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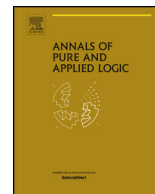For additional information about this publication click this link.
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Fixed point theorems for precomplete numberings

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A R T I C L E   I N F O

A B S T R A C T

In the context of his theory of numberings, Ershov showed that Kleene's recursion theorem holds for any precomplete numbering. We discuss various generalizations of this result. Among other things, we show that Arslanov's completeness criterion also holds for every precomplete numbering, and we discuss the relation with Visser's ADN theorem, as well as the uniformity or nonuniformity of the various fixed point theorems. Finally, we base numberings on partial combinatory algebras and prove a generalization of Ershov's theorem in this context.

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1. Introduction

In this paper we discuss various fixed point theorems in computability theory, and related areas such as λ-calculus and combinatory algebra. The starting point is Kleene's famous recursion theorem [15], which was generalized to precomplete numberings by Ershov [10]. These are discussed in section 3, after we first discuss Ershov's theory of numberings in section 2.

The recursion theorem was generalized in other ways by Visser [30] and Arslanov [2]. Visser proved the so-called 'anti diagonal normalization theorem' that we discuss in section 4. Arslanov extended the recursion theorem from computable functions to arbitrary functions computable from an incomplete c.e. Turing degree. The Arslanov completeness criterion states that a c.e. set is Turing complete if and only if it computes a fixed point free function. Recently, a joint generalization of Arslanov's completeness criterion and the ADN theorem was given by Terwijn [27]. We discuss Arslanov's completeness criterion in section 5.

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Finally, in sections 6 and 7, we discuss the relation with Feferman's version of the recursion theorem for partial combinatory algebras (pca's) [11]. Here we base the notion of numbering on pca's of arbitrary cardinality, and prove a fixed point theorem for these (Theorem 6.5). This generalizes Ershov's recursion theorem in this setting.

Our notation from computability theory is mostly standard. In the following, φ_n denotes the n-th partial computable (p.c.) function, in some standard numbering of the p.c. functions. Partial computable (p.c.) functions are denoted by lower case Greek letters, and (total) computable functions by lower case Roman letters. ω denotes the natural numbers. W_e denotes the domain of the p.c. function φ_e. We write $\varphi_e(n)\downarrow$ if this computation is defined, and $\varphi_e(n)\uparrow$ otherwise. We let $\langle e, n \rangle$ denote a computable pairing function. $\emptyset'$ denotes the halting set. For unexplained notions we refer to Odifreddi [21] or Soare [26].

2. Numberings and equivalence relations

The theory of numberings (also called numerations, after the German 'Numerierung') was initiated by Ershov. The following concepts were introduced by him in [8].

Definition 2.1. A *numbering* of a set S is a surjection $\gamma \colon \omega \to S$. Given γ, define an equivalence relation on ω by $n \sim_\gamma m$ if $\gamma(n) = \gamma(m)$.

A numbering γ is *precomplete* if for every partial computable unary function ψ there exists a computable unary f such that for every n

$$\psi(n)\downarrow \implies f(n) \sim_\gamma \psi(n). \tag{1}$$

Following Visser, we say that f *totalizes ψ modulo $\sim_\gamma$* if (1) holds.

A precomplete numbering γ is *complete* if there is a *special element* $a \in \omega$ such that next to (1) also $f(n) \sim_\gamma a$ for every n with $\psi(n)\uparrow$.

The prime example of a numbering is $n \mapsto \varphi_n$ for the set of unary p.c. functions. This numbering is precomplete: by the S-m-n-theorem, for any p.c. ψ there is a (total) computable f such that $\varphi_{f(n)} = \varphi_{\psi(n)}$ for every n such that $\psi(n)\downarrow$. The numbering is even complete: as required special element we can take an index of the totally undefined function.

The numbering $n \mapsto W_n$ of the c.e. sets is closely related (and for our purposes below equivalent) to the numbering of the p.c. functions. It is also complete, with as special element the empty set.

Other examples of numberings come from λ-calculus. For example, the closed λ-terms, modulo β-equality, can be enumerated as a precomplete numbering,[1] cf. Visser [30, p261,264], referring to Barendregt. If moreover unsolvable λ-terms are equated, then this numbering even becomes complete. Other examples can be found in [30], and still more examples come from pca's, that we discuss in section 6 below.

