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Nonlinear dynamics of delay systems: an overview

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Time delays play an important role in many fields such as engineering, physics or biology. Delays occur due to finite velocities of signal propagation or processing delays leading to memory effects and, in general, infinite dimensional systems. Time delay systems can be described by delay differential equations and often include non-negligible nonlinear effects. This overview article introduces the theme issue 'Nonlinear dynamics of delay systems', which contains new fundamental results in this interdisciplinary field as well as recent developments in applications. Fundamentally new results were obtained especially for systems with time-varying delay and state-dependent delay and for delay system with noise, which do often appear in real systems in engineering and nature. The applications range from climate modelling over network dynamics and laser systems with feedback to human balancing and machine tool chatter.

1. Introduction

Delayed interactions are an ubiquitous feature of many dynamical systems in science and engineering. Hence time delays play a fundamental role in fields as diverse as biology, chemistry, engineering, or social sciences [1]. In many applications systems normally involve interacting subunits, and depending on the time scales of the subunits, the propagation speed of information becomes relevant for the dynamics. Nonlinear properties play a prevalent role, causing complex dynamical behaviour which cannot be explained by just looking at the constituents of a system. Therefore nonlinear time delay dynamics is not just of crucial importance in applications but still poses a fundamental challenge for theoretical and mathematical studies.

A typical and famous example of a nonlinear delay system is given by the Mackey-Glass equation describing the production of blood cells [2]. The control of the cellular production at time t is based on the current number of blood cells $x(t)$. However, there exists a significant delay τ between the initiation of cellular production and the release of mature cells into the blood. This means that the change $\dot{x}(t)$ of the number of blood cells at time t depends not only on the instantaneous number of blood cells $x(t)$ but also on the retarded configuration $x(t-\tau)$. In general, a nonlinear delay system can be described by the delay differential equation (DDE)

$$x'(t) = f(x(t), x(t-\tau)), \tag{1.1}$$

where f denotes a nonlinear function. For example, in the Mackey-Glass equation, we have $f(x, x_\tau) = \frac{\beta x_\tau}{1+x_\tau^n} - \gamma x$ with the fixed parameters β, n, and γ. To capture the effects of the time delay one needs a description which takes the history of the system into account, and thus requires a dynamical system with an infinite dimensional phase space. In fact, it turns out that the state of the DDE (1.1) at time t is given by the function $x(\theta), \theta \in [t-\tau, t]$ (see [3,4] for the mathematical theory on DDEs). Therefore DDEs such as Eq. (1.1) are able to produce extremely complex behaviour and bifurcation scenarios. For example, albeit its relatively simple scalar description the Mackey-Glass equation shows periodic, aperiodic as well as chaotic behavior. Moreover, it has been shown that the dimension of the chaotic attractors of delay systems grows linearly with the delay [5]. This means that extremely high-dimensional chaos is possible for DDEs with large delay, which makes them interesting for applications. In general, time delay is able to induce instabilities, multistability, and chaotic motion on the one hand, but on the other hand can as well suppress instabilities, stabilise unstable stationary or periodic states, and render an initially complex and chaotic system predictable.

The classical studies of time delay dynamics were driven by engineering and mathematics focusing on linear dynamical systems and the basic mathematical theory for infinite dimensional dynamical systems. With the advent of nonlinear and chaotic dynamical systems the focus has shifted in the last three decades to the study of complex dynamical scenarios and the topic has become more and more interdisciplinary. While substantial theoretical foundations have been laid the picture is still incomplete, in particular, with respect to network dynamics, spatially extended time delay dynamics or systems with variable and state-dependent delay. Within the last decade the focus has shifted towards applications of nonlinear time delay dynamics in various disciplines such as laser systems, information processing and computing, climate research, or neuroscience. Nonlinear dynamics with time delay is going to play a vital role in new emerging fields of science and technology, for instance, because the relevant time scales in contemporary complex systems do not allow to neglect finite propagation speed of signals any more.

