FINITELY GENERATED ABELIAN GROUPS OF UNITS

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Abstract. In 1960 Fuchs posed the problem of characterizing the groups which are the groups of units of commutative rings. In the following years, some partial answers have been given to this question in particular cases.

In this paper we address Fuchs' question for *finitely generated abelian* groups and we consider the problem of characterizing those groups which arise in some fixed classes of rings $\mathcal{C}$, namely the integral domains, the torsion free rings and the reduced rings.

Most of the paper is devoted to the study of the class of torsion-free rings, which needs a substantially deeper study.

1. Introduction

1.1. **General introduction to the problem.** The study of the group of units of a ring is an old problem. The first general result is the classical Dirichlet's Unit Theorem (1846), which describes the group of units of the ring of integers $\mathcal{O}_K$ of a number field K: the group of units $\mathcal{O}_K^*$ is a finitely generated abelian group of the form $C_{2n} \times \mathbb{Z}^g$ where $n \geq 1$ and g is determined by the structure of the field K.

In 1940 G. Higman discovered a perfect analogue of Dirichlet's Unit Theorem for a group ring $\mathbb{Z}T$ where T is a finite abelian group: $(\mathbb{Z}T)^* \cong \pm T \times \mathbb{Z}^g$ for a suitable explicit constant g.

In 1960 Fuchs in [Fuc60, Problem 72] posed the following problem.

> *Characterize the groups which are the groups of all units in a commutative and associative ring with identity.*

In the subsequent years, this question has been considered by many authors. A first result is due to Gilmer [Gil63], who considered the case of *finite commutative rings*, classifying the possible cyclic groups that arise in this case. An important contribution to the problem can be derived from the results by Hallett and Hirsch [HH65], and subsequently by Hirsch and Zassenhaus [HZ66], combined with [Cor63]. From their study it is possible to deduce that if a finite group is the group of units of a reduced and torsion free ring, then it must satisfy

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some necessary conditions, namely, it must be a subgroup of a direct product of groups of a given family.

Later on, Pearson and Schneider [PS70] combined the result of Gilmer and the result of Hallett and Hirsch to describe explicitly all possible *finite cyclic groups* that can occur as A^* for some ring A.

Recently, Chebolu and Lockridge [CL15] were able to classify the *indecomposable abelian groups* which occur as groups of units of a ring.

In the papers [DCD18a] [DCD18b] R. Dvornicich and the author studied Fuchs' question for *finite abelian groups* and for a general ring of any characteristic, obtaining necessary conditions for a group to be realizable, and producing infinite families of both realizable and non-realizable groups. Moreover, they got a complete classification of the group of units realizable in some particular classes of rings (integral domains, torsion-free rings and reduced rings).

The study of groups of units has been investigated also for non abelian groups. Much has been said about the units of group rings. Recently, the finite dihedral groups and the simple groups that are realizable as the group of units of a ring have been classified (see [CL17] and [DO14]).

1.2. **The questions studied in the paper.** In this paper we consider Fuchs' question for *finitely generated abelian* groups and we consider the problem of characterizing those groups which arise in some fixed classes of rings $\mathcal{C}$, namely the integral domains, the torsion free rings and the reduced rings.

This question is twofold: on the one hand, we have to establish which finite abelian groups T (up to isomorphism) occur as the torsion subgroup of A^* when A varies in $\mathcal{C}$. On the other hand, we have to determine the possible values of the rank of A^* when $(A^*)_{tors} \cong T$. Therefore, the situation becomes substantially different from the case when the group of units is finite and abelian, which has been studied already in [DCD18a] and [DCD18b].

1.3. **Integral domains: result and idea of proof.** In Section 3 we focus on the study of groups of units of integral domains. Our main tools are Dirichlet's Unit Theorem and the properties of cyclotomic extensions. The principal result is the following theorem in which we collect the results of Theorems 3.1 and 3.4.

Theorem A: *The finitely generated abelian groups that occur as groups of units of integral domains are:*

i) the groups of the form $C_{2n} \times \mathbb{Z}^g$*, with* $n \in \mathbb{N}$*,* $g \geq \frac{\phi(2n)}{2} - 1$*, for domains of characteristic zero;*

ii) the groups of the form $\mathbb{F}_{p^n}^* \times \mathbb{Z}^g$ *with* $n \geq 1$ *and* $g \geq 0$*, for domains of finite characteristic.*

As a particular case we get the characterization of the finite abelian groups which are realizable as group of units of an integral domain (see Corollary 3.2).

Finally, in Proposition 3.3 we describe the finitely generated abelian groups that occur as group of units of an integral domain A which is integral over $\mathbb{Z}$.

1.4. **Torsion-free rings: result and idea of proof.** The most relevant part of the paper is the classification of the finitely generated abelian groups of units realizable with torsion-free rings (Sections 4 and 5). We remark that the study of the group of units of torsion free rings has become classical in the literature (see the aforementioned papers by Hallett, Hirsch and Zassenhaus) and that the finitely generated abelian group rings belong to this class.

In Theorem 5.1 we prove the following

Theorem B: *Let T be a finite abelian group of even order. Then there exists an explicit constant $g(T)$ depending on T (see (12) for the explicit value of $g(T)$) such that the following holds: the group $T \times \mathbb{Z}^r$ is the group of units of a torsion free ring if and only if $r \geq g(T)$.*

The proof is rather long and requires many steps. The first step is the reduction to the study of the subring of A generated over $\mathbb{Z}$ by the torsion units. This ring has the same torsion units as A and is finitely generated and integral over $\mathbb{Z}$. Restricting to study these rings, in Proposition 4.2 we show that the $\mathbb{Q}$-algebra $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is semisimple and is a finite product of cyclotomic fields (for short, a cyclotomic $\mathbb{Q}$-algebra). The next step is the study of the units of the subrings of A of type $\mathbb{Z}[\alpha]$, with α a torsion unit of A, in some particular cases (see Propositions 4.6 and 4.7). Once these preliminary results are established, we pass to the proof of the theorem, which requires two parts.

On the one hand, we have to show that if A is a torsion-free ring with $(A^*)_{tors} \cong T$, then $\mathrm{rank}(A^*) \geq g(T)$. This is done through the analysis of the possible maximal order of T-admissible cyclotomic $\mathbb{Q}$-algebras (namely, cyclotomic $\mathbb{Q}$-algebras which could admit a subring with $(A^*)_{tors} \cong T$). This gives a first lower bound on the rank of the group of units (Proposition 5.5). This "natural" bound works only if the 2-Sylow subgroup of T has "enough" cyclic factors of minimal order in its decomposition. If not, the actual bound is bigger then the natural one: this is described in Proposition 5.6.

On the other hand, for each T we have to construct a torsion-free ring A with $A^* \cong T \times \mathbb{Z}^{g(T)}$: the construction of orders with a bigger rank can then be obtained via localization. In the previous part for a given T we have identified a maximal order $\mathcal{M}_T$ of a cyclotomic $\mathbb{Q}$-algebra with $\mathrm{rank}(\mathcal{M}_T^*) = g(T)$. We construct A as an order of $\mathcal{M}_T$, hence $\mathrm{rank}(A) = \mathrm{rank}(\mathcal{M}_T^*) = g(T)$ (see Lemma 4.4). The group $(\mathcal{M}_T^*)_{tors}$ contains a subgroup isomorphic to T and it differs from T only in the

2-Sylow subgroup: our task is to construct an order with a 2-Sylow as small as possible.

We note that also in this case the results of [DCD18b] on finite abelian groups of units are recovered as a corollary of this more general result.

1.5. **Reduced rings: result and idea of proof.** In Section 6 we deal with the units of reduced rings.

For a non reduced ring R with nilradical $\mathcal{N}$, it is known that the R^* is an extension of $(R/\mathcal{N})^*$ by $1+\mathcal{N}$ (see Proposition 6.1). So the study of units of reduced rings is also a step towards the understanding of the units of general rings.

In Theorem 6.4 we prove the following.

Theorem C: *The finitely generated abelian groups that occur as group of units of a reduced ring are those of the form*

$$\prod_{i=1}^{k} \mathbb{F}^*_{p_i^{n_i}} \times T \times \mathbb{Z}^g$$

where $k, n_1, \dots, n_k$ are positive integers, $\{p_1, \dots, p_k\}$ are not necessarily distinct primes, T is any finite abelian group of even order and $g \geq g(T)$.

The proof is achieved by using a result by Pearson and Schneider [PS70, Prop. 1] which allows one to split a generic reduced ring A as a direct sum $A_1 \oplus A_2$ where A_1 is finite and A_2 is torsion-free. Putting together our previous results on torsion-free rings with some properties of the finite rings we get the classification of the groups of units in this case.

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2. Notation and preliminary results

Let A be a ring with 1: throughout the paper we will assume that its group of units A^* is finitely generated and abelian. Let $(A^*)_{tors}$ denote its torsion subgroup and let g_A be its rank so that

$$A^* \cong (A^*)_{tors} \times \mathbb{Z}^{g_A}.$$

Let A_0 be the fundamental subring of A, namely $A_0 = \mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ depending on whether the characteristic of A is 0 or n. It is immediate to check that the ring $A_0[A^*]$ has the same group of units as A. Since we are interested in the classification of the possible groups of units, we can assume without loss of generality that A is a ring of type $A_0[A^*]$.

In particular, we will always assume that A is commutative and that it is finitely generated over A_0.

Let B the subring of A generated over A_0 by the torsion units of A, namely $B \cong A_0[(A^*)_{tors}]$. It is important to note that all the elements of $(A^*)_{tors}$ are integral over A_0, since they have finite order. This ensures that B is *commutative, finitely generated and integral over* A_0.

Lemma 2.1. *$B^* \cong (A^*)_{tors} \times \mathbb{Z}^{g_B}$ and $g_B \leq g_A$. Moreover, if the characteristic of A is positive, then $B^* = (A^*)_{tors}$.*

Proof. B is a subring of A, hence $B^* < A^*$: in particular B^* is finitely generated and $g_B \leq g_A$. On the other hand, $(A^*)_{tors} < (B^*)_{tors} < (A^*)_{tors}$ and equality holds.

Moreover, when the characteristic of A_0 is positive, then B, being integral and finitely generated over A_0, is itself finite, so $B^* = (A^*)_{tors}$. □

Remark 2.2. The previous lemma shows that all possible torsion parts occur already when restricting to consider rings which are generated over A_0 by a finite number of integral elements verifying an equation of type $x^n - 1$ for some n.

The lemma also shows that there is a completely different behavior between the characteristic zero and positive characteristic rings. In fact, a finite abelian group T can be isomorphic to the torsion subgroup of the group of units of a ring A of positive characteristic only if it is also the group of units of a finite ring and all the results of [DCD18a] apply in this case. In particular, not all finite abelian groups can occur.

Instead, when $A_0 = \mathbb{Z}$ it will turn out that the torsion subgroup of A^* can be any finite abelian groups of even order, whereas this is not true if we also require that A^* is finite (see Theorem 5.1 and [DCD18b]). Nevertheless, to determine the minimum rank $g(T)$ such that $T \times \mathbb{Z}^{g(T)}$ is the group of units of some ring A, it is sufficient to consider the finitely generated integral extensions of $\mathbb{Z}$.

In the following subsections we collect some classical results we will need in the paper.

2.1. **Units of Laurent polynomials.**

Let R be a reduced ring, namely a ring without non-zero nilpotents. Then the polynomial ring $R[x]$ is reduced and has the same units as R and the ring of Laurent polynomials $R[x, x^{-1}]$ has group of units $\langle R^*, x \rangle$. Inductively we get that the group of units of the ring of Laurent polynomials in k indeterminates $R[x_1 \ldots, x_k, x_1^{-1}, \ldots, x_k^{-1}]$ is isomorphic to $R^* \times \mathbb{Z}^k$.

2.2. **The Chinese Remainder Theorem.**

Let R be a commutative ring with 1 and let $I, J \subseteq R$ be ideals. Then the map

$$\psi \colon R \to R/I \times R/J$$

defined by $r \mapsto (r+I, r+J)$ is a ring homomorphism with kernel $I \cap J$ and image $\{(r+I, s+J) \mid r-s \in I+J\}$. The well known Chinese Remainder Theorem (in the following CRT) ensures that ψ is surjective if and only if $I+J=R$.

More generally, if $I_1, \dots, I_n$ are ideals of R we can define the homomorphism

$$\psi\colon R \to R/I_1 \times \cdots \times R/I_n$$

by $\psi(r) = (r+I_1, \dots, r+I_n)$. We will refer to the map ψ or to the map induced by ψ on $R/\cap_{i=1}^n I_n$ as to the CRT map.

