$. Since $(q-p)/q \ge 0$, $w^{(q-p)/q}$ is decreasing on $\mathbf{R}^d_+$ and $\Pi(t)^{(q-p)/q}$ is increasing on $\mathbf{R}^d_+$. These facts allow us to compute

$$\begin{aligned}
\|f\|^p_{L^p_{w_{pq}}} &= \int_{\mathbf{R}^d_*} |f(t)|^p \, |\Pi(t)\, w(t)|^{(q-p)/q} \frac{dt}{|\Pi(t)|} \\
&= \sum_{n,\sigma} \int_{\sigma[2^n,2^{n+1}]} |f(t)|^p \, |\Pi(t)\, w(t)|^{(q-p)/q} \frac{dt}{|\Pi(t)|} \\
&\le \sum_{n,\sigma} \int_{\sigma[2^n,2^{n+1}]} |f(t)|^p \, |\Pi(2^{n+1})\, w(2^n)|^{(q-p)/q} \frac{dt}{|\Pi(t)|} \\
&\le \left(\sum_{n,\sigma} \left(\int_{\sigma[2^n,2^{n+1}]} |f(t)|^p \frac{dt}{|\Pi(t)|}\right)^{q/p}\right)^{p/q} \\
&\qquad \times \left(\sum_{n,\sigma} |\Pi(2^{n+1})\, w(2^n)|^{(\frac{q-p}{q})(q/p)'}\right)^{1/(q/p)'} \\
&= \|f\|^p_{W_*(L^p,L^q)} \left(\sum_{n,\sigma} \Pi(2^{n+1})\, w(2^n)\right)^{(q-p)/q} \\
&\le \|f\|^p_{W_*(L^p,L^q)} \left(2^{2d} \sum_{n} \Pi(2^n)\, w(2^n)\right)^{(q-p)/q} \\
&\le 2^{2d(q-p)/q} \left(\int_{\mathbf{R}^d} w(t)\, dt\right)^{(q-p)/q} \|f\|^p_{W_*(L^p,L^q)},
\end{aligned}$$

where we have used Hölder's inequality on the summations (possible since $q/p \ge 1$), the fact that $(q/p)' = q/(q-p)$, and Lemma 3.3.5c. Therefore,

$$\|f\|_{L^p_{w_{pq}}} \left(\int_{\mathbf{R}^d} w(t)\, dt\right)^{(p-q)/pq} \le 2^{2d(q-p)/pq} \|f\|_{W_*(L^p,L^q)}.$$

Taking the supremum over $w \in \Lambda(\mathbf{R}^d)$ we obtain

$$\|f\|_{X(p,q)} \le 2^{2d(q-p)/pq} \|f\|_{W_*(L^p,L^q)}.$$

b. Fix $f \in W_*(L^p, L^q)$ and $w \in \Lambda(\mathbf{R}^d)$. Since $(q-p)/q \le 0$, $w^{(q-p)/q}$ is increasing on $\mathbf{R}_+^d$ and $\Pi(t)^{(q-p)/q}$ is decreasing on $\mathbf{R}_+^d$. Therefore,

(3.4.1)

$$
\begin{aligned}
\|f\|_{W_*(L^p,L^q)}^q &= \sum_{n,\sigma} \left(\int_{\sigma[2^n,2^{n+1}]} |f(t)|^p \, \frac{dt}{|\Pi(t)|} \right)^{q/p} \\
&\leq \sum_{n,\sigma} \left(\int_{\sigma[2^n,2^{n+1}]} |f(t)|^p \left| \frac{\Pi(t)\, w(t)}{\Pi(2^{n+1})\, w(2^n)} \right|^{(q-p)/q} \frac{dt}{|\Pi(t)|} \right)^{q/p} \\
&\leq \left(\sum_{n,\sigma} \int_{\sigma[2^n,2^{n+1}]} |f(t)|^p \, |\Pi(t)\, w(t)|^{(q-p)/q} \, \frac{dt}{|\Pi(t)|} \right)^{q/p} \\
&\qquad \times \left(\sum_{n,\sigma} |\Pi(2^{n+1})\, w(2^n)|^{(\frac{p-q}{q})(p/q)'} \right)^{1/(p/q)'} \\
&= \|f\|_{L^p_{w_{pq}}}^q \left(\sum_{n,\sigma} |\Pi(2^{n+1})\, w(2^n)|^{p/q} \right)^{(p-q)/p},
\end{aligned}
$$

where we have used Hölder's inequality (possible since $p/q \geq 1$) and the fact that $(p/q)' = p/(p-q)$. Since $p/q \geq 1$ we have

(3.4.2)

$$
\begin{aligned}
\left(\sum_{n,\sigma} |\Pi(2^{n+1})\, w(2^n)|^{p/q} \right)^{q/p} &\leq \sum_{n,\sigma} \Pi(2^{n+1})\, w(2^n) \\
&= 2^{2d} \sum_{n} \Pi(2^n)\, w(2^n) \\
&\leq 2^{2d} \int_{\mathbf{R}^d} w(t)\, dt,
\end{aligned}
$$

the last line following from Lemma 3.3.5c. Combining (3.4.1) and (3.4.2),

$$
\|f\|_{W_*(L^p,L^q)} \leq 2^{2d(p-q)/pq} \, \|f\|_{L^p_{w_{pq}}} \left(\int_{\mathbf{R}^d} w(t)\, dt \right)^{(p-q)/pq}.
$$

Taking the infimum over $w \in \Lambda(\mathbf{R}^d)$ we obtain

$$
\|f\|_{W_*(L^p,L^q)} \leq 2^{2d(p-q)/pq} \, \|f\|_{X(p,q)}. \quad \blacksquare
$$

REMARK 3.4.7. a. From Remark 3.4.2, $B(p,1) = X(p,1)$, $B(p,p) = X(p,p)$, and $B(p,\infty) = X(p,\infty)$. From Proposition 3.4.6, $B(p,q) = W_*(L^p, L^q) \subset X(p,q)$ if $p \le q$, and $B(p,q) = W_*(L^p, L^q) \supset X(p,q)$ if $p \ge q$. We therefore strongly suspect, although we have not proved, that $B(p,q) = X(p,q)$ for all p, q.

b. In the deep paper [He], Herz introduced spaces related to $X(p,q)$. Using the notation of [Jo], the Herz space ${}_pL_q$ is defined as follows. Let Φ be the spherical analogue of $\Lambda(\mathbf{R}^d)$, i.e., Φ consists of all weights w on $\mathbf{R}^d$ which are positive, radial, integrable with respect to Lebesgue measure, and radially decreasing. Then

$$\|f\|_{{}_pL_q} = \begin{cases} \sup\limits_{\varphi\in\Phi} \left(\int_{\mathbf{R}^d} |f(t)|^p\, \varphi(t)^{1-\frac{p}{q}}\, dt\right)^{1/p} \left(\int_{\mathbf{R}^d} \varphi(t)\, dt\right)^{\frac{1}{q}-\frac{1}{p}}, & p \le q, \\ \inf\limits_{\varphi\in\Phi} \left(\int_{\mathbf{R}^d} |f(t)|^p\, \varphi(t)^{1-\frac{p}{q}}\, dt\right)^{1/p} \left(\int_{\mathbf{R}^d} \varphi(t)\, dt\right)^{\frac{1}{q}-\frac{1}{p}}, & p \ge q. \end{cases}$$

The space K^0_{pq} is defined by the norm

$$\|f\|_{K^0_{pq}} = \left\| |t|^{(\frac{1}{p}-\frac{1}{q})d} f(t) \right\|_{{}_pL_q}.$$

Since

$$\int_{\mathbf{R}^d} \left| |t|^{(\frac{1}{p}-\frac{1}{q})d} f(t) \right|^p \varphi(t)^{1-\frac{p}{q}}\, dt = \int_{\mathbf{R}^d} |f(t)|^p \left| |t|^d \varphi(t) \right|^{1-\frac{p}{q}} dt,$$

K^0_{pq} is the analogue of $X(p,q)$ obtained by using Lebesgue measure on $\mathbf{R}^d$ instead of Haar measure on $\mathbf{R}^d_*$, and using a spherical approach to higher dimensions rather than a rectangular approach.

The space K^{α}_{pq} is defined by the norm

$$\|f\|_{K^{\alpha}_{pq}} = \| |t|^{\alpha/d} f(t)\|_{K^0_{pq}}.$$

Therefore $K^{-p}_{pq} = |t|^{-p/d} K^0_{pq}$ is the exact spherical analogue of the rectangular $X(p,q)$.

Section 3.5. Weighted Besicovitch spaces.

In this section, we examine the effect of replacing the factor $1/|R_T|$ in the definition of $\|\cdot\|_{B(p,q)}$ by a general function $\rho(T)$. We show that the resulting space, denoted $B_\rho(p,q)$, equals a Wiener amalgam space $W_*(L^p_v, L^q)$ for an appropriate weight v.

The spaces $B_\rho(p,q)$, especially $B_\rho(p,\infty)$, have appeared in various places in the literature. For example, Wiener considered the one-dimensionsal case $1/(2T)^\alpha$, e.g., [**W3**], as did Lau and Chen, e.g., [**CL1**]. Strichartz considered higher-dimensional spherical analogues of this, e.g., [**St1; St2**]. Evans considered general functions, e.g., [**E2**].

DEFINITION 3.5.1. Given $1 \le p,\ q < \infty$ and a weight $\rho\colon \mathbf{R}^d_+ \to \mathbf{R}_+$, the **weighted Besicovitch space** $B_\rho(p,q)$ is the space of functions $f\colon \mathbf{R}^d \to \mathbf{C}$ for which the norm

$$\|f\|_{B_\rho(p,q)} = \left(\int_{\mathbf{R}^d_+} \left(\rho(T)\int_{R_T} |f(t)|^p\,dt\right)^{q/p} \frac{dT}{\Pi(T)}\right)^{1/q}$$

is finite. The standard adjustments are made if p or q is infinity, cf., Definition 3.2.1.

The following result is similar to part of Theorem 3.2.4.

PROPOSITION 3.5.2. *Given $1 \le p < \infty$, $1 \le q \le \infty$, and an even, moderate weight v on $\mathbf{R}^d_*$. Define*

$$\rho(t) = v(t)/|\Pi(t)|.$$

Then there is a constant $C > 0$ such that

$$\|\cdot\|_{B_\rho(p,q)} \geq C\|\cdot\|_{W_*(L^p_v,L^q)}.$$

Proof: Assume for simplicity that $1 \leq p,\ q < \infty$ (the case $q = \infty$ is similar). By Theorem 2.1.6e there is a constant $B > 0$ such that $\sup_{t[1,2]} v \leq B \inf_{t[1,2]} v$ for $t \in \mathbf{R}^d_*$. Therefore, if $T \in [2^n, 2^{n+1}]$ then

$$\rho(T) = \frac{v(T)}{\Pi(T)} \leq \frac{B\, v(2^{n+1})}{\Pi(2^n)} = 2^d B\, \rho(2^{n+1})$$

and

$$\rho(T) = \frac{v(T)}{\Pi(T)} \geq \frac{v(2^n)}{B\, \Pi(2^{n+1})} = \frac{\rho(2^n)}{2^d B}.$$

Fix now any $\sigma \in \Omega^d$. Then

$$\begin{aligned}
\|f\|^q_{B_\rho(p,q)} &= \int_{\mathbf{R}^d_+} \left(\rho(T) \int_{R_T} |f(t)|^p\, dt\right)^{q/p} \frac{dT}{\Pi(T)} \\
&= \sum_n \int_{[2^n,2^{n+1}]} \left(\rho(T) \int_{R_T} |f(t)|^p\, dt\right)^{q/p} \frac{dT}{\Pi(T)} \\
&\geq \sum_n \int_{[2^n,2^{n+1}]} \left(\frac{\rho(2^n)}{2^d B} \int_{R_{2^n}} |f(t)|^p\, dt\right)^{q/p} \frac{dT}{\Pi(T)} \\
&= (\log 2)^d \sum_n \left(\frac{\rho(2^n)}{2^d B} \int_{R_{2^n}} |f(t)|^p\, dt\right)^{q/p} \\
&\geq (\log 2)^d \sum_n \left(\frac{\rho(2^n)}{2^d B} \int_{\sigma[2^{n-1},2^n]} |f(t)|^p\, dt\right)^{q/p} \\
&\geq (\log 2)^d \sum_n \left(\frac{1}{2^{2d} B^2} \int_{\sigma[2^{n-1},2^n]} |f(t)|^p\, |\Pi(t)|\, \rho(t)\, \frac{dt}{|\Pi(t)|}\right)^{q/p},
\end{aligned}$$

where the summations in n run over $\mathbf{Z}^d$. Therefore,

$$
\begin{aligned}
2^d \, \|f\|^q_{B_\rho(p,q)} & \\
&= \sum_{\sigma \in \Omega^d} \|f\|^q_{B_\rho(p,q)} \\
&\geq (\log 2)^d \, 2^{-2dq/p} \, B^{-2q/p} \sum_{n,\sigma} \left(\int_{\sigma[2^{n-1},2^n]} |f(t)|^p \, v(t) \, \frac{dt}{|\Pi(t)|} \right)^{q/p} \\
&= (\log 2)^d \, 2^{-2dq/p} \, B^{-2q/p} \, \|f\|^q_{W_*(L^p_v, L^q)},
\end{aligned}
$$

from which the result follows. ∎

REMARK 3.5.3. a. The opposite inequality to the one in Proposition 3.5.2 can be proven just as in Theorem 3.2.4. Precisely, given $1 \leq p < \infty$ and $1 \leq q \leq \infty$ and given an even moderate weight $\rho: \mathbf{R}^d_* \to \mathbf{R}_+$, define $v(t) = |\Pi(t)| \, \rho(t)$. Then there exists a constant $D > 0$ such that $\| \cdot \|_{B_\rho(p,q)} \leq D \, \| \cdot \|_{W_*(L^p_v, L^q)}$.

b. From Proposition 3.5.2 and part a we have $B_\rho(p,q) = W_*(L^p_v, L^q)$ with equivalent norms.

c. By Theorem 2.3.1, if v is moderate then $W_*(L^p_v, L^q) = W_*(L^p, L^q_{v^{q/p}})$, i.e., the weight may be placed on either the local or global component.

CHAPTER 4

THE WIENER TRANSFORM

In this chapter, we prove that the Wiener transform W is a topological isomorphism of the Besicovitch space $B(2,q)$ onto the variation space $V(2,q)$ for each $1 \leq q \leq \infty$.

The definition of the Wiener transform and the symmetric difference operator Δ_λ used in this thesis follow the higher-dimensional, rectangular definitions of [**BBE**]. Many basic ideas in this chapter are from [**BBE**]; we thank those authors for making higher-dimensional calculations possible. In addition, the one-dimensional results of Beurling and Lau are critical in that they directly lead to our isomorphism theorem. Our method of proof includes new techniques based on amalgams, combined with the techniques of Beurling and Lau.

We begin in Section 4.1 by defining the Wiener transform, and showing that its domain of definition includes the spaces $B(2,q)$ for each $1 \leq q \leq \infty$.

In Section 4.2 we define the symmetric difference operators Δ_λ, and compute $\Delta_\lambda Wf$.

In Section 4.3 we define the higher-dimensional variation spaces $V(p,q)$.

In Section 4.4 we prove that the Wiener transform maps $B(2,q)$ continuously into $V(2,q)$. We prove this for the case $q = \infty$ using Lau's method, for $q = 1$ using Beurling's method, and for the general case using amalgam spaces.

In Section 4.5 we prove that the Wiener transform is invertible for each $1 \le q \le \infty$. We prove this for $1 \le q \le \infty$ by generalizing Lau's $q = \infty$ technique, and compare this to an amalgam space proof for $1 \le q < \infty$.

Throughout this chapter, k and K will be as in Example 3.3.4c. That is, $k(t) = \left((\sin 2\pi t)/(\pi t)\right)^2$ for $t \in \mathbf{R}$ and

$$K(t) = \Pi\left(\frac{\sin 2\pi t}{\pi t}\right)^2 = \prod_{j=1}^{d}\left(\frac{\sin 2\pi t_j}{\pi t_j}\right)^2 = \prod_{j=1}^{d} k(t_j)$$

$k^\star$ denotes the least decreasing majorant of k on $\mathbf{R}_+$, extended evenly to $\mathbf{R}$, and $k_\star$ the greatest decreasing minorant on $\mathbf{R}_+$, extended evenly to $\mathbf{R}$. $k^\star$ is even, positive, integrable, and decreasing on $\mathbf{R}_+$, and therefore is an element of $\Lambda(\mathbf{R})$. $k_\star$ is even, nonnegative, integrable, and decreasing on $\mathbf{R}_+$, but is not an element of $\Lambda(\mathbf{R})$ as it has zeroes. In fact, $k_\star = k \cdot \chi_{[0,1/2]}$. Numerically,

$$\int_0^\infty k^\star(t)\,dt \approx 1.068, \quad \int_0^\infty k(t)\,dt = 1, \quad \text{and} \quad \int_0^\infty k_\star(t)\,dt \approx 0.903.$$

Similarly, $K^\star$ denotes the least decreasing majorant of K on $\mathbf{R}_+^d$, extended evenly to $\mathbf{R}^d$, and $K_\star$ the greatest decreasing minorant on $\mathbf{R}_+^d$, extended evenly to $\mathbf{R}^d$. $K^\star$ is rectangular, even, positive, integrable, and decreasing on $\mathbf{R}_+^d$, and therefore is an element of $\Lambda(\mathbf{R}^d)$. $K_\star$ is rectangular, even, nonnegative, integrable, and decreasing on $\mathbf{R}_+^d$, but is not an element of $\Lambda(\mathbf{R}^d)$.

Section 4.1. Definitions.

In this section we define the Wiener transform and show that its domain of definition includes each Besicovitch space $B(2,q)$ for $1 \le q \le \infty$.

DEFINITION 4.1.1. Given $t \in \mathbf{R}^d$ and $\gamma \in \hat{\mathbf{R}}^d$ we define

$$\mathcal{E}(t,\gamma) = \prod_{j=1}^{d} \frac{e^{-2\pi i t_j \gamma_j} - \chi_{[-1,1]}(t_j)}{-2\pi i t_j}.$$

Note that if $|t_j| > 1$ for all j then $\mathcal{E}(t,\gamma) = E_{-\gamma}(t)/\Pi(-2\pi i t)$.

DEFINITION 4.1.2. Given a function $f\colon \mathbf{R}^d \to \mathbf{C}$, its **Wiener transform** is (formally)

$$Wf(\gamma) = \int_{\mathbf{R}^d} f(t)\,\mathcal{E}(t,\gamma)\,dt,$$

for $\gamma \in \hat{\mathbf{R}}^d$.

Wiener denoted Wf by s, a notation retained in [BBE], where it is called the *Wiener s-function.*

The integral defining the Wiener transform may converge in various senses, depending on the function f. For example, it may converge absolutely or only in mean, cf., Example 4.1.4 and the proof of Theorem 4.1.7.

LEMMA 4.1.3. *Given* $(t,\gamma) \in \mathbf{R}^d \times \hat{\mathbf{R}}^d$.

a. $|\Pi(t)\,\mathcal{E}(t,\gamma)| \le \pi^{-d}$.

b. $|\mathcal{E}(t,\gamma)| \le (2\pi)^{|\alpha(t)|-d} \prod_{j\in\alpha(t)} |\gamma_j|$, *where* $\alpha(t) = \{j : |t_j| \le 1\}$.

c. $\sup_{t\in[-1,1]} |\mathcal{E}(t,\gamma)| \le |\Pi(\gamma)|$.

PROOF: a. We compute

$$|\Pi(t)\,\mathcal{E}(t,\gamma)| \;=\; \prod_{j=1}^{d} \left|\frac{e^{-2\pi i t_j \gamma_j} - \chi_{[-1,1]}(t_j)}{-2\pi i}\right| \;\le\; \prod_{j=1}^{d} \frac{2}{2\pi} \;=\; \pi^{-d}.$$

b. Fix $t \in \mathbf{R}^d$ and $\gamma \in \hat{\mathbf{R}}^d$. If $t_j \in [-1,1]$ then

$$\left|\frac{e^{-2\pi i t_j \gamma_j} - 1}{-2\pi i t_j}\right| \;\le\; \left|\frac{2\pi t_j \gamma_j}{2\pi t_j}\right| \;=\; |\gamma_j|.$$

On the other hand, if $t_j \notin [-1,1]$ then

$$\left|\frac{e^{-2\pi i t_j \gamma_j}}{-2\pi i t_j}\right| \;\le\; \frac{1}{|2\pi t_j|} \;\le\; \frac{1}{2\pi}.$$

The result therefore follows by multiplication.

c. Follows immediately from b since $t \in [-1,1]$ implies $\alpha(t) = \{1,\dots,d\}$. ∎

We give examples of the various senses in which the Wiener transform may converge.

EXAMPLE 4.1.4. a. Given $f \in L^1(\mathbf{R}^d_*)$, we have from Lemma 4.1.3a that

$$\int_{\mathbf{R}^d} |f(t)\,\mathcal{E}(t,\gamma)|\,dt \;\le\; \pi^{-d} \int_{\mathbf{R}^d_*} |f(t)|\,\frac{dt}{|\Pi(t)|} \;=\; \pi^{-d}\,\|f\|_{L^1(\mathbf{R}^d_*)} \;<\; \infty.$$

Thus Wf converges absolutely and $\|Wf\|_\infty \le \pi^{-d}\,\|f\|_1$, so W is a continuous map of $L^1(\mathbf{R}^d_*)$ into $L^\infty(\hat{\mathbf{R}}^d_*)$.

Note that since $B(p,1) \subset L^p(\mathbf{R}^d_*) \cap L^1(\mathbf{R}^d_*)$, the Wiener transform converges absolutely for functions in $B(p,1)$.

b. Set $d = 1$ and fix $f \in L^\infty(\mathbf{R})$. Since

$$\int_{-1}^{1} \left|f(t)\,\frac{e^{-2\pi i t\gamma} - 1}{-2\pi i t}\right| dt \;\le\; |\gamma| \int_{-1}^{1} |f(t)|\,dt \;\le\; 2\,|\gamma|\,\|f\|_\infty,$$

the integral

$$(4.1.1) \qquad \int_{-1}^{1} f(t)\, \frac{e^{-2\pi i t\gamma} - 1}{-2\pi i t}\, dt$$

converges absolutely. Now define

$$g(t) = \frac{f(t)}{-2\pi i t}\, \chi_{(-\infty,-1]\cup[1,\infty)}(t).$$

Then

$$\int_{-\infty}^{\infty} |g(t)|^2\, dt \le \frac{\|f\|_\infty^2}{2\pi^2} \int_1^\infty t^{-2}\, dt < \infty,$$

so $g \in L^2(\mathbf{R})$. Therefore, its Fourier transform $\hat{g}$ converges in mean. Evaluating,

$$(4.1.2) \qquad \hat{g}(\gamma) = \int_{|t|>1} f(t)\, \frac{e^{-2\pi i t\gamma} - 1}{-2\pi i t}\, dt.$$

The Wiener transform of f is the sum of the two integrals (4.1.1) and (4.1.2), so is well-defined. Moreover, $Wf \in L^\infty_{\text{loc}}(\hat{\mathbf{R}}) + L^2(\hat{\mathbf{R}}) \subset L^2_{\text{loc}}(\hat{\mathbf{R}})$.

We partition $\mathbf{R}^d$ and $\mathbf{Z}^d$ into subsets $\mathbf{R}^d_\alpha$ and $\mathbf{Z}^d_\alpha$ as follows.

Definition 4.1.5. Given a subset $\alpha \subset \{1, \dots, d\}$.

a. $\mathbf{R}^d_\alpha = \{t \in \mathbf{R}^d : |t_j| < 1 \text{ for } j \in \alpha,\ |t_j| \ge 1 \text{ for } j \notin \alpha\}$.

b. $\mathbf{Z}^d_\alpha = \{n \in \mathbf{Z}^d : n_j < 0 \text{ for } j \in \alpha,\ n_j \ge 0 \text{ for } j \notin \alpha\}$.

Remark 4.1.6. a. $\{\mathbf{R}^d_\alpha\}$ is a partition of $\mathbf{R}^d$ and $\{\mathbf{Z}^d_\alpha\}$ is a partition of $\mathbf{Z}^d$.

b. If $\alpha = \{1, \dots, d\}$ then $\mathbf{R}^d_\alpha = (-1, 1)$. All other $\mathbf{R}^d_\alpha$ are unbounded and disconnected, consisting of $2^{d-|\alpha|}$ connected components.

c. $\mathbf{R}^d_\alpha = \bigcup_{n\in\mathbf{Z}^d_\alpha,\,\sigma\in\Omega^d} \sigma[2^n, 2^{n+1})$.

THEOREM 4.1.7. *The Wiener transform is defined on $W_*(L^2, L^\infty)$ and is a continuous linear map of $W_*(L^2, L^\infty)$ into $L^2_{\text{loc}}(\hat{\mathbf{R}}^d)$.*

PROOF: Fix $f \in W_*(L^2, L^\infty)$. It suffices to show that

$$F_\alpha(\gamma) = \int_{\mathbf{R}^d_\alpha} f(t)\,\mathcal{E}(t,\gamma)\,dt$$

is well-defined and an element of $L^2_{\text{loc}}(\hat{\mathbf{R}}^d)$ for each $\alpha \subset \{1,\dots,d\}$, and that the mapping $f \mapsto F_\alpha$ is continuous. Recall that

$$\|f\|_{W_*(L^2,L^\infty)} = \sup_{n\in\mathbf{Z}^d,\,\sigma\in\Omega^d} \left(\int_{\sigma[2^n,2^{n+1}]} |f(t)|^2\,\frac{dt}{|\Pi(t)|}\right)^{1/2}.$$

a. Assume first that $\alpha = \{1,\dots,d\}$, and note the following facts.

a1. $\mathbf{R}^d_\alpha = (-1,1)^d = \bigcup_{n\in\mathbf{Z}^d_\alpha,\,\sigma\in\Omega^d} \sigma[2^n, 2^{n+1})$.

a2. $t \in \mathbf{R}^d_\alpha \Rightarrow |\mathcal{E}(t,\gamma)| \le |\Pi(\gamma)|$.

a3. $n \in \mathbf{Z}^d_\alpha \Rightarrow n_j < 0$ for all j.

a4. $\sum_{n\in\mathbf{Z}^d_\alpha} \Pi(2^n) = 1$.

In the following calculation, the summations are over $n \in \mathbf{Z}^d_\alpha$ and $\sigma \in \Omega^d$.

$$\begin{aligned}
|F_\alpha(\gamma)| &\le \int_{\mathbf{R}^d_\alpha} |f(t)\,\mathcal{E}(t,\gamma)|\,dt \\
&\le |\Pi(\gamma)| \sum_{n,\sigma} \int_{\sigma[2^n,2^{n+1}]} |f(t)|\,dt \\
&\le |\Pi(\gamma)| \sum_{n,\sigma} \left(\Pi(2^n) \int_{\sigma[2^n,2^{n+1}]} |f(t)|^2\,dt\right)^{1/2}
\end{aligned}$$

$$\leq |\Pi(\gamma)| \sum_{n,\sigma} \left(\Pi(2^n)^2 \int_{\sigma[2^n,2^{n+1}]} |f(t)|^2 \frac{dt}{|\Pi(t)|} \right)^{1/2}$$

$$\leq |\Pi(\gamma)| \sup_{n,\sigma} \left(\int_{\sigma[2^n,2^{n+1}]} |f(t)|^2 \frac{dt}{|\Pi(t)|} \right)^{1/2} \sum_{n,\sigma} \Pi(2^n)$$

$$\leq 2^d \, |\Pi(\gamma)| \, \|f\|_{W_*(L^2,L^\infty)}.$$

Therefore F_α converges absolutely and is an element of $L^\infty_{\text{loc}}(\hat{\mathbf{R}}^d) \subset L^2_{\text{loc}}(\hat{\mathbf{R}}^d)$.

The mapping $f \mapsto F_\alpha$ is clearly continuous.

b. Assume now that $\alpha = \emptyset$, and note the following facts.

b1. $\mathbf{R}^d_\emptyset \;=\; \left[(-\infty,-1] \cup [1,\infty)\right]^d \;=\; \bigcup_{n \in \mathbf{Z}^d_\emptyset, \sigma \in \Omega^d} \sigma[2^n, 2^{n+1})$.

b2. $t \in \mathbf{R}^d_\emptyset \;\Rightarrow\; \mathcal{E}(t,\gamma) = E_{-\gamma}(t)/\Pi(-2\pi i t)$.

b3. $n \in \mathbf{Z}^d_\emptyset \;\Rightarrow\; n_j \geq 0$ for all j.

b4. $\sum_{n \in \mathbf{Z}^d_\emptyset} \Pi(2^{-n}) = 2^d$.

In the following calculation, the summations are over $n \in \mathbf{Z}^d_\emptyset$ and $\sigma \in \Omega^d$.

$$\begin{aligned}
\int_{\mathbf{R}^d_\emptyset} \left|\frac{f(t)}{\Pi(t)}\right|^2 dt &= \sum_{n,\sigma} \int_{\sigma[2^n,2^{n+1}]} \frac{|f(t)|^2}{|\Pi(t)|} \frac{dt}{|\Pi(t)|} \\
&\leq \sum_{n,\sigma} \frac{1}{\Pi(2^n)} \int_{\sigma[2^n,2^{n+1}]} |f(t)|^2 \frac{dt}{|\Pi(t)|} \\
&\leq \sup_{n,\sigma} \left(\int_{\sigma[2^n,2^{n+1}]} |f(t)|^2 \frac{dt}{|\Pi(t)|} \right) \sum_{n,\sigma} \frac{1}{\Pi(2^n)} \\
&\leq 2^{2d} \, \|f\|^2_{W_*(L^2,L^\infty)}.
\end{aligned}$$

Therefore,

$$G(t) \;=\; \frac{f(t)}{\Pi(-2\pi i t)} \chi_{\mathbf{R}^d_\emptyset}(t) \;\in\; L^2(\mathbf{R}^d).$$

The Fourier transform of G therefore converges in mean square. Moreover,

$$\begin{aligned}\hat{G}(\gamma) &= \int_{\mathbf{R}^d} G(t)\, E_{-\gamma}(t)\, dt \\ &= \int_{\mathbf{R}^d_\emptyset} f(t)\, \frac{E_{-\gamma}(t)}{\Pi(-2\pi i t)}\, dt \\ &= \int_{\mathbf{R}^d_\emptyset} f(t)\, \mathcal{E}(t,\gamma)\, dt \\ &= F_\emptyset(\gamma).\end{aligned}$$

Thus $F_\emptyset$ converges in mean square, and $F_\emptyset \in L^2(\hat{\mathbf{R}}^d)$.

c. Finally, assume α is a proper, nonempty subset of $\{1,\ldots,d\}$. Without loss of generality, let $\alpha = \{1,\ldots,k\}$ and $\beta = \{k+1,\ldots,d\}$. For notational convenience, given $t \in \mathbf{R}^d$ we write $t_\alpha = (t_1,\ldots,t_k)$ and $t_\beta = (t_{k+1},\ldots,t_d)$, so $t = (t_\alpha, t_\beta)$. Conversely, given $t_\alpha \in \mathbf{R}^k$ and $t_\beta \in \mathbf{R}^{d-k}$ we understand that $t \in \mathbf{R}^d$ is (t_α, t_β).

Note the following facts, cf., a1–a4.

c1. $\mathbf{R}^k_\alpha = (-1,1)^k = \bigcup_{n_\alpha \in \mathbf{Z}^k_\alpha,\, \sigma_\alpha \in \Omega^k} \sigma_\alpha[2^{n_\alpha}, 2^{n_\alpha+1})$.

c2. $t_\alpha \in \mathbf{R}^k_\alpha \Rightarrow |\mathcal{E}(t_\alpha,\gamma_\alpha)| \le |\Pi(\gamma_\alpha)|$.

c3. $n_\alpha \in \mathbf{Z}^k_\alpha \Rightarrow (n_\alpha)_j < 0$ for $j = 1,\ldots,k$.

c4. $\sum_{n_\alpha \in \mathbf{Z}^k_\alpha} \Pi(2^{n_\alpha}) = 1$.

The following calculation is similar to the one in part a. Given $t_\beta \in \mathbf{R}^{d-k}$ and $\gamma_\alpha \in \hat{\mathbf{R}}^k$,

$$\int_{\mathbf{R}^k_\alpha} |f(t)\,\mathcal{E}(t_\alpha,\gamma_\alpha)|\,dt_\alpha$$

$$\leq |\Pi(\gamma_\alpha)| \sum_{n_\alpha,\sigma_\alpha} \int_{\sigma_\alpha[2^{n_\alpha},2^{n_\alpha+1}]} |f(t)|\,dt_\alpha$$

$$\leq |\Pi(\gamma_\alpha)| \sum_{n_\alpha,\sigma_\alpha} \left(\Pi(2^{n_\alpha}) \int_{\sigma_\alpha[2^{n_\alpha},2^{n_\alpha+1}]} |f(t)|^2\,dt_\alpha\right)^{1/2}$$

$$\leq |\Pi(\gamma_\alpha)| \sum_{n_\alpha,\sigma_\alpha} \left(\Pi(2^{n_\alpha})^2 \int_{\sigma_\alpha[2^{n_\alpha},2^{n_\alpha+1}]} |f(t)|^2 \frac{dt_\alpha}{|\Pi(t_\alpha)|}\right)^{1/2}$$

$$\leq |\Pi(\gamma_\alpha)| \sup_{n_\alpha,\sigma_\alpha} \left(\int_{\sigma_\alpha[2^{n_\alpha},2^{n_\alpha+1}]} |f(t)|^2 \frac{dt_\alpha}{|\Pi(t_\alpha)|}\right)^{1/2} \sum_{n_\alpha,\sigma_\alpha} \Pi(2^{n_\alpha})$$

$$= 2^k\,|\Pi(\gamma_\alpha)| \sup_{n_\alpha,\sigma_\alpha} \left(\int_{\sigma_\alpha[2^{n_\alpha},2^{n_\alpha+1}]} |f(t)|^2 \frac{dt_\alpha}{|\Pi(t_\alpha)|}\right)^{1/2}.$$

Therefore,

$$G_{\gamma_\alpha}(t_\beta) = \left(\int_{\mathbf{R}^k_\alpha} f(t)\,\mathcal{E}(t_\alpha,\gamma_\alpha)\,dt_\alpha\right) \frac{\chi_{\mathbf{R}^{d-k}_\emptyset}(t_\beta)}{\Pi(-2\pi i t_\beta)}$$

is well-defined a.e. Note the following facts, cf., b1–b4.

c5. $\mathbf{R}^{d-k}_\emptyset = [(-\infty,1]\cup[1,\infty)]^{d-k} = \bigcup_{n_\beta\in\mathbf{Z}^{d-k}_\emptyset,\sigma_\beta\in\Omega^d} \sigma_\beta[2^{n_\beta},2^{n_\beta+1})$.

c6. $t_\beta\in\mathbf{R}^{d-k}_\emptyset \Rightarrow \mathcal{E}(t_\beta,\gamma_\beta) = E_{-\gamma_\beta}(t_\beta)/\Pi(-2\pi i t_\beta)$.

c7. $n_\beta\in\mathbf{Z}^{d-k}_\emptyset \Rightarrow (n_\beta)_j\geq 0$ for $j=k+1,\ldots,d$.

c8. $\sum_{n_\beta\in\mathbf{Z}^{d-k}_\emptyset} \Pi(2^{-n_\beta}) = 2^{d-k}$.

