Moments of Phase-type aging modeling for health dependent costs *

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Abstract

In this paper, we use a discrete time Phase-type process to model the health care cost of an insurance contract by considering all possible critical health states of an individual with constant interest rate. From the moment generating function of the NPV, we derive a recursive formula of this Markov Reward Model (MRM).

Keywords: Health dependent costs, Health dependent costs, net present value, phase-type aging process, Markov reward model, recursive moments.

JEL: C650, I130

1. Introduction

Many life insurance companies are exposed to long-term care risks. In contrast to classical health insurance where contracts are renewed every year and most lines of business are short-tailed, management of long-term care contracts is complicated as it involves uncertainty about the life and health of an individual over a long-term horizon. It is not only the uncertainty about the remaining lifetime, which complicates the estimation of future health costs, but also the uncertainty related to the quality of this remaining lifetime, which is highly affected by the progress of medical science, economic factors, etc.

Modelling health care costs is a problem of great interest in health insurance and health economics (Zhao and Zhou [2012]). The estimation of the costs plays a key role in pricing, reserving and risk assessment in health insurance, as well as performing cost-effectiveness and cost-utility analyses in health economics.

In Castelli et al. [2007], Gardiner et al. [2006] and Zhao and Zhou [2012] the Markov chain has a fixed number of states, which is a subjectively chosen parameter, and does not depend on the age of an individual. The health care cost related quantities that are studied in Castelli et al. [2007], Gardiner et al. [2006] and Zhao and Zhou [2012] slightly differ from each other, but all are computed as expected values. In particular, Gardiner et al. [2006] work in continuous time and determine the expected net present value (abbreviated as "NPV'') of health care costs over a fixed time horizon.

In this paper, we use a multi-states Markovian model to estimate the health care costs over a fixed period of time, where the changes of individual health states are taken into account. Specifically, it is assumed that health care costs depend not only on an individual health state, but also on the random interest rate, which is modelled by a Markov chain.

Our underlying assumption is that the lifetime and health of an individual are described by the phase-type distribution. Our motivation to use a phase-type representation for the lifetime of an individual comes from Lin and Liu [2007]. Define a finite-state continuous-time Markov process to represent the hypothetical aging process of an individual. They call it a phase-type aging model (″PH-aging model″). Aging is described as a process of consecutive transitions from one health state to another until death. One important property of this model is that the states have some physical interpretation, and their number are not chosen arbitrarily, they depend on mortality data through a well specified algorithmic procedure. Another important characteristic of the model, which makes it different from other phase-type models for health, and very relevant for actuarial applications at the same time, is that it provides a connection between the health state of an individual and his/her age.

The first moment of the discounted aggregate claims or its mathematical expectation intuitively represents the central tendency of that random variable, as well as the average of its distribution. The justification for the popularity of the notion of mathematical expectation comes from the Law of Large Numbers which essentially says that the average of the successive realizations of a random variable tends towards the expectation of this random variable when the number of realisations tends to infinity. This result gives an almost experimental status to the mathematical notion of mathematical expectation.

The mathematical expectation plays an important role in determining the pure premium. The expectation of a random variable gives information on the central tendency of the distribution, but no information on the dispersion of values around their average value. A natural idea to quantify this dispersion would be to measure how far from the mean a realization of that random variable falls. We could thus consider the expectation of the square of the distance from its mean, which is the second central moment.

The paper is organized as follows: in Section 2, we present the theoretical background of the study; in Section 3, we present the probability generating function and moment generating function of the net present value of the Markov reward process. The Recursive formula and examples for the moments of this process are derived in Sections 4, and 5. In Section 6, the conclusion follows.

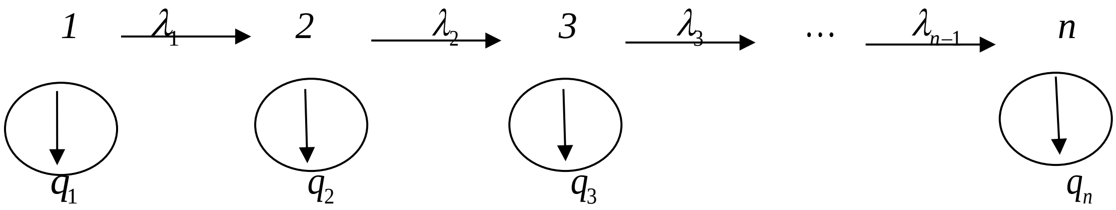
2. THE MODEL

2.1 Aging process and Lifetime

2.1.1 Definition

Lin and Liu [2007] define a finite-state continuous-time Markov process to model the hypothetical aging process of an individual. Aging is described as a process of consecutive transitions from one health state to another until death, as shown on Figure 1. There, the system has n phases with the transition rates λ_i from state i to $i + 1$, for $i = 1, \cdots, n-1$, and the transition rates q_i to the absorbing phase, which is interpreted as the state of death of the individual: the time to reach the absorbing phase is interpreted as the lifetime of the individual.