In the following we will consider the CRT map when $R = \mathbb{Z}[x]$, $I = (f(x))$ and $J = (g(x))$. If $f(x)$ and $g(x)$ are coprime polynomials, then $I \cap J = IJ = (f(x)g(x))$, so the CRT map

$$\psi\colon \mathbb{Z}[x]/(f(x)g(x)) \to \mathbb{Z}[x]/(f(x)) \times \mathbb{Z}/(g(x))$$

is an injection and it is an isomorphism if and only if $(f(x), g(x)) = \mathbb{Z}[x]$.

2.3. **Dirichlet's Unit Theorem.** Let K be a number field, and let $\mathcal{O}_K$ be its ring of integers; the classical Dirichlet's Theorem describes the groups of units of all *orders* of K (we recall that an order of K is a subring of $\mathcal{O}_K$ which spans K over $\mathbb{Q}$).

Proposition 2.3 (Dirichlet's Unit Theorem)**.** *Let K be a number field such that $[K:\mathbb{Q}] = n$ and assume that among the n embeddings of K in $\bar{\mathbb{Q}}$, r are real (namely map K into $\mathbb{R}$) and $2s$ are non-real ($n = r+2s$). Let R be an order of K. Then*

$$R^* \cong T \times \mathbb{Z}^{r+s-1}$$

where T is the group of the roots of unity contained in R.

For a proof see [Neu99, Ch.1,§12].

2.4. **Cyclotomic polynomials.** For $n \geq 1$ let $\zeta_n = e^{2\pi i/n}$, then ζ_n is a primitive n-th root of unity. Denote by $\Phi_n(x)$ its minimal polynomial over $\mathbb{Q}$: as it is well known, $\Phi_n(x) \in \mathbb{Z}[x]$ and

$$\Phi_n(x) = \prod_{\substack{j=1,\dots,n \\ (j,n)=1}} (x - \zeta_n^j).$$

Moreover, $\mathbb{Q}(\zeta_n)$ is a Galois extension of $\mathbb{Q}$ of degree $\phi(n)$, where ϕ is the Euler totient function, and its ring of integers is $\mathbb{Z}[\zeta_n]$.

The roots of unity contained in $\mathbb{Z}[\zeta_n]$ are the n-th roots of unity if n is even and the $2n$-roots of unity if n is odd and by Dirichlet's Unit Theorem $\mathbb{Z}[\zeta_n]^* \cong \langle -\zeta_n \rangle \times \mathbb{Z}^{\frac{\phi(n)}{2}-1}$ for each $n \geq 3$. In the following we will use the notation $(\frac{\phi(n)}{2} - 1)^*$ for the rank of $\mathbb{Z}[\zeta_n]^*$, namely, $(\frac{\phi(n)}{2} - 1)^* = \frac{\phi(n)}{2} - 1$ for $n \geq 3$ and $(\frac{\phi(n)}{2} - 1)^* = 0$ for $n = 1, 2$. We will omit the $*$ when $n > 2$.

In this paper we will need the following classical property of cyclotomic fields and cyclotomic polynomials. Most of the results could be generalized, but we give only those necessary for our purposes.

Lemma 2.4. .

(1) *Suppose that n has at least two distinct prime factors. Then $1-\zeta_n$ is a unit of $\mathbb{Z}[\zeta_n]$ and*

$$\Phi_n(1) = \prod_{\substack{j=1,\dots,n\\(j,n)=1}} (1-\zeta_n^j) = 1.$$

(2) *For p prime and $e>0$, then $1-\zeta_{p^e}$ is a generator of the prime ideal of $\mathbb{Z}[\zeta_{p^e}]$ lying over (p),*

$$p\mathbb{Z}[\zeta_{p^e}] = (1-\zeta_{p^e})^{\phi(p^e)}$$

and

$$\Phi_{p^e}(1) = \prod_{\substack{j=1,\dots,p^e\\(j,p^e)=1}} (1-\zeta_{p^e}^j) = p.$$

Proof. For part (1) see [Was87, Lemma 2.8]. For part (2) see [Lan94, IV, 1, Thm 1]. □

Lemma 2.5. *Let $l>1$ and let $\Psi_{n,l}(x)$ denote the minimal polynomial of ζ_n over $K=\mathbb{Q}(\zeta_l)$.*

(1) *Suppose that n has at least two distinct prime factors. Then the algebraic integer $\Psi_{n,l}(1)$ is a unit.*
(2) *If $n=p^a$, where p is a prime and $a>0$, and $l=l_1p^b$, with $(l_1,p)=1$ and $0\le b\le a$, then $\Psi_{p^a,l}=\Psi_{p^a,p^b}$ and $\Psi_{p^a,l}(1)$ is a generator of the prime ideal of $\mathbb{Z}[\zeta_{p^b}]$ lying over (p).*

Proof. $\Psi_{n,l}(x)$ divides $\Phi_n(x)$, hence $\Psi_{n,l}(1)$ is a unit since it divides the unit $\Phi_n(1)$ (this actually holds for any number field K).

For part (2) note that $\Psi_{p^a,l}(x)=\Phi_{p^a}$ if $b=0$ and $\Psi_{p^a,l}(x)=\Psi_{p^a,p^b}(x)=x^{p^{a-b}}-\zeta_{p^b}$ if $b>0$ (a divisibility relation is obvious and equality follows from a degree argument). It follows that $\Psi_{p^a,l}(1)$ is equal to p or $1-\zeta_{p^b}$ according to $b=0$ or $b>0$, namely it is a generator for the prime ideal of $\mathbb{Z}[\zeta_{p^b}]$ lying over (p). □

Lemma 2.6. *Let $n>m\ge 1$. The algebraic integer $\Phi_n(\zeta_m)$ is a unit in $\mathbb{Z}[\zeta_m]$ if n/m is not a prime power.*

In the case when $n/m=p^a$ for a prime p and an integer $a>0$, then $\Phi_n(\zeta_m)$ is associated to p.

Proof. The first part of the proof is [BHPM18, Corollary 8] (see also [Apo70]).

For the second part, we note that

$$\Phi_n(\zeta_m) = \prod_{\substack{j=1,\dots,n\\(j,n)=1}} (\zeta_m-\zeta_n^j) = \prod_{\substack{j=1,\dots,n\\(j,n)=1}} \zeta_m(1-\zeta_n^{j-p^a}).$$

From Lemma 2.4 we have that $(1-\zeta_n^{j-p^a})$ is invertible if $\frac{n}{(n,j-p^a)}$ is not a prime power. On the other hand, $\frac{n}{(n,j-p^a)}$ is a prime power only if it is a power of p and $j \equiv p^a \pmod{m_1}$, where $m = p^b m_1$ and $(m_1, p) = 1$. Taking into account that $(j, n) = 1$, an easy computation shows that there are $\phi(p^{a+b})$ values of j with this property. For these values $1 - \zeta_n^{j-p^a}$ is a generator of the ideal $(1-\zeta_{p^{a+b}})$, namely

$$(\Phi_n(\zeta_m)) = (1-\zeta_{p^{a+b}})^{\phi(p^{a+b})} = p\mathbb{Z}[\zeta_{p^{a+b}}].$$

□

In Sections 4 and 5 we will need to study the ring

$$\mathbb{Z}[x]/(\Phi_{m_1}(x)\dots\Phi_{m_r}(x))$$

when $m_1, \dots, m_r$ are distinct positive integers. Denote by ψ the CRT map, and, by abuse of notation, also its composition with the isomorphism given by the identifications $\mathbb{Z}[x]/(\Phi_{m_i}(x)) \cong \mathbb{Z}[\zeta_{m_i}]$, namely

$$\psi\colon \mathbb{Z}[x]/(\Phi_{m_1}(x)\cdots\Phi_{m_r}(x)) \to \prod_{i=1}^{r} \mathbb{Z}[x]/(\Phi_{m_i}(x)) \cong \prod_{i=1}^{r} \mathbb{Z}[\zeta_{m_i}]. \quad (1)$$

Then ψ is always an injection and we ask when it is also surjective.

The following lemma gives the answer for $r = 2$, in Proposition 2.8 we will give the general answer.

Lemma 2.7. *Let $n > m \geq 1$. The following are equivalent:*

i) $\psi\colon \mathbb{Z}[x]/(\Phi_m(x)\Phi_n(x)) \to \mathbb{Z}[\zeta_m]\times\mathbb{Z}[\zeta_n]$ *is an isomorphism.*
ii) $(\Phi_m(x), \Phi_n(x)) = \mathbb{Z}[x]$;
iii) $\Phi_n(\zeta_m)$ *is invertible;*
iv) n/m *is not a prime power.*

Proof. (i) is equivalent to (ii) by the CRT.

The equivalence between (ii) and (iii) follows from the following chain of isomorphisms

$$\frac{\mathbb{Z}[x]}{(\Phi_m(x), \Phi_n(x))} \cong \frac{\mathbb{Z}[x]/(\Phi_m(x))}{(\Phi_m(x), \Phi_n(x))/(\Phi_m(x))} \cong \frac{\mathbb{Z}[\zeta_m]}{(\Phi_n(\zeta_m))}.$$

Finally, Lemma 2.6 gives the equivalence between (iii) and (iv). □

Proposition 2.8. *Let $r \geq 2$ and $m_1 < \cdots < m_r$ be distinct positive integers. The following are equivalent*

a) the CRT map
$\psi\colon \mathbb{Z}[x]/(\Phi_{m_1}(x)\cdots\Phi_{m_r}(x)) \to \prod_{i=1}^{r}\mathbb{Z}[x]/(\Phi_{m_i}(x)) \cong \prod_{i=1}^{r}\mathbb{Z}[\zeta_{m_i}]$
is an isomorphism;
b) for all $1 \leq i < j \leq r$ the ratio m_j/m_i is not a prime power.

Proof. For each t with $2 \leq t \leq r$, consider the CRT map's

$$\psi_t\colon \mathbb{Z}[x]/(\Phi_{m_1}(x)\cdots\Phi_{m_t}(x)) \to \prod_{i=1}^{t}\mathbb{Z}[x]/(\Phi_{m_i}(x))$$

and

$\rho_t\colon \mathbb{Z}[x]/(\Phi_{m_1}(x)\cdots\Phi_{m_t}(x)) \to \mathbb{Z}[x]/(\Phi_{m_1}(x)\cdots\Phi_{m_{t-1}}(x))\times\mathbb{Z}[x]/(\Phi_{m_t}(x))$.

With this notation, we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[x]/(\prod_{i=1}^{t}\Phi_{m_i}(x)) & \xrightarrow{\psi_t} & \prod_{i=1}^{t}\mathbb{Z}[x]/(\Phi_{m_i}(x)) \\ \downarrow{\scriptstyle \rho_t} & \nearrow{\scriptstyle \psi_{t-1}\times id} & \\ \mathbb{Z}[x]/(\prod_{i=1}^{t-1}\Phi_{m_i}(x))\times\mathbb{Z}[x]/(\Phi_{m_t}(x)) & & \end{array} \tag{2}$$

namely,

$$\psi_t = (\psi_{t-1}\times id)\circ\rho_t. \tag{3}$$

We will prove that (a) is equivalent to (b) by induction on r.

For $r = 2$ the equivalence is given in Lemma 2.7. We now assume that $r > 2$ and that the equivalence holds in the case of $r-1$ integers $m_1 < \cdots < m_{r-1}$ and we prove it for $m_1 < \cdots < m_{r-1} < m_r$.

Assume (a), so $\psi = \psi_r$ is an isomorphism. From equation (3) we get that $\psi_{r-1}\times id$ is surjective; this ensures that also ψ_{r-1} is surjective and hence it is an isomorphism since it is always injective. Therefore, by inductive hypothesis we get that m_j/m_i is not a prime power for $1 \le i < j \le r-1$ and we are left to prove that m_r/m_i is not a prime power for $1 \le i < r$. We note that since both ψ_r and ψ_{r-1} are isomorphisms, equation (3) ensures that also the CRT map ρ_r is an isomorphism so $(\Phi_{m_1}(x)\cdots\Phi_{m_{r-1}}(x),\Phi_{m_r}(x)) = \mathbb{Z}[x]$, which in turns implies $(\Phi_{m_i}(x),\Phi_{m_r}(x)) = \mathbb{Z}[x]$ for each $1 \le i < r$. By Lemma 2.7 the last condition ensures that, for each i, the ratio m_r/m_i is not a prime power, proving (b).