The following calculation is similar to the one in part b.

$$\int_{\mathbf{R}^{d-k}} |G_{\gamma_\alpha}(t_\beta)|^2\,dt_\beta$$

$$= \int_{\mathbf{R}^{d-k}_\emptyset} \frac{1}{|\Pi(-2\pi i t_\beta)|^2} \left|\int_{\mathbf{R}^k_\alpha} f(t)\,\mathcal{E}(t_\alpha,\gamma_\alpha)\,dt_\alpha\right|^2 dt_\beta$$

$$\leq (2\pi)^{-2(d-k)} \sum_{n_\beta, \sigma_\beta} \int_{\sigma_\beta[2^{n_\beta}, 2^{n_\beta+1}]} \frac{2^{2k}}{|\Pi(t_\beta)|^2} |\Pi(\gamma_\alpha)|^2$$

$$\sup_{n_\alpha, \sigma_\alpha} \int_{\sigma_\alpha[2^{n_\alpha}, 2^{n_\alpha+1}]} |f(t)|^2 \frac{dt_\alpha}{|\Pi(t_\alpha)|} \, dt_\beta$$

$$\leq (2\pi)^{2(k-d)} \, 2^{2k} \, |\Pi(\gamma_\alpha)|^2 \sum_{n_\beta, \sigma_\beta} \frac{1}{\Pi(2^{n_\beta})} \int_{\sigma_\beta[2^{n_\beta}, 2^{n_\beta+1}]}$$

$$\sup_{n_\alpha, \sigma_\alpha} \int_{\sigma_\alpha[2^{n_\alpha}, 2^{n_\alpha+1}]} |f(t)|^2 \frac{dt_\alpha}{|\Pi(t_\alpha)|} \frac{dt_\beta}{|\Pi(t_\beta)|}$$

$$\leq (2\pi)^{2(k-d)} \, 2^{2k} \, |\Pi(\gamma_\alpha)|^2 \sup_{n_\beta, \sigma_\beta} \sup_{n_\alpha, \sigma_\alpha} \int_{\sigma_\beta[2^{n_\beta}, 2^{n_\beta+1}]}$$

$$\int_{\sigma_\alpha[2^{n_\alpha}, 2^{n_\alpha+1}]} |f(t)|^2 \frac{dt_\alpha}{|\Pi(t_\alpha)|} \frac{dt_\beta}{|\Pi(t_\beta)|} \sum_{n_\beta, \sigma_\beta} \frac{1}{\Pi(2^{n_\beta})}$$

$$= (2\pi)^{2(k-d)} \, 2^{2k} \, 2^{2(d-k)} \, |\Pi(\gamma_\alpha)|^2 \sup_{n, \sigma} \int_{\sigma[2^n, 2^{n+1}]} |f(t)|^2 \frac{dt}{|\Pi(t)|}$$

$$\leq \pi^{2k-2d} \, 2^{2k} \, |\Pi(\gamma_\alpha)|^2 \, \|f\|^2_{W_*(L^2, L^\infty)}.$$

Thus $G_{\gamma_\alpha} \in L^2(\mathbf{R}^{d-k})$. Its Fourier transform therefore converges in mean square, and we compute

$$\begin{aligned}
\hat{G}_{\gamma_\alpha}(\gamma_\beta) &= \int_{\mathbf{R}^{d-k}} G_{\gamma_\alpha}(t_\beta) \, E_{-\gamma_\beta}(t_\beta) \, dt_\beta \\
&= \int_{\mathbf{R}^{d-k}_\beta} \int_{\mathbf{R}^k_\alpha} f(t) \, \mathcal{E}(t_\alpha, \gamma_\alpha) \frac{E_{-\gamma_\beta}(t_\beta)}{\Pi(-2\pi i t_\beta)} \, dt_\alpha \, dt_\beta \\
&= \int_{\mathbf{R}^{d-k}_\beta} \int_{\mathbf{R}^k_\alpha} f(t) \, \mathcal{E}(t_\alpha, \gamma_\alpha) \, \mathcal{E}(t_\beta, \gamma_\beta) \, dt_\alpha \, dt_\beta \\
&= \int_{\mathbf{R}^d_\alpha} f(t) \, \mathcal{E}(t, \gamma) \, dt \\
&= F_\alpha(\gamma),
\end{aligned}$$

since $\mathbf{R}^d_\alpha = \mathbf{R}^k_\alpha \times \mathbf{R}^{d-k}_\beta$. Thus F_α is well-defined. Given $\gamma \in \hat{\mathbf{R}}^d$ we have by the Plancherel theorem that

$$\begin{aligned}\int_{\hat{\mathbf{R}}^{d-k}} |F_\alpha(\gamma)|^2 \, d\gamma_\beta &= \int_{\hat{\mathbf{R}}^{d-k}} |\hat{G}_{\gamma_\alpha}(\gamma_\beta)|^2 \, d\gamma_\beta \\ &= \int_{\mathbf{R}^{d-k}} |G_{\gamma_\alpha}(t_\beta)|^2 \, dt_\beta \\ &\leq \pi^{2k-2d} \, 2^{2k} \, |\Pi(\gamma_\alpha)|^2 \, \|f\|^2_{W_*(L^2,L^\infty)}.\end{aligned}$$

Therefore, if $T \in \mathbf{R}^d_+$ then

$$\begin{aligned}\int_{R_T} |F_\alpha(\gamma)|^2 \, d\gamma &= \int_{R_{T_\alpha}} \int_{R_{T_\beta}} |F_\alpha(\gamma)|^2 \, d\gamma_\beta \, d\gamma_\alpha \\ &\leq \pi^{2k-2d} \, 2^{2k} \, \|f\|^2_{W_*(L^2,L^\infty)} \int_{R_{T_\alpha}} |\Pi(\gamma_\alpha)|^2 \, d\gamma_\alpha \\ &= \pi^{2k-2d} \, 2^{2k} \, \|f\|^2_{W_*(L^2,L^\infty)} \, (2/3)^{d-k} \, \Pi(T_\alpha)^3.\end{aligned}$$

Thus $F_\alpha \in L^2_{\mathrm{loc}}(\hat{\mathbf{R}}^d)$, and the mapping $f \mapsto F_\alpha$ is continuous. ∎

REMARK 4.1.8. a. It is shown in [BBE] that W maps $L^2_v(\mathbf{R}^d)$ into $L^2_{\mathrm{loc}}(\hat{\mathbf{R}}^d)$, where $v(t) = \prod_1^d (1 + |t_j|)^{-2}$, i.e., v is as in Theorem 3.3.1 with $a = 2$. From Theorems 3.2.4 and 3.3.1 (or 3.3.9), $W_*(L^2, L^\infty) = B(2,\infty) \subset L^2_v(\mathbf{R}^d)$.

b. Since $B(2,q) = W_*(L^2, L^q) \subset W_*(L^2, L^\infty) = B(2,\infty)$, the Wiener transform is well-defined on each $B(2,q)$. It is not difficult to modify the proof of Theorem 4.1.7 to show this directly.

Section 4.2. The symmetric difference operator.

In this section we define the rectangular, higher-dimensional symmetric difference operators Δ_λ, and compute $\Delta_\lambda W f$, where W is the Wiener transform.

DEFINITION 4.2.1.

a. Given $\lambda \in \hat{\mathbf{R}}^d$, the **symmetric difference operator** Δ_λ is

$$\Delta_\lambda = 2^{-d} \sum_{\sigma \in \Omega^d} \Pi(\sigma) T_{-\sigma\lambda},$$

where T is the translation operator. That is,

$$\Delta_\lambda F(\gamma) = 2^{-d} \sum_{\sigma \in \Omega^d} \Pi(\sigma) F(\gamma + \sigma\lambda).$$

b. Given $\lambda \in \hat{\mathbf{R}}^d$, the **one-sided difference operator** Δ_λ^+ is

$$\Delta_\lambda^+ = 2^{-d} \sum_{\sigma \in \{0,1\}^d} \Pi(\sigma) T_{-\sigma\lambda}.$$

c. Given $\lambda \in \hat{\mathbf{R}}$ and $1 \le j \le d$, the **directional difference operator** Δ_λ^j is

$$\Delta_\lambda^j = \tfrac{1}{2}(T_{-\lambda e_j} - T_{\lambda e_j}),$$

where $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the j^{th} unit vector.

d. Given $\lambda \in \hat{\mathbf{R}}$ and $1 \le j \le d$, the **one-sided directional difference operator** Δ_λ^{j+} is

$$\Delta_\lambda^{j+} = \tfrac{1}{2}(T_{-\lambda e_j} - I),$$

where I is the identity operator.

REMARK 4.2.2. a. For $d = 1$,

$$\Delta_\lambda F(\gamma) = \tfrac{1}{2}\big[F(\gamma+\lambda) - F(\gamma-\lambda)\big],$$

$$\Delta_\lambda^+ F(\gamma) = \tfrac{1}{2}\big[F(\gamma+\lambda) - F(\gamma)\big],$$

and $\Delta_\lambda^1 = \Delta_\lambda$, $\Delta_\lambda^{1+} = \Delta_\lambda^+$.

b. For $d = 2$,

$$\Delta_\lambda F(\gamma) = \tfrac{1}{4}\big[F(\gamma_1+\lambda_1,\gamma_2+\lambda_2) - F(\gamma_1+\lambda_1,\gamma_2-\lambda_2) - F(\gamma_1-\lambda_1,\gamma_2+\lambda_2) + F(\gamma_1-\lambda_1,\gamma_2-\lambda_2)\big],$$

$$\Delta_\lambda^+ F(\gamma) = \tfrac{1}{4}\big[F(\gamma_1+\lambda_1,\gamma_2+\lambda_2) - F(\gamma_1+\lambda_1,\gamma_2) - F(\gamma_1,\gamma_2+\lambda_2) + F(\gamma_1,\gamma_2)\big],$$

for $\lambda \in \hat{\mathbf{R}}^2$. Also,

$$\Delta_\lambda^1 F(\gamma) = \tfrac{1}{2}\big[F(\gamma_1+\lambda,\gamma_2) - F(\gamma_1-\lambda,\gamma_2)\big],$$

$$\Delta_\lambda^{1+} F(\gamma) = \tfrac{1}{2}\big[F(\gamma_1+\lambda,\gamma_2) - F(\gamma_1,\gamma_2)\big],$$

for $\lambda \in \hat{\mathbf{R}}$.

c. The operators $\{\Delta_\lambda\}$ commute, since

$$\begin{aligned}\Delta_\mu \Delta_\lambda &= 2^{-d} \sum_{\tau\in\Omega^d} \Pi(\tau) T_{-\tau\mu}\, \Delta_\lambda \\ &= 2^{-d} \sum_{\tau\in\Omega^d} \Pi(\tau) T_{-\tau\mu}\; 2^{-d} \sum_{\sigma\in\Omega^d} \Pi(\sigma) T_{-\sigma\lambda} \\ &= 2^{-d} \sum_{\sigma\in\Omega^d} \Pi(\sigma) T_{-\sigma\lambda}\; 2^{-d} \sum_{\tau\in\Omega^d} \Pi(\tau) T_{-\tau\mu} \\ &= \Delta_\lambda \Delta_\mu.\end{aligned}$$

Similarly, the operators $\{\Delta_\lambda^+\}$, $\{\Delta_\lambda^j\}$, and $\{\Delta_\lambda^{j+}\}$ commute. Moreover, all four types commute with each other.

d. Δ_λ equals the composition of the operators $\Delta_{\lambda_1}^1, \ldots, \Delta_{\lambda_d}^d$ and, similarly, Δ_λ^+ is the composition of $\Delta_{\lambda_1}^{1+}, \ldots, \Delta_{\lambda_d}^{d+}$, i.e.,

$$\Delta_\lambda = \Delta_{\lambda_1}^1 \cdots \Delta_{\lambda_d}^d,$$

$$\Delta_\lambda^+ = \Delta_{\lambda_1}^{1+} \cdots \Delta_{\lambda_d}^{d+},$$

and since these operators commute, the composition may be taken in any order. For $d = 2$ this follows from the computation

$$\begin{aligned}
\Delta_{\lambda_1}^1 \Delta_{\lambda_2}^2 &= \tfrac{1}{2}\left[T_{(-\lambda_1,0)} - T_{(\lambda_1,0)}\right] \tfrac{1}{2}\left[T_{(0,-\lambda_2)} - T_{(0,\lambda_2)}\right] \\
&= \tfrac{1}{4}\left[T_{(-\lambda_1,-\lambda_2)} - T_{(-\lambda_1,\lambda_2)} - T_{(\lambda_1,-\lambda_2)} + T_{(\lambda_1,\lambda_2)}\right] \\
&= \Delta_{(\lambda_1,\lambda_2)}.
\end{aligned}$$

The general case is similar.

e. For $\lambda \in \hat{\mathbf{R}}$,

$$\Delta_\lambda^{j+} = \tfrac{1}{2}(T_{-\lambda e_j} - I) = T_{-\lambda e_j/2}\, \tfrac{1}{2}(T_{-\lambda e_j/2} - T_{\lambda e_j/2}) = T_{-\lambda e_j/2}\, \Delta_{\lambda/2}^j.$$

Therefore, for $\lambda \in \hat{\mathbf{R}}^d$,

$$\Delta_\lambda^+ = \Delta_{\lambda_1}^{1+} \cdots \Delta_{\lambda_d}^{d+} = T_{-\lambda_1 e_1/2} \cdots T_{-\lambda_d e_d/2}\, \Delta_{\lambda_1/2}^1 \cdots \Delta_{\lambda_d/2}^d = T_{-\lambda/2}\, \Delta_{\lambda/2}.$$

f. Assume F is rectangular in the sense of Section 1.3d, i.e., $F(\gamma) = F_1(\gamma_1) \cdots F_d(\gamma_d)$. Then

$$\begin{aligned}\Delta_\lambda F(\gamma) &= \Delta^1_{\lambda_1}\cdots\Delta^d_{\lambda_d} F_1(\gamma_1)\cdots F_d(\gamma_d)\\ &= \prod_{j=1}^{d} \Delta^j_{\lambda_j} F_j(\gamma_j)\\ &= 2^{-d}\prod_{j=1}^{d}\left[F_j(\gamma_j+\lambda_j) - F_j(\gamma_j-\lambda_j)\right].\end{aligned}$$

g. Since Δ_λ is a sum of translation operators, it is a tempered distribution. Therefore, its Fourier transform exists and is a sum of modulation operators. We denote this operator by $\hat{\Delta}_\lambda$. Since the act of modulation is multiplication by an exponential, $\hat{\Delta}_\lambda$ acts by multiplying by a function which is a sum of exponentials. We denote this function by $\hat{\Delta}_\lambda$, i.e.,

$$(\Delta_\lambda F)^\wedge(\gamma) = \hat{\Delta}_\lambda(\gamma)\,\hat{F}(\gamma).$$

Similarly, $\check{\Delta}_\lambda$ is the function which is the inverse Fourier transform of Δ_λ. We evaluate these functions in Proposition 4.2.5.

DEFINITION 4.2.3. Given $\lambda \in \hat{\mathbf{R}}^d$.

a. The **sine product function** s_λ is

$$s_\lambda(t) = \Pi(\sin 2\pi\lambda t) = \prod_{j=1}^{d} \sin 2\pi\lambda_j t_j.$$

b. The **Dirichlet kernel** d_λ is

$$d_\lambda(t) = \frac{s_\lambda(t)}{\Pi(\pi t)} = \Pi\Big(\frac{\sin 2\pi\lambda t}{\pi t}\Big) = \prod_{j=1}^{d}\frac{\sin 2\pi\lambda_j t_j}{\pi t_j}.$$

REMARK 4.2.4. a. $\|s_\lambda\|_\infty = 1$.

b. $\|d_\lambda\|_\infty = 2^d\,|\Pi(\lambda)|$.

c. $d_\lambda \in L^p(\mathbf{R}^d)$ for $1 < p \leq \infty$, and $d_\lambda \notin L^1(\mathbf{R}^d)$.

d. $K(t) = |d_1(t)|^2$ and $|d_\lambda(t)|^2 = \Pi(\lambda)^2\, K(\lambda t)$.

PROPOSITION 4.2.5. *$\hat{\Delta}_\lambda(t) = i^d s_\lambda(t)$ and $\check{\Delta}_\lambda(t) = (-i)^d s_\lambda(t)$.*

PROOF: We prove only the first statement as the second is similar. We compute

$$\begin{aligned}
\hat{\Delta}_\lambda(t) &= \Big(2^{-d} \sum_{\sigma\in\Omega^d} \Pi(\sigma) T_{-\sigma\lambda}\Big)^\wedge(t) \\
&= 2^{-d} \sum_{\sigma\in\Omega^d} \Pi(\sigma)\, E_{\sigma\lambda}(t) \\
&= 2^{-d} \sum_{\sigma_1\in\{-1,1\}} \cdots \sum_{\sigma_d\in\{-1,1\}} \sigma_1\cdots\sigma_d\, e^{2\pi i\sigma_1\lambda_1 t_1}\cdots e^{2\pi i\sigma_d\lambda_d t_d} \\
&= 2^{-d} \prod_{j=1}^{d} \sum_{\sigma_j\in\{-1,1\}} \sigma_j\, e^{2\pi i\sigma_j\lambda_j t_j} \\
&= 2^{-d} \prod_{j=1}^{d} \big(e^{2\pi i\lambda_j t_j} - e^{-2\pi i\lambda_j t_j}\big) \\
&= 2^{-d} \prod_{j=1}^{d} 2i\sin 2\pi\lambda_j t_j \\
&= i^d s_\lambda(t). \ \blacksquare
\end{aligned}$$

We characterize in the following proposition those functions F such that $\Delta_\lambda F(\gamma) \equiv 0$ for all λ. For example, constant functions satisfy this condition. In one dimension there are no other examples.

PROPOSITION 4.2.6. a. *A function $F\colon\hat{\mathbf{R}}^d \to \mathbf{C}$ with*

$$\Delta_\lambda F \equiv 0 \text{ for all } \lambda \in \hat{\mathbf{R}}^d \tag{4.2.1}$$

is completely determined by its values on the coordinate hyperplanes $\mathbf{A}_d =$

$\{\gamma \in \hat{\mathbf{R}}^d : \Pi(\gamma) = 0\}$. *Conversely, every function* $F: \mathbf{A}_d \to \mathbf{C}$ *uniquely determines a function* F *on* $\hat{\mathbf{R}}^d$ *which satisfies* (4.2.1).

b. *For* $d = 1$, *a function* F *satisfies* (4.2.1) *if and only if it is constant.*

c. *For* $d > 1$, *all constant functions satisfy* (4.2.1), *but they do not exhaust the class of* F *satisfying* (4.2.1).

PROOF: Assume first that $d = 1$, and recall that $\Delta_\lambda F(\gamma) = \frac{1}{2}\big[F(\gamma + \lambda) - F(\gamma - \lambda)\big]$. Setting $\gamma = \lambda$, we find $0 \equiv 2\,\Delta_\gamma F(\gamma) = F(2\gamma) - F(0)$. Thus $F(\gamma) \equiv F(0)$ for all γ, so F is constant.

Now assume that $d = 2$ (the general case is similar). Recall that

$$\begin{aligned}\Delta_\lambda F(\gamma) \;=\; \tfrac{1}{4}\big[F(\gamma_1+\lambda_1,\gamma_2+\lambda_2) \;-\; F(\gamma_1+\lambda_1,\gamma_2-\lambda_2) \\ -\; F(\gamma_1-\lambda_1,\gamma_2+\lambda_2) \;+\; F(\gamma_1-\lambda_1,\gamma_2-\lambda_2)\big].\end{aligned}$$

Setting $\gamma = \lambda$, we find

$$0 \;\equiv\; 4\,\Delta_\gamma F(\gamma) \;=\; F(2\gamma_1,2\gamma_2) - F(2\gamma_1,0) - F(0,2\gamma_2) - F(0,0).$$

The last three terms of this expression lie on the coordinate axes, so the value of $F(2\gamma_1,2\gamma_2)$ is completely determined by the values of F on the coordinate axes.

Conversely, assume $F: \mathbf{A}_2 \to \mathbf{C}$ is given. Extend F to $\mathbf{R}^2$ by defining

$$F(\gamma_1,\gamma_2) \;=\; F(\gamma_1,0) \;+\; F(0,\gamma_2) \;-\; F(0,0).$$

Given any γ, $\lambda \in \hat{\mathbf{R}}^2$, we then have

$$\begin{aligned}4\,\Delta_\lambda F(\gamma) \;=\; F(\gamma_1+\lambda_1,\gamma_2+\lambda_2) \;-\; F(\gamma_1+\lambda_1,\gamma_2-\lambda_2) \\ -\; F(\gamma_1-\lambda_1,\gamma_2+\lambda_2) \;+\; F(\gamma_1-\lambda_1,\gamma_2-\lambda_2)\end{aligned}$$

$$
\begin{aligned}
&= F(\gamma_1+\lambda_1,\gamma_2+\lambda_2) \;-\; F(\gamma_1+\lambda_1,0) \\
&\qquad -\; F(0,\gamma_2+\lambda_2) \;+\; F(0,0) \\
&\quad -\; F(\gamma_1+\lambda_1,\gamma_2-\lambda_2) \;+\; F(\gamma_1+\lambda_1,0) \\
&\qquad +\; F(0,\gamma_2-\lambda_2) \;-\; F(0,0) \\
&\quad -\; F(\gamma_1-\lambda_1,\gamma_2+\lambda_2) \;+\; F(\gamma_1-\lambda_1,0) \\
&\qquad +\; F(0,\gamma_2+\lambda_2) \;-\; F(0,0) \\
&\quad +\; F(\gamma_1-\lambda_1,\gamma_2-\lambda_2) \;-\; F(\gamma_1-\lambda_1,0) \\
&\qquad -\; F(0,\gamma_2-\lambda_2) \;+\; F(0,0) \\
&\equiv 0.
\end{aligned}
$$

Thus F satisfies (4.2.1). ∎

We apply the difference operator to $\mathcal{E}(t,\cdot)$.

Lemma 4.2.7 [**BBE**]. $\Delta_\lambda \mathcal{E}(t,\gamma) = 2^{-d}\, E_{-\gamma}(t)\, d_\lambda(t)$.

Proof: By Remark 4.2.2d, it suffices to show $\Delta_\lambda^j \mathcal{E}(t,\gamma) = \frac{1}{2}\, E_{-\gamma_j}(t_j)\, d_{\lambda_j}(t_j)$. Since this calculation is the same as the one-dimensional case, we assume $d=1$ and compute

$$
\begin{aligned}
2\,\Delta_\lambda \mathcal{E}(t,\gamma) &= \mathcal{E}(t,\gamma+\lambda) \;-\; \mathcal{E}(t,\gamma-\lambda) \\
&= \frac{e^{-2\pi i t(\gamma+\lambda)} \;-\; \chi_{[-1,1]}(t) \;-\; e^{-2\pi i t(\gamma-\lambda)} \;+\; \chi_{[-1,1]}(t)}{-2\pi i t} \\
&= e^{-2\pi i t\gamma}\, \frac{e^{2\pi i t\lambda} - e^{-2\pi i t\lambda}}{2\pi i t}
\end{aligned}
$$

$$= e^{-2\pi i t\gamma}\,\frac{\sin 2\pi\lambda t}{\pi t}$$

$$= e^{-2\pi i t\gamma}\,d_\lambda(t). \quad \blacksquare$$

PROPOSITION 4.2.8. *Given $1 \le q \le \infty$, $f \in B(2,q)$, and $\lambda \in \hat{\mathbf{R}}^d_+$.*

a. $f \cdot d_\lambda \in L^2(\mathbf{R}^d)$.

b. $\Delta_\lambda W f = 2^{-d}(f \cdot d_\lambda)^\wedge \in L^2(\hat{\mathbf{R}}^d)$.

PROOF: a. Since $B(2,q) \subset B(2,\infty) = W_*(L^2, L^\infty)$, it suffices to prove the result for $f \in W_*(L^2, L^\infty)$.

Note that

$$(4.2.2) \qquad |d_\lambda(t)|^2 = \Pi(\lambda)^2\,K(\lambda t) \le \Pi(\lambda)^2\,K^\star(\lambda t).$$

Since $K^\star(\lambda t) \in \Lambda(\mathbf{R}^d)$ for each fixed λ, we have by Lemma 3.3.5c that

$$(4.2.3) \qquad \sum_{n\in\mathbf{Z}^d} \Pi(2^n)\,K^\star(2^n\lambda) \le 2^d \int_{\mathbf{R}^d_+} K^\star(\lambda t)\,dt = \frac{2^d}{\Pi(\lambda)} \int_{\mathbf{R}^d_+} K^\star(t)\,dt.$$

From (4.2.2), (4.2.3), and the fact that $K^\star$ is even and decreasing on $\mathbf{R}^d_+$, we therefore have

$$\begin{aligned}
\int_{\mathbf{R}^d} |f(t)\,d_\lambda(t)|^2\,dt &\le \Pi(\lambda)^2 \int_{\mathbf{R}^d} |f(t)|^2\,K^\star(\lambda t)\,dt \\
&= \Pi(\lambda)^2 \sum_{n,\sigma} \int_{\sigma[2^n,2^{n+1}]} |f(t)|^2\,K^\star(\lambda t)\,dt \\
&\le \Pi(\lambda)^2 \sum_{n,\sigma} \int_{\sigma[2^n,2^{n+1}]} |f(t)|^2\,K^\star(2^n\lambda)\,\frac{\Pi(2^{n+1})}{|\Pi(t)|}\,dt \\
&\le \Pi(\lambda)^2\,\|f\|^2_{W_*(L^2,L^\infty)} \sum_{n,\sigma} \Pi(2^{n+1})\,K^\star(2^n\lambda)
\end{aligned}$$

$$\leq 2^{3d}\,\Pi(\lambda)\,\|f\|^2_{W_*(L^2,L^\infty)} \int_{\mathbf{R}^d_+} K^\star(t)\,dt$$

$$< \infty.$$

b. Fix $f \in B(2,q)$. From Theorem 4.1.7, Wf is well-defined and is an element of $L^2_{\text{loc}}(\hat{\mathbf{R}}^d)$. Using part a and Lemma 4.2.7, we compute

$$\begin{aligned}
\Delta_\lambda W f(\gamma) &= \Delta_\lambda \int_{\mathbf{R}^d} f(t)\,\mathcal{E}(t,\gamma)\,dt \\
&= \int_{\mathbf{R}^d} f(t)\,\Delta_\lambda\,\mathcal{E}(t,\gamma)\,dt \\
&= 2^{-d} \int_{\mathbf{R}^d} f(t)\,d_\lambda(t)\,E_{-\gamma}(t)\,dt \\
&= 2^{-d}(f \cdot d_\lambda)^\wedge(\gamma).
\end{aligned}$$

The fact that Δ_λ can be interchanged with the integral in the above calculation follows immediately from the fact that Δ_λ is a sum of translation operators acting only on γ. ∎

REMARK 4.2.9. In [BBE], Proposition 4.2.8 is proved (using different estimates) for all $f \in L^2_v(\mathbf{R}) \supset B(2,q)$, where v is as in Theorem 3.3.1 with $a = 2$.

Section 4.3. The variation spaces.

In this section we define the variation spaces $V(p,q)$. Definitions for $d=1$ were given in (0.2.16).

DEFINITION 4.3.1. Given $1 \le p,\ q < \infty$, the **variation space** $V(p,q)$ is the space of functions $F\colon \hat{\mathbf{R}}^d \to \mathbf{C}$ for which the seminorm

$$\|F\|_{V(p,q)} = \left(\int_{\hat{\mathbf{R}}^d_+} \left(\frac{2^d}{\Pi(\lambda)} \int_{\hat{\mathbf{R}}^d} |\Delta_\lambda F(\gamma)|^p \, d\gamma\right)^{q/p} \frac{d\lambda}{\Pi(\lambda)}\right)^{1/q}$$

is finite. The standard adjustments are made if p or q is infinity, namely,

$$\|F\|_{V(p,\infty)} = \operatorname*{ess\,sup}_{\lambda \in \hat{\mathbf{R}}^d_+} \left(\frac{2^d}{\Pi(\lambda)} \int_{\hat{\mathbf{R}}^d} |\Delta_\lambda F(\gamma)|^p \, d\gamma\right)^{1/p},$$

$$\|F\|_{V(\infty,q)} = \left(\int_{\hat{\mathbf{R}}^d_+} \left(\frac{2^d}{\Pi(\lambda)} \operatorname*{ess\,sup}_{\gamma \in \hat{\mathbf{R}}^d} |\Delta_\lambda F(\gamma)|\right)^{q} \frac{d\lambda}{\Pi(\lambda)}\right)^{1/q},$$

$$\|F\|_{V(\infty,\infty)} = \operatorname*{ess\,sup}_{\lambda \in \hat{\mathbf{R}}^d_+} \left(\frac{2^d}{\Pi(\lambda)} \operatorname*{ess\,sup}_{\gamma \in \hat{\mathbf{R}}^d} |\Delta_\lambda F(\gamma)|\right).$$

We also define

$$V(p,\lim) = \left\{F : \lim_{\lambda \to 0} \frac{2^d}{\Pi(\lambda)} \int_{\hat{\mathbf{R}}^d} |\Delta_\lambda F(\gamma)|^p \, d\gamma \text{ exists}\right\},$$

where the limit is the d-dimensional limit defined in Section 3.1.

REMARK 4.3.2. $\|\cdot\|_{V(p,q)}$ is not a norm, since $\|F\|_{V(p,q)} = 0$ implies only that $\Delta_\lambda F(\gamma) = 0$ for a.e. γ and λ. For example, all constant functions F satisfy $\|F\|_{V(p,q)} = 0$, cf., Proposition 4.2.6. However, $\|\cdot\|_{V(p,q)}$ is a seminorm, and therefore becomes a norm once we identify functions $F,\ G \in V(p,q)$ such that $\|F - G\|_{V(p,q)} = 0$. We adopt this convention for the remainder of this thesis, so $V(p,q)$ is at least a normed linear space. We prove in Theorem 5.2.3 that $V(p,q)$ is complete, hence a Banach space. The proof of this fact is complicated by the fact that $V(p,q)$ is not solid, i.e., given F, $G \in V(p,q)$ with $|F| \leq |G|$ a.e., we cannot conclude that $\|F\|_{V(p,q)} \leq \|G\|_{V(p,q)}$, cf., Example 4.3.3. We will not need the completeness of $V(p,q)$ for any results in this chapter.

EXAMPLE 4.3.3. Set $d = 1$, $q = \infty$, $F = \chi_{[0,1]}$, and $G = 1$. Then we have $\|G\|_{V(p,\infty)} = 0$, while $\|F\|_{V(p,\infty)} > 0$ since $d = 1$ and F is not identically constant (Proposition 4.2.6). In fact, since

$$\Delta_\lambda F(\gamma) = \chi_{[-\lambda,1-\lambda]} - \chi_{[\lambda,1+\lambda]},$$

we have

$$\int_{-\infty}^{\infty} |\Delta_\lambda F(\gamma)|^p \, d\gamma = \begin{cases} 4\lambda, & 0 < \lambda \leq 1/2, \\ 2, & 1/2 \leq \lambda, \end{cases}$$

whence

$$\|F\|_{V(p,\infty)}^p = \sup_{\lambda>0} \frac{2}{\lambda} \int_{-\infty}^{\infty} |\Delta_\lambda F(\gamma)|^p \, d\gamma = 8.$$

Thus $|F| \leq |G|$, but $\|F\|_{V(p,\infty)} = 8^{1/p} > 0 = \|G\|_{V(p,\infty)}$.

REMARK 4.3.4. a. The Wiener–Plancherel formula, as proved by Benedetto, Benke, and Evans in higher-dimensions, states that the Wiener transform W is an isometry of the nonlinear space $B(2, \lim)$ onto $V(2, \lim)$.

b. Lau and Chen proved that, for $d = 1$, W is a topological isomorphism of $B(2,\infty)$ onto $V(2,\infty)$. We discuss this result in Sections 4.4–4.5.

c. Beurling proved that, for $d = 1$, the Fourier transform is a topological isomorphism of A^2 onto $V(2,1)$. We show in Sections 4.4–4.5 that this implies that the Wiener transform W is a topological isomorphism of $B(2,1)$ onto $V(2,1)$.

d. We prove in Sections 4.4–4.5, for arbitrary $d \geq 1$, that W is a topological isomorphism of $B(2,q)$ onto $V(2,q)$, for each $1 \leq q \leq \infty$.

EXAMPLE 4.3.5. Assume f is given and its Wiener transform Wf is well-defined, e.g., $f \in B(2,q)$. Note that $|d_\lambda(t)|^2 = \Pi(\lambda)^2\, K(\lambda t)$. Therefore, by Proposition 4.2.8 and the Plancherel theorem,

$$\begin{aligned}
\|Wf\|_{V(2,q)} &= \left(\int_{\hat{\mathbf{R}}^d_+} \left(\frac{2^d}{\Pi(\lambda)} \int_{\hat{\mathbf{R}}^d} |\Delta_\lambda Wf(\gamma)|^2\, d\gamma\right)^{q/2} \frac{d\lambda}{\Pi(\lambda)}\right)^{1/q} \\
&= \left(\int_{\hat{\mathbf{R}}^d_+} \left(\frac{2^d}{\Pi(\lambda)} \int_{\hat{\mathbf{R}}^d} |2^{-d}(f \cdot d_\lambda)^\wedge(\gamma)|^2\, d\gamma\right)^{q/2} \frac{d\lambda}{\Pi(\lambda)}\right)^{1/q} \\
&= \left(\int_{\hat{\mathbf{R}}^d_+} \left(\frac{1}{2^d\,\Pi(\lambda)} \int_{\mathbf{R}^d} |f(t)\, d_\lambda(t)|^2\, dt\right)^{q/2} \frac{d\lambda}{\Pi(\lambda)}\right)^{1/q} \\
&= \left(\int_{\hat{\mathbf{R}}^d_+} \left(\frac{\Pi(\lambda)}{2^d} \int_{\mathbf{R}^d} |f(t)|^2\, K(\lambda t)\, dt\right)^{q/2} \frac{d\lambda}{\Pi(\lambda)}\right)^{1/q}.
\end{aligned}$$

Section 4.4. Continuity of the Wiener transform.

In this section we prove that the Wiener transform is a continuous mapping of $B(2,q)$ into $V(2,q)$ for each $1 \le q \le \infty$. We begin by examining Lau's proof for the case $q = \infty$. Next, we show that Beurling's proof that the Fourier transform is a continuous map of A^2 into $V(2,1)$ implies that the Wiener transform is a continuous linear map of $B(2,1)$ into $V(2,1)$. Finally, we prove the general case by using amalgam space techniques.

The following proposition, for the case $d = 1$, is due to Lau and Chen.

PROPOSITION 4.4.1. *Given a rectangular, positive, even function w, and given $1 \le p < \infty$. Then*

$$\sup_{\lambda \in \hat{\mathbf{R}}^d_+} \frac{\Pi(\lambda)}{2^d} \int_{\mathbf{R}^d} |f(t)|^p \, w(\lambda t) \, dt \le \|f\|^p_{B(p,\infty)} \int_{\mathbf{R}^d_+} w^\star(t) \, dt$$

for all measurable functions $f\colon \mathbf{R}^d \to \mathbf{C}$.

PROOF: Assume without loss of generality that $\int_{\mathbf{R}^d_+} w^\star(t)\,dt < \infty$ and that $f \in B(p,\infty)$. Extend $w^\star$ evenly to $\mathbf{R}^d$ and note that

(4.4.1)
$$\begin{aligned} \frac{\Pi(\lambda)}{2^d} \int_{\mathbf{R}^d} |f(t)|^p \, w(\lambda t) \, dt &= 2^{-d} \int_{\mathbf{R}^d} |f(t/\lambda)|^p \, w(t) \, dt \\ &\le 2^{-d} \int_{\mathbf{R}^d} |D_\lambda f(t)|^p \, w^\star(t) \, dt \end{aligned}$$

for all $\lambda \in \mathbf{R}^d_+$, where D_λ is the usual dilation operator.

Recall from Theorem 3.3.9 that $B(p,\infty) = B^p = \bigcap_{w \in \Lambda} L^p_w(\mathbf{R}^d)$, with norm equality, i.e.,

$$\|g\|^p_{B(p,\infty)} = \sup_{w \in \Lambda} \frac{\int_{\mathbf{R}^d} |g(t)|^p \, w(t) \, dt}{\int_{\mathbf{R}^d} w(t) \, dt}.$$

Since $B(p,\infty)$ is dilation isometric and $w^\star \in \Lambda(\mathbf{R}^d)$, we therefore have

(4.4.2)
$$\int_{\mathbf{R}^d} |D_\lambda f(t)|^p\, w^\star(t)\, dt \;\le\; \|D_\lambda f\|_{B(p,\infty)}^p \int_{\mathbf{R}^d} w^\star(t)\, dt$$
$$= 2^d\, \|f\|_{B(p,\infty)}^p \int_{\mathbf{R}_+^d} w^\star(t)\, dt.$$

The result follows upon combining (4.4.1) and (4.4.2). ∎

REMARK 4.4.2. Lau and Chen prove in **[CL1]** that $\int_{\mathbf{R}_+^d} w^\star(t)\, dt$ is the best possible constant in Proposition 4.4.1. Their proof of this fact is intricate, and will be omitted. We point out, however, that it carries over immediately to higher dimensions.

COROLLARY 4.4.3. *The Wiener transform W is a continuous linear map of $B(2,\infty)$ into $V(2,\infty)$, with*

$$\|W\| = \left(\int_0^\infty k^\star(t)\, dt\right)^{d/2} \approx (1.033)^d > 1.$$

PROOF: From Example 4.3.5,

$$\|Wf\|_{V(2,\infty)}^2 = \sup_{\lambda \in \hat{\mathbf{R}}_+^d} \frac{\Pi(\lambda)}{2^d} \int_{\mathbf{R}^d} |f(t)|^2\, K(\lambda t)\, dt.$$

The result therefore follows from Proposition 4.4.1, Remark 4.4.2, and the fact that $\int_{\mathbf{R}_+^d} K^\star(t)\, dt = \left(\int_0^\infty k^\star(t)\, dt\right)^d$. ∎

We turn now to Beurling's proof that the Fourier transform is a continuous linear mapping of A^2 into $V(2,1)$, which we recast as showing that the Wiener transform maps $B(2,1)$ continuously into $V(2,1)$. The critical fact, and our starting point, is the following nontrivial result, also due to Beurling, e.g., **[Be2]**.

LEMMA 4.4.4. *Given $w \in \Lambda(\mathbf{R})$ and given $0 < a < 1 < b < \infty$, there exists a function w^* such that*

a. $w^* \geq w$,

b. $t^a w^*(t)$ *is decreasing on* $\mathbf{R}_+$,

c. $t^b w^*(t)$ *is increasing on* $\mathbf{R}_+$,

d. $\int_0^\infty w^*(t)\,dt \leq \frac{b}{(1-a)(b-1)} \int_0^\infty w(t)\,dt$.

