Stochastic Cahn-Hilliard Equations and Their Sharp Interface Limits

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Preface

This thesis is about the well-posedness and sharp interface limits of stochastic Cahn-Hilliard equations. We are concerned with the following three related problems:

(1) We consider the stochastic 2-dimensional Cahn-Hilliard equation which is driven by the derivative in space of a space-time white noise:

$$\partial_t u = \Delta(-\Delta u + f(u)) + \nabla \cdot \xi.$$

We use two different approaches to study this equation. First we prove that there exists a unique solution to the stochastic Cahn-Hilliard equation. Moreover, we use the Dirichlet form approach in [AR91] to construct the probabilistically weak solution. By clarifying the precise relation between the two solutions, we also get the restricted Markov uniqueness of the generator and the uniqueness of the martingale solutions. Furthermore, we also obtain exponential ergodicity of the solutions.

(2) We study the the sharp interface limit of ε-dependent two dimensional stochastic Cahn-Hilliard equation as $\varepsilon \to 0$:

$$\begin{cases} \partial_t u^\varepsilon = \Delta v^\varepsilon + \varepsilon^\sigma \dot{\mathcal{W}}_t, \\ \quad v^\varepsilon = -\varepsilon \Delta u^\varepsilon + \dfrac{1}{\varepsilon} f(u^\varepsilon), \end{cases}$$

where $\dot{\mathcal{W}}$ is space-time white noise or conservative noise. In the case when the noise is sufficiently small, by comparing the solutions with the approximation solution constructed in [ABC94], we show that the limit of the solutions is also solutions to the deterministic Hele-Shaw problem.

(3) We study the asymptotic limit, as $\varepsilon \searrow 0$, of solutions of the stochastic Cahn-Hilliard equation:

$$\partial_t u^\varepsilon = \Delta \left(-\varepsilon \Delta u^\varepsilon + \frac{1}{\varepsilon} f(u^\varepsilon) \right) + \varepsilon^\sigma \dot{\mathcal{W}}^\varepsilon_t,$$

where $\mathcal{W}^\varepsilon = W$ or $\mathcal{W}^\varepsilon = W^\varepsilon$, W is a Q-Wiener process and W^ε is smooth in time and converges to W as $\varepsilon \searrow 0$. In the case that $\mathcal{W}^\varepsilon = W$, we prove that for all $\sigma > \frac{1}{2}$, the solution u^ε converges to a weak solution to an appropriately defined limit of the deterministic Cahn-Hilliard equation. In radial symmetric case we prove that for all $\sigma \geq \frac{1}{2}$, u^ε converges to the deterministic Hele-Shaw model. In the case that $\mathcal{W}^\varepsilon = W^\varepsilon$, we prove that for all $\sigma > 0$, u^ε converges to the weak solution to the deterministic limit Cahn-Hilliard equation. In radial symmetric case we prove that u^ε converges to deterministic Hele-Shaw model when $\sigma > 0$ and converges to a stochastic model related to stochastic Hele-Shaw model when $\sigma = 0$.

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Contents

Chapter 1

Introduction and Main Results

This thesis is concerned on the well-posedness and sharp interface limits of stochastic Cahn-Hilliard equations.

1.1 The deterministic case

The Cahn-Hilliard equation on a smooth domain $\mathcal{D}$ is given by

$$\begin{cases} \partial_t u = \Delta v, \\ \quad v = -\Delta u + f(u), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 \ \text{ on } \ \partial\mathcal{D}, \end{cases} \tag{1.1}$$

which was introduced by Cahn and Hilliard [CH58] to study the phase separation of binary alloys. Here $f(u) = u^3 - u$. The equation (1.1) is the H^{-1}-gradient flow of the energy functional

$$\mathcal{E}(u) := \frac{1}{2}\int_{\mathcal{D}} |\nabla u(x)|^2 dx + \int_{\mathcal{D}} F(u(x))dx, \tag{1.2}$$

where $F(u) = \frac{1}{4}(u^2 - 1)^2$ is the double-well potential.

If u is a solution to equation (1.1), then

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(u(t,\cdot)) &= \int_{\mathcal{D}} \partial_t u(t,x) \left(-\Delta u(t,x) + f(u(t,x))\right) dx \\ &= \int_{\mathcal{D}} v(t,x)\Delta v(t,x)dx = -\int_{\mathcal{D}} |\nabla v(t,x)|^2 dx \le 0. \end{aligned} \tag{1.3}$$

Clearly, the minimizers of the energy (1.2) are the constant functions $u \equiv 1$ and $u \equiv -1$, which represent the "pure phases" of the system. However, these "pure phases" cannot be reached unless the initial value u_0 satisfies $\int_{\mathcal{D}} u_0(x)dx = \pm|\mathcal{D}|$ because of the conservation law, i.e.

$$\frac{d}{dt}\int_{\mathcal{D}} u(t,x)dx = 0$$

for any solution u to (1.1). Instead, what will be produced is a "mixed phase", more precisely, which is a region in which $u \approx +1$ with $u \approx -1$ in its complement. Moreover, a transition occurs across its boundary. This is referred as *phase segregation* and the boundary is the interface between the two phases. If we look at the solution in a large scale ("stand far enough back"), the transition is on an invisibly small scale. All we see is just the interface. The evolution of u under the Cahn-Hilliard equation (1.1) derives an evolution of the interface. One of the most important problems is to determine how the interface evolves. To see any evolution of the interface, we must wait for a long time. More specifically, let ε be a small parameter and $u^\varepsilon(t,x) := u(\frac{t}{\varepsilon^3}, \frac{x}{\varepsilon})$, where u is a solution to (1.1). Then u^ε satisfies the following equation:

$$\begin{cases} \partial_t u^\varepsilon = \Delta v^\varepsilon, \\ \quad v = -\varepsilon \Delta u^\varepsilon + \dfrac{1}{\varepsilon} f(u^\varepsilon), \\ \dfrac{\partial u^\varepsilon}{\partial n} = \dfrac{\partial v^\varepsilon}{\partial n} = 0 \text{ on } \partial\mathcal{D}. \end{cases} \tag{1.4}$$

It was formally derived by Pego [Peg89] and rigorous proved by [ABC94] by using the method of matched asymptotic expansions that the equation (1.4) converges to the Hele-Shaw model. That is, as $\varepsilon \searrow 0$, the chemical potential v^ε tends to a limit v which, together with a free boundary $\Gamma := \cup_{0\le t\le T}(\{t\} \times \Gamma_t)$, solves the following deterministic Hele-Shaw model:

$$\begin{cases} \Delta v = 0 \text{ in } \mathcal{D} \setminus \Gamma_t,\ t > 0, \\ \dfrac{\partial v}{\partial n} = 0 \text{ on } \partial\mathcal{D}, \\ \quad v = SH \text{ on } \Gamma_t, \\ \quad \mathcal{V} = \dfrac{1}{2}\left[\dfrac{\partial}{\partial n}\right]_{\Gamma_t} v \text{ on } \Gamma_t, \end{cases} \tag{1.5}$$

where

$$S = \int_{-1}^{1} \sqrt{\frac{F(s)}{2}} ds = \frac{2}{3},$$

$$\left[\frac{\partial}{\partial n}\right]_{\Gamma_t} v := (\partial_n v^+ - \partial_n v^-),$$

H is the scalar mean curvature of Γ_t with the sign convention that convex hypersurfaces have positive mean curvature, $\mathcal{V}$ is the normal velocity of the interface with the sign convention that the normal velocity of expanding hypersurfaces is positive, n is the unit outward normal either to $\partial\mathcal{D}$ or to Γ_t. Denote $\mathcal{D}^+$ and $\mathcal{D}^-$ are the exterior and interior of Γ_t. v^+, v^- are respectively the restriction of v on $[0,t] \times \mathcal{D}^+$ and $[0,t] \times \mathcal{D}^-$.

Later in [Che96], the author formulated a weak solution to the free boundary problem (1.5) (see Definition 5.2) and showed that the solutions of (1.4) approach, as $\varepsilon \searrow 0$, to weak solutions of (1.5) by using a compactness argument. In fact the energy functional of (1.4) is given by

$$\mathcal{E}^\varepsilon(u^\varepsilon) := \int_{\mathcal{D}} e^\varepsilon(u^\varepsilon) dx, \quad e^\varepsilon(u^\varepsilon) := \frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon}F(u^\varepsilon). \tag{1.6}$$

One can directly verify that for any solution $(u^\varepsilon, v^\varepsilon)$ to equation (1.4),

$$\frac{d}{dt}\mathcal{E}^\varepsilon(u^\varepsilon(t,\cdot)) = -\int_{\mathcal{D}} |\nabla v^\varepsilon(t,x)|^2 dx \le 0, \tag{1.7}$$

which is also called the Lyapunov property for equation (1.4). Thus $\mathcal{E}^\varepsilon(u^\varepsilon)$ is uniformly bounded in $t, \varepsilon > 0$ if the energy of the initial value is bounded uniformly in ε. Note that as $\varepsilon \to 0$, $F(u^\varepsilon) \to 0$, which is equivalent to $u^\varepsilon \to -1 + 2\mathbb{1}_E$ for some $E \subset [0,T] \times \mathcal{D}$ where $\mathbb{1}_E$ is the characteristic function of E, i.e. $\mathbb{1}_E(x) = 1$ when $x \in E$ and $\mathbb{1}_E(x) = 0$ when $x \notin E$. $\Gamma_t := \partial E_t$ is the interface. By using a varifold approach, Chen in [Che96] analyzed the property of the limit of the solutions to equation (1.4) and then proposed a definition of weak solution of this limit. Any classical smooth solutions to (1.5) are weak solutions. In some special case, the smooth weak solutions are also classical solutions to (1.5).

1.2 The stochastic case

We are interested in the global well-posedness and the sharp interface limit of the stochastic Cahn-Hilliard equation:

$$\partial_t u^\varepsilon = \Delta \left(-\varepsilon \Delta u^\varepsilon + \frac{1}{\varepsilon} f(u^\varepsilon) \right) + \varepsilon^\sigma \dot{\mathcal{W}}_t^\varepsilon, \tag{1.8}$$

where $\dot{\mathcal{W}}_t^\varepsilon$ is the noise which may depend on ε.

1.2.1 Well-posedness for stochastic Cahn-Hilliard equation

For the well-posedness, the stochastic Cahn-Hilliard equation was first studied in [PM83], where Petschek and Metiu performed some numerical experiments for the stochastic Cahn-Hilliard equation driven by space-time white noise. In [EM91], Elezovic and Mikelic proved the existence and uniqueness of a strong solution to the stochastic Cahn-Hilliard equation driven by trace-class noise. Then Da Prato and Debussche [DPD96] proved existence and uniqueness of solutions for space-time white noise and obtained the existence and uniqueness of an invariant measure for trace-class noise. Later there are many papers in which the authors study the properties of the solutions to the stochastic Cahn-Hilliard equations driven by trace-class noise (e.g. [DG11, Sca17]).

In Chapter 3 we show the well-posedness for the conservative stochastic Cahn-Hilliard equation

$$\begin{cases} dX_t = -\dfrac{1}{2} A \left(AX - :X^3: \right) dt + B dW_t, \\ X(0) = z \in V_0^{-1}, \end{cases} \tag{1.9}$$

on $\mathbb{T}^2$ in the probabilistically strong sense where $A = \Delta$, $B = \mathrm{div}$. W_t is an $L_0^2(\mathbb{T}^2, \mathbb{R}^2)$-cylindrical Wiener process, which is defined in Section 3.2. $:X^3:$ denotes the Wick power, which is introduced in Section 3.2 and the space V_0^{-1} is similar to the Sobolev space of order -1, which is introduced in Section 3.1.

For the conservative-type equation (1.9), the Gibbs measure ν is formally given by the following Φ_2^4-field:

$$\nu(d\phi) = c \exp\left(-\int_{\mathbb{T}^2} \frac{1}{4} :\phi^4: dx \right) \mu(d\phi),$$

where μ is the Gaussian free field, c is a normalization constant, and $:\phi^4:$ is the fourth order Wick power of ϕ. Equation (1.9) can be interpreted as the natural "Kawasaki" dynamics (see [GLP99]) associated to the Euclidean Φ_2^4-quantum field. In [PW81] Parisi and Wu proposed a program for Euclidean quantum field theory based on getting Gibbs states of classical statistical mechanics as limiting distributions of stochastic processes, especially as solutions to non-linear stochastic differential equations. Then one can use the stochastic differential equations to study properties of the Gibbs states. This procedure is called stochastic field quantization (see [JLM85]). The equation (1.9) can also be viewed as a stochastic quantization equation for the Φ^4_2-field.

Over the years, there is a lot of literature (see [JLM85, AR91, DPD03, MW17, RZZ17a, RZZ17b]) on the stochastic quantization of the Φ^4_2-field. The authors in these papers considered the following non-conservative stochastic quantization equation:

$$dX_t = (AX - :X^3:)dt + dW_t. \tag{1.10}$$

First results are due to Jona-Lasinio and Mitter [JLM85]. Using the Girsanov theorem, they constructed solutions to a modified equation on $\mathbb{T}^2$:

$$dX_t = (-\triangle + 1)^{-\varepsilon}(\triangle X - :X^3: + aX) + (-\triangle + 1)^{-\frac{\varepsilon}{2}} dW_t \tag{1.11}$$

for $\frac{9}{10} < \varepsilon < 1$. They also proved the ergodicity for (1.11). In [AR91] Albeverio and Röckner studied (1.10) using Dirichlet forms and constructed probabilistically weak solutions to (1.10). In [MR99], Mikulevicius and Rozovskii constructed martingale solutions to (1.10) but the uniqueness remained open. In [DPD03] Da Prato and Debussche considered the associated shifted equation to (1.10) on $\mathbb{T}^2$ and proved the local existence and uniqueness of solutions in the probabilistically strong sense via a fixed point argument and then showed the non-explosion for almost every initial point by using the invariant measure. Recently Mourrat and Weber [MW17] showed the global existence and uniqueness for the shifted equation both on $\mathbb{T}^2$ and $\mathbb{R}^2$ for every initial point. Combining the results from the weak approach and strong approach, Röckner, Zhu and Zhu [RZZ17b] proved the restricted Markov uniqueness for the generator of (1.10) and the uniqueness of the martingale problem to (1.10) arised in [MR99] on $\mathbb{T}^2$ and $\mathbb{R}^2$. Furthermore, the ergodicity of (1.10) on $\mathbb{T}^2$ has been obtained in [HM18, RZZ17a, TW16].

For the conservative case, Funaki [Fun89] proved the existence and uniqueness of equation (1.9) on $\mathbb{R}$ and in [DZ07] Debussche and Zambotti studied equation (1.9) on $[0, 1]$ with reflection. But for the higher dimensional case, even though the linear operator Δ^2 gives much more regularity, the noise and hence the solutions are still so singular that the non-linear terms in (1.9) are not well-defined in the classical sense. This difficulty is similar as in equation (1.10).

To overcome this difficulty, we use two approaches to study (1.9). First we follow the idea in [DPD03], [MW17] and [RZZ17b] to split the solution to $X = Y + Z$, where $Z(t) = \int_0^t e^{-\frac{(t-s)}{2}A^2} B dW_s$. Similarly as in the Φ^4_2 case, Y has better regularity than the solution to (1.9) and satisfies the following *shifted equation*:

$$\begin{cases} \dfrac{dY}{dt} = -\dfrac{1}{2}A^2 Y + \dfrac{1}{2}A \displaystyle\sum_{k=0}^{3} C_3^k Y^{3-k} :Z^k: \\ Y(0) = z \end{cases} \tag{1.12}$$

where $Z(t) = \int_0^t e^{-\frac{t-s}{2}A^2} B dW_s$. In Chapter 3 we obtain the existence and uniqueness of the solution to (1.12). The fixed point arguments for local well-posedness in [DPD03] and [MW17] only hold for initial values in $\mathcal{C}^{-\frac{4}{3}+}$. Due to the singularity of the noise and the lack of a maximum principle and a uniform L^p-estimate, we only have a uniform

H^{-1}-estimate (see Theorem 3.7), which is not strong enough to combine it with local well-posedness (see Remark 3.11). Instead, our argument is based on a classical compactness argument. We obtain the existence of global solutions starting from the uniform H^{-1}-estimate directly. Moreover we consider the solutions in H^{-1} and use the L^4-integrability to obtain uniqueness for (1.12).

In addition, we use the method in [AR91] to construct the Dirichlet form for (1.9) (see Theorem 3.15), which is given by

$$\Lambda(\varphi,\psi) = \frac{1}{2}\int \langle \nabla\varphi, \nabla\psi\rangle_{V_0^{-1}} d\nu, \varphi, \psi \in \mathcal{F}C_b^\infty,$$

where $\mathcal{F}C_b^\infty$ is defined in Section 3.4. We note that the tangent space is chosen as V_0^{-1} and the gradient operator ∇ is also defined in V_0^{-1}. This is different from the Dirichlet form for (1.10), where the tangent space is chosen as L^2 and the gradient is the L^2-derivative. By the integration by parts formula for ν we also obtain the closability fo the bilinear form $(\Lambda,\mathcal{F}C_b^\infty)$. The closure $(\Lambda,D(\Lambda))$ is a quasi-regular Dirichlet form, which enables us to construct a probabilistically weak solution to (1.9). Then by clarifying the relation between this solution and the solution to (1.12), we prove that $X - Z$, where X is the solution obtained by the Dirichlet form approach, also satisfies the shifted equation (1.12). It follows that Φ^4_2 field is an invariant measure for X. Then we obtain the Markov uniqueness in the restricted sense for the generator of the Dirichlet form restricted to $\mathcal{F}C_b^\infty$ and the uniqueness of probabilistically weak solutions to (1.9) having ν as an invariant measure.

We also prove exponential ergodicity by two approaches. One simple and short way by the Dirichlet form approach is presented in Remark 3.31. Using a uniform estimate, an invariant measure can also be constructed by the Krylov-Bogoliubov method. We follow an idea from [TW16] to prove the strong Feller property of the semigroup of the solution to the equation (1.9). Then we obtain exponential convergence to the unique invariant measure of the semigroup for every starting point.

1.2.2 Sharp interface limit for big $\sigma > 0$

In Chapter 4 we obtain the convergence results arising in the study of the sharp interface limit, as $\varepsilon \searrow 0$, of the solutions to the stochastic Cahn-Hilliard equation on $\mathcal{D} := (0,1)^2$,

$$\begin{cases} \partial_t u^\varepsilon = \Delta v^\varepsilon + \varepsilon^\sigma \dot{\mathcal{W}}_t, \\ v^\varepsilon = -\varepsilon\Delta u^\varepsilon + \dfrac{1}{\varepsilon} f(u^\varepsilon), \\ u^\varepsilon(0) = z, \end{cases} \tag{1.13}$$

with Neumann boundary conditions,

$$\frac{\partial u^\varepsilon}{\partial n} = \frac{\partial v^\varepsilon}{\partial n} = 0 \text{ on } \partial\mathcal{D}. \tag{1.14}$$

Here $f(u) = F'(u)$ and $F(u) = \frac{1}{4}(u^2-1)^2$ is the double-well potential , $\sigma > 0$ is a constant, and $\dot{\mathcal{W}}$ is a singular noise which represents the space-time white noise in Section 4.2 and the conservative noise in Section 4.4.

In [ABC94], the authors study the deterministic Cahn-Hilliard equation

$$\begin{cases} \partial_t u^\varepsilon = \Delta v^\varepsilon \text{ in } \mathcal{D}_T, \\ v^\varepsilon = \dfrac{1}{\varepsilon} f(u^\varepsilon) - \varepsilon\Delta u^\varepsilon \text{ in } \mathcal{D}_T, \end{cases} \tag{1.15}$$

where $\mathcal{D}_T := (0,T) \times \mathcal{D}$. Assume that the interface has been formed initially. That is, there exists a smooth closed curve $\Gamma_{00} \subset\subset \mathcal{D}$ such that $u^\varepsilon(0) \approx -1$ in $\mathcal{D}^-$, the region enclosed by Γ_{00}, and $u^\varepsilon(0) \approx 1$ in $\mathcal{D}^+ := \mathcal{D} \setminus (\Gamma_{00} \cup \mathcal{D}^-)$. Formally as $\varepsilon \to 0$, the solutions to equation (1.15) reach the stable state u^* such that $f(u^*) = 0$, i.e. $\lim_{\varepsilon\to 0} u^\varepsilon(t,x) = \pm 1$. Hence there is an interface Γ_t between these two states.

The authors in [ABC94] use a new matched asymptotics to constructed approximation solutions. They construct a pair of approximation solutions $(u_A^\varepsilon, v_A^\varepsilon)$, such that Γ_t is the zero level set of $u_A^\varepsilon(t)$, which satisfies

$$\begin{cases} \partial_t u_A^\varepsilon = \Delta v_A^\varepsilon \text{ in } \mathcal{D}_T, \\ \quad v_A^\varepsilon = \dfrac{1}{\varepsilon} f(u_A^\varepsilon) - \varepsilon \Delta u_A^\varepsilon + r_A^\varepsilon \text{ in } \mathcal{D}_T, \end{cases} \tag{1.16}$$

for boundary conditions

$$\frac{\partial u_A^\varepsilon}{\partial n} = \frac{\partial \Delta u_A^\varepsilon}{\partial n} = 0 \text{ on } \partial\mathcal{D}.$$

They also showed that as $\varepsilon \to 0$, both v^ε and v_A^ε tend to v in $C(\mathcal{D}_T)$, which, together with a free boundary $\Gamma \equiv \cup_{0\le t\le T}(\Gamma_t \times \{t\})$, satisfies the following deterministic Hele-Shaw problem (1.5), starting from Γ_{00}:

$$\begin{cases} \Delta v = 0 \text{ in } \ \mathcal{D} \setminus \Gamma_t, \ \ t > 0, \\ \partial_n v = 0 \ \text{ on } \ \partial\mathcal{D}, \\ \quad v = SH \text{ on } \ \Gamma_t, \\ \quad \mathcal{V} = \dfrac{1}{2}\left[\dfrac{\partial}{\partial n}\right]_{\Gamma_t} v \ \text{ on } \ \Gamma_t, \\ \Gamma_0 = \Gamma_{00}, \end{cases} \tag{1.17}$$

where

$$S = \int_{-1}^{1} \sqrt{\frac{F(s)}{2}} ds = \frac{2}{3},$$

H is the mean curvature of Γ_t with the sign convention that convex hypersurfaces have positive mean curvature, $\mathcal{V}$ is the normal velocity of the interface with the sign convention that the normal velocity of expanding hypersurfaces is positive, n is the unit outward normal either to $\partial\mathcal{D}$ or to Γ_t, v^+ and v^- are respectively the restriction of v on $[0,t] \times \mathcal{D}^+$ and $[0,t] \times \mathcal{D}^-$.