Fig. 1 Phase-type aging process



The generator is:

$$Q = \begin{pmatrix} \Lambda & \underline{q} \\ \underline{0}^T & 0 \end{pmatrix}, \tag{1}$$

where $\underline{q} = \begin{pmatrix} q_1 & q_2 & \cdots & q_n \end{pmatrix}^T$ is the absorbing vector and Λ is the transition rate matrix:

$$\Lambda = \begin{pmatrix} -(\lambda_1 + q_1) & \lambda_1 & 0 & \cdots & 0 \\ 0 & -(\lambda_2 + q_2) & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -(\lambda_{n-1} + q_{n-1}) & \lambda_{n-1} \\ 0 & 0 & 0 & 0 & -q_n \end{pmatrix}. \tag{2}$$

Newborns start from state 1, thus the initial probability vector is $\left(\underline{\alpha}^T, 0\right)$ with $\underline{\alpha}^T = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^T$. The transition rates have a special structure, it suffices here to mention that they accommodate a developmental period for very young ages and a period of higher accident probability in the mid-range of childhood. Some of the rates are constant, thus although the model potentially consists of a large number of parameters, 9 to 13 different parameters are often enough to give a good approximation of mortality data. For a more detailed explanation of the parameters structure and fitting, we refer the reader to Lin and Liu [2007] and Govorun and Latouche [2014].

2.2 Health distribution

The distribution of the Markov process at time t given by:

$$\underline{\alpha}^T e^{Qt} = \left(\underline{\alpha}^T e^{\Lambda t},\ 1 - \underline{\alpha}^T e^{\Lambda t} \underline{1}\right), \tag{3}$$

means the following:

- $\left(\underline{\alpha}^T e^{\Lambda t}\right)_i$ is the probability to survive for t years and to be in phase i at time t.
- $1 - \underline{\alpha}^T e^{\Lambda t} \underline{1}$ is the probability to have died before time t.

One important feature of this model, which makes it different from other phase-type models for health, is that it provides a connection between the age of an individual and his/her health states.

Specifically, if we denote the health distribution at age x as $\underline{\tau}_x$, we obtain from the model that:

$$\underline{\tau}_x^T = \left(\underline{\alpha}^T e^{\Lambda x} \underline{1}\right)^{-1} \underline{\alpha}^T e^{\Lambda x}. \tag{4}$$

2.2.1 Lifetime distribution

The time until death T_x of an individual of age x follows the discrete phase-type distribution with parameters $\left(\underline{\tau}_x, e^{\Lambda}\right)$ (see Latouche and Ramaswami [1999]) and the phase distribution of the individual after *t* units of time is $\underline{\tau}_x^T e^{\Lambda t}$.

The probability that such an individual ages x survives for t units of time is given by:

$$S_X\left(t\right)=P\left(T_X>t\right)=\underline{\tau}_x^T e^{\Lambda t}\underline{1}\ . \tag{5}$$

2.2.2 Expectation

- Denote by $\overline{L}^{(i)}$ the expected remaining life for an individual in physiological state i.

It is given by:

$$\overline{L}^{(i)}=\underline{\alpha}^{(i)}\left(I-e^{\Lambda}\right)^{-1}\underline{1}, \tag{6}$$

where $\underline{\alpha}^{(i)}$ is a row vector of size n with $\alpha_i^{(i)}=1$ and $\alpha_j^{(i)}=0$ for $i\neq j$.

- Denote by $\overline{L}_X$ the expected remaining lifetime for an individual aged x. Then we have:

$$\overline{L}_X=\underline{\tau}_x^T\left(I-e^{\Lambda}\right)^{-1}\underline{1}\ . \tag{7}$$

2.2.3 Markov Reward Model for health costs

This section summarizes the important result by Govorun- Latouche- Loisel (2014) that calculate the expected present value of health care costs when the impact of the transition from one health state to other until the death, and a constant interest rate are taken into account. Our contribution is in the detailed proofs and explanations that we provide that are otherwise omitted in their paper.

The health care contract is represented by:

$$S=\sum_{t=1}^{\lfloor L\rfloor}v^{t-1}X_t\ , \tag{8}$$

where $\lfloor L\rfloor$ is the integer number of remaining years of life, and X_t is the health care cost in year *t*. The coefficient *v* is allowed to take any positive value including greater than one, so as to include inflation, interest force, the increase of health care prices, etc.

The health state of the individual ages x at time t is denoted by ψ_t and its distribution is given by:

$$p(\psi_t = i) = \left(\underline{\tau}_x^T e^{\Lambda t}\right)_i, \; i = 1, 2, ..., n. \tag{9}$$

For annual costs X_t we define a Markov reward model (abbreviated as “MRM”). It is convenient to introduce the model as a triplet $(A \cup \{D\}, \underline{W}, P)$, where:
1. $A \cup \{D\}$ is the set of possible states: A is the set of n health states from the aging model, $\{D\}$ is one absorbing state for death;

2. $\underline{W}$ is the set of $n+1$ variables representing annual cost in different health states. We assume that, for $i \in A, W_i, i = 1, ..., n$ is a discrete random: it is defined on a given set $C = \{c_1, c_2, \ldots, c_M\}$ of non-negative values with a distribution which may depend on *i*. For $i \in \{D\}$, W_{n+1} is zero with probability one.