PROPOSITION 4.4.5. *The Wiener transform W is a continuous linear map of $B(2,1)$ into $V(2,1)$, with*

$$\|W\| \;\leq\; (48/\pi)^{d/2} \;\approx\; (3.909)^d.$$

PROOF: Fix $f \in B(2,1)$; then $f(t)/\Pi(t) \in A^2$ by Corollary 3.3.11, and

$$\begin{aligned} \|f\|_{B(2,1)} &= 2^{-d}\,\|f(t)/\Pi(t)\|_{A^2} \\ &= 2^{-d} \inf_{w\in\Lambda} \left(\int_{\mathbf{R}^d} \frac{|f(t)/\Pi(t)|^2}{w(t)}\,dt\right)^{1/2} \left(\int_{\mathbf{R}^d} w(t)\,dt\right)^{1/2}. \end{aligned}$$

Fix any $w \in \Lambda(\mathbf{R}^d)$ with $\|w\|_1 = \int_{\mathbf{R}^d} w(t)\,dt = 1$. By definition, $w(t) = \prod_1^d w_j(t_j)$ for some $w_j \in \Lambda(\mathbf{R})$. Let w_j^* be the functions given by Lemma 4.4.4 applied to w_j with $a = 1/2$ and $b = 3/2$. Define $w^*(t) = \prod_1^d w_j^*(t_j)$. Then

$$\begin{aligned} \text{(4.4.3)} \qquad \int_{\mathbf{R}_+^d} w^*(t)\,dt &= \prod_{j=1}^d \int_0^\infty w_j^*(t_j)\,dt_j \\ &\leq \prod_{j=1}^d \frac{3/2}{(1/2)(1/2)} \int_0^\infty w_j(t_j)\,dt_j \end{aligned}$$

$$= 6^d \int_{\mathbf{R}^d_+} w(t)\,dt$$

$$= 3^d.$$

Now, by Proposition 4.2.8 and the Plancherel theorem,

(4.4.4)
$$\begin{aligned}\eta(\lambda) &= \frac{2^d}{\Pi(\lambda)} \int_{\hat{\mathbf{R}}^d} |\Delta_\lambda W f(\gamma)|^2\, d\gamma \\ &= \frac{1}{2^d\,\Pi(\lambda)} \int_{\mathbf{R}^d} |f(t)\, d_\lambda(t)|^2\, dt \\ &= \frac{1}{(2\pi^2)^d\,\Pi(\lambda)} \int_{\mathbf{R}^d} |s_\lambda(t)\, f(t)/\Pi(t)|^2\, dt.\end{aligned}$$

Using (4.4.3) and (4.4.4) we therefore compute

(4.4.5)
$$\begin{aligned}\|Wf\|_{V(2,1)} &= \int_{\hat{\mathbf{R}}^d_+} \eta(\lambda)^{1/2}\, \frac{d\lambda}{\Pi(\lambda)} \\ &= \int_{\hat{\mathbf{R}}^d_+} \left(\frac{\eta(\lambda)}{w^*(1/2\pi\lambda)}\right)^{1/2} \left(\frac{w^*(1/2\pi\lambda)}{\Pi(\lambda)^2}\right)^{1/2} d\lambda \\ &\le \left(\int_{\hat{\mathbf{R}}^d_+} \frac{\eta(\lambda)}{w^*(1/2\pi\lambda)}\, d\lambda\right)^{1/2} \left(\int_{\hat{\mathbf{R}}^d_+} \frac{w^*(1/2\pi\lambda)}{\Pi(\lambda)^2}\, d\lambda\right)^{1/2} \\ &= \left(\int_{\hat{\mathbf{R}}^d_+} \frac{1}{(2\pi^2)^d\,\Pi(\lambda)} \int_{\mathbf{R}^d} \frac{|s_\lambda(t)\, f(t)/\Pi(t)|^2}{w^*(1/2\pi\lambda)}\, dt\, d\lambda\right)^{1/2} \\ &\qquad \times \left((2\pi)^d \int_{\mathbf{R}^d_+} w^*(t)\, dt\right)^{1/2} \\ &= \left(\frac{3}{\pi}\right)^{d/2} \left(\int_{\mathbf{R}^d} \left|\frac{f(t)}{\Pi(t)}\right|^2 \int_{\hat{\mathbf{R}}^d_+} \frac{|s_\lambda(t)|^2}{w^*(1/2\pi\lambda)}\, \frac{d\lambda}{\Pi(\lambda)}\, dt\right)^{1/2}.\end{aligned}$$

Now, ${t_j}^{1/2} w_j^*(t_j)$ is decreasing on $\mathbf{R}_+$ and ${t_j}^{3/2} w_j^*(t_j)$ is increasing on $\mathbf{R}_+$ for each j. Therefore, given t_j, $\beta_j \in \mathbf{R}_+$,

$$0 < \beta_j \le 1 \quad \Rightarrow \quad {t_j}^{3/2}\, w_j^*(t_j) \le (t_j/\beta_j)^{3/2}\, w_j^*(t_j/\beta_j),$$

$$1 \le \beta_j \quad \Rightarrow \quad {t_j}^{1/2}\, w_j^*(t_j) \le (t_j/\beta_j)^{1/2}\, w_j^*(t_j/\beta_j),$$

whence

$$0 < \beta_j \le 1 \quad \Rightarrow \quad w_j^*(t_j/\beta_j) \ \ge\ {\beta_j}^{3/2}\, w_j^*(t_j),$$

$$1 \le \beta_j \quad \Rightarrow \quad w_j^*(t_j/\beta_j) \ \ge\ {\beta_j}^{1/2}\, w_j^*(t_j).$$

Therefore,

(4.4.6)

$$\begin{aligned}
&\int_0^\infty \frac{|s_{\lambda_j}(t_j)|^2}{w_j^*(1/2\pi\lambda_j)}\,\frac{d\lambda_j}{\lambda_j} \\
&\quad= \int_0^\infty \frac{\sin^2 2\pi\lambda_j t_j}{w_j^*(1/2\pi\lambda_j)}\,\frac{d\lambda_j}{\lambda_j} \\
&\quad= \int_0^\infty \frac{\sin^2 \beta_j}{w_j^*(t_j/\beta_j)}\,\frac{d\beta_j}{\beta_j} \\
&\quad\le \int_0^1 \frac{\sin^2 \beta_j}{{\beta_j}^{3/2}\, w_j^*(t_j)}\,\frac{d\beta_j}{\beta_j} \;+\; \int_1^\infty \frac{\sin^2 \beta_j}{{\beta_j}^{1/2}\, w_j^*(t_j)}\,\frac{d\beta_j}{\beta_j} \\
&\quad= \frac{1}{w_j^*(t_j)} \int_0^1 \left|\frac{\sin\beta_j}{\beta_j}\right|^2 \frac{d\beta_j}{{\beta_j}^{1/2}} \;+\; \frac{1}{w_j^*(t_j)} \int_1^\infty \sin^2\beta_j\, \frac{d\beta_j}{{\beta_j}^{3/2}} \\
&\quad\le \frac{1}{w_j(t_j)} \int_0^1 \frac{d\beta_j}{{\beta_j}^{1/2}} \;+\; \frac{1}{w_j(t_j)} \int_1^\infty \frac{d\beta_j}{{\beta_j}^{3/2}} \\
&\quad= \frac{4}{w_j(t_j)},
\end{aligned}$$

from which it follows that

$$\text{(4.4.7)} \qquad \int_{\hat{\mathbf{R}}^d_+} \frac{|s_\lambda(t)|^2}{w^*(1/2\pi\lambda)}\,\frac{d\lambda}{\Pi(\lambda)} \ \le\ \frac{4^d}{w(t)}.$$

Substituting (4.4.7) into (4.4.5),

$$\|Wf\|_{V(2,1)} \ \le\ \left(\frac{12}{\pi}\right)^{d/2} \left(\int_{\mathbf{R}^d} \frac{|f(t)/\Pi(t)|^2}{w(t)}\,dt\right)^{1/2}.$$

Since this is true for all $w \in \Lambda(\mathbf{R}^d)$ with $\|w\|_1 = 1$,

$$\|Wf\|_{V(2,1)} \ \le\ \left(\frac{12}{\pi}\right)^{d/2} \|f(t)/\Pi(t)\|_{A^2} \ =\ \left(\frac{48}{\pi}\right)^{d/2} \|f\|_{B(2,1)}. \quad \blacksquare$$

REMARK 4.4.6. a. If we repeat the calculations in the proof of Proposition 4.4.5, keeping all estimates the same but not fixing a and b, we find that

$$\|Wf\|_{V(2,1)} \leq \left(\frac{2b(2+a-b)}{\pi a(b-1)(1-a)(2-b)}\right)^{d/2} \|f\|_{B(2,1)}.$$

The expression

$$F(a,b) = \frac{2b(2+a-b)}{\pi a(b-1)(1-a)(2-b)}$$

is clearly is not minimized at $a = 1/2$, $b = 3/2$, but at

$$a_0 = \frac{1}{2}\left(-\frac{7}{2} + \frac{8+\frac{7}{2}A_1}{A_2} - \sqrt{\left(\frac{7}{2} - \frac{8+\frac{7}{2}A_1}{A_2}\right)^2 + 2A_2 - 2A_1}\right),$$

$$b_0 = \frac{1}{2}\left(\frac{9}{2} - \frac{-4-\frac{9}{2}B_1}{B_2} - \sqrt{\left(-\frac{9}{2} + \frac{-4-\frac{9}{2}B_1}{B_2}\right)^2 - 2B_2 - 2B_1}\right),$$

where

$$A_1 = \frac{4}{3} + \frac{208}{9}\left(\frac{2971}{27} + \frac{\sqrt{-6373}}{3\sqrt{3}}\right)^{-1/3} + \left(\frac{2971}{27} + \frac{\sqrt{-6373}}{3\sqrt{3}}\right)^{1/3},$$

$$A_2 = 2\sqrt{-2 + \tfrac{1}{4}{A_1}^2},$$

$$B_1 = \frac{14}{3} + \frac{160}{9}\left(-\frac{1846}{27} + \frac{2\sqrt{-6373}}{3\sqrt{3}}\right)^{-1/3} + \left(-\frac{1846}{27} + \frac{2\sqrt{-6373}}{3\sqrt{3}}\right)^{1/3},$$

$$B_2 = 2\sqrt{12 + \tfrac{1}{4}{B_1}^2}.$$

This follows from solving for the critical points of F, using that

$$\partial_a F(a,b) = \frac{-4b + 8ab + 2a^2b + 2b^2 - 4ab^2}{\pi\, a^2\,(a-1)^2\,(2-b)\,(b-1)},$$

$$\partial_b F(a,b) = \frac{8 + 4a - 8b + 2b^2 - 2ab^2}{\pi\, a\,(a-1)\,(b-2)^2\,(b-1)^2}.$$

A_1, A_2, B_1, and B_2 are real. Numerically, $a_0 \approx 0.352$ and $b_0 \approx 1.528$, and

$$F(a_0, b_0) \approx 43.904/\pi < 48/\pi = F(1/2, 3/2).$$

Therefore,

$$\|Wf\|_{V(2,1)} \leq (3.738)^d \, \|f\|_{B(2,1)}.$$

b. If we repeat the calculations in the proof of Proposition 4.4.5 but without fixing a and b and without approximating $(\sin t)/t$ and $\sin t$ in (4.4.6), we find that

$$\|Wf\|_{V(2,1)} \leq G(a,b)^{d/2} \, \|f\|_{B(2,1)},$$

where

$$G(a,b) = \frac{2b}{\pi\,(b-1)\,(1-a)} \left(\int_0^1 \frac{\sin^2 t}{t^{b+1}}\, dt + \int_1^\infty \frac{\sin^2 t}{t^{a+1}}\, dt \right).$$

Using numerical integration, we compute

$$G(1/2, 3/2) \approx 36.85/\pi \qquad \text{and} \qquad G(a_0, b_0) \approx 32.30/\pi,$$

both of which improve on the estimates in part a. G is minimized at $a_1 \approx 0.30$, $b_1 \approx 1.54$, with $G(a_1, b_1) \approx 31.92/\pi$. Thus,

$$\|Wf\|_{V(2,1)} \leq (3.19)^d \, \|f\|_{B(2,1)}.$$

We do not know if this is the best possible constant.

Beurling's and Lau's results establish the "endpoints" of our isomorphism theorem. The "midpoint" is proved in the following proposition.

PROPOSITION 4.4.7. *The Wiener transform is an isometry of $B(2,2)$ into $V(2,2)$.*

PROOF: We compute, with the help of Example 4.3.5 and Proposition 3.2.2,

$$\begin{aligned}
\|Wf\|_{V(2,2)}^2 &= \int_{\hat{\mathbf{R}}_+^d} \frac{\Pi(\lambda)}{2^d} \int_{\mathbf{R}^d} |f(t)|^2\, K(\lambda t)\, dt\, \frac{d\lambda}{\Pi(\lambda)} \\
&= 2^{-d} \int_{\mathbf{R}^d} |f(t)|^2 \int_{\hat{\mathbf{R}}_+^d} K(\lambda t)\, d\lambda\, dt \\
&= 2^{-d} \int_{\mathbf{R}^d} |f(t)|^2 \int_{\hat{\mathbf{R}}_+^d} K(\lambda)\, d\lambda\, \frac{dt}{|\Pi(t)|} \\
&= 2^{-d} \int_{\mathbf{R}^d} |f(t)|^2\, \frac{dt}{|\Pi(t)|} \\
&= 2^{-d}\, \|f\|_{L^2(\mathbf{R}_*^d)}^2 \\
&= \|f\|_{B(2,2)}^2. \qquad \blacksquare
\end{aligned}$$

REMARK 4.4.8. W is surjective by Theorem 4.5.5, so is actually a unitary map of $B(2,2)$ onto $V(2,2)$. As $\|\cdot\|_{B(2,2)} = 2^{-d/2}\,\|\cdot\|_{L^2(\mathbf{R}_*^d)}$, W is therefore a multiple of a unitary map of $L^2(\mathbf{R}_*^d)$ onto $V(2,2)$.

Next we prove the continuity of the Wiener transform on $B(2,q)$ for $2 \leq q \leq \infty$ by using amalgam space methods. The constants we obtain are not best possible, cf., Remark 4.4.16.

PROPOSITION 4.4.9. *Given $1 \leq p \leq q \leq \infty$ with $p \neq \infty$, and given a nonnegative, even function w on $\mathbf{R}^d$. Then*

$$\left(\int_{\hat{\mathbf{R}}^d_+}\left(\frac{\Pi(\lambda)}{2^d}\int_{\mathbf{R}^d}|f(t)|^p\,w(\lambda t)\,dt\right)^{q/p}\frac{d\lambda}{\Pi(\lambda)}\right)^{1/q}$$
$$\leq\ (\log 2)^{d/q}\,2^{3d/p}\left(\int_{\mathbf{R}^d_+} w^\star(t)\,dt\right)^{1/p}\|f\|_{W_*(L^p,L^q)}$$

for all measurable functions $f\colon \mathbf{R}^d \to \mathbf{C}$ *(with the standard adjustments if* $q = \infty$*).*

PROOF: Extend $w^\star$ evenly to $\mathbf{R}^d$ and assume $q < \infty$ (the case $q = \infty$ is similar). The summations in the following calculation are over $m,\ n \in \mathbf{Z}^d$ and $\sigma \in \Omega^d$. Using the fact that $w^\star$ is even and decreasing on $\mathbf{R}^d_+$, we compute

(4.4.8)
$$\int_{\hat{\mathbf{R}}^d_+}\left(\frac{\Pi(\lambda)}{2^d}\int_{\mathbf{R}^d}|f(t)|^p\,w(\lambda t)\,dt\right)^{q/p}\frac{d\lambda}{\Pi(\lambda)}$$
$$\leq \sum_n \int_{[2^n,2^{n+1}]}\left(\frac{\Pi(\lambda)}{2^d}\sum_{m,\sigma}\int_{\sigma[2^m,2^{m+1}]}|f(t)|^p\,w^\star(\lambda t)\,dt\right)^{q/p}\frac{d\lambda}{\Pi(\lambda)}$$
$$\leq \sum_n \int_{[2^n,2^{n+1}]}\left(\frac{\Pi(2^{n+1})}{2^d}\sum_{m,\sigma}\int_{\sigma[2^m,2^{m+1}]}|f(t)|^p\,w^\star(2^{m+n})\,dt\right)^{q/p}\frac{d\lambda}{\Pi(\lambda)}$$
$$= (\log 2)^d\sum_n\left(\sum_{m,\sigma}\Pi(2^n)\,w^\star(2^{m+n})\int_{\sigma[2^m,2^{m+1}]}|f(t)|^p\,dt\right)^{q/p}$$
$$\leq (\log 2)^d\sum_n\left(\sum_{m,\sigma}\Pi(2^{m+n+1})\,w^\star(2^{m+n})\int_{\sigma[2^m,2^{m+1}]}|f(t)|^p\,\frac{dt}{|\Pi(t)|}\right)^{q/p}$$
$$= (\log 2)^d\sum_n\left(\sum_{m,\sigma}\Pi(2^{m+1})\,w^\star(2^m)\int_{\sigma[2^{m-n},2^{m-n+1}]}|f(t)|^p\,\frac{dt}{|\Pi(t)|}\right)^{q/p}$$
$$= (\log 2)^d\,2^{dq/p}\sum_n\left|\sum_{m,\sigma}F_{m,\sigma}(n)\right|^{q/p}$$
$$= (\log 2)^d\,2^{dq/p}\left\|\sum_{m,\sigma}F_{m,\sigma}\right\|_{\ell^{q/p}}^{q/p},$$

where $F_{m,\sigma}$ is the sequence

$$F_{m,\sigma}(n) \;=\; \Pi(2^m)\, w^\star(2^m) \int_{\sigma[2^{m-n}, 2^{m-n+1}]} |f(t)|^p \,\frac{dt}{|\Pi(t)|}.$$

Since $q/p \ge 1$, we can apply Minkowski's inequality in the Banach space $\ell^{q/p}$ to estimate $\left\|\sum F_{m,\sigma}\right\|_{\ell^{q/p}}$, i.e.,

(4.4.9)

$$\begin{aligned}
\Big\|\sum_{m,\sigma} F_{m,\sigma}\Big\|_{\ell^{q/p}}
&\le \sum_{m,\sigma} \|F_{m,\sigma}\|_{\ell^{q/p}} \\
&= \sum_{m,\sigma} \Big(\sum_n |F_{m,\sigma}(n)|^{q/p}\Big)^{p/q} \\
&= \sum_{m,\sigma} \Big(\sum_n \Big(\Pi(2^m)\, w^\star(2^m) \int_{\sigma[2^{m-n}, 2^{m-n+1}]} |f(t)|^p \,\frac{dt}{|\Pi(t)|}\Big)^{q/p}\Big)^{p/q} \\
&\le \sum_{m,\sigma} \Pi(2^m)\, w^\star(2^m) \Big(\sum_{n,\tau} \Big(\int_{\tau[2^{m-n}, 2^{m-n+1}]} |f(t)|^p \,\frac{dt}{|\Pi(t)|}\Big)^{q/p}\Big)^{p/q} \\
&= 2^d \sum_{m} \Pi(2^m)\, w^\star(2^m) \Big(\sum_{n,\tau} \Big(\int_{\tau[2^{n}, 2^{n+1}]} |f(t)|^p \,\frac{dt}{|\Pi(t)|}\Big)^{q/p}\Big)^{p/q} \\
&\le 2^{2d} \Big(\int_{\mathbf{R}^d_+} w^\star(t)\, dt\Big) \|f\|^p_{W_*(L^p, L^q)},
\end{aligned}$$

the last line following from Lemma 3.3.5c.

The result follows upon combining (4.4.8) and (4.4.9). ∎

COROLLARY 4.4.10. *Given $2 \le q \le \infty$, the Wiener transform W is a continuous linear map of $B(2,q)$ into $V(2,q)$, with*

$$\|W\| \;\le\; 2^{(3+\frac{1}{q})d} \Big(\int_0^\infty k^\star(t)\, dt\Big)^{d/2} \;\approx\; (8.268 \cdot 2^{1/q})^d.$$

PROOF: From Example 4.3.5 and Proposition 4.4.9,

(4.4.10)
$$\|Wf\|_{V(2,q)} = \left(\int_{\hat{\mathbf{R}}^d_+} \left(\frac{\Pi(\lambda)}{2^d} \int_{\mathbf{R}^d} |f(t)|^2 K(\lambda t)\, dt\right)^{q/2} \frac{d\lambda}{\Pi(\lambda)}\right)^{1/q}$$
$$\leq (\log 2)^{d/q}\, 2^{3d/2} \left(\int_{\mathbf{R}^d_+} K^\star(t)\, dt\right)^{1/2} \|f\|_{W_*(L^2,L^q)}.$$

Now,

$$\int_{\mathbf{R}^d_+} K^\star(t)\, dt = \left(\int_0^\infty k^\star(t)\, dt\right)^d, \tag{4.4.11}$$

and, by Theorem 3.2.4,

$$\|f\|_{W_*(L^2,L^q)} \leq \frac{2^{(\frac{3}{2}+\frac{1}{q})d}}{(\log 2)^{d/q}} \|f\|_{B(2,q)}. \tag{4.4.12}$$

The result follows upon combining (4.4.10), (4.4.11), and (4.4.12). ∎

REMARK 4.4.11. For $q = 2$ and $q = \infty$ we know the actual value of $\|W\|$, which we can compare to the estimate for $\|W\|$ given by Corollary 4.4.10.

For $q = 2$, $\|W\| = 1$ by Proposition 4.4.7, while Corollary 4.4.10 implies only that $\|W\| \leq \left(2^7 \int_0^\infty k^\star(t)\, dt\right)^{d/2} \approx (11.69)^d$.

For $q = \infty$, $\|W\| = \left(\int_0^\infty k^\star(t)\, dt\right)^{d/2} \approx (1.03)^d$ by Corollary 4.4.3, while Corollary 4.4.10 implies only that $\|W\| \leq \left(2^6 \int_0^\infty k^\star(t)\, dt\right)^{d/2} \approx (8.27)^d$.

Finally, we prove the continuity of the Wiener transform on $B(2,q)$ for $1 < q \leq 2$ using amalgam space methods.

PROPOSITION 4.4.12. *Given $1 \leq q \leq p < \infty$, and given a nonnegative, even function w on $\mathbf{R}^d$. Then*

$$\left(\int_{\hat{\mathbf{R}}_+^d}\left(\frac{\Pi(\lambda)}{2^d}\int_{\mathbf{R}^d}|f(t)|^p\,w(\lambda t)\,dt\right)^{q/p}\frac{d\lambda}{\Pi(\lambda)}\right)^{1/q}$$
$$\le (\log 2)^{d/q}\,2^{d/p}\left(\sum_{m\in\mathbf{Z}^d}\left(\Pi(2^m)\,w^\star(2^m)\right)^{q/p}\right)^{1/q}\|f\|_{W_*(L^p,L^q)}$$

for all measurable functions $f\colon\mathbf{R}^d\to\mathbf{C}$.

PROOF: Extend $w^\star$ evenly to $\mathbf{R}^d$. Just as in (4.4.8), we have

(4.4.13)
$$\int_{\hat{\mathbf{R}}_+^d}\left(\frac{\Pi(\lambda)}{2^d}\int_{\mathbf{R}^d}|f(t)|^p\,w(\lambda t)\,dt\right)^{q/p}\frac{d\lambda}{\Pi(\lambda)} \le (\log 2)^d\,2^{dq/p}\left\|\sum_{m,\sigma}F_{m,\sigma}\right\|_{\ell^{q/p}}^{q/p},$$

where $F_{m,\sigma}$ is the sequence

$$F_{m,\sigma} = \Pi(2^m)\,w^\star(2^m)\int_{\sigma[2^{m-n},2^{m-n+1}]}|f(t)|^p\,\frac{dt}{|\Pi(t)|},$$

and m, n range over $\mathbf{Z}^d$ while σ ranges over Ω^d. Since $0<q/p\le 1$, we can apply the triangle inequality in the metric space $\ell^{q/p}$ to estimate $\left\|\sum F_{m,\sigma}\right\|_{\ell^{q/p}}$, i.e.,

(4.4.14)
$$\begin{aligned}\left\|\sum_{m,\sigma}F_{m,\sigma}\right\|_{\ell^{q/p}}^{q/p}
&\le \sum_{m,\sigma}\|F_{m,\sigma}\|_{\ell^{q/p}}^{q/p}\\
&= \sum_{m,\sigma}\sum_{n}|F_{m,\sigma}(n)|^{q/p}\\
&= \sum_{m,\sigma}\sum_{n}\left(\Pi(2^m)\,w^\star(2^m)\int_{\sigma[2^{m-n},2^{m-n+1}]}|f(t)|^p\,\frac{dt}{|\Pi(t)|}\right)^{q/p}\\
&= \sum_{m}\left(\Pi(2^m)\,w^\star(2^m)\right)^{q/p}\sum_{n,\sigma}\left(\int_{\sigma[2^n,2^{n+1}]}|f(t)|^p\,\frac{dt}{|\Pi(t)|}\right)^{q/p}\\
&= \|f\|_{W_*(L^p,L^q)}^q\sum_{m}\left(\Pi(2^m)\,w^\star(2^m)\right)^{q/p}.\end{aligned}$$

The result follows upon combining (4.4.13) and (4.4.14). ∎

LEMMA 4.4.13. *Given a nonnegative, decreasing function w on $\mathbf{R}_+$ and given $0 < p \leq 1$,*

$$\sum_{n \in \mathbf{Z}} \left(2^n\, w(2^n)\right)^p \;\leq\; \frac{2^p}{2^p - 1} \left(\sup_{0 \leq t \leq 1} \, w(t)^p \;+\; \int_1^\infty w(t)^p \, dt \right).$$

PROOF: Set $M = \sup_{0 \leq t \leq 1} w(t)^p$. Then

$$\sum_{n \leq 0} \left(2^n\, w(2^n)\right)^p \;\leq\; M \sum_{n \leq 0} 2^{np} \;=\; M \, \frac{2^p}{2^p - 1}.$$

If $n > 0$ then $2^{np} \leq 2^n$. Since w is decreasing, we therefore have

$$\begin{aligned}
\sum_{n > 0} \left(2^n\, w(2^n)\right)^p \;&=\; \sum_{n > 0} \frac{2^p}{2^p - 1} \int_{2^{(n-1)p}}^{2^{np}} w(2^n)^p \, dt \\
&\leq\; \frac{2^p}{2^p - 1} \sum_{n > 0} \int_{2^{(n-1)p}}^{2^{np}} w(t)^p \, dt \\
&=\; \frac{2^p}{2^p - 1} \int_1^\infty w(t)^p \, dt. \;\blacksquare
\end{aligned}$$

LEMMA 4.4.14. *Given $1/2 < p \leq 1$,*

$$\sum_{m \in \mathbf{Z}^d} \left(\Pi(2^m)\, K^\star(2^m)\right)^p \;\leq\; \left(\frac{2^p}{2^p - 1} \left(4^p \;+\; \int_1^\infty k^\star(t)^p \, dt \right) \right)^d \;<\; \infty.$$

PROOF: First note that since $p > 1/2$,

$$\int_1^\infty k^\star(t)^p \, dt \;\leq\; \pi^{-2p} \int_1^\infty t^{-2p} \, dt \;=\; \frac{1}{\pi^{2p}\,(1 - 2p)} \;<\; \infty.$$

Also,

$$\sup_{0 \leq t \leq 1} \, k^\star(t)^p \;=\; k^\star(0)^p \;=\; 4^p.$$

Since

$$\sum_{m \in \mathbf{Z}^d} \left(\Pi(2^m)\, K^\star(2^m)\right)^p \;=\; \left(\sum_{n \in \mathbf{Z}} \left(2^n\, k^\star(2^n)\right)^p \right)^d,$$

the result follows from Lemma 4.4.13. $\blacksquare$

COROLLARY 4.4.15. *Given $1 < q \leq 2$, the Wiener transform W is a continuous linear map of $B(2,q)$ into $V(2,q)$, with*

$$\|W\| \leq \left(\frac{2^{\frac{5q}{2}+1}}{2^{\frac{q}{2}} - 1} \left(2^q + \int_1^\infty k^\star(t)^{q/2}\, dt \right) \right)^{d/q}.$$

PROOF: From Example 4.3.5 and Proposition 4.4.12,

(4.4.15)

$$\begin{aligned} \|Wf\|_{V(2,q)} &= \left(\int_{\hat{\mathbf{R}}_+^d} \left(\frac{\Pi(\lambda)}{2^d} \int_{\mathbf{R}^d} |f(t)|^2\, K(\lambda t)\, dt \right)^{q/2} \frac{d\lambda}{\Pi(\lambda)} \right)^{1/q} \\ &\leq (\log 2)^{d/q}\, 2^{d/2} \left(\sum_{m \in \mathbf{Z}^d} \left(\Pi(2^m)\, K^\star(2^m) \right)^{q/2} \right)^{1/q} \|f\|_{W_*(L^2, L^q)}. \end{aligned}$$

From Corollary 4.4.14,

$$\text{(4.4.16)} \quad \sum_{m \in \mathbf{Z}^d} \left(\Pi(2^m)\, K^\star(2^m) \right)^{q/2} \leq \left(\frac{2^{q/2}}{2^{q/2} - 1} \left(2^q + \int_1^\infty k^\star(t)^{q/2}\, dt \right) \right)^d.$$

And, by Theorem 3.2.4,

$$\text{(4.4.17)} \qquad \|f\|_{W_*(L^2, L^q)} \leq \frac{2^{(\frac{8}{2} + \frac{1}{q})d}}{(\log 2)^{d/q}} \|f\|_{B(2,q)}.$$

The result follows upon combining (4.4.15), (4.4.16), and (4.4.17). ∎

REMARK 4.4.16. a. The combination of Corollary 4.4.3, Proposition 4.4.5, Corollary 4.4.10, and Corollary 4.4.15 establish that the Wiener transform W is a continuous linear mapping of $B(2,q)$ into $V(2,q)$ for each $1 \leq q \leq \infty$. In summary, we used techniques due to Beurling for the case $q = 1$, techniques due to Lau for $q = \infty$, and amalgam space techniques for $1 < q \leq \infty$. Our amalgam space estimate for $\|W\|$ goes to infinity as $q \to 1$, and is inferior to Lau's exact estimate at the other endpoint, $q = \infty$. It is undoubtably possible to improve the estimates for W which we derived using amalgam space methods. For $1 < q < 2$, it is likely that there is an amalgam space proof which does not exhibit the "blowing up" effect of the norm as $q \to 1$.

b. We do not believe that either Beurling's or Lau's methods can be adapted to prove that the Wiener transform is continuous when $1 < q < \infty$. In the next section, we prove the W is invertible and derive estimates for $\|W^{-1}\|$ for each $1 \leq q \leq \infty$. Again, Beurling's methods suffice for $q = 1$, Lau's for $q = \infty$, and amalgam spaces for $1 \leq q < \infty$. However, Lau's method generalizes easily to all $1 \leq q \leq \infty$.

Section 4.5. Invertibility of the Wiener transform.

In the preceding section, we proved that the Wiener transform W is a continuous linear map of $B(2,q)$ into $V(2,q)$ for each $1 \le q \le \infty$. We proved this for $q = 1$ using a technique due to Beurling, for $q = \infty$ using a technique due to Lau and Chen, and for $1 < q \le \infty$ using amalgam space techniques. Lau's method for $q = \infty$ gave the exact value of $\|W\|$, while the amalgam space method for $q = \infty$ gave an inferior estimate.

In this section we prove that W is invertible, and estimate $\|W^{-1}\|$ for each $1 \le q \le \infty$. Again, Beurling's method would suffice for $q = 1$ and amalgam spaces for $1 \le q < \infty$; instead we generalize a variant of Lau's method to all $1 \le q \le \infty$. We prove the surjectivity of W for $1 \le q \le \infty$ using the same method Beurling used for $q = 1$ and Lau for $q = \infty$.

The following proposition is similar to one proved by Lau and Chen for the special case $d = 1$ and $q = \infty$. They did not make use of the minorant $w_\star$, but rather assumed that w itself was decreasing on some interval $[0, b]$.

PROPOSITION 4.5.1 [**CL1**]. *Given $1 \le p < \infty$ and $1 \le q \le \infty$, and given a nonnegative, even function w on $\mathbf{R}^d$. Then*

$$\left(\sup_{T \in \mathbf{R}_+^d} \Pi(T)\, w_\star(T)\right)^{1/p} \|f\|_{B(p,q)}$$
$$\le \left(\int_{\hat{\mathbf{R}}_+^d} \left(\frac{\Pi(\lambda)}{2^d} \int_{\mathbf{R}^d} |f(t)|^p\, w(\lambda t)\, dt\right)^{q/p} \frac{d\lambda}{\Pi(\lambda)}\right)^{1/q},$$

for all measurable functions $f: \mathbf{R}^d \to \mathbf{C}$ (with the standard adjustments if $q = \infty$).

PROOF: Assume $q < \infty$, the $q = \infty$ case being similar. Given b, $T \in \mathbf{R}_+^d$, we compute

$$\Pi(b)\, w_\star(b)\, \frac{1}{\Pi(T)} \int_{[0,T]} |f(t)|^p\, dt \;=\; \frac{\Pi(b)}{\Pi(T)} \int_{[0,T]} |f(t)|^p\, w_\star(b)\, dt$$

$$\leq\; \Pi(b/T) \int_{[0,T]} |f(t)|^p\, w_\star(bt/T)\, dt,$$

since $bt/T \leq b$ for $t \in [0,T]$ and $w_\star$ is decreasing on $\mathbf{R}_+^d$. Combining this with similar inequalities for the other quadrants (possible since w is even), we have

$$\begin{aligned}
&\Pi(b)\, w_\star(b)\, \frac{1}{|R_T|} \int_{[0,T]} |f(t)|^p\, dt \\
&= \sum_{\sigma \in \Omega^d} \Pi(b)\, w_\star(b)\, \frac{1}{2^d\, \Pi(T)} \int_{\sigma[0,T]} |f(t)|^p\, dt \\
&\leq 2^{-d} \sum_{\sigma \in \Omega^d} \Pi(b/T) \int_{\sigma[0,T]} |f(t)|^p\, w_\star(bt/T)\, dt \\
&= 2^{-d}\, \Pi(b/T) \int_{R_T} |f(t)|^p\, w_\star(bt/T)\, dt \\
&\leq 2^{-d}\, \Pi(b/T) \int_{\mathbf{R}^d} |f(t)|^p\, w_\star(bt/T)\, dt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left(\Pi(b)\, w_\star(b)\right)^{1/p} \|f\|_{B(p,q)} \\
&= \left(\int_{\mathbf{R}_+^d} \left(\Pi(b)\, w_\star(b)\, \frac{1}{|R_T|} \int_{R_T} |f(t)|^p\, dt \right)^{q/p} \frac{dT}{\Pi(T)} \right)^{1/q} \\
&\leq \left(\int_{\mathbf{R}_+^d} \left(\frac{\Pi(b/T)}{2^d} \int_{\mathbf{R}^d} |f(t)|^p\, w_\star(bt/T)\, dt \right)^{q/p} \frac{dT}{\Pi(T)} \right)^{1/q}
\end{aligned}$$

$$= \left(\int_{\hat{\mathbf{R}}^d_+} \left(\frac{\Pi(\lambda)}{2^d} \int_{\mathbf{R}^d} |f(t)|^p \, w_\star(\lambda t) \, dt \right)^{q/p} \frac{d\lambda}{\Pi(\lambda)} \right)^{1/q}$$

$$\leq \left(\int_{\hat{\mathbf{R}}^d_+} \left(\frac{\Pi(\lambda)}{2^d} \int_{\mathbf{R}^d} |f(t)|^p \, w(\lambda t) \, dt \right)^{q/p} \frac{d\lambda}{\Pi(\lambda)} \right)^{1/q},$$

where we have made the substitution $\lambda = b/T$ and used the fact that $dT/\Pi(T)$ is dilation invariant. Taking the supremum over all $b \in \mathbf{R}^d_+$ therefore gives the desired inequality. ∎

REMARK 4.5.2. For the case $d = 1$ and $q = \infty$, Lau and Chen prove that if $\sup_{t \in \mathbf{R}_+} tw(t) = \sup_{t \in \mathbf{R}_+} tw_\star(t)$ then the constant in Proposition 4.5.1 is best possible. We extend this to higher dimensions as follows.

Fix $\varepsilon > 1$. It suffices to show that there exists an $f \in B(p, \infty)$ with $\|f\|_{B(p,\infty)} = 1$ such that

$$\sup_{\lambda \in \hat{\mathbf{R}}^d_+} \frac{\Pi(\lambda)}{2^d} \int_{\mathbf{R}^d} |f(t)|^p \, w(\lambda t) \, dt \;\leq\; \varepsilon \, C,$$

where $C = \sup_{T \in \mathbf{R}^d_+} \Pi(T) \, w(T)$. Fix $\delta \in (0,1) \subset \mathbf{R}^d_+$ and define

$$f \;=\; \left(\frac{2^d}{\Pi(\delta)} \right)^{1/p} \chi_{[1-\delta, 1]}.$$

For each $j = 1, \ldots, d$ we have

$$\sup_{T_j > 0} \frac{1}{T_j \, \delta_j} \int_{-T_j}^{T_j} \chi_{[1-\delta_j, 1]}(t_j) \, dt_j \;=\; 1.$$

Therefore,

$$\begin{aligned} \|f\|^p_{B(p,\infty)} &= \sup_{T \in \mathbf{R}^d_+} \frac{1}{|R_T|} \int_{R_T} |f(t)|^p \, dt \\ &= \prod_{j=1}^{d} \sup_{T_j \in \mathbf{R}^d_+} \frac{1}{2T_j} \int_{-T_j}^{T_j} \frac{2}{\delta_j} \chi_{[1-\delta_j, 1]}(t_j) \, dt_j \\ &= 1. \end{aligned}$$

Now let the components of δ be small enough that $1/\Pi(1-\delta) \leq \varepsilon$. Then

$$\begin{aligned}
\frac{\Pi(\lambda)}{2^d} \int_{\mathbf{R}^d} |f(t)|^p \, w(\lambda t) \, dt &= \frac{\Pi(\lambda)}{\Pi(\delta)} \int_{[1-\delta,1]} w(\lambda t) \, dt \\
&= \frac{1}{\Pi(\delta)} \int_{[1-\delta,1]} \frac{\Pi(\lambda t) \, w(\lambda t)}{\Pi(t)} \, dt \\
&\leq \frac{1}{\Pi(\delta)} \int_{[1-\delta,1]} \frac{C}{\Pi(1-\delta)} \, dt \\
&= \frac{C}{\Pi(1-\delta)} \\
&\leq \varepsilon \, C.
\end{aligned}$$

EXAMPLE 4.5.3. a. The function $tk(t)$ is continuous on $\mathbf{R}$. If $t \geq 1/4$ then

$$tk(t) = \frac{\sin^2 2\pi t}{\pi^2 t} \leq \frac{1}{\pi^2 t} \leq \frac{4}{\pi^2} = \frac{1}{4} k\left(\frac{1}{4}\right).$$

Therefore $tk(t)$ achieves its maximum somewhere in the interval $[0, 1/4]$. We compute

$$k'(t) = \frac{2 \sin 2\pi t}{\pi^2 t^3} (2\pi t \cos 2\pi t - \sin 2\pi t)$$

and

$$[tk(t)]' = k(t) + tk'(t) = \frac{\sin 2\pi t}{\pi^2 t^2} (4\pi t \cos 2\pi t - \sin 2\pi t).$$

The maximum of $tk(t)$ therefore occurs at the point $b \in (0, 1/4)$ such that $\tan 2\pi b = 4\pi b$. There is a unique such point in the interval $(0, 1/4)$; numerically, $b \approx 0.186$ and $bk(b) \approx 0.461$. Since $k_\star = k \cdot \chi_{[0,1/2]}$, we have $\sup_{t \in \mathbf{R}_+} tk_\star(t) = bk(b) = \sup_{t \in \mathbf{R}_+} tk(t)$.

b. Let b be as in part a. Since $K(t) = \prod_1^d k(t_j)$, it follows from part a that $(bk(b))^d = \sup_{T \in \mathbf{R}_+^d} \Pi(T) \, K(T) = \sup_{T \in \mathbf{R}_+^d} \Pi(T) \, K_\star(T)$.