For the stochastic Cahn-Hilliard equation, the authors in [ABK18] proved that for large $\sigma > 0$ the sharp interface limit of equation (1.13) also satisfies the deterministic Hele-Shaw model if $\dot{\mathcal{W}}$ is a trace-class noise. For $\sigma = 1$, the sharp interface limit is also conjectured to satisfy the following stochastic Hele-Shaw model:

$$\begin{cases} \Delta v = 0 \text{ in } \mathcal{D} \setminus \Gamma_t, \ t > 0, \\ \partial_n v = 0 \text{ on } \partial\mathcal{D}, \\ \quad v = \lambda H + \dot{\mathcal{W}} \text{ on } \Gamma_t, \\ \quad \mathcal{V} = \dfrac{1}{2}\left[\dfrac{\partial}{\partial n}\right]_{\Gamma_t} v \text{ on } \Gamma_t, \\ \Gamma_0 = \Gamma_{00}, \end{cases} \tag{1.18}$$

In [AKO14], the authors prove that the sharp interface limit of generalized Cahn-Hilliard equation: $\partial_t u = \Delta(-\varepsilon\Delta u + \frac{1}{\varepsilon}f(u) - G_2^\varepsilon) + G_1^\varepsilon$ satisfies the following Hele-Shaw model:

$$\begin{cases} \Delta v = -\lim\limits_{\varepsilon\to 0} G_1^\varepsilon \text{ in } \mathcal{D}\setminus\Gamma_t,\ t>0, \\ \partial_n v = 0 \text{ on } \partial\mathcal{D}, \\ \quad v = \lambda H - \lim\limits_{\varepsilon\to 0} G_2^\varepsilon \text{ on } \Gamma_t, \\ \quad \mathcal{V} = \dfrac{1}{2}\left[\dfrac{\partial}{\partial n}\right]_{\Gamma_t} v \text{ on } \Gamma_t, \\ \Gamma_0 = \Gamma_{00}, \end{cases} \tag{1.19}$$

Since they require some regularity conditions for G_1^ε, G_2^ε w.r.t time, which are not satisfied by Brownian motions, it is not clear how to obtain the stochastic Hele-Shaw model rigorously. *Until now, the rigorous complete description of the motion of interfaces in dimensions two and three in stochastic case stands for many years as a wide open problem.*

We mention that in [Fun99] and [Web10], the authors consider the following stochastic Allen-Cahn equation

$$\partial_t u = \Delta u - \frac{1}{\varepsilon^2} f(u) + \frac{1}{\varepsilon}\Xi_t^\varepsilon. \tag{1.20}$$

The noise Ξ^ε is constant in space and smooth in time. For $\varepsilon \to 0$ the correlation length goes to zero at a precise rate and $\int_0^t \Xi_s^\varepsilon ds$ converges to a Brownian motion pathwisely. They prove that the dynamics of the phase-separating hyperplane Γ_t appearing in the limit is given by stochastic mean curvature flow (see also in [Fun16, Chapter 4]). For space-time white noise, in [TW18] the authors prove the "exponential loss of memory property". But for sharp interface limit, there is still no result for space-time white noise.

In Chapter 4, we consider the sharp interface limit of stochastic Cahn-Hilliard equation driven by singular noise. The stochastic Cahn-Hilliard equation is a model for the non-equilibrium dynamics of metastable states in phase transitions, [Coo70, HH77, Lan71]. In Section 4.2, we consider the Cahn-Hilliard-Cook model which is generated by Cook, [Coo70] (see also in [HH77]), incorporating thermal fluctuations in the form of an additive noise. In our case the noise is chosen as $\mathcal{W} = W_1$ or $\mathcal{W} = \nabla\cdot W_2$, where W_1 is mass-conserved $L^2(\mathcal{D},\mathbb{R})$-cylindrical Wiener process and W_2 is an $L^2(\mathcal{D},\mathbb{R}^2)$-cylindrical Wiener process. In the case that $\mathcal{W} = \nabla\cdot W_2$, the equation is also well-known as time-dependent Ginzburg-Landau (TDGL) equation. This equation is also related to the stochastic quantization for Φ_2^4-quantum field. For the existence and uniqueness results for these two kinds of equations, we refer to [DPD96, RYZ18] and the reference therein.

To analyze the sharp interface limit of the solution $(u^\varepsilon, v^\varepsilon)$ to equation (1.13), we estimate the difference of $(u^\varepsilon, v^\varepsilon)$ to $(u_A^\varepsilon, v_A^\varepsilon)$ which is the solution to equation (1.16). For the case $\mathcal{W} = W_1$, we follow the idea in [ABK18]. Let u^ε be the solutions to equation (1.13) and u_A^ε be the approximation solution in Theorem 4.2. We consider the equation that the residual $R^\varepsilon := u^\varepsilon - u_A^\varepsilon$ satisfies. Then we prove that R^ε converges to 0 for $\sigma > \frac{107}{12}$ by obtaining a uniform estimate of R^ε . Moreover, we prove that $v^\varepsilon - v_A^\varepsilon$ also converges to 0, where v_A^ε is the potential defined in (1.16). Hence we obtain that the sharp interface limit of the equation (1.13) satisfies the deterministic Hele-Shaw model (1.17) if $\sigma > \frac{107}{12}$. We mention that since the noise is rougher, we cannot apply Itô's formulae to R^ε directly. Hence the trick in [ABK18] fails in our case. Instead, we make use of the Da prato-Debussche's trick (see [DPD03]). That is, let $Z^\varepsilon = \varepsilon^\sigma \int_0^t e^{-\varepsilon(t-s)\Delta^2} d\mathcal{W}_s$ and $Y^\varepsilon = R^\varepsilon - Z^\varepsilon$. Compared with Z^ε and u^ε, Y^ε has better regularity, which enables us to apply Newton-Leibniz formula and obtain uniform estimate for Y^ε instead.

For the case $\mathcal{W} = \nabla \cdot W_2$ the equation (1.13) is ill-posed in the classical sense, since the solution is not a function. To define the nonlinear terms, a renormalization method is required. As the solution is a distribution, we do not consider the sharp interface limit for the solutions of (1.13) directly. Instead we do suitable approximation for the noise with $W^h := W_2 * \rho_h$, where ρ_h approximates to identity (as $h \to 0$) and we consider the following renormalized equation:

$$du^{\varepsilon,h} = \Delta\left(-\varepsilon\Delta u^{\varepsilon,h} + \frac{1}{\varepsilon}\left(f(u^{\varepsilon,h}) - 3c^{\varepsilon}_{h,t}u^{\varepsilon,h}\right)\right)dt + \varepsilon^{\sigma}\nabla \cdot dW^h_t, \tag{1.21}$$

where $3c^{\varepsilon}_{h,t}u^{\varepsilon,h}$ is the renormalization term (see (refc4a.5)). As $h \to 0$, $u^{\varepsilon,h}$ converges to u^{ε}, which is the unique solution to equation (1.13). Similarly we consider the residual $R^{\varepsilon,h} = u^{\varepsilon,h} - u^{\varepsilon}_A$ and do a similar estimate as before. We mention that for fixed $\varepsilon > 0$, $c^{\varepsilon}_{h,t} \to \infty$ as $h \to 0$, which makes the term $c^{\varepsilon}_{h,t}u^{\varepsilon,h}$ hard to control. Thus we consider the case that $\varepsilon \lesssim h^{\iota}$ for some $\iota > 0$ and h goes to 0 (see Theorem 4.14). In this case, $c^{\varepsilon}_{h,t}$ can be very small as $h \to 0$. Thus the term $c^{\varepsilon}_{h,t}u^{\varepsilon,h}$ is small. For other terms in (1.21), the method is similar as the case that $\mathcal{W} = W_1$. Finally we prove that $R^{\varepsilon,h}$ and $v^{\varepsilon,h} - v^{\varepsilon}_A$ converge to 0 if $\sigma > \frac{26}{3}$. This also implies that the sharp interface limit of the solution to equation (1.21) is given by (1.17).

1.2.3 Sharp interface limit for small $\sigma \geq 0$

In Chapter 5, we continue to consider the sharp interface limit of stochastic Cahn-Hilliard equation (1.8) for small $\sigma \geq 0$. As what was showed in Chapter 4, for large $\sigma > 0$, the stochastic Cahn-Hilliard equation (1.8) converges to the deterministic Hele-Shaw model (1.5). However, for $\sigma \geq 0$ small, the perturbation by the noise become much stronger. It is reasonable to think that the solutions to equation (1.8) do not converge to deterministic Hele-Shaw model (1.5) when σ is small. But the method in Chapter 4 can be only applied to prove the convergence to (1.5) and also seems not easy to obtain the convergence for small σ.

To overcome the difficulty, we use a weak approach which is motivated by [Che96], where the author considered the deterministic Cahn-Hilliard equation

$$\begin{cases} \partial_t u^{\varepsilon} = \Delta v^{\varepsilon}, \quad (t,x) \in [0,T] \times \mathcal{D}, \\ v^{\varepsilon} = -\varepsilon\Delta u^{\varepsilon}(t) + \dfrac{1}{\varepsilon}f(u^{\varepsilon}(t)), \quad (t,x) \in [0,T] \times \mathcal{D}, \\ \dfrac{\partial u^{\varepsilon}}{\partial n} = \dfrac{\partial v^{\varepsilon}}{\partial n} = 0, \quad (t,x) \in [0,T] \times \partial\mathcal{D}, \\ u^{\varepsilon}(0,x) = u^{\varepsilon}_0(x), \quad x \in \mathcal{D}, \end{cases} \tag{1.22}$$

and formulated a weak solution to the deterministic Hele-Shaw model (1.5) (see Definition 5.2) and showed that the solutions to (1.22) approach, as $\varepsilon \searrow 0$, to weak solutions to (1.5) by using a compactness argument. In fact, the Cahn-Hilliard equation (1.22) is an H^{-1}-gradient flow with the van der Waals-Cahn-Hilliard energy functional

$$\mathcal{E}^{\varepsilon}(u^{\varepsilon}) := \int_{\mathcal{D}} e^{\varepsilon}(u^{\varepsilon})dx, \quad e^{\varepsilon}(u^{\varepsilon}) := \frac{\varepsilon}{2}|\nabla u^{\varepsilon}|^2 + \frac{1}{\varepsilon}F(u^{\varepsilon}). \tag{1.23}$$

Denote by $(u^{\varepsilon}_D, v^{\varepsilon}_D)$ the solution to the deterministic Cahn-Hilliard equation (1.22). One can directly verify that

$$\frac{d}{dt}\mathcal{E}^{\varepsilon}(u^{\varepsilon}_D) = -\int_{\mathcal{D}} |\nabla v^{\varepsilon}_D|^2 \leq 0, \tag{1.24}$$

which is also called the Lyapunov property for equation (1.22). Thus $\mathcal{E}^\varepsilon(u_D^\varepsilon)$ is uniformly bounded in $t, \varepsilon > 0$ if the energy of initial value is uniformly bounded in ε. Note that as $\varepsilon \to 0$, $F(u_D^\varepsilon) \to 0$, which is equivalent to $u_D^\varepsilon \to -1 + 2\mathbb{1}_E$ for some $E \subset [0,T] \times \mathcal{D}$ where $\mathbb{1}_E$ is the characteristic function of E, i.e. $\mathbb{1}_E(x) = 1$ when $x \in E$ and $\mathbb{1}_E(x) = 0$ when $x \notin E$. $\Gamma_t := \partial E_t$ is the interface. By using a varifold approach, Chen in [Che96] analyzed the property of the limit of the solutions to equation (1.22) and then proposed a definition of weak solution of this limit. Any classical smooth solutions to (1.22) are weak solutions. In some special case, the smooth weak solutions are also classical solutions to (1.5). We need to mention that in [ABC94] and Chapter 4, the convergence of solutions to Cahn-Hilliard equation (1.22) to (1.5) is proved under the assumption on the existence of smooth solution to (1.5). While in [Che96], Chen proved the convergence of the solution to equation (1.22) and analyzed the limit directly. No assumption on existence of solution to (1.5) is required in [Che96].

In our case, we consider the sharp interface limit of the following stochastic Cahn-Hilliard equation on a bounded smooth open domain $\mathcal{D} \subset \mathbb{R}^d$ $(d = 2, 3)$:

$$\begin{cases} du^\varepsilon = \Delta v^\varepsilon dt + \varepsilon^\sigma dW_t, \quad (t,x) \in [0,T] \times \mathcal{D}, \\ v^\varepsilon = -\varepsilon \Delta u^\varepsilon(t) + \dfrac{1}{\varepsilon} f(u^\varepsilon(t)), \quad (t,x) \in [0,T] \times \mathcal{D}, \\ \dfrac{\partial u^\varepsilon}{\partial n} = \dfrac{\partial v^\varepsilon}{\partial n} = 0, \quad (t,x) \in [0,T] \times \partial\mathcal{D}, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x), \quad x \in \mathcal{D}. \end{cases} \tag{1.25}$$

Here W is a Q-Wiener process where Q satisfies (5.4) and (5.5). $f(u) = F'(u)$ where $F(u) = \frac{1}{4}(u^2-1)^2$ is the double well potential and the initial data u_0^ε satisfies

$$\begin{cases} \sup\limits_{0<\varepsilon\le 1} \displaystyle\int_{\mathcal{D}} \left(\frac{\varepsilon}{2} |\nabla u_0^\varepsilon(x)|^2 + \frac{1}{\varepsilon} F(u_0^\varepsilon(x)) \right) dx \le \mathcal{E}_0 < \infty, \\ \dfrac{1}{|\mathcal{D}|} \displaystyle\int_{\mathcal{D}} u_0^\varepsilon(x) dx = m_0 \in (-1,1) \quad \forall \varepsilon \in (0,1]. \end{cases} \tag{1.26}$$

For small $\sigma \ge 0$, we extend the method in [Che96] to equation (1.25) and obtain weak solutions to the limit of equation (1.25). Then we consider the limit of the solution to equation (1.25) directly, which enables us to analyze different models the limit should satisfy. We mainly consider (1.25) with two types of driven noise: Q-Wiener process and “smeared” noise which is smooth in time.

The equation with Q-Wiener process for $\boldsymbol{\sigma \ge \frac{1}{2}}$. In this case, we can obtain that for $\sigma > \frac{1}{2}$, the solutions to equation (1.25) converge to the weak solutions defined in Definition 5.2. In fact, motivated by [DPD96], we apply the Itô's formula to $\mathcal{E}^\varepsilon(u^\varepsilon)$ and prove the Lyapunov property of equation (1.25) for all $\boldsymbol{\sigma \ge \frac{1}{2}}$ (see Lemma 5.6). By tightness argument, we prove that for all $\sigma > \frac{1}{2}$, the solutions to equation (1.25) converge to the weak solution of the limit of deterministic Cahn-Hilliard equation (1.22) defined by Chen [Che96] (see Theorem 5.3). For $\sigma = \frac{1}{2}$, the tightness and convergence results are still true. But we cannot conclude that the limit is a weak solution defined in Definition 5.2.

Particularly in radial symmetric case, we prove that for all $\sigma \ge \frac{1}{2}$, the limit of solutions to equation (1.25) satisfy (1.22) in the weak sense. Thus we conjecture that in general for $\mathbb{P} - a.s.\ \omega$, as $\varepsilon \searrow 0$, the chemical potential $v^\varepsilon(\omega)$ tends to a limit $v(\omega)$ which, together with a free boundary $\Gamma(\omega) := \cup_{0\le t\le T}(\{t\} \times \Gamma_t(\omega))$, $(v(\omega), \Gamma(\omega))$ satisfies (1.5).

The equation with "smeared" noise for $\boldsymbol{\sigma \geq 0}$. Moreover, we consider stochastic Cahn-Hilliard equation driven by"smeared" noise which is smooth in time. This kind of noise was considered also for stochastic Allen-Cahn equation in [Fun99, Web10, FY19].

We smoothen the noise in time and consider the following random PDE:

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \Delta v^\varepsilon + \varepsilon^\sigma \xi_t^\varepsilon, \quad (t,x) \in [0,T] \times \mathcal{D}, \\ v^\varepsilon = -\varepsilon \Delta u^\varepsilon(t) + \dfrac{1}{\varepsilon} f(u^\varepsilon(t)), \quad (t,x) \in [0,T] \times \mathcal{D}, \\ \dfrac{\partial u^\varepsilon}{\partial n} = \dfrac{\partial v^\varepsilon}{\partial n} = 0, \quad (t,x) \in [0,T] \times \partial\mathcal{D}, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x), \quad x \in \mathcal{D}, \end{cases} \tag{1.27}$$

where $\xi_t^\varepsilon = \frac{dW^\varepsilon}{dt}$, $W_t^\varepsilon := \int_{-\infty}^{\infty} \rho_\varepsilon(t-s) W_s ds$ and ρ_ε is an approximate delta function on $\mathbb{R}$. Formally as $\varepsilon \to 0$, $\xi^\varepsilon \to \frac{dW}{dt}$. Since ξ_t^ε is smooth in time, this enables us to apply the Newton-Leibniz formula to $\mathcal{E}^\varepsilon(u^\varepsilon)$ and obtain the Lyapunov property. Thus the tightness and the convergence results hold for all $\boldsymbol{\sigma \geq 0}$. Similar as before, for all $\sigma > 0$, the solutions to (1.27) converge to the weak solution to Definition 5.2 (see Theorem 5.21). For the interesting case that $\boldsymbol{\sigma = 0}$, when $\varepsilon \searrow 0$, we have that $u^\varepsilon \to -1 + 2\mathbb{1}_E$ for some $E \in [0,T] \times \mathcal{D}$, $v^\varepsilon \to v$ and

$$2d\mathbb{1}_E = \Delta v dt + dW_t. \tag{1.28}$$

(1.28) actually gives a weak formula to describe how the evolution of the interface $\Gamma_t := \partial E_t$ is governed by the noise W (see Theorem 5.24). This gives the first rigorous result of the sharp interface limit of stochastic Cahn-Hillliard limit to a stochastic model. Similar as before, we conjecture that for $\mathbb{P} - a.s.\ \omega$, as $\varepsilon \searrow 0$, the chemical potential $v^\varepsilon(\omega)$ tends to a limit $v(\omega)$ which, together with a free boundary $\Gamma(\omega) := \cup_{0 \leq t \leq T}(\{t\} \times \Gamma_t(\omega))$, $(v(\omega), \Gamma(\omega))$ satisfies the following stochastic problem:

$$\begin{cases} \Delta v dt = -dW_t \text{ in } \mathcal{D} \setminus \Gamma_t,\ t > 0, \\ \quad \dfrac{\partial v}{\partial n} = 0 \text{ on } \partial\mathcal{D}, \\ \quad v = SH \text{ on } \Gamma_t, \\ \quad \mathcal{V} dt = \dfrac{1}{2} \left[\dfrac{\partial}{\partial n}\right]_{\Gamma_t} (v dt + \Delta^{-1} dW_t). \end{cases} \tag{1.29}$$

We also mention that Chen's definition for weak solution in Definition 5.2 is not so "perfect", since it is still unknown whether in general such a smooth weak solution is a classical solution to (1.5). The problems come from that a "good" weak formula for the third equation in (1.5) is still missing. Moreover, in [ABK18] the authors also give some different conjectures about the sharp interface limit of equation (1.25) via a formal calculation, especially in the case that $\sigma = 1$. In their case the value of v on the interface is different from ours. As what we analyze in Remark 5.27, our model (1.29) fit quite well in radial symmetric case. But in general case, we still cannot give a fully rigorous proof.

In fact, identifying the value of v on the interface Γ_t is the main task of varifold approach to study the sharp interface limit of both Cahn-Hilliard equation and Allen-Cahn equation (cf. [HT00, Ton02, Ton05, RS06, Le08, RT08]). In these literature, the authors give a weak formula for the third equation in (1.5). But they are unable to prove the limit of the solutions to equation (1.22) satisfy such weak formula. Until now, a fully rigorous proof of the sharp interface limit of Cahn-Hilliard equation is still missing.

Finally, as what we mentioned before, the methods in [Che96] and also in this chapter are deeply related to the theory of varifolds. We recall some related definitions in Section 5.1. In fact, varifolds represent very natural generalizations of classical n-surfaces, as they encode, loosely speaking, a joint distribution of mass and tangents. More technically, varifolds are Radon measures defined on the Grassmann bundle $\mathbb{R}^d \times G(n, d)$, whose elements are pairs (x, S) specifying a position in space and an unoriented n-plane. Varifolds have been proposed more than 50 years ago by Almgren [Alm65] as a mathematical model for soap films, bubble clusters, crystals, and grain boundaries. After Allard's fundamental work [All72], varifolds have been successfully used in the context of Geometric Measure Theory, Geometric Analysis, and Calculus of Variations. One successful application of varifolds resulted in the definition and the study of a general weak mean curvature flow in [Bra78], which allowed to prove existence of mean curvature evolution with singularities in [KT17]. Beyond the theory of rectifiable varifolds, the flexibility of the varifold structure has been proved to be relevant to model diffuse interfaces, e.g., phase field approximations, and a crucial part in the proof of the convergence of the Allen-Cahn equation to Brakke's mean curvature flow [Ilm93, Ton03, TT15], or in the proof of the Γ-convergence of Cahn-Hilliard type energies to the Willmore energy (up to an additional perimeter term) [Ton05, RS06, Le08, RT08].

1.3 Structure of the thesis

This thesis is organised in the following:

In Chapter 2 we collect some preliminaries for later chapters.

In Chapter 3, we obtain the global well-posedness of stochastic Cahn-Hilliard equation (1.4) driven by conservative white noise and prove the ergodicity. This chapter is organized as follows: In Section 3.1 we collect some results related to Besov spaces. In Section 3.2 we study the solution to the linear equation and define the Wick power. In Section 3.3 we obtain the global existence and uniqueness of solutions to the shifted equation (1.12). In Section 3.4 we obtain existence of probabilistically weak solutions via the Dirichlet form approach. By clarifying the relation between the two solutions we obtain Φ^4_2-field ν is an invariant measure of X Markov uniqueness in the restricted sense for the generator of the Dirichlet form restricted to $\mathcal{F}C_b^\infty$ and uniqueness of the probabilistically weak solutions to (1.9). Moreover, using the Yamada-Watanabe Theorem in [Kur07] we obtain a probabilistically strong solution to (1.9) in the stationary case. Finally we prove the strong Feller property and exponential ergodicity of the Markov semigroup associated to the solution to (1.9) in Section 3.5. This part is based on the joint work [RYZ18] with Prof. Michael Röckner and Prof. Rongchan Zhu.

In Chapter 4, we consider the stochastic Cahn-Hilliard equation (1.4) driven by space-time white noise and conservative noise and prove that for large $\sigma > 0$, the sharp interface limit satisfies the deterministic Hele-Shaw model (1.5). This chapter is organized as follows: In Section 4.1, we collect some results related to Besov spaces. The theorem about the sharp interface limit for space-time white noise is stated in Section 4.2 and we prove it in Section 4.3. In Section 4.4 we use a similar argument as we used in Section 4.3 to prove the results for conservative noise. This part is based on the joint work [BYZ19] with Prof. Lubomir Banas and Prof. Rongchan Zhu.