Conversely, assume that (b) holds, then by applying the inductive hypothesis to $m_1 < \cdots < m_{r-1}$ we get that ψ_{r-1} is an isomorphism. On the other hand, since m_r/m_i is not a prime power, by Lemma 2.7 we get $(\Phi_{m_i}(x),\Phi_{m_r}(x)) = \mathbb{Z}[x]$ for all $i = 1,\dots,r-1$, so there exist $a_i(x), b_i(x) \in \mathbb{Z}[x]$ such that

$$a_i(x)\Phi_{m_i}(x) + b_i(x)\Phi_{m_r}(x) = 1.$$

Multiplying the $r-1$ equations we get

$$a(x)\Phi_{m_1}(x)\cdots\Phi_{m_{r-1}}(x) + b(x)\Phi_{m_r}(x) = 1,$$

for some $a(x), b(x) \in \mathbb{Z}[x]$, or equivalently

$$(\Phi_{m_1}(x)\dots\Phi_{m_{r-1}}(x),\Phi_{m_r}(x)) = \mathbb{Z}[x].$$

This ensures that the CRT map

$\rho_r\colon \mathbb{Z}[x]/(\Phi_{m_1}(x)\dots\Phi_{m_r}(x)) \cong \mathbb{Z}[x]/(\Phi_{m_1}(x)\dots\Phi_{m_{r-1}}(x))\times\mathbb{Z}[x]/(\Phi_{m_r}(x))$

is an isomorphism and using equation (3) we can conclude that also ψ_r is an isomorphism, proving (a). $\square$

We conclude this section with an arithmetical lemma which will be useful in Proposition 5.5.

Lemma 2.9. *Let $q_1, \dots, q_k$ be pairwise distinct odd primes and let $e_1, \dots e_k > 0$. Then, for $\delta \geq 1$*

$$\frac{\phi(2^\delta q_1^{e_1} \cdots q_k^{e_k})}{2} - 1 \geq \sum_{i=1}^{k} (\frac{\phi(2^\delta q_i^{e_i})}{2} - 1) \tag{4}$$

and, if $\delta \geq 2$,

$$\frac{\phi(2^\delta q_1^{e_1} \cdots q_k^{e_k})}{2} - 1 \geq \sum_{i=1}^{k} (\frac{\phi(2^{\delta-1} q_i^{e_i})}{2} - 1) + \frac{\phi(2^\delta)}{2} - 1. \tag{5}$$

Proof. Both inequalities are trivial for $k = 0$, so let $k \geq 1$. Since $\phi(q_i^{e_i}) \geq 2$ for all i, the obvious relation $mn \geq m + n$ for all $m, n \geq 2$, gives

$$\phi(q_1^{e_1} \cdots q_k^{e_k}) = \prod_{i=1}^{k} \phi(q_i^{e_i}) \geq \sum_{i=1}^{k} \phi(q_i^{e_i}),$$

from which we get

$$\phi(2^\delta q_1^{e_1} \cdots q_k^{e_k}) = 2^{\delta-1} \phi(q_1^{e_1} \cdots q_k^{e_k}) \geq \sum_{i=1}^{k} 2^{\delta-1} \phi(q_i^{e_i}) = \sum_{i=1}^{k} \phi(2^\delta q_i^{e_i}) \tag{6}$$

and (4) follows.

On the other hand, if $\delta \geq 2$

$$\phi(2^\delta q_1^{e_1} \cdots q_k^{e_k}) = \phi(2^{\delta-1} q_1^{e_1} \cdots q_k^{e_k}) + \phi(2^{\delta-1} q_1^{e_1} \cdots q_k^{e_k});$$

using (6) on both summands and then the trivial estimate $\phi(2^{\delta-1} q_i^{e_i}) \geq \phi(2^\delta)$, we get

$$\phi(2^\delta q_1^{e_1} \cdots q_k^{e_k}) \geq \sum_{i=1}^{k} \phi(2^{\delta-1} q_i^{e_i}) + \sum_{i=1}^{k} \phi(2^{\delta-1} q_i^{e_i}) \geq \sum_{i=1}^{k} \phi(2^{\delta-1} q_i^{e_i}) + k\phi(2^\delta)$$

and (5) follows a fortiori. □

3. Integral domains

In this section we characterize the finitely generated groups which occur as group of units of an integral domain of any characteristic and in Proposition 3.3 those which are the group of units of integral extensions of $\mathbb{Z}$.

Theorem 3.1. *The finitely generated abelian groups that occur as groups of units of integral domains of characteristic zero are the groups of the form $C_{2n} \times \mathbb{Z}^g$, with $n \in \mathbb{N}$, $g \geq \frac{\phi(2n)}{2} - 1$.*

Proof. Suppose A is an integral domain of characteristic zero whose group of units A^* is finitely generated, so that $A^* \cong T \times \mathbb{Z}^{g_A}$ where T denotes the (finite) torsion subgroup. Let K be the quotient field of A, then T is a finite multiplicative subgroup of K^*, hence it is a cyclic group.

As noted in Section 2, the ring $B = \mathbb{Z}[T]$ has group of units isomorphic to $T \times \mathbb{Z}^{g_B}$ with $g_B \leq g_A$. Hence, to prove that A^* has the required form it is enough to restrict to the case when $A = B$, namely it is finitely generated and integral over $\mathbb{Z}$. In this case, its quotient field K is a number field and A is an order of K. By Dirichlet's Unit Theorem $A^* \cong T \times \mathbb{Z}^{r+s-1}$ where T is the (cyclic) group of roots of unity contained in A and r and $2s$ are the number of real and non-real embeddings of K, respectively. Clearly, $|T|$ is even since $-1 \in A^*$. Let $T = \langle \zeta_{2n} \rangle$, then $\mathbb{Z}[\zeta_{2n}] \subseteq A$, so $\mathbb{Q}(\zeta_{2n}) \subseteq K$. For $n = 1$ we have nothing to prove. If $n > 1$, then all embeddings of K in $\bar{\mathbb{Q}}$ must be non-real, so $r = 0$ and $2s = [K : \mathbb{Q}]$. Since $\mathbb{Q}(\zeta_{2n}) \subseteq K$ then $\frac{\phi(2n)}{2} \mid s$ so the rank of A^* is $g = s - 1 \geq \frac{\phi(2n)}{2} - 1$.

As to the converse, let $n \geq 1$ and let $K = \mathbb{Q}(\zeta_{2n})$. Then $\mathcal{O}_K^* \cong C_{2n} \times \mathbb{Z}^{\frac{\phi(2n)}{2}-1}$ and for any $k \geq 1$ the ring of Laurent polynomials in k indeterminates $\mathcal{O}_K[x_1, \dots, x_k, x_1^{-1}, \dots, x_k^{-1}]$ has group of units isomorphic to $C_{2n} \times \mathbb{Z}^{\frac{\phi(2n)}{2}-1+k}$. □

As a corollary we recover the characterization of the finite abelian groups which are groups of units of an integral domain.

Corollary 3.2. *The finite abelian groups that occur as groups of units of integral domains of characteristic 0 are the cyclic groups of order 2,4, or 6.*

Proof. From Theorem 3.1 we know that if A is a domain such that A^* is finitely generated, then $A^* \cong C_{2n} \times \mathbb{Z}^g$ with $g \geq (\frac{\phi(2n)}{2} - 1)^*$, so we can have $g = 0$ only for $n = 1, 2, 3$. □

In Theorem 3.1 we have seen that among the rings with finitely generated group of units and torsion subgroup isomorphic to C_{2n}, the ring $A = \mathbb{Z}[\zeta_{2n}]$ has the minimum possible rank. The example of rings whose group of units has the same torsion subgroup, but a greater rank are constructed in the theorem by localizing polynomial rings. In particular, the rings of our examples are no longer integral over $\mathbb{Z}$. Actually, only some of these groups can also be obtained with units that are integral over $\mathbb{Z}$. The following proposition characterizes these cases.

Proposition 3.3. *The finitely generated abelian groups that can be realized as group of units of an integral domain A, with A integral over $\mathbb{Z}$, are the groups of the type $C_{2n} \times \mathbb{Z}^g$, with $n \geq 1$, $g \geq 0$ and $\phi(2n) \mid 2(g+1)$.*

Proof. Up to replacing A with $\mathbb{Z}[A^*]$ we can assume that its quotient field K is a number field and that A is an order of K. Then the necessity of the condition follows from Theorem 3.1 and from its proof, where it is shown that $\phi(2n)$ divides $2s = 2(g+1)$.

As for the converse, we have to construct examples of orders in number fields realizing all the listed groups. One possible construction is the following.

For $n = 1$ and $d \geq 1$, let m be any integer such that $2d|\phi(m)$. This condition guarantees that the field $\mathbb{Q}(\zeta_m + \zeta_m^{-1})$ contains a subfield K_d of degree d over $\mathbb{Q}$. Cleary, K_d is totally real, so $r = d$, $s = 0$ and the only roots of unity in K_d are ± 1, hence the group of units of the integers of K_d is isomorphic to $C_2 \times \mathbb{Z}^{d-1}$.

Consider now the case $n > 1$. Let $d \geq 1$ and let p be a prime such that

$$p \equiv 1 \pmod{2d}; \tag{7}$$

since there are infinitely many such primes (see for example [Was87, Corollary 2.11] or use Dirichlet's Prime Number Theorem) we can assume $p \nmid n$. The congruence condition guarantees that inside the cyclotomic extension $\mathbb{Q}(\zeta_p)$ there is a (unique) subextension, $K_{d,p}$, of degree d over $\mathbb{Q}$, which is indeed contained in the real subfield $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$. Put $L = L_{d,p,n} = K_{d,p}\mathbb{Q}(\zeta_{2n})$ and denote by $\mathcal{O}_L = \mathcal{O}_{L_{d,p,n}}$ its ring of integers. We claim that

$$\mathcal{O}_L^* \cong C_{2n} \times \mathbb{Z}^{\frac{d\phi(2n)}{2}-1}.$$

In fact, $(\mathcal{O}_L^*)_{tors} = \langle \zeta_{2n} \rangle$ since $\zeta_{2n} \in \mathcal{O}_L^*$, $\zeta_p \notin \mathcal{O}_L^*$.

To compute the rank of $\mathcal{O}_L^*$, we note that $\mathbb{Q}(\zeta_p)$ is arithmetically disjoint from $\mathbb{Q}(\zeta_{2n})$ since $(p, 2n) = 1$, hence also $K_{d,p}$ is arithmetically disjoint from $\mathbb{Q}(\zeta_{2n})$ and $[L : \mathbb{Q}] = [K_{d,p} : \mathbb{Q}][\mathbb{Q}(\zeta_{2n}) : \mathbb{Q}] = d\phi(2n)$. Moreover, L is Galois over $\mathbb{Q}$ and all its embeddings are non-real, so the rank of its group of units is $s - 1 = \frac{d\phi(2n)}{2} - 1$. □

To complete the description of the finitely generated groups of units of integral domains, in the following theorem we present the simple result for finite characteristic rings.

Theorem 3.4. *The finitely generated abelian groups that occur as groups of units of an integral domain of characteristic p are the groups of the form* $\mathbb{F}_{p^n}^* \times \mathbb{Z}^g$ *with* $n \geq 1$ *and* $g \geq 0$.

Proof. Let A be a domain and let $A^* \cong (A^*)_{tors} \times \mathbb{Z}^g$ with $(A^*)_{tors}$ finite and $g \geq 0$. By Lemma 2.1, for $B = \mathbb{F}_p[(A^*)_{tors}]$ we have $B^* = (A^*)_{tors}$. Now, B is a finite integral domain (it is a finitely generated integral extension of $\mathbb{F}_p$), whence it is a finite field, namely $B \cong \mathbb{F}_{p^n}$ for some $n \geq 1$. It follows that $(A^*)_{tors} = B^* \cong \mathbb{F}_{p^n}^*$, and $A^* \cong \mathbb{F}_{p^n}^* \times \mathbb{Z}^g$ as required.

Conversely, for $n \geq 1$ and $g \geq 0$, the group $\mathbb{F}_{p^n}^* \times \mathbb{Z}^g$ is isomorphic to the group of units of the ring of Laurent polynomials with coefficients in $\mathbb{F}_{p^n}$ and g indeterminates. □

4. Torsion-free rings: preliminary results

A commutative ring A is called torsion-free if its only element of finite additive order is 0. Clearly, a torsion-free ring has characteristic zero.

For a torsion-free ring A we put $Q_A = A \otimes_{\mathbb{Z}} \mathbb{Q}$. We note that in this case the map

$$\iota \colon A \to Q_A$$

defined by $a \mapsto a \otimes 1$ is an embedding, so we will say that $A \subseteq Q_A$.

As noted in Section 2 (Lemma 2.1 and Remark 2.2), to characterize the finitely generated abelian groups $T \times \mathbb{Z}^g$ that arise as groups of units of torsion-free rings, a substantial step is the study of the subrings that are generated over $\mathbb{Z}$ by units of finite order. In fact, in this subclass all possible torsion subgroups T are realized and, for each T, the minimum possible rank $g(T)$ is attained. This case is much easier to study since if A is integral over $\mathbb{Z}$ then Q_A is a finite dimensional $\mathbb{Q}$-algebra and A is an order of Q_A. In this section and in the first part of the next one we will restrict to this case; then it will be easy to deal with the general case.