COROLLARY 4.5.4. *Given $1 \le q \le \infty$, the Wiener transform W is an injective mapping of $B(2,q)$ into $V(2,q)$, and the inverse mapping W^{-1}:* Range$(W) \to$ *$B(2,q)$ is continuous, with*

$$\|W^{-1}\| \le \left(\sup_{t\in\mathbf{R}_+} t\,k(t)\right)^{-d/2} \approx (1.472)^d. \tag{4.5.1}$$

If $q = \infty$ then (4.5.1) is equality.

PROOF: From Example 4.3.5 and Proposition 4.5.1,

$$\begin{aligned}\|Wf\|_{V(2,q)} &= \left(\int_{\hat{\mathbf{R}}^d_+} \left(\frac{\Pi(\lambda)}{2^d} \int_{\mathbf{R}^d} |f(t)|^2\, K(\lambda t)\, dt\right)^{q/2} \frac{d\lambda}{\Pi(\lambda)}\right)^{1/q} \\ &\ge \left(\sup_{T\in\mathbf{R}^d_+} \Pi(T)\, K_\star(T)\right)^{1/2} \|f\|_{B(2,q)}.\end{aligned}$$

From Example 4.5.3b,

$$\sup_{T\in\mathbf{R}^d_+} \Pi(T)\, K_\star(T) = \left(\sup_{t\in\mathbf{R}_+} t\,k(t)\right)^d.$$

Therefore W is injective, and $\|W^{-1}\| \le \left(\sup_{t\in\mathbf{R}_+} tk(t)\right)^{-d/2}$, which from Example 4.5.3a is approximately $(1.472)^d$. It follows from Remark 4.5.2 that this is equality if $q = \infty$. ∎

We now complete the proof of the major result of this thesis.

THEOREM 4.5.5. *Given $1 \le q \le \infty$, the Wiener transform W is a topological isomorphism of $B(2,q)$ onto $V(w,q)$.*

PROOF: The combination of Corollary 4.4.3, Proposition 4.4.5, Corollary 4.4.10, and Corollary 4.4.15 establish that the Wiener transform is a continuous linear mapping of $B(2,q)$ into $V(2,q)$ for each $1 \le q \le \infty$. Corollary

4.5.4 establishes that W is injective, and that $W^{-1}\colon \text{Range}(W) \to B(2,q)$ is continuous for each $1 \le q \le \infty$. It therefore remains only to show that W is surjective.

Fix any $G \in V(2,q)$. Then $\Delta_\lambda G \in L^2(\hat{\mathbf{R}}^d)$ for a.e. λ. Since $\check{\Delta}_\lambda(t) = (-i)^d\, s_\lambda(t) \neq 0$ a.e. (Proposition 4.2.5), we can define a function f_λ by

$$\check{\Delta}_\lambda \cdot f_\lambda \;=\; (\Delta_\lambda G)^\vee.$$

Since $\Delta_\lambda G \in L^2(\hat{\mathbf{R}}^d)$, $\Delta_\mu \Delta_\lambda G \in L^2(\hat{\mathbf{R}}^d)$ as well. Therefore, from Remark 4.2.2g,

$$\begin{aligned}
\check{\Delta}_\mu \cdot \check{\Delta}_\lambda \cdot f_\lambda \;&=\; \check{\Delta}_\mu \cdot (\Delta_\lambda G)^\vee \\
&=\; (\Delta_\mu \Delta_\lambda G)^\vee \\
&=\; (\Delta_\lambda \Delta_\mu G)^\vee \\
&=\; \check{\Delta}_\lambda \cdot (\Delta_\mu G)^\vee \\
&=\; \check{\Delta}_\lambda \cdot \check{\Delta}_\mu \cdot f_\mu.
\end{aligned}$$

As $\check{\Delta}_\mu \cdot \check{\Delta}_\lambda \neq 0$ a.e., it follows that f_λ is independent of λ, and is therefore denoted hereafter by f. Now,

$$\begin{aligned}
\text{(4.5.2)} \qquad (\Delta_\lambda G)^\vee(t) \;&=\; \check{\Delta}_\lambda(t)\, f(t) \\
&=\; (-i)^d\, s_\lambda(t)\, f(t) \\
&=\; \Pi(-\pi i t)\, d_\lambda(t)\, f(t).
\end{aligned}$$

By Proposition 4.2.8, if $h \in B(2,q)$ then $h \cdot d_\lambda \in L^2(\mathbf{R}^d)$, and

$$(4.5.3) \qquad (\Delta_\lambda W h)^\vee(t) = 2^{-d} h(t)\, d_\lambda(t).$$

Comparing (4.5.2) and (4.5.3) we therefore define

$$g(t) = \Pi(-2\pi i t)\, f(t).$$

Using the Plancherel theorem, the fact that $|d_\lambda(t)|^2 = \Pi(\lambda)^2 K(\lambda t)$, and Proposition 4.5.1, we compute

$$\begin{aligned}
&\|G\|_{V(2,q)} \\
&= \left(\int_{\hat{\mathbf{R}}^d_+} \left(\frac{2^d}{\Pi(\lambda)} \int_{\hat{\mathbf{R}}^d} |\Delta_\lambda G(\gamma)|^2 \, d\gamma \right)^{q/2} \frac{d\lambda}{\Pi(\lambda)} \right)^{1/q} \\
&= \left(\int_{\hat{\mathbf{R}}^d_+} \left(\frac{2^d}{\Pi(\lambda)} \int_{\hat{\mathbf{R}}^d} |\check{\Delta}_\lambda(t)\, f(t)|^2 \, dt \right)^{q/2} \frac{d\lambda}{\Pi(\lambda)} \right)^{1/q} \\
&= \left(\int_{\hat{\mathbf{R}}^d_+} \left(\frac{2^d}{\Pi(\lambda)} \int_{\hat{\mathbf{R}}^d} |\Pi(-\pi i t)\, f(t)\, d_\lambda(t)|^2 \, dt \right)^{q/2} \frac{d\lambda}{\Pi(\lambda)} \right)^{1/q} \\
&= \left(\int_{\hat{\mathbf{R}}^d_+} \left(\frac{1}{2^d\,\Pi(\lambda)} \int_{\hat{\mathbf{R}}^d} |\Pi(-2\pi i t)\, f(t)|^2\, \Pi(\lambda)^2 K(\lambda t) \, dt \right)^{q/2} \frac{d\lambda}{\Pi(\lambda)} \right)^{1/q} \\
&= \left(\int_{\hat{\mathbf{R}}^d_+} \left(\frac{\Pi(\lambda)}{2^d} \int_{\hat{\mathbf{R}}^d} |g(t)|^2 K(\lambda t) \, dt \right)^{q/2} \frac{d\lambda}{\Pi(\lambda)} \right)^{1/q} \\
&\geq C\, \|g\|_{B(2,q)}.
\end{aligned}$$

Since $\|G\|_{V(2,q)} < \infty$, it follows that $g \in B(2,q)$, and therefore $Wg \in V(2,q)$. Finally,

$$\Delta_\lambda W g = 2^{-d} (g \cdot d_\lambda)^\wedge = (\check{\Delta}_\lambda \cdot f)^\wedge = \Delta_\lambda G$$

for a.e. λ, so $\|G - Wg\|_{V(2,q)} = 0$. Since we identify functions in $V(2,q)$ whose difference has zero norm, $Wg = G$ in $V(2,q)$, and therefore W is surjective. ∎

Since the Wiener transform is a topological isomorphism of the Banach space $B(2,q)$ onto the normed linear space $V(2,q)$, it follows that $V(2,q)$ is complete. We prove this in detail in the following corollary. We devote Chapter 5 to proving that $V(p,q)$ is complete for all p, q.

COROLLARY 4.5.6. *$V(2,q)$ is a Banach space for each $1 \le q \le \infty$.*

PROOF: Fix $1 \le q \le \infty$. By Theorem 4.5.5, the Wiener transform W is a topological isomorphism of the Banach space $B(2,q)$ onto the normed linear space $V(2,q)$. Assume $\{G_n\}_{n\in\mathbf{Z}_+}$ is a Cauchy sequence in $V(2,q)$. Then $\|W^{-1}G_m - W^{-1}G_n\|_{B(2,q)} \le C\,\|G_m - G_n\|_{V(2,q)}$, so $\{W^{-1}G_n\}$ forms a Cauchy sequence in $B(2,q)$. Therefore, $W^{-1}G_n \to g$ in $B(2,q)$ for some $g \in B(2,q)$. The continuity of W implies then that $G_n = WW^{-1}G_n \to Wg$ in $V(2,q)$, so $V(2,q)$ is complete. ∎

We illustrate now that the value $\sup_{T\in\mathbf{R}^d_+} \Pi(T)\, w_\star(T)$ appearing in Proposition 4.5.1 also arises naturally when amalgam space methods are used. However, the conversion from the continuous norm to a discrete approximation in the proof results, as usual, in an inferior estimate.

PROPOSITION 4.5.7. *Given $1 \le p$, $q < \infty$ and a nonnegative, even function w on $(\mathbf{R}^d)$. Then*

$$\left(\int_{\hat{\mathbf{R}}^d_+} \left(\frac{\Pi(\lambda)}{2^d}\int_{\mathbf{R}^d} |f(t)|^p\, w(\lambda t)\, dt\right)^{q/p} \frac{d\lambda}{\Pi(\lambda)}\right)^{1/q}$$
$$\ge (\log 2)^{d/q}\, 2^{-(\frac{4}{p}+\frac{1}{q})d} \left(\sup_{T\in\mathbf{R}^d_+} \Pi(T)\, w_\star(T)\right)^{1/p} \|f\|_{W_*(L^p,L^q)}$$

for all measurable functions $f: \mathbf{R}^d \to \mathbf{C}$.

PROOF: Fix any $\sigma \in \Omega^d$. The summations in the following calculation are over m, $n \in \mathbf{Z}^d$. Using the fact that $w_\star$ is even and decreasing, we compute

$$\int_{\hat{\mathbf{R}}^d_+} \left(\frac{\Pi(\lambda)}{2^d} \int_{\mathbf{R}^d} |f(t)|^p \, w(\lambda t) \, dt \right)^{q/p} \frac{d\lambda}{\Pi(\lambda)}$$

$$\geq \sum_n \int_{[2^n, 2^{n+1}]} \left(\frac{\Pi(\lambda)}{2^d} \sum_m \int_{\sigma[2^m, 2^{m+1}]} |f(t)|^p \, w_\star(\lambda t) \, dt \right)^{q/p} \frac{d\lambda}{\Pi(\lambda)}$$

$$\geq \sum_n \int_{[2^n, 2^{n+1}]} \left(\frac{\Pi(2^n)}{2^d} \sum_m \int_{\sigma[2^m, 2^{m+1}]} |f(t)|^p \, w_\star(2^{m+n+2}) \, dt \right)^{q/p} \frac{d\lambda}{\Pi(\lambda)}$$

$$\geq (\log 2)^d \sum_n \left(\sum_m \Pi(2^{m+n-1}) \, w_\star(2^{m+n+2}) \int_{\sigma[2^m, 2^{m+1}]} |f(t)|^p \frac{dt}{|\Pi(t)|} \right)^{q/p}$$

$$= (\log 2)^d \sum_n \left(\sum_m \Pi(2^{m-3}) \, w_\star(2^m) \int_{\sigma[2^{m-n-2}, 2^{m-n-1}]} |f(t)|^p \frac{dt}{|\Pi(t)|} \right)^{q/p}$$

$$= (\log 2)^d \, 2^{-3dq/p} \sum_n \left| \sum_m F_{n,\sigma}(m)^{p/q} \right|^{q/p},$$

where $F_{n,\sigma}$ is the sequence

$$F_{n,\sigma}(m) = \left(\Pi(2^m) \, w_\star(2^m) \int_{\sigma[2^{m-n-2}, 2^{m-n-1}]} |f(t)|^p \frac{dt}{|\Pi(t)|} \right)^{q/p}.$$

Therefore,

(4.5.4)

$$\int_{\hat{\mathbf{R}}^d_+} \left(\frac{\Pi(\lambda)}{2^d} \int_{\mathbf{R}^d} |f(t)|^p \, w(\lambda t) \, dt \right)^{q/p} \frac{d\lambda}{\Pi(\lambda)}$$

$$= 2^{-d} \sum_{\sigma \in \Omega^d} \int_{\hat{\mathbf{R}}^d_+} \left(\frac{\Pi(\lambda)}{2^d} \int_{\mathbf{R}^d} |f(t)|^p \, w(\lambda t) \, dt \right)^{q/p} \frac{d\lambda}{\Pi(\lambda)}$$

$$\geq 2^{-d} (\log 2)^d \sum_{n,\sigma} \left| \sum_m F_{n,\sigma}(m)^{p/q} \right|^{q/p}$$

$$= 2^{-d} (\log 2)^d \sum_{n,\sigma} \| F_{n,\sigma} \|_{\ell^{p/q}}.$$

Since $0 < p/q < \infty$, we have $\|\cdot\|_{\ell^{p/q}} \geq \|\cdot\|_{\ell^\infty}$. Therefore,

(4.5.5)

$$\begin{aligned}
\sum_{n,\sigma} \|F_{n,\sigma}\|_{\ell^{p/q}} \\
&\geq \sum_{n,\sigma} \|F_{n,\sigma}\|_{\ell^\infty} \\
&= \sum_{n,\sigma} \sup_m F_{n,\sigma}(m) \\
&\geq \sup_m \sum_{n,\sigma} F_{n,\sigma}(m) \\
&= \sup_m \sum_{n,\sigma} \left(\Pi(2^m)\, w_\star(2^m) \int_{\sigma[2^{m-n-2}, 2^{m-n-1}]} |f(t)|^p \frac{dt}{|\Pi(t)|} \right)^{q/p} \\
&= \sup_m \left(\Pi(2^m)\, w_\star(2^m) \right)^{q/p} \sum_{n,\sigma} \left(\int_{\sigma[2^n, 2^{n+1}]} |f(t)|^p \frac{dt}{|\Pi(t)|} \right)^{q/p} \\
&= \left(\sup_m \Pi(2^m)\, w_\star(2^m) \right)^{q/p} \|f\|_{W_*(L^p, L^q)} \\
&\geq \left(2^{-d} \sup_{T \in \mathbf{R}_+^d} \Pi(T)\, w_\star(T) \right)^{q/p} \|f\|_{W_*(L^p, L^q)},
\end{aligned}$$

the last inequality following from Lemma 3.3.5b.

The result follows upon combining (4.5.4) and (4.5.5). ∎

CHAPTER 5

COMPLETENESS OF THE VARIATION SPACES

In this chapter, we prove that the higher-dimensional variation spaces $V(p, q)$ defined in Section 4.3 are complete. Because these spaces are not solid, the completeness is difficult to prove by ordinary techniques. Lau and Chen overcame this difficulty in the one dimensional $V(p, \infty)$ case by using *helices*, a concept developed by Masani. Lau and Chen's proof generalizes immediately to the one dimensional $V(p, q)$ case. We prove the completeness in higher dimensions by using an iterated helix technique.

In Section 5.1 we review the basic definitions and properties of helices, first on abstract topological groups and then specifically on the real line.

In Section 5.2 we prove that $V(p, q)$ is complete. We review Lau and Chen's proof for one dimension, then extend it to higher dimensions by using an iterated helix technique.

Throughout this chapter, we make use of the symmetric, one-sided, directional, and one-sided directional difference operators defined in Definition 4.2.1. We let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ denote the j^{th} unit vector in $\mathbf{R}^d$. We make use of the group representation definitions given in Section 1.9, and use vector-valued integration, following the definitions in [**HP**]. We use the shorthand notation of writing $U(t)$ as U_t for representations U and related maps.

Section 5.1. Helices.

In this section we define helices and derive their basic properties. We begin with the general definition on abstract topological groups, then turn to the specific case of helices on the real line. The results in this section are taken directly from [**M1**] and [**M3**], and therefore the credit for this section is due to Masani, except for some remarks and examples.

We let G denote an arbitrary locally compact abelian group, written additively with identity element 0, and let X be an arbitrary Banach space.

DEFINITION 5.1.1 [**M1**]. A continuous function $\gamma: G \to X$ is a **variety in** X **parameterized by** G. Given such a variety we define the following terms.

a. γ is a **curve** in X if $G = \mathbf{R}$. γ is a **surface** in X if $G = \mathbf{R}^d$.

b. Given a, $b \in G$, $\gamma_b - \gamma_a$ is a **chord** of γ.

c. The **chordal length function** of γ is $L_\gamma(a) = \|\gamma_a - \gamma_0\|$ for $a \in G$.

d. The **subspace generated by** γ is $S(\gamma) = \overline{\operatorname{span}}\{\gamma_a : a \in G\}$.

e. The **chordal subspace generated by** γ is $CS(\gamma) = \overline{\operatorname{span}}\{\gamma_b - \gamma_a : a, b \in G\}$.

f. γ is **stationary** if there exists a strongly continuous unitary representation U of G on M_γ such that $U_t\gamma_a = \gamma_{a+t}$ for a, $t \in G$. U is the **shift group** of γ.

g. γ is a **helix** if there exists a strongly continuous unitary representation U of G on $S(\gamma)$ such that $U_t(\gamma_b - \gamma_a) = \gamma_{b+t} - \gamma_{a+t}$ for a, $t \in G$. U is the **shift group** of γ. Helices will usually be denoted by the symbol $\hbar$.

h. A function $\varphi: G \to \mathbf{R}$ is a **screw function** if it is the chordal length function of some helix in X.

EXAMPLE 5.1.2. a. Set $d = 1$ and let $X \subset L^1_{\text{loc}}(\hat{\mathbf{R}})$ be a homogeneous Banach function space. Assume $F: \hat{\mathbf{R}} \to \mathbf{C}$ is such that $\Delta^+_\lambda F \in X$ for all $\lambda \in \hat{\mathbf{R}}$, and define $\hbar: \hat{\mathbf{R}} \to X$ by

$$\hbar_\lambda = \Delta^+_\lambda F = \tfrac{1}{2}(T_{-\lambda}F - F).$$

Then $\hbar$ is a helix in X, parameterized by $\hat{\mathbf{R}}$, with shift group $\{T_{-\lambda}\}_{\lambda \in \hat{\mathbf{R}}}$.

To see this, first note that

$$\lim_{b \to a} \|\hbar_b - \hbar_a\| = \lim_{b \to a} \tfrac{1}{2}\|T_{-b}F - T_{-a}F\| = 0,$$

since translation is strongly continuous in X. Thus $\hbar$ is continuous. Since

$$\begin{aligned} T_{-\lambda}(\hbar_b - \hbar_a) &= \tfrac{1}{2}T_{-\lambda}(T_{-b}F - T_{-a}F) \\ &= \tfrac{1}{2}(T_{-\lambda-b}F - T_{-\lambda-a}F) \\ &= \hbar_{b+\lambda} - \hbar_{a+\lambda}, \end{aligned}$$

it remains only to show that $\{T_{-\lambda}\}$ is a unitary, strongly continuous representation of $\hat{\mathbf{R}}$ on X. It clearly is a representation, and the unitarity follows from the fact that X is translation isometric. The strong continuity of the representation follows from the fact that translation is strongly continuous in X, i.e., $\lim_{b \to a} \|T_b g - T_a g\| = 0$ for all $g \in X$.

b. Let $d = 1$ and $X = L^p(\hat{\mathbf{R}})$ and fix $F \in V(p,q)$. Then, by part a, $\hbar_\lambda = \Delta^+_\lambda F$ is a helix in $L^p(\hat{\mathbf{R}})$ since $L^p(\hat{\mathbf{R}})$ is homogeneous and $\Delta^+_\lambda F \in L^p(\hat{\mathbf{R}})$ for a.e. λ.

c. Let $d \geq 1$ be arbitrary, and let $X \subset L^1_{\mathrm{loc}}(\hat{\mathbf{R}}^d)$ be a homogeneous Banach function space. Fix $1 \leq j \leq d$, and assume $F: \hat{\mathbf{R}}^d \to \mathbf{C}$ is such that $\Delta_\lambda^{j+} F \in X$ for $\lambda \in \hat{\mathbf{R}}$. Define $\hbar: \hat{\mathbf{R}} \to X$ by

$$\hbar_\lambda \;=\; \Delta_\lambda^{j+} F \;=\; \tfrac{1}{2}\,(T_{-\lambda e_j} F \;-\; F).$$

Then, just as in part a, $\hbar$ is a helix in X, parameterized by $\hat{\mathbf{R}}$, with shift group $\{T_{-\lambda e_j}\}_{\lambda \in \hat{\mathbf{R}}}$. As in part b, a typical example is formed by taking $X = L^p(\hat{\mathbf{R}}^d)$ and $F \in V(p,q)$.

d. Set $d = 2$, fix $F \in V(p,q)$, and define $\hbar_\lambda = \Delta_\lambda^+ F$ for $\lambda \in \hat{\mathbf{R}}^2$. Given a, b, $\lambda \in \hat{\mathbf{R}}^2$, we compute

$$\begin{aligned}
&4\,(\hbar_{b+\lambda} - \hbar_{a+\lambda})(\gamma) \\
&\quad= 4\,\Delta_{b+\lambda}^+ F(\gamma) \;-\; 4\,\Delta_{a+\lambda}^+ F(\gamma) \\
&\quad= F(\gamma_1 + b_1 + \lambda_1, \gamma_2 + b_2 + \lambda_2) \;-\; F(\gamma_1 + b_1 + \lambda_1, \gamma_2) \\
&\qquad\quad - F(\gamma_1, \gamma_2 + b_2 + \lambda_2) \;+\; F(\gamma_1, \gamma_2) \\
&\qquad - F(\gamma_1 + a_1 + \lambda_1, \gamma_2 + a_2 + \lambda_2) \;+\; F(\gamma_1 + a_1 + \lambda_1, \gamma_2) \\
&\qquad\quad + F(\gamma_1, \gamma_2 + a_2 + \lambda_2) \;-\; F(\gamma_1, \gamma_2),
\end{aligned}$$

while

$$\begin{aligned}
&4\,T_{-\lambda}(\hbar_b - \hbar_a)(\gamma) \\
&\quad= 4\,T_{-\lambda}\,\Delta_b^+ F(\gamma) \;-\; 4\,T_{-\lambda}\,\Delta_a^+ F(\gamma)
\end{aligned}$$

$$= F(\gamma_1 + b_1 + \lambda_1, \gamma_2 + b_2 + \lambda_2) - F(\gamma_1 + b_1 + \lambda_1, \gamma_2 + \lambda_2)$$

$$- F(\gamma_1 + \lambda_1, \gamma_2 + b_2 + \lambda_2) + F(\gamma_1 + \lambda_1, \gamma_2 + \lambda_1)$$

$$- F(\gamma_1 + a_1 + \lambda_1, \gamma_2 + a_2 + \lambda_2) + F(\gamma_1 + a_1 + \lambda_1, \gamma_2 + \lambda_2)$$

$$+ F(\gamma_1 + \lambda_1, \gamma_2 + a_2 + \lambda_2) - F(\gamma_1 + \lambda_1, \gamma_2 + \lambda_2).$$

Thus, in general, $\hbar_{b+\lambda} - \hbar_{a+\lambda} \neq T_{-\lambda}(\hbar_b - \hbar_a)$, so $\hbar$ is not a helix. The same considerations hold for any $d > 1$, i.e., $\hbar_\lambda = \Delta_\lambda^+ F$ is not a helix over $\hat{\mathbf{R}}^d$ when $d > 1$.

LEMMA 5.1.3 [**M1**].

a. *The shift group of a stationary variety γ is unique (on $S(\gamma)$).*

b. *The shift group of a helix $\hbar$ is unique (on $CS(\hbar)$).*

PROOF: We prove only a as b is similar. Assume U, V are two shift groups for a stationary variety γ. Then $U_t\gamma_a = \gamma_{a+t} = V_t\gamma_a$ for all a, $t \in G$. By linearity and continuity we therefore have $U_t f = V_t f$ for every $f \in S(\gamma) = \overline{\text{span}}\{\gamma_a : a \in G\}$, so $U = V$ on $S(\gamma)$. ∎

LEMMA 5.1.4 [**M1**]. *Given a helix $\hbar$, the chordal length function $L_\hbar$ is symmetric, subadditive, and continuous. Further, $L_\hbar(0) = 0$, and $\|\hbar_b - \hbar_a\| = L_\hbar(b-a)$ for a, $b \in G$.*

PROOF: Recall that $L_\hbar(a) = \|\hbar_a - \hbar_0\|$. $L_\hbar$ is therefore continuous since $\hbar$ is continuous.

Given $a \in G$ we have

$$L_\hbar(a) = \|\hbar_a - \hbar_0\| = \|U_a(\hbar_0 - \hbar_{-a})\| = \|\hbar_0 - \hbar_{-a}\| = L_\hbar(-a),$$

since the shift group U is unitary. Thus $\hbar$ is symmetric.

Given $a,\, b \in G$ we compute

$$\|\hbar_b - \hbar_a\| \;=\; \|U_a(\hbar_{b-a} - \hbar_0)\| \;=\; \|\hbar_{b-a} - \hbar_0\| \;=\; L_\hbar(b-a).$$

Therefore,

$$L_\hbar(a+b) \;=\; \|\hbar_{a+b} - \hbar_0\| \;\leq\; \|\hbar_{a+b} - \hbar_a\| \;+\; \|\hbar_a - \hbar_0\| \;=\; L_\hbar(b) \;+\; L_\hbar(a),$$

so $L_\hbar$ is subadditive and $L_\hbar(0) = 0$. ∎

We turn now to the specific case of helices parameterized by **R**. We assume for the remainder of this section that all helices are over **R**. The following proposition limits the growth of a screw function.

PROPOSITION 5.1.5 [**M1**]. *Given a helix $\hbar$ and $a \in \mathbf{R}$,*

$$L_\hbar(a) \;\leq\; |a|\, L_\hbar(1) \;+\; \max_{0 \leq t \leq 1} L_\hbar(t).$$

PROOF: Set $M = \max_{0\leq t\leq 1} L_\hbar(t)$. Given $N \in \mathbf{Z}_+$, note that

$$L_\hbar(N) \;=\; L_\hbar(1 + \cdots + 1) \;\leq\; L_\hbar(1) + \cdots + L_\hbar(1) \;=\; N\, L_\hbar(1).$$

Given $a \geq 0$ let $N = \lfloor a \rfloor$, the largest integer $N \leq a$. Since $0 \leq a - N < 1$,

$$L_\hbar(a) \;\leq\; L_\hbar(N) \;+\; L_\hbar(a-N) \;\leq\; N\, L_\hbar(1) \;+\; M \;\leq\; a\, L_\hbar(1) \;+\; M.$$

If $a < 0$ then, by the symmetry of $L_\hbar$,

$$L_\hbar(a) \;=\; L_\hbar(-a) \;\leq\; (-a)\, L_\hbar(1) \;+\; M \;=\; |a|\, L_\hbar(1) \;+\; M. \quad ∎$$

Proposition 5.1.6 [**M1**]. *Given a helix* $\hbar$.

a. $\int_0^\infty e^{-t}\,\|\hbar_0 - \hbar_t\|\,dt < \infty$.

b. $e^{-t}\,(\hbar_0 - \hbar_t)$ *is Lebesgue–Bochner integrable on* $\mathbf{R}_+$.

c. $\int_0^\infty e^{-t}\,(\hbar_0 - \hbar_t)\,dt \in CS(\hbar)$.

Proof: a. From Proposition 5.1.5,

$$\begin{aligned}\int_0^\infty e^{-t}\,\|\hbar_0 - \hbar_t\|\,dt &= \int_0^\infty e^{-t}\,L_\hbar(t)\,dt \\ &\le L_\hbar(1)\int_0^\infty t\,e^{-t}\,dt \;+\; M\int_0^\infty e^{-t}\,dt \\ &= L_\hbar(1) \;+\; M \\ &< \infty,\end{aligned}$$

where $M = \max_{0\le t\le 1} L_\hbar(t) < \infty$.

b, c. Follow immediately from a and [**HP**, Theorem 3.7.4]. ∎

Definition 5.1.7 [**M1**]. Given a helix $\hbar$, the vector

$$\alpha_\hbar \;=\; \int_0^\infty e^{-t}\,(\hbar_0 - \hbar_t)\,dt \;\in\; CS(\hbar) \;\subset\; X$$

is the **average vector** of $\hbar$.

Let U be the shift group of a helix $\hbar$. Then, by definition, $U\colon \mathbf{R} \to L(X)$ is continuous in the strong topology of $L(X)$. Therefore, by [**HP**, Theorem 3.3.4], $\int_a^b U_s\,ds$ exists as a Riemann integral in the strong topology of $L(X)$ for each $a \le b$. We use the following notation:

$$T_U(a,b) \;=\; U_b \;-\; U_a \;-\; \int_a^b U_s\,ds.$$

PROPOSITION 5.1.8 [**M1**]. *Given a helix $\hbar$,*

$$\int_0^\infty e^{-t}\, T_U(0,t)\, dt \;=\; -I.$$

PROOF: First note that $e^{-t}\, T_U(0,t)\colon \mathbf{R} \to L(X)$ is strongly continuous and therefore Riemann integrable on every finite interval by [**HP**, Theorem 3.3.4]. Since

$$\|T_U(0,t)\| \;\leq\; \|U_t\| \;+\; \|U_0\| \;+\; \int_0^t \|U_s\|\, ds \;\leq\; 2+t,$$

the integral $\int_0^\infty e^{-t}\, T_U(0,t)\, dt$ exists as an improper Riemann integral. Evaluating,

$$\begin{aligned}
\int_0^\infty e^{-t}\, T_U(0,t)\, dt &= \int_0^\infty e^{-t}\left(U_t \;-\; U_0 \;-\; \int_0^t U_s\, ds\right)\, dt \\
&= \int_0^\infty e^{-t}\, U_t\, dt \;-\; I \int_0^\infty e^{-t}\, dt \;-\; \int_0^\infty \int_0^t e^{-t}\, U_s\, ds\, dt \\
&= \int_0^\infty e^{-t}\, U_t\, dt \;-\; I \;-\; \int_0^\infty U_s \int_s^\infty e^{-t}\, dt\, ds \\
&= \int_0^\infty e^{-t}\, U_t\, dt \;-\; I \;-\; \int_0^\infty U_s\, e^{-s}\, ds \\
&= -I. \quad \blacksquare
\end{aligned}$$

LEMMA 5.1.9. *Given $f \in L^1_{\text{loc}}(\mathbf{R})$ and $a \leq b$, $c \leq d$,*

$$\int_{a+d}^{b+d} f(t)\, dt \;-\; \int_{a+c}^{b+c} f(t)\, dt \;=\; \int_{b+c}^{b+d} f(t)\, dt \;-\; \int_{a+c}^{a+d} f(t)\, dt.$$

PROOF: There are only two possibilities: either $a+c \leq a+d \leq b+c \leq b+d$ or $a+c \leq b+c \leq a+d \leq b+d$. The result follows immediately in either case. $\blacksquare$

The following is known as the *Switching Lemma.*

Proposition 5.1.10 [M1]. *Given a helix $\hbar$,*

$$T_U(a,b)(\hbar_d - \hbar_c) = T_U(c,d)(\hbar_b - \hbar_a)$$

for every $a, b, c, d \in \mathbf{R}$.

Proof: By definition of T_U,

$$T_U(a,b)(\hbar_d - \hbar_c) = \left(U_b - U_a - \int_a^b U_s\,ds\right)(\hbar_d - \hbar_c), \tag{5.1.1}$$

and

$$T_U(c,d)(\hbar_b - \hbar_a) = \left(U_d - U_c - \int_c^d U_s\,ds\right)(\hbar_b - \hbar_a). \tag{5.1.2}$$

Since $\hbar$ is a helix,

(5.1.3)
$$\begin{aligned}(U_b - U_a)(\hbar_d - \hbar_c) &= (\hbar_{d+b} - \hbar_{c+b}) - (\hbar_{d+a} - \hbar_{c+a}) \\ &= (\hbar_{d+b} - \hbar_{d+a}) - (\hbar_{c+b} - \hbar_{c+a}) \\ &= (U_d - U_c)(\hbar_b - \hbar_a).\end{aligned}$$

From Lemma 5.1.9,

(5.1.4)
$$\begin{aligned}\left(\int_a^b U_s\,ds\right)(\hbar_d - \hbar_c) &= \int_a^b U_s(\hbar_d - \hbar_c)\,ds \\ &= \int_a^b (\hbar_{d+s} - \hbar_{c+s})\,ds \\ &= \int_{a+d}^{b+d} \hbar_s\,ds - \int_{a+c}^{b+c} \hbar_s\,ds \\ &= \int_{b+c}^{b+d} \hbar_s\,ds - \int_{a+c}^{a+d} \hbar_s\,ds\end{aligned}$$

$$= \int_c^d (\hbar_{b+s} - \hbar_{a+s})\, ds$$

$$= \int_c^d U_s(\hbar_b - \hbar_a)\, ds$$

$$= \left(\int_c^d U_s\, ds\right)(\hbar_b - \hbar_a).$$

The result follows upon combining (5.1.1) through (5.1.4). ∎

The following proposition shows that the chords of a helix can be recovered from knowledge of the average vector.

PROPOSITION 5.1.11 [M1]. *Given a helix $\hbar$ and given a, $b \in \mathbf{R}$,*

$$T_U(a,b)\,\alpha_\hbar = \hbar_b - \hbar_a.$$

PROOF: From the switching lemma (Proposition 5.1.10) and Proposition 5.1.8,

$$\begin{aligned}
T_U(a,b)\,\alpha_\hbar &= T_U(a,b)\left(\int_0^\infty e^{-t}(\hbar_0 - \hbar_t)\, dt\right) \\
&= -\int_0^\infty e^{-t}\, T_U(a,b)(\hbar_t - \hbar_0)\, dt \\
&= -\int_0^\infty e^{-t}\, T_U(0,t)(\hbar_b - \hbar_a)\, dt \\
&= -\left(\int_0^\infty e^{-t}\, T_U(0,t)\, dt\right)(\hbar_b - \hbar_a) \\
&= -(-I)(\hbar_b - \hbar_a) \\
&= \hbar_b - \hbar_a. \ \blacksquare
\end{aligned}$$

REMARK 5.1.12. It is not difficult to extend Masani's results on helices over $\mathbf{R}$ (Proposition 5.1.5 through Proposition 5.1.11) to higher dimensions, i.e., to helices over $\mathbf{R}^d$.

In particular, given a helix $\hbar$ parameterized by $\mathbf{R}^d$, the chordal length function will satisfy the growth condition

$$L_\hbar(a) \;\leq\; R(|a_1| + \cdots + |a_d|) \;+\; dM,$$

where $R = \max\{L_\hbar(e_1), \ldots, L_\hbar(e_d)\}$ and $M = \max_{t\in[0,1]} L_\hbar(t)$, cf., Proposition 5.1.5. The average vector

$$\alpha_\hbar \;=\; \int_{\mathbf{R}^d_+} \Pi(e^{-t})\,(\hbar_0 - \hbar_t)\,dt$$

will converge to an element of $CS(\hbar)$, cf., Proposition 5.1.6 and Definition 5.1.7. Defining

$$T_U(a,b) \;=\; U_b \;-\; U_a \;-\; \int_{[a,b]} U_s\,ds,$$

we have

$$\int_{\mathbf{R}^d_+} \Pi(e^{-t})\,T_U(0,t)\,dt \;=\; -I,$$

cf., Proposition 5.1.8. The switching lemma takes the form

$$T_U(a,b)\,(\hbar_d - \hbar_c) \;=\; T_U(c,d)\,(\hbar_b - \hbar_a)$$

for a, b, c, $d \in \mathbf{R}^d$, cf., Proposition 5.1.10. And, finally, given a, $b \in \mathbf{R}^d$, we can recover the chords of the helix from the average vector by

$$T_U(a,b)\,\alpha_\hbar \;=\; \hbar_b - \hbar_a,$$

cf., Proposition 5.1.11.

Section 5.2. Completeness.

In this section we prove that each of the variation spaces $V(p,q)$ is complete. We begin by presenting a theorem due to Lau and Chen (Theorem 5.2.1) from which they derived the completeness of the one dimensional $V(p,\infty)$, cf., Example 5.2.2a. The completeness of $V(p,q)$ for one dimension follows immediately from that theorem as well, cf., Example 5.2.2b.

Lau and Chen's theorem takes advantage of the fact that $\hbar_\lambda = \Delta_\lambda^+ F$ is a helix when $d = 1$, cf., Example 5.1.2a. The existence of the helix average vector allows a Cauchy sequence to be "pulled back" from $V(p,\infty)$ to L^p, where it will converge. A candidate limit vector for the original Cauchy sequence is then constructed using the fact that helix chords are determined by the average vector (Proposition 5.1.11). Masani's results on helices parameterized by **R**, Proposition 5.1.5 through Proposition 5.1.11, are the critical facts which make Lau and Chen's proof possibile.