In Chapter 5, we consider the stochastic Cahn-Hilliard equation (1.4) where $\mathcal{W}^\varepsilon = W$ or $\mathcal{W}^\varepsilon = W^\varepsilon$. Here W is a Q-Wiener process and W^ε is smooth in time and converges to W as $\varepsilon \searrow 0$. This chapter is organized as follows: In Section 5.1 we give some basic notations and assumptions. In subsection 5.1.3 we give the main results for (1.25) driven by Q-Wiener process. In Section 5.2, we establish certain ε-independent estimates for the solution to (1.25), which allow us obtain tightness and then apply Skorokhod's theorem to obtain a convergence subsequence for all $\sigma \geq \frac{1}{2}$. Moreover for $\sigma > \frac{1}{2}$, we prove that this limit is actually a weak solution to (1.5). Similar as in [Che96], in Section 5.3, we study the radially symmetric case and prove that for all $\sigma \geq \frac{1}{2}$, the limit of the solution to equation (1.25) satisfies the deterministic Hele-Shaw model (1.5). The rigorous proof of Theorem 5.4 in radial symmetric case is given in Section 5.4. Finally in Section 5.5, we consider the case for "smeared" noise ξ_t^ε and obtain the convergence result for all $\sigma \geq 0$. For $\sigma > 0$, the limit of the solution to (1.27) is a weak solution to equation (1.5). For $\sigma = 0$, we obtain a stochastic characterisation of the evolution of the interface (1.28) and partially prove in radial symmetric case that it satisfies the stochastic Hele-Shaw model (1.29). This part is based on the joint work [YZ19] with Prof. Rongchan Zhu.

Chapter 2

Preliminary

2.1 Besov spaces

In the following we recall the definition of Besov spaces which will be frequently used in Chapter 3 and 4. For a general introduction to the theory of Besov spaces we refer to [BCD11, Tri78, Tri06]. First we introduce the following notations. Throughout the thesis, we use the notation $a \lesssim b$ if there exists a constant $c > 0$ such that $a \leq cb$, and we write $a \backsimeq b$ if $a \lesssim b$ and $b \lesssim a$. The space of real valued infinitely differentiable functions of compact support is denoted by $\mathscr{D}(\mathbb{R}^d)$ or $\mathscr{D}$. The space of Schwartz functions is denoted by $\mathcal{S}(\mathbb{R}^d)$. Its dual, the space of tempered distributions, is denoted by $\mathcal{S}'(\mathbb{R}^d)$. The Fourier transform and the inverse Fourier transform are denoted by $\mathcal{F}$ and $\mathcal{F}^{-1}$, respectively.

Let $\chi, \theta \in \mathcal{D}$ be nonnegative radial functions on $\mathbb{R}^d$, such that

(i). the support of χ is contained in a ball and the support of θ is contained in an annulus;

(ii). $\chi(z) + \sum_{j\geq 0} \theta(2^{-j}z) = 1$ for all $z \in \mathbb{R}^d$.

(iii). $\mathrm{supp}(\chi) \cap \mathrm{supp}(\theta(2^{-j}\cdot)) = \emptyset$ for $j \geq 1$ and $\mathrm{supp}\theta(2^{-i}\cdot) \cap \mathrm{supp}\theta(2^{-j}\cdot) = \emptyset$ for $|i-j| > 1$.

We call such a pair (χ, θ) dyadic partition of unity, and for the existence of dyadic partitions of unity we refer to [BCD11, Proposition 2.10]. The Littlewood-Paley blocks are now defined as

$$\Delta_{-1}u = \mathcal{F}^{-1}(\chi \mathcal{F}u) \quad \Delta_j u = \mathcal{F}^{-1}(\theta(2^{-j}\cdot)\mathcal{F}u).$$

Besov spaces

For $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$, $u \in \mathscr{D}$ we define

$$\|u\|_{B^\alpha_{p,q}} := (\sum_{j\geq -1} (2^{j\alpha}\|\Delta_j u\|_{L^p})^q)^{1/q},$$

with the usual interpretation as l^∞ norm in case $q = \infty$. The Besov space $B^\alpha_{p,q}$ consists of the completion of $\mathscr{D}$ with respect to this norm and the Hölder-Besov space $\mathcal{C}^\alpha$ is given by $\mathcal{C}^\alpha(\mathbb{R}^d) = B^\alpha_{\infty,\infty}(\mathbb{R}^d)$. For $p, q \in [1, \infty)$,

$$B^\alpha_{p,q}(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) : \|u\|_{B^\alpha_{p,q}} < \infty\},$$

$$\mathcal{C}^\alpha(\mathbb{R}^d) \subsetneq \{u \in \mathcal{S}'(\mathbb{R}^d) : \|u\|_{\mathcal{C}^\alpha(\mathbb{R}^d)} < \infty\}.$$

We point out that everything above and everything that follows can be applied to distributions on the torus (see [Sic85], [SW72]). More precisely, let $\mathcal{S}'(\mathbb{T}^d)$ be the space of distributions on $\mathbb{T}^d$. Besov spaces on the torus with general indices $p, q \in [1, \infty]$ are defined as the completion of $C^\infty(\mathbb{T}^d)$ with respect to the norm

$$\|u\|_{B^\alpha_{p,q}(\mathbb{T}^d)} := (\sum_{j\geq -1} (2^{j\alpha}\|\Delta_j u\|_{L^p(\mathbb{T}^d)})^q)^{1/q},$$

and the Hölder-Besov space $\mathcal{C}^\alpha$ is given by $\mathcal{C}^\alpha = B^\alpha_{\infty,\infty}(\mathbb{T}^d)$. We write $\|\cdot\|_\alpha$ instead of $\|\cdot\|_{B^\alpha_{\infty,\infty}(\mathbb{T}^d)}$ in the following for simplicity. For $p, q \in [1, \infty)$

$$B^\alpha_{p,q}(\mathbb{T}^d) = \{u \in \mathcal{S}'(\mathbb{T}^d) : \|u\|_{B^\alpha_{p,q}(\mathbb{T}^d)} < \infty\}.$$

$$\mathcal{C}^\alpha \subsetneq \{u \in \mathcal{S}'(\mathbb{T}^d) : \|u\|_\alpha < \infty\}. \tag{2.1}$$

Here we choose Besov spaces as completions of smooth functions, which ensures that the Besov spaces are separable which has a lot of advantages for our analysis below.

Wavelet analysis

We will also use wavelet analysis to determine the regularity of a distribution in a Besov space. In the following we briefly summarize wavelet analysis below and we refer to work of Meyer [Mey95], Daubechies [Dau92] and [Tri06] for more details on wavelet analysis. For every $r > 0$, there exists a compactly supported function $\varphi \in C^r(\mathbb{R})$ such that:

(i). We have $\langle \varphi(\cdot), \varphi(\cdot - k)\rangle = \delta_{k,0}$ for every $k \in \mathbb{Z}$;

(ii). There exist Ëœ$\tilde{a}_k, k \in \mathbb{Z}$ with only finitely many non-zero values, and such that $\varphi(x) = \sum_{k\in\mathbb{Z}} \tilde{a}_k \varphi(2x - k)$ for every $x \in \mathbb{R}$;

(iii). For every polynomial P of degree at most r and for every $x \in \mathbb{R}$, $\sum_{k\in\mathbb{Z}} \int P(y)\varphi(y-k)dy\varphi(x-k) = P(x)$.

Given such a function φ, we define for every $x \in \mathbb{R}^d$ the recentered and rescaled function φ^n_x as follows

$$\varphi^n_x(y) := \Pi^d_{i=1} 2^{\frac{n}{2}} \varphi(2^n(y_i - x_i)).$$

Observe that this rescaling preserves the L^2-norm. We let V_n be the subspace of $L^2(\mathbb{R}^d)$ generated by $\{\varphi^n_x : x \in \Lambda_n\}$, where

$$\Lambda_n := \{(2^{-n}k_1, ..., 2^{-n}k_d) : k_i \in \mathbb{Z}\}.$$

An important property of wavelets is the existence of a finite set Ψ of compactly supported functions in $\mathcal{C}^r$ such that, for every $n \geq 0$, the orthogonal complement of V_n inside V_{n+1} is given by the linear span of all the $\psi^n_x, x \in \Lambda_n, \psi \in \Psi$. For every $n \geq 0$

$$\{\varphi^n_x, x \in \Lambda_n\} \cup \{\psi^m_x : m \geq n, \psi \in \Psi, x \in \Lambda_m\},$$

forms an orthonormal basis of $L^2(\mathbb{R}^d)$. This wavelet analysis allows one to identify a countable collection of conditions that determine the regularity of a distribution.

Setting $\Psi_\star = \Psi \cup \{\varphi\}$, by some methods in weighted Besov space (see [RZZ17b, (2.2), (2.3), (2.4)] and its reference for details), we know that for $p \in (1, \infty)$, $\alpha \in \mathbb{R}$, $f \in \mathcal{C}^\alpha$

$$\|f\|^p_\alpha \lesssim \sum_{n=0}^{\infty} 2^{n(\alpha+1)p} \sum_{\psi\in\Psi_\star} \sum_{x\in\Lambda_n} |\langle f, \psi^n_x\rangle|^p w(x)^p. \tag{2.2}$$

where $w(x) = (1 + |x|^2)^{-\frac{\sigma}{2}}, \sigma > 0$.

Estimates on the torus

In this part we give estimates on the torus for later use. Set $\mathfrak{A} = (I - \Delta)^{\frac{1}{2}}$. For $s \geq 0, p \in [1, +\infty]$ we use H_p^s to denote the subspace of $L^p(\mathbb{T}^d)$, consisting of all f which can be written in the form $f = \mathfrak{A}^{-s}g, g \in L^p(\mathbb{T}^d)$ and the H_p^s norm of f is defined to be the L^p norm of g, i.e. $\|f\|_{H_p^s} := \|\mathfrak{A}^s f\|_{L^p(\mathbb{T}^d)}$.

To study (1.1) in the finite volume case, we will need several important properties of Besov spaces on the torus and we recall the following Besov embedding theorems on the torus first (c.f. [Tri78, Theorem 4.6.1], [GIP15, Lemma A.2], [Tri92, Remark 3, Section 2.3.2]):

Lemma 2.1. *(i) Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$, and let $\alpha \in \mathbb{R}$. Then $B^\alpha_{p_1,q_1}(\mathbb{T}^d)$ is continuously embedded in $B^{\alpha-d(1/p_1-1/p_2)}_{p_2,q_2}(\mathbb{T}^d)$.*

(ii) Let $s \geq 0$, $1 < p < \infty$, $\varepsilon > 0$. Then $H_p^{s+\varepsilon} \subset B^s_{p,1}(\mathbb{T}^d) \subset B^s_{1,1}(\mathbb{T}^d)$.

(iii) Let $1 \leq p_1 \leq p_2 < \infty$ and let $\alpha \in \mathbb{R}$. Then $H^\alpha_{p_1}$ is continuously embedded in $H^{\alpha-d(1/p_1-1/p_2)}_{p_2}$.

(iv) Let $0 < q \leq \infty$, $1 \leq p \leq \infty$ and $s > 0$. Then $B^s_{p,q} \subset L^p$.

Here $\subset$ means that the embedding is continuous and dense.

We recall the following Schauder estimates, i.e. the smoothing effect of the heat flow, for later use.

Lemma 2.2. *([GIP15, Lemma A.7]) Let $u \in B^\alpha_{p,q}(\mathbb{T}^d)$ for some $\alpha \in \mathbb{R}, p, q \in [1, \infty]$. Then for every $\delta \geq 0$*

$$\|e^{-tA^2}u\|_{B^{\alpha+\delta}_{p,q}(\mathbb{T}^d)} \lesssim t^{-\delta/4}\|u\|_{B^\alpha_{p,q}(\mathbb{T}^d)}.$$

One can extend the multiplication on suitable Besov spaces and also have the duality properties of Besov spaces from [Tri78, Chapter 4]:

Lemma 2.3. *(i) The bilinear map $(u; v) \mapsto uv$ extends to a continuous map from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{C}^{\alpha\wedge\beta}$ if and only if $\alpha + \beta > 0$.*

(ii) Let $\alpha \in (0, 1)$, $p, q \in [1, \infty]$, p' and q' be their conjugate exponents, respectively. Then the mapping $(u; v) \mapsto \int uvdx$ extends to a continuous bilinear form on $B^\alpha_{p,q}(\mathbb{T}^d) \times B^{-\alpha}_{p',q'}(\mathbb{T}^d)$.

We recall the following interpolation inequality and multiplicative inequality for the elements in H_p^s, which is required for the a-priori estimate in section 3.3 (cf. [Tri78, Theorem 4.3.1], [RZZ15, Lemma 2.1], [BCD11, Theorem 2.80]):

Lemma 2.4. *(i) Suppose that $s \in (0, 1)$ and $p \in (1, \infty)$. Then for $u \in H_p^1$*

$$\|u\|_{H_p^s} \lesssim \|u\|^{1-s}_{L^p(\mathbb{T}^d)}\|u\|^s_{H_p^1}.$$

(ii) Suppose that $s > 0$ and $p \in (1, \infty)$. If $u, v \in C^\infty(\mathbb{T}^2)$ then

$$\|\mathfrak{A}^s(uv)\|_{L^p(\mathbb{T}^d)} \lesssim \|u\|_{L^{p_1}(\mathbb{T}^d)}\|\mathfrak{A}^s v\|_{L^{p_2}(\mathbb{T}^d)} + \|v\|_{L^{p_3}(\mathbb{T}^d)}\|\mathfrak{A}^s u\|_{L^{p_4}(\mathbb{T}^d)},$$

with $p_i \in (1, \infty], i = 1, ..., 4$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

(iii) Suppose that $s_1 < s_2$ *and* $1 \le p, q \le \infty$. *Then for* $u \in B^{s_2}_{p,q}$ *and* $\forall \theta \in (0,1)$

$$||u||_{B^{\theta s_1+(1-\theta)s_2}_{p,q}} \le ||u||^{\theta}_{B^{s_1}_{p,q}} ||u||^{1-\theta}_{B^{s_2}_{p,q}}.$$

We also collect some important properties for the multiplicative structure of Besov spaces from [MW17] and [Tri06].

Lemma 2.5. *([MW17, Corollary 3.19, Corollary 3.21]) (1) For* $\alpha > 0, p_1, p_2, p, q \in [1,\infty], \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, *the bilinear map* $(u; v) \mapsto uv$ *extends to a continuous bilinear map from* $B^{\alpha}_{p_1,q} \times B^{\alpha}_{p_2,q}$ *to* $B^{\alpha,}_{p,q}$.

(2) For $\alpha < 0, \alpha+\beta > 0, p_1, p_2, p, q \in [1,\infty], \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, *the bilinear map* $(u; v) \mapsto uv$ *extends to a continuous bilinear map from* $B^{\alpha}_{p_1,q} \times B^{\beta}_{p_2,q}$ *to* $B^{\alpha}_{p,q}$.

2.2 Symmetric quasi regular Dirichlet forms and Markov Processes

In this section we recall some general Dirichlet form results from [MR92] which is used in Chapter 3. Let E be a Hausdorff topological space, m a σ-finite measure on E, and let $\mathcal{B}$ the smallest σ-algebra of subsets of E with respect to which all continuous functions on E are measurable. Let Λ be a symmetric Dirichlet form acting in the real $L^2(m)$-space, i.e. Λ is a positive, symmetric, bilinear, closed form with domain $D(\Lambda)$ dense in $L^2(m)$, and such that $\Lambda(\Phi(u),\Phi(u)) \le \Lambda(u,u)$, for any $u \in D(\Lambda)$, where $\Phi(t) = (0 \vee t) \wedge 1, t \in \mathbb{R}$. The latter condition is known to be equivalent with the condition that the associated C_0-contraction semigroup $T_t, t \ge 0$, is submarkovian (i.e. $0 \le u \le 1$ m-a.e. implies $0 \le T_t u \le 1$ m-a.e., for all $u \in L^2(m)$); association means that $\lim_{t\downarrow 0} \frac{1}{t}\langle u - T_t u, v\rangle_{L^2(m)} = \Lambda(u,v), \forall u, v \in D(\Lambda)$.

Definition 2.6. *(cf. [MR92, Chap. IV, Defi. 3.1]) A symmetric Dirichlet form is called* quasi-regular *if the following holds:*

(i) There exists a sequence $(F_k)_{k\in\mathbb{N}}$ *of compact subsets of* E *such that* $\cup_k D(\Lambda)_{F_k}$ *is* $\Lambda_1^{1/2}$*-dense in* $D(\Lambda)$ *(where* $D(\Lambda)_{F_k} := \{u \in D(\Lambda) | u = 0$ *m-a.e. on* $E - F_k\}$; $\Lambda_1^{1/2}$ *is the norm given by the scalar product in* $L^2(m)$ *defined by* Λ_1, *where* $\Lambda_1(u,v) := \Lambda(u,v) + \langle u, v\rangle$, $\langle , \rangle$ *being the scalar product in* $L^2(m)$. *Such a sequence* $(F_k)_{k\in\mathbb{N}}$ *is called an* Λ*-nest.*

(ii) There exists an $\Lambda_1^{1/2}$*-dense subset of* $D(\Lambda)$ *whose elements have* Λ*-quasi continuous* m*-versions. A real function* u *on* E *is called quasi continuous when there exists an* Λ*-nest* (F_k) *s.t.* u *restricted to* F_k *is continuous.*

(iii) There exists $u_n \in D(\Lambda), n \in \mathbb{N}$, *with* Λ*-quasi continuous* m*-versions* $\tilde{u}_n$ *and there exists an* Λ*-exceptional subset* N *of* E *s.t.* $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ *separates the points of* $E - N$. *An* Λ*-exceptional subset of* E *is a subset* $N \subset \cap_k (E - F_k)$ *for some* Λ*-nest* (F_k).

To recall the main results in [MR92] we recall the definitions of a Markov process and a right process. Here we consider only Markov processes with life time ∞.

Definition 2.7. *(cf. [MR92, Chap. IV Defi. 1.5] A collection* $\mathbf{M} := (\Omega, \mathcal{M}, (X_t)_{t\ge 0}, (\mathbb{P}^z)_{z\in E})$ *is called a Markov process (with state space* E*) if it has the following properties.*

(i) There exists a filtration $(\mathcal{M}_t)$ *on* $(\Omega, \mathcal{M})$ *such that* $(X_t)_{t\ge 0}$ *is an* $(\mathcal{M}_t)_{t\ge 0}$ *adapted stochastic process with state space* E.

(ii) For each $t \geq 0$ *there exists a shift operator* $\theta_t : \Omega \to \Omega$ *such that* $X_s \circ \theta_t = X_{s+t}$ *for all* $s, t \geq 0$

(iii) $\mathbb{P}^z, z \in E$, *are probability measures on* $(\Omega, \mathcal{M})$ *such that* $z \mapsto \mathbb{P}^z(A)$ *is* $\mathcal{B}(E)^*$-*measurable for each* $A \in \mathcal{M}$ *resp.* $\mathcal{B}(E)$-*measurable if* $A \in \sigma\{X_s | s \in [0, \infty)\}$, *where* $\mathcal{B}(E)^* := \cap_{\mathbb{P} \in \mathcal{P}(E)} \mathcal{B}^{\mathbb{P}}(E)$ *for* $\mathcal{P}(E)$ *denoting the family of all probability measures on* $(E, \mathcal{B}(E))$ *and* $\mathcal{B}^{\mathbb{P}}(E)$ *denotes the completion of the* σ-*algebra* $\mathcal{B}(E)$ *w.r.t. a probability* $\mathbb{P}$.

(iv) (Markov property) For all $A \in \mathcal{B}(E)$ *and any* $t, s \geq 0$

$$\mathbb{P}^z[X_{s+t} \in A | \mathcal{M}_s] = \mathbb{P}^{X_s}[X_t \in A] \quad \mathbb{P}^z - a.s., z \in E.$$

Definition 2.8. *(cf. [MR92, Chap. IV Defi. 1.8]) Let* $\mathbf{M} := (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (\mathbb{P}^z)_{z\in E})$ *be a Markov process with state space* E *and corresponding filtration* $(\mathcal{M}_t)$. $\mathbf{M}$ *is called a* right process *if it has the following additional properties.*

(i) (Normal property) $\mathbb{P}^z(X_0 = z) = 1$ *for all* $z \in E$.

(ii) (Right continuity) For each $\omega \in \Omega$, $t \mapsto X_t(\omega)$ *is right continuous on* $[0, \infty)$.

(iii) (Strong Markov property) $(\mathcal{M}_t)$ *is right continuous and for every* $(\mathcal{M}_t)$-*stopping time* σ *and every* $\nu \in \mathcal{P}(E)$

$$\mathbb{P}^\nu[X_{\sigma+t} \in A | \mathcal{M}_\sigma] = \mathbb{P}^{X_\sigma}[X_t \in A] \quad P^\nu - a.s.$$

for all $A \in \mathcal{B}(E)$, $t \geq 0$.

Theorem 2.9. *([MR92, Chap. IV Thm 6.7]) Let* E *be a metrizable Lusin space. Then a Dirichlet form* $(\Lambda, D(\Lambda))$ *on* $L^2(E, m)$ *is quasi-regular if and only if there exists a right process* $\mathbf{M}$ *associated with* $(\Lambda, D(\Lambda))$, *i.e. the semigroup of* $\mathbf{M}$ *is an* m-*version of the semigroup associated with* $(\Lambda, D(\Lambda))$. *In this case* $\mathbf{M}$ *is always properly associated with* $(\Lambda, D(\Lambda))$.

Remark 2.10. *The results in [MR92, Chap. IV] are more general and can be applied for general Hausdorff topological spaces and more general Markov processes. Lusin spaces are enough for our use in this thesis.*

2.3 Geometric measure theory

In this section, we recall some definitiosn and results of geometric measure theory which is used in Chapter 5.

In this thesis, we denote by $n \otimes n$ the matrix $(n^i n^j)_{d\times d}$ for $n = (n^1, \cdots, n^d)$. We use "I" to denote the identity matrix $(\delta_{ij})_{d\times d}$. For any $d \times d$ matrices $A = (a_{ij})$ and $B = (b_{ij})$,

$$A : B := \mathrm{Trace}(A^T B) = \sum_{i,j=1}^{d} a_{ij} b_{ij}.$$

We denote by $C_c^m(\mathcal{O})$ the space of m-th differentiable functions with compact support in $\mathcal{O}$ where $\mathcal{O}$ can be open or closed. Note that if $\mathcal{O}$ is compact, $C_c^m(\mathcal{O}) = C^m(\mathcal{O})$. Moreover, we say a vector function $\vec{Y} = (Y^1, \cdots, Y^d) \in C_c^m(\mathcal{O}; \mathbb{R}^d)$ if $Y^i \in C_c^m(\mathcal{O})$ for any $i = 1, \cdots, d$. For any $t > 0$, we denote $\mathcal{O}_t := [0, t] \times \mathcal{O}$. We also denote by $\mathbb{1}_E$ the characteristic function of a set E, which is defined by $\mathbb{1}_E(x) = 1$ for $x \in E$ and $\mathbb{1}_E(x) = 0$ for $x \notin E$.