P is the one-year transition probability matrix for the states, which is constructed as follows

$$P = \begin{pmatrix} e^{\Lambda} & \underline{y} \\ \underline{0}^T & 1 \end{pmatrix}, \tag{10}$$

where $\underline{Y}$ is the conditional probability to die in the first year, given the state at time 0. It is a complement to the one year survival probability. So that

$$\underline{y} = \underline{1} - e^{\Lambda}\underline{1}. \tag{11}$$

3. Probability Generating Function of S.

Let $g(\zeta)$ be the probability generation function of S. It is often easier to calculate the moments of a random variable S than finding distribution. If the probability generation function of S or its moment generating function (mgf) S exists, it is possible to obtain the corresponding distribution of S by inversion of its mgf. In Govorun et al. (2014), the expressions obtained for the distribution of the Net Present Value of the Markov Reward Model are hardly simple. We could then think about another technique other than the one proposed by the above authors by studying the moments of S.

Theorem 1

The probability generation function of S is given by:

$$g(\zeta) = P_S(\zeta) = \underline{\tau}_x^T \underline{\sigma}(\zeta), \tag{12}$$

where

$$p(\psi_0 = i) = \underline{\tau}_x \text{ and } \underline{\sigma}(\zeta) = E\left(\zeta^S \middle| \psi_0 = i\right) = \underline{\Upsilon}(\zeta) e^{\Lambda} \underline{\sigma}(\zeta^v) + \underline{\Upsilon}(\zeta)\underline{Y}, \tag{13}$$

with

$$\underline{\gamma}(\zeta)=\begin{pmatrix} f_1(\zeta) & 0 & 0 & \cdots & 0 & 0 \\ 0 & f_2(\zeta) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & f_i(\zeta) & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & f_n(\zeta) \end{pmatrix},$$

and $f_i(\zeta)=E\left(\zeta^{X_1}\left|\psi_0=i\right.\right)=E\left(\zeta^{W_i}\right)$.

Proof.

$$\begin{aligned} g(\zeta) &= P_S(\zeta)=E\left(\zeta^S\right) \\ &= E\left(\zeta^S\left|\psi_0=i\right.\right)p(\psi_0=i)=\underline{\tau}_x^T\underline{\sigma}(\zeta) \\ &= \underline{\sigma}(\zeta):\sigma_i(\zeta)=E\left(\zeta^S\left|\psi_0=i\right.\right) \\ &= E\left(\zeta^S\left|\lfloor L\rfloor=1,\psi_0=i\right.\right)p\left(\lfloor L\rfloor=1\left|\psi_0=i\right.\right)+E\left(\zeta^S\left|\lfloor L\rfloor\geq 2,\psi_0=i\right.\right)p\left(\lfloor L\rfloor\geq 2\left|\psi_0=i\right.\right) \\ &= E\left(\zeta^S\left|\lfloor L\rfloor=1,\psi_0=i\right.\right)y_i+\sum_{s\in\{A\}}E\left(\zeta^{X_1+vS}\left|\psi_1=s,\psi_0=i\right.\right)p\left(\psi_1=s\left|\psi_0=i\right.\right) \\ &= E\left(\zeta^{X_1}\left|\psi_0=i\right.\right)Y_i+\sum_{s\in\{A\}}\left(e^{\Lambda}\right)_{(i,s)}E\left(\zeta^{X_1}\left|\psi_1=s,\psi_0=i\right.\right)E\left(\zeta^{vS}\left|\psi_1=s,\psi_0=i\right.\right) \\ &= E\left(\zeta^{X_1}\left|\psi_0=i\right.\right)Y_i+\sum_{s\in\{A\}}\left(e^{\Lambda}\right)_{(i,s)}E\left(\zeta^{X_1}\left|\psi_0=i\right.\right)E\left(\left(\zeta^{v}\right)^{\widetilde{S}}\left|\psi_1=s,\psi_0=i\right.\right) \\ &= E\left(\zeta^{X_1}\left|\psi_0=i\right.\right)Y_i+E\left(\zeta^{X_1}\left|\psi_0=i\right.\right)\sum_{s\in\{A\}}\left(e^{\Lambda}\right)_{(i,s)}E\left(\left(\zeta^{v}\right)^{\widetilde{S}}\left|\psi_1=s,\psi_0=i\right.\right). \end{aligned}$$

Then,

$$\sigma_i(\zeta)=f_i(\zeta)Y_i+f_i(\zeta)\sum_{s\in\{A\}}\left(e^{\Lambda}\right)_{(i,s)}\sigma_s\left(\zeta^{v}\right).$$

In the matrix representation we obtain:

$$\underline{\sigma}(\zeta)=E\left(\zeta^S\left|\psi_0=i\right.\right)=\underline{\Upsilon}(\zeta)e^{\Lambda}\underline{\sigma}\left(\zeta^{v}\right)+\underline{\Upsilon}(\zeta)\underline{Y}\ .$$

Hence (13).