As mentioned in Remark 5.1.12, Masani's results on helices over **R** can be extended to helices over $\mathbf{R}^d$. However, by Example 5.1.2d, $\hbar_\lambda = \Delta_\lambda^+ F$ is not a helix when $d > 1$. Therefore, helices over $\mathbf{R}^d$ are not appropriate for proving the completeness of $V(p,q)$ in higher dimensions. Instead, we use the fact that $\hbar_{\lambda_j} = \Delta_{\lambda_j}^{j+} F$ is a helix over **R** for each $j = 1, \ldots, d$, and that $\Delta_\lambda^+ = \Delta_{\lambda_1}^{1+} \cdots \Delta_{\lambda_d}^{d+}$. An iterated averaging technique allows us to "pull back" a Cauchy sequence from $V(p,q)$ to L^p. An iterated chord reconstruction then gives the candidate limit vector for the Cauchy sequence.

THEOREM 5.2.1 [**CL1**]. *Given a homogeneous Banach function space* $X \subset L^1_{\mathrm{loc}}(\hat{\mathbf{R}})$ *and given* $1 \leq q \leq \infty$. *Assume* $\varphi \colon \hat{\mathbf{R}}_+ \to \mathbf{R}_+$ *satisfies* $\lambda e^{-\lambda}/\varphi(\lambda) \in L^{q'}(\hat{\mathbf{R}}_+)$, *where this space is taken with the Haar measure* $d\lambda/\lambda$ *for* $\hat{\mathbf{R}}_+$. *Let* Y *be the space of functions* F *such that*

$$\|F\|_Y = \big\|\varphi(\lambda) \cdot \|\Delta^+_\lambda F\|_X\big\|_q < \infty.$$

Then Y *is a Banach space, once we identify functions* F, $G \in Y$ *such that* $\|F - G\|_Y = 0$.

PROOF: The seminorm properties of $\|\cdot\|_Y$ are evident, so Y is a normed linear space once we make the identification of functions whose difference has zero norm. It remains to show that Y is complete.

Assume that $\{G_n\}_{n \in \mathbf{Z}_+}$ is a Cauchy sequence in Y.

a. Given $n \in \mathbf{Z}_+$, define $\hbar^n \colon \hat{\mathbf{R}} \to X$ by

$$\hbar^n_\lambda = \Delta^+_\lambda G_n = \tfrac{1}{2}(T_{-\lambda} G_n - T_\lambda G_n).$$

By Example 5.1.2a, $\hbar^n$ is a helix in X, parameterized by $\hat{\mathbf{R}}$, with shift group $\{T_{-\lambda}\}_{\lambda \in \hat{\mathbf{R}}}$. This helix has an average vector defined by

$$\alpha_n = \int_0^\infty e^{-\lambda} (\hbar^n_0 - \hbar^n_\lambda)\, d\lambda = -\int_0^\infty e^{-\lambda}\, \Delta^+_\lambda G_n\, d\lambda.$$

By Proposition 5.1.6, $\alpha_n \in CS(\hbar^n) \subset X$. The sequence $\{\alpha_n\}$ is Cauchy in X since

$$\|\alpha_m - \alpha_n\|_X$$

$$= \Big\|\int_0^\infty e^{-\lambda}\, \Delta^+_\lambda (G_m - G_n)\, d\lambda\Big\|_X$$

$$\leq \int_0^\infty \lambda\, e^{-\lambda}\, \|\Delta_\lambda^+(G_m - G_n)\|_X \,\frac{d\lambda}{\lambda}$$

$$\leq \left(\int_0^\infty \varphi(\lambda)^q\, \|\Delta_\lambda^+(G_m - G_n)\|_X^q \,\frac{d\lambda}{\lambda}\right)^{1/q} \left(\int_0^\infty \left|\frac{\lambda\, e^{-\lambda}}{\varphi(\lambda)}\right|^{q'} \frac{d\lambda}{\lambda}\right)^{1/q'}$$

$$= C\, \|G_m - G_n\|_Y$$

$$\to 0 \quad \text{as } m, n \to \infty$$

(the cases $q = 1, \infty$ are similar). Therefore, $\alpha_n \to \alpha$ for some $\alpha \in X$.

By Proposition 5.1.11,

$$\Delta_\lambda^+ G_n = \hbar_\lambda^n = \hbar_\lambda^n - \hbar_0^n = T_U(0,\lambda)\,\alpha_n = T_{-\lambda}\alpha_n - T_0\alpha_n - \int_0^\lambda T_{-s}\alpha_n\, ds = 2\,\Delta_\lambda^+\alpha_n - \int_0^\lambda T_{-s}\alpha_n\, ds. \tag{5.2.1}$$

Since $\alpha \in X \subset L^1_{\text{loc}}(\hat{\mathbf{R}})$, we can define

$$G(\gamma) = 2\left(\alpha(\gamma) - \int_0^\gamma \alpha(s)\, ds\right).$$

We compute

$$\Delta_\lambda^+ G(\gamma) = \tfrac{1}{2}\left[G(\gamma+\lambda) - G(\gamma)\right] = \alpha(\gamma+\lambda) - \int_0^{\gamma+\lambda} \alpha(s)\, ds - \alpha(\gamma) + \int_0^\gamma \alpha(s)\, ds = \alpha(\gamma+\lambda) - \alpha(\gamma) - \int_\gamma^{\gamma+\lambda} \alpha(s)\, ds \tag{5.2.2}$$

$$= 2\,\Delta_\lambda^+\alpha(\gamma) \;-\; \int_0^\lambda \alpha(\gamma+s)\,ds$$

$$= \Big(2\,\Delta_\lambda^+\alpha \;-\; \int_0^\lambda T_{-s}\alpha\,ds\Big)(\gamma).$$

From (5.2.1) and (5.2.2),

$$\begin{aligned}
&\|\Delta_\lambda^+(G-G_n)\|_X \\
&\quad = \Big\|2\,\Delta_\lambda^+(\alpha-\alpha_n) \;-\; \int_0^\lambda T_{-s}(\alpha-\alpha_n)\,ds\Big\|_X \\
&\quad \le \|T_{-\lambda}(\alpha-\alpha_n)\|_X \;+\; \|\alpha-\alpha_n\|_X \;+\; \int_0^\lambda \|T_{-s}(\alpha-\alpha_n)\|_X\,ds \\
&\quad = (2+\lambda)\,\|\alpha-\alpha_n\|_X.
\end{aligned}$$

Therefore, for each fixed λ,

$$\begin{aligned}
\text{(5.2.3)} \qquad \theta_n(\lambda) &= \varphi(\lambda)\,\|\Delta_\lambda^+(G-G_n)\|_X \\
&\le (2+\lambda)\,\varphi(\lambda)\,\|\alpha-\alpha_n\|_X \\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

b. We show now that $G \in Y$. Define

$$\beta_n(\lambda) = \varphi(\lambda)\,\|\Delta_\lambda^+ G_n\|_X \qquad \text{and} \qquad \beta(\lambda) = \varphi(\lambda)\,\|\Delta_\lambda^+ G\|_X.$$

By definition, $G \in Y$ if and only if $\beta \in L^q(\hat{\mathbf{R}}_+)$. Now, $G_n \in Y$, so $\beta_n \in L^q(\hat{\mathbf{R}}_+)$. By the triangle inequality and (5.2.3),

$$\text{(5.2.4)} \qquad |\beta(\lambda)-\beta_n(\lambda)| \;\le\; \theta_n(\lambda) \;\to\; 0 \quad \text{as } n\to\infty.$$

Moreover,

(5.2.5)

$$\begin{aligned}
\|\beta_m - \beta_n\|_q^q &= \int_0^\infty \varphi(\lambda)^q \left| \|\Delta_\lambda^+ G_m\|_X - \|\Delta_\lambda^+ G_m\|_X \right|^q \frac{d\lambda}{\lambda} \\
&\le \int_0^\infty \varphi(\lambda)^q \|\Delta_\lambda^+ (G_m - G_n)\|_X^q \frac{d\lambda}{\lambda} \\
&= \|G_m - G_n\|_Y^q \\
&\to 0 \quad \text{as } m, n \to \infty.
\end{aligned}$$

Thus $\{\beta_n\}$ forms a Cauchy sequence in $L^q(\hat{\mathbf{R}}_+)$, so must converge to some element of $L^q(\hat{\mathbf{R}}_+)$. Since $\beta_n \to \beta$ pointwise by (5.2.4) we must have $\beta_n \to \beta$ in $L^q(\hat{\mathbf{R}}_+)$. Thus $\beta \in L^q(\hat{\mathbf{R}}_+)$, so $G \in Y$.

c. We show now that $G_n \to G$ in Y for the case $1 \le q < \infty$. Since $\theta_n(\lambda) = \varphi(\lambda) \|\Delta_\lambda^+(G - G_n)\|_X$, it suffices to show that $\|\theta_n\|_q \to 0$ as $n \to \infty$. By (5.2.3), $\theta_n \to 0$ pointwise, and, by (5.2.4) and (5.2.5), $\beta_n \to \beta$ both pointwise and in $L^q(\hat{\mathbf{R}}_+)$. Also, $\theta_n \le \beta_n + \beta$, so

$$\theta_n(\lambda)^q \le \left(\beta(\lambda) + \beta_n(\lambda)\right)^q \le 2^q \beta(\lambda)^q + 2^q \beta_n(\lambda)^q.$$

Thus $2^q \beta(\lambda)^q + 2^q \beta_n(\lambda)^q - \theta_n(\lambda)^q \ge 0$, so we can apply Fubini's thoerem in the following calculation:

$$\begin{aligned}
&2^{q+1} \int_0^\infty \beta(\lambda)^q \frac{d\lambda}{\lambda} \\
&= \int_0^\infty \left(2^q \beta(\lambda)^q + 2^q \beta(\lambda)^q + 0\right) \frac{d\lambda}{\lambda} \\
&= \int_0^\infty \liminf_{n\to\infty} \left(2^q \beta(\lambda)^q + 2^q \beta_n(\lambda)^q - \theta_n(\lambda)^q\right) \frac{d\lambda}{\lambda} \\
&\le \liminf_{n\to\infty} \int_0^\infty \left(2^q \beta(\lambda)^q + 2^q \beta_n(\lambda)^q - \theta_n(\lambda)^q\right) \frac{d\lambda}{\lambda} \\
&= 2^q \int_0^\infty \beta(\lambda)^q \frac{d\lambda}{\lambda} + 2^q \int_0^\infty \beta(\lambda)^q \frac{d\lambda}{\lambda} - \limsup_{n\to\infty} \int_0^\infty \theta_n(\lambda)^q \frac{d\lambda}{\lambda}.
\end{aligned}$$

Thus $\limsup_{n\to\infty} \int_0^\infty \theta_n(\lambda)^q \frac{d\lambda}{\lambda} \leq 0$, whence $\lim_{n\to\infty} \|\theta_n\|_q = 0$, and therefore $G_n \to G$ in Y.

d. Finally, we show that $G_n \to G$ in Y for the case $q = \infty$. Fix $\varepsilon > 0$. Then there exists an $N > 0$ such that

$$\|G_m - G_n\|_Y \leq \varepsilon \quad \text{for } m, n \geq N.$$

Also, by definition of $\|\cdot\|_Y$ when $q = \infty$, there must exist a $\lambda > 0$ such that

$$\|G - G_N\|_Y \leq \varphi(\lambda)\, \|\Delta_\lambda^+(G - G_N)\|_X + \varepsilon.$$

From (5.2.3), there then exists an $M \geq N$ such that

$$\varphi(\lambda)\, \|\Delta_\lambda^+(G - G_n)\|_X \leq \varepsilon \quad \text{for } n \geq M.$$

Therefore,

$$\begin{aligned}
\|G - G_n\|_Y &\leq \|G - G_N\|_Y + \|G_N - G_n\|_Y \\
&\leq \varphi(\lambda)\, \|\Delta_\lambda^+(G - G_N)\|_X + \varepsilon + \varepsilon \\
&\leq \varphi(\lambda)\, \|\Delta_\lambda^+(G - G_M)\|_X + \varphi(\lambda)\, \|\Delta_\lambda^+(G_M - G_N)\|_X + 2\varepsilon \\
&\leq \varepsilon + \|G_M - G_N\|_Y + \varepsilon \\
&\leq 4\varepsilon
\end{aligned}$$

for $n \geq M$. Thus $G_n \to G$ in Y. ∎

EXAMPLE 5.2.2. a. Set $X = L^p(\hat{\mathbf{R}})$, $\varphi(\lambda) = (4/\lambda)^{1/p}$, and $q = \infty$. Since $\lambda e^{-\lambda}/\varphi(\lambda) = 4^{-1/p}\, \lambda^{(p+1)/p}\, e^{-\lambda} \in L^1(\hat{\mathbf{R}}_+)$, the space Y defined in Theorem

5.2.1 is complete. To evaluate $\|\cdot\|_Y$, recall from Remark 4.2.2e that $\Delta_\lambda^+ = T_{-\lambda/2}\Delta_{-\lambda/2}$, so $\|\Delta_\lambda^+ F\|_p = \|\Delta_{-\lambda/2}F\|_p$. Therefore,

$$\begin{aligned}
\|F\|_Y &= \Big\|\varphi(\lambda)\cdot\|\Delta_\lambda^+ F\|_{L^p(\hat{\mathbf{R}})}\Big\|_{L^\infty(\hat{\mathbf{R}}_+)} \\
&= \sup_{\lambda>0}\left(\frac{4}{\lambda}\right)^{1/p}\left(\int_{-\infty}^{\infty}|\Delta_{\lambda/2}F(\gamma)|^p\,d\gamma\right)^{1/p} \\
&= \sup_{\lambda>0}\left(\frac{2}{\lambda}\int_{-\infty}^{\infty}|\Delta_\lambda F(\gamma)|^p\,d\gamma\right)^{1/p} \\
&= \|F\|_{V(p,\infty)}.
\end{aligned}$$

Thus $V(p,\infty)$ is complete (for $d=1$).

b. The completeness of $V(p,q)$ follows exactly as in part a. Set $X = L^p(\hat{\mathbf{R}})$, $\varphi(\lambda) = (4/\lambda)^{1/p}$, and fix $1 \le q < \infty$. We have $\lambda e^{-\lambda}/\varphi(\lambda) = 4^{-1/p}\,\lambda^{(p+1)/p}\,e^{-\lambda} \in L^{q'}(\hat{\mathbf{R}}_+)$, so Y is complete. Evaluating,

$$\begin{aligned}
\|F\|_Y &= \Big\|\varphi(\lambda)\cdot\|\Delta_\lambda^+ F\|_{L^p(\hat{\mathbf{R}})}\Big\|_{L^q(\hat{\mathbf{R}}_+)} \\
&= \left(\int_0^\infty\left(\frac{4}{\lambda}\int_{-\infty}^{\infty}|\Delta_{\lambda/2}F(\gamma)|^p\,d\gamma\right)^{q/p}\frac{d\lambda}{\lambda}\right)^{1/q} \\
&= \left(\int_0^\infty\left(\frac{2}{\lambda}\int_{-\infty}^{\infty}|\Delta_\lambda F(\gamma)|^p\,d\gamma\right)^{q/p}\frac{d\lambda}{\lambda}\right)^{1/q} \\
&= \|F\|_{V(p,q)},
\end{aligned}$$

so $V(p,q)$ is complete (for $d=1$).

We now extend Theorem 5.2.1 to higher dimensions.

THEOREM 5.2.3. *Given a homogeneous Banach function space $X \subset L^1_{\text{loc}}(\hat{\mathbf{R}}^d)$ and given $1 \le q \le \infty$. Assume $\varphi: \hat{\mathbf{R}}^d_+ \to \mathbf{R}_+$ satisfies $\Pi(\lambda e^{-\lambda})/\varphi(\lambda) \in$*

$L^{q'}(\hat{\mathbf{R}}_+^d)$, *where this space is taken with the Haar measure* $d\lambda/\Pi(\lambda)$ *for* $\hat{\mathbf{R}}_+^d$. *Let* Y *be the space of functions* F *such that*

$$\|F\|_Y \;=\; \left\|\varphi(\lambda)\cdot\|\Delta_\lambda^+ F\|_X\right\|_q \;<\; \infty.$$

Then Y *is a Banach space, once we identify functions* F, $G \in Y$ *such that* $\|F - G\|_Y = 0$.

PROOF: The seminorm properties of $\|\cdot\|_Y$ are evident, so Y is a normed linear space once we make the identification of functions whose difference has zero norm. It remains to show that Y is complete.

Assume that $\{G_n\}_{n\in\mathbf{Z}_+}$ is a Cauchy sequence in Y. Fix $n \in \mathbf{Z}_+$, let $\alpha_{0n} = G_n$, and define $\hbar^{1n}: \hat{\mathbf{R}} \to X$ by

$$\hbar^{1n}_{\lambda_1} \;=\; \Delta^{1+}_{\lambda_1}\alpha_{0n}$$

for $\lambda_1 \in \hat{\mathbf{R}}$. By Example 5.1.2c, $\hbar^{1n}$ is a helix in X, parameterized by $\hat{\mathbf{R}}$, with shift group $U^1 = \{T_{-\lambda_1 e_1}\}_{\lambda_1\in\hat{\mathbf{R}}}$. This helix has an average vector α_{1n} defined by

$$\alpha_{1n} \;=\; \int_0^\infty e^{-\lambda_1}\,(\hbar^{1n}_0 - \hbar^{1n}_{\lambda_1})\,d\lambda_1 \;=\; -\int_0^\infty e^{-\lambda_1}\,\Delta^{1+}_{\lambda_1}\alpha_{0n}\,d\lambda_1.$$

By Proposition 5.1.6, $\alpha_{1n} \in CS(\hbar^{1n}) \subset X$.

Since X is closed under translations, $\Delta^{2+}_{\lambda_2}\alpha_{1n} \in X$ for $\lambda_2 \in \hat{\mathbf{R}}$. Therefore

$$\hbar^{2n}_{\lambda_2} \;=\; \Delta^{2+}_{\lambda_2}\alpha_{1n}$$

is also a helix in X, parameterized by $\hat{\mathbf{R}}$, with shift group $U^2 = \{T_{-\lambda_2 e_2}\}_{\lambda_2\in\hat{\mathbf{R}}}$. This helix has an average vector α_{2n} defined by

$$\alpha_{2n} \;=\; -\int_0^\infty e^{-\lambda_2}\,\Delta^{2+}_{\lambda_2}\alpha_{1n}\,d\lambda_2 \;\in\; CS(\hbar^{2n}) \;\subset\; X.$$

Continuing in this way we obtain helices $\hbar^{1n}, \ldots, \hbar^{dn}$ and average vectors $\alpha_{1n}, \ldots, \alpha_{dn}$ such that

$$\hbar^{jn}_{\lambda_j} = \Delta^{j+}_{\lambda_j} \alpha_{(j-1)n}$$

and

$$\alpha_{jn} = -\int_0^\infty e^{-\lambda_j} \, \Delta^{j+}_{\lambda_j} \alpha_{(j-1)n} \, d\lambda_j.$$

Define

$$\begin{aligned}
\alpha_n &= \alpha_{dn} \\
&= -\int_0^\infty e^{-\lambda_d} \, \Delta^{d+}_{\lambda_d} \alpha_{(d-1)n} \, d\lambda_d \\
&= -\int_0^\infty e^{-\lambda_d} \, \Delta^{d+}_{\lambda_d} \left(-\int_0^\infty e^{-\lambda_{d-1}} \, \Delta^{(d-1)+}_{\lambda_{d-1}} \alpha_{(d-2)n} \, d\lambda_{d-1} \right) d\lambda_d \\
&= (-1)^2 \int_0^\infty \int_0^\infty e^{-\lambda_d} \, e^{-\lambda_{d-1}} \, \Delta^{d+}_{\lambda_d} \, \Delta^{(d-1)+}_{\lambda_{d-1}} \alpha_{(d-2)n} \, d\lambda_{d-1} \, d\lambda_d \\
&\vdots \\
&= (-1)^d \int_0^\infty \cdots \int_0^\infty e^{-\lambda_d} \cdots e^{-\lambda_1} \, \Delta^{d+}_{\lambda_d} \cdots \Delta^{1+}_{\lambda_1} \alpha_{0n} \, d\lambda_1 \cdots d\lambda_d \\
&= (-1)^d \int_{\mathbb{R}^d_+} \Pi(e^{-\lambda}) \, \Delta^+_\lambda G_n \, d\lambda.
\end{aligned}$$

Then

$$\begin{aligned}
\|\alpha_m - \alpha_n\|_X &= \left\| \int_{\mathbb{R}^d_+} \Pi(e^{-\lambda}) \, \Delta^+_\lambda (G_m - G_n) \, d\lambda \right\|_X \\
&\le \int_{\mathbb{R}^d_+} \Pi(\lambda \, e^{-\lambda}) \, \|\Delta^+_\lambda (G_m - G_n)\|_X \, \frac{d\lambda}{\Pi(\lambda)} \\
&\le \left\| \varphi(\lambda) \cdot \|\Delta^+_\lambda (G_m - G_n)\|_X \right\|_q \, \|\Pi(\lambda \, e^{-\lambda}) / \varphi(\lambda)\|_{q'} \\
&= C \, \|G_m - G_n\|_Y \\
&\to 0 \quad \text{as } m, n \to \infty.
\end{aligned}$$

Thus $\{\alpha_n\}$ forms a Cauchy sequence in X, and therefore $\alpha_n \to \alpha$ for some $\alpha \in X$.

By Proposition 5.1.11,

$$\begin{aligned}
\Delta^{j+}_{\lambda_j} \alpha_{(j-1)n} &= \hbar^{jn}_{\lambda_j} \\
&= \hbar^{jn}_{\lambda_j} - \hbar^{jn}_{0} \\
&= T_{U^j}(0, \lambda_j)\, \alpha_{jn} \\
&= T_{-\lambda_j e_j} \alpha_{jn} - \alpha_{jn} - \int_0^{\lambda_j} T_{-s_j e_j} \alpha_{jn}\, ds_j \\
&= 2\,\Delta^{j+}_{\lambda_j} \alpha_{jn} - \int_0^{\lambda_j} T_{-s_j e_j} \alpha_{jn}\, ds_j \\
&= \left(2\,\Delta^{j+}_{\lambda_j} - \int_0^{\lambda_j} T_{-s_j e_j}\, ds_j\right) \alpha_{jn}.
\end{aligned}$$

Therefore,

(5.2.6)

$$\begin{aligned}
\Delta^{+}_{\lambda} G_n &= \Delta^{d+}_{\lambda_d} \cdots \Delta^{1+}_{\lambda_1} \alpha_{0n} \\
&= \Delta^{d+}_{\lambda_d} \cdots \Delta^{2+}_{\lambda_2} \left(2\,\Delta^{1+}_{\lambda_1} - \int_0^{\lambda_1} T_{-s_1 e_1}\, ds_1\right) \alpha_{1n} \\
&= \left(2\,\Delta^{1+}_{\lambda_1} - \int_0^{\lambda_1} T_{-s_1 e_1}\, ds_1\right) \Delta^{d+}_{\lambda_d} \cdots \Delta^{2+}_{\lambda_2} \alpha_{1n} \\
&\;\;\vdots \\
&= \left(2\,\Delta^{1+}_{\lambda_1} - \int_0^{\lambda_1} T_{-s_1 e_1}\, ds_1\right) \cdots \left(2\,\Delta^{d+}_{\lambda_d} - \int_0^{\lambda_d} T_{-s_d e_d}\, ds_d\right) \alpha_n.
\end{aligned}$$

Since $\alpha \in X \subset L^1_{\text{loc}}(\hat{\mathbf{R}}^d)$, we can, by Fubini's theorem, define

$$\begin{aligned}
F_0(\gamma) &= \alpha(\gamma) \\
F_1(\gamma) &= 2\left(F_0(\gamma) - \int_0^{\gamma_1} F_0(s_1, \gamma_2, \ldots, \gamma_d)\, ds_1\right),
\end{aligned}$$

$$F_2(\gamma) = 2\left(F_1(\gamma) - \int_0^{\gamma_2} F_1(\gamma_1, s_2, \gamma_3, \ldots, \gamma_d)\, ds_2\right),$$

$$\vdots$$

$$G(\gamma) = F_d(\gamma) = 2\left(F_{d-1}(\gamma) - \int_0^{\gamma_d} F_{d-1}(\gamma_1, \ldots, \gamma_{d-1}, s_d)\, ds_d\right).$$

We compute

$$\begin{aligned}
&\Delta_{\lambda_j}^{j+} F_j(\gamma) \\
&\quad = \tfrac{1}{2}\left[F_j(\gamma + \lambda_j e_j) - F_j(\gamma)\right] \\
&\quad = F_{j-1}(\gamma + \lambda_j e_j) - \int_0^{\gamma_j + \lambda_j} F_{j-1}(\gamma_1, \ldots, \gamma_{j-1}, s_j, \gamma_{j+1}, \ldots, \gamma_j)\, ds_j \\
&\qquad\qquad - F_{j-1}(\gamma) + \int_0^{\gamma_j} F_{j-1}(\gamma_1, \ldots, \gamma_{j-1}, s_j, \gamma_{j+1}, \ldots, \gamma_j)\, ds_j \\
&\quad = F_{j-1}(\gamma + \lambda_j e_j) - F_{j-1}(\gamma) \\
&\qquad\qquad - \int_{\gamma_j}^{\gamma_j + \lambda_j} F_{j-1}(\gamma_1, \ldots, \gamma_{j-1}, s_j, \gamma_{j+1}, \ldots, \gamma_j)\, ds_j \\
&\quad = 2\,\Delta_{\lambda_j}^{j+} F_{j-1}(\gamma) - \int_0^{\lambda_j} F_{j-1}(\gamma_1, \ldots, \gamma_{j-1}, \gamma_j + s_j, \gamma_{j+1}, \ldots, \gamma_j)\, ds_j \\
&\quad = 2\,\Delta_{\lambda_j}^{j+} F_{j-1}(\gamma) - \int_0^{\lambda_j} T_{-s_j e_j} F_{j-1}(\gamma)\, ds_j \\
&\quad = \left(2\,\Delta_{\lambda_j}^{j+} - \int_0^{\lambda_j} T_{-s_j e_j}\, ds\right) F_{j-1}(\gamma).
\end{aligned}$$

Therefore, just as in (5.2.6),

(5.2.7)

$$\Delta_\lambda^+ G = \left(2\,\Delta_{\lambda_1}^{1+} - \int_0^{\lambda_1} T_{-s_1 e_1}\, ds_1\right) \cdots \left(2\,\Delta_{\lambda_d}^{d+} - \int_0^{\lambda_d} T_{-s_d e_d}\, ds_d\right) \alpha.$$

Define $H_0 = \alpha - \alpha_n$ and, for $j = 1, \ldots, d$,

$$H_j = \left(2\,\Delta_{\lambda_j}^{j+} - \int_0^{\lambda_j} T_{-s_j e_j}\, ds_j\right) \cdots \left(2\,\Delta_{\lambda_1}^{1+} - \int_0^{\lambda_1} T_{-s_1 e_1}\, ds_1\right)(\alpha - \alpha_n).$$

Then,

(5.2.8)

$$\begin{aligned}\|H_j\|_X &= \left\|\left(2\,\Delta_{\lambda_j}^{j+} - \int_0^{\lambda_j} T_{-s_j e_j}\,ds_j\right)H_{j-1}\right\|_X \\ &\le \|T_{-\lambda_j e_j} H_{j-1}\|_X + \|H_{j-1}\|_X + \int_0^{\lambda_j} \|T_{-s_j e_j} H_{j-1}\|_X\,ds_j \\ &= (2+\lambda_j)\,\|H_{j-1}\|_X.\end{aligned}$$

Combining (5.2.6), (5.2.7), and (5.2.8),

$$\begin{aligned}\|\Delta_\lambda^+(G-G_n)\|_X &= \|H_d\|_X \\ &\le (2+\lambda_d)\,\|H_{d-1}\|_X \\ &\ \ \vdots \\ &\le (2+\lambda_d)\cdots(2+\lambda_1)\,\|H_0\|_X \\ &= \Pi(2+\lambda)\,\|\alpha-\alpha_n\|_X.\end{aligned}$$

Hence, for each fixed λ,

$$\begin{aligned}\theta_n(\lambda) &= \varphi(\lambda)\,\|\Delta_\lambda^+(G-G_n)\|_X \\ &\le \Pi(2+\lambda)\,\varphi(\lambda)\,\|\alpha-\alpha_n\|_X \\ &\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

The remainder of the proof is now precisely similar to parts b, c, and d of the proof of Theorem 5.2.1. ∎

PART IV

WAVELET THEORY

CHAPTER 6

FRAMES

Frames were invented by Duffin and Schaeffer in their work on nonharmonic Fourier series as an alternative to orthonormal bases in Hilbert spaces, cf., [**DS**]. They were later used by Daubechies, Grossmann, and Meyer to formulate wavelet theory in $L^2(\mathbf{R})$, cf., [**DGM; D1**]. Gröchenig has extended the notion of frames (and the related concept of sets of atoms) to Banach spaces, cf., [**G**]. This chapter is an essentially expository review of basic results on frames and sets of atoms, especially in Hilbert spaces. We have combined results from many sources, including [**D1; DGM; DS; G; GK; Y**], with remarks, examples, and minor results of our own, into a single survey chapter.

In Section 6.1 we recall the definitions and basic properties of bases in Banach and Hilbert spaces.

In Section 6.2 we define frames for Hilbert spaces and discuss their basic properties. The primary result is that given a frame $\{x_n\}$, any element $x \in H$ can be written as $x = \sum c_n x_n$, where the scalars $\{c_n\}$ are explicitly known (although not necessarily unique), and the series converges unconditionally.

In Section 6.3 we characterize those frames which are bases, i.e., those frames for which the representations $x = \sum c_n x_n$ are unique for all x.

In Section 6.4 we discuss sets of atoms, which are a dual concept to frames. The term *atoms* is an unfortunate terminology, since this word is heavily overused in the literature. In particular, the sets of atoms discussed here are not related to the atoms and atomic decompositions appearing in Littlewood–Paley theory. We discuss in this section the exact relationship between frames and sets of atoms, and show that, while atoms are a more general concept, in most practical applications atoms and frames in Hilbert spaces are equivalent.

In Section 6.5 we discuss the formulation of frames and sets of atoms in Banach spaces.

Finally, in Section 6.6 we prove a stability result for sets of atoms in Banach spaces. In particular, we prove that the elements of a set of atoms may be perturbed by a small amount without destroying the atomic properties.

Section 6.1. Bases.

In this section we review the basic definitions and properties of bases in Banach and Hilbert spaces.

DEFINITION 6.1.1. Given a sequence $\{x_n\}_{n\in\mathbf{Z}_+}$ of elements of a Banach space X.

a. The **span** of $\{x_n\}$, denoted $\mathrm{span}\{x_n\}$, is the set of finite linear combinations of elements of $\{x_n\}$. The **closed linear span** of $\{x_n\}$, denoted $\overline{\mathrm{span}}\{x_n\}$, is the closure in X of $\mathrm{span}\{x_n\}$.

b. $\{x_n\}$ is **complete** if $\overline{\mathrm{span}}\{x_n\} = X$, or, equivalently, if $\mu \in X'$ and $\mu(x_n) = 0$ for all n implies $\mu = 0$.

c. $\{x_n\}$ is **minimal** if $x_m \neq \overline{\mathrm{span}}\{x_n\}_{n\neq m}$ for each m.

d. $\{x_n\}$ is a **basis** if for each $x \in X$ there exist unique scalars $a_n(x)$ such that $x = \sum a_n(x)\, x_n$. The basis is **unconditional** if the series $\sum a_n(x)\, x_n$ converges unconditionally for each x, cf., Section 1.4. The basis is **bounded** if $0 < \inf \|x_n\| \leq \sup \|x_n\| < \infty$.

REMARK 6.1.2. a. Bases are complete and minimal, but the reverse need not be true.

b. Every basis is a **Schauder basis**, i.e., each coefficient functional a_n is continuous and therefore an element of X'.

c. If $\{x_n\}$ is a basis then $\{x_n\}$ and $\{a_n\}$ are **biorthonormal**, i.e., $a_m(x_n) = \delta_{mn}$. The following proposition states that the existence of a biorthonormal sequence is equivalent to minimality.

PROPOSITION 6.1.3 [**S**]. *Given a sequence $\{x_n\}_{n\in\mathbf{Z}_+}$ in a Banach space X.*

a. *$\{x_n\}$ is minimal if and only if there exists a sequence $\{a_n\} \subset X'$ which is biorthonormal to $\{x_n\}$.*

b. *$\{x_n\}$ is minimal and complete if and only if there exists a unique sequence $\{a_n\} \subset X'$ which is biorthonormal to $\{x_n\}$.*

PROPOSITION 6.1.4 [**S**]. *Given a complete sequence $\{x_n\}_{n\in\mathbf{Z}_+}$ in a Banach space X with every $x_n \neq 0$, the following statements are equivalent.*

a. *$\{x_n\}$ is an unconditional basis for X.*

b. *There exists $C_1 > 0$ such that for all scalars $c_1, \dots, c_N$ and all signs $\sigma_1, \dots, \sigma_N = \pm 1$,*

$$\Big\|\sum_1^N \sigma_n c_n x_n\Big\| \leq C_1 \Big\|\sum_1^N c_n x_n\Big\|.$$

c. *There exists $C_2 > 0$ such that for all scalars $b_1, \dots, b_N$ and $c_1, \dots, c_N$ with $|b_n| \leq |c_n|$,*

$$\Big\|\sum_1^N b_n x_n\Big\| \leq C_2 \Big\|\sum_1^N c_n x_n\Big\|.$$

d. *There exist $C_3, C_4 > 0$ such that for all scalars $c_1, \dots, c_N$,*

$$C_3 \Big\|\sum_1^N |c_n|\, x_n\Big\| \leq \Big\|\sum_1^N c_n x_n\Big\| \leq C_4 \Big\|\sum_1^N |c_n|\, x_n\Big\|.$$

DEFINITION 6.1.5. Two bases $\{x_n\}$ and $\{y_n\}$ for a Banach space X are **equivalent** if there exists a topological isomorphism $U: X \to X$ such that

$Ux_n = y_n$ for all n, or, equivalently, if $\sum c_n x_n$ converges if and only if $\sum c_n y_n$ converges.

We list some additional facts about bases in Hilbert spaces. The inner product in a Hilbert space H is written $\langle \cdot, \cdot \rangle$.

DEFINITION 6.1.6. A sequence $\{e_n\}$ of elements of a Hilbert space H is an **orthonormal basis** if

a. $\{e_n\}$ is orthonormal, i.e., $\langle e_m, e_n \rangle = \delta_{mn}$,

b. the Plancherel formula holds, i.e., $\sum |\langle x, e_n \rangle|^2 = \|x\|^2$ for $x \in H$.

All orthonormal bases are bases (in the sense of Definition 6.1.1), with $x = \sum \langle x, e_n \rangle e_n$ for $x \in H$.

DEFINITION 6.1.7. A basis for a Hilbert space H is a **Riesz basis** if it is equivalent to some orthonormal basis for H.

PROPOSITION 6.1.8 [**Y**; **GK**]. *Given a sequence $\{x_n\}_{n \in \mathbf{Z}_+}$ in a Hilbert space H, the following statements are equivalent.*

a. *$\{x_n\}$ is a Riesz basis for H.*

b. *$\{x_n\}$ is a bounded unconditional basis for H.*

c. *$\{x_n\}$ is a basis for H, and*

$$\sum c_n x_n \text{ converges } \Leftrightarrow \sum |c_n|^2 < \infty.$$

d. *$\{x_n\}$ is complete and there exist A, $B > 0$ such that for all scalars $c_1, \dots, c_N$,*

$$A \sum_{1}^{N} |c_n|^2 \leq \left\| \sum_{1}^{N} c_n x_n \right\| \leq B \sum_{1}^{N} |c_n|^2.$$

The following is known as *Orlicz' Theorem.*

PROPOSITION 6.1.9 [O; LT; S]. *Given a sequence* $\{x_n\}$ *in a Hilbert space* H. *If* $\sum x_n$ *converges unconditionally then* $\sum \|x_n\|^2 < \infty$.

The converse of Proposition 6.1.9 is not true.

6.2 Frames in Hilbert Spaces.

In this section we define and describe the basic properties of frames in Hilbert spaces.

DEFINITION 6.2.1. A sequence $\{x_n\}_{n\in J}$ in a Hilbert space H is a **frame** if there exist A, $B > 0$ such that for all $x \in H$,

$$A\,\|x\|^2 \;\le\; \sum_{n\in J} |\langle x, x_n\rangle|^2 \;\le\; B\,\|x\|^2. \tag{6.2.1}$$

The numbers A, B are the **frame bounds**, A being the **lower** bound and B the **upper** bound. The frame is **tight** if $A = B$. The frame is **exact** if it ceases to be a frame whenever any single element is deleted from the sequence.

REMARK 6.2.2. a. A sequence $\{x_n\}$ for which $\sum |\langle x, x_n\rangle|^2 < \infty$ for all $x \in H$ is a **Bessel sequence** (cf., [Y]). By the Uniform Boundedness Principle, a Bessel sequence will possess an upper frame bound $B > 0$, i.e., $\sum |\langle x, x_n\rangle|^2 \le B\|x\|^2$ for $x \in H$. In applications, a sequence which is a frame is often easily shown to be a Bessel sequence, while the lower frame bound is more difficult to establish.

b. From the Plancherel formula, every orthonormal basis is a frame with $A = B = 1$. Any orthonormal sequence which satisfies the Plancherel formula is an orthonormal basis, and therefore gives a decomposition of the Hilbert space in terms of the basis elements. The pseudo-Plancherel formula (6.2.1) for frames also implies a decomposition in terms of the frame elements, although the representations induced need not be unique (Proposition 6.2.8c).

c. Since $\sum |\langle x, x_n \rangle|^2$ is a series of nonnegative real numbers, it converges absolutely, hence unconditionally. That is, every rearrangement of the sum also converges, and converges to the same value. Thus, every rearrangement of a frame is also a frame, and all sums involving frames converge unconditionally. Therefore, we can use any countable index set to specify a frame. For this reason we suppress the index set in the remainder of this chapter.

d. Frames are complete, for if $x \in H$ and $\langle x, x_n \rangle = 0$ for all n, then $A \|x\|^2 \leq \sum |\langle x, x_n \rangle|^2 = 0$, so $x = 0$. Therefore, any Hilbert space which possesses a frame must be separable, for the set of finite linear combinations of $\{x_n\}$ with rational coefficients (i.e., rational real and imaginary parts) is a countable dense subset of H. Every separable Hilbert spaces does possess frames since it possesses orthonormal bases.

e. Frames were introduced in 1952 by Duffin and Schaeffer in connection with nonharmonic Fourier series [**DS**]. Much of the general theory of frames was laid out in that paper, although frames were apparently not used in any other context until the paper [**DGM**] by Daubechies, Grossmann, and Meyer.