Numberings and equivalence relations are mutually related [6]. For every numbering γ we have the corresponding equivalence $\sim_\gamma$. Conversely, given an equivalence relation R on ω (or any other countable set), we have the numbering $n \mapsto [n]$ of the equivalence classes of R. Hence the study of numberings is equivalent to that of (countable) equivalence relations. In particular we can also apply the terminology of Definition 2.1 to such relations, and talk about precomplete and complete equivalence relations.

A class of countable equivalence relations that is of particular interest is the class of computably enumerable equivalence relations, simply called *ceers*. These were studied by Ershov [9] in the context of the theory of numberings (though examples of them occurred earlier in the literature), and in early writings

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[1] By $\gamma(n) = \mathsf{E}\mathbf{c}_n$, where E is a λ-term enumerating closed terms and $\mathbf{c}_n$ is the n-th numeral adequately representing natural numbers in λ-calculus.

were called *positive* equivalence relations. Bernardi and Sorbi [6] proved that every precomplete ceer is m-complete (even with an extra uniformity condition). They also showed that this implies 1-completeness [6, p532]. This result was later strengthened by Lachlan [18] (see also [1, p425]), who showed that all precomplete ceers are computably isomorphic. For a recent survey about ceers we refer the reader to Andrews, Badaev, and Sorbi [1].

An interesting example of a ceer (discussed in Bernardi and Sorbi [6, p534]) is the Lindenbaum algebra of PA (Peano arithmetic). Identify formulas φ and ψ in the language of PA with their Gödel numbers. Let $\varphi \sim_{\mathrm{PA}} \psi$ if these formulas are provably equivalent in PA. Then $\sim_{\mathrm{PA}}$ is obviously a ceer. This relation is not precomplete, as can be seen using Theorem 3.1 below: the function $\varphi \mapsto \neg\varphi$ is computable, but does not have a fixed point modulo $\sim_{\mathrm{PA}}$. By contrast, the analogous ceer $\sim_{\Sigma_n}$, obtained by considering the fragment of PA of Σ_n-formulas, is precomplete, cf. Visser [30, p263].

3. The recursion theorem

Kleene's recursion theorem [15] states that every computable function h has a fixed point, in the sense that there exists a number n such that $\varphi_{h(n)} = \varphi_n$. This result holds uniformly, meaning that the fixed point can be found computably from a code of h. For an extensive discussion of this fundamental theorem, and the many applications it has found in logic, see Moschovakis [20].

Using Ershov's terminology, we can phrase Kleene's result by saying that h has a fixed point modulo $\sim_\gamma$, where γ is the numbering $n \mapsto \varphi_n$ of the p.c. functions. Ershov showed that the recursion theorem holds for every precomplete numbering γ in the following way.

Theorem 3.1. (Ershov's recursion theorem [10]) *Let γ be a precomplete numbering, and let h be a computable function. Then h has a fixed point modulo $\sim_\gamma$, i.e. there exists a number n such that $h(n) \sim_\gamma n$.*

As is the case for Kleene's recursion theorem, this result holds uniformly. For later reference we explicitly state the following version:

Theorem 3.2. (Ershov's recursion theorem with parameters) *Let γ be a precomplete numbering, and let $h(x, n)$ be a computable binary function. Then there exists a computable function f such that for all n, $f(n) \sim_\gamma h(f(n), n)$.*

Proof. By precompleteness, let d be a computable function such that

$$d(x, n) \sim_\gamma \varphi_x(x, n)$$

for every x and n where the latter is defined.[2] Let e be a code such that $\varphi_e(x, n) = h(d(x, n), n)$ for all x and n. Then

$$d(e, n) \sim_\gamma \varphi_e(e, n) = h(d(e, n), n),$$

so that $d(e, n)$ is a fixed point for every n. $\square$

Theorem 3.2 is equivalent with the following form, given in Andrews, Badaev, and Sorbi [1, p423].

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[2] Note that we can generalize precompleteness (Definition 2.1) to functions with multiple arguments, which is allowed by the usual coding of sequences.

Theorem 3.3. *Let γ be a precomplete numbering. There exists a computable function f such that for every n, if $\varphi_n(f(n))\downarrow$ then*

$$\varphi_n(f(n)) \sim_\gamma f(n).$$

Theorem 3.2 and Theorem 3.3 are equivalent, *for precomplete numberings*. To see that Theorem 3.3 implies Theorem 3.2, observe that, given a computable function h as in the latter theorem, there is a computable function g such that $\varphi_{g(n)}(x) = h(x,n)$ for every x and n. For f as in Theorem 3.3 we then have

$$h(f(g(n)),n) = \varphi_{g(n)}(f(g(n))) \sim_\gamma f(g(n))$$

for every n, so $f \circ g$ is the desired computable function producing fixed points.