4. Recursive formula for the moments of S.

The mathematical expectation of total claims plays an important role in the determination of the pure premium, in addition to giving a measure of the central tendency of its distribution. The moments centered at the average of order 2, 3 and 4 are the other moments usually considered because they usually give a good indication of the pace of distribution, and these give us respectively a measure of the dispersion of the distribution around its mean, a measure of the asymmetry and flattening of the distribution considered.

Moments, whether simple, joined or conditional, may eventually be used to construct approximations of the distribution of the Markov Reward Model.

Theorem 1: According to the hypotheses of Section 2, for any $t \geq 0,\ \delta \geq 0$, the n^{th} moment of S is given by

$$E\left[S^n\right] = \tau_x^T \left\{ \sum \frac{n!}{k_1!k_2!...k_n!} \sigma^{(k)}(1) \prod_{j=1}^{n} \left(\frac{1}{j!}\right)^{k_j} \right\}, \tag{14}$$

where

$k_1 + 2k_2 + ... + nk_n = n;\ k_1 + k_2 + ... + k_n = k$, and where $\sigma^{(k)}(1)$ are solutions of the equations :

$$\sigma^{(m)}(1) = \sum_{n=0}^{m} \binom{m}{n} \Upsilon^{(m-n)}(1) e^{\wedge} \left(\sigma\left(\zeta^{v}\right)\right)^{(n)}\Big|_{\zeta=1} + \Upsilon^{(m)}(1) \underline{Y}, \tag{15}$$

with

$$\left(\sigma\left(\zeta^{v}\right)\right)^{(n)}\Big|_{\zeta=1} = \sum \frac{n!}{k_1!k_2!...k_n!} \sigma^{(k)}(1) \prod_{j=1}^{n} \left(\frac{v(v-1)...(v-j+1)\zeta^{v-j}}{j!}\right)^{k_j}. \tag{16}$$

Proof. Taking derivative of $M_S(\xi) = \tau_x^T \sigma\left(e^{\xi}\right)$ and using Faà di Bruno's (1855) rule's yield

$$\left(\sigma\left(e^{\xi}\right)\right)^{(n)} = \sum \frac{n!}{k_1!k_2!...k_n!} f^{(k)}(g(\xi)) \prod_{j=1}^{n} \left(\frac{g^{(j)}(\xi)}{j!}\right)^{k_j}, \tag{17}$$

where the sum is over all nonnegative integer solutions of the Diophantine equation $k_1 + 2k_2 + ... + nk_n = n;\ k_1 + k_2 + ... + k_n = k$, and $g(\xi) = e^{\xi}$.

Otherwise

$$f^{(k)}(g(\xi)) = \sigma^{(k)}(g(\xi)), \tag{18}$$

and

$$g^{(k)}(\xi) = e^{\xi}, \tag{19}$$

it follows that

$$f^{(k)}(1) = \sigma^{(k)}(1), \tag{20}$$

and

$$g^{(k)}(0) = 1. \tag{21}$$

Substituting (20) and (21) into (17) with $\zeta = 0$ yields

$$\left(\sigma\left(e^{\xi}\right)\right)^{(n)}\Big|_{\zeta=0} = \sum \frac{n!}{k_1!k_2!...k_n!}\sigma^{(k)}(1)\prod_{j=1}^{n}\left(\frac{1}{j!}\right)^{k_j}.$$

Multiplying both sides of the above equation by τ_x^T, yields

$$\tau_x^T\left(\sigma\left(e^{\xi}\right)\right)^{(n)}\Big|_{\zeta=0} = \tau_x^T\left\{\sum \frac{n!}{k_1!k_2!...k_n!}\sigma^{(k)}(1)\prod_{j=1}^{n}\left(\frac{1}{j!}\right)^{k_j}\right\},$$ to finally have:

$$E\left[S^n\right] = \tau_x^T\left\{\sum \frac{n!}{k_1!k_2!...k_n!}\sigma^{(k)}(1)\prod_{j=1}^{n}\left(\frac{1}{j!}\right)^{k_j}\right\}. \tag{22}$$

We have:

$$M_S(\xi) = \tau_x^T\sigma\left(e^{\xi}\right),$$

Taking the (m) derivative on the right and left side of equation (13), yields :

$$\sigma^{(m)}(\zeta) = \sum_{n=0}^{m}\binom{m}{n}\Upsilon^{(m-n)}(\zeta)e^{\Lambda}\left(\sigma\left(\zeta^{v}\right)\right)^{(n)} + \Upsilon^{(m)}(\zeta)\underline{Y}.$$

5. Examples

Since it is important to evaluate the mean, variance and skewness of the present value associated with our risk process, the following formulas will give us the basic tools to calculate the first three moments of S.

Example 1 : First moment of S

If $v \neq \rho^{-1}\left(e^{\Lambda}\right)$, the expectation of S in the Markov reward model $\left(A\cup\{D\},\underline{W},P\right)$ is

$$E(S) = \frac{\partial g(\zeta)}{\partial\zeta}\Big|_{\zeta=1} = \underline{\tau}_x^T\left(I - ve^{\Lambda}\right)^{-1}E\left(\underline{W}_A^T\right), \tag{23}$$

where $\underline{\tau}_x$ is the initial health state distribution, Λ is the generator of the corresponding phase-type aging process and $^{-1}\underline{\tau}_x^T E(S) = \underline{\sigma}'(1)$.