Moreover, we denote $\mathcal{H}^n$ as the n-dimensional Hausdorff measure on $\mathbb{R}^d$ for any $n \in [0, d]$. For $n = d$, $\mathcal{H}^d$ is just the Lebesgue measure on $\mathbb{R}^d$. For any $\mathcal{H}^n$-measurable function θ, we denote a measure $\mathcal{H}^n\lfloor\theta$ by

$$\mathcal{H}^n\lfloor\theta(A) = \int_A \theta(x) d\mathcal{H}^n(x).$$

For any $\mathcal{H}^n$-measurable set M, $\mathcal{H}^n\lfloor M := \mathcal{H}^n\lfloor\mathbb{1}_M$ is the restriction of $\mathcal{H}^n$ on M.

We denote by $B_r(x)$ the ball in $\mathbb{R}^d$ centered at the origin x with radius r and $B_r := B_r(0)$. We also denote by S_r the sphere of radius r in $\mathbb{R}^d$ and by ω_d the area of unit sphere S_1.

In the following, we recall several definitions from geometric measure theory (cf. [AFP00, Fed14, Sim83]).

Radon measures

Let $\mathcal{D}$ be either an open or a closed domain. If L is a bounded linear functional on $C_c(\mathcal{D})$ satisfying $\langle L, \psi\rangle \geq 0$ whenever $\psi \geq 0$ and $\psi \in C_c(\mathcal{D})$, a measure μ generated by

$$\mu(A) = \sup_{\psi \in C_c(A), |\psi| \leq 1} \langle L, \psi\rangle \quad \text{for all } A \text{ open in } \mathcal{D}$$

is called a Radon measure on $\mathcal{D}$. We use $\langle \mu, \psi\rangle$ $\psi \in C_c(\mathcal{D})$ to denote the value $\int_{\mathcal{D}} \psi d\mu (= \langle L, \psi\rangle)$.

Let $\mathfrak{M}(\mathcal{D}_T)$ be the space of all finite signed measures on $\mathcal{D}_T$ and $\mathfrak{M}_R(\mathcal{D}_T) \subset \mathfrak{M}(\mathcal{D}_T)$ is the space of all Radon measures on $\mathcal{D}_T$. $\mathfrak{M}_R(\mathcal{D}_T)$ and $\mathfrak{M}(\mathcal{D}_T)$ are equipped with the total variation norm $\|\cdot\|_{TV}$ and weak topology, respectively. Now we give a criterion theorem for a compactness sequence in $\mathfrak{M}_R(\mathcal{D}_T)$,

Theorem 2.11. *([Sim83, Theorem 4.4]) Suppose $\{\mu_k\}_{k\geq 1}$ is a sequence of Radon measures on $\mathcal{D}$ with $\sup_{k\geq 1} \mu_k(U) < \infty$ for each open $U \subset \mathcal{D}$ with $\bar{U}$ compact in $\mathcal{D}$. Then there exists a subsequence $\{\mu_{k'}\}$ which weakly converges to a Radon measure on $\mathcal{D}$ in the sense that*

$$\lim_{k'\to\infty} \mu_{k'}(f) = \mu(f) \ \text{ for each } \ f \in C_c(\mathcal{D}),$$

where we used the notation

$$\mu(f) := \int_{\mathcal{D}} f d\mu.$$

BV functions

Let $u \in L^1_{\text{loc}}(\mathcal{D})$. If the distributional gradient Du defined by

$$\langle Du, \vec{Y}\rangle := \langle u, -\text{div}\vec{Y}\rangle \quad \forall \vec{Y} \in C^1_c(\mathcal{D}; \mathbb{R}^d)$$

can be extended as a bounded linear functional over $C_c(\mathcal{D}; \mathbb{R}^d)$, then we say that u is a function of bounded variation, denoted by $u \in BV(\mathcal{D})$. If $u \in BV(\mathcal{D})$, we use $D_i u$ to denote the measure on $C_c(\mathcal{D})$ generated by the functional $\langle u, -\partial_{x_i}\psi\rangle$ for all $\psi \in C^1_c(\mathcal{D})$. We denote by $|Du|$ the Radon measure generated by

$$|Du|(A) := \sup_{\vec{Y} \in C_c(A; \mathbb{R}^d), |\vec{Y}| \leq 1} \int_A u \, \text{div}\vec{Y} dx, \quad \forall A \text{ open } \subset \mathcal{D}.$$

One can show in [Fed14] that $D_i u$ is absolutely continuous with respect to $|Du|$ and there exists a $|Du|$-measurable unit vector valued function $\vec{\nu}$ such that $Du = \vec{\nu}|Du|$, $|Du|-a.e.$.

We say that a set $E \subset \mathcal{D}$ is a BV set if $\mathbb{1}_E \in BV(\mathcal{D})$. We denote $\vec{\nu}_E$ by

$$D\mathbb{1}_E = \vec{\nu}_E|D\mathbb{1}_E| \text{ or } \vec{\nu}_E(x) := \frac{D\mathbb{1}_E(x)dx}{|D\mathbb{1}_E|(x)dx}. \tag{2.3}$$

Clearly, in the case that ∂E is smooth, $\vec{\nu}_E$ is the unit inward normal of E on ∂E.

In the following, we introduce the several concepts of varifold, which can be found in [Sim83, Chapter 8].

Rectifiable set

Definition 2.12. *(rectifiable set) A set $M \subset \mathbb{R}^d$ is said to be a countably $(d-1)$-rectifiable set if $M \subset M_0 \cup \left(\bigcup_{j=1}^{\infty} F_j\left(\mathbb{R}^{d-1}\right)\right)$, where $\mathcal{H}^{d-1}(M_0) = 0$ and $F_j : \mathbb{R}^{d-1} \to \mathbb{R}^d$ are Lipschitz functions for $i = 1, 2, \cdots$.*

Now we will give an important characterization of countably rectifiable sets in terms of "approximate tangent spaces".

Definition 2.13. *If M is an $\mathcal{H}^{d-1}$-measurable subset of $\mathbb{R}^d$ and θ is positive lacally $\mathcal{H}^{d-1}$-integrable function on M, then we say that a $(d-1)$-dimensional subspace $T \subset \mathbb{R}^d$ is the approximate tangent space for M at x with respect to theta if for any $f \in C_c(\mathbb{R}^d)$*

$$\lim_{\lambda \searrow 0} \int_{\eta_{x,\lambda}(M)} f(y)\theta(x+\lambda y)d\mathcal{H}^{d-1}(y) = \theta(x)\int_T f(y)d\mathcal{H}^{d-1}(y),$$

where $\eta_{x,\lambda} : \mathbb{R}^d \to \mathbb{R}^d$ is defined by $\eta_{x,\lambda}(y) = \lambda^{-1}(y-x)$, $x, y \in \mathbb{R}^d$, $\lambda > 0$. We denote $T = T_x M$.

The following theorem gives the important characterization of countably rectifiable sets in terms of existence of approximate tangent spaces.

Theorem 2.14. *([Sim83, Theorem 11.6]) Suppose M is $\mathcal{H}^{d-1}$-measurable. Then M is countably $(d-1)$-rectifiable if and only if there is a positive locally $\mathcal{H}^{d-1}$-integrable function θ on M with respect to which the approximate tangent space $T_x M$ exists for $\mathcal{H}^{d-1} - a.e.\ x \in M$.*

In more general case, we have

Theorem 2.15. *([Sim83, Theorem 11.8]) Suppose μ is a Radon measure on $\mathbb{R}^d$, and for $x \in \mathbb{R}^d$, $\lambda > 0$, let $\mu_{x,\lambda}$ be the measure given by $\mu_{x,\lambda}(A) = \lambda^{1-d}\mu(x+\lambda A)$. Suppose that for $\mu - a.e.\ x$, there is $\theta(x) \in (0,\infty)$ and a $(d-1)$-dimensional subspace $T \subset \mathbb{R}^d$ with*

$$\lim_{\lambda \searrow 0} \int_{\mathbb{R}^d} f(y)d\mu_{x,\lambda}(y) = \theta(x)\int_T f(y)d\mathcal{H}^{d-1}(y). \tag{2.4}$$

(T is called the approximate tangent space for μ at x, and θ is called the multiplicity, such μ is also called rectifiable measure.) Let

$$M := \{x : (2.4) \text{ holds for some } T \text{ and some } \theta(x) \in (0,\infty)\},$$

and set $\theta(x) = 0$, $x \in \mathbb{R}^d \setminus M$.

Then M is countably $(d-1)$-rectifiable, θ is $\mathcal{H}^d$-measurable and $\mu = \mathcal{H}^d \lfloor \theta$. In particular

$$\theta(x) = \lim_{\rho \searrow 0} \frac{\mu(B_\rho(x))}{\omega_{d-1}\rho^{d-1}}, \quad \mu - a.e.\ x \in \mathbb{R}^d.$$

Sets of locally finite perimeter

An important class of countably $(d-1)$-rectifiable sets in $\mathbb{R}^d$ comes from the sets of locally finite perimeter (or Caccipoli sets).

We say that a $\mathcal{H}^d$-measurable subset E in $\mathbb{R}^d$ has *locally finite perimeter* in $\mathcal{D}$ if the characteristic function $\mathbb{1}_E$ of E is a BV function in $\mathcal{D}$. Thus $|D\mathbb{1}_E|$ is a Radon measure on $\mathcal{D}$.

We can also define the *reduced boundary* $\partial^* E$ of E by (see [Sim83, Section 14] for details)

$$\begin{aligned} \partial^* E :=& \{x \in \mathcal{D} : |\vec{\nu}_E(x)| = 1\} = \mathrm{supp}(|D\mathbb{1}_E|) \\ =& \left\{ x \in \mathcal{D} : \vec{\nu}_E = \lim_{\rho \searrow 0} \frac{D\mathbb{1}_E(B_\rho(x))}{|D\mathbb{1}_E|(B_\rho(x))} \text{ exists and has length } 1 \right\}. \end{aligned} \tag{2.5}$$

Moreover we have

Theorem 2.16. *([Sim83, Theorem 14.3]) Suppose E has locally finite perimeter in $\mathcal{D}$. Then $\partial^* E$ is countably $(d-1)$-rectifiable and $|D\mathbb{1}_E| = \mathcal{H}^{d-1}\lfloor \partial^* E$. In fact at each point $x \in \partial^* E$, the approximate tangent space T_x of $|D\mathbb{1}_E|$ exists, has multiplicity 1, and is given by*

$$T_x = \left\{ y \in \mathbb{R}^d : y \cdot \vec{\nu}_E(x) = 0 \right\}.$$

Varifolds

Let $G(d, d-1)$ be the Grassmannian space which parametrizes of all $(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$, which is a compact smooth manifold. For any $T \in G(d, d-1)$, T can be identified with its unit normal vector $\vec{\nu}$. More precisely, $G(d, d-1) \cong P$, where $P := S^{d-1}/\{\vec{\nu}, -\vec{\nu}\}$ is the set of unit normals of unoriented $(d-1)$-planes in $\mathbb{R}^d$.

Definition 2.17. *(varifold). A varifold (or, more precisely a $(d-1)$-varifold) V is a non-negative Radon measure on $G_{d-1}(\mathcal{D}) := \mathcal{D} \times G(d, d-1)$. The convergence of a sequence of varifolds is defined as the weak convergence in the sense of Radon measure.*

Definition 2.18. *(mass). Given a $(d-1)$-varifold V, there corresponds a Radon measure $\|V\|$ on $\mathcal{D}$ defined by*

$$\|V\|(A) := V(\pi^{-1}(A)),$$

where π is the projection $G_{d-1}(\mathcal{D}) \ni (x, T) \longmapsto x$ onto $\mathcal{D}$.

Definition 2.19. *(rectifiable varifold) Let M be a countably $(d-1)$–rectifiable set and θ be a non negative function with $\theta > 0, \mathcal{H}^{d-1} -$ a.e. in M. A rectifiable $(d-1)$–varifold $V = \underline{v}(M, \theta)$ in $\mathcal{D}$ is a non-negative Radon measure on $G_{d-1}(\mathcal{D})$ of the form $V = \theta \mathcal{H}^{d-1}\lfloor M \otimes \delta_{T_x M}$, i.e.*

$$\int_{G_{d-1}(\mathcal{D})} \varphi(x, T) dV(x, T) = \int_M \varphi\left(x, T_x M\right) \theta(x) d\mathcal{H}^{d-1}(x) \quad \forall \varphi \in C_c\left(G_{d-1}(\mathcal{D})\right).$$

First variation of a varifold

Definition 2.20. *The first variation of a $(d-1)$-varifold V in $\mathcal{D}$ is the vector-valued distribution δV defined for any vector field $\vec{Y} = (Y^1, \cdots, Y^d) \in C_c^1(\mathcal{D}, \mathbb{R}^d)$ as*

$$\langle \delta V, \vec{Y} \rangle := \int_{G_{d-1}(\mathcal{D})} \mathrm{div}_T \vec{Y}(x) dV(x, T).$$

Here for any $T \in G(d, d-1)$,

$$\mathrm{div}_T \vec{Y} = \sum_{i=1}^{d} \nabla_i^T Y^i,$$

where $\nabla_i^T := \varphi_i \cdot \nabla^T$, $\{\varphi_i\}_{i=1}^d$ is ONB in $\mathbb{R}^d$, with

$$\nabla^T f(x) = P_T(\nabla f(x)), \quad f \in C_c^1(\mathcal{D}),$$

and P_T is the orthogonal projection of $\mathbb{R}^d$ onto T.

For any $T \in G(d, d-1)$ with $p \in P$ the unit normal of T, we have that

$$\begin{aligned}
\mathrm{div}_T \vec{Y} &= \sum_{i=1}^{d} \nabla_i^T Y^i = \sum_{i=1}^{d} \varphi_i \cdot (P_T(\nabla Y^i)) \\
&= \sum_{i=1}^{d} \varphi_i \cdot \big(\nabla Y^i - (\nabla Y^i \cdot p)p\big) \\
&= \sum_{i=1}^{d} \left(\partial_{x_i} Y^i - \sum_{j=1}^{d} \partial_{x_j} Y^i p_j p_i \right) \\
&= \nabla \vec{Y} : (\mathrm{I} - p \otimes p) \,.
\end{aligned}$$

We simply denote $P \equiv G(d, d-1)$. Hence the first variation formula becomes

$$\langle \delta V, \vec{Y} \rangle = \int \int_{\mathcal{D} \times P} \nabla \vec{Y}(x) : (\mathrm{I} - p \otimes p) \, dV(x, p), \tag{2.6}$$

Moreover V is said to have *locally bounded first variation* in $\mathcal{D}$ if for each U compactly embedded in $\mathcal{D}$, i.e. U is open in $\mathcal{D}$ and $\bar{U}$ is compact in $\mathcal{D}$, there exists a constant $c > 0$ such that

$$\langle \delta V, \vec{Y} \rangle \leq c \sup_U |\vec{Y}|, \quad \forall \vec{Y} \in C_c^1(U, \mathbb{R}^d).$$

By the general Riesz representation [Sim83, Theorem 4.1], this is equivalent to that there exists a Radon measure $|\delta V|$ on $\mathcal{D}$ characterized by

$$|\delta V|(U) = \sup_{\vec{Y} \in C_c(U; \mathbb{R}^d), |\vec{Y}| \leq 1} |\langle \delta V, \vec{Y} \rangle| < \infty.$$

The following theorem is called the rectifiability theorem.

Theorem 2.21. *([Sim83, Theorem 42.4]) Suppose V is a $(d-1)$-varifold which has locally bounded first variation in $\mathcal{D}$ and satisfies*

$$\lim_{\rho \searrow 0} \frac{\|V\|(B_\rho(x))}{\omega_{d-1} \rho^{d-1}} > 0,$$

for $\|V\|$ − a.e. $x \in \mathcal{D}$. Then V is a $(d-1)$-rectifiable varifold.

Mean curvature vector

Definition 2.22. *Let V be a varifold which has locally bounded first variation in $\mathcal{D}$ such that $|\delta V|$ is absolutely continuous w.r.t. $\|V\|$. A $\|V\|$-measurable vector-valued function $\vec{H}_V$ is called a (generalized) mean curvature vector of V, if*

$$-\langle \delta V, \vec{Y} \rangle = \langle \|V\|, \vec{H}_V \cdot \vec{Y} \rangle := \int_{\mathcal{D}} \vec{H}_V(x) \cdot \vec{Y}(x) d\|V\|(x). \tag{2.7}$$

Remark 2.23. *Consider E which has locally finite perimeter in $\mathcal{D}$. Then by Theorem 2.16, $V = \underline{v}(\partial^* E, 1)$ is a $(d-1)$-rectifiable varifold and $\|V\| = \mathcal{H}^{d-1} \lfloor \partial^* E$. By Definition 2.20,*

$$\langle \delta V, \vec{Y} \rangle := \int_{\partial^* E} \mathrm{div}_{T_x} \vec{Y}(x) d\mathcal{H}^{d-1}(x), \tag{2.8}$$

where T_x is the tangent space of $\mathcal{H}^{d-1} \lfloor \partial^* E$ at $x \in \partial^* E$.

In the case that E is smooth, then $\partial^* E = \partial E$, $\vec{\nu}_E(x) = \vec{\nu}(x) \mathbb{1}_{\partial E}(x)$ where $\vec{\nu}(x)$ is the inward normal vector of ∂E. By Theorem 2.16, T_x is the orthogonal complement space of ν, which coincides with the definition of the classical tangent space. Note that in smooth case, the mean curvature vector $\vec{H}_{\partial E}$ of ∂E can be identified by

$$\int_{\partial E} \vec{Y} \cdot \vec{H}_{\partial E} d\mathcal{H}^{d-1} = -\int_{\partial E} \mathrm{div}_{T_x} \vec{Y}(x) d\mathcal{H}^{d-1}.$$

Combining with (2.8), we obtain that

$$\langle \delta V, \vec{Y} \rangle = -\int_{\partial E} \vec{Y} \cdot \vec{H}_{\partial E} d\mathcal{H}^{d-1}.$$

By Definition 2.22, $\vec{H}_V = \vec{H}_{\partial E}, \mathcal{H}^{d-1} - a.e.$.

Chapter 3

Conservative stochastic 2-dimensional Cahn-Hilliard equation

In this chapter, we consider the conservative stochastic Cahn-Hilliard equation

$$\begin{cases} dX_t = -\frac{1}{2}A\left(AX- :X^3:\right)dt + BdW_t, \\ X(0) = z \in V_0^{-1}, \end{cases} \tag{3.1}$$

on $\mathbb{T}^2$ in the probabilistically strong sense where $A = \Delta$, $B = \text{div}$. W_t is an $L_0^2(\mathbb{T}^2, \mathbb{R}^2)$-cylindrical Wiener process, which is defined in Section 3.2. $:X^3:$ denotes the Wick power, which is introduced in Section 3.2 and the space V_0^{-1} is similar to the Sobolev space of order -1, which is introduced in Section 3.1.

3.1 Notations and preliminaries

Let L denote the space $L^2(\mathbb{T}^2)$, where $\mathbb{T}^2 = (0,1)^2$ is the 2 dimensional torus and we use $\langle\cdot,\cdot\rangle$ to denote the inner product in L. A is the Laplacian operator on L, that is,

$$D(A) = H_2^2(\mathbb{T}^2), A = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{3.2}$$

A is a self-adjoint operator in L, with complete orthonormal system $(e_n)_n$ of eigenvectors in L, given by

$$e_0(x) := 1, e_{(k_1,0)}(x) = \sqrt{2}e^{i\pi k_1 x_1}, e_{(0,k_2)}(x) = \sqrt{2}e^{i\pi k_2 x_2},$$
$$e_k(x) := 2e^{i\pi(k_1x_1+k_2x_2)}, k_1k_2 \neq 0.$$

Then we have $Ae_k = -\lambda_k e_k$, where $\lambda_k = |k|^2\pi^2, k = (k_1,k_2) \in \mathbb{Z}^2, |k|^2 = k_1^2 + k_2^2$. We also introduce a notation for the average of $h \in \mathcal{S}'(\mathbb{T}^2)$:

$$m(h) := {}_{\mathcal{S}'}\langle h, e_0\rangle_{\mathcal{S}}.$$

For any $\alpha \in \mathbb{R}$, we define

$$V^\alpha := \{u \in \mathcal{S}' : \sum_k \lambda_k^\alpha |{}_{\mathcal{S}'}\langle u, e_k\rangle_{\mathcal{S}}|^2 < \infty\}.$$

For any $u, v \in V^\alpha$, define

$$\langle u, v\rangle_{V^\alpha} := m(u)m(v) + \sum_k \lambda_k^\alpha{}_{\mathcal{S}'}\langle u, e_k\rangle_{\mathcal{S}\mathcal{S}'}\langle v, e_k\rangle_{\mathcal{S}}.$$

It's easy to see that $(V^\alpha, \langle\cdot,\cdot\rangle_{V^\alpha})$ is a Hilbert space and $V^\alpha \simeq H_2^\alpha$. Then for any $s, \alpha \in \mathbb{R}$, we can define a bounded operator $(-A)^s : V^\alpha \to V^{\alpha-2s}$ by:

$$(-A)^s u = \sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} \lambda_k^s u_k e_k,$$

where $u = \sum_k u_k e_k \in V^\alpha$. In particular, we set $Q := (-A)^{-1}$ and extend it to a one-to-one bounded operator $\bar{Q}$ by

$$\bar{Q}h := Qh + m(h)e_0. \tag{3.3}$$

Note that

$$Qe_k = \begin{cases} \frac{1}{\lambda_k}e_k & k \neq (0,0), \\ 0 & k = (0,0), \end{cases} \tag{3.4}$$

and

$$\bar{Q}e_k = \begin{cases} \frac{1}{\lambda_k}e_k & k \neq (0,0), \\ e_0 & k = (0,0). \end{cases} \tag{3.5}$$

Then we have

$$\langle u, v\rangle_{V^\alpha} := \langle \bar{Q}^{-\alpha/2}u, \bar{Q}^{-\alpha/2}v\rangle,$$

and $\bar{Q}^s : V^\alpha \to V^{\alpha+2s}$ is an isomorphism for any $\alpha, s \in \mathbb{R}$, since

$$\langle \bar{Q}^s u, \bar{Q}^s v\rangle_{V^{\alpha+2s}} = \langle u, v\rangle_{V^\alpha}.$$

We also set

$$V_0^\alpha := \{h \in V^\alpha : \langle h, e_0\rangle_{V^\alpha} = 0\},$$

and denote $L_0^2 := V_0^0$. Let Π denote the symmetric projector of V^α on V_0^α, that is,

$$\Pi : V^\alpha \to V_0^\alpha, \Pi h := h - m(h). \tag{3.6}$$

Moreover, we define

$$V^\alpha(\mathbb{T}^2, \mathbb{R}^2) := \{f = (f_1, f_2) : f_i \in V^\alpha, i = 1, 2\},$$

and similarly

$$V_0^\alpha(\mathbb{T}^2, \mathbb{R}^2) := \{f = (f_1, f_2) : f_i \in V_0^\alpha, i = 1, 2\}.$$

In this chapter, we consider the initial value and the reference measure on V_0^α for simplicity. For general case, we refer to [DZ07].