Conversely, Theorem 3.2 implies Theorem 3.3. By precompleteness of γ, there is a computable function h that totalizes the universal p.c. function modulo $\sim_\gamma$, i.e. such that

$$\varphi_n(x)\downarrow \implies h(x,n) = \varphi_n(x)$$

for every x and n. Now Theorem 3.2 provides the required fixed points $f(n)$.

The converse of Theorem 3.3 also holds. The statement of the theorem holds for a numbering γ if and only if γ is precomplete (cf. [1, p423]). Since the equivalence of Theorem 3.2 and Theorem 3.3 above uses that γ is precomplete, it is not clear whether the converse of Theorem 3.2 also holds. Hence we ask the following.

Question 3.4. *Suppose that an arbitrary numbering γ satisfies the statement of Theorem 3.2. Does it follow that γ is precomplete?*

4. The ADN theorem

The ADN theorem (Theorem 4.2 below) is an extension of the recursion theorem, proved in Visser [30]. It was motivated by developments in early proof theory, in particular Rosser's extension of Gödel's incompleteness theorem. Visser mentions the work of Smoryński and Shepherdson's fixed point as further motivation, cf. [25]. The analogy between the ADN theorem and Rosser's theorem was neatly summarized in Barendregt [5] by the following mock equation.

$$\frac{\text{Gödel}}{\text{Rosser}} = \frac{\text{recursion theorem}}{\text{ADN theorem}}$$

The analogy is further illustrated by the proof of the ADN theorem below.

The ADN theorem has several interesting applications.

- Visser himself discusses some consequences of the ADN theorem for the λ-calculus in [30].
- Theorem 1 (about the m-completeness of precomplete ceers) in Bernardi and Sorbi [6] uses $\omega + 1$ applications of the ADN theorem. The construction in the proof uses the ADN theorem ω times, plus one more for Lemma 2.
- Barendregt [5] uses the ADN theorem to prove a result of Statman.
- The notion of diagonal function used in the ADN theorem relates nicely to the concept of fixed point free function and similar concepts that figure prominently in computability theory, cf. the discussion below.

Definition 4.1. A partial function δ is a *diagonal function* for the numbering γ if $\delta(x) \not\sim_\gamma x$ for every x in the domain of δ.

N.B. Note that in this definition we do not require δ to be p.c., in contrast to the original definition in Visser [30]. This is because it is also interesting to discuss the Turing degrees of diagonal functions in general.

By Jockusch et al. [12], the Turing degrees of diagonal functions for the numberings $n \mapsto \varphi_n$ and $n \mapsto W_n$ coincide. They also coincide with the degrees of *diagonally noncomputable*, or DNC, functions, i.e. functions g with $g(e) \neq \varphi_e(e)$ for every e. Diagonal functions for the numbering $n \mapsto W_n$ of the c.e. sets are called *fixed point free* (or simply FPF) in the literature. (Usually these are total functions, though in [27] and [28] also partial FPF functions were considered.) DNC and FPF functions play an important part in computability theory, for example in the work of Kučera [17]. See Astor [3] for a recent example of their use, or Downey and Hirschfeldt [7] for many more. They are also closely related to the study of complete extensions of Peano Arithmetic, see e.g. the work of Jockusch and Soare [13].

Theorem 4.2. (ADN theorem, Visser [30]) *Let γ be a precomplete numbering, and suppose that δ is a partial computable diagonal function for γ. Then for every partial computable function ψ there exists a computable function f such that for every n,*

$$\psi(n)\downarrow \implies f(n) \sim_\gamma \psi(n) \tag{2}$$

$$\psi(n)\uparrow \implies \delta(f(n))\uparrow \tag{3}$$

Definition 4.3. Note that (2) expresses that f totalizes ψ modulo $\sim_\gamma$. If both (2) and (3) hold, we say that f *totalizes ψ avoiding δ.*

Note that the ADN theorem implies Ershov's recursion theorem (Theorem 3.1). Indeed, suppose towards a contradiction that some total computable d has no fixed point modulo $\sim_\gamma$. Then d is a total computable diagonal function. Then a p.c. function ψ with $\psi(0)\uparrow$ cannot be totalized modulo $\sim_\gamma$ avoiding d by any f, as we will not have $d(f(0))\uparrow$, by the totality of d. This contradicts the ADN Theorem.