The following lemma allows us to describe the ring A when it is generated by one torsion unit and it is a generalization of [DCD18b, Lemma 4.2].

Lemma 4.1. *Let K be a number field and let $\mathcal{O}_K$ be its ring of integers. Assume that $\mathcal{O}_K \subseteq A$. Let $\alpha \in A^*$ be an element of order n, let*

$$\varphi_\alpha \colon \mathcal{O}_K[x] \to A$$

be the evaluation homomorphism $p(x) \mapsto p(\alpha)$.

Then $\ker(\varphi_\alpha) = (\mu_\alpha(x))$ with

$$\mu_\alpha(x) = \Psi_{m_1}(x) \cdots \Psi_{m_r}(x)$$

where, for each i, $\Psi_{m_i}(x) \in \mathcal{O}_K[x]$ denotes the minimal polynomial over K of a primitive m_i-th root of unity. Moreover, the $\Psi_{m_i}(x)$'s are pairwise distinct and $[m_1, \ldots, m_r] = lcm\{m_1, \ldots, m_r\} = n$.

Proof. The element α has order n, so $x^n - 1 \in \ker(\varphi_\alpha)$. Denote by $\tilde{\varphi}_\alpha \colon K[x] \to Q_A$ the extension of φ_α. Then, there exists a monic polynomial $\mu_\alpha(x) \in K[x]$ such that $\ker(\tilde{\varphi}_\alpha) = (\mu_\alpha(x))$. Clearly, $\mu_\alpha(\alpha) = 0$ and $\mu_\alpha(x)$ divides the separable polynomial $x^n - 1$ in $K[x]$,

$$\mu_\alpha(x) \mid (x^n - 1) = \prod_{m|n} \Phi_m(x).$$

Now, each Φ_m factors as a product of distinct cyclotomic polynomials over K, hence $\mu_\alpha(x)$ factors in $K[x]$ as

$$\mu_\alpha(x) = \Psi_{m_1}(x)\cdots\Psi_{m_r}(x),$$

where $\Psi_{m_i}(x)$ denotes the minimal polynomial over K of a primitive m_i-th root of unity. The $\Psi_{m_i}(x)$'s are pairwise distinct since x^n-1 is separable; moreover, $\Psi_{m_i}(x)\in\mathcal{O}_K[x]$ for all i, so $\mu_\alpha(x)\in\mathcal{O}_K[x]$ and $\mu_\alpha(x)\in\ker(\varphi_\alpha)$.

On the other hand, for each $f(x)\in\ker(\varphi_\alpha)$ we have $\mu_\alpha(x)|f(x)$ in $K[x]$ and since $\mu_\alpha(x)\in\mathcal{O}_K[x]$ is a monic polynomial, then it divides $f(x)$ in $\mathcal{O}_K[x]$. This proves that $\ker(\varphi_\alpha)=(\mu_\alpha(x))$.

Let $[m_1,\dots,m_r]=m$. Since $m_i|n$ for all i, then $m\mid n$. In fact $m=n$, since otherwise $\mu_\alpha(x)\mid x^m-1$ and therefore $\alpha^m=1$, contrary to our assumption. □

Proposition 4.2. *Let $A=\mathbb{Z}[\alpha_1,\dots,\alpha_s]$, where, for all i, α_i is a unit of finite order and assume that A is torsion free. Then the $\mathbb{Q}$-algebra $Q_A = A\otimes_{\mathbb{Z}}\mathbb{Q}$ is a finite direct product of cyclotomic fields. In particular, Q_A is a semisimple $\mathbb{Q}$-algebra.*

Proof. For $\alpha=\alpha_i$, in the notation of Lemma 4.1, let $\ker(\varphi_\alpha)=(\mu_\alpha(x))$ and assume

$$\mu_\alpha(x)=\Phi_{m_1}(x)\cdots\Phi_{m_r}(x)$$

for some distinct $m_1,\dots m_r$. Then the CRT gives

$$\mathbb{Q}[\alpha]=\mathbb{Z}[\alpha]\otimes_{\mathbb{Z}}\mathbb{Q}\cong\mathbb{Q}[x]/(\mu_\alpha(x))\cong\prod_{i=1}^r\mathbb{Q}[x]/(\Phi_{m_i}(x))\cong\prod_{i=1}^r\mathbb{Q}(\zeta_{m_i}).$$

Now, the degree of ζ_m over $\mathbb{Q}(\zeta_n)$ is $\phi(m)/\phi((n,m))$, so m-th cyclotomic polynomial $\Phi_m(x)$ splits into $\phi((n,m))$ of factors in $\mathbb{Q}(\zeta_n)[x]$. It follows that

$$\mathbb{Q}(\zeta_n)\otimes_{\mathbb{Q}}\mathbb{Q}(\zeta_m)\cong\mathbb{Q}(\zeta_n)[x]/(\Phi_m(x))\cong\mathbb{Q}(\zeta_{[n,m]})^{\phi((n,m))},$$

so the $\mathbb{Q}$-algebra $\mathcal{Q}=\mathbb{Q}[\alpha_1]\otimes_{\mathbb{Q}}\cdots\otimes_{\mathbb{Q}}\mathbb{Q}[\alpha_s]$ is a product of cyclotomic fields. It turn out that the same is true for $Q_A=\mathbb{Q}[\alpha_1,\dots,\alpha_s]$ since it is the epimorphic image of $\mathcal{Q}$ via the $\mathbb{Q}$-algebra homomorphism defined by $\alpha_1\otimes\cdots\otimes\alpha_s\mapsto\alpha_1\cdots\alpha_s$. □

Remark 4.3. The last proposition shows that the $\mathbb{Q}$-algebra Q_A is isomorphic to $\prod_{i=1}^t\mathbb{Q}(\zeta_{n_i})$ for some $n_1,\dots,n_t$, namely, it is semisimple and of finite dimension over the perfect field $\mathbb{Q}$, hence it is separable (see for example [CR81, Cor. 7.6]). Moreover, Q_A is clearly commutative, so by [CR81, Prop. 26.10] it has a unique maximal order $\mathcal{M}_A$, which is the integral closure of $\mathbb{Z}$ in Q_A, namely

$$\mathcal{M}_A\cong\prod_{i=1}^t\mathbb{Z}[\zeta_{n_i}].$$

Since A is an order of Q_A, then A is a subring of $\mathcal{M}_A$, therefore the rings we are taking into account are subrings of finite products of cyclotomic rings.

The next lemma shows that the groups of units of all orders of Q_A have the same rank (see also [Seh93, Prop. 2.5] or [BL17, Lemma 3.7]).

Lemma 4.4. *Let R be an order of a commutative and finitely generated semisimple $\mathbb{Q}$-algebra Q and let $\mathcal{M}$ denote its maximal order. Then R^* has the same rank of $\mathcal{M}^*$.*

Proof. Each order R of Q is a subring of finite index of $\mathcal{M}$, since both are $\mathbb{Z}$-modules of the same finite rank. Let $[\mathcal{M} : R] = m$, then the ideal $m\mathcal{M}$ is contained in R and $\mathcal{M}/m\mathcal{M}$ is a finite ring.

Consider the projection $\pi\colon \mathcal{M} \to \mathcal{M}/m\mathcal{M}$. Since π is a ring homomorphism, it sends the unit of $\mathcal{M}$ into the unit of the quotient and the restriction of $\pi\colon \mathcal{M}^* \to (\mathcal{M}/m\mathcal{M})^*$ is a group homomorphism.

Let $|(\mathcal{M}/m\mathcal{M})^*| = c$. For each $\varepsilon \in \mathcal{M}^*$ we have that $\varepsilon^c \equiv 1 \pmod{m\mathcal{M}}$ so $\varepsilon^c - 1 \in m\mathcal{M} \subset R$. Now, R and $\mathcal{M}$ have the same identity, hence $\varepsilon^c \in R$ and $(\mathcal{M}^*)^c \subseteq R^* \subseteq \mathcal{M}^*$. Finally, since $(\mathcal{M}^*)^c$ and $\mathcal{M}^*$ have the same rank, this is also the rank of R^*. □

Corollary 4.5. *In the notation of Proposition 4.2, let $Q_A = \prod_{i=1}^t \mathbb{Q}(\zeta_{n_i})$. Then, $A^* \cong T \times \mathbb{Z}^g$ where*

$$g = \sum_{i=1}^t \left(\frac{\phi(n_i)}{2} - 1\right)^*$$

and T is a subgroup of even order of $U = \prod_{i=1}^t \langle -\zeta_{n_i} \rangle$.

Proof. The order A is contained in the maximal order $\mathcal{M}_A \cong \prod_{i=1}^t \mathbb{Z}[\zeta_{n_i}]$, hence $\{\pm 1\} < A^* < \mathcal{M}_A^*$ and by Lemma 4.4 the two groups have the same rank g. The result follows since

$$\mathcal{M}_A^* \cong \prod_{i=1}^t \mathbb{Z}[\zeta_{n_i}]^* \cong \prod_{i=1}^t \left(\langle -\zeta_{n_i} \rangle \times \mathbb{Z}^{(\frac{\phi(n_i)}{2} - 1)^*} \right) \cong U \times \mathbb{Z}^g.$$

□

The next proposition classifies the cases when $\mathbb{Z}[\alpha]$ coincides with $\mathcal{M}$.

Proposition 4.6. *Let $\alpha \in A^*$ be an element of finite order. Denote by $\varphi_\alpha\colon \mathbb{Z}[x] \to A$ the evaluation homomorphism and let $\mu_\alpha(x) = \Phi_{m_1}(x) \dots \Phi_{m_r}(x)$ be a generator of $\ker(\varphi_\alpha)$. Then*

$$\mathbb{Z}[\alpha] \cong \prod_{i=1}^r \mathbb{Z}[\zeta_{m_i}]$$

if and only if , for all i, j, the ratio m_i/m_j is not a prime power.

In this case

$$\mathbb{Z}[\alpha]^* \cong \prod_{i=1}^{r} \langle -\zeta_{m_i} \rangle \times \mathbb{Z}^{\sum_{i=1}^{r}(\frac{\phi(m_i)}{2}-1)^*}.$$

Proof. Consider the following commutative diagram, where the vertical arrows are the obvious isomorphisms

$$\begin{array}{ccc} \mathbb{Z}[\alpha] & \hookrightarrow & \mathcal{M} = \prod_{i=1}^{r} \mathbb{Z}[\zeta_{m_i}] \\ \cong \uparrow & & \cong \uparrow \\ \mathbb{Z}[x]/(\prod_{i=1}^{r} \Phi_{m_i}(x)) & \overset{\psi}{\hookrightarrow} & \prod_{i=1}^{r} \mathbb{Z}[x]/(\Phi_{m_i}(x)) \end{array} \tag{8}$$

The diagram shows that $\mathbb{Z}[\alpha] = \mathcal{M}$ if and only if the CRT map is onto and this is classified in Proposition 2.8. The description of $\mathbb{Z}[\alpha]^*$ follows immediately. ☐

Example 1. Let $\mathcal{M} = \mathbb{Z}[\zeta_3] \times \mathbb{Z}[i]$ and let $\alpha = (\zeta_3, i) \in \mathcal{M}$. The element α is a unit of order 12, $\mu_\alpha(x) = \Phi_3(x)\Phi_4(x)$ and $\mathbb{Z}[\alpha] \cong \mathbb{Z}[x]/(\Phi_3(x)\Phi_4(x))$. By last proposition $\mathbb{Z}[\alpha] \cong \mathcal{M}$ and $(\mathbb{Z}[\alpha])^*_{tors} = (\mathcal{M}^*)_{tors} \cong C_6 \times C_4$.

Example 2. Let $\mathcal{M} = \mathbb{Z}[\zeta_3] \times \mathbb{Z}[\zeta_9]$ and let $\alpha = (\zeta_3, \zeta_9) \in \mathcal{M}$. Clearly, α is a unit of order 9 and $\mathbb{Z}[\alpha] \cong \mathbb{Z}[x]/(\Phi_3(x)\Phi_9(x))$. Proposition 4.6 shows that $\mathbb{Z}[\alpha] \subsetneq \mathcal{M}$ and it is easy to see that $(\mathbb{Z}[\alpha])^*_{tors} \cong C_9$, in fact $(\zeta_3, 1) \notin \mathbb{Z}[\alpha]$.

In the following proposition we compute the groups of units of torsion free rings of a particular form which will be useful in the next section. Actually, using the results of this section together with those of §2.4 one could prove more general results, substantially with the same methods, but this would require a greater technical effort. However, this is beyond our scope, so we decided to limit the generality to what is necessary for our application.