3.2 The Linear Equation and Wick Powers

We consider the O-U process

$$\begin{cases} dZ_t = -\frac{1}{2}A^2 Z dt + BdW_t, \\ Z(0) = 0, \end{cases} \tag{3.7}$$

where W is a U-cylindrical Wiener process and $U := L_0^2(\mathbb{T}^2, \mathbb{R}^2)$. For $f \in L_0^2(\mathbb{T}^2, \mathbb{R}^2)$ we denote its component functions by $f_1, f_2 \in L_0^2(\mathbb{T}^2)$ i.e. $f(x) = (f_1(x), f_2(x)), \forall x \in \mathbb{T}^2$. There exist two independent $L^2(\mathbb{T}^2)$-cylindrical Wiener processes W^1 and W^2 such that $W = (W^1, W^2)$. Set

$$D(B) = H^1(\mathbb{T}^2, \mathbb{R}^2), B = \mathrm{div}, D(B^*) = H_2^1(\mathbb{T}^2), B^* = -\nabla. \tag{3.8}$$

We know that

$$Z_t(x) = \int_0^t e^{-\frac{t-s}{2}A^2} BdW_s = \int_0^t \langle K(t-s, x-\cdot), dW_s \rangle_U,$$

where $K(t,x) = -\nabla_x M(t,x) = (K^1, K^2)$, and $M(t,x)$ is the kernel of $e^{-\frac{t}{2}A^2}$, that is, $M(t,x) = \sum_k e^{-\frac{t}{2}\lambda_k^2} e_k(x)$.

For any function f on $\mathbb{T}^2$, we can view it as a periodic function on $\mathbb{R}^2$ by defining $\bar{f}(x) := f(x+m)$, when $x+m \in \mathbb{T}^2$, $x \in \mathbb{R}^2$, $m \in \mathbb{Z}^2$. Moreover, define

$$\bar{K}^j(t,x) := -\mathcal{F}^{-1}(\pi i \xi_j e^{-\frac{t}{2}|\pi\xi|^4})(x), j = 1, 2.$$

and $\bar{K} := (\bar{K}^1, \bar{K}^2)$. By the Poisson summation formula (see [SW72, Section VII.2]) we know that

$$K(t,x) = \sum_m \bar{K}(t, x+m), \forall t \tag{3.9}$$

and for any $f \in L^2(\mathbb{T}^2)$, $j = 1, 2$, $x \in \mathbb{T}^2$

$$\begin{aligned} \partial_j e^{-\frac{t}{2}A^2} f(x) &= \int_{\mathbb{T}^2} K^j(t, x-y) f(y) dy \\ &= \int_{\mathbb{R}^2} K^j(t, x-y) f(y) \mathbb{1}_{\mathbb{T}^2}(y) dy \\ &= \sum_m \int_{\mathbb{R}^2} \bar{K}^j(t, x-y+m) f(y) \mathbb{1}_{\mathbb{T}^2}(y) dy \qquad , \\ &= \int_{\mathbb{R}^2} \bar{K}^j(t, x-y) \sum_m \mathbb{1}_{\mathbb{T}^2}(y+m) f(y+m) dy \\ &= (\bar{K}^j(t, \cdot) * \bar{f})(x) \end{aligned} \tag{3.10}$$

where we used (3.9) in the third inequality and $\mathbb{1}_{\mathbb{T}^2}$ is the indicator function of $\mathbb{T}^2$. Since

$$\bar{K}^j(t,x) = -\mathcal{F}^{-1}(\pi i \xi_j e^{-\frac{t}{2}|\pi\xi|^4})(x) = t^{-\frac{3}{4}} \bar{K}^j(1, t^{-\frac{1}{4}}x)$$

and

$$|\bar{K}^j(1, t^{-\frac{1}{4}}x)| \lesssim |\mathcal{F}^{-1}(\pi i \xi_j e^{-\frac{1}{2}|\pi\xi|^4})(t^{-\frac{1}{4}}x)| \lesssim |1 + t^{-\frac{1}{4}}x|^{-3},$$

we have the following estimate:

$$|\bar{K}(t,x)| \lesssim t^{-\frac{\varepsilon}{4}} |x|^{-3+\varepsilon}, \forall \varepsilon \in [0, 3]. \tag{3.11}$$

Lemma 3.1. *$Z \in C([0,T];\mathcal{C}^{-\alpha})$ $\mathbb{P}$-almost-surely, for all $\alpha > 0$.*

Proof By the factorization method in [DP04] we have that for $\kappa \in (0,1)$

$$Z(t) = \frac{\sin(\pi\kappa)}{\pi}\int_0^t (t-s)^{\kappa-1}\langle M(t-s,x-\cdot),U(s)\rangle ds,$$

and

$$U(s,\cdot) = \int_0^s (s-r)^{-\kappa}e^{-\frac{s-r}{2}A^2}BdW_r.$$

A similar argument as in the proof of Lemma 2.7 in [DP04] implies that it suffices to prove that for $p > 1/(2\kappa)$,

$$\mathbb{E}\|U\|_{L^{2p}(0,T;\mathcal{C}^{-\alpha})} < \infty. \tag{3.12}$$

In fact, by (2.2) we have that

$$\begin{aligned}\mathbb{E}\|U(s)\|_{-\alpha}^{2p} &\lesssim \sum_{\psi\in\Psi_\star}\sum_{n\geq 0}\sum_{x\in\Lambda_n}\mathbb{E}2^{-2\alpha pn+2np}|\langle U(s),\psi_x^n\rangle|^{2p}w(x)^{2p}\\ &\lesssim \sum_{\psi\in\Psi_\star}\sum_{n\geq 0}\sum_{x\in\Lambda_n}2^{-2\alpha pn+2np}(\mathbb{E}|\langle U(s),\psi_x^n\rangle|^2)^p w(x)^{2p}.\end{aligned}$$

Here $\sigma > 0$ in $w(x)$ and we used that $\langle U(s),\psi_x^n\rangle$ belongs to the first order Wiener-chaos and Gaussian hypercontractivity (cf. [Nua13, Section 1.4.3] and [Nel73]) in the second inequality. Moreover, we obtain that

$$\begin{aligned}\mathbb{E}|\langle U(s),\psi_x^n\rangle|^2 =&\mathbb{E}|\langle U^1(s),\psi_x^n\rangle|^2 + \mathbb{E}|\langle U^2(s),\psi_x^n\rangle|^2\\ \leq&\sum_{j=1}^2\int\int|\psi_x^n(y)\psi_x^n(\bar{y})|\int_0^s (s-r)^{-2\kappa}\bar{K}^j * \bar{K}^j(s-r,y-\bar{y})drdyd\bar{y}\\ \lesssim&\int\int|\psi_x^n(y)\psi_x^n(\bar{y})|\int_0^s (s-r)^{-\frac{\varepsilon}{2}-2\kappa}|y-\bar{y}|^{-4+2\varepsilon}drdyd\bar{y}\\ \lesssim&2^{2n-2\varepsilon n}s^{1-2\kappa-\frac{\varepsilon}{2}},\end{aligned}$$

where

$$U^j(y) = \int_0^s (t-s)^{\kappa-1}\langle K^j(s-r,y-\cdot),dW_r^j\rangle, j=1,2$$

and we used (3.10) in the second inequality and we also used [Hai14, Lemma 10.17] and (3.11) to deduce that $|\bar{K}^j * \bar{K}^j(s-r,y-\bar{y})| \lesssim |s-r|^{-\frac{\varepsilon}{2}}|y-\bar{y}|^{-4+2\varepsilon}$ in the second inequality.

In fact, we can decompose $\bar{K}$ into $\bar{K} := \bar{K}_\delta + \bar{K}_\delta^c$, where $\bar{K}_\delta$ is a compactly supported function and satisfies (3.11), $\bar{K}_\delta^c$ is a Schwartz function. Then $\bar{K}*\bar{K} = \bar{K}_\delta*\bar{K}_\delta + H$, where H is a Schwartz function. By [Hai14, Lemma 10.17] we have $\bar{K}_\delta * \bar{K}_\delta(t,x) \lesssim t^{-\frac{\varepsilon}{2}}|x|^{-4+2\varepsilon}$ and $\bar{K}*\bar{K}$ satisfies the same inequality.

Thus, we have

$$\mathbb{E}\|U(s)\|_{-\alpha}^{2p} \lesssim \sum_{n\geq 0}2^{(4-2\varepsilon-2\alpha)pn}s^{(1-2\kappa-\frac{\varepsilon}{2})p}.$$

Let κ be so small that $2-\alpha < \varepsilon < 2-4\kappa+\frac{2}{p}$, which implies that

$$4-2\varepsilon-2\alpha < 0, (1-2\kappa-\frac{\varepsilon}{2})p > -1.$$

Then (3.12) follows.

□

Note that $BB^* = -A$. Then by Fourier expansion it is easy to see that $Z_t \sim \mathcal{N}(0, Q_t)$, i.e. for any $h \in \mathcal{S}(\mathbb{T}^2)$

$$\mathbb{E}e^{i_{\mathcal{S}}\langle h, Z_t\rangle_{\mathcal{S}'}} = \exp(-\frac{1}{2}\langle Q_t h, h\rangle),$$

where $Q_t = (-A)^{-1}(I - e^{-\frac{t}{2}A^2})$.

According to the definition of V^α and Lemma 2.1, we have $\mathcal{C}^{-\alpha} \subset V^{-\alpha-\varepsilon}$ for any $\alpha, \varepsilon > 0$. Then by Lemma 3.1, μ_t is supported on $V_0^{-\alpha}$ for any $\alpha > 0$ and letting $t \to \infty$, by [Bog98, 3.8.13, Example], the law of Z_t converges to the Gaussian measure $\mu = \mathcal{N}(0, Q)$, which is also supported on $V_0^{-\alpha}$.

In the following we are going to define the Wick powers both in the state space and the path space.

Firstly, we define the Wick powers on $L^2(\mathcal{S}'(\mathbb{T}^2), \mu)$.

Wick powers on $L^2(\mathcal{S}'(\mathbb{T}^2), \mu)$

μ is of course also a measure supported on $\mathcal{S}'(\mathbb{T}^2)$. We have the well-known (Wiener-Itô) chaos decomposition

$$L^2(\mathcal{S}'(\mathbb{T}^2), \mu) = \bigoplus_{n\geq 0} \mathcal{H}_n,$$

where $\mathcal{H}_n$ is the Wiener chaos of order n (cf. [Nua13, Section 1.1.1]). Now we define the Wick powers by using approximations: for $\phi \in \mathcal{S}'(\mathbb{T}^2)$ define

$$\phi_\varepsilon := \rho_\varepsilon * \phi,$$

with ρ_ε an approximate delta function on $\mathbb{R}^2$ given by

$$\rho_\varepsilon(x) = \varepsilon^{-2}\rho(\frac{x}{\varepsilon}) \in \mathcal{D}, \int \rho = 1.$$

Here the convolution means that we view ϕ as a periodic distribution in $\mathcal{S}'(\mathbb{R}^2)$ and convolve on $\mathbb{R}^2$. For every $n \in \mathbb{N}$ we set

$$:\phi_\varepsilon^n:_Q := c_\varepsilon^{n/2} P_n(c_\varepsilon^{-1/2}\phi_\varepsilon),$$

where $P_n, n = 0, 1, ...,$ are the Hermite polynomials defined by the formula

$$P_n(x) = \sum_{j=0}^{[n/2]} (-1)^j \frac{n!}{(n-2j)!j!2^j} x^{n-2j},$$

and $c_\varepsilon = \int \phi_\varepsilon^2 \mu(d\phi) = \int\int G(z-y)\rho_\varepsilon(y)dy\rho_\varepsilon(z)dz = \|1_{[0,t]}K_\varepsilon\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2;\mathbb{R}^2)}$. Then

$$:\phi_\varepsilon^n:_Q \in \mathcal{H}_n.$$

Here and in the following G is the Green function associated with $-A$ on $\mathbb{T}^2$. In fact by [SW72, Section 6.1, Chapter VII],

$$G(x) = \sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{\lambda_k} e_k(x) \simeq -\log|x|, |x| \to 0,$$

and G is continuously differentiable except outside $\{0\}$.

For Hermite polynomial P_n we have that for $s, t \in \mathbb{R}$

$$P_n(s+t) = \sum_{m=0}^{n} C_n^m P_m(s) t^{n-m}, \tag{3.13}$$

where $C_n^m = \frac{n!}{m!(n-m)!}$.

A direct calculation yields the following:

Lemma 3.2. *Let $\alpha > 0$, $n \in \mathbb{N}$ and $p > 1$. $: \phi_\varepsilon^n :_Q$ converges to some element in $L^p(\mathcal{S}'(\mathbb{T}^2), \mu; \mathcal{C}^{-\alpha})$ as $\varepsilon \to 0$. This limit is called the n-th Wick power of ϕ with respect to the covariance Q and denoted by $: \phi^n :_Q$.*

Proof The proof in similar to that of [RZZ17b, Lemma 3.1] since the Green function G has the same regularity.

In fact, for any $p > 1, \varepsilon_1, \varepsilon_2 > 0, m \in \mathbb{N}$, by (2.2), we have that

$$\begin{aligned}
&\int \left\| : \phi_{\varepsilon_1}^m :_Q - : \phi_{\varepsilon_2}^m :_Q \right\|_{-\alpha}^{2p} \mu(d\phi) \\
\lesssim &\sum_{\psi \in \Psi_\star} \sum_{n \geq 0} \sum_{x \in \Lambda_n} 2^{-2\alpha pn + 2np} \int \left| \left\langle : \phi_{\varepsilon_1}^m :_Q - : \phi_{\varepsilon_2}^m :_Q, \psi_x^n \right\rangle \right|^{2p} \mu(d\phi) w(x)^{2p} \\
\lesssim &\sum_{\psi \in \Psi_\star} \sum_{n \geq 0} \sum_{x \in \Lambda_n} 2^{-2\alpha pn + 2np} w(x)^{2p} \left(\int \left| \left\langle : \phi_{\varepsilon_1}^m :_Q - : \phi_{\varepsilon_2}^m :_Q, \psi_x^n \right\rangle \right|^2 \mu(d\phi) \right)^p,
\end{aligned}$$

where $\sigma > 0$ in $\omega(x)$ and in the last inequality we used the hypercontractivity of the Gaussian measure. Moreover, we obtain that

$$\begin{aligned}
&\int \left| \left\langle : \phi_{\varepsilon_1}^m :_Q - : \phi_{\varepsilon_2}^m :_Q, \psi_x^n \right\rangle \right|^2 \mu(d\phi) \\
\lesssim &\iint |\psi_x^n(y) \psi_x^n(\overline{y})| \left| \left(\int \phi_{\varepsilon_1}(y) \phi_{\varepsilon_1}(\overline{y}) \mu(d\phi) \right)^m - 2 \left(\int \phi_{\varepsilon_1}(y) \phi_{\varepsilon_2}(\overline{y}) \mu(d\phi) \right)^m \right. \\
&\left. + \left(\int \phi_{\varepsilon_2}(y) \phi_{\varepsilon_2}(\overline{y}) \mu(d\phi) \right)^m \right| dy d\overline{y} \\
\lesssim &\iint |\psi_x^n(y) \psi_x^n(\overline{y})| \left| \left(\iint \rho_{\varepsilon_1}(y - x_1) \rho_{\varepsilon_1}(\overline{y} - x_2) G(x_1 - x_2) dx_1 dx_2 \right)^m \right. \\
&- 2 \left(\iint \rho_{\varepsilon_1}(y - x_1) \rho_{\varepsilon_2}(\overline{y} - x_2) G(x_1 - x_2) dx_1 dx_2 \right)^m \\
&\left. + \left(\iint \rho_{\varepsilon_2}(y - x_1) \rho_{\varepsilon_2}(\overline{y} - x_2) (x_1 - x_2) dx_1 dx_2 \right)^m \right| dy d\overline{y} \\
\lesssim &(\varepsilon_1^\kappa + \varepsilon_2^\kappa) \iint |\psi_x^n(y) \psi_x^n(\overline{y})| |y - \overline{y}|^{-\delta} dy d\overline{y} \lesssim (\varepsilon_1^\kappa + \varepsilon_2^\kappa) 2^{-2n + n\delta},
\end{aligned}$$

where $\delta > \kappa > 0, -2\alpha + \delta < 0$. Here in the first inequality we used [Sim74, Theorem I.3], in the third inequality we have used [Hai14, Lemma 10.17] and the fact that $|G(x)| \lesssim |\log x|$. Thus the results follow from a direct calculation.

□

Wick powers on a fixed probability space

Now we fix a probability space $(\Omega, \mathcal{F}, P)$ and consider a U-cylindrical Wiener process W. In the following we assume that $\mathcal{F}$ is the σ-field generated by $\{\langle W_t, h\rangle, h \in U, t \in \mathbb{R}^+\}$. We also have the well-known (Wiener-Itô) chaos decomposition

$$L^2(\Omega, \mathcal{F}, P) = \bigoplus_{n\geq 0} \mathcal{H}'_n,$$

where $\mathcal{H}'_n$ is the Wiener chaos of order n (cf. [Nua13, Section 1.1.1]). We can define Wick powers of $Z(t)$ with respect to different covariances by approximations: Let

$$\begin{aligned} Z_\varepsilon(t,x) = \rho_\varepsilon * Z_t &= \int_0^t \langle B^* e^{-\frac{t-s}{2}A^2}\rho_{\varepsilon,x}, dW_s\rangle_U \\ &= \int_0^t \langle K_\varepsilon(t-s, x-\cdot), dW_s\rangle_U, \end{aligned},$$

where $\rho_{\varepsilon,x} = \rho_\varepsilon(x-\cdot)$, $K_\varepsilon(t,x) = (\rho_\varepsilon * K_t^1, \rho_\varepsilon * K_t^2)$ and

$$K_t^j = -\sum_k (i\pi k_j) e^{-\frac{t}{2}\lambda_k^2} e_k, j = 1, 2.$$

For any $n\mathbb{N}$, we set

$$: Z_\varepsilon^n(t) :_{Q_t} := (c_{\varepsilon,t})^{\frac{n}{2}} P_n\left((c_{\varepsilon,t})^{-\frac{1}{2}} Z_\varepsilon(t)\right) \in \mathcal{H}'_n,$$

where $P_n, n = 0, 1, \cdots$, are the Hermite polynomials and $c_{\varepsilon,t} = \left\|\mathbb{1}_{[0,t]} K_\varepsilon\right\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2;\mathbb{R}^2)}$

Lemma 3.3. *([RZZ17b, Lemma 3.3]) For $\alpha > 0$, $p > 1$, $n \in \mathbb{N}$, $: Z_\varepsilon^n :_{Q_t}$ converges in $L^p(\Omega, C([0,T]; \mathcal{C}^{-\alpha}))$. The limit is called Wick power of $Z(t)$ of order n with respect to the covariance Q_t and is denoted by $: Z^n(t) :_{Q_t}$.*

Proof By (3.1) we already proved $Z_\varepsilon \in C([0,T], \mathcal{C}^{-\alpha})$, $\mathbb{P} - a.s.$. Now we prove that $: Z^m :_{Q_t}$ is a Cauchy sequence. For every $p > 1$, by (2.2) we have for $t_1, t_2 \geq 0$ that

$$\begin{aligned} &\mathbb{E}\|(: Z_{\varepsilon_1}^m :_{Q_{t_1}} - : Z_{\varepsilon_2}^m :_{Q_{t_1}})(t_1,\cdot) - (: Z_{\varepsilon_1}^m :_{Q_{t_2}} - : Z_{\varepsilon_2}^m :_{Q_{t_2}})(t_2,\cdot)\|_{-\alpha}^{2p} \\ &\leq \sum_{\psi\in\Psi_\star}\sum_{n\geq 0}\sum_{x\in\Lambda_n} \mathbb{E} 2^{-2\alpha pn+2np} |\langle (: Z_{\varepsilon_1}^m :_{Q_{t_1}} - : Z_{\varepsilon_2}^m :_{Q_{t_1}})(t_1,\cdot) \\ &\qquad\qquad - (: Z_{\varepsilon_1}^m :_{Q_{t_2}} - : Z_{\varepsilon_2}^m :_{Q_{t_2}})(t_2,\cdot), \psi_x^n\rangle|^{2p} w(x)^{2p} \\ &\lesssim \sum_{\psi\in\Psi_\star}\sum_{n\geq 0}\sum_{x\in\Lambda_n} 2^{-2\alpha pn+2np} (\mathbb{E}|\langle (: Z_{\varepsilon_1}^m :_{Q_{t_1}} - : Z_{\varepsilon_2}^m :_{Q_{t_1}})(t_1,\cdot) \\ &\qquad\qquad - (: Z_{\varepsilon_1}^m :_{Q_{t_2}} - : Z_{\varepsilon_2}^m :_{Q_{t_2}})(t_2,\cdot), \psi_x^n\rangle|^2)^p w(x)^{2p}, \end{aligned}$$

where we used Gaussian hypercontractivity in the second inequality. For convenience we use ξ^l, $l = 1, 2$ to denote space-time white noise given by $\int \phi(s,y)\xi^l(ds,dy) = \int_{\mathbb{R}^+}\langle \phi, dW_s^l\rangle$ for $\phi \in L^2(\mathbb{R}^+ \times \mathbb{T}^2)$. Then by [Nua13, Proposition 1.1.4] we obtain that for $k = 1, 2$ and $j = 1, 2$

$$: Z_{\varepsilon_k}^m(t_j) :_{Q_{t_j}} = \sum_{l=1}^2 \int \Pi_{i=1}^m \bar{K}_{\varepsilon_k}^l(t_j - s_i, y - y_i) 1_{s_i\in[0,t_j]} \xi^l(d\eta_1)\ldots\xi^l(d\eta_m),$$

where $\eta_a = (s_a, y_a)$, and $\int f(\eta_{1\dots n})\,\xi(d\eta_1)\dots\xi(d\eta_m)$ denote a generic element of the n-th chaos of ξ for $\eta_{1\cdots n} = \eta_1\cdots\eta_n$. Moreover, for $t_1 \le t_2$ to estimate

$$\mathbb{E}\left|\left\langle\left(: Z^m_{\varepsilon_1} :_{Q_{t_1}} - : Z^m_{\varepsilon_2} :_{Q_{t_1}}\right)(t_1,\cdot) - \left(: Z^m_{\varepsilon_1} :_{Q_{t_2}} - : Z^m_{\varepsilon_2} :_{Q_{t_2}}\right)(t_2,\cdot), \psi^n_x\right\rangle\right|^2,$$

since ξ^1 is independent to ξ^2, it suffices to calculate

$$\begin{aligned}\int &|\langle \Pi^m_{i=1}\bar{K}^l_{\varepsilon_1}(t_1-s_i,\cdot-y_i)\mathbb{1}_{s_i\in[0,t_1]} - \Pi^m_{i=1}\bar{K}^l_{\varepsilon_2}(t_1-s_i,\cdot-y_i)\mathbb{1}_{s_i\in[0,t_1]}\\ &- [\Pi^m_{i=1}\bar{K}^l_{\varepsilon_1}(t_2-s_i,\cdot-y_i)\mathbb{1}_{s_i\in[0,t_2]} - \Pi^m_{i=1}\bar{K}^l_{\varepsilon_2}(t_2-s_i,\cdot-y_i)\mathbb{1}_{s_i\in[0,t_2]}], \psi^n_x\rangle|^2 d\eta_{1\dots m},\end{aligned}$$

which is bounded by

$$\begin{aligned}2\int &|\langle(\Pi^m_{i=1}\bar{K}^l_{\varepsilon_1}(t_1-s_i,\cdot-y_i) - \Pi^m_{i=1}\bar{K}_{\varepsilon_1}(t_2-s_i,\cdot-y_i))\mathbb{1}_{s_i[0,t_1]}\\ &- (\Pi^m_{i=1}\bar{K}^l_{\varepsilon_2}(t_1-s_i,\cdot-y_i) - \Pi^m_{i=1}\bar{K}^l_{\varepsilon_2}(t_2-s_i,\cdot-y_i))\mathbb{1}_{s_i\in[0,t_1]}, \psi^n_x\rangle|^2 d\eta_{1\dots m}\\ +2\int &\left|\left\langle\left[\Pi^m_{i=1}\bar{K}_{\varepsilon_1}(t_2-s_i,\cdot-y_i)\mathbb{1}_{s_i\in[t_1,t_2]} - \Pi^m_{i=1}\bar{K}_{\varepsilon_2}(t_2-s_i,\cdot-y_i)\mathbb{1}_{s_i\in[t_1,t_2]}\right], \psi^n_x\right\rangle\right|^2 d\eta_{1\dots m}.\end{aligned} \tag{3.14}$$