Proof of Theorem 4.2. We use Ershov's recursion theorem with parameters (Theorem 3.2). Let η be p.c. such that for all x and n,

$$\eta(x,n) = \begin{cases} \delta(x) & \text{if } \delta(x)\downarrow < \ \psi(n)\downarrow \text{ (by this we mean that the first stage at which } \delta(x) \text{ converges} \\ & \text{is less than that of } \psi(n), \text{ if the latter converges at all),} \\ \psi(n) & \text{if } \psi(n)\downarrow \leq \ \delta(x)\downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

By precompleteness of γ, there is a computable function h that totalizes η modulo $\sim_\gamma$. Let f be as in Ershov's recursion theorem with parameters (Theorem 3.2). Then for every n,

$$f(n) \sim_\gamma h(f(n), n) \sim_\gamma \eta(f(n), n),$$

whenever the latter is defined. Now $\delta(f(n))\downarrow$ and $f(n) \sim_\gamma \delta(f(n))$ is impossible, since δ is a diagonal for γ, and hence f totalizes ψ avoiding δ. $\square$

By taking ψ in Theorem 4.2 universal, we see that the following uniform version holds.

Theorem 4.4. (ADN theorem, uniform version) *Let γ be a precomplete numbering, and suppose that δ is a partial computable diagonal function for γ. Then there exists a computable function f such that for every fixed e the function $f(\langle e, n \rangle)$ totalizes φ_e avoiding δ.*

Proof. Consider the universal function $\psi(\langle e, n \rangle) = \varphi_e(n)$. By Theorem 4.2, there exists a computable f that totalizes ψ avoiding δ. Hence

$$\psi(\langle e, n \rangle) = \varphi_e(n)\!\downarrow \implies f(\langle e, n \rangle) \sim_\gamma \varphi_e(n)$$
$$\psi(\langle e, n \rangle) = \varphi_e(n)\!\uparrow \implies \delta(f(\langle e, n \rangle))\!\uparrow,$$

and therefore $f(\langle e, n \rangle)$ totalizes φ_e avoiding δ. □

Theorem 4.4 shows that Theorem 4.2 is uniform in a code of ψ. A careful reading of the proof of the ADN theorem above shows that it is also uniform in a code d of δ.[3] It is shown in Terwijn [28] that (for the numbering $n \mapsto W_n$) neither Arslanov's completeness criterion nor the ADN theorem have a version with parameters analogous to the recursion theorem with parameters. (Note that the ADN theorem is in a way a contrapositive formulation of the recursion theorem, so that some care is needed in how to formulate the parameterized version.) A fortiori, the same holds in the context of arbitrary precomplete numberings.

5. Arslanov's completeness criterion for precomplete numberings

Arslanov's completeness criterion [2] states that a c.e. set A is Turing complete if and only if A can compute a FPF function, i.e. a function f such that $W_{f(n)} \neq W_n$ for every n. Note that this vastly extends Kleene's recursion theorem, namely from computable sets to incomplete c.e. sets. The condition that A is c.e. is necessary, as by the low basis theorem [14] there exist FPF functions of low Turing degree.

In the next theorem we formulate Arslanov's completeness criterion for arbitrary precomplete numberings. The usual version of the completeness criterion corresponds to the case where γ is the standard numbering of the c.e. sets $n \mapsto W_n$. (Or equivalently, by the aforementioned result of Jockusch et al. [12], the numbering of the p.c. functions $n \mapsto \varphi_n$.) Below, $\emptyset'$ denotes the halting set, and $\emptyset'_s$ denotes its s-step approximation.

Theorem 5.1. *Suppose γ is a precomplete numbering, and $A <_T \emptyset'$ is an incomplete c.e. set. If g is an A-computable function, then g has a fixed point modulo γ, i.e. there exists $n \in \omega$ such that $g(n) \sim_\gamma n$.*

Proof. The following proof is a modification of the proof in Soare [26].