Proposition 4.7. *Let p be a prime and let l be a positive even integer such that $l = l_1 p^b$ with $(l_1, p) = 1$. Let $a > b$ and let $\Psi_{p^a,p^b}(x)$ denote the minimal polynomial of ζ_{p^a} over $\mathbb{Z}[\zeta_{p^b}]$.*[†] *Then*

$$\left(\frac{\mathbb{Z}[\zeta_l][x]}{((x-1)\Psi_{p^a,p^b}(x))}\right)^* \cong C_l \times C_{p^a} \times \mathbb{Z}^g$$

where $g = (\frac{\phi(l)}{2} - 1)^* + (\frac{\phi(l_1 p^a)}{2} - 1)$.

Proof. The ring $\mathbb{Z}[\zeta_l][x]/((x-1)\Psi_{p^a,p^b}(x))$ embeds into the maximal order $\mathcal{M} = \mathbb{Z}[\zeta_l] \times \mathbb{Z}[\zeta_l][\zeta_{p^a}] \cong \mathbb{Z}[\zeta_l] \times \mathbb{Z}[\zeta_{l_1 p^a}]$ via the CRT map:

$$\psi\colon \mathbb{Z}[\zeta_l][x]/((x-1)\Psi_{p^a,p^b}(x)) \to \mathbb{Z}[\zeta_l] \times \mathbb{Z}[\zeta_l][x]/(\Psi_{p^a,p^b}(x)) \cong \mathcal{M},$$

[†] $\Psi_{p^a,p^b}(x)$ is also the minimal polynomial of ζ_{p^a} over $\mathbb{Z}[\zeta_l]$

then

$$\operatorname{rank}\left(\left(\frac{\mathbb{Z}[\zeta_l][x]}{((x-1)\Psi_{p^a,p^b}(x))}\right)^*\right) = (\frac{\phi(l)}{2}-1)^* + (\frac{\phi(l_1p^a)}{2}-1).$$

As for the torsion units, let

$$T = \psi\left(\left(\frac{\mathbb{Z}[\zeta_l][x]}{((x-1)\Psi_{p^a,p^b}(x))}\right)^*_{tors}\right).$$

Clearly, T is the subgroup of $U = \langle\zeta_l\rangle \times \langle\zeta_{l_1p^a}\rangle \cong C_l \times C_{l_1p^a}$ made by the units belonging to $\mathrm{Im}(\psi) = \{(a(1), a(\zeta_{p^a})) \mid a(x) \in \mathbb{Z}[\zeta_l][x]\}$. We will show that all of them are *trivial units*, namely they belong to the subgroup T_0 generated by $\psi(\zeta_l) = (\zeta_l, \zeta_l)$ and $\psi(x) = (1, \zeta_{p^a})$. We note that $T_0 \cong C_l \times C_{p^a}$, since $\langle(\zeta_l, \zeta_l)\rangle \cap \langle(1, \zeta_{p^a})\rangle = (1, 1)$.

Let $u = (\zeta_l^i, \zeta_l^j\zeta_{p^a}^k) \in U$, then u is equivalent to $v = (\zeta_l^{i-j}, 1)$ modulo T_0, so $u \in T$ if and only if $v - (1,1) = (\zeta_l^{i-j} - 1, 0) \in \mathrm{Im}(\psi)$.

This means that there exists $a(x) \in \mathbb{Z}[\zeta_l][x]$ such that

$$\zeta_l^{i-j} - 1 = a(1)\Psi_{p^a,p^b}(1).$$

By Lemma 2.5, $(\Psi_{p^a,p^b}(1)) = P_b$ where $P_b = (1 - \zeta_{p^b})$ if $b \geq 1$ and $P_0 = (p)$, hence last equation implies

$$\zeta_l^{i-j} - 1 \in P_b. \tag{9}$$

Let $\nu = l/(l, i-j)$, then (9) can be rewritten as $\zeta_\nu - 1 \in P_b$ and, using Lemma 2.4, we get that this holds if and only if $\nu | p^b$.

If $b \geq 1$, $\nu \mid p^b$ exactly when $i \equiv j \pmod{l_1}$. Let $j = i + hl_1$, then $u = (\zeta_l^i, \zeta_l^i\zeta_{p^b}^h\zeta_{p^a}^k)$ and clearly this element is in T_0.

If $b = 0$ equation (9) can hold only for $p = 2$, so $\nu = 1$ or 2 and $i \equiv j \pmod{l_1 2^{b-1}}$. Letting $j = i + tl_1 2^{b-1}$ $(t = 0, 1)$ the unit $u = (\zeta_l^i, (-1)^t\zeta_l^i\zeta_{2^a}^k) = (\zeta_l^i, \zeta_l^i\zeta_{2^a}^{k+t2^{a-1}})$ and clearly it belongs to T_0.

This proves that $T = T_0$ and hence it has the required decomposition. □

5. Torsion-free rings: the classification theorem

Our aim is to classify the abelian and finitely generated groups which arise as groups of units of torsion-free rings. This question is twofold: on the one hand, we have to establish which finite groups T (up to isomorphism) can be the torsion subgroup of A^* when A is a torsion-free ring. On the other hand, we have to determine the possible values of the rank, $g(A)$, of A^* when $(A^*)_{tors} \cong T$. Theorem 5.1 gives a complete answer to both questions.

Let T be a finite abelian group of even order. In this section we will use the following notation for the decomposition of T as a product of cyclic factors that we fix once and for all. We will refer to this notation as to the "standard" notation for T, or we will call (10) the "standard" decomposition of T

Standard notation for T. Let $\varepsilon = \varepsilon(T)$ be the minimum exponent of 2 in the decomposition of T as direct sum of cyclic groups. Then T can be uniquely written as

$$T \cong \prod_{i=1}^{s} C_{p_i^{a_i}} \times \prod_{\iota=1}^{\rho} C_{2^{\varepsilon_\iota}} \times C_{2^\varepsilon}^{\sigma} \tag{10}$$

where $s, \rho \geq 0$, $\sigma \geq 1$ and
- for all $i = 1, \ldots, s$ the p_i's are odd prime numbers not necessarily distinct and $a_i \geq 1$;
- $\varepsilon = \varepsilon(T) \geq 1$ and $\varepsilon_\iota > \varepsilon$ for all $\iota = 1, \ldots, \rho$.

Assume that $p_1, \ldots, p_{s_0}$ are the distinct primes in the set $\{p_1, \ldots, p_s\}$. Denoting by T_{p_i} the p_i-Sylow of T, for $i = 1, \ldots, s_0$, and by T_2 its 2-Sylow, we can also write T as

$$T \cong \prod_{i=1}^{s_0} T_{p_i} \times T_2. \tag{11}$$

As usual, we call the decomposition in (11) the Sylow decomposition.

Theorem 5.1. *Let T be a finite abelian group of even order. Referring to the "standard" notation for T, we define*

$$g(T) = \sum_{i=1}^{s} \left(\frac{\phi(2^\varepsilon p_i^{a_i})}{2} - 1\right) + \sum_{\iota=1}^{\rho} \left(\frac{\phi(2^{\varepsilon_\iota})}{2} - 1\right) + c(T) \tag{12}$$

where

$$c(T) = \begin{cases} (\sigma - s)(\frac{\phi(2^\varepsilon)}{2} - 1)^* & \text{for } s < \sigma \\ 0 & \text{for } s_0 \leq \sigma \leq s \\ (\frac{\phi(2^\varepsilon)}{2} - 1)^* & \text{for } \sigma < s_0. \end{cases}$$

Then there exists a torsion free ring A with

$$A^* \cong T \times \mathbb{Z}^r$$

if and only if $r \geq g(T)$.

As a particular case of this theorem we re-obtain the classification of finite groups which occur as groups of units of torsion-free rings, already found in [DCD18b, Thm 4.1].

Corollary 5.2. *The finite abelian groups which are the groups of units of torsion-free rings are all those of the form*

$$C_2^a \times C_4^b \times C_3^c$$

where $a, b, c \in \mathbb{N}$, $a + b \geq 1$ and $a \geq 1$ if $c \geq 1$.

Proof. A finite abelian group T of even order is the group of units of a torsion-free ring if and only if $g(T) = 0$. In the "standard" notation for T, this means that $\frac{\phi(2^\varepsilon p_i^{a_i})}{2} - 1 = 0$ for all $i = 1, \ldots, s$, $\frac{\phi(2^{\varepsilon_\iota})}{2} - 1 = 0$ for each $\iota = 1, \ldots, \rho$ and $c(T) = 0$. If $s = 0$ this gives $\varepsilon = 1$ and $\varepsilon_\iota \leq 2$

for all ι, or $\varepsilon = 2$ and $\rho = 0$. If $s > 0$, then $p_i = 3$ for all i, $\varepsilon = 1$ and $\varepsilon_\iota \leq 2$ for all ι. □

Before proceeding with the proof we point out that all the difficulties relative to the realization of a group T come from its 2-torsion part. The following examples show a phenomenon which at first sight may seem paradoxical: it may happen that a group T has a subgroup T' for which $g(T) < g(T')$.

Example 3. Let $T = C_2 \times C_8 \times C_5$. In this case $\varepsilon = 1$ and $g(T) = 2$: in fact, choosing A equal to the maximal order $\mathcal{M} = \mathbb{Z}[\zeta_8] \times \mathbb{Z}[\zeta_5]$ we have $A^* \cong T \times \mathbb{Z}^2$.

Example 4. Let $T \cong C_8 \times C_5$ and let A be a torsion-free ring such that $(A^*)_{tors} \cong T$. Then, A contains a unit α of order 8 and a unit β of order 5. Then in the notation of Lemma 4.1, we have that $\Phi_5(x) \mid \mu_\alpha(x)$ and $\Phi_8(x) \mid \mu_\beta(x)$, so $\mathcal{M}$, the maximal order of A, must contain a direct factor with a subring isomorphic to $\mathbb{Z}[\zeta_8]$ and one which contains $\mathbb{Z}[\zeta_5]$. There are two minimal possibilities: $\mathcal{M} = \mathbb{Z}[\zeta_8] \times \mathbb{Z}[\zeta_5]$ or $\mathcal{M} = \mathbb{Z}[\zeta_{40}]$. The first possibility has to be excluded since each order of a maximal order containing $\mathbb{Z}[\zeta_8] \times \mathbb{Z}[\zeta_5]$ has at least 3 units of order 2 (this will be clear after Lemma 5.3). In this case Theorem 5.1 shows that $g(T) = \phi(40)/2 - 1 = 7$.

The proof of Theorem 5.1 is quite long. For the convenience of the reader, we separate the “only if” part and the “if” part. Both parts require a number of auxiliary results that we will prove separately, in order to make it easier to follow the main argument.

5.1. **Proof of Theorem 5.1: the “only if” part.** Let A be a torsion free ring with finitely generated group of units, such that $(A^*)_{tors} \cong T$. We have to prove the rank$(A^*) \geq g(T)$.

To this aim, by Lemma 2.1, we can assume that $A = \mathbb{Z}[(A^*)_{tors}]$ and Proposition 4.2 says that there exist $n_1, \dots, n_t$ such that $Q_A = A \otimes_{\mathbb{Z}} \mathbb{Q} \cong \prod_{j=1}^t \mathbb{Q}(\zeta_{n_j})$. Now, by Lemma 4.4, the rank of A^* is equal to the rank of the maximal order $\mathcal{M}_A = \prod_{j=1}^t \mathbb{Z}[\zeta_{n_j}]$ which is known by Dirichlet's Unit Theorem.

In order that $\mathcal{M} = \prod_{j=1}^t \mathbb{Z}[\zeta_{n_j}]$ contains an order $\mathcal{O}$ such that $(\mathcal{O}^*)_{tors} \cong T$, the n_j's must fulfill the following necessary conditions (see Lemma 5.3 below):

i) $t \geq \rho + \sigma$;
ii) $2^\varepsilon \mid n_j$ for all $j = 1, \dots, t$;
iii) for each $i = 1, \dots, s$ there exists an index $j_i \in \{1, \dots, t\}$ such that $p_i^{a_i} | n_{j_i}$; moreover, $j_i \neq j_h$ if $p_i = p_h$ and $i \neq h$;
iv) for each $\iota = 1, \dots, \rho$ there exists an index $l_\iota \in \{1, \dots, t\}$ such that $2^{\varepsilon_\iota} | n_{l_\iota}$ and $l_\iota \neq l_h$ if $\iota \neq h$.

We will say that the maximal order $\mathcal{M} = \prod_{j=1}^{t} \mathbb{Z}[\zeta_{n_j}]$ is *T-admissible* if $\{n_1, \dots, n_t\}$ fulfills the conditions (i)-(iv), where the parameters are those of the “standard” decomposition of T.