To calculate the bound (3.14) above, we need several estimations of $\bar{K}^l(t,x)$ and $\bar{K}^l_\varepsilon(t,x)$. For convenience, we denote

$$\|(t,x)\|_s := t^{\frac{1}{4}} + |x|.$$

Then by [Hai14, Lemma 10.17, Lemma 10.18], we have that for any $\delta\in(0,1)$

$$|K^l_\varepsilon(t,x) - K^l(t,x)| \lesssim \varepsilon^\delta \|(t,x)\|_s^{-3-\delta} \lesssim \varepsilon^\delta t^{\frac{-1+\delta}{2}}|x|^{-1-3\delta}, \tag{3.15}$$

and

$$|K^l(t,x) - K^l(s,y)| \lesssim \|(t-s,x-y)\|^\delta_s\left(\|(t,x)\|_s^{-3-\delta} + \|(s,y)\|_s^{-3-\delta}\right). \tag{3.16}$$

In particular, for any $\delta\in(0,1)$, we obtain that

$$\begin{aligned}&\left|\bar{K}^l_{\varepsilon_1}(t_1-s_i,y-y_i) - \bar{K}^l_{\varepsilon_1}(t_2-s_i,y-y_i)\right|\\ \lesssim &\left|\bar{K}^l_{\varepsilon_1}(t_1-s_i,y-y_i) - \bar{K}^l(t_1-s_i,y-y_i)\right| + \left|\bar{K}^l(t_1-s_i,y-y_i) - \bar{K}^l(t_2-s_i,y-y_i)\right|\\ &+ \left|\bar{K}^l(t_2-s_i,y-y_i) - \bar{K}^l_{\varepsilon_1}(t_2-s_i,y-y_i)\right|\\ \lesssim &\left(\varepsilon_1^\delta + |t_1-t_2|^{\frac{\delta}{4}}\right)\left(\|(t_1-s_i,y-y_i)\|_s^{-3-\delta} + \|(t_2-s_i,y-y_i)\|^{-3-\delta}\right)\\ \lesssim &|t_1-t_2|^{\frac{\delta}{4}}\left(|t_2-s_i|^{-\frac{1}{2}+\frac{\delta}{2}} + |t_1-s_i|^{-\frac{1}{2}+\frac{\delta}{2}}\right)|y-y_i|^{-1-3\delta},\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}&\left|\bar{K}^l_{\varepsilon_1}(t_1-s_i,y-y_i) - \bar{K}^l_{\varepsilon_2}(t_1-s_i,y-y_i)\right|\\ \lesssim &\left|\bar{K}^l_{\varepsilon_1}(t_1-s_i,y-y_i) - \bar{K}^l(t_1-s_i,y-y_i)\right| + \left|\bar{K}^l_{\varepsilon_2}(t_1-s_i,y-y_i) - \bar{K}^l(t_1-s_i,y-y_i)\right|\\ \lesssim &\left(\varepsilon_1^\delta + \varepsilon_2^\delta\right)|t_1-s_i|^{-\frac{1}{2}+\frac{\delta}{2}}|y-y_i|^{-1-3\delta}.\end{aligned} \tag{3.18}$$

Then the estimate of (3.14) can be obtained by using the interpolation of the form:

$$\Pi^m_{i=1}a_i - \Pi^m_{i=1}b_i = \sum_{k=1}^m(\Pi^{k-1}_{i=1}b_i)(a_k-b_k)(\Pi^m_{i=k+1}a_i),$$

and the estimates (3.17), (3.18). For the third line in (3.14), since K^l_ε are integrable, by using the interpolation and (3.18), we obtain that

$$\int \left|\left\langle \left[\Pi_{i=1}^m \bar{K}^l_{\varepsilon_1}(t_2-s_i,\cdot-y_i)\mathbb{1}_{s_i\in[t_1,t_2]} - \Pi_{i=1}^m \bar{K}^l_{\varepsilon_2}(t_2-s_i,\cdot-y_i)\mathbb{1}_{s_i\in[t_1,t_2]}\right], \psi^n_x\right\rangle\right|^2 d\eta_{1\ldots m}$$
$$\lesssim(\varepsilon_1^\delta+\varepsilon_2^\delta)\sum_{i=1}^m\int \psi^n_x(z)\psi^n_x(\bar{z})|t_2-s_i|^{-1+\delta}\mathbb{1}_{s_i\in[t_1,t_2]}|z-y^i|^{-1-3\delta}|\bar{z}-y^i|^{-1-3\delta}dzd\bar{z}d\eta_{1\ldots m}$$
$$\lesssim(\varepsilon_1^\delta+\varepsilon_2^\delta)|t_1-t_2|^\delta\int \psi^n_x(z)\psi^n_x(\bar{z})|z-\bar{z}|^{-6\delta}dzd\bar{z}$$
$$\lesssim(\varepsilon_1^\delta+\varepsilon_2^\delta)|t_1-t_2|^\delta 2^{-2n+6n\delta}.$$

For the first two lines in (3.14), we consider $m=1$ for simplicity. The general $m\in\mathbb{N}$ can be obtained in a similar way by using the interpolation. For $m=1$, the first two lines can be rewrote as

$$\int((a_1(z)-b_1(z))-(a_2(z)-b_2(z)))((a_1(\bar{z})-a_2(\bar{z}))-(b_1(\bar{z})-b_2(\bar{z})))\psi^n_x(z)\psi^n_x(\bar{z})dzd\bar{z}dsdy,$$

where for $i=1,2$

$$a_i(z)=\bar{K}^l_{\varepsilon_i}(t_1-s,z-y)\mathbb{1}_{s\in[0,t_1]},\quad b_i(z)=\bar{K}^l_{\varepsilon_i}(t_2-s,z-y)\mathbb{1}_{s\in[0,t_1]}.$$

We only calculate

$$\int(a_1(z)-b_1(z))(b_1(z)-b_2(\bar{z}))\psi^n_x(z)\psi^n_x(\bar{z})dzd\bar{z}dsdy,$$

the rest term can be estimated similarly. In fact, by (3.17) and (3.18)

$$|(a_1(z)-b_1(z))(b_1(\bar{z})-b_2(\bar{z}))|$$
$$\lesssim\left(\varepsilon_1^\delta+\varepsilon_2^\delta\right)|t_1-t_2|^{\frac{\delta}{4}}\left(|t_2-s|^{-\frac{1}{2}+\frac{\delta}{2}}+|t_1-s|^{-\frac{1}{2}+\frac{\delta}{2}}\right)^2|z-y|^{-1-3\delta}|\bar{z}-y|^{-1-3\delta}\mathbb{1}_{s\in[0,t_1]}.$$

Thus we obtain

$$\int|(a_1(z)-b_1(z))(b_1(\bar{z})-b_2(\bar{z}))||\psi^n_x(z)\psi^n_x(\bar{z})|dzd\bar{z}dyds$$
$$\lesssim\left(\varepsilon_1^\delta+\varepsilon_2^\delta\right)|t_1-t_2|^{\frac{\delta}{4}}\int|\psi^n_x(z)\psi^n_x(\bar{z})||z-\bar{z}|^{-6\delta}dzd\bar{z}$$
$$\lesssim(\varepsilon_1^\delta+\varepsilon_2^\delta)|t_1-t_2|^{\frac{\delta}{4}}2^{-2n+6n\delta}.$$

Hence the first two lines in (3.14) are bounded by $(\varepsilon_1^\delta+\varepsilon_2^\delta)|t_1-t_2|^{\frac{\delta}{4}}2^{-2n+6n\delta}$. Then we deduce that

$$\mathbb{E}\left|\left\langle\left(:Z^m_{\varepsilon_1}:_{Q_{t_1}}-:Z^m_{\varepsilon_2}:_{Q_{t_1}}\right)(t_1,\cdot)-\left(:Z^m_{\varepsilon_1}:_{Q_{t_2}}-:Z^m_{\varepsilon_2}:_{Q_{t_2}}\right)(t_2,\cdot),\psi^n_x\right\rangle\right|^2$$
$$\lesssim(\varepsilon_1^\delta+\varepsilon_2^\delta)|t_1-t_2|^{\frac{\delta}{4}}2^{-2n+6n\delta}.$$

Then the above estimates yield that

$$\mathbb{E}\left\|\left(:Z^m_{\varepsilon_1}:_{C_1}-:Z^m_{\varepsilon_2}:_{C_{t_1}}\right)(t_1,\cdot)-\left(:Z^m_{\varepsilon_1}:_{C_{t_2}}-:Z^m_{\varepsilon_2}:_{C_{t_2}}\right)(t_2,\cdot)\right\|^{2p}_{-\alpha}$$

$$\lesssim \sum_{\psi\in\Psi_\star}\sum_{n\geq 0} 2^{-2\alpha np+2np+2n}\left(\varepsilon_1^{2\delta}+\varepsilon_2^{2\delta}\right)^p |t_2-t_1|^{\frac{\delta p}{4}}\, 2^{-2np+6np\delta}.$$

Thus the results follow from Kolmogorov's continuity test (in time) if we choose $\delta > 0$ small enough and p sufficiently large.

□

By this lemma we can also define the Wick powers with respect to another covariance $: Z_\varepsilon^n(t) :_Q := c_\varepsilon^{\frac{n}{2}} P_n\left(c_\varepsilon^{-\frac{1}{2}} Z_\varepsilon(t)\right)$.

Lemma 3.4. *([RZZ17b, Lemma 3.4]) For $\alpha > 0$, $p > 1$, $n \in \mathbb{N}$, $: Z_\varepsilon^n(t) :_Q$ converges in $L^p(\Omega, C((0,T];\mathcal{C}^{-\alpha}))$. Here $C((0,T];\mathcal{C}^{-\alpha})$ is equipped with the norm $\sup_{t\in[0,T]} t^\rho \|\cdot\|_{-\alpha}$ for $\rho > 0$. The limit is called Wick power of $Z(t)$ of order n with respect to the covariance Q and is denoted by $: Z^n(t) :_Q$.*

Proof By the definition of Hermite polynomials, we have that

$$: Z_\varepsilon^n(t) :_Q = \sum_{l=0}^{[n/2]} (c_{\varepsilon,t} - c_\varepsilon)^l \frac{n!}{(n-2l)!l!2^l} : Z_\varepsilon^{n-2l}(t) :_{Q_t} .$$

Then the theorem follows directly from Lemma 3.3 and the fact that

$$|c_{\varepsilon,t} - c_\varepsilon| \lesssim t^{-\rho},$$

for any $\rho > 0$.

□

Remark 3.5. *Here we do not combine the initial value with the Wick powers as in [MW17, RZZ17b], since we can obtain existence of solutions to the shifted equation (3.19) for any initial value in V_0^{-1} (see Section 3.3).*

In the following, we only use Wick powers $: \cdot :_Q$ and we write $: \cdot :$ for simplicity.

Relations between two different Wick powers

We introduce the following probability measure. Set $: q(\phi) := \frac{1}{4} : \phi^4 :$, $: p(\phi) :=: \phi^3 :$. Let

$$\nu = c\exp(-N)\mu,$$

where c is a normalization constant and $N = {}_{\mathcal{S}'}\langle : q :, e_0\rangle_{\mathcal{S}}$. According to [Sim74, Lemma V.5 and Theorem V.7] we have that for every $p \in [1,\infty)$, $\varphi(\phi) := e^{-N} \in L^p(\mathcal{S}'(\mathbb{T}^2), \mu)$.

The following result is about the relation between the two different Wick powers.

Lemma 3.6. *Let ϕ be a measurable map from $(\Omega, \mathcal{F}, \mathbb{P})$ to $C([0,T], B_{2,2}^{-\gamma})$ with $\gamma > 2$, $\mathbb{P}\circ\phi(t)^{-1} = \nu$ for every $t \in [0,T]$ and let $Z(t)$ be defined as above. Assume in addition that $y = \phi - Z \in C((0,T];\mathcal{C}^\beta)$ $\mathbb{P}$-a.s. for some $\beta > \alpha > 0$. Here $C((0,T];\mathcal{C}^\beta)$ is equipped with the norm $\sup_{t\in[0,T]} t^{\frac{\beta+\alpha}{4}} \|\cdot\|_\beta$. Then for every $t > 0$, $n \in \mathbb{N}$*

$$: \phi^n(t) := \sum_{k=0}^n C_n^k y^{n-k}(t) : Z^k(t) : \quad \mathbb{P}-a.s..$$

Here the Wick power on the left hand side is the limit obtained and defined in Lemma 3.2.

Proof By Lemma 3.4 it follows that for every $k \in \mathbb{N}$, $p > 1$

$$:Z_\varepsilon^k: \to :Z^k: \quad \text{in } L^p(\Omega, C((0,T];\mathcal{C}^{-\alpha})), \text{ as } \varepsilon \to 0.$$

Since $y_\varepsilon = \phi_\varepsilon - Z_\varepsilon = \rho_\varepsilon * y$ and $y \in C((0,T];\mathcal{C}^\beta)$ $\mathbb{P}$-a.s., it is obvious that $y_\varepsilon \to y$ in $C((0,T];\mathcal{C}^{\beta-\kappa})$ $\mathbb{P}$-a.s. for every $\kappa > 0$ with $\beta - \kappa - \alpha > 0$, which combined with Lemma 2.3 implies that for $k \in \mathbb{N}$, $k \leq n$,

$$y_\varepsilon^{n-k} :Z_\varepsilon^k: \to^{\mathbb{P}} y^{n-k} :Z^k: \quad \text{in } C((0,T];\mathcal{C}^{-\alpha}), \text{ as } \varepsilon \to 0.$$

Here $\to^{\mathbb{P}}$ means convergence in probability. Since $e^{-N} \in L^p(\mathcal{S}'(\mathbb{T}^2), \mu)$ for every $p \geq 1$, by Hölder's inequality and Lemma 3.2 we get that for $t > 0$ and $p > 1$

$$:\phi_\varepsilon^n(t): \to :\phi^n(t): \quad \text{in } L^p(\Omega, \mathcal{C}^{-\alpha}), \text{ as } \varepsilon \to 0.$$

Moreover, by (3.13) we have

$$\begin{aligned} :\phi_\varepsilon^n: &=: (y_\varepsilon + Z_\varepsilon)^n := c_\varepsilon^{n/2} P_n(c_\varepsilon^{-1/2}(y_\varepsilon + Z_\varepsilon)) \\ &= \sum_{k=0}^n C_n^k c_\varepsilon^{n/2} P_k(c_\varepsilon^{-1/2} Z_\varepsilon)(c_\varepsilon^{-1/2} y_\varepsilon)^{n-k} \\ &= \sum_{k=0}^n C_n^k :Z_\varepsilon^k: y_\varepsilon^{n-k}, \end{aligned}$$

which implies the result by letting $\varepsilon \to 0$. □

3.3 The Solution to the Shifted Equation

Now we fix a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty)}, \mathbb{P})$ and on it a U-cylindrical Wiener process W. Define $Z(t) = \int_0^t e^{-(t-s)A^2/2} B dW(s)$ as in Section 3.2. Now we consider the following shifted equation:

$$\begin{cases} \dfrac{dY}{dt} = -\dfrac{1}{2}A^2 Y + \dfrac{1}{2}A \displaystyle\sum_{k=0}^3 C_3^k Y^{3-k} :Z^k:, \\ Y(0) = x. \end{cases} \tag{3.19}$$

Generally we consider initial data x that are $\mathcal{F}_0$ measurable and belong to $V_0^{-1}, a.s.$. To prove the existence of the solution to equation (3.19), we use a smooth approximation on each path:

$$\begin{cases} \dfrac{dY_\varepsilon}{dt} = -\dfrac{1}{2}A^2 Y_\varepsilon + \dfrac{1}{2}A \displaystyle\sum_{k=0}^3 C_3^k Y_\varepsilon^{3-k} :Z_\varepsilon^k:, \\ Y_\varepsilon(0) = x_\varepsilon, \end{cases} \tag{3.20}$$

where $Z_\varepsilon = Z * \rho_\varepsilon$, $x_\varepsilon = x * \rho_\varepsilon$, and ρ_ε is as introduced in Section 3.2. Note that the solution Y to equation (3.19) and the solution Y_ε to (3.20) also satisfy:

$$\frac{dm(Y(t))}{dt} = 0, m(Y(0)) = 0, \tag{3.21}$$

which means that $m(Y(t)) = m(Y_\varepsilon(t)) = 0$.

From Lemma 3.2 we know that there exists a $\Omega' \subset \Omega$, $\mathbb{P}(\Omega')=1$, such that for any $\omega \in \Omega'$, $Z(\omega), : Z^n : (\omega) \in C((0,T];\mathcal{C}^{-\alpha}), n=2,3$, $\forall \alpha>0$. Since $Z_\varepsilon(\omega)$ is smooth, by monotone trick in [LR15, Theorem 5.2.2 and Theorem 5.2.4], there exists a unique solution $Y_\varepsilon(\omega)$ to equation (3.20) in $L^2(0,T;V_0^2)\cap C([0,T];L_0^2)$ for each $\omega\in\Omega'$. We are going to find a convergent subsequence of $\{Y_\varepsilon(\omega)\}$, which converge to a solution to equation (3.19) and prove uniqueness of solutions to (3.19). Then we obtain a unique $\mathcal{F}_t$-adapted solution to equation (3.19).

In this section we never distinguish V^α, H_2^α and $B_{2,2}^\alpha$ since they have equivalent norms. For convenience we denote all of them as H^α.

Theorem 3.7 (a-priori estimate)**.** *If Y is a solution to equation (3.19), then there exists a constant C_T which only depends on T and $Z(\omega)$, such that for $\forall t\in[0,T]$*

$$||Y||_{H^{-1}}^2 - ||x||_{H^{-1}} + \frac{1}{2}\int_0^t \left(||Y(s)||_{H^1}^2 + ||Y(s)||_{L^4}^4\right) ds \le C_T. \tag{3.22}$$

Moreover there exist constants $C>0$, $\lambda_k>0$, $k=1,2,3$, for every $t\in(0,T]$

$$\|Y_t\|_{H^{-1}}^2 \le C\left(t^{-1} \vee \left(\sum_{k=1}^3 t^{-\rho\lambda_k} \sup_{0\le r\le t}\left(r^{\rho\lambda_k}\| : Z_r : \|_{-\alpha}^{\lambda_k}\right)\right)^{\frac{1}{2}}\right), \tag{3.23}$$

where $\rho>0$ is a small enough constant introduced in Lemma 3.4.

Proof Since

$$\frac{dY}{dt} = -\frac{1}{2}A(AY - \sum_{k=0}^3 C_3^k Y^{3-k} : Z^k :),$$

and $m(Y)=0$, taking scalar product with $(-A)^{-1}Y$ we obtain that

$$\frac{d}{dt}||Y||_{H^{-1}}^2 + ||Y||_{H^1}^2 + ||Y||_{L^4}^4 = -\langle \sum_{k=1}^3 C_3^k Y^{3-k} : Z^k :, Y\rangle,$$

that is

$$\frac{d}{dt}||Y||_{H^{-1}}^2 + ||Y||_{H^1}^2 + ||Y||_{L^4}^4 \lesssim |\langle Y, : Z^3 :\rangle| + |\langle Y^2, : Z^2 :\rangle| + |\langle Y^3, Z\rangle|. \tag{3.24}$$

So, we only need to estimate the right hand side of (3.24). We only consider $|\langle Y^3, Z\rangle|$. The other terms can be estimated similarly. Lemma 2.3 implies

$$|\langle Y^3, Z\rangle| \lesssim ||Z||_{-\alpha}||Y^3||_{B_{1,1}^\alpha}, \quad \forall \alpha>0.$$

Moreover, by Lemma 2.1 and Lemma 2.4. Then

$$||Y^3||_{B_{1,1}^\alpha} \lesssim ||\mathfrak{A}^{\beta_0}Y^3||_{L^{p_0}} \lesssim ||\mathfrak{A}^{\beta_0}Y^{\frac{3}{2}}||_{L^{p_1}}||Y^{\frac{3}{2}}||_{L^{q_1}},$$

where $\beta_0>\alpha, p_0>1$ and $\frac{1}{p_0}=\frac{1}{p_1}+\frac{1}{q_1}$.