Since $g \leq_T \emptyset'$, by Shoenfield's limit lemma [24] there is a computable approximation $\hat{g}$ such that

$$g(n) = \lim_{s \to \infty} \hat{g}(n, s)$$

for every n. Because A is c.e., this approximation has a modulus $m \leq_T A$, that is, for all $s \geq m(n)$ we have $g(n) = \hat{g}(n, s)$. Now let η be partial computable such that

$$\eta(x, n) = \begin{cases} \hat{g}(x, s_n) & \text{if } s_n \text{ is the least number } s \text{ such that } n \in \emptyset'_s, \\ \uparrow & \text{if such } s \text{ does not exist.} \end{cases}$$

By the precompleteness of γ, let h be a computable function that totalizes η modulo $\sim_\gamma$, so that $h(x, n) \sim_\gamma \eta(x, n)$ whenever the latter is defined. By Ershov's recursion theorem with parameters (Theorem 3.2), let

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[3] This means that there is a computable function $f = f(d, n)$ such that, if $\delta = \varphi_d$ is a diagonal function for γ, then $f(d, n)$ totalizes ψ avoiding δ.

f be a computable function such that $f(n) \sim_\gamma h(f(n), n)$ for every n. In particular we have $f(n) \sim_\gamma \eta(f(n), n) \sim_\gamma \hat{g}(f(n), s_n)$ when $n \in \emptyset'$.

We claim that there exists $n \in \emptyset'$ such that $\hat{g}(f(n), s_n) = g(f(n))$, so that $f(n)$ is a fixed point of g. Otherwise we would have that for every n, if $n \in \emptyset'$ then $\hat{g}(f(n), s_n) \neq g(f(n))$, and hence $m(f(n)) > s_n$. It follows that $n \in \emptyset' \Leftrightarrow n \in \emptyset'_{m(f(n))}$, and hence $\emptyset' \leq_T A$, contrary to assumption. $\square$

A joint generalization of the ADN theorem and Arslanov's completeness criterion for the numbering $n \mapsto W_n$ of the c.e. sets was given in Terwijn [27]. At this point it is not clear that the proof in [27] generalizes to arbitrary precomplete numberings, though we conjecture that it is possible to adapt the proof.

Question 5.2. *Does the joint generalization Theorem 5.1 in [27] hold for arbitrary precomplete numberings?*

6. Numberings and partial combinatory algebra

In this section we discuss the relation of the theory of numberings with partial combinatory algebra. Combinatory algebra was introduced by Schönfinkel [23], and partial combinatory algebra in Feferman [11]. We begin by repeating some relevant definitions. A fuller account of partial combinatory algebra can be found in van Oosten [22], from which we also borrow some of the terminology.

A *partial applicative structure* (pas) is a set $\mathcal{A}$ together with a partial map from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$. We denote the image of (a, b), if it is defined, by ab, and think of this as 'a applied to b'. If this is defined we denote this by $ab\downarrow$. By convention, application associates to the left. We write abc instead of $(ab)c$. *Terms* over $\mathcal{A}$ are built from elements of $\mathcal{A}$, variables, and application. If t_1 and t_2 are terms then so is $t_1 t_2$. If $t(x_1, \ldots, x_n)$ is a term with variables x_i, and $a_1, \ldots, a_n \in \mathcal{A}$, then $t(a_1, \ldots, a_n)$ is the term obtained by substituting the a_i for the x_i. For closed terms (i.e. terms without variables) t and s, we write $t \simeq s$ if either both are undefined, or both are defined and equal. Here application is *strict* in the sense that for $t_1 t_2$ to be defined, it is necessary (but not sufficient) that both t_1, t_2 are defined. We say that an element $f \in \mathcal{A}$ is *total* if $fa\downarrow$ for every $a \in \mathcal{A}$.

Definition 6.1. A partial applicative structure $\mathcal{A}$ is *combinatory complete* if for any term $t(x_1, \ldots, x_n, x)$, with free variables among $x_1, \ldots, x_n, x$ and $0 \leq n$, there exists a $b \in \mathcal{A}$ such that for all $a_1, \ldots, a_n, a \in \mathcal{A}$,

  (i) $ba_1 \cdots a_n \downarrow$,
  (ii) $ba_1 \cdots a_n a \simeq t(a_1, \ldots, a_n, a)$.

A pas $\mathcal{A}$ is a *partial combinatory algebra* (pca) if it is combinatory complete.

The property of combinatory completeness allows for the following definition in any pca.[4] For every term $t(x_1, \ldots, x_n, x)$, $0 \leq n$, with free variables among $x_1, \ldots, x_n, x$, one can explicitly define a term $\lambda^* x.t$ with variables among $x_1, \ldots, x_n$, with the property that for all $a_1, \ldots, a_n, a \in \mathcal{A}$,

  (i) $(\lambda^* x.t)(a_1, \ldots, a_n) \downarrow$,
  (ii) $(\lambda^* x.t)(a_1, \ldots, a_n)a \simeq t(a_1, \ldots, a_n, a)$.