The following example shows that tightness and exactness are not related.

EXAMPLE 6.2.3. Given an orthonormal basis $\{e_n\}_{n \in \mathbf{Z}_+}$ for a Hilbert space H.

a. $\{e_n\}$ is a tight exact frame for H with bounds $A = B = 1$.

b. $\{e_1, e_1, e_2, e_2, e_3, e_3, \ldots\}$ is a tight inexact frame with bounds $A = B = 2$ but is not orthogonal and is not a basis, although it contains an orthonormal basis.

c. $\{e_1, e_2/2, e_3/3, \ldots\}$ is a complete orthogonal sequence and is a basis, but is not a frame.

d. $\{e_1, e_2/\sqrt{2}, e_2/\sqrt{2}, e_3/\sqrt{3}, e_3/\sqrt{3}, e_3/\sqrt{3}, \ldots\}$ is a tight inexact frame with bounds $A = B = 1$, and no nonredundant subsequence is a frame.

e. $\{2e_1, e_2, e_3, \ldots\}$ is a nontight exact frame with bounds $A = 1$, $B = 2$.

EXAMPLE 6.2.4. The frames used in wavelet theory (e.g., [**DGM**]) are *coherent state frames*, i.e., they are generated from a single fixed element by the action of a group representation. Precisely, they have the form $\{U_{\gamma_n} g\}$, where $g \in H$ is fixed, U is a representation of a locally compact group G on H, and $\{\gamma_n\} \subset G$. Typically, γ_n will be a regular lattice of points in G, though this is not necessary. For example, in Chapter 7 we discuss the situation $H = L^2(\mathbf{R}^d)$, G is the Heisenberg group, U is the Schroedinger representation, and $U_{\gamma_{mn}} g = T_{na} E_{mb} g$ for m, $n \in \mathbf{Z}^d$.

The structure inherent in coherent state frames provides a means for analyzing them. For example, assume that G is compact, U is unitary and square-integrable, and $\{U_{\gamma_n} g\}_{n \in J}$ is a Bessel sequence in H. By definition of square-integrability, there then exists an admissible vector f, i.e., an element $f \in H$ such that $\int_G |\langle U_\gamma f, f\rangle|^2 \, d\gamma < \infty$, where $d\gamma$ is the left Haar measure on G. Since $\{U_{\gamma_n} g\}$ is a Bessel sequence, we therefore have

$$\begin{aligned}\sum \int_G |\langle U_\gamma f, U_{\gamma_n} g\rangle|^2 \, d\gamma &= \int_G \sum |\langle U_\gamma f, U_{\gamma_n} g\rangle|^2 \, d\gamma \\ &\leq \int_G B \, \|U_\gamma f\|^2 \, d\gamma\end{aligned}$$

$$= B\,\|f\|^2\,|G|.$$

Note that $|G| < \infty$ since G is compact. By Proposition 1.9.1,

$$\begin{aligned}\int_G |\langle U_\gamma f, U_{\gamma_n} g\rangle|^2\,d\gamma &= \int_G |\langle U_{\gamma_n^{-1}\gamma} f, g\rangle|^2\,d\gamma \\ &= \int_G |\langle U_\gamma f, g\rangle|^2\,d\gamma \\ &= \frac{\|g\|^2}{\|f\|^2}\int_G |\langle U_\gamma f, f\rangle|^2\,d\gamma,\end{aligned}$$

independent of n. Therefore J must be finite, and hence H must be finite-dimensional.

We now prove some basic properties of frames. Part a of the following lemma is proved in [**DS**].

LEMMA 6.2.5. *Given a Bessel sequence $\{x_n\}$ with upper bound B.*

a. *$\sum c_n x_n$ converges unconditionally in H for every $\{c_n\} \in \ell^2$, and*

$$\left\|\sum c_n x_n\right\|^2 \le B \sum |c_n|^2.$$

b. *Define $Ux = \{\langle x, x_n\rangle\}$ for $x \in H$. Then $U\colon H \to \ell^2$ continuously, and its adjoint $U^*\colon \ell^2 \to H$ is given by $U^*\{c_n\} = \sum c_n x_n$.*

c. *If $\{x_n\}$ is a frame then U is injective and U^* surjective.*

PROOF: a. Let F be any finite subset of the index set J. Then

(6.2.2)
$$\begin{aligned}\left\|\sum_{n\in F} c_n x_n\right\|^2 &= \sup_{\|y\|=1}\left|\left\langle \sum_{n\in F} c_n x_n,\ y\right\rangle\right|^2 \\ &= \sup_{\|y\|=1}\left|\sum_{n\in F} c_n\,\langle x_n, y\rangle\right|^2\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{\|y\|=1} \Big(\sum_{n\in F} |c_n|^2\Big)\Big(\sum_{n\in F} |\langle x_n, y\rangle|^2\Big) \\
&\leq \sup_{\|y\|=1} \Big(\sum_{n\in F} |c_n|^2\Big) B\|y\|^2 \\
&= B \sum_{n\in F} |c_n|^2.
\end{aligned}
$$

Since $\sum |c_n|^2$ converges absolutely and unconditionally, it follows from (6.2.2) that $\sum c_n x_n$ converges unconditionally in H, cf., Lemma 1.4.2c. Therefore we can replace F by J in (6.2.2), i.e., $\| \sum c_n x_n\|^2 \leq B \sum |c_n|^2$.

b. That U is well-defined and continuous follows from the definition of Bessel sequence, for $\|Ux\|_2^2 = \sum |\langle x, x_n\rangle|^2 \leq B\, \|x\|^2$. Its adjoint $U^*: \ell^2 \to H$ is therefore well-defined and continuous, so we need only verify that it has the correct form. If $\{c_n\} \in \ell^2$ then $\sum c_n x_n$ converges to an element of H by part a, so given $x \in H$ we can compute

$$
\begin{aligned}
\langle x, U^*\{c_n\}\rangle &= \langle Ux, \{c_n\}\rangle \\
&= \big\langle \{\langle x, x_n\rangle\},\ \{c_n\}\big\rangle \\
&= \sum \langle x, x_n\rangle\, \bar{c}_n \\
&= \Big\langle x,\ \sum c_n x_n \Big\rangle,
\end{aligned}
$$

whence $U^*\{c_n\} = \sum c_n x_n$.

c. Follows from the fact that frames are complete. ∎

PROPOSITION 6.2.6 [DS]. *Given a sequence $\{x_n\}$ in a Hilbert space H, the following statements are equivalent.*

a. *$\{x_n\}$ is a frame with bounds A, B.*

b. *$Sx = \sum \langle x, x_n \rangle x_n$ is a bounded linear operator with $AI \le S \le BI$.*

In case these hold, the series in b *converge unconditionally.*

PROOF: b $\Rightarrow$ a. If b holds then $\langle AIx, x\rangle \le \langle Sx, x\rangle \le \langle BIx, x\rangle$ for $x \in H$. As $\langle Ix, x\rangle = \|x\|^2$ and $\langle Sx, x\rangle = \sum |\langle x, x_n\rangle|^2$, it follows that $\{x_n\}$ is a frame.

a $\Rightarrow$ b. Assume $\{x_n\}$ is a frame and fix $x \in H$. Then $\sum |\langle x, x_n\rangle|^2 < \infty$, so $Sx = \sum \langle x, x_n\rangle x_n$ converges unconditionally by Lemma 6.2.5. The lemma also implies that $\|Sx\|^2 \le B \sum |\langle x, x_n\rangle|^2 \le B^2 \|x\|^2$, so S is bounded with $\|S\| \le B$. The relations $AI \le S \le BI$ follow immediately from the definition of frame. ∎

DEFINITION 6.2.7. Given a frame $\{x_n\}$, the operator $Sx = \sum \langle x, x_n\rangle x_n$ is the **frame operator** for $\{x_n\}$.

From $AI \le S \le BI$ it follows that $A\|x\| \le \|Sx\| \le B\|x\|$ for $x \in H$. S is therefore continuous and injective, and $S^{-1}\colon \mathrm{Range}(S) \to H$ is continuous. The following proposition shows that S is surjective, hence a topological isomorphism of H.

PROPOSITION 6.2.8 [**DS**]. *Given a frame $\{x_n\}$.*

a. *S is invertible and $B^{-1}I \le S^{-1} \le A^{-1}I$.*

b. *$\{S^{-1}x_n\}$ is a frame with bounds B^{-1}, A^{-1}.*

c. *Given $x \in H$,*

$$x = \sum \langle x, S^{-1}x_n\rangle\, x_n = \sum \langle x, x_n\rangle\, S^{-1}x_n,$$

and these series converge unconditionally.

PROOF: a. Note that $0 \le I - B^{-1}S \le I - \frac{A}{B}I = \frac{B-A}{B}I$ since $AI \le S \le BI$. Therefore $\|I - B^{-1}S\| \le \|\frac{B-A}{B}I\| = \frac{B-A}{B} < 1$, whence $B^{-1}S$, and therefore S, is invertible. The operator S^{-1} is positive since

$$\langle S^{-1}x, x\rangle = \langle S^{-1}x, S(S^{-1}x)\rangle \ge A\,\|S^{-1}x\|^2 \ge 0.$$

As S^{-1} commutes with both I and S we can therefore multiply through by S^{-1} in the equation $AI \le S \le BI$, obtaining $B^{-1}I \le S^{-1} \le A^{-1}I$, cf., **[Heu,** p. 269].

b. The operator S^{-1} is self-adjoint since it is positive. Therefore,

$$\begin{aligned}\sum \langle x, S^{-1}x_n\rangle\, S^{-1}x_n &= \sum \langle S^{-1}x, x_n\rangle\, S^{-1}x_n \\ &= S^{-1}\Big(\sum \langle S^{-1}x, x_n\rangle\, x_n\Big) \\ &= S^{-1}S(S^{-1}x) \\ &= S^{-1}x.\end{aligned}$$

That $\{S^{-1}x_n\}$ is a frame now follows from part a and Proposition 6.2.6.

c. We compute

$$x = S(S^{-1}x) = \sum \langle S^{-1}x, x_n\rangle\, x_n = \sum \langle x, S^{-1}x_n\rangle\, x_n$$

and

$$x = S^{-1}(Sx) = S^{-1}\Big(\sum \langle x, x_n\rangle\, x_n\Big) = \sum \langle x, x_n\rangle\, S^{-1}x_n.$$

The unconditionality of the convergence follows from the fact that $\{x_n\}$ and $\{S^{-1}x_n\}$ are both frames. ∎

DEFINITION 6.2.9. Given a frame $\{x_n\}$ with frame operator S, the frame $\{S^{-1}x_n\}$ is the **dual frame** of $\{x_n\}$.

REMARK 6.2.10. a. The expressions in Proposition 6.2.8c are what we mean when we informally say that a frame $\{x_n\}$ gives a decomposition of the Hilbert space.

b. In case $\{x_n\}$ is a tight frame, i.e., $A = B$, the conclusions of Proposition 6.2.8 reduce to $S = AI$, $S^{-1} = A^{-1}I$, and $x = A^{-1} \sum \langle x, x_n \rangle x_n$ for $x \in H$.

We now prove some results relating to the uniqueness of the decomposition given by a frame. The following proposition shows that the scalars given in Proposition 6.2.8c have the minimal ℓ^2 norm among all choices of scalars $\{c_n\}$ for which $x = \sum c_n x_n$.

PROPOSITION 6.2.11 [DS]. *Given a frame $\{x_n\}$ and given $x \in H$. If $x = \sum c_n x_n$ for some scalars $\{c_n\}$, then*

$$\sum |c_n|^2 = \sum |\langle x, S^{-1}x_n \rangle|^2 + \sum |\langle x, S^{-1}x_n \rangle - c_n|^2.$$

PROOF: Define $a_n = \langle x, S^{-1}x_n \rangle$; then $x = \sum a_n x_n$ by Proposition 6.2.8c. Since $\sum |a_n|^2 < \infty$, assume without loss of generality that $\sum |c_n|^2 = \infty$. Then

$$\begin{aligned} \langle x, S^{-1}x \rangle &= \left\langle \sum a_n x_n,\ S^{-1}x \right\rangle \\ &= \sum a_n \langle S^{-1}x_n, x \rangle \end{aligned}$$

$$= \sum a_n \bar{a}_n$$

$$= \langle \{a_n\},\ \{a_n\} \rangle$$

and

$$\begin{aligned} \langle x, S^{-1}x \rangle &= \left\langle \sum c_n x_n,\ S^{-1}x \right\rangle \\ &= \sum c_n \langle S^{-1}x_n, x \rangle \\ &= \sum c_n \bar{a}_n \\ &= \langle \{c_n\},\ \{a_n\} \rangle. \end{aligned}$$

Therefore $\{c_n - a_n\}$ is orthogonal to $\{a_n\}$ in ℓ^2, whence

$$\|\{c_n\}\|_2^2 = \|\{c_n - a_n\} + \{a_n\}\|_2^2 = \|\{c_n - a_n\}\|_2^2 + \|\{a_n\}\|_2^2. \quad \blacksquare$$

PROPOSITION 6.2.12 [**DS**]. *The removal of a vector from a frame leaves either a frame or an incomplete set. Precisely,*

$$\langle x_m, S^{-1}x_m \rangle \neq 1 \Rightarrow \{x_n\}_{n \neq m} \text{ is a frame,}$$

$$\langle x_m, S^{-1}x_m \rangle = 1 \Rightarrow \{x_n\}_{n \neq m} \text{ is incomplete.}$$

PROOF: a. Fix m and define $a_n = \langle x_m, S^{-1}x_n \rangle$. By Proposition 6.2.8c, $x_m = \sum a_n x_n$. However, $x_m = \sum \delta_{mn} x_n$ as well, so by Proposition 6.2.11,

$$\begin{aligned} 1 = \sum_n |\delta_{mn}|^2 &= \sum_n |a_n|^2 + \sum_n |a_n - \delta_{mn}|^2 \\ &= |a_m|^2 + \sum_{n \neq m} |a_n|^2 + |a_m - 1|^2 + \sum_{n \neq m} |a_n|^2. \end{aligned}$$

Therefore,

$$\sum_{n \neq m} |a_n|^2 = \frac{1 - |a_m|^2 - |a_m - 1|^2}{2} < \infty.$$

b. Suppose that $a_m = 1$. Then $\sum_{n \neq m} |a_n|^2 = 0$, so $a_n = \langle S^{-1}x_m, x_n \rangle = 0$ for $n \neq m$. Thus $S^{-1}x_m$ is orthogonal to x_n for $n \neq m$. However, $S^{-1}x_m \neq 0$ since $\langle S^{-1}x_m, x_m \rangle = a_m = 1 \neq 0$. Therefore $\{x_n\}_{n \neq m}$ is incomplete in this case.

c. On the other hand, suppose $a_m \neq 1$. Then $x_m = \frac{1}{1-a_m} \sum_{n \neq m} a_n x_n$, so for $x \in H$,

$$|\langle x, x_m \rangle|^2 = \left| \frac{1}{1 - a_m} \sum_{n \neq m} a_n \langle x, x_n \rangle \right|^2 \leq C \sum_{n \neq m} |\langle x, x_n \rangle|^2,$$

where $C = |1 - a_m|^{-2} \sum_{n \neq m} |a_n|^2$. Therefore,

$$\sum_n |\langle x, x_n \rangle|^2 = |\langle x, x_m \rangle|^2 + \sum_{n \neq m} |\langle x, x_n \rangle|^2 \leq (1 + C) \sum_{n \neq m} |\langle x, x_n \rangle|^2,$$

whence

$$\frac{A}{1+C} \|x\|^2 \leq \frac{1}{1+C} \sum_n |\langle x, x_n \rangle|^2 \leq \sum_{n \neq m} |\langle x, x_n \rangle|^2 \leq B \|x\|^2.$$

Thus $\{x_n\}_{n \neq m}$ is a frame with bounds $A/(1 + C)$, B. ∎

In the course of the proof of Proposition 6.2.12 we proved the following.

COROLLARY 6.2.13. *Given a frame $\{x_n\}$ and given m,*

$$\sum_{n \neq m} |\langle x_m, S^{-1}x_n \rangle|^2 = \frac{1 - |\langle x_m, S^{-1}x_m \rangle|^2 - |1 - \langle x_m, S^{-1}x_m \rangle|^2}{2}.$$

In particular, if $\langle x_m, S^{-1}x_m\rangle = 1$ *then* $\langle x_m, S^{-1}x_n\rangle = 0$ *for* $n \neq m$.

COROLLARY 6.2.14. *Given a frame* $\{x_n\}$*, the following three statements are equivalent.*

a. $\{x_n\}$ *is exact.*

b. $\{x_n\}$ *and* $\{S^{-1}x_n\}$ *are biorthonormal.*

c. $\langle x_n, S^{-1}x_n\rangle = 1$ *for all* n.

PROOF: a $\Rightarrow$ c. If $\{x_n\}$ is exact, then, by definition, $\{x_n\}_{n\neq m}$ is not a frame for any m. Therefore, by Proposition 6.2.12, $\langle x_m, S^{-1}x_m\rangle = 1$ for every m.

c $\Rightarrow$ a. If $\langle x_m, S^{-1}x_m\rangle = 1$ then $\{x_n\}_{n\neq m}$ is not a frame by Proposition 6.2.12. By definition, $\{x_n\}$ is exact if this is true for all m.

c $\Rightarrow$ b. Follows from Corollary 6.2.13. ∎

COROLLARY 6.2.15. *Given a tight frame* $\{x_n\}$ *with bounds* $A = B$*, the following statements are equivalent.*

a. $\{x_n\}$ *is exact.*

b. $\{x_n\}$ *is an orthogonal sequence.*

c. $\|x_n\|^2 = A$ *for all* n.

PROOF: Follows from Corollary 6.2.14 and the fact that $S = AI$. ∎

PROPOSITION 6.2.16.

a. *Frames are norm bounded above, with* $\sup \|x_n\|^2 \leq B$.

b. *Exact frames are norm bounded below, with* $A \leq \inf \|x_n\|^2$.

PROOF: a. Fix m; then

$$\|x_m\|^4 = |\langle x_m, x_m\rangle|^2 \le \sum_n |\langle x_m, x_n\rangle|^2 \le B\,\|x_m\|^2.$$

b. If $\{x_n\}$ is an exact frame then $\{x_n\}$ and $\{S^{-1}x_n\}$ are biorthonormal by Corollary 6.2.14. Therefore, for m fixed,

$$\begin{aligned} A\,\|S^{-1}x_m\|^2 &\le \sum_n |\langle S^{-1}x_m, x_n\rangle|^2 \\ &= |\langle S^{-1}x_m, x_m\rangle|^2 \\ &\le \|S^{-1}x_m\|^2\,\|x_m\|^2. \end{aligned}$$

As $\{x_n\}$ is exact we have $x_m \neq 0$, so $S^{-1}x_m \neq 0$ and the result follows. ∎

REMARK 6.2.17. Example 6.2.3d shows that inexact frames need not be bounded below.

We collect now some remarks on the convergence of $\sum c_n x_n$ for arbitrary sequences of scalars.

EXAMPLE 6.2.18. In general, it is not true that $x = \sum c_n x_n$ implies that $\sum |c_n|^2 < \infty$. For example, let $\{x_n\}$ be any frame which includes infinitely many zero elements and take the coefficients of the zero elements to be 1. Less trivially, let $\{e_n\}_{n\in\mathbf{Z}_+}$ be an orthonormal basis for H and define $f_n = n^{-1}e_n$ and $g_n = (1-n^{-2})^{1/2}e_n$. Then $\{f_n, g_n\}$ is a tight frame with $A = B = 1$. Now consider the element $x = \sum n^{-1}e_n$; we have $x = \sum(1\cdot f_n + 0\cdot g_n)$ while $\sum(1^2 + 0^2) = \infty$.

PROPOSITION 6.2.19. *Given a frame $\{x_n\}$ which is norm bounded below,*

$$\sum |c_n|^2 < \infty \quad \Leftrightarrow \quad \sum c_n x_n \text{ converges unconditionally.}$$

PROOF: Assume $\sum c_n x_n$ converges unconditionally. Then, by Proposition 6.1.9, $\sum |c_n|^2 \, \|x_n\|^2 = \sum \|c_n x_n\|^2 < \infty$. Since $\{x_n\}$ is norm bounded below it follows that $\sum |c_n|^2 < \infty$. The converse is Lemma 6.2.5. ∎

EXAMPLE 6.2.20. There exist frames $\{x_n\}$ which are norm bounded below and scalars $\{c_n\}$ such that $\sum c_n x_n$ converges but $\sum |c_n|^2 = \infty$.

Let $\{e_n\}_{n \in \mathbf{Z}_+}$ be an orthonormal basis for H, and consider the frame $\{e_1, e_1, e_2, e_2, \ldots\}$, which is norm bounded below. The series

$$e_1 - e_1 + \frac{e_2}{\sqrt{2}} - \frac{e_2}{\sqrt{2}} + \frac{e_3}{\sqrt{3}} - \frac{e_3}{\sqrt{3}} + \cdots \tag{6.2.3}$$

converges strongly to 0. However, the series

$$e_1 + e_1 + \frac{e_2}{\sqrt{2}} + \frac{e_2}{\sqrt{2}} + \frac{e_3}{\sqrt{3}} + \frac{e_3}{\sqrt{3}} + \cdots$$

does not converge. Therefore the series (6.2.3) converges conditionally, cf., Lemma 1.4.2e. Since $\{n^{-1/2}\} \notin \ell^2$, the conditionality of the convergence also follows from Proposition 6.2.19.

Section 6.3. Frames and bases.

In this section we determine the exact relationship between frames and bases in Hilbert spaces.

PROPOSITION 6.3.1. *Inexact frames are not bases.*

PROOF: Assume $\{x_n\}$ is an inexact frame, with frame operator S. Then, by definition, $\{x_n\}_{n\neq m}$ is a frame for some m, and is therefore complete, while no subset of a basis can be complete. In particular, define $a_n = \langle x_m, S^{-1}x_n\rangle$; then $\sum \delta_{mn}x_n = x_m = \sum a_n x_n$ (Proposition 6.2.8c). Since $a_m \neq 1$ by Proposition 6.2.12, these are two different representations of x_m. ∎

LEMMA 6.3.2. *Frames are preserved by topological isomorphisms. Precisely, we have the following. Let H_1, H_2 be Hilbert spaces, and let $\{x_n\}$ be a frame for H_1 with bounds A, B and frame operator S. Assume $T\colon H_1 \to H_2$ is a topological isomorphism. Then $\{Tx_n\}$ is a frame for H_2 with bounds $A\,\|T^{-1}\|^{-2}$, $B\,\|T\|^2$ and frame operator TST^*. Moreover, $\{Tx_n\}$ is exact if and only if $\{x_n\}$ is exact.*

PROOF: First note that for each $y \in H_2$,

$$TST^*y = T\Big(\sum \langle T^*y, x_n\rangle\, x_n\Big) = \sum \langle y, Tx_n\rangle\, Tx_n.$$

By Proposition 6.2.6, it therefore suffices to show that $A\,\|T^{-1}\|^{-2}I \le TST^* \le B\,\|T\|^2 I$. Given $y \in H$ we have $\langle TST^*y, y\rangle = \langle S(T^*y), (T^*y)\rangle$, so

$$A\,\|T^*y\|^2 \le \langle TST^*y, y\rangle \le B\,\|T^*y\|^2, \tag{6.3.1}$$

since $AI \le S \le BI$. Since T is a topological isomorphism,

$$(6.3.2) \qquad \frac{\|y\|}{\|T^{-1}\|} = \frac{\|y\|}{\|T^{*-1}\|} \le \|T^*y\| \le \|T^*\|\,\|y\| = \|T\|\,\|y\|.$$

Combining (6.3.1) and (6.3.2),

$$\frac{A\,\|y\|^2}{\|T^{-1}\|^2} \le \langle TST^*y, y\rangle \le B\,\|T\|^2\,\|y\|^2,$$

as desired. The statement about exactness follows immediately from the fact that topological isomorphisms preserve complete and incomplete sequences. ∎

The statement and a different proof of the following can be found in [**Y**].

PROPOSITION 6.3.3. *A sequence $\{x_n\}$ in a Hilbert space H is an exact frame if and only if it is a bounded unconditional basis.*

PROOF: ⇒. Assume $\{x_n\}$ is an exact frame. Then $\{x_n\}$ is bounded in norm by Proposition 6.2.16. By Proposition 6.2.8c, $x = \sum \langle x, S^{-1}x_n\rangle\, x_n$ for all x, and this series converges unconditionally. This representation is unique, for if $x = \sum c_n x_n$ then

$$\langle x, S^{-1}x_m\rangle = \left\langle \sum c_n x_n,\ S^{-1}x_m \right\rangle = \sum c_n \langle x_n, S^{-1}x_m\rangle = c_m,$$

since $\{x_n\}$ and $\{S^{-1}x_n\}$ are biorthonormal (Corollary 2.3.14). Thus $\{x_n\}$ is a bounded unconditional basis.

⇐. Assume $\{x_n\}$ is a bounded unconditional basis for H. Then by Proposition 6.1.8, $\{x_n\}$ is equivalent to an orthonormal basis for H, i.e., there exists an orthonormal basis $\{e_n\}$ and a topological isomorphism $U: H \to H$ such that $Ue_n = x_n$ for all n. Since $\{e_n\}$ is an exact frame, $\{x_n\}$ must also be an exact frame by Lemma 6.3.2. ∎

Remark 6.3.4. We can exhibit directly the topological isomorphism U used in the proof of Proposition 6.3.3. First note that $S^{-1/2}$ exists and is a positive topological isomorphism of H since both S and S^{-1} are positive topological isomorphisms, e.g., [We, Theorem 7.20]. Since $\{x_n\}$ is exact, $\{x_n\}$ and $\{S^{-1}x_n\}$ are biorthonormal. Therefore,

$$\langle S^{-1/2}x_m, S^{-1/2}x_n\rangle = \langle x_m, S^{-1/2}S^{-1/2}x_n\rangle = \langle x_m, S^{-1}x_n\rangle = \delta_{mn}.$$

Thus $\{S^{-1/2}x_n\}$ is orthonormal. It is complete since topological isomorphisms preserve complete sequences. Thus, $\{S^{-1/2}x_n\}$ is an orthonormal basis for H, and the topological isomorphism $U = S^{1/2}$ maps this orthonormal basis onto the frame $\{x_n\}$.

For inexact frames, $\{S^{-1/2}x_n\}$ will not be an orthonormal basis, but will be a tight frame.

Corollary 6.3.5. *Any frame in a Hilbert space is equivalent to a tight frame. Precisely, if $\{x_n\}$ is a frame with frame operator S then $S^{-1/2}$ is a positive topological isomorphism of H and $\{S^{-1/2}x_n\}$ is a tight frame with bounds $A = B = 1$.*

Proof: It follows from Lemma 6.3.2 that $\{S^{-1/2}x_n\}$ is a frame. Since

$$\sum \langle x, S^{-1/2}x_n\rangle\, S^{-1/2}x_n = S^{-1/2}SS^{-1/2}x = x,$$

the frame is tight by Proposition 6.2.6. ∎

Example 6.3.6. From Propositions 6.1.8 and 6.3.3, if $\{x_n\}$ is an exact frame then

(6.3.3)
$$\sum |c_n|^2 < \infty \quad \Leftrightarrow \quad \sum c_n x_n \text{ converges}$$
$$\Leftrightarrow \quad \sum c_n x_n \text{ converges unconditionally.}$$

Now let $\{e_n\}_{n \in \mathbf{Z}_+}$ be an orthonormal basis for H, and consider the frame $\{x_n\} = \{e_1, e_1, e_2, e_3, \ldots\}$. The series $\sum c_n x_n$ will converge if and only if $\sum |c_n|^2 < \infty$ since $\{x_n\}$ is obtained from an orthonormal basis by the addition of a single element. Since $\{x_n\}$ is norm bounded below, it follows from Proposition 6.2.19 that $\sum |c_n|^2 < \infty$ if and only if $\sum c_n x_n$ converges unconditionally. Therefore (6.3.3) holds for this nontight, inexact frame.

Section 6.4. Atoms in Hilbert spaces.

By definition, a sequence $\{x_n\}$ is a frame if there exists a norm equivalence between $\|x\|_H$ and $\|\{\langle x, x_n\rangle\}\|_{\ell^2}$ (Definition 2.3.1). Given such a frame, it follows that there exist coefficients $\{a_n(x)\}$ such that $x = \sum a_n(x)\, x_n$, in particular, $a_n(x) = \langle x, S^{-1}x_n\rangle$ (Proposition 6.2.8). Since $\{S^{-1}x_n\}$ is also a frame, there is also a norm equivalence between $\|x\|_H$ and $\|\{a_n(x)\}\|_{\ell^2}$. In this section, we examine a dual concept to frames, due to Gröchenig, which begins from the existence of coefficients $\{a_n(x)\}$ which reproduce x and satisfy a norm equivalence. We establish in this section the exact relationship between frames and Gröchenig's *sets of atoms*, in the Hilbert space setting.

DEFINITION 6.4.1 [G]. Given a sequence $\{x_n\}$ in a Hilbert space H, and given a sequence $\{a_n\}$ of linear functionals on H. If

a. $x = \sum a_n(x)\, x_n$ for every $x \in H$,

b. there exist constants $A, B > 0$ such that for each $x \in H$,

$$A\,\|x\|^2 \;\le\; \sum |a_n(x)|^2 \;\le\; B\,\|x\|^2,$$

then $\{x_n; a_n\}$ is a **set of atoms** for H. A, B are the **atomic bounds**, and the functionals $\{a_n\}$ are the **atomic coefficient functionals.**

REMARK 6.4.2. a. We do not assume that the representation $x = \sum a_n(x)\, x_n$ in Definition 6.4.1 is unique, i.e., $\{x_n\}$ need not be a basis for H.

b. Since $|a_m(x)|^2 \le \sum |a_n(x)|^2 \le B\,\|x\|^2$, each functional a_m is continuous, and is therefore given by the inner product with a unique $y_m \in H$, i.e.,

$a_m(\cdot) = \langle \cdot, y_m \rangle$. We identify the functional a_m with the element y_m, and refer to $\{y_n\}$ as the **atomic coefficients.**

c. If $\{x_n\}$ is a frame, then $\{x_n; S^{-1}x_n\}$ is a set of atoms by Proposition 6.2.8, where S is the frame operator for $\{x_n\}$. We establish a partial converse to this result in this section (Proposition 6.4.5).

From Definition 6.4.1 and Remark 6.4.2b we immediately obtain the following.

PROPOSITION 6.4.3. *If $\{x_n; y_n\}$ is a set of atoms with atomic bounds A, B then $\{y_n\}$ is a frame with frame bounds A, B.*

EXAMPLE 6.4.4. It need not be true that $\{x_n\}$ is a frame for H if $\{x_n; y_n\}$ is a set of atoms. For example, if $\{e_n\}_{n \in \mathbf{Z}_+}$ is an orthonormal basis for H then $\{e_n, ne_n\}$ is not a frame since it is not bounded in norm. However, it does form a set of atoms for H if we define the atomic coefficents to be $\{e_n, 0\}$ or $\{e_n/2, e_n/2n\}$.

PROPOSITION 6.4.5. *Given a set of atoms $\{x_n; y_n\}$, with atomic bounds A, B.*

a. *$\{x_n\}$ satisfies a lower frame bound of B^{-1}, i.e., $B^{-1} \|x\|^2 \le \sum |\langle x, x_n \rangle|^2$ for all $x \in H$.*

b. *If $\{x_n\}$ is a Bessel sequence with upper bound C then it is a frame, with frame bounds B^{-1}, C. Moreover, $\{y_n; x_n\}$ is in this case a set of atoms, with atomic bounds B^{-1}, C.*

PROOF: a. Assume $\{x_n; y_n\}$ is a set of atoms. Given x, $y \in H$ we have

$$
\begin{aligned}
|\langle x,y\rangle|^2 &= \left|\left\langle x,\ \sum \langle y,y_n\rangle\, x_n\right\rangle\right|^2 \\
&= \left|\sum \langle x,x_n\rangle\, \langle y,y_n\rangle\right|^2 \\
&\le \left(\sum |\langle x,x_n\rangle|^2\right)\left(\sum |\langle y,y_n\rangle|^2\right) \\
&\le B\,\|y\|^2 \sum |\langle x,x_n\rangle|^2.
\end{aligned}
$$

Therefore,

$$
\|x\|^2 = \sup_{\|y\|=1} |\langle x,y\rangle|^2 \le B \sum |\langle x,x_n\rangle|^2,
$$

so $\{x_n\}$ possesses a lower frame bound of B^{-1}.

b. Assume $\{x_n\}$ is also a Bessel sequence. Then, by definition, it possesses an upper frame bound. Since it possesses a lower frame bound by part a, $\{x_n\}$ is a frame.

It remains to show that $\{y_n; x_n\}$ is a set of atoms. The norm equivalence is satisfied since $\{x_n\}$ is a frame, so we need only show that $x = \sum\langle x,x_n\rangle\, y_n$ for all x. Now, both $\{x_n\}$ and $\{y_n\}$ are Bessel sequences (by assumption for $\{x_n\}$ and by Propostion 6.4.3 for $\{y_n\}$), so by Lemma 6.2.5 the mappings $U,\ V\colon H \to \ell^2$ defined by $Ux = \{\langle x,x_n\rangle\}$ and $Vx = \{\langle x,y_n\rangle\}$ are linear and continuous, with adjoints $U^*, V^*\colon \ell^2 \to H$ given by $U^*\{c_n\} = \sum c_n x_n$ and $V^*\{c_n\} = \sum c_n y_n$. Since $\{y_n; x_n\}$ is a set of atoms we have by definition that

$$
U^*Vx = \sum \langle x,y_n\rangle\, x_n = x,
$$

i.e., $U^*V = I$. Therefore, $V^*U = (U^*V)^* = I^* = I$, whence

$$
x = V^*Ux = \sum \langle x,x_n\rangle\, y_n. \quad ∎
$$

REMARK 6.4.6. a. In summary, by Remark 6.4.2c all frames are sets of atoms, while by Proposition 6.4.5b all atoms which are also Bessel sequences are frames. By Example 6.4.4, atoms which are not Bessel sequences need not be frames.

b. In practice, most sets of atoms are clearly Bessel sequences and therefore are frames.

c. Given a set of atoms $\{x_n; y_n\}$ such that $\{x_n\}$ is a Bessel sequence, we have by Proposition 6.4.5 that $\{x_n\}$ is a frame. We also have from Proposition 6.4.3 that $\{y_n\}$ is a frame. However, it need not be true that $\{y_n\}$ is the dual frame of $\{x_n\}$ or vice versa, as this would imply that atomic coefficients are unique.

We gave an example of nonunique coefficient functions in Example 6.4.4; however, that example did not satisfy the Bessel condition. An example in which the Bessel condition is satisfied is the following. Let $\{e_n\}_{n\in\mathbf{Z}_+}$ be an orthonormal basis for H. Then $\{e_n, e_n\}$ is a frame with bounds $A = B = 2$, and is therefore a Bessel sequence. The dual frame $\{e_n/2, e_n/2\}$ gives one immediate choice for atomic coefficients. However, we can also define atomic coefficients by $\{e_n, 0\}$, so they are not unique.

d. Nonuniqueness of the atomic coefficients means more than nonuniqueness of the individual representations $x = \sum\langle x, y_n\rangle\, x_n$. Nonuniqueness of the individual representations means only that given x there exist some other scalars $\{c_n\}$ such that $x = \sum c_n x_n$. Nonuniqueness of the atomic coefficients means that there exists another entire fixed set of vectors $\{z_n\}$ such that $x = \sum \langle x, z_n \rangle \, x_n$ for all x, moreover, with norm equivalence between $\| \cdot \|_H$ and $\| \{ \langle \cdot, z_n \rangle \} \|_{\ell^2}$.

e. There are a few remarks to be made about the history of Proposition 6.4.5. It was originally believed by Gröchenig that all atoms in Hilbert spaces were frames, and he communicated privately a proof of this result to Walnut. We realized that Gröchenig's proof implied that the atomic coefficients $\{y_n\}$ are the dual frame of $\{x_n\}$, and therefore are unique. These results were reported in [**Wa**], where they are used in a noncritical way for some minor results. Feichtinger later pointed out to us by example that atomic coefficients are not unique. We therefore re-examined Gröchenig's proof, and isolated the subtle error. Walnut then gave examples of atoms which were not frames, and suggested the independence of the assumption of the upper frame bound. Finally, we proved Proposition 6.4.5. A special case of Proposition 6.4.5 is proved in [**Wa**, Theorem 2.6.1].

The following proposition gives a condition under which the atomic coefficient functionals $\{y_n\}$ will be the dual frame of $\{x_n\}$.