The recently held Wilhelm und Else Heraeus Seminar: Delayed Complex Systems, the third instalment in series of events, held at the Physikzentrum Bad Honnef, 2nd to 5th July 2018, provided a forum where participants from all relevant disciplines discussed recent advances in this field. We took this opportunity to ask world leading experts to contribute to this theme issue and to provide a brief overview of the state of art in their respective field. Due to its interdisciplinary character the broad field of nonlinear systems with time delay is discussed from various different points of view. The issue covers aspects ranging from mathematical foundations and theoretical results to applications and experiments where the presentations are kept accessible for a large audience. Theoretical topics covered by the theme issue are, for example, the control, stability and bifurcation analysis of DDEs, the relationship between the Doppler effect and variable delays, as well as the effect of time delays on network dynamics, slow-fast systems, Langevin equations and reaction-diffusion systems. On the other hand, the relevant applications are the El Niño Southern Oscillation (ENSO) phenomenon, synchronisation, consensus and Chimera states in different types of networks or multi-agent systems, reservoir computing, machine-tool chatter, and human balancing, to mention only a few.

2. Content of the issue

The first six contributions focus on different fundamental aspects of time delay systems.

The first contribution by Adrián Ramírez et al. [6] introduces main aspects of the subject and demonstrates how the long standing issue of delay in control engineering has advanced and which new challenges are posed in the context of large scale multi agent systems. New results are presented for the tuning of proportional retarded controllers for reaching fast consensus in multi-agent networks. The authors demonstrate that the problem is equivalent to the problem of pole placement in scalar linear DDEs with complex coefficients.

The next contribution by Luca Giuiolli and Zohar Neu [7] shifts the focus to the interplay between noise and time delay, that is, the dynamics of the DDE Eq.(1.1) with additive noise. Such systems are known as delayed Langevin equation or generalised Langevin equation. The authors introduce the topic and review existing results on the challenging problem of finding a suitable Fokker-Planck representation for such non-Markovian stochastic dynamics. The authors present a new Fokker-Planck representation in form of a hierarchy of coupled Fokker-Planck equations, which even enables them to derive analytic results.

Another extension of DDEs can be done by introducing an additional diffusion term in Eq.(1.1). Such systems are known as delayed reaction-diffusion systems and are the topic of the third contribution by Jan Rombouts et al. [8]. In particular, a genuine feature of spatially extended time delay dynamics namely the impact of time delay on front propagation in scalar reaction diffusion systems is studied. The authors present results on the determination of the shape, the propagation speed and the stability behaviour of the travelling fronts.

David Müller-Bender et al. [9] introduce novel dynamical features caused by time-dependent delay times. In particular, they study Eq. (1.1) with a time-varying delay, $\tau = \tau(t)$, and show that such a system can be interpreted as a Doppler effect with feedback. The authors review recent developments in this field, such as the existence of a universal dichotomy between conservative and dissipative delays as well as laminar chaos which is an ubiquitous feature of systems with dissipative delays. The authors extend the theory of laminar chaos to generalised laminar chaos and present new time-multiplexed dynamics, which can be generated via dissipative delays.

The fifth contribution by Andrew Keane et al. [10] deals with a specific extension of Eq. (1.1), where the delay depends on the state of the system leading to a DDE with state-dependent delay. In particular, the authors point out the relevance of state-dependent delays in a delay model for the El Niño Southern Oscillation phenomenon. They perform a numerical bifurcation analysis of their new model with state-dependent delay via DDE-Biftool and compare them with bifurcation diagrams from the existing model with constant delay. The contribution also demonstrates the capability of the DDE-Biftool package, which is a powerful tool for the numerical analysis and continuation of periodic solution of DDEs with constant and/or state-dependent delay.

The sixth contribution by Stefan Ruschel and Serhiy Yanchuk [11] focusses on analytic perturbation schemes for time delay dynamics. The particular emphasis is on a singular perturbation expansions which can capture delay dynamics operating on hugely different time scales and lead to delay-induced switched states. The mathematical studies of these slow-fast systems have relevance for optoelectronic experiments.

The remaining seven articles are related to applications, where time delay systems are highly relevant.

The article from Joseph D. Hart et al. [12] reviews several interesting experiments and applications, which can be realised via optoelectronic oscillators with time-delayed feedback. Examples are reservoir computing and the experimental realisation of chimera states. The authors show how the spatio-temporal representation of time delay systems can be used for an experimental realisation of complex delay systems via a single oscillator with delayed feedback and time-multiplexing. The advantage of this approach lies in the fact that the complexity of the experimental setup is manageable and the virtual nodes of the network representing independent coupled oscillators are truly identical.