Define

$$\mathcal{M}_{0,T} = \prod_{i=1}^{s} \mathbb{Z}[\zeta_{2^\varepsilon p_i^{a_i}}] \times \prod_{\iota=1}^{\rho} \mathbb{Z}[\zeta_{2^{\varepsilon_\iota}}] \times \mathbb{Z}[\zeta_{2^\varepsilon}]^d, \tag{13}$$

where $d = \max\{\sigma - s, 0\}$. $\mathcal{M}_{0,T}$ is T-admissible and in Proposition 5.5 we prove that $\mathcal{M}_{0,T}^*$ has minimum rank among the groups of units of all T-admissible maximal orders. This ensures that

$$\operatorname{rank}(A^*) = \operatorname{rank}(\mathcal{M}_A^*) \geq \operatorname{rank}(\mathcal{M}_{0,T}^*).$$

Now,

$$\operatorname{rank}(\mathcal{M}_{0,T}^*) = \sum_{i=1}^{s} (\frac{\phi(2^\varepsilon p_i^{a_i})}{2} - 1) + \sum_{\iota=1}^{\rho} (\frac{\phi(2^{\varepsilon_\iota})}{2} - 1) + d(\frac{\phi(2^\varepsilon)}{2} - 1)^*,$$

hence

$$\operatorname{rank}(\mathcal{M}_{0,T}^*) = \begin{cases} g(T) & \text{for } \sigma \geq s_0 \\ g(T) - (\frac{\phi(2^\varepsilon)}{2} - 1)^* & \text{for } \sigma < s_0. \end{cases}$$

If $\sigma \geq s_0$ or if $\varepsilon = 1$ we get the required bound on $\operatorname{rank}(A^*)$.

On the other hand, by Proposition 5.6 if $\sigma < s_0$, then $\mathcal{M}_{0,T}$ does not contain any order A with $(A^*)_{tors} \cong T$, so $\mathcal{M}_A \neq \mathcal{M}_{0,T}$. Now, by Proposition 5.5, for $\varepsilon > 1$, $\mathcal{M}_{0,T}$ is the only T-admissible maximal order of minimum rank , hence, if $\sigma < s_0$ and $\varepsilon > 1$, then $\operatorname{rank}(A^*) > \operatorname{rank}(\mathcal{M}_{0,T}^*)$ and, using again Proposition 5.5, we get

$$\operatorname{rank}(A^*) \geq \operatorname{rank}(\mathcal{M}_{0,T}^*) + (\frac{\phi(2^\varepsilon)}{2} - 1)^* = g(T).$$

□

We now state and prove the results quoted above.

Lemma 5.3. , *Let $\mathcal{M} = \prod_{j=1}^{t} \mathbb{Z}[\zeta_{n_j}]$. If $\mathcal{M}$ contains a subring A with $(A^*)_{tors} \cong T$, then $\mathcal{M}$ is T-admissible.*

Proof. For each prime q, the q-Sylow subgroup of $\mathcal{M}^*$ is the direct product of the (cyclic) q-Sylow subgroups of its cyclic factors $\langle \zeta_{n_j} \rangle$, hence every of its q-Sylow has at most t cyclic components. Looking at the 2-Sylow of T we get $t \geq \sigma + \rho$, proving (i). Moreover, if T has an element of order q^k, for some $k \geq 1$, then the q-Sylow of $\mathcal{M}^*$ has a cyclic component of order at least q^k, namely, $q^k | n_j$ for some $j \in \{1, \dots, t\}$; this proves the first part of (iii) and (iv). The last part of these statements follows by noticing that the q-Sylow of $\langle \zeta_{n_j} \rangle$ is cyclic.

We are now left to prove (ii). By identifying A with its image in $\prod_{j=1}^{t} \mathbb{Z}[\zeta_{n_j}]$, we have that the opposite $(-1, \dots, -1)$ of the identity is an element of order 2 in $(A^*)_{tors} = T$ which is in turn a subgroup of

$\prod_{j=1}^{t}\langle\zeta_{n_j}\rangle$. Now, the 2-Sylow of $(A^*)_{tors}$ is isomorphic to $C_{2^\varepsilon}^{\sigma}\times\prod_{\iota=1}^{\rho}C_{2^{\varepsilon_\iota}}$ and all the elements of order 2 of such a group belong to the subgroup $(C_{2^\varepsilon}^{2^{\varepsilon-1}})^{\sigma}\times\prod_{\iota=1}^{\rho}C_{2^{\varepsilon_\iota}}^{2^{\varepsilon_\iota-1}}$, hence they are $2^{\varepsilon-1}$-powers since $\varepsilon_\iota>\varepsilon$ for all ι. In particular,

$$(-1,\dots,-1)=\gamma^{2^{\varepsilon-1}}=(\gamma_1^{2^{\varepsilon-1}},\dots,\gamma_t^{2^{\varepsilon-1}})$$

with $\gamma_j\in\langle\zeta_{n_j}\rangle$, $\forall\, j$. It follows that $\mathrm{ord}(\gamma_j)=2^\varepsilon$ since $\mathrm{ord}(\gamma_j)\mid 2^\varepsilon$ and $\mathrm{ord}(\gamma_j)\nmid 2^{\varepsilon-1}$, so $2^\varepsilon|n_j$ for all j. □

Remark 5.4. According to point (ii) of the definition of T-admissible maximal order, each T-admissible maximal order is a $\mathbb{Z}[\zeta_{2^\varepsilon}]$-algebra.

Proposition 5.5. *Let $\mathcal{M}=\prod_{j=1}^{t}\mathbb{Z}[\zeta_{n_j}]$ be T-admissible. Then,*

$$\mathrm{rank}(\mathcal{M}^*)\geq\sum_{i=1}^{s}(\frac{\phi(2^\varepsilon p_i^{a_i})}{2}-1)+\sum_{\iota=1}^{\rho}(\frac{\phi(2^{\varepsilon_\iota})}{2}-1)+d(\frac{\phi(2^\varepsilon)}{2}-1)^*$$

and equality holds only for $\mathcal{M}=\mathcal{M}_{0,T}$ or, in the case when $\varepsilon=1$, for $\mathcal{M}=\mathcal{M}_{0,T}\times\mathbb{Z}^k$ and $k\geq 0$.

Moreover, if $\mathcal{M}\neq\mathcal{M}_{0,T}$, then $\mathrm{rank}(\mathcal{M}^*)\geq\mathrm{rank}(\mathcal{M}_{0,T}^*)+(\frac{\phi(2^\varepsilon)}{2}-1)^*$.

Proof. For $\mathcal{M}=\prod_{j=1}^{t}\mathbb{Z}[\zeta_{n_j}]$, we have

$$\mathrm{rank}(\mathcal{M}^*)=\sum_{j=1}^{t}\mathrm{rank}(\mathbb{Z}[\zeta_{n_j}]^*)=\sum_{j=1}^{t}(\frac{\phi(n_j)}{2}-1)^*.\qquad(14)$$

Our first step is to bound the rank of $\mathcal{M}^*$, by estimating from below the summands $\frac{\phi(n_j)}{2}-1$ for all j, using Lemma 2.9.

Since $\mathcal{M}$ is T-admissible, all the n_j's are divisible at least by 2^ε and, up to reordering, we can assume that $n_1,\dots,n_\rho$ are divisible by $2^{\varepsilon_1},\dots,2^{\varepsilon_\rho}$, respectively.

Now, $\varepsilon_j>\varepsilon$ for $j=1,\dots,\rho$, so using the inequality (5) we get

$$\mathrm{rank}(\mathbb{Z}[\zeta_{n_j}]^*)=\frac{\phi(n_j)}{2}-1\geq\sum_{\substack{q \text{ odd prime}\\ q^e||n_j}}(\frac{\phi(2^\varepsilon q^e)}{2}-1)+\frac{\phi(2^{\varepsilon_j})}{2}-1.\quad(15)$$

For $j=\rho+1,\dots t$ we can use (4), which gives

$$\mathrm{rank}(\mathbb{Z}[\zeta_{n_j}]^*)\geq\sum_{\substack{q \text{ odd prime}\\ q^e||n_j}}(\frac{\phi(2^\varepsilon q^e)}{2}-1)^{\ddagger}.\qquad(16)$$

These inequalities allow to prove that

$$\mathrm{rank}(\mathcal{M}^*)\geq\sum_{i=1}^{s}(\frac{\phi(2^\varepsilon p_i^{a_i})}{2}-1)+\sum_{\iota=1}^{\rho}(\frac{\phi(2^{\varepsilon_\iota})}{2}-1)+d(\frac{\phi(2^\varepsilon)}{2}-1)^*.\quad(17)$$

‡This inequality holds also if $n_j=2$ since $\mathrm{rank}(\mathbb{Z}[\zeta_{n_j}]^*)=0$ and on the RHS we have an empty sum which is 0.

In fact, it is enough to show that each term on the RHS of (17) appears at least once in (15) or (16), for some j. This is trivially the case for the terms in the second sum since each of them appears in (15).

As for the first sum, we note that since $\mathcal{M}$ is T-admissible, then each $p_i^{a_i}$ divides some n_j. This ensures that, for all i, the RHS of (15) or (16) contains a term of type $\frac{\phi(2^\varepsilon p_i^{b_i})}{2} - 1$ with $b_i \geq a_i$: we can estimate this term by $\frac{\phi(2^\varepsilon p_i^{a_i})}{2} - 1$.

Finally, the term $d(\frac{\phi(2^\varepsilon)}{2} - 1)^*$ can be explained as follows. The two sums on the RHS of (17) involve only $s + \rho$ summands, so they can be obtained by considering the contribution to the rank of $\tau \leq s + \rho$ of the $(\mathbb{Z}[\zeta_{n_j}])^*$'s. We estimate the rank of the $t - \tau$ remaining $(\mathbb{Z}[\zeta_{n_j}])^*$'s simply by

$$\operatorname{rank}(\mathbb{Z}[\zeta_{n_j}]^*) \geq (\frac{\phi(2^\varepsilon)}{2} - 1)^*.$$

Since $t - \tau \geq 0$ and $t - \tau \geq t - \rho - s \geq \sigma - s$ we have $t - \tau \geq d$ and we get (17).

The RHS of (17) is equal to the rank of $\mathcal{M}_{0,T}^*$, so $\mathcal{M}_{0,T}^*$ has the minimum possible rank among the groups of units of the T-admissible maximal orders. When $\varepsilon = 1$, the same is clearly true for the units of $\mathcal{M} = \mathcal{M}_{0,T} \times \mathbb{Z}^k$.

Finally, if $\mathcal{M} = \prod_{j=1}^t \mathbb{Z}[\zeta_{n_j}]$ is T-admissible, but $\mathcal{M} \neq \mathcal{M}_{0,T}$, then either $\mathcal{M}$ has more direct summands than $\mathcal{M}_{0,T}$ (hence $t > s + \rho + d$) or at least one of the following holds:

- $2^\varepsilon p_{i_1}^{a_{i_1}} p_{i_2}^{a_{i_2}} | n_j$ for some j and two coprime factors $p_{i_1}^{a_{i_1}}, p_{i_2}^{a_{i_2}}$;
- $2^{\varepsilon_\iota} p_i^{a_i} | n_j$ for some ι, i and j,

and in both cases we get $\tau < s + \rho$.

In conclusion we always have $t - \tau > d$, so on the RHS of (17) we have at least one extra summand of type $(\frac{\phi(2^\varepsilon)}{2} - 1)^*$, giving

$$\operatorname{rank}(\mathcal{M}^*) \geq \operatorname{rank}(\mathcal{M}_{0,T}^*) + (\frac{\phi(2^\varepsilon)}{2} - 1)^*.$$

□

The last proposition shows that the group of units of $\mathcal{M}_{0,T}$ has minimum rank among the T-admissible maximal orders. However, for some T, no order of $\mathcal{M}_{0,T}$ has T as the group of torsion units.

Proposition 5.6. *Let T be a finite abelian group of even order with its "standard" notation. If $\sigma < s_0$, then $\mathcal{M}_{0,T}$ contains no order A with $(A^*)_{tors} \cong T$.*

Proof. In this proof, for brevity, we will write $\mathcal{M}$ for $\mathcal{M}_{0,T}$. From (13), we obtain $(\mathcal{M}^*)_{tors} \cong T \times C_{2^\varepsilon}^{s-\sigma}$. Assume, by contradiction, that $\mathcal{M}$ contains an order A with $(A^*)_{tors} \cong T$. In the notation of (10) and

(11), we have $T \cong \prod_{i=1}^{s_0} T_{p_i} \times T_2$, where

$$T_{p_i} = \prod_{j=1}^{v_i} C_{p_i^{b_{ij}}} \text{ and } T_2 = \prod_{\iota=1}^{\rho} C_{2^{\varepsilon_\iota}} \times C_{2^\varepsilon}^{\sigma} \tag{18}$$

for some b_{ij}'s.