Proof.

If $n=1$, then $k_1=k_2=1$ and from Theorem 1, we have:

$$E[S]=\tau_x^T\underline{\sigma}'(1),$$

$$\begin{aligned}\sigma'(1)&=\sum_{n=0}^{1}\binom{1}{n}\Upsilon^{(1-n)}(1)e^{\Lambda}\left(\sigma\left(\zeta^{v}\right)\right)^{(n)}\Big|_{\zeta=1}+\Upsilon'(1)\underline{Y}\\&=\Upsilon'(1)e^{\Lambda}\sigma(1)+\Upsilon(1)e^{\Lambda}\left(\sigma\left(\zeta^{v}\right)\right)'\Big|_{\zeta=1}+\Upsilon'(1)\underline{Y},\end{aligned}\tag{24}$$

with,

$$\left(\sigma\left(\zeta^{v}\right)\right)'\Big|_{\zeta=1}=v\sigma'(1).\tag{25}$$

Substituting equation (25) in (24), we obtain: $v\neq\rho^{-1}\left(e^{\Lambda}\right)$,

$$\sigma'(1)=\Upsilon'(1)e^{\Lambda}\sigma(1)+\Upsilon(1)e^{\Lambda}v\sigma'(1)+\Upsilon'(1)\underline{Y}.$$

Solving the above equation in $\sigma'(1)$, we get:

$$\sigma'(1)\left(1-v\Upsilon(1)e^{\Lambda}\right)=\Upsilon'(1)\left(\underline{Y}+e^{\Lambda}\sigma(1)\right),$$

with $\left(\underline{Y}+e^{\Lambda}\sigma(1)\right)=\underline{1}$ and $\Upsilon(1)=\underline{1}$, $\Upsilon'(1)=E\left(\underline{W}_A\right)$, we finally obtain:

$\sigma'(1)=\left(1-ve^{\Lambda}\right)^{-1}E\left(\underline{W}_A\right)\underline{1}$, which gives the result in example 1.

In the following example, we derive formulas to calculate the second moment of S.

Example 2 : Second moment of S

If $v\neq\rho^{-1}\left(e^{\Lambda}\right)$ and $v^2\neq\rho^{-1}\left(e^{\Lambda}\right)$, the second moment of S in the Markov reward model $\left(A\cup\{D\},\underline{W},P\right)$ is:

$$E\left(S^2\right)=\frac{d^2}{d\zeta^2}M_S(\zeta)\Big|_{\zeta=0}=\underline{\tau}_x^T\left(I-v^2e^{\Lambda}\right)^{-1}\left[E\left(\underline{W}_A^2\right)\underline{1}+2vE\left(\underline{W}_A\right)e^{\Lambda}\left({}^{-1}\underline{\tau}_x^T\right)E(S)\right],\tag{26}$$

Where $\underline{\tau}_x$ is the initial health state distribution, Λ is the generator of the corresponding phase-type aging process and $\left(\underline{\tau}_x^T\right)^{-1}E\left(S^2\right)=\left[\underline{\sigma}'(1)+\underline{\sigma}''(1)\right]$.

Proof

If $n=2$, we find that the nonnegative integer solutions of the equation $k_1+2k_2=2$ are $(k_1;\, k_2)=(2;\, 0)$ or $(0;\, 1)$ with corresponding values of k being 2 or 1 respectively, we deduce:

$$E\left[S^2\right]=\tau_x^T\left\{\underline{\sigma}'(1)+\underline{\sigma}''(1)\right\},$$

$$\begin{aligned}\sigma''(1)&=\sigma^{(2)}(1)=\sum_{n=0}^{2}\binom{2}{n}\Upsilon^{(2-n)}(1)e^{\Lambda}\left(\sigma\left(\zeta^{v}\right)\right)^{(n)}\Big|_{\zeta=1}+\Upsilon^{(2)}(1)\underline{Y}\\&=\Upsilon''(1)e^{\Lambda}\sigma(1)+2\Upsilon'(1)e^{\Lambda}\left(\sigma\left(\zeta^{v}\right)\right)'\Big|_{\zeta=1}+\Upsilon(1)e^{\Lambda}\left(\sigma\left(\zeta^{v}\right)\right)''\Big|_{\zeta=1}+\Upsilon''(1)\underline{Y},\end{aligned}\qquad(27)$$

with,

$$\begin{aligned}\left(\sigma\left(\zeta^{v}\right)\right)''\Big|_{\zeta=1}&=\sum\frac{2!}{k_1!k_2!}\sigma^{(k)}(1)\prod_{j=1}^{2}\left(\frac{v(v-1)...(v-j+1)\zeta^{v-j}}{j!}\right)^{k_j}\\&=v^2\sigma''(1)+v(v-1)\sigma'(1).\end{aligned}\qquad(28)$$