This is noted in Feferman [11, p95], and makes use of the Curry combinators k and s familiar from combinatory logic. In fact, for any pas, the existence of such combinators is equivalent to being a pca [22, p3].

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[4] Because the lambda-terms in combinatory algebra do not have the same substitution properties as in the lambda calculus, we use the notation λ^* rather than λ.

The prime example of a pca is *Kleene's first model* $\mathcal{K}_1$, consisting of ω, with application defined by $nm = \varphi_n(m)$. This structure is combinatory complete by the S-m-n-theorem. However, there are many other examples, including uncountable structures, see Section 1.4 of [22].

Another important example of a pca is *Kleene's second model* $\mathcal{K}_2$ [16]. This is a pca defined on Baire space ω^ω (often informally referred to as the 'reals'), with application defined by coding partial continuous functionals by reals. The application $\alpha\beta$ is then the result of applying the functional with code α to the real β. We will use this model below in section 7. It also plays an important role in the theory of realizability and higher-order computability. For a more elaborate discussion see for example Longley and Normann [19].

The structures $\mathcal{K}_1$ and $\mathcal{K}_2$ can also be considered as total combinatory algebras if one restricts them to combinators corresponding to λI-calculus, in which the formation of $\lambda x.M$ only is allowed if x is a free variable of M, see Barendregt [4, Exercises 9.5.13-14].

We note the following about the pca $\mathcal{K}_1$.

- The notion of precompleteness (1) generalizes the property that one can totalize any p.c. function ψ on codes. This property gives Ershov's form of the recursion theorem (Theorem 3.1).
- Pca's generalize the applicative structure of $nm = \varphi_n(m)$. The property of combinatory completeness may be seen as an abstraction of Kleene's S-m-n-theorem. This property *also* gives rise to a fixed point theorem (due to Feferman), see Theorem 6.2 below.

These two generalizations of properties of Kleene's model are more or less orthogonal. For numberings, there is no notion of application, and pca's need not be countable.

The following is Feferman's form of the recursion theorem in pca's, inspired by the fixed point theorem in combinatory logic.

Theorem 6.2. (Feferman's recursion theorem [11]) *Let $\mathcal{A}$ be a pca. Then there exists $f \in \mathcal{A}$ such that for all $g \in \mathcal{A}$*

$$g(fg) \simeq fg.$$

Comparing this to Ershov's recursion theorem (in the form of Theorem 3.3), we see that Feferman's version is more general in that it applies to arbitrary pca's, but that it is also weaker in that the 'function' f giving the fixed point fg does not have to be total.[5] In some cases f may be total (as for example in Theorem 3.3, or in the case that $\mathcal{A}$ is a combinatory algebra, i.e. a pca in which application is total), but in general f cannot be total. This is obviously the case when the pca has a totally undefined element g.[6] We will comment further on this at the end of section 7.

We now proceed by showing how a combination of the fixed point theorems of Ershov and Feferman can be obtained. We extend the notions of numbering and precompleteness of numberings from ω to arbitrary pca's as follows.

Definition 6.3. Suppose that $\mathcal{A}$ is a pca, S is a set, and $\gamma \colon \mathcal{A} \to S$ is surjective. We call γ a *(generalized) numbering*. Define an equivalence relation on $\mathcal{A}$ by $a \sim_\gamma b$ if $\gamma(a) = \gamma(b)$.

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[5] There is a second version of the recursion theorem for pca's in [22], namely that there is a term $f \in \mathcal{A}$ such that $fg\downarrow$ for every g, and such that $g(fg)a \simeq fga$ for every $a \in \mathcal{A}$. Since f is total, this version does imply Theorem 3.3, but only for the special case of the numbering $n \mapsto \varphi_n$.

[6] This is in fact the case in every nontotal combinatory algebra. As soon as there is one application ab that is undefined, $\mathcal{A}$ has a totally undefined element, namely $f = \lambda^* x.ab = s(ka)(kb)$. In this case f clearly satisfies Theorem 6.2. Although the theorem is thus quite weak as an extension from combinatory algebra, its use for us is that it suggests the generalization of Ershov's recursion theorem to pca's that we prove below (Theorem 6.5).