PROPOSITION 6.4.7. *Given a set of atoms $\{x_n; y_n\}$ such that $\{x_n\}$ is a Bessel sequence. Define $U, V: H \to \ell^2$ by $Ux = \{\langle x, x_n \rangle\}$ and $Vx = \{\langle x, y_n \rangle\}$. If* $\text{Range}(U) = \text{Range}(V)$ *then $\{y_n\}$ is the dual frame of $\{x_n\}$.*

PROOF: As is the proof of Proposition 6.4.5 we have $U^*V = V^*U = I$, the identity map on H. Let $K = \text{Range}(U) = \text{Range}(V)$. Since $UV^*U = UI =$

U, we have $(UV^*)|_K = I|_K$. Since Range(U) = Range(V) = K this implies $UV^*V = I|_K V = V$. Now, $V^*Vx = \sum \langle x, y_n \rangle y_n = Sx$, where S is the frame operator for the frame $\{y_n\}$. Therefore, given $x \in H$,

$$\{\langle x, y_n \rangle\} = Vx = UV^*Vx = USx = \{\langle Sx, x_n \rangle\} = \{\langle x, Sx_n \rangle\}.$$

Since this is true for all x we have $y_n = Sx_n$, i.e., $x_n = S^{-1}y_n$. Thus $\{x_n\}$ is the dual frame of $\{y_n\}$. ∎

Section 6.5. Frames and atoms in Banach spaces.

In this section we extend the notions of frames and atoms to Banach spaces, following the ideas of Gröchenig, e.g., [G].

DEFINITION 6.5.1 [G]. Given a Banach space X and a related Banach space X_d of sequences of scalars. Let $\{x_n\}$ be a sequence of elements of X, and let $\{a_n\}$ be a sequence of linear functionals on X such that

a. $x = \sum a_n(x) x_n$ for all $x \in X$,

b. there exist $A, B > 0$ such that for all $x \in X$,

$$A \|x\|_X \le \|\{a_n(x)\}\|_{X_d} \le B \|x\|_X.$$

Then $\{x_n; a_n\}$ is a **set of (Banach) atoms** for (X, X_d). A, B are the **atomic bounds**, and $\{a_n\}$ are the **atomic coefficient functionals.**

REMARK 6.5.2. a. Often the sequence space X_d will be understood and therefore not specifically mentioned.

b. We assume for the remainder of this chapter that each a_n is continuous, i.e., $a_n \in X'$, and therefore write $a_n(x) = \langle x, a_n \rangle$.

This is true, for example, if X_d is solid and contains each of the sequences $\{\delta_{mn}\}_n$, for then

$$|a_m(x)| \, \|\{\delta_{mn}\}_n\|_{X_d} = \|\{a_m(x) \cdot \delta_{mn}\}_n\|_{X_d} \le \|\{a_n(x)\}\|_{X_d} \le B \|x\|_X,$$

so a_m is continuous on X.

c. If $\{x_n\}$ is a basis then $\{x_n\}$ and $\{a_n\}$ are biorthonormal, i.e., $a_m(x_n) = \delta_{mn}$.

DEFINITION 6.5.3. Given a Banach space X and a related Banach space X_d of sequences of scalars. A sequence $\{y_n\}$ of elements of X' is a **(Banach) frame** for X if there exist A, $B > 0$ such that

$$A \|x\|_X \leq \left\|\{\langle x, y_n \rangle\}\right\|_{X_d} \leq B \|x\|_X$$

for $x \in X$. A, B are the **frame bounds.** If only the upper bound holds then the sequence is a **(Banach) Bessel sequence.**

REMARK 6.5.4. a. Definitions 6.5.1 and 6.5.3 for frames and atoms in Banach spaces are consistent with Definitions 6.2.1 and 6.4.1 for frames and atoms in Hilbert spaces. For, the Hilbert space definitions are the special cases of the Banach space definitions obtained by taking $X = H$ and $X_d = \ell^2$ (except for a square-root factor in the bounds).

b. Walnut, in [**Wa**], discussed the existence of Banach atoms in $L^2_w(\mathbf{R}^d)$, where w is a moderate weight. Although $L^2_w(\mathbf{R}^d)$ is a Hilbert space, his Banach atoms are not Hilbert atoms since the sequence space is not ℓ^2 but rather an appropriate weighted ℓ^2_w.

Comparing Definitions 6.5.1 and 6.5.3, we obtain the following.

PROPOSITION 6.5.5. *If $\{x_n; y_n\}$ is a set of atoms for a Banach space X then $\{y_n\} \subset X'$ is a Banach frame for X.*

REMARK 6.5.6. Given a set of atoms $\{x_n; y_n\}$ for a Banach space X. As usual, we identify X with its canonical embedding in X'', i.e., $X \subset X''$. Therefore, it possible for $\{x_n\}$ to be a Banach frame for X. The following results give conditions under which this will be true. The situation is similar to the Hilbert space case, in particular, the same considerations about upper and lower frame bounds apply.

We assume for the remainder of this chapter that X_d is such that X'_d is also a sequence space, with duality between X_d and X'_d given by $\langle \{b_n\}, \{c_n\} \rangle = \sum b_n \bar{c}_n$. This is true, for example, if X_d is a weighted ℓ^p-space.

PROPOSITION 6.5.7. *Given a set of atoms $\{x_n; y_n\}$ for (X, X_d), with atomic bounds A, B.*

a. *$\{x_n\}$ satisfies a lower frame bound for X of B^{-1}.*

b. *If $\{x_n\}$ is a Bessel sequence for X with upper bound C then it is a Banach frame for X', with frame bounds B^{-1}, C.*

PROOF: a. Assume $\{x_n; y_n\}$ is a set of atoms. Given $x \in X$ and $y \in X'$ we compute

$$\begin{aligned}
|\langle x, y\rangle| &= \left|\left\langle \sum \langle x, y_n\rangle\, x_n,\ y \right\rangle\right| \\
&= \left|\sum \langle x, y_n\rangle \langle x_n, y\rangle\right| \\
&= |\langle \{\langle x, y_n\rangle\},\ \{\langle x_n, y\rangle\}\rangle| \\
&\le \|\{\langle x, y_n\rangle\}\|_{X_d}\ \|\{\langle x_n, y\rangle\}\|_{X'_d} \\
&\le B\,\|x\|_X\ \|\{\langle x_n, y\rangle\}\|_{X'_d}.
\end{aligned}$$

Therefore,

$$\|y\|_{X'} = \sup_{\|x\|=1} |\langle x, y\rangle| \le B\,\|\{\langle x_n, y\rangle\}\|_{X'_d}.$$

This establishes the lower frame bound for $\{x_n\}$.

b. Follows immediately from a. ∎

Section 6.6. Stability of atoms.

There are many results on the stability of bases in Banach spaces, e.g., [**DE**; **GK**; **PW**; **Po**]. Typically, these give conditions on the amounts the elements of a basis may be perturbed without affecting the basis property. We formulate in this section an analogous stability theorem for atoms.

We continue to assume that X'_d is a sequence space of scalars, with duality between X_d and X'_d given by $\langle\{b_n\},\{c_n\}\rangle = \sum b_n\bar{c}_n$.

PROPOSITION 6.6.1. *Given a set of atoms $\{x_n; y_n\}$ for (X, X_d), with atomic bounds A, B. Assume $w_n \in X$ satisfy*

$$R = \big\|\{\|x_n - w_n\|\}\big\|_{X'_d} < B^{-1}.$$

Then there exist $z_n \in X'$ such that $\{w_n; z_n\}$ is a set of atoms for (X, X_d) with atomic bounds $A/(1+RB)$, $B/(1-RB)$. Moreover, $\{w_n\}$ is a basis if and only if $\{x_n\}$ is a basis.

PROOF: Given $x \in X$,

$$\begin{aligned} \text{(6.6.1)} \qquad \sum |\langle x, y_n\rangle|\, \|x_n - w_n\|_X &\le \big\|\{\langle x, y_n\rangle\}\big\|_{X_d}\, \big\|\{\|x_n - w_n\|_X\}\big\|_{X'_d} \\ &\le RB\,\|x\|. \end{aligned}$$

Thus $\sum \langle x, y_n\rangle\,(x_n - w_n)$ converges absolutely in X. Since $x = \sum \langle x, y_n\rangle\, x_n$ also converges in X, the series $Tx = \sum\langle x, y_n\rangle\, w_n$ must therefore converge.

Clearly T is linear, and, by (6.6.1), $\|I - T\| \leq RB < 1$. Therefore T is invertible, whence

$$x = TT^{-1}x = \sum \langle T^{-1}x, y_n \rangle w_n$$

for $x \in X$. By definition of atoms,

$$(6.6.2) \qquad \frac{A}{\|T\|} \|x\|_X \leq A \|T^{-1}x\|_X$$

$$\leq \left\| \{\langle T^{-1}x, y_n \rangle\} \right\|_{X_d}$$

$$\leq B \|T^{-1}x\|_X$$

$$\leq B \|T^{-1}\| \, \|x\|_X.$$

Define the functional $z_n \in X'$ by $z_n = T^{-1}y_n$, i.e., $\langle x, z_n \rangle = \langle T^{-1}x, y_n \rangle$ for $x \in X$. By (6.6.2),

$$A \|T\|^{-1} \|x\|_X \leq \left\| \{\langle x, z_n \rangle\} \right\|_{X_d} \leq B \|T^{-1}\| \, \|x\|_X,$$

so $\{w_n; z_n\}$ is a set of atoms for (X, X_d). Since

$$\|T\| \leq \|I\| + \|T - I\| \leq 1 + RB$$

and

$$\|T^{-1}\| \leq \frac{1}{1 - \|T - I\|} \leq \frac{1}{1 - RB},$$

the bounds are as claimed.

Finally, assume $\{x_n\}$ is a basis for X. Then $\{x_n\}$ and $\{y_n\}$ are biorthonormal, so

$$Tx_m = \sum \langle T^{-1}Tx_m, y_n \rangle w_n = \sum \langle x_m, y_n \rangle w_n = w_m.$$

Since topological isomorphisms preserve bases, $\{w_n\}$ must therefore be a basis. Conversely, if $\{w_n\}$ is a basis then T^{-1} is a topological isomorphism which maps $\{w_n\}$ onto $\{x_n\}$, so $\{x_n\}$ must be a basis. ∎

CHAPTER 7

GABOR SYSTEMS AND THE ZAK TRANSFORM

This chapter is an essentially expository survey of results obtained by using the Zak transform to analyze Gabor systems. We have combined results from [**D1**; **DGM**; **J2**] and others with remarks, examples, and results of our own.

In Section 7.1 we define Gabor systems, and give necessary and sufficient conditions under which a Gabor system will be a frame, if the mother wavelet has compact support.

In Section 7.2 we define the Zak transform and prove that it is a unitary map of $L^2(\mathbf{R}^d)$ onto $L^2(Q)$, where Q is any unit cube in $\mathbf{R}^d \times \hat{\mathbf{R}}^d$.

In Section 7.3 we analyze Gabor systems at the critical value $ab = 1$ through the use of the Zak transform. We characterize those systems which are frames by a condition on the Zak transform of the mother wavelet.

In Section 7.4 we prove that the Zak transform maps $L^p(\mathbf{R}^d)$ into $L^p(Q)$ for $1 \le p \le 2$ but cannot be defined on $L^p(\mathbf{R}^d)$ if $p > 2$.

In Section 7.5 we prove that the Zak transform maps the Wiener amalgam space $W(L^p, L^1)$ into $L^p(Q)$ for each $1 \le p \le \infty$. As a corollary, we obtain a variant of the Balian–Low theorem: if (g, a, b) generates a Gabor frame at the critical value $ab = 1$ then g is either not smooth or does not decay quickly at infinity.

Finally, in Section 7.6 we address some questions similar to ones which arise in Section 7.3 from the application of the Zak transform to Gabor frames. In particular, we generalize slightly a result of Boas and Pollard which shows that if finitely many elements are removed from an orthonormal basis for $L^2(X)$ then it is always possible to find a single function to multiply the remaining elements by so that the resulting sequence is complete. We show this need not be true if infinitely many elements are deleted, and discuss some related results by other authors.

Section 7.1. Gabor systems.

In this section we define Gabor systems in $L^2(\mathbf{R}^d)$, and prove an existence theorem for Gabor frames generated by compactly supported mother wavelets.

DEFINITION 7.1.1. Given $g \in L^2(\mathbf{R}^d)$ and given a, $b \in \mathbf{R}^d_+$, the **Gabor system** generated by (g, a, b) is $\{T_{na}E_{mb}g\}_{m,n\in\mathbf{Z}^d}$. The function g is the **mother wavelet** and the vectors a, b are the **system parameters.** The set of points $\{(na, mb)\}_{m,n\in\mathbf{Z}^d}$ is the **system lattice.** When a, b are understood we use the abbreviation $g_{mn} = T_{na}E_{mb}g$.

REMARK 7.1.2. a. Since $T_{na}E_{mb} = e^{-2\pi i na mb}E_{mb}T_{na}$, we also refer to $\{E_{mb}T_{na}g\}_{m,n\in\mathbf{Z}^d}$ as a Gabor system.

b. Since $(T_{na}E_{mb}g)^\wedge = E_{-na}T_{mb}\hat{g}$, the Gabor system generated by $(\hat{g}, b, a)$ consists of the Fourier transforms of the elements of the Gabor system generated by (g, a, b). Since the Fourier transform is a unitary mapping of $L^2(\mathbf{R}^d)$ onto $L^2(\hat{\mathbf{R}}^d)$ there is a duality between properties held by the system generated by (g, a, b) and the system generated by $(\hat{g}, b, a)$.

c. Assume that the Gabor system generated by (g, a, b) is a frame. The frame operator is then, by definition, $Sf = \sum\langle f, g_{mn}\rangle\, g_{mn}$, and the dual frame is $\{S^{-1}g_{mn}\}$. A straightforward calculation shows that $ST_{na}E_{mb} = T_{na}E_{mb}S$, whence $S^{-1}g_{mn} = (S^{-1}g)_{mn}$. Therefore the dual frame is also a Gabor frame, generated by $(S^{-1}g, a, b)$.

The following is an elaboration of a basic result from [**DGM**].

Proposition 7.1.3. *Given $g \in L^2(\mathbf{R}^d)$ with compact support and given $a \in \mathbf{R}^d_+$. Let $I \supset \mathrm{supp}(g)$ be any compact rectangle, and let $1/b$ be the side lengths of I. Define*

$$\lambda(t) = \sum_{n \in \mathbf{Z}^d} |g(t - na)|^2, \quad A = \operatorname*{ess\,inf}_{t \in \mathbf{R}^d} \lambda(t), \quad \text{and} \quad B = \operatorname*{ess\,sup}_{t \in \mathbf{R}^d} \lambda(t).$$

Then (g, a, b) generates a Gabor frame if and only if $A > 0$ and $B < \infty$. In this case, the following also hold.

a. *The bounds for the frame are $\Pi(1/b)\, A$, $\Pi(1/b)\, B$.*

b. *The frame operator is $Sf = \Pi(1/b)\, \lambda f$.*

c. $0 < ab \leq 1$.

d. *The frame is exact if and only if $ab = 1$.*

e. *If $ab \neq 1$ then the frame has infinite excess, i.e., there exist finite sets $F \subset \mathbf{Z}^d \times \mathbf{Z}^d$ of arbitrarily large cardinality such that $\{g_{mn}\}_{(m,n) \notin F}$ is a frame.*

Proof: We use a Fourier series argument. Set $I_n = I + na$ for $n \in \mathbf{Z}^d$. Then $\{\Pi(b)^{1/2} E_{mb} \chi_{I_n}\}_{m \in \mathbf{Z}^d}$ is an orthonormal basis for $L^2(I_n) = \{f \in L^2(\mathbf{R}^d) : \mathrm{supp}(f) \subset I_n\}$. Given $f \in L^2(\mathbf{R}^d)$ we have $f \cdot \overline{T_{na} g} \in L^1(I_n)$, so

$$\begin{aligned}
\sum_m |\langle f, g_{mn} \rangle|^2 &= \sum_m |\langle f \cdot \overline{T_{na} g}, E_{mb} \chi_{I_n} \rangle|^2 \\
&= \Pi(1/b)\, \|f \cdot \overline{T_{na} g}\|_2^2 \\
&= \Pi(1/b) \int_{\mathbf{R}^d} |f(t)\, g(t - na)|^2 \, dt,
\end{aligned}$$

where either both sides are finite and equal or both sides are infinite. Thus

$$\sum_{m,n} |\langle f, g_{mn}\rangle|^2 \;=\; \Pi(1/b) \int_{\mathbf{R}^d} |f(t)|^2\, \lambda(t)\, dt, \tag{7.1.1}$$

so (g, a, b) forms a frame if and only if λ is essentially constant.

Assume now that (g, a, b) generates a frame.

a. Follows from (7.1.1).

b. For each $f \in L^2(\mathbf{R}^d)$ we have

$$\langle Sf, f\rangle \;=\; \sum_{m,n} |\langle f, g_{mn}\rangle|^2 \;=\; \Pi(1/b) \int_{\mathbf{R}^d} |f(t)|^2\, \lambda(t)\, dt \;=\; \Pi(1/b)\, \langle \lambda f, f\rangle.$$

It follows immediately from elementary Hilbert space results that $Sf = \Pi(1/b)\, \lambda f$.

c. If $a_j b_j > 1$ for some j then $\{I_n\}$, and therefore $\{\text{supp}(g_{mn})\}$, does not cover $\mathbf{R}^d$. Hence $\{g_{mn}\}$ is incomplete and therefore not a frame.

d. Assume $ab = 1$. In this case the sets $\{I_n\}$ are disjoint. Therefore $\lambda(t) = |g(t - na)|^2$ if $t \in I_n$, whence $|T_{na} g|$ is bounded above and below on I_n. As $\{\Pi(b)^{1/2} E_{mb} \chi_{I_n}\}_m$ forms an orthonormal basis for $L^2(I_n)$, it follows that $\{E_{mb} T_{na} g\}_m$ is a bounded unconditional basis for $L^2(I_n)$. Since $\{I_n\}$ is a partition of $\mathbf{R}^d$, it follows that $\{E_{mb} T_{na} g\}_{m,n}$ is a bounded unconditional basis, and hence an exact frame, for $L^2(\mathbf{R}^d)$.

Conversely, assume $ab \neq 1$. From part c, every coordinate of ab is at most 1. There are two possibilities: either $\text{supp}(g) \neq I$ or $\text{supp}(g) \cap \text{supp}(T_{ka} g) \neq \emptyset$ for some $k \in \mathbf{Z}^d$. We claim that in either case it is possible to remove one element from the frame $\{g_{mn}\}$ and retain a complete set, from which it follows that the frame is inexact. In particular, we remove the element $g = g_{00}$.

To show that $\{g_{mn}\}_{(m,n)\neq(0,0)}$ is complete, assume $f \in L^2(\mathbf{R}^d)$ satisfies $\langle f, g_{mn}\rangle = 0$ for $(m,n) \neq (0,0)$. Note that $\text{supp}(f\cdot\bar{g}) \subset I$ and that $\langle f\cdot\bar{g}, E_{mb}\rangle = \langle f, g_{m0}\rangle = 0$ for $m \neq 0$. As $\{\Pi(b)^{1/2} E_{mb}\chi_I\}_m$ is an orthonormal basis for $L^2(I)$, it follows that $f\cdot\bar{g} = cE_0 = c$ for some constant c.

If $\text{supp}(g) \neq I$ then $c = 0$ since $f\cdot\bar{g} = 0$ on $I\backslash\,\text{supp}(g)$.

On the other hand, assume $\text{supp}(g) \cap \text{supp}(T_{ka}g) \neq \emptyset$ for some k. Then $\langle f\cdot\overline{T_{ka}g}, E_{mb}\rangle = \langle f, g_{mk}\rangle = 0$ for all m, whence $f\cdot\overline{T_{ka}g} = 0$ on I_k. Therefore $f = 0$ on $\text{supp}(T_{ka}g) \supset \text{supp}(g) \cap \text{supp}(T_{ka}g)$, so again $c = 0$.

Thus, in any case, $f\cdot\bar{g} = 0$ on I, whence $\langle f, g\rangle = 0$ since $\text{supp}(g) \subset I$. Thus f is orthogonal to every element of $\{g_{mn}\}$. As this set is complete, it follows that $f = 0$, and therefore $\{g_{mn}\}_{(m,n)\neq(0,0)}$ is complete.

Alternatively, recall from Corollary 6.2.14 that the frame $\{g_{mn}\}$ is exact if and only if $\langle g_{mn}, S^{-1}g_{mn}\rangle = 1$ for all m, n. By part b,

$$\begin{aligned}(7.1.2)\qquad \langle g_{mn}, S^{-1}g_{mn}\rangle &= \Pi(b)\,\langle g_{mn}, g_{mn}/\lambda\rangle \\ &= \Pi(b)\int_{\mathbf{R}^d} \frac{|g(t-na)|^2}{\sum_k |g(t-ka)|^2}\,dt.\end{aligned}$$

Now,

$$(7.1.3)\qquad \frac{|g(t-na)|^2}{\sum_k |g(t-ka)|^2} \leq \chi_{I_n}(t).$$

If $ab = 1$ then there is equality in (7.1.3), and therefore $\langle g_{mn}, S^{-1}g_{mn}\rangle = 1$ by (7.1.2). This is true for all m, n, so the frame is exact.

Assume, on the other hand, that $ab \neq 1$, and set $m = n = 0$. If $\text{supp}(g) \neq I$ then $g(t) = 0$ for $t \in E = I \setminus \text{supp}(g)$. If $\text{supp}(g) \cap \text{supp}(T_{ka}g) \neq \emptyset$ for some $k \in \mathbf{Z}^d$ then $|g(t)|^2 < |g(t)|^2 + |g(t - ka)|^2$ for $t \in E = \text{supp}(g) \cap \text{supp}(T_{ka}g)$. In either case there is strict inequality in (7.1.3) for $t \in E$. As $|E| > 0$, it follows that $\langle g, S^{-1}g \rangle < 1$, whence $\{g_{mn}\}$ is inexact.

e. Assume $ab \neq 1$. From part d, $\{g_{mn}\}_{(m,n)\neq(0,0)}$ is complete, and therefore is a frame (Proposition 6.2.12). The argument in part d used only the function g and those g_{mn} whose support intersected that of g. Therefore the argument can be repeated using some g_{kl} whose support is far distant from that of g and its immediate neighbors, i.e., we can remove some a second function from the frame and still have a complete set and therefore a frame. This process can be repeated arbitrarily many times, so the frame has infinite excess. ∎

EXAMPLE 7.1.4. Functions in $C_c(\mathbf{R}^d)$ satisfy the hypothesis of Proposition 7.1.3 for all a and b whose components are small enough. Therefore, any function in $C_c(\mathbf{R}^d)$ will generate a Gabor frame for some choice of a and b.

It is possible to prove sufficient conditions under which functions without compact support will generate Gabor frames, e.g., [**D1**; **HW**; **Wa**].

Section 7.2. The Zak transform.

In this section we define the Zak transform and prove that it is a unitary mapping of $L^2(\mathbf{R}^d)$ onto $L^2(Q)$, where Q is any unit cube in $\mathbf{R}^d \times \hat{\mathbf{R}}^d$.

DEFINITION 7.2.1. The **Zak transform** of a function $f: \mathbf{R}^d \to \mathbf{C}$ is (formally)

$$Zf(t,\omega) = \sum_{k \in \mathbf{Z}^d} f(t+k)\, E_k(\omega)$$

for $(t,\omega) \in \mathbf{R}^d \times \hat{\mathbf{R}}^d$.

The series defining Zf may converge in various senses, e.g., pointwise, L^p_{loc}, etc.

Formally, Zf is quasiperiodic, in the following sense.

DEFINITION 7.2.2. A function $F: \mathbf{R}^d \times \hat{\mathbf{R}}^d \to \mathbf{C}$ is **quasiperiodic** if

$$F(t+j, \omega+k) = E_{-j}(\omega)\, F(t,\omega) = e^{-2\pi i j \cdot \omega}\, F(t,\omega)$$

for j, $k \in \mathbf{Z}^d$ and $(t,\omega) \in \mathbf{R}^d \times \hat{\mathbf{R}}^d$.

REMARK 7.2.3. a. A quasiperiodic function is completely determined by its values on any unit cube Q in $\mathbf{R}^d \times \hat{\mathbf{R}}^d$.

b. If F, G are quasiperiodic then $F\bar{G}$ is 1-periodic.

c. If F is quasiperiodic then the norm

$$\|F\|_p = \|F\|_{p,Q} = \left(\iint_Q |F(t,\omega)|^p \, d\omega\, dt \right)^{1/p}$$

is independent of the unit cube Q. Hence

$$\{F: Q \to C : \|F\|_{2,Q} < \infty\}$$

can be identified with

$$\{F : \mathbf{R}^d \times \hat{\mathbf{R}}^d \to C : F \text{ is quasiperiodic and } \|F\|_{2,Q} < \infty\}$$

via quasiperiodic extension. We refer to both of these spaces as $L^p(Q)$.

Without loss of generality, we let $Q = [0,1] \times [0,1] \subset \mathbf{R}^d \times \hat{\mathbf{R}}^d$ for the remainder of this chapter.

d. $\{E_{(m,n)}\}_{m,n \in \mathbf{Z}^d}$ is an orthonormal basis for the Hilbert space $L^2(Q)$, where

$$E_{(m,n)}(t,\omega) = e^{2\pi i (m,n)\cdot(t,\omega)} = e^{2\pi i m\cdot t} e^{2\pi i n\cdot\omega}.$$

e. Quasiperiodicity is not a translation invariant property. For example, set $d = 1$ and assume F is quasiperiodic. If b is not an integer then

$$\begin{aligned}
(T_{(0,b)}F)(t+1,\omega) &= F(t+1,\omega-b) \\
&= e^{-2\pi i(\omega-b)} F(t,\omega-b) \\
&= e^{2\pi i b} e^{-2\pi i\omega} (T_{(0,b)}F)(t,\omega) \\
&\neq e^{-2\pi i\omega} (T_{(0,b)}F)(t,\omega).
\end{aligned}$$

Thus $T_{(0,b)}F$ is not quasiperiodic.

The following proposition and its proof is a generalization to higher dimensions of a result appearing in [J2].

PROPOSITION 7.2.4. *The Zak transform is a unitary map of* $L^2(\mathbf{R}^d)$ *onto* $L^2(Q)$.

PROOF: Fix $f \in L^2(\mathbf{R}^d)$. For $k \in \mathbf{Z}^d$ define $F_k(t,\omega) = f(t+k)\, E_k(\omega)$. Since

$$\|F_k\|_2^2 \;=\; \iint_Q |f(t+k)\, E_k(\omega)|^2 \, d\omega\, dt \;=\; \int_{[0,1]} |f(t+k)|^2 \, dt \;<\; \infty,$$

we have $F_k \in L^2(Q)$ for each k. The sequence $\{F_k\}$ is orthogonal, for if $k \neq l$ then

$$\langle F_k, F_l \rangle \;=\; \int_{[0,1]} f(t+k)\, \overline{f(t+l)} \left(\int_{[0,1]} E_{k-l}(\omega)\, d\omega \right) dt \;=\; 0.$$

Given a finite subset $F \subset \mathbf{Z}^d$ we therefore have

$$\Big\| \sum_{k \in F} F_k \Big\|_2^2 \;=\; \sum_{k \in F} \|F_k\|_2^2 \;=\; \sum_{k \in F} \int_{[0,1]} |f(t+k)|^2 \, dt.$$

Since

$$\sum_{k \in \mathbf{Z}^d} \int_{[0,1]} |f(t+k)|^2 \, dt \;=\; \int_{\mathbf{R}^d} |f(t)|^2 \, dt \;=\; \|f\|_2^2 \;<\; \infty,$$

it follows that $Zf = \sum F_k$ converges in $L^2(Q)$ and $\|Zf\|_2 = \|f\|_2$, so Z is continuous and norm-preserving.

Now define $g = \chi_{[0,1]}$ and set $a = b = 1$. The Gabor system $\{g_{mn}\}$ is then an orthonormal basis for $L^2(\mathbf{R}^d)$. We easily compute $Zg_{mn} = E_{(m,n)}$. Thus Z maps the orthonormal basis $\{g_{mn}\}$ for $L^2(\mathbf{R}^d)$ onto the orthonormal basis $\{E_{(m,n)}\}$ for $L^2(Q)$. As Z is continuous, it follows immediately that Z is unitary. ∎

PROPOSITION 7.2.5. *Given $f \in L^2(\mathbf{R}^d)$,*

$$Zf(t,\omega) \;=\; e^{-2\pi i t \cdot \omega}\, Z\hat{f}(\omega, -t).$$

PROOF: a. Fix φ in the Schwartz space $\mathcal{S}(\mathbf{R}^d)$; then we can apply the Poisson summation formula in the calculation below, cf., Section 1.8b.

$$\begin{aligned} Z\varphi(t,\omega) &= \sum \varphi(t+k)\, E_k(\omega) \\ &= \sum T_{-t}\varphi(k)\, E_\omega(k) \\ &= \sum (E_\omega T_{-t}\varphi)(k) \\ &= \sum (E_\omega T_{-t}\varphi)^\wedge(k) \\ &= \sum T_\omega E_t \hat{\varphi}(k) \\ &= \sum E_t \hat{\varphi}(k-\omega) \\ &= E_t(-\omega) \sum \hat{\varphi}(k-\omega)\, E_t(k) \\ &= e^{-2\pi i t\cdot\omega} Z\hat{\varphi}(-\omega, t). \end{aligned}$$

b. Now fix $f \in L^2(\mathbf{R}^d)$. Then there exist $\varphi_n \in \mathcal{S}(\mathbf{R}^d)$ such that $\varphi_n \to f$ in $L^2(\mathbf{R}^d)$. By the Plancherel formula, $\hat{\varphi}_n \to \hat{f}$ in $L^2(\hat{\mathbf{R}}^d)$. By Proposition 7.2.4, it follows that $Z\varphi_n \to Zf$ and $Z\hat{\varphi}_n \to Z\hat{f}$ in $L^2(Q)$. By passing to subsequences if necessary we may assume that all four of these convergences hold pointwise a.e. Therefore, pointwise a.e.,

$$\begin{aligned} Zf(t,\omega) &= \lim_{n\to\infty} Z\varphi_n(t,\omega) \\ &= \lim_{n\to\infty} e^{-2\pi i t\cdot\omega} Z\hat{\varphi}_n(-\omega, t) \\ &= e^{-2\pi i t\cdot\omega} Z\hat{f}(-\omega, t). \ \blacksquare \end{aligned}$$

EXAMPLE 7.2.6 [DGM]. Set $d = 1$. We compute the Zak transform of the Gaussian $g(t) = e^{-rt^2}$, where $r > 0$.

We make use of the Jacobi theta function θ_3. There are four Jacobi theta functions, defined by

$$\begin{aligned}
\theta_1(z,q) &= 2\sum_{k=0}^{\infty} (-1)^k\, q^{(k+1/2)^2} \sin 2\pi(2k+1)z \\
&= -i \sum_{k=-\infty}^{\infty} (-1)^k\, q^{(k+1/2)^2}\, e^{2\pi i(2k+1)z},
\end{aligned}$$

$$\begin{aligned}
\theta_2(z,q) &= 2\sum_{k=0}^{\infty} q^{(k+1/2)^2} \cos 2\pi(2k+1)z \\
&= \sum_{k=-\infty}^{\infty} (-1)^k\, q^{(k+1/2)^2}\, e^{2\pi i(2k+1)z},
\end{aligned}$$

$$\begin{aligned}
\theta_3(z,q) &= 1 + 2\sum_{k=1}^{\infty} q^{k^2} \cos 4\pi kz \\
&= \sum_{k=-\infty}^{\infty} q^{k^2}\, e^{4\pi ikz},
\end{aligned}$$

$$\begin{aligned}
\theta_4(z,q) &= 1 + 2\sum_{k=1}^{\infty} (-1)^k\, q^{k^2} \cos 4\pi kz \\
&= \sum_{k=-\infty}^{\infty} (-1)^k\, q^{k^2}\, e^{4\pi ikz},
\end{aligned}$$

for $0 \le q < 1$ and $z \in \mathbf{C}$, cf., [**Ra**]. We compute

$$\begin{aligned}
Zg(t,\omega) &= \sum g(t+k)\, e^{2\pi ik\omega} \\
&= \sum e^{-rt^2}\, e^{-2rtk}\, e^{-rk^2}\, e^{2\pi ik\omega} \\
&= e^{-rt^2} \sum (e^{-r})^{k^2}\, e^{4\pi ik(\frac{\omega}{2}+\frac{irt}{2\pi})} \\
&= e^{-rt^2}\, \theta_3(\tfrac{\omega}{2} + \tfrac{irt}{2\pi}, e^{-r}).
\end{aligned}$$

Zg is therefore continuous. The zeroes of $\theta_3(\cdot, q)$ occur precisely at the points

$$z_{mn} = \frac{1}{4} + \frac{\tau}{4} + \frac{m}{2} + \frac{n\tau}{2},$$

where $q = e^{\pi i \tau}$, $\mathrm{Im}(\tau) > 0$. Since $e^{-r} = e^{\pi i (ir)}$, it follows that $Zg(t, \omega) = 0$ if and only if

$$\frac{\omega}{2} + \frac{irt}{2\pi} = \frac{1}{4} + \frac{ir}{4\pi} + \frac{m}{2} + \frac{irn}{2\pi},$$

i.e., $(t, \omega) = (n + 1/2, m + 1/2)$. Thus Zg has a single zero in any unit cube in $\mathbf{R}^d \times \hat{\mathbf{R}}^d$.

Section 7.3. Gabor systems and the Zak transform.

In this section we use the Zak transform to analyze Gabor systems satisfying $ab = 1$.

REMARK 7.3.1. Let D_a denote the dilation operator which is isometric on $L^2(\mathbf{R}^d)$, i.e., $D_a f(t) = |\Pi(a)|^{-1/2} f(t/a)$, and assume $ab = 1$. Then

$$D_{1/a}(T_{na}E_{mb}g) = T_n E_{mab}(D_{1/a}g) = T_n E_m(D_{1/a}g).$$

Since dilation is a unitary operator on $L^2(\mathbf{R}^d)$ it therefore suffices to consider Gabor systems satisfying $a = b = 1$. We assume these values for the remainder of this chapter, i.e., $g_{mn} = T_n E_m g$.

LEMMA 7.3.2. *Given a function g on $\mathbf{R}^d$ and $a = b = 1$,*

$$Zg_{mn} = E_{(m,n)}\, Zg.$$

PROOF: We compute

$$\begin{aligned} Zg_{mn}(t,\omega) &= \sum g_{mn}(t+k)\, E_k(\omega) \\ &= \sum T_n E_m g(t+k)\, E_k(\omega) \\ &= \sum E_m(t+k-n)\, g(t+k-n)\, E_k(\omega) \\ &= E_m(t) \sum g(t+k)\, E_{k+n}(\omega) \\ &= E_m(t)\, E_n(\omega)\, Zg(t,\omega) \\ &= E_{(m,n)}(t,\omega)\, Zg(t,\omega). \ \blacksquare \end{aligned}$$

REMARK 7.3.3. Since the Zak transform is a unitary map of $L^2(\mathbf{R}^d)$ onto $L^2(Q)$, the Gabor system $\{g_{mn}\}$ will form a frame for $L^2(\mathbf{R}^d)$ if and only if $\{Zg_{mn}\}$ forms a frame for $L^2(Q)$. By Lemma 7.3.2, $Zg_{mn} = E_{(m,n)}\, Zg$. Since $\{E_{(m,n)}\}$ forms an orthonormal basis for $L^2(Q)$, the requirement that $\{E_{(m,n)}\, Zg\}$ be a frame therefore places severe restrictions on Zg, which we examine in the following proposition.

The study of Gabor systems satisfying $ab = 1$ is thus reduced via the Zak transform to the study of the effect of multiplying the elements of a particular orthonormal basis, $\{E_{(m,n)}\}$, by a single fixed function, Zg. There are many related questions which have appeared in the literature; we discuss some of these in Section 7.6.

Parts a and d and the frame statement of part c of the following proposition have appeared in print several times, e.g., [DGM].

PROPOSITION 7.3.4. *Given $g \in L^2(\mathbf{R}^d)$ and $a = b = 1$.*

a. *$\{g_{mn}\}$ is complete in $L^2(\mathbf{R}^d)$ if and only if $Zg \neq 0$ a.e.*

b. *$\{g_{mn}\}$ is minimal and complete in $L^2(\mathbf{R}^d)$ if and only if $1/Zg \in L^2(Q)$.*

c. *$\{g_{mn}\}$ is a frame for $L^2(\mathbf{R}^d)$ (with frame bounds A, B) if and only if*

$$0 < A \leq |Zg|^2 \leq B < \infty \text{ a.e.}$$

In this case the frame is exact.

d. *$\{g_{mn}\}$ is an orthonormal basis for $L^2(\mathbf{R}^d)$ if and only if $|Zg| = 1$ a.e.*

PROOF: a. Assume that $\{g_{mn}\}$ is complete in $L^2(\mathbf{R}^d)$; then $\{Zg_{mn}\}$ is complete in $L^2(Q)$ by the unitarity of Z. Define F on Q by

$$F(t,\omega) = \begin{cases} 1, & Zg(t,\omega) = 0, \\ 0, & Zg(t,\omega) \neq 0. \end{cases}$$

Then, by Lemma 7.3.2, for $m, n \in \mathbf{Z}^d$,

$$\langle F, Zg_{mn} \rangle = \langle F, E_{(m,n)} Zg \rangle = \langle F \cdot \overline{Zg}, E_{(m,n)} \rangle = 0.$$

Therefore $F = 0$ a.e. since $\{Zg_{mn}\}$ is complete, whence $Zg \neq 0$ a.e.