Choose $q_1\le\frac{8}{3}$ and $p_1>\frac{8}{5}$, we have

$$||Y^{\frac{3}{2}}||_{L^{q_1}} = ||Y||_{L^{\frac{3}{2}q_1}}^{\frac{3}{2}} \lesssim ||Y||_{L^4}^{\frac{3}{2}}.$$

For $||\mathfrak{A}^{\beta_0}Y^{\frac{3}{2}}||_{L^{p_1}}$, we have

$$||\mathfrak{A}^{\beta_0}Y^{\frac{3}{2}}||_{L^{p_1}} \lesssim ||\mathfrak{A}^{\beta_1}Y^{\frac{3}{2}}||_{L^{p_2}} \lesssim ||\mathfrak{A}Y^{\frac{3}{2}}||_{L^{p_2}}^{\beta_1}||Y^{\frac{3}{2}}||_{L^{p_2}}^{1-\beta_1},$$

where $1 < p_2 < p_1 < 2$, $\beta_0 = \beta_1 - 2(\frac{1}{p_2} - \frac{1}{p_1})$, $\beta_1 < 1$ and we used Lemma 2.1 in the first inequality and Lemma 2.5 in the second inequality. For $||\mathfrak{A}Y^{\frac{3}{2}}||_{L^{p_2}}$, let $p_2 < \frac{8}{5}$, we have

$$||\mathfrak{A}Y^{\frac{3}{2}}||_{L^{p_2}} \lesssim ||Y^{\frac{1}{2}}\nabla Y||_{L^{p_2}} \lesssim ||Y||_{H^1}||Y^{\frac{1}{2}}||_{L^{\frac{2p_2}{2-p_2}}} \lesssim ||Y||_{H^1}||Y||^{\frac{1}{2}}_{L^{\frac{p_2}{2-p_2}}} \lesssim ||Y||_{H^1}||Y||^{\frac{1}{2}}_{L^4},$$

where we used Hölder's inequality in the second inequality. Furthermore

$$||Y^{\frac{3}{2}}||_{L^{p_2}} \lesssim ||Y||^{\frac{3}{2}}_{L^{\frac{3p_2}{2}}} \lesssim ||Y||^{\frac{3}{2}}_{L^4}.$$

Combining the above estimates we get

$$||Y^3||_{B^{\alpha}_{1,1}} \lesssim ||Y||^{3-\beta_1}_{L^4}||Y||^{\beta_1}_{H^1}.$$

Combining this with Lemma 3.4, we have

$$|\langle Y^3, Z\rangle| \lesssim ||Y||^{3-\beta_1}_{L^4}||Y||^{\beta_1}_{H^1}t^{-\frac{\rho}{4}} \lesssim t^{-\frac{\rho}{4}\lambda} + \kappa\left(||Y||^4_{L^4} + ||Y||^2_{H^1}\right),$$

where $\lambda = \frac{4}{1-\beta_1}$ and we used the Young's inequality. Choosing ρ to be so small that $\frac{\rho}{4}\lambda < 1$, we can conclude that there exists $\lambda_k > 0$, $k = 1, 2, 3$ such that $\frac{\lambda_k \rho}{4} < 1$

$$\frac{d}{dt}||Y||^2_{H^{-1}} + \frac{1}{2}\left(||Y||^2_{H^1} + ||Y||^4_{L^4}\right) \lesssim \sum_{k=1}^{3} \| : Z^k : \|^{\lambda_k}_{-\alpha} \lesssim \sum_{k=1}^{3} t^{-\frac{\lambda_k \rho}{4}},$$

and (3.22) follows. Moreover, since $\|Y\|_{H^{-1}} \lesssim \|Y\|_{L^4}$ we have that

$$\frac{d}{dt}||Y||^2_{H^{-1}} + \frac{1}{2}||Y||^4_{H^{-1}} \lesssim \sum_{k=1}^{3} \| : Z^k : \|^{\lambda_k}_{-\alpha}.$$

By [TW16, Lemma 3.8] we have

$$\|Y_t\|^2_{H^{-1}} \lesssim t^{-1} \vee \left(\sum_{k=1}^{3} t^{-\rho\lambda_k} \sup_{0\le r\le t}\left(r^{\rho\lambda_k}\| : Z_r : \|^{\lambda_k}_{-\alpha}\right)\right)^{\frac{1}{2}}.$$

□

Since the approximation equation (3.20) have the same prior estimate as (3.19). By (3.22) we deduce that the sequence $\{Y_\varepsilon\}$ is bounded in $L^\infty(0, T; H^{-1}) \cap L^4([0, T] \times \mathbb{T}^2) \cap L^2(0, T; H^1)$. This implies that $\{AY_\varepsilon\}$ is bounded in $L^2(0, T; H^{-1})$ and $\{Y_\varepsilon^3\}$ is bounded in $L^{4/3}([0, T] \times \mathbb{T}^2)$. Moreover, Lemma 2.1 and Lemma 3.4 imply that $\{: Z_\varepsilon^3 :\}$ is bounded in $L^p(0, T; H^{-\alpha})$ for any $\alpha > 0, \varepsilon > 0$ and $p > 1$. Then we can prove the following lemma:

Lemma 3.8. $\{\frac{dY_\varepsilon}{dt}\}$ *is bounded in* $L^p(0, T; H^{-3})$, *where* $p \in (1, \frac{4}{3})$.

Proof According to the argument before, we only need to show that $\{Y_\varepsilon^2 Z_\varepsilon\}$ and $\{Y_\varepsilon : Z_\varepsilon^2 :\}$ are bounded in $L^p(0,T;H^{-1})$ when $p\in(1,\frac{4}{3})$.

We omit ε if there is no confusion in this proof.

For Y^2Z we have

$$||Y^2Z||_{B^{-\alpha}_{2,\infty}} \lesssim ||Y^2||_{B^{\beta_0}_{2,\infty}}||Z||_{-\alpha} \lesssim ||Y^2||_{B^{\beta_0}_{2,1}}||Z||_{-\alpha},$$

where $\beta_0>\alpha>0$, we used Lemma 2.5 in the first inequality and Lemma 2.1 in the second inequality. Furthermore,

$$||Y^2||_{B^{\beta_0}_{2,1}} \lesssim ||\mathfrak{A}^{\beta_1}Y^2||_{L^2} \lesssim ||\mathfrak{A}^{\beta_1}Y||_{L^{p_0}}||Y||_{L^{q_0}},$$

where $\beta_1>\beta_0$, $\frac{1}{p_0}+\frac{1}{q_0}=\frac{1}{2}$, we used Lemma 2.1 in the first inequality and Lemma 2.4 in the second inequality. By Lemma 2.1, $B^s_{q,2}\subset L^q$ for any $q\geq 1$ and $s>0$. Since $H^\delta\simeq B^\delta_{2,2}\subset B^{\delta-1+\frac{2}{q}}_{q,2}$ for $q\geq 2$, the Besov interpolation in Lemma 2.4 implies that

$$||Y||_{L^{q_0}} \lesssim ||Y||_{B^s_{q_0,2}} \lesssim ||Y||^{1-\frac{1}{q_0}+\frac{s}{2}}_{B^{\frac{2}{q_0}}_{q_0,2}}||Y||^{\frac{1}{q_0}-\frac{s}{2}}_{B^{\frac{2}{q_0}-2}_{q_0,2}} \lesssim ||Y||^{1-\frac{1}{q_0}+\frac{s}{2}}_{H^1}||Y||^{\frac{1}{q_0}-\frac{s}{2}}_{H^{-1}}. \tag{3.25}$$

For $||\mathfrak{A}^{\beta_1}Y||_{L^{p_0}}$, let $p_0\geq 2$. Then we use Lemma 2.1 and Sobolev interpolation to get

$$||\mathfrak{A}^{\beta_1}Y||_{L^{p_0}} \lesssim ||Y||_{H^{\beta_2}} \lesssim ||Y||^{\frac{1+\beta_2}{2}}_{H^1}||Y||^{\frac{1-\beta_2}{2}}_{H^{-1}},$$

where $\beta_1=\beta_2+\frac{2}{p_0}-1=\beta_2-\frac{2}{q_0}$. Thus we have

$$||Y^2||_{B^{\beta_0}_{2,1}} \lesssim ||Y||^{\frac{3}{2}+\frac{\beta_1}{2}+\frac{s}{2}}_{H^1}||Y||^{\frac{1}{2}-\frac{\beta_1}{2}-\frac{s}{2}}_{H^{-1}}. \tag{3.26}$$

By Lemma 3.4 we deduce that

$$||Y^2Z||_{B^{-\alpha}_{2,\infty}} \lesssim ||Y||^{\frac{3}{2}+\frac{\beta_1}{2}+\frac{s}{2}}_{H^1}||Y||^{\frac{1}{2}-\frac{\beta_1}{2}-\frac{s}{2}}_{H^{-1}}t^{-\frac{\rho}{4}}.$$

For any $p\in(1,\frac{4}{3})$, let β_1 and s be small enough such that $(\beta_1+s+3)p<4$. Then Young's inequality implies that there exists $\lambda>0$ such that

$$||Y^2Z||^p_{B^{-\alpha}_{2,\infty}} \lesssim ||Y||^2_{H^1}+||Y||^{\frac{4}{3}\lambda(\frac{1}{2}-\frac{\beta_1}{2}-\frac{s}{2})}_{H^{-1}}t^{-\frac{\rho}{3}\lambda}.$$

For ρ small enough, $\{Y_\varepsilon^2Z_\varepsilon\}$ is bounded in $L^p(0,T;B^{-\alpha}_{2,\infty})$.

On the other hand,

$$||Y:Z^2:||_{B^{-\alpha}_{2,\infty}} \lesssim ||Y||_{B^1_{2,\infty}}||:Z^2:||_{-\alpha} \lesssim ||Y||_{H^1}t^{-\frac{\rho}{4}},$$

where we used Lemma 2.5 in the first inequality and Lemma 2.1, Lemma 3.4 in the second inequality. Then by Young's inequality

$$||Y:Z^2:||^{\frac{4}{3}}_{B^{-\alpha}_{2,\infty}} \lesssim ||Y||^2_{H^1}+t^{-\rho}.$$

Choosing ρ small enough we deduce that $\{Y_\varepsilon : Z_\varepsilon^2 :\}$ is bounded in $L^{\frac{4}{3}}(0,T;B^{-\alpha}_{2,\infty})$. By Lemma 2.1 we have $B^{-\alpha}_{2,\infty}\subset H_2^{-\alpha-\delta}$ for any $\delta>0$. Hence $\{Y_\varepsilon^2Z_\varepsilon\}$ and $\{Y_\varepsilon : Z_\varepsilon^2 :\}$ are bounded in $L^p(0,T;H^{-1})$, $\forall p\in(1,\frac{4}{3})$, which implies the results. □

Theorem 3.9. *For every $x \in V_0^{-1}$, there exists at least one solution to equation (3.19) in $C([0,T];V_0^{-1}) \cap L^4([0,T]\times\mathbb{T}^2) \cap L^2(0,T;V_0^1)$.*

Proof Since $H^1 \subset H^\delta$ compactly for any $\delta < 1$ (see [Tri06, Proposition 4.6]), a classical compactness argument (cf. [GRZ09, Lemma C.2] or [Tem01, Theorem 2.1, Chapter III]) implies that there exists a sequence $\{\varepsilon_k\}$ and $Y \in L^\infty(0,T,H^{-1}) \cap L^2(0,T;H^1) \cap L^4([0,T]\times\mathbb{T}^2)$, such that $Y_{\varepsilon_k} \to Y$ in $L^2(0,T;H^\delta) \cap C([0,T];H^{-3})$, $\forall \delta < 1$.

It is sufficient to show that for a suitable $\delta \in (0,1)$, the limit Y we obtained above is a solution in H^{-3}.

In fact, if Y is a solution in H^{-3}, i.e. for any $h \in H^3$

$$ {}_{H^{-3}}\langle Y_t - Y_0, h\rangle_{H^3} = -\frac{1}{2}\int_0^t {}_{H^{-1}}\langle A^2 h, Y_s\rangle_{H^1} ds + \frac{1}{2}\int_0^t {}_{H^{-1}}\langle \sum_{k=0}^{3} C_3^k Y_s^{3-k} : Z_s^k :, Ah\rangle_{H^1} ds. \tag{3.27} $$

Y is in $L^\infty(0,T,H^{-1}) \cap L^2(0,T;H^1) \cap L^4([0,T]\times\mathbb{T}^2)$. Then we take the scalar product of $\frac{dY}{dt}$ and $(-A)^{-1}Y$, which is just the duality in H^{-3} and H^3. Hence

$$ \frac{d}{dt}||Y||_{H^{-1}}^2 + ||Y||_{H^1}^2 + ||Y||_{L^4}^4 = -\langle \sum_{k=1}^{3} C_3^k Y^{3-k} : Z^k :, Y\rangle. $$

Thus $||Y||_{H^{-1}}$ is continuous w.r.t t. Moreover, [Tem01, Lemma 1.4, Chapter III] implies that Y is weakly continuous in H^{-1}, then $Y \in C([0,T];H^{-1})$.

We still write ε instead of ε_k if there is no confusion. Since Y_ε is a solution to equation (3.20), letting $\varepsilon \to 0$, it's easy to see that

$$ \lim_{\varepsilon\to 0} {}_{H^{-3}}\langle Y_\varepsilon, h\rangle_{H^3} = {}_{H^{-3}}\langle Y, h\rangle_{H^3}, \quad \lim_{\varepsilon\to 0} {}_{H^{-1}}\langle A^2 h, Y_\varepsilon\rangle_{H^1} = {}_{H^{-1}}\langle A^2 h, Y\rangle_{H^1}, $$
$$ \lim_{\varepsilon\to 0} {}_{H^{-1}}\langle : \bar{Z}_\varepsilon^3 :, Ah\rangle_{H^1} = {}_{H^{-1}}\langle : \bar{Z}^3 :, Ah\rangle_{H^1}. $$

It remains to show for any $h \in H^1$

$$ \lim_{\varepsilon\to 0} |\int_0^t \langle Y_\varepsilon^3(s) - Y^3(s), h\rangle ds| = 0, \tag{3.28} $$

$$ \lim_{\varepsilon\to 0} |\int_0^t \langle Y_\varepsilon^2(s) Z_\varepsilon(s) - Y^2(s) Z(s), h\rangle ds| = 0, \tag{3.29} $$

$$ \lim_{\varepsilon\to 0} |\int_0^t \langle Y_\varepsilon(s) : Z_\varepsilon^2 : (s) - Y(s) : Z^2 : (s), h\rangle ds| = 0. \tag{3.30} $$

Since $Y_\varepsilon \to Y$ in $L^4([0,T]\times\mathbb{T}^2)$, which is equivalent to $||Y_\varepsilon||_{L^4([0,T]\times\mathbb{T}^2)} \to ||Y||_{L^4([0,T]\times\mathbb{T}^2)}$ and $Y_\varepsilon \Rightarrow^m Y$, where $\Rightarrow^m$ means convergence in Lebesgue measure m on $[0,T]\times\mathbb{T}^2$, we have $||Y_\varepsilon^3||_{L^{\frac{4}{3}}([0,T]\times\mathbb{T}^2)} \to ||Y^3||_{L^{\frac{4}{3}}([0,T]\times\mathbb{T}^2)}$ and $Y_\varepsilon^3 \Rightarrow^m Y^3$. Then (3.28) holds by uniform integrability.

For (3.29), let $R_\varepsilon = Y_\varepsilon - Y$. By the triangle inequality

$$ |\langle Y_\varepsilon^2 Z_\varepsilon - Y^2 Z, h\rangle| \lesssim |\langle R_\varepsilon (Y + Y_\varepsilon) h, Z\rangle| + |\langle Z_\varepsilon - Z, Y^2 h\rangle|. $$

For the second term on the right hand side of the above inequality, we have

$$ |\langle Z_\varepsilon - Z, Y^2 h\rangle| \lesssim ||Z_\varepsilon - Z||_{-\alpha} ||Y^2 h||_{B_{1,1}^\alpha} \lesssim ||Z_\varepsilon - Z||_{-\alpha} ||Y^2||_{B_{2,1}^\alpha} ||h||_{B_{2,1}^\alpha}, $$

where we used Lemma 2.3 in the first inequality and Lemma 2.5 in the second inequality. By [Tri92, Remark 2, Section 3.2, Chapter 2] we have $H^1 \subset B^\alpha_{2,1}$ for any $\alpha < 1$. Hence

$$|\langle Z_\varepsilon - Z, Y^2 h\rangle| \lesssim ||Z_\varepsilon - Z||_{-\alpha}||Y^2||_{B^\alpha_{2,1}}||h||_{H^1}.$$

Combining with (3.26), we have

$$|\langle Z_\varepsilon - Z, Y^2 h\rangle| \lesssim ||Z_\varepsilon - Z||_{-\alpha}||h||_{H^1}||Y||_{H^1}^{\frac{3}{2}+\frac{\beta_3}{2}+\frac{s}{2}}||Y||_{H^{-1}}^{\frac{1}{2}-\frac{\beta_3}{2}-\frac{s}{2}},$$

where $\beta_3 > \alpha > 0, s > 0$. Let $\frac{3}{2} + \frac{\beta_3}{2} + \frac{s}{2} < 2$. Then Lemma 3.4 and Hölder's inequality imply that

$$|\int_0^t \langle Z_\varepsilon - Z, Y^2 h\rangle ds| \to 0, \quad \varepsilon \to 0.$$

Similarly

$$|\langle R_\varepsilon Y h, Z\rangle| \lesssim ||R_\varepsilon Y||_{B^\alpha_{2,1}}||h||_{H^1}||Z||_{-\alpha}.$$

For $||R_\varepsilon Y||_{B^\alpha_{2,1}}$, we have

$$||R_\varepsilon Y||_{B^\alpha_{2,1}} \lesssim ||R_\varepsilon Y||_{B^{\beta_0}_{2,2}} \lesssim ||\mathfrak{A}^{\beta_0} R_\varepsilon||_{L^4}||Y||_{L^4} + ||\mathfrak{A}^{\beta_0} Y||_{L^4}||R_\varepsilon||_{L^4},$$

where $\beta_0 > \alpha > 0$ and we used Lemma 2.1 in the first inequality and Lemma 2.4 in the second inequality. By Lemma 2.1 we have the Sobolev embedding $H_2^{\beta+\frac{1}{2}} \subset H_4^\beta$. Hence

$$||R_\varepsilon Y||_{B^\alpha_{2,1}} \lesssim ||R_\varepsilon||_{H^{\beta_0+\frac{1}{2}}}||Y||_{L^4} + ||Y||_{H^{\beta_0+\frac{1}{2}}}||R_\varepsilon||_{H^{\frac{1}{2}}}.$$

By Sobolev interpolation, choosing $\delta > \frac{1}{2} + \beta_0$, we have

$$||Y||_{H^{\beta_0+\frac{1}{2}}} \lesssim ||Y||_{H^1}^{\frac{3}{4}+\frac{\beta_0}{2}}||Y||_{H^{-1}}^{\frac{1}{4}-\frac{\beta_0}{2}}.$$

Moreover, since $\delta > \frac{1}{2} + \beta_0$, we have $||R_\varepsilon||_{H^{\frac{1}{2}}} \lesssim ||R_\varepsilon||_{H^\delta}$ and $||Y||_{H^{\frac{1}{2}+\beta_0}} \lesssim ||Y||_{H^\delta}$. Then we deduce that

$$||R_\varepsilon Y||_{B^\alpha_{2,1}} \lesssim ||R_\varepsilon||_{H^\delta}||Y||_{L^4} + ||Y||_{H^1}^{\frac{3}{4}+\frac{\beta_0}{2}}||R_\varepsilon||_{H^\delta}||Y||_{H^{-1}}^{\frac{1}{4}-\frac{\beta_0}{2}}.$$

Let $\beta_0 < \frac{1}{2}$ such that

$$\frac{3}{4} + \frac{\beta_0}{2} + 1 < 2.$$

Then by Hölder inequality, we get

$$\int_0^t ||R_\varepsilon Y||_{B^\alpha_{2,1}}||h||_{H^1}||\bar{Z}||_{-\alpha}ds \lesssim \left(\int_0^t ||R_\varepsilon||^2_{H^\delta}ds\right)^{\frac{1}{2}} \left(\int_0^t (||Y||^2_{H^1})F ds\right)^{\frac{1}{2}} \left(\int_0^t ||Y||^4_{L^4}ds\right)^{\frac{1}{4}} \to 0,$$

where $F \in L^\infty(0, T)$.

Moreover, we have

$$|\langle Y_\varepsilon : Z_\varepsilon^2 : - Y : Z^2 :, h\rangle| \lesssim |\langle Y_\varepsilon(: Z_\varepsilon^2 : - : Z^2 :), h\rangle| + |\langle R_\varepsilon : Z^2 :, h\rangle|.$$

By essentially the same argument as above, (3.30) also follows.

Then we have got a solution Y in $C([0, T]; H^{-1}) \cap L^4([0, T] \times \mathbb{T}^2) \cap L^2(0, T; H^1)$. Combining this with (3.21), we have $Y \in C([0, T]; V_0^{-1}) \cap L^4([0, T] \times \mathbb{T}^2) \cap L^2(0, T; V_0^1)$. □

Now we have obtained the existence of solutions of equation (3.19). The following is the uniqueness result.

Theorem 3.10. *For every $x \in V_0^{-1}$, there exists a unique solution to equation (3.19) in $C([0,T];V_0^{-1}) \cap L^4([0,T]\times\mathbb{T}^2) \cap L^2(0,T;V_0^1)$.*

Proof Suppose u, v are two solutions of (3.19) with the same initial value. Let $r = u-v$, then r satisfies:

$$\begin{cases} \dfrac{dr}{dt} = -\dfrac{1}{2}A^2 r + \dfrac{1}{2}A\displaystyle\sum_{k=0}^{3} C_3^k(u^{3-k}-v^{3-k}) : Z^k :, \\ r(0)=0. \end{cases}$$

Similarly to (3.24) we have:

$$\frac{d}{dt}||r||_{H^{-1}}^2 + ||r||_{H^1}^2 \lesssim |\langle r^2(u+v), Z\rangle| + |\langle r^2, : Z^2 :\rangle|. \tag{3.31}$$

By Lemma 2.3 and Lemma 3.4 we know

$$|\langle r^2, : Z^2 :\rangle| \lesssim ||r^2||_{B_{1,1}^{\alpha}} t^{-\rho},$$

where $\beta > \alpha > 0$. Then Lemma 2.1 and Lemma 2.4 imply that

$$||r^2||_{B_{1,1}^{\alpha}} \lesssim ||\mathfrak{A}^{\beta_0} r^2||_{L^{\frac{4}{3}}} \lesssim ||\mathfrak{A}^{\beta_0} r||_{L^2}||r||_{L^4} \lesssim ||r||_{H^1}^{\frac{\beta_0+3}{2}} ||r||_{H^{-1}}^{\frac{1-\beta_0}{2}},$$

where $1 > \beta_0 > \alpha > 0$ and we used the Sobolev interpolation and Sobolev embedding theorem in the last inequality. Then by Young's inequality, there exists a $\lambda_1 > 0$ such that for any $\varepsilon > 0$

$$|\langle r^2, : Z^2 :\rangle| \lesssim \varepsilon||r||_{H^1}^2 + ||r||_{H^{-1}}^2 t^{-\rho\lambda_1}. \tag{3.32}$$

Let ρ be small enough. Then $g := t^{-\rho\lambda_1} \in L^1(0,T)$.