For $i = 1, \dots, s_0$, put $\mathcal{M}_{p_i} = \prod_{j=1}^{v_i} \mathbb{Z}[\zeta_{2^\varepsilon p_i^{b_{ij}}}]$ and let $\mathcal{M}_2 = \prod_{\iota=1}^{\rho} \mathbb{Z}[\zeta_{2^{\varepsilon_\iota}}]$. The condition $\sigma < s_0$ yields $d = 0$, hence

$$\mathcal{M} \cong \left(\prod_{i=1}^{s_0} \mathcal{M}_{p_i} \right) \times \mathcal{M}_2.$$

We first consider the case when $\rho = 0$, so $\mathcal{M}_2$ is trivial.

For each $i = 1, \dots, s_0$ let $\alpha_{p_i} = (\zeta_{p_i}, \dots \zeta_{p_i}) \in \mathcal{M}_{p_i}$ and put $\boldsymbol{\alpha} = (\alpha_{p_1}, \dots, \alpha_{p_{s_0}}) \in \mathcal{M}$. Clearly, $\boldsymbol{\alpha}$ is a unit of $\mathcal{M}$ of order $p_1 \cdots p_{s_0}$, therefore $\boldsymbol{\alpha}$ also belongs to A^*, having $(A^*)_{tors}$ the same p_i-Sylow subgroups of $(\mathcal{M}^*)_{tors}$ for all $i = 1, \dots s_0$.

Now, if $\varphi_\alpha \colon \mathbb{Z}[x] \to A$ is the substitution homomorphism $x \mapsto \alpha$ we have $\mathbb{Z}[\boldsymbol{\alpha}] \cong \mathbb{Z}[x]/(\ker \varphi_{\boldsymbol{\alpha}})$ and it is easy to check that $\ker \varphi_{\boldsymbol{\alpha}}$, which by Lemma 4.1 is principal and generated by a product of cyclotomic polynomials, is generated by

$$\Phi_{p_1}(x) \dots \Phi_{p_{s_0}}(x),$$

and, the primes p_i's being distinct, Proposition 4.6 ensures that

$$\mathbb{Z}[\boldsymbol{\alpha}] \cong \prod_{i=1}^{s_0} \mathbb{Z}[\zeta_{p_i}].$$

and

$$(\mathbb{Z}[\boldsymbol{\alpha}]^*)_{tors} \cong C_2^{s_0} \times C_{p_1} \times \dots \times C_{p_{s_0}}.$$

This gives a contradiction since $(\mathbb{Z}[\boldsymbol{\alpha}]^*)_{tors} < (A^*)_{tors} \cong T$ and $\sigma < s_0$.

In the case when $\rho > 0$, we have to slightly modify the previous argument to find a contradiction.

As in the previous case, for each $i = 1, \dots, s_0$ let $\alpha_{p_i} = (\zeta_{p_i}, \dots \zeta_{p_i}) \in \mathcal{M}_{p_i}$; also denote by v_0 the unit element of $\mathcal{M}_2$ and by $\alpha_2 = -v_0 = (-1, \dots, -1)$ its opposite.

In $\mathcal{M}$ consider the elements

$$\boldsymbol{\alpha}' = (\alpha_{p_1}, \dots, \alpha_{p_{s_0}}, v_0), \text{ and } \boldsymbol{\delta} = (1, \dots, 1, \alpha_2).$$

Both of them belong to A: in fact, $\boldsymbol{\alpha}' \in A^*$ since it is a unit of $\mathcal{M}$ of odd order; $\boldsymbol{\delta}$ is a 2^ε power in $\mathcal{M}^*$ and $(\mathcal{M}^*)_{tors}^{2^\varepsilon} = T^{2^\varepsilon} = (A^*)_{tors}^{2^\varepsilon}$.

It follows that also $\boldsymbol{\alpha} = (\alpha_{p_1}, \dots, \alpha_{p_{s_0}}, \alpha_2) = \boldsymbol{\alpha}'\boldsymbol{\delta}$ belongs to A, so that $\mathbb{Z}[\boldsymbol{\alpha}] \subseteq A$. As before, we have

$$\mathbb{Z}[\boldsymbol{\alpha}] \cong \mathbb{Z}[x]/(\Phi_{p_1}(x) \dots \Phi_{p_{s_0}}(x)\Phi_2(x))$$

and, again by Proposition 4.6, we get

$$\mathbb{Z}[\boldsymbol{\alpha}] \cong \mathbb{Z} \times \prod_{i=1}^{s_0} \mathbb{Z}[\zeta_{p_i}]$$

and therefore

$$(\mathbb{Z}[\boldsymbol{\alpha}]^*)_{tors} \cong C_2^{s_0+1} \times C_{p_1} \times \cdots \times C_{p_{s_0}}$$

is a subgroup of $(A^*)_{tors}$.

If $\rho = 1$ this gives a contradiction, since A^* has exactly $\sigma + 1$ cyclic factors of order a power of 2, and $s_0 > \sigma$.

If $\rho > 1$, for each $\iota = 1, \ldots, \rho - 1$, let $\boldsymbol{\beta}_\iota$ be the element of $\mathcal{M}$ with all coordinates 1, but whose coordinate in $\mathbb{Z}[\zeta_{2^{\varepsilon_\iota}}]$ is equal to -1. These elements generate a subgroup of $(\mathcal{M})^*$ isomorphic to $C_2^{\rho-1}$. Moreover, all the $\boldsymbol{\beta}_\iota$'s belong to A^*: in fact, $\boldsymbol{\beta}_\iota$ is a 2^ε power of an element of $(\mathcal{M}^*)_{tors}$ so it belongs to $(\mathcal{M}^*)_{tors}^{2^\varepsilon} = (A^*)_{tors}^{2^\varepsilon}$.

Now,

$$(\mathbb{Z}[\boldsymbol{\alpha}]^*)_{tors} \cap \langle \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{\rho-1} \rangle = \{(1, \ldots, 1)\},$$

in fact, the torsion units of $\mathbb{Z}[\boldsymbol{\alpha}]$ are of type $((-\alpha_{p_1})^{e_1}, \ldots, (-\alpha_{p_{s_0}})^{e_{s_0}}, \alpha_2^{e_0})$ with $e_0, e_1 \ldots, e_{s_0} \in \mathbb{Z}$, so their coordinates in $\mathcal{M}_2$ are all 1 or all -1. It follows that $(A^*)_{tors}$ contains a subgroup isomorphic to

$$(\mathbb{Z}[\boldsymbol{\alpha}]^*)_{tors} \times \langle \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{\rho-1} \rangle \cong C_2^{s_0+\rho} \times C_{p_1} \times \cdots \times C_{p_{s_0}}$$

and this is not possible since A^* has $\sigma + \rho$ cyclic factors of order a power of 2, and $s_0 + \rho > \sigma + \rho$. □

5.2. **Proof of Theorem 5.1: the "if" part.** Let T be any finite abelian group of even order; consider on T its "standard" notation as in (10). For each $g \geq g(T)$ we will construct an example of a torsion free ring A with $A^* \cong T \times \mathbb{Z}^g$.

The first and most substantial step is the construction for $g = g(T)$. The following propositions deal with two particular cases.

Proposition 5.7. *Let p be an odd prime and let $\varepsilon, b_1, \ldots, b_v$ be integers, with $1 \leq b_1 \leq b_2 \leq \cdots \leq b_v$ and $\varepsilon \geq 1$. The maximal order $\mathcal{M} = \prod_{j=1}^v \mathbb{Z}[\zeta_{2^\varepsilon p^{b_j}}]$ contains an order A with $(A^*)_{tors} \cong C_{2^\varepsilon} \times \prod_{j=1}^v C_{p^{b_j}}$.*

Proof. For $j = 2, \ldots, v$, let $\boldsymbol{\beta}^{(j)} = (\beta_1^{(j)}, \ldots, \beta_v^{(j)}) \in \mathcal{M}$, where $\beta_i^{(j)} = 1$ for $i \neq j$ and $\beta_j^{(j)} = \zeta_{p^{b_j}}$, and put

$$A = \mathbb{Z}[\zeta_{2^\varepsilon p^{b_1}}][\boldsymbol{\beta}^{(2)}, \ldots, \boldsymbol{\beta}^{(v)}]$$

where we are identifying A with a subring of $\mathcal{M}$ via the diagonal embedding of $\mathbb{Z}[\zeta_{2^\varepsilon p^{b_1}}]$. This means that we identify $\zeta_{2^\varepsilon p^{b_1}}$ with $\boldsymbol{\alpha} = (\zeta_{2^\varepsilon p^{b_1}}, \ldots, \zeta_{2^\varepsilon p^{b_1}})$.

We claim that $(A^*)_{tors} \cong V = C_{2^\varepsilon} \times \prod_{j=1}^v C_{p^{b_j}}$.

It is clear that the elements $\boldsymbol{\alpha}, \boldsymbol{\beta}^{(2)}, \ldots, \boldsymbol{\beta}^{(v)} \in A$ are multiplicatively independent units and that they generate a subgroup of $(A^*)_{tors}$ isomorphic to V. On the other hand, $(\mathcal{M}^*)_{tors} \cong \prod_{j=1}^{v} C_{2^\varepsilon p^{b_j}}$, hence, up to isomorphism,

$$(A^*)_{tors} \leq V \times C_{2^\varepsilon}^{v-1}.$$

To prove that $(A^*)_{tors} \cong V$ it is enough to show that the 2-Sylow of $(A^*)_{tors}$ is cyclic, or equivalently, that $(-1, \ldots, -1)$ is the only element of order 2 of $(A^*)_{tors}$.

For each $i = 2, \ldots, s$ define $\mathcal{M}_i = \mathbb{Z}[\zeta_{2^\varepsilon p^{b_1}}] \times \mathbb{Z}[\zeta_{2^\varepsilon p^{b_i}}]$ and denote by $\pi_i \colon \mathcal{M} \to \mathcal{M}_i$ the canonical projection. Put $A_i = \pi_i(A)$ and $\beta_{0,i} = (1, \zeta_{p^{b_i}})$, then

$$A_i = \mathbb{Z}[\zeta_{2^\varepsilon p^{b_1}}][\pi_i(\boldsymbol{\beta}^{(2)}), \ldots, \pi_i(\boldsymbol{\beta}^{(v)})] = \mathbb{Z}[\zeta_{2^\varepsilon p^{b_1}}][\beta_{0,i}].$$

Let $\varphi_{\beta_{0,i}}$ be the evaluation homomorphism defined on $\mathbb{Z}[\zeta_{2^\varepsilon}][x]$. It is easily checked that its kernel is generated by $(x-1)\Psi_{p^{b_i}, p^{b_1}}(x)$, so

$$A_i = \mathbb{Z}[\zeta_{2^\varepsilon p^{b_1}}][\beta_{0,i}] \cong \mathbb{Z}[\zeta_{2^\varepsilon p^{b_1}}][x]/((x-1)\Psi_{p^{b_i}, p^{b_1}}(x))$$

and, by Proposition 4.7, $(A_i^*)_{tors} \cong C_{2^\varepsilon p^{b_1}} \times C_{p^{b_i}}$. This ensures that, for all indices i, the 2-Sylow of $\pi_i((A^*)_{tors})$, which is a subgroup of $(A_i^*)_{tors}$, is cyclic and this allows us to conclude the proof. In fact, let $\mathbf{u} = (u_1, \ldots, u_v) \in \mathcal{M}^*$ be such that $\mathbf{u}^2 = (1, \ldots, 1)$; if $\mathbf{u} \in A$, then $\pi_i(\mathbf{u}) = (u_1, u_v)$ is an element of exponent 2 of $(A_i^*)_{tors}$, so (u_1, u_v) must be equal to $(1,1)$ or $(-1,-1)$, in particular, $u_i = u_1$ for all $i = 1, \ldots, v$. This yields $\mathbf{u} = (1, \ldots, 1)$ or $\mathbf{u} = (-1, \ldots, -1)$, so A^* has only one element of order 2, therefore

$$(A^*)_{tors} \cong C_{2^\varepsilon} \times C_{p^{b_1}} \times \cdots \times C_{p^{b_v}} = V.$$

□

When the group T has too few 2-cyclic factors of minimal order, Proposition 5.6 shows that no order of $\mathcal{M}_{0,T}$, has torsion units isomorphic to T. In this case, to find an order A with $(A^*)_{tors} \cong T$, we have to consider a bigger maximal order obtained by adding to $\mathcal{M}_{0,T}$ an extra direct factor, which works as a "control" factor on the 2-torsion. The following proposition deals with the case $\sigma = 1$.