Conversely, assume $Zg \neq 0$ a.e. Assume that $F \in L^2(Q)$ is such that $\langle F, Zg_{mn} \rangle = 0$ for all m, n. Then $\langle F \cdot \overline{Zg}, E_{(m,n)} \rangle = \langle F, Zg_{mn} \rangle = 0$ for all m, n, so $F \cdot \overline{Zg} = 0$ a.e. since $F \cdot \overline{Zg} \in L^1(Q)$ and $\{E_{(m,n)}\}$ is complete in $L^1(Q)$. As $Zg \neq 0$ a.e., this implies $F = 0$ a.e., so $\{Zg_{mn}\}$ is complete.

b. Assume $\{g_{mn}\}$ is minimal and complete in $L^2(\mathbf{R}^d)$; then the same is true of $\{Zg_{mn}\}$ in $L^2(Q)$. By part a, $Zg \neq 0$ a.e. By Proposition 6.1.3, the minimality of $\{Zg_{mn}\}$ implies that there exist functions $F_{mn} \in L^2(Q)$ which are biorthonormal to $\{Zg_{mn}\}$, i.e., $\langle F_{mn}, Zg_{m'n'} \rangle = \delta_{mm'} \delta_{nn'}$. Thus

$$\langle F_{mn} \cdot \overline{Zg}, E_{(m',n')} \rangle = \langle F_{mn}, Zg_{m'n'} \rangle = \delta_{mm'} \delta_{nn'} = \langle E_{(m,n)}, E_{(m',n')} \rangle.$$

Since $F_{mn} \cdot \overline{Zg} \in L^1(Q)$ and $\{E_{(m',n')}\}$ is complete in $L^1(Q)$, $F_{mn} \cdot \overline{Zg} = E_{(m,n)}$ a.e. for all m, n. Thus $E_{(m,n)}/\overline{Zg} = F_{mn} \in L^2(Q)$ for all m, n; in particular, $1/Zg \in L^2(Q)$.

Conversely, assume $1/Zg \in L^2(Q)$. Then $Zg \neq 0$ a.e., so $\{g_{mn}\}$ is complete by part a. Let $\tilde{g} = Z^{-1}(1/\overline{Zg}) \in L^2(\mathbf{R}^d)$. Then

$$\begin{aligned} \langle g_{mn}, \tilde{g}_{m'n'} \rangle &= \langle Zg_{mn}, Z\tilde{g}_{m'n'} \rangle \\ &= \langle E_{(m,n)} Zg, E_{(m',n')} Z\tilde{g} \rangle \end{aligned}$$

$$= \langle E_{(m,n)}, E_{(m',n')} \rangle$$

$$= \delta_{mm'}\, \delta_{nn'}.$$

Thus $\{g_{mn}\}$ and $\{\tilde{g}_{mn}\}$ are biorthonormal. The existence of a biorthonormal sequence implies by Proposition 6.1.3 that $\{g_{mn}\}$ is minimal.

c. Assume $\{g_{mn}\}$ is a frame for $L^2(\mathbf{R}^d)$ with frame bounds A, B; then the same is true of $\{Zg_{mn}\}$ in $L^2(Q)$. Therefore, by definition,

$$A\,\|F\|_2^2 \;\le\; \sum |\langle F, Zg_{mn}\rangle|^2 \;\le\; B\,\|F\|_2^2$$

for $F \in L^2(Q)$. Since $\{E_{(m,n)}\}$ is an orthonormal basis for $L^2(Q)$,

$$\sum |\langle F, Zg_{mn}\rangle|^2 \;=\; \sum |\langle F \cdot \overline{Zg}, E_{(m,n)}\rangle|^2 \;=\; \|F \cdot \overline{Zg}\|_2^2.$$

It follows immediately that $A \le \inf |Zg|^2$ and $B \ge \sup |Zg|^2$.

Conversely, assume Zg is essentially constant, i.e., $A \le |Zg|^2 \le B$ a.e. Then the mapping $UF = F \cdot \overline{Zg}$ is a topological isomorphism of $L^2(Q)$ onto itself. Since $\{E_{(m,n)}\}$ is an exact frame for $L^2(Q)$ and exact frames are preserved by topological isomorphisms (Lemma 6.3.2), it follows that $\{UE_{(m,n)}\} = \{Zg_{mn}\}$ is an exact frame for $L^2(Q)$, whence $\{g_{mn}\}$ is an exact frame for $L^2(\mathbf{R}^d)$.

d. Follows immediately from c. ∎

Example 7.3.5. Set $d = 1$. In Example 7.2.6 we determined that the Zak transform of the Gaussian $g(t) = e^{-rt^2}$ is continuous and has a zero. Therefore $|Zg|$ is not bounded below a.e., so by Proposition 7.3.4 and Remark 7.3.1,

$\{g_{mn}\}$ is not a frame for $L^2(\mathbf{R}^d)$ when $ab = 1$. However, as Zg is nonzero a.e., this Gabor system is complete. The question of the completeness of this Gabor system was Zak's original motivation for introducing the Zak transform.

We prove now that Zg must have a zero if it is continuous. Therefore, by Proposition 7.3.3, no function with a continuous Zak transform can generate a Gabor frame at the critical value $ab = 1$. By saying Zg is continuous we mean that Zg is continuous on all of $\mathbf{R}^d \times \hat{\mathbf{R}}^d$, not just inside a unit cube Q. Equivalently, we require continuity both inside and at the edges of Q. The proof of the following proposition is adapted from the one-dimensional version found in [**J2**]. The first published proofs were [**AT2; BZ**].

PROPOSITION 7.3.6. *Every continuous quasiperiodic function has a zero. Precisely, given a continuous quasiperiodic function F, fix any $j = 1, \dots, d$, $T \in \mathbf{R}^d$, and $\Omega \in \hat{\mathbf{R}}^d$ and define*

$$
\begin{aligned}
T(t) &= (T_1, \dots, T_{j-1}, t, T_{j+1}, \dots, T_d), \\
\Omega(\omega) &= (\Omega_1, \dots, \Omega_{j-1}, \omega, \Omega_{j+1}, \dots, \Omega_d).
\end{aligned}
$$

Then there exist $t \in \mathbf{R}$ and $\omega \in \hat{\mathbf{R}}$ such that $F\big(T(t), \Omega(\omega)\big) = 0$.

PROOF: Assume F is continuous, quasiperiodic, and nonvanishing. Then $f(t,\omega) = F(T(t), \Omega(\omega))$ is continuous and nonvanishing on $\mathbf{R} \times \hat{\mathbf{R}}$, so by [**RR**, Lemma VI.1.7] there is a continuous real-valued function φ such that $f(t,\omega) = |f(t,\omega)|\, e^{i\varphi(t,\omega)}$ for $(t,\omega) \in [0,1] \times [0,1]$. It follows immediately from the quasiperiodicity of F that

$$f(t,1) = f(t,0) \qquad \text{and} \qquad f(1,\omega) = e^{-2\pi i\omega} f(0,\omega).$$

Therefore, for t, $\omega \in [0,1]$,

$$\begin{aligned} |f(t,1)|\, e^{i\varphi(t,1)} &= f(t,1) \\ &= f(t,0) \\ &= |f(t,0)|\, e^{i\varphi(t,0)} \\ &= |f(t,1)|\, e^{i\varphi(t,0)} \end{aligned}$$

and

$$\begin{aligned} |f(1,\omega)|\, e^{i\varphi(1,\omega)} &= f(1,\omega) \\ &= e^{-2\pi i\omega}\, f(0,\omega) \\ &= e^{-2\pi i\omega}\, |f(0,\omega)|\, e^{i\varphi(0,\omega)} \\ &= e^{-2\pi i\omega}\, |f(1,\omega)|\, e^{i\varphi(0,\omega)}. \end{aligned}$$

As f is nonvanishing, it follows that

$$e^{i\varphi(t,1)} = e^{i\varphi(t,0)} \qquad \text{and} \qquad e^{i\varphi(1,\omega)} = e^{i\varphi(0,\omega)-2\pi i\omega}.$$

Therefore, for each t, $\omega \in [0,1]$ there exist integers k_t and l_ω such that

$$\varphi(t,1) = \varphi(t,0) + 2\pi k_t \qquad \text{and} \qquad \varphi(1,\omega) = \varphi(0,\omega) - 2\pi\omega + 2\pi l_\omega.$$

The functions $\varphi(t,1) - \varphi(t,0)$ and $\varphi(1,\omega) - \varphi(0,\omega) + 2\pi\omega$ are continuous functions of t and ω, respectively. Therefore, the integers k_t must equal a single integer k, and the integers l_ω must equal a single integer l. That is,

$$\varphi(t,1) = \varphi(t,0) + 2\pi k \qquad \text{and} \qquad \varphi(1,\omega) = \varphi(0,\omega) - 2\pi\omega + 2\pi l.$$

Therefore,

$$\begin{aligned} 0 &= \big(\varphi(0,0) - \varphi(1,0)\big) + \big(\varphi(1,0) - \varphi(1,1)\big) \\ &\qquad + \big(\varphi(1,1) - \varphi(0,1)\big) + \big(\varphi(0,1) - \varphi(0,0)\big) \\ &= (-2\pi l) + (-2\pi k) + (-2\pi + 2\pi l) + (2\pi k) \\ &= -2\pi \\ &\neq 0. \ \blacksquare \end{aligned}$$

Section 7.4. The Zak transform on $L^p(\mathbf{R}^d)$.

In this section we consider the convergence of the Zak transform on $L^p(\mathbf{R}^d)$ for $1 \le p \le \infty$, extending one-dimensional results of Janssen to higher dimensions. Janssen observes that the Zak transform is well-defined on $L^1(\mathbf{R})$ and $L^2(\mathbf{R})$, and therefore by interpolation on $L^p(\mathbf{R})$ for $1 \le p \le 2$. He also proves that the Zak transform cannot be defined on $L^p(\mathbf{R})$ for $p > 2$.

PROPOSITION 7.4.1. *The Zak transform is a linear, continuous, injective mapping of $L^1(\mathbf{R}^d)$ into $L^1(Q)$ with $\|Z\| = 1$. Moreover,* Range(Z) *is dense in $L^1(Q)$ but Z is not surjective, and Z^{-1}:* Range$(Z) \to L^1(\mathbf{R}^d)$ *is not continuous.*

PROOF: Fix $f \in L^1(\mathbf{R}^d)$. For $k \in \mathbf{Z}^d$ define $F_k(t,\omega) = f(t+k)\,E_k(\omega)$. Since

$$\|F_k\|_1 \;=\; \iint_Q |F_k(t,\omega)|\,d\omega\,dt \;=\; \int_{[0,1]} |f(t+k)|\,dt \;<\; \infty,$$

$F_k \in L^1(Q)$. Moreover, $\sum \|F_k\|_1 = \|f\|_1$, so Zf converges absolutely and $\|Zf\|_1 \le \sum \|F_k\|_1 = \|f\|_1$. Thus Z is continuous and $\|Z\| \le 1$. If $g = \chi_{[0,1]}$ then $Zg_{mn} = E_{(m,n)}$, so $\|Z\| = 1$ since $\|Zg_{mn}\|_1 = \|E_{(m,n)}\|_1 = 1 = \|g_{mn}\|_1$. Also, Range$(Z)$ is dense since $\{E_{(m,n)}\}$ is complete in $L^1(Q)$.

For a.e. $t \in \mathbf{R}^d$,

$$\begin{aligned}\int_{[0,1]} Zf(t,\omega)\,d\omega &= \int_{[0,1]} \sum f(t+k)\,E_k(\omega)\,d\omega \\ &= \sum f(t+k) \int_{[0,1]} E_k(\omega)\,d\omega \\ &= \sum f(t+k)\,\delta_{0k} \\ &= f(t),\end{aligned}$$

the interchange of summation and integration is justified by the fact that

$$\sum \int_{[0,1]} |f(t+k)\, E_k(\omega)|\, d\omega \;=\; \sum |f(t+k)| \;<\; \infty \text{ a.e.}$$

Therefore, $Zf = 0$ a.e. implies $f = 0$ a.e., so Z is injective.

If Z was surjective then $Z(L^1(\mathbf{R}^d)) = L^1(Q) \supset L^2(Q) = Z(L^2(\mathbf{R}^d))$, which implies $L^1(\mathbf{R}^d) \supset L^2(\mathbf{R}^d)$, a contradiction. Therefore Z is not surjective.

Finally, we show Z^{-1} is not continuous. Given $R > 0$ there exists a bounded, 1-periodic function $g \in L^\infty[0,1)$ such that $\|g\|_\infty \le 1$ and $R \le \sum |\hat{g}(k)| < \infty$, e.g., [**K**, p. 99], where $\{\hat{g}(k)\}$ are the Fourier coefficients of g, i.e., $\hat{g}(k) = \int_{[0,1)} g(t)\, E_{-k}(t)\, dt$ for $k \in \mathbf{Z}^d$. Define $f = \sum \hat{g}(k)\, \chi_{[k,k+1]}$; then $\|f\|_1 = \sum |\hat{g}(k)| < \infty$. Since g has an absolutely convergent Fourier series,

$$Zf(t,\omega) \;=\; \sum f(t+k)\, E_k(\omega) \;=\; \sum \hat{g}(k)\, E_k(\omega) \;=\; g(\omega) \text{ a.e.}$$

Thus $\|Zf\|_1 = \|g\|_1 \le \|g\|_\infty \le 1$ while $\|f\|_1 \ge R$. As R is arbitrary, this implies Z^{-1} is unbounded and therefore not continuous. ∎

In the course of the proof of Proposition 7.4.1 we proved the following, cf., [**J2**].

Corollary 7.4.2. *If $f \in L^1(\mathbf{R}^d)$ then*

$$f(t) \;=\; \int_{[0,1]} Zf(t,\omega)\, d\omega$$

for a.e. $t \in \mathbf{R}^d$.

Corollary 7.4.3. *If $f \in L^1(\mathbf{R}^d)$ and Zf is continuous then f is continuous.*

PROOF: If Zf is continuous then it is uniformly continuous on Q, so

$$\begin{aligned} |f(t) - f(s)| &\le \int_{[0,1]} |Zf(t,\omega) - Zf(s,\omega)|\, d\omega \\ &\le \sup_{\omega\in[0,1]} |Zf(t,\omega) - Zf(s,\omega)| \\ &\to 0 \quad \text{as } s \to t. \ \blacksquare \end{aligned}$$

EXAMPLE 7.4.4. The converse of Corollary 7.4.3 does not hold.

To see this, set $d = 1$ and let φ be a continuous, 1-periodic function on $\mathbf{R}$ whose Fourier series does not converge at zero, e.g., [K, p. 99]. For $k \in \mathbf{Z}$ define

$$f(k) = \hat{\varphi}(k) = \int_0^1 \varphi(t)\, e^{-2\pi ikt}\, dt.$$

Let $f(t) = 0$ for $t \notin \bigcup[k - \varepsilon_k, k + \varepsilon_k]$, and let f be linear on $[k - \varepsilon_k, k]$ and $[k, k + \varepsilon_k]$. Then f is continuous and integrable if the ε_k are small enough. Since $\varphi \in L^1[0,1)$, it follows from the Riemann-Lebesgue lemma that $\hat{\varphi}(k) \to 0$ as $k \to \pm\infty$, so $f \in C_0(\mathbf{R}) \cap L^1(\mathbf{R})$. We have $Zf(0,\omega) = \sum \hat{\varphi}(k)\, e^{2\pi ik\omega}$. Since this series does not converge for $\omega = 0$, Zf cannot be continuous at $(0,0)$.

LEMMA 7.4.5. *Let $\{x_n\}$ be a basis for a Banach space X and $\{y_n\}$ a basis for a Banach space Y. If $S: X \to Y$ is continuous and linear, and $Sx_n = y_n$ for all n, then S is injective and* Range(S) *is dense in Y. If, in addition, S^{-1}:* Range$(S) \to X$ *is continuous then S is surjective and hence a topological isomorphism of X onto Y.*

PROOF: Since Range(S) $\supset \{y_n\}$, a complete set, it must be dense in Y. Assume $x \in X$ and $Sx = 0$. By definition there exist unique scalars c_n such that $x = \sum c_n x_n$. As S is continuous we therefore have

$$\sum c_n y_n = \sum c_n S x_n = S\Big(\sum c_n x_n\Big) = Sx = 0.$$

As $\{y_n\}$ is a basis, it follows that $c_n = 0$ for every n. Thus $x = 0$, so S is injective.

Assume now that S^{-1} is continuous. We claim then that Range(S) is closed. To see this, assume $z_n \in$ Range(S) and $z \in Y$ with $z_n \to z$ in Y. Then $\{z_n\}$ is a Cauchy sequence in Y, whence $\{S^{-1} z_n\}$ is a Cauchy sequence in X, so must converge to some $w \in X$. Therefore $Sw = \lim z_n = z$, so $z \in$ Range(S) and therefore Range(S) is closed. As it is also dense, it must be all of Y, and therefore S is surjective. ∎

PROPOSITION 7.4.6. *Given $1 < p < 2$, the Zak transform is a linear, continuous, injective mapping of $L^p(\mathbf{R}^d)$ into $L^p(Q)$ with $\|Z\| = 1$. Moreover,* Range(Z) *is dense in $L^p(Q)$, but Z is not surjective, and Z^{-1}:* Range(Z) $\to$ *$L^p(\mathbf{R}^d)$ is not continuous.*

PROOF: By Propositions 7.2.4 and 7.4.1, Z maps $L^2(\mathbf{R}^d)$ onto $L^2(Q)$ and $L^1(\mathbf{R}^d)$ into $L^1(Q)$, both with $\|Z\| = 1$. Standard interpolation results therefore imply that Z maps $L^p(\mathbf{R}^d)$ into $L^p(Q)$ with $\|Z\| \le 1$ for $1 \le p \le 2$.

Fix $1 < p < 2$ and set $g = \chi_{[0,1]}$. Then $\{g_{mn}\}_m$ is a basis for $L^p[n, n+1]$, cf., [Ma, p. 51] or [K, p. 50]. Therefore the Gabor system $\{g_{mn}\}$ is a basis for $L^p(\mathbf{R}^d)$. Moreover, $Zg_{mn} = E_{(m,n)}$ and $\{E_{(m,n)}\}$ is a basis for $L^p(Q)$. Therefore $\|Z\| = 1$, and, by Lemma 7.4.5, Z is injective and its range is dense. If Z was surjective then $Z(L^p(\mathbf{R}^d)) = L^p(Q) \supset L^2(Q) = Z(L^2(\mathbf{R}^d))$, whence $L^p(\mathbf{R}^d) \supset L^2(\mathbf{R}^d)$, a contradiction. Therefore Z is not surjective, and hence Z^{-1} is not continuous by Lemma 7.4.5. ∎

Since $\{E_{(m,n)}\}$ is not a basis for $L^1(Q)$, the method of Proposition 7.4.6 cannot directly be used to prove Proposition 7.4.1.

EXAMPLE 7.4.7. The Zak transform cannot be defined as a map of $L^q(\mathbf{R}^d)$ onto any $L^r(Q)$ when $q > 2$. For example, by [**K**, p. 100], there exist scalars $\{c_n\}$ such that $\sum |c_k|^q < \infty$ for every $q > 2$ but $\sum c_k E_k$ is not a Fourier series. Define $f = \sum c_k \chi_{[k,k+1]}$; then $f \in L^q(\mathbf{R}^d)$ but $Zf(t,\omega) = \sum f(t+k)E_k(\omega) = \sum c_k E_k(\omega)$ does not converge in any $L^r(Q)$.

Section 7.5. Amalgam spaces and the Zak transform.

In this section we examine the convergence of the Zak transform on Wiener amalgam spaces, in particular, on the amalgam space $W(L^p, L^1)$ on the additive group $\mathbf{R}^d$. We prove that the Zak transform maps $W(L^p, L^1)$ into $L^p(Q)$ for each $1 \le p \le \infty$, and maps $W(C_0, L^1)$ into the space of continuous quasiperiodic functions. This gives us a variant of the Balian–Low theorem, i.e., if (g, a, b) generates a Gabor frame at the critical value $ab = 1$ then $g \notin W(C_0, L^1)$, whence g is either not continuous or decays slowly at infinity.

From Example 2.4.4, the Wiener amalgam space $W(L^p, L^1)$ is defined by the norm

$$\|f\|_{W(L^p,L^1)} = \sum \|f \cdot \chi_{[k,k+1]}\|_p.$$

We adopt this as the standard norm for $W(L^p, L^1)$. Also, from Example 2.2.4,

$$W(C_0, L^1) = \{f \in W(L^\infty, L^1) : f \text{ is continuous}\},$$

with the $W(L^\infty, L^1)$ norm.

PROPOSITION 7.5.1. *Given $1 \le p \le \infty$, the Zak transform is a continuous, linear, injective map of $W(L^p, L^1)$ into $L^p(Q)$, with $\|Z\| = 1$.*

PROOF: Fix $f \in W(L^p, L^1)$. For $k \in \mathbf{Z}^d$ define $F_k(t, \omega) = f(t + k)\, E_k(\omega)$. Then $F_k \in L^p(Q)$ since $\|F_k\|_p = \|f \cdot \chi_{[k,k+1]}\|_p < \infty$. Moreover,

$$\sum \|F_k\|_p = \sum \|f \cdot \chi_{[k,k+1]}\|_p = \|f\|_{W(L^p,L^1)} < \infty,$$

so $Zf = \sum F_k$ converges absolutely in $W(L^p, L^1)$, and $\|Z\| \le 1$. We have $\|Z\| = 1$ as $\|Zg_{mn}\|_p = \|E_{(m,n)}\|_p = 1 = \|g_{mn}\|_{W(L^p,L^1)}$, where $g = \chi_{[0,1]}$.

Z is injective as $W(L^p, L^1) \subset L^1(\mathbf{R}^d)$ and Z is injective on $L^1(\mathbf{R}^d)$ (Proposition 7.4.1). ∎

COROLLARY 7.5.2. *If $f \in W(C_0, L^1)$ then Zf is continuous on $\mathbf{R}^d \times \hat{\mathbf{R}}^d$.*

PROOF: If $f \in W(C_0, L^1)$ then the series defining Zf converges in $L^\infty(Q)$, i.e., uniformly on Q, by Proposition 7.5.1, since $W(C_0, L^1) \subset W(L^\infty, L^1)$. As each term $f(t+k)\,E_k(\omega)$ in the series defining Zf is continuous, it follows that the series must converge to a function which is continuous on Q, and therefore, by quasiperiodicity, on all of $\mathbf{R}^d \times \hat{\mathbf{R}}^d$. ∎

Corollary 7.5.2 can also be proved by noting that translation and modulation are both strongly continuous in $W(C_0, L^1)$.

The following is a variant of the *Balian–Low theorem.*

COROLLARY 7.5.3. *Given $g \in L^2(\mathbf{R}^d)$ and $a = b = 1$. If (g, a, b) generates a Gabor frame for $L^2(\mathbf{R}^d)$ then g, $\hat{g} \notin W(C_0, L^1)$.*

PROOF: If $g \in W(C_0, L^1)$ then Zg is continuous by Corollary 7.5.2. By Proposition 7.3.5, Zg therefore has a zero, so $|Zg|$ is not bounded below a.e. Therefore (g, a, b) cannot generate a frame (Proposition 7.3.3c).

Similarly, if $\hat{g} \in W(C_0, L^1)$ then $(\hat{g}, b, a)$ cannot generate a frame for $L^2(\hat{\mathbf{R}}^d)$, and therefore, by Remark 7.1.2b, (g, a, b) cannot generate a frame for $L^2(\mathbf{R}^d)$. ∎

REMARK 7.5.4. The usual Balian–Low theorem states that if (g, a, b) generates a Gabor frame for $L^2(\mathbf{R})$ and $ab = 1$ then $\|tg(t)\|_2\,\|\gamma\hat{g}(\gamma)\|_2 = \infty$, cf., [Bal; Bat; BHW; D1; DJ; Low].

EXAMPLE 7.5.5. The Gaussian function $g(t) = e^{-\pi t \cdot t}$, $t \in \mathbf{R}^d$, is an element of $W(C_0, L^1)$, therefore does not generate a Gabor frame at the critical value $ab = 1$. We proved this directly (for $d = 1$) in Example 7.3.4.

EXAMPLE 7.5.6. Note that $W(C_0, L^1) \subset C_0(\mathbf{R}^d) \cap L^1(\mathbf{R}^d)$. Although the Zak transform of any element of $W(C_0, L^1)$ is continuous by Corollary 7.5.3, there exist elements of $C_0(\mathbf{R}^d) \cap L^1(\mathbf{R}^d)$ whose Zak transform is not continuous. We constructed an example in Example 7.4.4.

Section 7.6. Multiplicative completion.

In this section we address some questions similar to ones which arose during our study of Gabor frames and the Zak transform, cf., Remark 7.3.3. Our motivation is the following question asked by Boas and Pollard, e.g., **[BPo]**. Given an incomplete sequence $\{f_n\}$ of functions in $L^2(a,b)$, where (a,b) is an interval in $\mathbf{R}$, when is it possible or impossible to find a function m such that $\{m \cdot f_n\}$ is complete in $L^2(a,b)$? They proved that if $\{f_n\}$ is obtained by deleting finitely many elements from an orthonormal basis for $L^2(a,b)$ then it is always possible to find such a function m, while for the orthonormal sequence $\{E_{2n}\}_{n\in\mathbf{Z}}$ in $L^2(0,1)$ it is impossible to find such a function m. We elaborate on these two results, then comment on related work by other authors.

In this section we use the following definition of *solid*, which differs slightly from the one in Section 1.7c. Given a measure space (X,μ), a Banach space A of functions on X is **solid** if $g \in A$ and $|f| \leq |g|$ a.e. implies $f \in A$ and $||f||_A \leq ||g||_A$.

LEMMA 7.6.1. *Given a measure space (X,μ) and given a solid Banach space A of functions on X. Assume that for any set $E \subset X$ there exists a function ψ on X such that* $\text{supp}(\psi) \subset E$, *$\psi$ is finite a.e., and $\psi \notin A$. Then, given any $f_1,\dots,f_N \in A$ there exists a function $g \in L^\infty(X)$ with $g \neq 0$ a.e. such that $f/g \notin A$ if $f \in \text{span}\{f_1,\dots,f_N\} \setminus \{0\}$.*

PROOF: Without loss of generality we assume $f_n \neq 0$ for all n. We proceed by induction.

a. Set $N = 1$, and assume $f = cf_1$, where $c \neq 0$. As $f \neq 0$, there exists a set $E \subset X$ with positive measure such that $|f(t)| \geq \varepsilon > 0$ for a.e. $t \in E$. By hypothesis there then exists a function $\psi \notin A$ with $\text{supp}(\psi) \subset E$ which is finite a.e. Set $\varphi(t) = \max\{|\psi(t)|, 1\}$, and let $g = 1/\varphi$. Then $g \leq 1$ a.e. as $\varphi \geq 1$ a.e., and $g \neq 0$ a.e. as φ is finite a.e. Moreover, if $t \in E$ then $|f(t)/g(t)| \geq \varepsilon\,\varphi(t) \geq \varepsilon\,|\psi(t)|$. This also holds for $t \notin E$ since $\text{supp}(\psi) \subset E$. As $\psi \notin A$ and A is solid, it follows that $f/g \notin A$.

b. Assume now that the conclusion of the lemma holds for some $N \geq 1$, and let $f_1, \ldots, f_{N+1} \in A$ be fixed. Then, by hypothesis, there exists a function $g \in L^\infty(X)$ with $g \neq 0$ a.e. such that

$$S_N = \{f \in \text{span}\{f_1, \ldots, f_N\} \setminus \{0\} : f/g \in A\} = \emptyset.$$

Define

$$S_{N+1} = \{f \in \text{span}\{f_1, \ldots, f_{N+1}\} \setminus \{0\} : f/g \in A\}.$$

If $S_{N+1} = \emptyset$ then the proof is complete, so assume $F = \sum_1^{N+1} c_n f_n \in S_{N+1}$. Note that $c_{N+1} \neq 0$, for otherwise $F \in S_N$. Assume also that $G = \sum_1^{N+1} b_n f_n \in S_{N+1}$; then $b_{N+1} \neq 0$ for the same reason. Clearly,

$$H = \frac{1}{c_{N+1}} F - \frac{1}{b_{N+1}} G \in \text{span}\{f_1, \ldots, f_N\}.$$

Moreover, $H/g \in A$ as both F/g, $G/g \in A$. As $S_N = \emptyset$, it follows that $H = 0$. Thus G is a multiple of F, so $S_N \subset \{cF : c \neq 0\}$. Now, $F \neq 0$ since

$F \in S_{N+1}$. Therefore there exists a set $E \subset X$ with positive measure such that $|F(t)| \geq \varepsilon > 0$ for a.e. $t \in E$. By hypothesis there then exists a function $\psi \notin A$ with $\mathrm{supp}(\psi) \subset E$ which is finite a.e. Set $\varphi(t) = \max\{|\psi(t)|, 1/|g(t)|\}$ and define $h = 1/\varphi$. Then h is finite a.e. since $g \neq 0$ a.e., and $h \leq |g|$ so $h \in L^\infty(X)$. Moreover, if $t \in E$ then $|F(t)/h(t)| \geq \varepsilon\,\varphi(t) \geq \varepsilon\,|\psi(t)|$. This also holds for $t \notin E$ since $\mathrm{supp}(\psi) \subset E$. As $\psi \notin A$ and A is solid, it follows that $F/h \notin A$.

Finally, to finish the proof, assume that $f \in \mathrm{span}\{f_1, \dots, f_{N+1}\} \setminus \{0\}$ is given. If $f/h \in A$ then $f/g \in A$ since $h \leq |g|$. Therefore $f \in S_{N+1}$, so $f = cF$ for some $c \neq 0$. However, $F/h \notin A$, a contradiction. Therefore $f/h \notin A$, so the result follows. ∎

PROPOSITION 7.6.2. *Given a measure space (X, μ) and a solid Banach space B of functions on X. Assume that B' is also a solid Banach function space on X which satisfies the hypotheses of Lemma 7.6.1. Given $S \subset B$, define*

$$S^\perp = \{g \in B' : \langle f, g\rangle = 0 \text{ for } f \in S\}.$$

Assume $\{f_n\}_{n \in \mathbf{Z}_+} \subset B$ and $g_1, \dots, g_N \in B'$ satisfy

$$\{f_n\}^\perp \subset \mathrm{span}\{g_1, \dots, g_N\}.$$

Then there exists a function $m \in L^\infty(X)$ with $m \neq 0$ a.e. such that $\{m \cdot f_n\}$ is complete in B.

PROOF: By Lemma 7.6.1 there exists a function $m \in L^\infty(X)$ with $m \neq 0$ a.e. such that

$$g \in \text{span}\{g_1, \ldots, g_N\} \setminus \{0\} \quad \Rightarrow \quad g/\bar{m} \notin B'. \tag{7.6.1}$$

Assume $h \in B'$ satisfies $\langle m \cdot f_n, h\rangle = 0$ for all n. Since $m \in L^\infty(X)$ we have $h \cdot \bar{m} \in B'$. Since $\langle f_n, h \cdot \bar{m}\rangle = 0$ for all n, $h \cdot \bar{m} \in \{f_n\}^\perp \subset \text{span}\{g_1, \ldots, g_N\}$. If $h \cdot \bar{m} \neq 0$ then $h = (h \cdot \bar{m})/\bar{m} \notin B'$, a contradiction. Therefore $h \cdot \bar{m} = 0$, which implies $h = 0$ a.e. as $m \neq 0$ a.e., so $\{m \cdot f_n\}$ is complete in B by Definition 6.1.1b. ∎

EXAMPLE 7.6.3. a. Assume $\{f_n\} \subset B$ and $\{g_n\} \subset B'$ satisfy $g = \sum\langle g, f_n\rangle\, g_n$ for $g \in B'$ (not necessarily uniquely), and fix $N > 0$. If $g \in \{f_n\}^\perp_{n>N}$ then $g = \sum_1^N \langle g, f_n\rangle\, g_n \in \text{span}\{g_1, \ldots, g_N\}$.

b. If $\{f_n\}_{n\in\mathbf{Z}_+}$ is a basis for B and B is reflexive, then there exists a *dual basis* $\{g_n\}_{n\in\mathbf{Z}_+} \subset B'$, i.e., $g = \sum\langle g, f_n\rangle\, g_n$, uniquely, for all $g \in B'$, e.g., [S], cf., Remark 6.1.2. Therefore, by part a and Proposition 7.6.2, given any $N > 0$ there exists a function $m \in L^\infty(X)$ such that $\{m \cdot f_n\}_{n>N}$ is complete in B.

c. If $\{g_n\}_{n\in\mathbf{Z}_+}$ is a frame for $B = L^2(X)$ and $\{f_n\}_{n\in\mathbf{Z}_+}$ is its dual frame, then $g = \sum\langle g, f_n\rangle\, g_n$ for all $g \in L^2(X)$. Therefore, by part a and Proposition 7.6.2, given any $N > 0$ there exists a function $m \in L^\infty(X)$ such that $\{m \cdot f_n\}_{n>N}$ is complete in $L^2(X)$.

d. Let X be a finite set and let μ be counting measure on X. Given $\emptyset \neq E \subset X$ and any finite function ψ on X with $\text{supp}(\psi) \subset E$,

$$\|\psi\|^p_{L^p(X)} = \sum_{t\in E} |\psi(t)|^p < \infty,$$

since X is finite. Thus $A = L^p(X)$ does not satisfy the hypotheses of Lemma 7.6.1.

By Remark 7.6.3b, if finitely many elements are removed from a basis (thereby leaving an incomplete set) then it is possible to find a single function m to multiply the remaining elements by to obtain a complete set. We now show by example this need not be true if infinitely many elements are removed, cf., [**BPo**].

We assume $d = 1$ in Lemma 7.6.4 and Proposition 7.6.5. Functions in $L^2[0,1)$ are considered to be extended 1-periodically to the entire real line.

LEMMA 7.6.4. *If $f \in L^2[0,1)$ is $1/N$-periodic, where $N \in \mathbf{Z}_+$, then $\langle f, E_n \rangle = 0$ for all $n \in \mathbf{Z}$ such that N does not divide n.*

PROOF: If f is $1/N$-periodic then

$$\begin{aligned}
\langle f, E_n \rangle &= \int_0^1 f(t)\, e^{-2\pi i n t}\, dt \\
&= \sum_{k=0}^{N-1} \int_0^{1/N} f(t + k/N)\, e^{-2\pi i n (t+k/N)}\, dt \\
&= \sum_{k=0}^{N-1} \int_0^{1/N} f(t)\, e^{-2\pi i n t}\, e^{-2\pi i n k/N}\, dt \\
&= \left(\int_0^{1/N} f(t)\, e^{-2\pi i n t}\, dt \right) \left(\sum_{k=0}^{N-1} e^{-2\pi i n k/N} \right)
\end{aligned}$$

Let $z = e^{-2\pi i n/N}$ and $w = \sum_0^{N-1} z^k$. Then $wz = w - 1 + z^N = w$. If N does not divide n then $z \neq 1$, so $w = 0$ and therefore $\langle f, E_n \rangle = 0$. ∎

PROPOSITION 7.6.5. *If $S \subset \mathbf{Z}$ contains an arithmetic progression then the* *orthonormal sequence $\{E_n\}_{n\notin S}$ cannot be completed in $L^2[0,1)$ by multiplication by an integrable function.*

PROOF: Without loss of generality, assume $S = \{nN\}_{n\in\mathbf{Z}}$ for some $N \in \mathbf{Z}_+$. Fix any $m \in L^1[0,1)$. If the measure of the zero set of m is positive then $\{m \cdot E_n\}_{n\notin S}$ is incomplete, so assume $m \neq 0$ a.e. Then there exists a set $E_1 \subset (0, 1/N)$ on which $|m|$ is bounded above and below, and then a set $E_2 \subset E_1 + 1/N$ on which $|m|$ is bounded above and below, and so forth. Define

$$F = E_N \cup (E_N - \tfrac{1}{N}) \cup \cdots \cup (E_N - \tfrac{N-1}{N}).$$

Then $F + 1/N = F$ (mod 1). Moreover, $F \subset E_N \cup \cdots \cup E_1$, so $|m|$ is bounded above and below on F. Therefore,

$$f(t) = \begin{cases} 1/\overline{m(t)}, & t \in F, \\ 0, & t \notin F, \end{cases}$$

is an nonzero element of $L^2[0,1)$. Further, $f \cdot \bar{m} = \chi_F$ is $1/N$-periodic, so $\langle f, m \cdot E_n\rangle = \langle f \cdot \bar{m}, E_n\rangle = 0$ for all $n \notin S$ by Lemma 7.6.4. Thus $\{m \cdot E_n\}_{n\notin S}$ is incomplete. ∎

REMARK 7.6.6. Given a sequence $\{f_n\}_{n\in\mathbf{Z}_+} \subset L^2(X)$, where (X,μ) is a finite separable measure space with $\mu(X) = 1$, Talalyan proved that the following statements are equivalent, e.g., [Ta].

a. Given $\varepsilon > 0$ there exists $S_\varepsilon \subset X$ such that $\mu(S_\varepsilon) > 1 - \varepsilon$ and $\{f_n \cdot \chi_{S_\varepsilon}\}$ is complete in $L^2(S_\varepsilon)$.

b. For every function f on X which is finite a.e. and every $\varepsilon > 0$ there exists $S_\varepsilon \subset X$ and $g \in \text{span}\{f_n\}$ such that $\mu(S_\varepsilon) > 1-\varepsilon$ and $|f-g| < \varepsilon$ on S_ε.

Price and Zink proved that a and b are also equivalent to the following, seemingly unrelated, Boas–Pollard property, e.g., [**Pr**; **PZ**].

c. There exists a bounded, nonnegative function m such that $\{m \cdot f_n\}$ is complete in $L^2(X)$.

REMARK 7.6.7. In [**By1**; **By2**; **BN**], Byrnes and Newman consider a problem similar to the one addressed by Boas and Pollard. Instead of deleting elements from a sequence and then multiplying the remaining elements by a function, they retain all elements of the sequence and multiply only a portion of the sequence by a function. In particular, they show in [BN] that if $\{f_n\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2[0,1)$ and $S \subset \mathbf{Z}$, then $\{f_n\}_{n\in S} \cup \{m\cdot f_n\}_{n\notin S}$ is complete in $L^2[0,1)$ if and only if there exists an $\alpha \in \mathbf{C}$ such that $\text{Re}(\alpha m) \geq 0$ a.e. and either $\text{Im}(\alpha m) > 0$ a.e. or $\text{Im}(\alpha m) < 0$ a.e. on the zero set of $\text{Re}(\alpha m)$.

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