Substituting equation (27) in (26), we obtain: $v\neq\rho^{-1}\left(e^{\Lambda}\right)$, $v^2\neq\rho^{-1}\left(e^{\Lambda}\right)$,

$$\begin{aligned}\sigma''(1)&=\Upsilon''(1)e^{\Lambda}\sigma(1)+2\Upsilon'(1)e^{\Lambda}\left\{v\sigma'(1)\right\}\\&\quad+\Upsilon(1)e^{\Lambda}\left\{v^2\sigma''(1)+v(v-1)\sigma'(1)\right\}+\Upsilon''(1)\underline{Y}\\&=\Upsilon''(1)\left\{\underline{Y}+e^{\Lambda}\sigma(1)\right\}+v^2\Upsilon(1)e^{\Lambda}\sigma''(1)+v(v-1)\Upsilon(1)e^{\Lambda}\sigma'(1)+2v\Upsilon'(1)e^{\Lambda}\sigma'(1),\end{aligned}$$

with $\left(\underline{Y}+e^{\Lambda}\sigma(1)\right)=\underline{1}$ and $\Upsilon(1)=\underline{1}$, $\Upsilon'(1)+\Upsilon''(1)=E\left(\underline{W}^2{}_A\right)$, we have:

$$\begin{aligned}\sigma''(1)&=\Upsilon''(1)+v^2e^{\Lambda}\sigma''(1)+v(v-1)e^{\Lambda}\sigma'(1)+2v\Upsilon'(1)e^{\Lambda}\sigma'(1)\\&=\Upsilon''(1)+v^2e^{\Lambda}\sigma''(1)+v^2e^{\Lambda}\sigma'(1)+2v\Upsilon'(1)e^{\Lambda}\sigma'(1)-ve^{\Lambda}\sigma'(1)+\sigma'(1)-\sigma'(1)\\\sigma''(1)+\sigma'(1)&=\left(1-ve^{\Lambda}\right)\sigma'(1)+v^2e^{\Lambda}\left(\sigma''(1)+\sigma'(1)\right)+2v\Upsilon'(1)e^{\Lambda}\sigma'(1)+\Upsilon''(1).\end{aligned}$$

For, $\left(1-ve^{\Lambda}\right)\sigma'(1)=E\left(\underline{W}_A\right)\underline{1}$, solving the above equation, we have:

$$\left(1-v^2e^{\Lambda}\right)\left(\sigma''(1)+\sigma'(1)\right)=E\left(\underline{W}^2{}_A\right)+2v\Upsilon'(1)e^{\Lambda}\sigma'(1),$$

which gives the result in example 2.

To study the skewness of S, we will need the following result.

Example 3: **Third moment of** S

If $v\neq\rho^{-1}\left(e^{\Lambda}\right)$, $v^2\neq\rho^{-1}\left(e^{\Lambda}\right)$ and $v^3\neq\rho^{-1}\left(e^{\Lambda}\right)$, the third moment $E\left(S^3\right)$ of S in the Markov Reward Model $\left(A\cup\{D\},\underline{W},P\right)$ is:

$$E\left(S^3\right)=\underline{\tau}_x^T\left(I-v^3e^{\Lambda}\right)^{-1}\left[E\left(\underline{W}_A^3\right)\underline{1}+3vE\left(\underline{W}_A^2\right)e^{\Lambda}\,{}^{-1}\underline{\tau}_x^T E\left(S\right)+3v^2E\left(\underline{W}_A\right)e^{\Lambda}\,{}^{-1}\underline{\tau}_x^T E\left(S^2\right)\right], \quad (29)$$

where $\underline{\tau}_x$ is the initial health state distribution, Λ is the generator of the corresponding phase-type aging process, ${}^{-1}\underline{\tau}_x^T E\left(S\right)=\underline{\sigma}'(1)$ and ${}^{-1}\underline{\tau}_x^T E\left(S^2\right)=\underline{\sigma}'(1)+\underline{\sigma}''(1)$, ${}^{-1}\underline{\tau}_x^T E\left(S^3\right)=\left[\underline{\sigma}'(1)+3\underline{\sigma}''(1)+\underline{\sigma}'''(1)\right]$.

Proof.

If $n=3$, we find that the nonnegative integer solutions of the equation $k_1+2k_2+3k_3=3$ are $(k_1;\,k_2;\,k_3)=(0;\,0;\,1)$, $(k_1;\,k_2;\,k_3)=(3;\,0;\,0)$ or $(1;\,1;\,0)$ with corresponding values of k being 1, 2 or 3 respectively, we deduce that:

$$E\left[S^3\right]=\tau_x^T\left\{\underline{\sigma'''(1)+3\sigma''(1)+\sigma'(1)}\right\},$$

$$\begin{aligned}\sigma'''(1)=\sigma^{(3)}(1)&=\sum_{n=0}^{3}\binom{3}{n}\Upsilon^{(3-n)}(1)e^{\Lambda}\left(\sigma\left(\zeta^v\right)\right)^{(n)}\Big|_{\zeta=1}+\Upsilon^{(3)}(1)\underline{Y}\\ &=\Upsilon'''(1)e^{\Lambda}\sigma(1)+3\Upsilon''(1)e^{\Lambda}\left(\sigma\left(\zeta^v\right)\right)'\Big|_{\zeta=1} \qquad (30)\\ &\quad+3\Upsilon'(1)e^{\Lambda}\left(\sigma\left(\zeta^v\right)\right)''\Big|_{\zeta=1}+\Upsilon(1)e^{\Lambda}\left(\sigma\left(\zeta^v\right)\right)'''\Big|_{\zeta=1}+\Upsilon'''(1)\underline{Y},\end{aligned}$$