Proposition 5.8. *Let $p_1, \ldots p_s$ be prime numbers and let $\varepsilon, a_1, \ldots a_s$ be positive integers. The maximal order $\mathcal{M} = \mathbb{Z}[\zeta_{2^\varepsilon}] \times \prod_{i=1}^{s} \mathbb{Z}[\zeta_{2^\varepsilon p_i^{a_i}}]$ contains a subring A with $(A^*)_{tors} \cong C_{2^\varepsilon} \times \prod_{i=1}^{s} C_{p_i^{a_i}}$.*

Proof. For each $i = 1, \ldots, s$, let $\boldsymbol{\beta}^{(i)} = (1, \beta_1^{(i)}, \ldots, \beta_s^{(i)}) \in \mathcal{M}$, where $\beta_j^{(i)} = 1$ for all $j \neq i$ and $\beta_i^{(i)} = \zeta_{p_i^{a_i}}$. Put

$$A = \mathbb{Z}[\zeta_{2^\varepsilon}][\boldsymbol{\beta}^{(1)}, \ldots, \boldsymbol{\beta}^{(s)}]$$

viewed as a subring of $\mathcal{M}$. We claim that $(A^*)_{tors} \cong C_{2^\varepsilon} \times \prod_{i=1}^{s} C_{p_i^{a_i}}$.

Clearly, the elements $\boldsymbol{\alpha} = (\zeta_{2^\varepsilon}, \dots, \zeta_{2^\varepsilon}), \boldsymbol{\beta}^{(1)}, \dots, \boldsymbol{\beta}^{(s)} \in A$ are multiplicatively independent units which generate a subgroup of $(A^*)_{tors}$ isomorphic to $C_{2^\varepsilon} \times \prod_{i=1}^{s} C_{p_i^{a_i}}$.

On the other hand, $(\mathcal{M}^*)_{tors} \cong C_{2^\varepsilon}^{s+1} \times \prod_{i=1}^{s} C_{p_i^{a_i}}$, then to prove our claim it is enough to show that the 2-Sylow of $(A^*)_{tors}$ is cyclic, or equivalently that $(-1, \dots, -1)$ is the only element of order 2 of $(A^*)_{tors}$.

This can be proved arguing as in the previous proposition. In fact, for each $i = 1, \dots, s$ define $\mathcal{M}_i = \mathbb{Z}[\zeta_{2^\varepsilon}] \times \mathbb{Z}[\zeta_{2^\varepsilon p_i^{a_i}}]$ and denote by $\pi_i \colon \mathcal{M} \to \mathcal{M}_i$ the canonical projection. Let $A_i = \pi_i(A)$ and $\beta_{0,i} = (1, \zeta_{p_i^{a_i}})$, then

$$A_i = \mathbb{Z}[\zeta_{2^\varepsilon}][\pi_i(\boldsymbol{\beta}^{(1)}), \dots, \pi_i(\boldsymbol{\beta}^{(s)})] = \mathbb{Z}[\zeta_{2^\varepsilon}][\beta_{0,i}].$$

The kernel of the evaluation homomorphism $\varphi_{\beta_{0,i}} \colon \mathbb{Z}[\zeta_{2^\varepsilon}][x] \to A$ is generated by $\Phi_1(x)\Phi_{p_i^{a_i}}(x)$: in fact, since $p_i^{a_i}$ is odd, the polynomial $\Phi_{p_i^{a_i}}(x)$ is irreducible in $\mathbb{Z}[\zeta_{2^\varepsilon}]$. Thus

$$A_i = \mathbb{Z}[\zeta_{2^\varepsilon}][\beta_{0,i}] \cong \mathbb{Z}[\zeta_{2^\varepsilon}][x]/(\Phi_1(x)\Phi_{p_i^{a_i}}(x))$$

and, by Proposition 4.7, $(A_i^*)_{tors} \cong C_{2^\varepsilon p_i^{a_i}}$. This implies that also its subgroup $\pi_i((A^*)_{tors})$ is cyclic and this allows us to conclude the proof.

In fact, let $\mathbf{u} = (u_0, \dots, u_s) \in \mathcal{M}^*$ be such that $\mathbf{u}^2 = (1, \dots, 1)$; if $\mathbf{u} \in A$, then, for all i, $\pi_i(\mathbf{u}) = (u_0, u_i)$ is an element of exponent 2 of the cyclic group $\pi_i((A^*)_{tors})$, so (u_0, u_i) must be equal to $(1, 1)$ or $(-1, -1)$. In particular, $u_i = u_0$ for all $i = 1, \dots, s$. This ensures that $\mathbf{u} = (1, \dots, 1)$ or $\mathbf{u} = (-1, \dots, -1)$, and A^* has only one element of order 2, as required. □

We are now ready for the general construction for $g = g(T)$.
Let

$$\mathcal{M}_T = \begin{cases} \mathcal{M}_{0,T} & \text{for } \sigma \geq s_0 \\ \mathcal{M}_{0,T} \times \mathbb{Z}[\zeta_{2^\varepsilon}] & \text{for } \sigma < s_0, \end{cases} \tag{19}$$

then $\operatorname{rank}(\mathcal{M}_T) = g(T)$ for all T. We will construct A as an order in $\mathcal{M}_T$.

The case when $s \leq \sigma$ is very easy: we can simply take $A = \mathcal{M}_T$ since $\mathcal{M}_T^* \cong T \times \mathbb{Z}^{g(T)}$.

Consider now the more general case when $\sigma \geq s_0$. We can write the group T as

$$T = V_2 \times \prod_{i=1}^{s_0} V_{p_i},$$

where $V_{p_i} = C_{2^\varepsilon} \times T_{p_i} = C_{2^\varepsilon} \times \prod_{j=1}^{v_i} C_{p_i^{b_{ij}}}$ and $V_2 = C_{2^\varepsilon}^{\sigma - s_0} \times \prod_{\iota=1}^{\rho} C_{2^{\varepsilon_\iota}}$.

For $i = 1, \dots, s_0$, let $\mathcal{M}_{p_i} = \prod_{j=1}^{v_i} \mathbb{Z}[\zeta_{2^\varepsilon p_i^{b_{ij}}}]$ and $\mathcal{M}_2 = \mathbb{Z}[\zeta_{2^\varepsilon}]^{\sigma - s_0} \times \prod_{i=1}^{\rho} \mathbb{Z}[\zeta_{2^{\varepsilon_i}}]$. Then

$$\mathcal{M}_T \cong \mathcal{M}_2 \times \prod_{i=1}^{s_0} \mathcal{M}_{p_i}.$$

By Proposition 5.7, for all $p = p_1, \dots, p_{s_0}$, the maximal order $\mathcal{M}_p$ contains an order A_p such that $(A_p^*)_{tors} \cong V_p$. It follows that $A = \mathcal{M}_2 \times \prod_{i=1}^{s_0} A_{p_i}$ is an order of $\mathcal{M}_T$ with $(A^*)_{tors} \cong T$.

Let now $\sigma < s_0$. We write the group T as $T_0 \times T_1$ where

$$T_0 = \prod_{i=1}^{\sigma-1} C_{2^\varepsilon p_i^{a_i}} \times \prod_{\iota=1}^{\rho} C_{2^{\varepsilon_\iota}} \text{ and } T_1 = C_{2^\varepsilon} \times C_{p_\sigma^{a_\sigma}} \times \cdots \times C_{p_s^{a_s}}.$$

By Proposition 5.8 the order $\mathcal{M}_1 = \mathbb{Z}[\zeta_{2^\varepsilon}] \times \prod_{i=\sigma}^{s} \mathbb{Z}[\zeta_{2^\varepsilon p_i^{a_i}}]$ contains a subring A_1 with $(A_1^*)_{tors} \cong T_1$.

On the other hand,

$$\mathcal{M}_T = \mathcal{M}_1 \times \prod_{i=1}^{\sigma-1} \mathbb{Z}[\zeta_{2^\varepsilon p_i^{a_i}}] \times \prod_{i=1}^{\rho} \mathbb{Z}[\zeta_{2_i^\varepsilon}]$$

and its subring

$$A = A_1 \times \prod_{i=1}^{\sigma-1} \mathbb{Z}[\zeta_{2^\varepsilon p_i^{a_i}}] \times \prod_{i=1}^{\rho} \mathbb{Z}[\zeta_{2_i^\varepsilon}]$$

is such that $(A^*)_{tors} \cong T$.

Moreover, $\operatorname{rank}(A^*) \leq \operatorname{rank}(\mathcal{M}_T^*)$. On the other hand, the rank of A^* is the same of the rank of $\mathcal{M}_A^*$, which is a T-admissible maximal order, and thus its rank is at least the rank of $\mathcal{M}_T$. This gives $\operatorname{rank}(A^*) = \operatorname{rank}(\mathcal{M}_T^*)$ and also proves that A is an order of $\mathcal{M}_T$.

The final step is the construction of torsion-free rings with group of units isomorphic to $T \times \mathbb{Z}^g$ for all $g > g(T)$. Also in this case if A is a torsion-free ring with $(A^*)_{tors} = T$ and minimal rank $g(T)$, then $\mathcal{A} = A[x_1, \dots, x_k, x_1^{-1}, \dots, x_k^{-1}]$ is torsion-free and has group of units isomorphic to $T \times \mathbb{Z}^{g(T)+k}$.

6. Reduced rings

In this section we classify the finitely generated abelian groups which arise as groups of units of reduced rings. The next proposition describes the relation between the units of a ring and those of its reduced quotient, showing that the study of reduced rings is a substantial step to the study of units of a general ring.

Proposition 6.1. *Let A be a commutative ring and let $\mathfrak{N}$ be its nilradical. Then the sequence*

$$1 \to 1 + \mathfrak{N} \hookrightarrow A^* \xrightarrow{\phi} (A/\mathfrak{N})^* \to 1, \tag{20}$$

where $\phi(x) = x + \mathfrak{N}$, is exact.

We note that for finite characteristic rings the exact sequence (20) always splits (see [DCD18a, Thm 3.1]). This is no longer true in general, as shown in [DCD18b, Ex 2]).

The units of a reduced ring of finite characteristic rings are characterized as follows.

Proposition 6.2. *The finitely generated abelian groups which are the groups of units of reduced rings of positive characteristic are exactly those of the form*

$$\prod_{i=1}^{k} \mathbb{F}^*_{p_i^{n_i}} \times \mathbb{Z}^g$$

where $k, n_1, \ldots, n_k$ are positive integers, $\{p_1, \ldots, p_k\}$ are not necessarily distinct prime numbers and $g \geq 0$.

Proof. Let A be a reduced ring of characteristic n, such that $A^* \cong (A^*)_{tors} \times \mathbb{Z}^g$, with $(A^*)_{tors}$ finite and $g \geq 0$. The ring $B = \mathbb{Z}/n\mathbb{Z}[(A^*)_{tors}]$ is a finite ring and by Lemma 2.1 $B^* = (A^*)_{tors}$. Since B is finite, B is artinian and so it is a product of local artinian rings. Moreover, a reduced local artinian ring is a field, hence B is a product of finite fields (see also [DCD18a, Corollary 3.2]) and we get that $(A^*)_{tors} = B^*$ has the required form.

On the other hand, let the p_i's, n_i's and g be as in the statement and put $R = \prod_{i=1}^{k} \mathbb{F}_{p_i^{n_i}}$. Then the ring $R[x_1, \ldots, x_g, x_1^{-1}, \ldots, x_g^{-1}]$ has group of units isomorphic to $\prod_{i=1}^{k} \mathbb{F}^*_{p_i^{n_i}} \times \mathbb{Z}^g$. □

The following proposition together with the results of the previous section allows us to classify the finitely generated abelian groups which arise as group of units of a reduced ring.

Proposition 6.3. ([PS70, Prop. 1]) *Let A be a commutative ring which is finitely generated and integral over its fundamental subring. Then $A = A_1 \oplus A_2$, where A_1 is a finite ring and the torsion ideal of A_2 is contained in its nilradical.*

Now, if A is reduced then the finite ring A_1 is reduced and A_2 is torsion-free. Then, Theorems 5.1 and 6.2 immediately gives the following.

Theorem 6.4. *The finitely generated abelian groups that occur as groups of units of reduced rings are those of the form*

$$\prod_{i=1}^{k} \mathbb{F}^*_{p_i^{n_i}} \times T \times \mathbb{Z}^g$$

where $k, n_1, \ldots, n_k$ are positive integers, $\{p_1, \ldots, p_k\}$ are not necessarily distinct prime numbers, T is any finite abelian group of even order and $g \geq g(T)$.

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