An alternative formulation of the recursion theorem in pca's, analogous to Theorem 3.3, would be: There exists a total $f \in \mathcal{A}$ such that for all $g \in \mathcal{A}$, if $g(fg)\downarrow$ then $g(fg) \sim fg$. This, however, does not hold in general by Proposition 7.3 below.

Call γ *precomplete* if for every term $t(x)$ with one variable x, there exists a total element $f \in \mathcal{A}$ such that

$$t(a)\downarrow \implies fa \sim_\gamma t(a) \tag{4}$$

for every $a \in \mathcal{A}$. In this case, we say that f *totalizes* t *modulo* $\sim_\gamma$.

As before, we say that a generalized precomplete numbering γ is *complete* if there is a special element $c \in \mathcal{A}$ such that in addition to (4), $fa \sim_\gamma c$ for every a with $t(a)\uparrow$.

Lemma 6.4. *Let $\mathcal{A}$ be a pca, and let $\gamma \colon \mathcal{A} \to S$ be a generalized numbering. Then the following are equivalent.*

  (i) *γ is precomplete.*
 (ii) *For every $b \in \mathcal{A}$ there exists a total element $f \in \mathcal{A}$ such that for all $a \in \mathcal{A}$,*

$$ba\downarrow \implies fa \sim_\gamma ba.$$

(iii) *For every $b \in \mathcal{A}$ there exists a total element $f \in \mathcal{A}$ such that for all $n \in \omega$ and $\vec{a} = a_1, \ldots, a_n \in \mathcal{A}$,*

$$b\vec{a}\downarrow \implies f\vec{a} \sim_\gamma b\vec{a}.$$

Proof. (i) $\Rightarrow$ (ii). Apply (i) to the term bx.

(ii) $\Rightarrow$ (iii). With the use of the λ^*-terms defined above for any pca, n-tuples $a_1, \ldots, a_n$ can be coded as a single element $\langle a_1, \ldots, a_n \rangle = \lambda^* z.za_1 \ldots a_k$, from which each a_i can be decoded. Indeed, for $\mathsf{U}_i^n = \lambda^* u_1 \ldots u_n.u_i$ we have

$$\langle a_1, \ldots, a_n \rangle \mathsf{U}_i^n = a_i.$$

Now given n, b define $b' = \lambda^* z.b(z\mathsf{U}_1^n) \cdots (z\mathsf{U}_n^n)$. Let f' totalize b' modulo γ. Then $f = \lambda^* x_1 \ldots x_n.f'\langle x_1, \ldots, x_n \rangle$ totalizes b: if $ba_1 \cdots a_n\downarrow$, then

$$\begin{aligned}
fa_1 \cdots a_n &= f'\langle a_1, \ldots, a_n \rangle \\
&\sim_\gamma b'\langle a_1, \ldots, a_n \rangle \\
&= b(\langle a_1, \ldots, a_n \rangle \mathsf{U}_1^n) \cdots (\langle a_1, \ldots, a_n \rangle \mathsf{U}_n^n) \\
&= ba_1 \cdots a_n.
\end{aligned}$$

(iii) $\Rightarrow$ (i). Given term $t(x)$, apply (iii) with $n=1$ to $b = \lambda^* x.t(x)$. $\quad\square$

By Lemma 6.4, the notion of precompleteness from Definition 2.1 is a special case of Definition 6.3, namely the case where $\mathcal{A}$ is the pca $\mathcal{K}_1$, with application $nm = \varphi_n(m)$. Hence we see that Ershov's recursion theorem (Theorem 3.3) is a special case of the following theorem.

Theorem 6.5. *Suppose $\mathcal{A}$ is a pca, and that $\gamma \colon \mathcal{A} \to S$ is a precomplete numbering. Then there exists a total $f \in \mathcal{A}$ such that for all $g \in \mathcal{A}$, if $g(fg)\downarrow$ then*

$$g(fg) \sim_\gamma fg.$$

Proof. The proof mimics $\Theta = (\lambda^* xy.y(xxy))(\lambda^* xy.y(xxy))$, the fixed point operator of Turing [29]. Let $t(x, y) = y(xxy)$. By Lemma 6.4 there is a $u \in \mathcal{A}$ that totalizes the term $t(x, y)$ modulo $\sim_\gamma$. Then $uab\downarrow$, for all $a, b \in \mathcal{A}$, and $b(aab)\downarrow$ implies $uab \sim_\gamma b(aab)$. Take $f = uu$. Then f is total, because $uua\downarrow$ for every $a \in \mathcal{A}$. Suppose for a $g \in \mathcal{A}$ one has $g(fg)\downarrow$. Then $g(uug)\downarrow$ and

$$fg = uug$$
$$\sim_\gamma g(uug)$$
$$= g(fg). \quad \square$$

In order to combine the fixed point theorems of Ershov and Feferman, we have extended the definition of numberings to pca's. It could also be interesting to investigate how much of the theory can be expressed in terms of realizability toposes derived from the pca's, cf. [22].