Whereas in optoelectronic oscillators the feedback is realised by a delayed electrical signal that is injected into an electro-optic intensity modulator, the work by Benjamin Lingnau et al. [13] is about the dynamics of a laser subjected to optical feedback. In such systems an external mirror is used to generate an optical feedback with a significant time delay leading to so-called external cavity modes. The authors have analysed the stability and bifurcations of the external cavity modes in a class-C laser, compared their results to known results for class-B lasers, and provides optimal parameter settings for stabilising the output.

The following four contributions explore time delay dynamics in the context of theoretical and applied neuroscience.

The ninth contribution by Hwayeon Ryu and Sue Ann Campbell [14] discusses synchronisation in neural networks with excitatory and inhibitory synapses and two different discrete delays in the coupling between them, which serves as a model for thalamic sleep rhythms. The authors use geometric singular perturbation methods to show that synchronous oscillatory periodic solutions can exist in the presence of delay but do not exists in the absence of delays.

The article by Denis Nikitin et al. [15] studies partially synchronised solutions in coupled FitzHugh-Nagumo oscillators. The authors found a new regime of coexisting slow and fast oscillations in a two-layer network with identical units, which coexist with Chimera states. Specifically, there is a time delay in the inter-layer coupling, and the intra-layer coupling is fractal in one layer and nonlocal in the other layer. The new regime is characterized by synchronised domains of fast neuron pairs and desynchronized domains of slow neuron pairs, which oscillate in-phase and in anti-phase, respectively. Moreover, the authors provide an approach to control the proportion between slow and fast classes in the network.

The contribution by Space Petkoski and Viktor K. Jirsa [16] focusses on modelling brain dynamics on the basis of the connectome, which is a map of neural connections in a healthy human brain. Chaotic oscillators, subjected to additive noise and coupled via heterogeneous time delays, are used to model neural activity in 68 cortical regions of the brain. This means that each connection between the ith and jth node of the network has its own delay τ_{ij}. The authors analyse the statistics of the amplitudes and phases of the nodes, which can be used to uncover the spatio-temporal structure of the brain.

The penultimate contribution by John Milton and Tamas Insperger [17] illustrate the relevance of time delay for biomechanics, in particular for the human balancing process. The authors study the effects of time delay, sensory dead zone, and maximum ankle torque on the balance time by assuming human control in the form of a delayed proportional derivative feedback. Quite counterintuitively, the three different effects, which on their own destabilise balance, are able to stabilise balance when acting together.

With the final contribution by Zoltan Dombovari et al. [18] we close the loop and come back to time delay dynamics in engineering, namely the impact of delays in metal cutting. In particular, undesired self-excited machine tool vibrations in milling can be explained by the regenerative effect, which is a closed-loop between tool vibrations at the present cut generated by a wavy surface left by the previous cut. The authors give a brief overview on the literature in this field and review results on a bi-stability between the stable cutting solution and the undesired chattering motion. They present measurements which prove the existence of the bistable domain (or unsafe zone) and are used to determine the size of these zones.

3. Conclusion

The first two contributions [6,7] already point out the complexity of the topic due to the fact that, even after decades of intensive research on time delay systems, there are still open fundamental questions for scalar systems with one fixed delay. Other contributions show that the research trend goes toward studying systems with multiple delays and distributed delays, and that only little is known for systems with time-varying delay or state-dependent delay. However, these delays are the most realistic ones because in many systems in engineering and nature the delays are not constant but rather variable. In other words, the question is not only if we can neglect the effect of time delays but also if we can neglect fluctuations of the delay or a state-dependence of the delay. We think this theme issue will advance the subject of complex time delay dynamics in this direction. Moreover, the theoretical contributions in this issue and the corresponding ongoing research will stimulate developments towards new applications.

In addition, the theme issue presents some quite realistic models and interesting applications related to time delay systems. In some cases the goal is a better understanding of the system dynamics and the stabilisation of unstable stationary or periodic behaviour. Such situations often appear in engineering applications, where time delay effects typically need to be suppressed. Other applications like reservoir computing make use of the high complexity of nonlinear delay dynamics. In these cases, the infinite dimensional state space of DDEs is used to generate complex behaviour for applications, which are based on multistability, time-multiplexing or high-dimensional chaos.

Altogether, the theme issue demonstrates that nonlinear delay systems and its applications can be found in nearly every research field, and we think that it will foster interdisciplinary research in this rapidly expanding vibrant area of research. Finally, we hope that the issue contributes to the dissemination of the time delay approach into other areas, where similar effects are still not taken into account.

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