with,

$$\begin{aligned}\left(\sigma\left(\zeta^v\right)\right)^{(3)}\Big|_{\zeta=1}&=\sum\frac{3!}{k_1!k_2!k_3!}\sigma^{(k)}(1)\prod_{j=1}^{3}\left(\frac{v(v-1)\ldots(v-j+1)\zeta^{v-j}}{j!}\right)^{k_j} \qquad (31)\\ &=v^3\sigma'''(1)+3v^2(v-1)\sigma''(1)+v(v-1)(v-2)\sigma'(1)\,.\end{aligned}$$

Substituting equation (31) in (30), we obtain: $v\neq\rho^{-1}\left(e^{\Lambda}\right)$, $v^2\neq\rho^{-1}\left(e^{\Lambda}\right)$, $v^3\neq\rho^{-1}\left(e^{\Lambda}\right)$

$$\begin{aligned}\sigma'''(1)=\Upsilon'''(1)e^{\Lambda}\sigma(1)&+3\Upsilon''(1)e^{\Lambda}\left\{v\sigma'(1)\right\}\\ &+3\Upsilon'(1)e^{\Lambda}\left\{v^2\sigma''(1)+v(v-1)\sigma'(1)\right\}+\Upsilon'''(1)\underline{Y}\\ &+\Upsilon(1)e^{\Lambda}\left\{v^3\sigma'''(1)+3v^2(v-1)\sigma''(1)+v(v-1)(v-2)\sigma'(1)\right\},\end{aligned}$$

then,

$$\begin{aligned}\sigma'''(1)+3\sigma''(1)+\sigma'(1)&=\left(\Upsilon'(1)+3\Upsilon''(1)+\Upsilon'''(1)\right)\left(\underline{y}+e^{\Lambda}\underline{\sigma}(1)\right)+3v\left(\Upsilon'(1)+\Upsilon''(1)\right)e^{\Lambda}\underline{\sigma}'(1)\\ &+3v^2\Upsilon'(1)e^{\Lambda}\underline{\sigma}'(1)+3v^2\Upsilon'(1)e^{\Lambda}\underline{\sigma}''(1)+v^3\underline{\gamma}(1)e^{\Lambda}\underline{\sigma}'(1)\\ &+3v^3\Upsilon(1)e^{\Lambda}\underline{\sigma}''(1)+v^3\Upsilon(1)e^{\Lambda}\underline{\sigma}'''(1)\,.\end{aligned}$$

We also show that $\Upsilon'''(1)=E\left(\underline{W}_A^3\right)-3E\left(\underline{W}_A^2\right)+2E\left(\underline{W}_A\right)$ or $\Upsilon''(1)=E\left(\underline{W}_A^2\right)-E\left(\underline{W}_A\right)$, $\Upsilon'(1)=E\left(\underline{W}_A\right)$ and $\underline{\sigma}(1)=\underline{1}$ then after simplification we obtain:

$$\sigma'''(1)+3\sigma''(1)+\sigma'(1)=\underline{\tau}_x^T\Big[E\left(\underline{W}_A^3\right)\underline{1}+3vE\left(\underline{W}_A^2\right)e^{\Lambda}\,\underline{\sigma}'(1)+3v^2E\left(\underline{W}_A\right)e^{\Lambda}\left(\underline{\sigma}'(1)+\underline{\sigma}''(1)\right)$$
$$+\,v^3e^{\Lambda}\,\underline{\sigma}'(1)+3v^3e^{\Lambda}\,\underline{\sigma}''(1)+v^3e^{\Lambda}\,\underline{\sigma}'''(1)\Big],$$

and:

$$\underline{\sigma}'(1)+3\underline{\sigma}''(1)+\underline{\sigma}'''(1)=E\left(\underline{W}_A^3\right)\underline{1}+3vE\left(\underline{W}_A^2\right)e^{\Lambda}\underline{\sigma}'(1)+3v^2E\left(\underline{W}_A\right)e^{\Lambda}\left(\underline{\sigma}'(1)+\underline{\sigma}''(1)\right)$$
$$+\,v^3e^{\Lambda}\underline{\sigma}'(1)+3v^3e^{\Lambda}\underline{\sigma}''(1)+v^3e^{\Lambda}\underline{\sigma}'''(1).$$

Then,

$$\left(I-v^3e^{\Lambda}\right)\underline{\sigma}'''(1)=E\left(\underline{W}_A^3\right)\underline{1}+3vE\left(\underline{W}_A^2\right)e^{\Lambda}\underline{\sigma}'(1)+3v^2E\left(\underline{W}_A\right)e^{\Lambda}\left(\underline{\sigma}'(1)+\underline{\sigma}''(1)\right)$$
$$-\left(I-v^3e^{\Lambda}\right)\underline{\sigma}'(1)-3\left(I-v^3e^{\Lambda}\right)\underline{\sigma}''(1),$$

$$\underline{\sigma}'''(1)=\left(I-v^3e^{\Lambda}\right)^{-1}\Big[E\left(\underline{W}_A^3\right)\underline{1}+3vE\left(\underline{W}_A^2\right)e^{\Lambda}\underline{\sigma}'(1)+3v^2E\left(\underline{W}_A\right)e^{\Lambda}\left(\underline{\sigma}'(1)+\underline{\sigma}''(1)\right)\Big]$$
$$-\underline{\sigma}'(1)-3\underline{\sigma}''(1).$$