7. Combinatory completeness and precompleteness

With every pca $\mathcal{A}$ we have an associated generalized numbering $\gamma_\mathcal{A} \colon \mathcal{A} \to \mathcal{A}$, which is just the identity. We will discuss the relation between combinatory completeness of $\mathcal{A}$ and the precompleteness of $\gamma_\mathcal{A}$.

Combinatory completeness is the property in pca's analogous to the S-m-n-theorem, and precompleteness of (generalized) numberings (Definition 6.3) generalizes the property that every p.c. function can be totalized modulo equivalence of codes, i.e. that the numbering $n \mapsto \varphi_n$ is precomplete. Now the latter fact is proved using the S-m-n-theorem, so one might think that perhaps the property of combinatory completeness of a pca $\mathcal{A}$ *implies* that of precompleteness of the associated numbering $\gamma_\mathcal{A}$. We now show that this is not the case, and hence that the assumption of precompleteness in Theorem 6.5 is not superfluous. Recall Kleene's second model $\mathcal{K}_2$ from section 6.

Proposition 7.1. *Kleene's second model $\mathcal{K}_2$ is not precomplete, meaning that its associated generalized numbering $\gamma_{\mathcal{K}_2}$ is not precomplete.*

Proof. According to Lemma 6.4, we have to prove that there is a partial continuous functional $\psi \colon \omega^\omega \to \omega^\omega$ that does not have a total continuous extension. For every finite string $\sigma \in \omega^{<\omega}$, denote by $[\sigma]$ the basic open set consisting of all $X \in \omega^\omega$ that have σ as an initial segment. Now define ψ on every basic open $[0^n 1]$ by mapping it continuously to $[0^n 1]$ if n is even, and to $[10^{n-1}]$ if n is odd. We let ψ be undefined on the rest of ω^ω. Then ψ is continuous on its domain. Now consider the all zero sequence 0^ω, and suppose that f is a total continuous extension of ψ. Since the reals $0^n 10^\omega$ converge to 0^ω for $n \to \infty$, their images under f should converge to $f(0^\omega)$. But for even n, $f(0^n 10^\omega)$ tends to 0^ω, and for odd n it tends to 10^ω. Hence every continuous extension f of ψ must have both $f(0^\omega) = 0^\omega$ and $f(0^\omega) = 10^\omega$, which is impossible. $\quad \square$

Corollary 7.2. *Combinatory completeness of a pca $\mathcal{A}$ does not imply precompleteness of the associated numbering $\gamma_\mathcal{A}$.*

Proof. As $\mathcal{K}_2$ is a pca, this is immediate from Proposition 7.1. $\quad \square$

We already noted that in general it is not possible to have the f in Feferman's recursion theorem (Theorem 6.2) total. For $\mathcal{K}_2$, we can in fact say a bit more.

Proposition 7.3. *In Kleene's second model $\mathcal{K}_2$, for every total element f there exists a total element g such that $g(fg) \not\simeq fg$.*

Proof. Given the code f of a total continuous functional on ω^ω, we define a total continuous functional g such that $g(fg) \not\simeq fg$.

The particulars of the coding of $\mathcal{K}_2$ are not essential to the proof. (The interested reader can find them in Longley and Normann [19].) What is needed is that if $fg\downarrow$, this computation uses only a *finite* part of the coding of g (this is precisely what it means for f to be continuous on its domain), and further that the

code of an element g can be equal to an initial part of the code of the totally undefined function, and later become defined on a given number. Informally, the strategy to define g is then as follows. First let g be totally undefined, until fg commits to a certain value on $(fg)(0)$. This has to happen since f is total. We can then diagonalize by letting the value $(g(fg))(0)$ be different from $(fg)(0)$, as well as make g total. □

Note that Proposition 7.3 gives another proof of Proposition 7.1. Namely, if $\mathcal{K}_2$ were precomplete, then by Theorem 6.5 there would be a total element f producing the fixed points, contradicting Proposition 7.3.

Conflict of interest statement

Declarations of interest: none.

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