But,

$$E\left(S^3\right)=M_S'''(\zeta)\Big|_{\zeta=0}=\underline{\tau}_x^T\left[\underline{\sigma}'(1)+3\underline{\sigma}''(1)+\underline{\sigma}'''(1)\right]$$
$$=\underline{\tau}_x^T\left(I-v^3e^{\Lambda}\right)^{-1}\Big[E\left(\underline{W}_A^3\right)\underline{1}+3vE\left(\underline{W}_A^2\right)e^{\Lambda}\underline{\sigma}'(1)+3v^2E\left(\underline{W}_A\right)e^{\Lambda}\left(\underline{\sigma}'(1)+\underline{\sigma}''(1)\right)\Big],$$

with ${}^{-1}\underline{\tau}_x^TE(S)=\underline{\sigma}'(1)$ and ${}^{-1}\underline{\tau}_x^TE\left(S^2\right)=\underline{\sigma}'(1)+\underline{\sigma}''(1)$ .To finally have the closed form expression of the third moment of the annual health care cost:

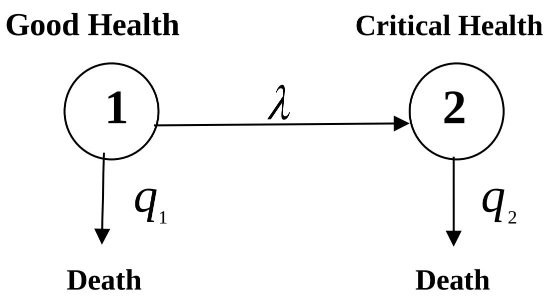
$$E\left(S^3\right)=\underline{\tau}_x^T\left(I-v^3e^{\Lambda}\right)^{-1}\Big[E\left(\underline{W}_A^3\right)\underline{1}+3vE\left(\underline{W}_A^2\right)e^{\Lambda}\,{}^{-1}\underline{\tau}_x^TE(S)+3v^2E\left(\underline{W}_A\right)e^{\Lambda}\,{}^{-1}\underline{\tau}_x^TE\left(S^2\right)\Big].$$

Example:

In this section, we will use the following tractable example, taken into account the interesting parametrization aspects of the MRM.

We consider one insurance contract for age X with two health's states $n=2$: {Good heath; Critical ill} and one absorbing phase (the death). This contract is modeled using the MRM by the following PH representation:

Fig. 2 Two states PH aging process



- The annual heath care cost depends on the insured health state $i \in \{1;2\}$ such as:

Table1: Conditional annual health care cost				
	i	j	c_j^i	$f^{(i)}(c_j)$
W_i	1	1	5	$\frac{3}{4}$
		2	25	$\frac{1}{4}$
	2	1	50	$\frac{2}{3}$
		2	100	$\frac{1}{3}$

Consider three insureds aged $X_1 = 30, X_2 = 40$ and $X_3 = 50$, we have the following results in this case $\lambda = 0.3$, $\alpha_1 = 1, \alpha_2 = 0$, $q_1 = 0.15$ and $q_2 = 0.45$ for different values of $v = 0.92$:

Table2: Contract analysis1

Insured age X	Expected Lifetime $\bar{L}_X$	Expected Health Cost	Cost Standard Deviation
30	2.905	154.429	107.902
40	2.872	156.008	108.596
50	2.851	156.995	109.015

Consider three insureds aged $X_1 = 30$, $X_2 = 40$ and $X_3 = 50$, we have the following results in this case where $\lambda = 0.3$, $\alpha_1 = 1, \alpha_2 = 0$, $q_1 = 0.15$ and $q_2 = 0.45$ for $v = 1.02$:

Table3: Contract analysis 2

Insured age X	Expected Lifetime $\bar{L}_X$	Expected Health Cost	Cost Standard Deviation
30	13.007	243.918	309.258
40	13.007	244.918	309.258
50	13.007	246.918	309.258

6. CONCLUSION

We have found recursive formulas for the moments of the Net Present Value of the Markov Reward Model, with a constant interest rate. These formulas have been obtained by giving first an integral expression for the moments generating function of our risk process and thereafter by taking the appropriate derivatives.

The expressions for the distribution function of Markov Reward Model derived by Govorum et al. (2014) are not very simple and be may be difficult to implement. One could then think of approximating this distribution function by another one whose parameters would have to be estimated, for example, by the method of moments. The first four moments often give a good indication of the shape of the distribution of S .

Possible extensions to this research include taking into account random interest rate of the Net Present Value of the Markov reward process. We also intend to collect data for the South African market and estimate not only the different transition probabilities but also the health Cost for a given health status.

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