For $|\langle r^2(u+v), Z\rangle|$, we similarly obtain that

$$|\langle r^2(u+v), Z\rangle| \lesssim ||r^2(u+v)||_{B_{1,1}^{\alpha}}||Z||_{-\alpha} \lesssim \left(||ur^2||_{B_{1,1}^{\alpha}} + ||vr^2||_{B_{1,1}^{\alpha}}\right) t^{-\rho}.$$

For $||ur^2||_{B_{1,1}^{\alpha}}$, we have

$$||ur^2||_{B_{1,1}^{\alpha}} \lesssim ||\mathfrak{A}^{\beta_0}(ur^2)||_{L^{p_0}} \lesssim ||\mathfrak{A}^{\beta_0}u||_{L^{p_1}}||r^2||_{L^{q_1}} + ||\mathfrak{A}^{\beta_0}r^2||_{L^{p_2}}||u||_{L^{q_2}} := (I)+(II),$$

with $p_0 > 1$, $\beta_0 > \alpha > 0$, and $\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$, $p_i, q_i > p_0, i = 1,2$. Here we used Lemma 2.1 in the first inequality and Lemma 2.5 in the second inequality.

For (I), according to (3.25) we know that for any $s > 0$

$$||r^2||_{L^{q_1}} = ||r||_{L^{2q_1}}^2 \lesssim ||r||_{H^1}^{2-\frac{1}{q_1}+s} ||r||_{H^{-1}}^{\frac{1}{q_1}-s}.$$

Moreover, let $p_1 \geq 4$. Then

$$||\mathfrak{A}^{\beta_0}u||_{L^{p_1}} \lesssim ||\mathfrak{A}^{\beta_1}u||_{L^4} \lesssim ||u||_{L^4}^{1-2\beta_1}||u||_{H_4^{\frac{1}{2}}}^{2\beta_1} \lesssim ||u||_{L^4}^{1-2\beta_1}||u||_{H^1}^{2\beta_1},$$

where $\beta_1 = \beta_0 + \frac{1}{2} - \frac{2}{p_1}$ and we used Lemma 2.1 in the first inequality and Sobolev interpolation in the second inequality and Besov embedding Lemma 2.1 in the last inequality. Combining these estimates above we have

$$(I) \lesssim ||r||_{H^1}^{2-\frac{1}{q_1}+s}||r||_{H^{-1}}^{\frac{1}{q_1}-s}||u||_{L^4}^{1-2\beta_1}||u||_{H^1}^{2\beta_1}.$$

Hence by Young's inequality

$$t^{-\rho}(I) \lesssim \varepsilon ||r||_{H^1}^2 + ||r||_{H^{-1}}^2 ||u||_{H^1}^{\frac{4\beta_1}{\frac{1}{q_1}-s}} ||u||_{L^4}^{\frac{2(1-2\beta_1)}{\frac{1}{q_1}-s}} t^{-\frac{2\rho}{\frac{1}{q_1}-s}}.$$

Let p_0 be close to 1 and β_0, s be so small enough such that $\frac{1}{p_1} > 1 - \frac{1}{p_0} + \beta_0 + s$, which is equivalent to $\frac{2\beta_1}{\frac{1}{q_1}-s} + \frac{(1-2\beta_1)}{\frac{1}{q_1}-s}\frac{1}{2} < 1$. Then the Hölder inequality yields for ρ small enough

$$\int_0^t ||u||_{H^1}^{\frac{4\beta_1}{\frac{1}{q_1}-s}} ||u||_{L^4}^{\frac{2(1-\beta_1)}{\frac{1}{q_1}-s}} \tau^{-\frac{2\rho}{\frac{1}{q_1}-s}} d\tau \lesssim \left(\int_0^t ||u||_{H^1}^2 d\tau\right)^{\frac{1}{2}} \left(\int_0^t ||u||_{L^4}^4 d\tau\right)^{\frac{1}{4}}.$$

Then we get

$$f_1^u := ||u||_{H^1}^{\frac{4\beta_1}{\frac{1}{q_1}-s}} ||u||_{L^4}^{\frac{2(1-\beta_1)}{\frac{1}{q_1}-s}} t^{-\frac{2\rho}{\frac{1}{q_1}-s}} \in L^1(0,T),$$

and for any $\varepsilon > 0$,

$$t^{-\rho}(I) \lesssim \varepsilon ||r||_{H^1}^2 + f_1^u ||r||_{H^{-1}}^2. \tag{3.33}$$

For (II), let $q_2 = 4$. Then we have $\frac{1}{p_2} + \frac{1}{4} = \frac{1}{p_0} \in (\frac{3}{4}, 1)$, which implies that $p_2 \in (\frac{4}{3}, 2)$. Similarly by Lemma 2.5

$$||\mathfrak{A}^{\beta_0} r^2||_{L^{p_2}} \lesssim ||\mathfrak{A}^{\beta_0} r||_{L^{p_3}} ||r||_{L^{q_3}},$$

where $\frac{1}{p_3} + \frac{1}{q_3} = \frac{1}{p_2}$, $p_3, q_3 > p_2$. From (3.25) we know that for every $s > 0$

$$||r||_{L^{q_3}} \lesssim ||r||_{H^1}^{1-\frac{1}{q_3}+\frac{s}{2}} ||r||_{H^{-1}}^{\frac{1}{q_3}-\frac{s}{2}}.$$

Let $p_3 \geq 2$. Then by Lemma 2.1 we have

$$||\mathfrak{A}^{\beta_0} r||_{L^{p_3}} \lesssim ||r||_{H^{\beta_2}} \lesssim ||r||_{H^1}^{\frac{1+\beta_2}{2}} ||r||_{H^{-1}}^{\frac{1-\beta_2}{2}},$$

where we used Sobolev interpolation in the second inequality and that $\beta_0 = \beta_2 - 1 + \frac{2}{p_3}$. Hence

$$||\mathfrak{A}^{\beta_0} r^2||_{L^{p_2}} \lesssim ||r||_{H^1}^{\frac{3}{2}+\frac{\beta_2}{2}-\frac{1}{q_3}+\frac{s}{2}} ||r||_{H^{-1}}^{\frac{1}{2}-\frac{\beta_2}{2}+\frac{1}{q_3}-\frac{s}{2}} = ||r||_{H^1}^{2+\frac{\beta_0}{2}-\frac{1}{p_2}+\frac{s}{2}} ||r||_{H^{-1}}^{\frac{1}{p_2}-\frac{s}{2}-\frac{\beta_0}{2}}.$$

Thus, we have

$$(II) \lesssim ||r||_{H^1}^{2+\frac{\beta_0}{2}-\frac{1}{p_2}+\frac{s}{2}} ||r||_{H^{-1}}^{\frac{1}{p_2}-\frac{s}{2}-\frac{\beta_0}{2}} ||u||_{L^4}.$$

Then by Young's inequality we have

$$t^{-\rho}(II) \lesssim \varepsilon ||r||_{H^1}^2 + ||r||_{H^{-1}}^2 ||u||_{L^4}^{\frac{2}{\frac{1}{p_2}-\frac{s}{2}-\frac{\beta_0}{2}}} t^{-\frac{2\rho}{\frac{1}{p_2}-\frac{s}{2}-\frac{\beta_0}{2}}}.$$

It is easy to see that $p_2 < 2$ yields $\frac{2}{\frac{1}{p_2}-\frac{s}{2}-\frac{\beta_0}{2}} \leq 4$ when s, β_0 are small enough. Then for small enough ρ we have $f_2^u := ||u||_{L^4}^{\frac{2}{\frac{1}{p_2}-\frac{s}{2}-\frac{\beta_0}{2}}} t^{-\frac{2\rho}{\frac{1}{p_2}-\frac{s}{2}-\frac{\beta_0}{2}}} \in L^1(0,T)$.

Then we obtain that for any $\varepsilon > 0$

$$|\langle r^2 u, Z\rangle| \lesssim \varepsilon ||r||_{H^1}^2 + f^u ||r||_{H^{-1}}^2,$$

where $f^u := f_1^u + f_2^u \in L^1(0,T)$.

The same holds with u replaced by v. Let $f = f^u + f^v \in L^1(0,T)$. Then

$$|\langle r^2(u+v), \bar{Z}\rangle| \lesssim \varepsilon ||r||_{H^1}^2 + f||r||_{H^{-1}}^2.$$

Hence we get

$$\frac{d}{dt}||r||_{H^{-1}}^2 + ||r||_{H^1}^2 \lesssim \varepsilon ||r||_{H^1}^2 + (f+g)||r||_{H^{-1}}^2.$$

Choose a suitable $\varepsilon > 0$ such that

$$\frac{d}{dt}||r||_{H^{-1}}^2 \lesssim (f+g)||r||_{H^{-1}}^2.$$

Then by Gronwall's inequality we have

$$||r(t)||_{H^{-1}}^2 \lesssim ||r(0)||_{H^{-1}}^2 \exp\left(\int_0^t f(s) + g(s) ds\right) = 0.$$

Since V_0^{-1} is a subspace of H^{-1}, we obtain the uniqueness.

□

Remark 3.11. *We emphasize that we cannot obtain global well-posedness of equation (3.19) by combining (3.22) with fixed point argument in [DPD03] and [MW17] since we only have an H^{-1}-estimate. In fact, in order to use fixed point arguments to obtain local solutions, the initial value should be in $\mathcal{C}^{-\frac{4}{3}+}$. An initial value in H^{-1}-norm is not enough to use mild formulation to obtain local solution.*

3.4 Relation to the solution given by Dirichlet forms

In this section, we are going to obtain a probabilistically weak solution of equation (3.1) via the Dirichlet form approach and compare this solution with the solution we obtain in Section 4.

According to the definition of V_0^α and [Hid80, Theorem 3.1], μ is supported on V_0^{-s} for any $s > 1$. So we fix a small enough $s_0 > 0$ and $V_0^{-1-s_0}$ as the state space and denote it by E for convenience. By identifying V_0^1 and V_0^{-1} via the Riesz isomrophism we have the following Gelfand triple:

$$E^* \subset V_0^{-1} \subset E \tag{3.34}$$

where $E^* = V_0^{s_0-1}$ and the dualization between E^* and E is ${}_{E^*}\langle u, v\rangle_E := {}_{V_0^{1+s_0}}\langle Qu, v\rangle_{V_0^{-1-s_0}}$ for any $u \in E^*, v \in E$. Here ${}_{V^s}\langle \cdot,\cdot\rangle_{V^{-s}}$ is denoted by

$${}_{V_0^s}\langle u, v\rangle_{V^{-s}} := \sum_k {}_{\mathcal{S}'}\langle u, e_k\rangle_{\mathcal{S}\mathcal{S}'}\langle v, e_k\rangle_{\mathcal{S}}, u \in V_0^s, v \in V_0^{-s}. \tag{3.35}$$

Then we have that

$${}_{E^*}\langle u, v\rangle_E = \langle u, v\rangle_{V_0^{-1}}, \forall u \in E^*, \forall v \in V_0^{-1}. \tag{3.36}$$

Moreover we define $\mathcal{F}C_b^\infty; = \{f({}_{E^*}\langle l_1, \cdot\rangle_E, \cdots, {}_{E^*}\langle l_m, \cdot\rangle_E) : m \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^m), l_1, \cdots, l_m \in E^*\}$. For all $\varphi = f({}_{E^*}\langle l_1, \cdot\rangle_E, \cdots, {}_{E^*}\langle l_m, \cdot\rangle_E) \in \mathcal{F}C_b^\infty$, we can define the directional derivative for $h \in V_0^{-1}$:

$$\partial_h \varphi(z) := \lim_{t\to 0} \frac{\varphi(z+th) - \varphi(z)}{t} = \sum_{i=1}^m \partial_i f({}_{E^*}\langle l_1, \cdot\rangle_E, \cdots, {}_{E^*}\langle l_m, \cdot\rangle_E)\langle l_i, h\rangle_{V_0^{-1}}.$$

Then by the Riesz representation theorem, there exists a map $\nabla\varphi : E \to V_0^{-1}$ such that

$$\langle \nabla\varphi(z), h\rangle_{V_0^{-1}} = \partial_h \varphi(z), h \in V_0^{-1}.$$

3.4.1 Solution given by Dirichlet forms

Since $Q^{-1-s_0} : V_0^{1+s_0} \to V_0^{-1-s_0}$ is the Riesz isomorphism for $V_0^{1+s_0}$, i.e.

$$ {}_{V_0^{1+s_0}}\langle h, k\rangle_{V_0^{-1-s_0}} = \langle Q^{-1-s_0}h, k\rangle_{V_0^{-1-s_0}}, $$

μ is in fact a Gaussian measure on Hilbert space $V_0^{-1-s_0}$, with covariance operator $C := Q^{2+s_0}$, that is

$$ \int_{V_0^{-1-s_0}} e^{i\langle h,z\rangle_{V_0^{-1-s_0}}} \mu(dz) = \langle Ch, h\rangle_{V_0^{-1-s_0}}. $$

Then we have the following integration by parts formula for μ:

Proposition 3.12. *For all $F \in \mathcal{F}C_b^\infty, h \in V_0^{3+s_0}$, we have*

$$ \int \partial_h F d\mu = \int {}_{E^*}\langle A^2 h, \phi\rangle_E F(\phi)\mu(d\phi). \tag{3.37} $$

Proof First, by [DPZ02, Section 1.2.4] we know the reproducing kernel of $(V_0^{-1-s_0}, \mu)$ is $V_\mu := C^{1/2}V_0^{-1-s_0} = V_0^1$. Then by [MR92, Theorem 3.1, Chapter II] we have

$$ \begin{aligned} \int \partial_h F d\mu &= \int \langle C^{-1}h, \phi\rangle_{V_0^{-1-s_0}} F(\phi)\mu(d\phi) \\ &= \int \langle Q^{-2-s_0}h, \phi\rangle_{V_0^{-1-s_0}} F(\phi)\mu(d\phi) \\ &= -\int {}_{V_0^{1+s_0}}\langle Ah, \phi\rangle_{V_0^{-1-s_0}} F(\phi)\mu(d\phi) \\ &= -\int {}_{E^*}\langle Q^{-1}Ah, \phi\rangle_E F(\phi)\mu(dz) \\ &= \int {}_{E^*}\langle A^2 h, \phi\rangle_E F(\phi)\mu(dz). \end{aligned} $$

□

Remark 3.13. *In fact, by a similar argument in [GJ12, (9.1.32)], (3.37) still holds for $F\exp(-N)$, where $N = {}_{\mathcal{S}'}\langle : q :, e_0\rangle_{\mathcal{S}}$ i.e. for all $F \in \mathcal{F}C_b^\infty, h \in V_0^{3+s_0}$*

$$ \int \partial_h (F\exp(-N))\, d\mu = \int {}_{E^*}\langle A^2 h, \phi\rangle_E F(\phi)\exp(-N(\phi))\mu(d\phi) $$

Then for the Gibbs measure ν defined in Section 3.2, we have the following integration by parts formula:

Proposition 3.14. *For all $F \in \mathcal{F}C_b^\infty, h \in V_0^{3+s_0}$, we have*

$$ \int \partial_h F d\nu = \int \left({}_{E^*}\langle A^2 h, \phi\rangle_E - {}_{E^*}\langle Ah, :\phi^3:\rangle_E \right) F(\phi)\nu(d\phi). \tag{3.38} $$

Proof According to Proposition 3.12 and Remark 3.13

$$ \int \partial_h F d\nu = c\int (\partial_h F)\exp(-N) d\mu $$

$$= c\int [\partial_h(F\exp(-N)) + F\exp(-N)\partial_h N]d\mu$$
$$= \int F(\phi)\left({}_{E^*}\langle A^2h,\phi\rangle_E - \partial_h N(\phi)\right)\nu(d\phi)$$

By [Oba94, Theorem 4.1.1],

$$\partial_h : \phi_\varepsilon^n(x) := n : \phi_\varepsilon^{n-1}(x) : (\rho_\varepsilon * h)(x).$$

Here $\partial_h : \phi_\varepsilon^n(x) :$ is defined as the directional derivative of the function $\phi \rightarrow : \phi_\varepsilon^n(x)$. Then

$$\partial_h N_\varepsilon(\phi) = \langle : \phi_\varepsilon^3 :, h * \rho_\varepsilon \rangle,$$

where $N_\varepsilon(\phi) := \langle \frac{1}{4} : \phi_\varepsilon^4 :, e_0\rangle$. Letting $\varepsilon \to 0$, due to the closability of $\partial_{\Pi h}$ in $L^2(E,\mu)$,

$$\partial_h N(\phi) = \langle : \phi^3 :, h\rangle = -{}_{E^*}\langle Ah, : \phi^3 :\rangle_E,$$

which implies

$$\int \partial_h F d\nu = \int \left({}_{E^*}\langle A^2h,\phi\rangle_E - {}_{E^*}\langle Ah, : \phi^3 :\rangle_E\right) F(\phi)\nu(d\phi).$$

□

Theorem 3.15. *The bilinear form*

$$\Lambda(\varphi,\psi) := \frac{1}{2}\int \langle \nabla\varphi, \nabla\psi\rangle_{V_0^{-1}} d\nu, \forall \varphi,\psi \in \mathcal{F}C_b^\infty,$$

is closable in $L^2(E,\nu)$. Its closure is a symmetric quasi-regular Dirichlet form denoted by $(\Lambda, D(\Lambda))$.

Proof Let $h_k = \sqrt{\lambda_k}e_k$, $\{h_k\}_{k\in\mathbb{Z}^2\setminus\{(0,0)\}}$ is an orthonormal basis of V_0^{-1}. Then

$$\Lambda(\varphi,\psi) = \frac{1}{2}\sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} \int \frac{\partial\varphi}{\partial h_k}\frac{\partial\psi}{\partial h_k} d\nu, \forall\varphi,\psi\in\mathcal{F}C_b^\infty,$$

By Proposition 3.14 we have $\int \frac{\partial\varphi}{\partial h_k} d\nu = -\int \varphi\beta_{h_k} d\nu$, where $\beta_{h_k} \in L^2(E,\nu)$ and

$$\beta_{h_k}(\phi) = -{}_{E^*}\langle A^2h_k,\phi\rangle_E + {}_{E^*}\langle Ah_k, : \phi^3 :\rangle_E, \forall k \neq (0,0).$$

According to [MR92, Proposition 3.5, Chapter II], $(\Lambda, \mathcal{F}C_b^\infty)$ is closable on $L^2(E,\nu)$ and its closure $(\Lambda, D(\Lambda))$ is a symmetric Dirichlet form. Moreover by [MR92, Proposition 4.2, Chapter IV], the capacity of $(\Lambda, D(\Lambda))$ is tight, and according to the fact that $\mathcal{F}C_b^\infty$ separates the points in $L^2(E,\nu)$, we obtain that $(\Lambda, D(\Lambda))$ is a quasi-regular Dirichlet form. □

Since $(\Lambda, D(\Lambda))$ is a quasi-regular Dirichlet form on $L^2(E,\nu)$, it is well-known that there is a conservative Markov diffusion processes

$$M = \left(\Omega, \mathcal{F}, \mathcal{M}_t, (X(t))_{t\geq 0}, (\mathbb{P}^z)_{z\in E}\right),$$

which is properly associated with $(\Lambda, D(\Lambda))$, i.e. for $u \in L^2(E,\nu)\cap\mathcal{B}_b(E)$, the transition semigroup $P_t u(z) := E^z[u(X(t))]$ is Λ-quasi-continuous for all $t > 0$ and is a ν-version of $T_t u$ where T_t is the semigroup associated with $(\Lambda, D(\Lambda))$. For the notion of Λ-quasi-continuity we refer to [MR92, Chapter III, Definition 3.2]. Then we have the following Fukushima decomposition for $X(t)$ under $\mathbb{P}^z$:

Theorem 3.16. *There exists a map* $W : \Omega \to C([0,\infty); C([0,\infty); V_0^{-1-s_0}(\mathbb{T}^2,\mathbb{R}^2)))$*, and a properly* Λ*-exceptional set* $S \subset E$*,i.e.* $\nu(S) = 0$ *and* $\mathbb{P}^z\left(X(t) \in E \setminus S, \forall t \geq 0\right) = 1$ *for* $z \in E \setminus S$*, such that* $\forall z \in E \setminus S$*,* W *is a* U*-cylindrical Wiener process on* $(\Omega, \mathcal{M}_t, \mathbb{P}^z)$ *and the sample paths of the associated process* $M = (\Omega, \mathcal{F}, \mathcal{M}_t, (X(t))_{t\geq 0}, (\mathbb{P}^z)_{z\in E})$ *on* E *satisfy the following: for* $h \in V^{3+s_0}$*,*

$$\begin{aligned} {}_{E^*}\langle h, X(t) - X(0)\rangle_E = &-\frac{1}{2}\int_0^t {}_{E^*}\langle A^2 h, X(s)\rangle_E ds \\ &+ \frac{1}{2}\int_0^t {}_{E^*}\langle Ah, : X(s)^3 :\rangle_E ds \\ &+ \int_0^t \langle B^* h, dW_s\rangle_{V_0^{-1}(\mathbb{T}^2,\mathbb{R}^2)}, \forall t \geq 0, \mathbb{P}^z - a.s., \end{aligned} \tag{3.39}$$

where B, B^* *are defined as in (3.8). Moreover,* ν *is an invariant measure for* M *in the sense that* $\int P_t u d\nu = \int u d\nu$ *for* $u \in L^2(E,\nu) \cap \mathcal{B}_b(E)$*.*

Proof Let $u_h(\phi) = {}_{E^*}\langle h, \phi\rangle_E, h \in V_0^{3+s_0}$, and let $\mathcal{L}$ be the generator of $(\Lambda, D(\Lambda))$. For any $v \in D(\Lambda)$

$$\begin{aligned} -\int \mathcal{L}u_h v d\nu &= \frac{1}{2}\int \langle \nabla u_h, \nabla v\rangle_{V_0^{-1}} d\nu \\ &= -\frac{1}{2}\int \partial_h v(\phi)\nu(d\phi) \\ &= \frac{1}{2}\int \left({}_{E^*}\langle A^2 h, \phi\rangle_E - {}_{E^*}\langle Ah, : \phi^3 :\rangle_E\right) v(\phi)\nu(d\phi). \end{aligned}$$

Hence $u_h \in D(\mathcal{L})$ and $\mathcal{L}u_h(\phi) = -\frac{1}{2}\left({}_{E^*}\langle A^2 h, \phi\rangle_E - {}_{E^*}\langle Ah, : \phi^3 :\rangle_E\right)$.

By Fukushima's decomposition, we have for q.e. $z \in E$,

$$u_h(X_t) - u_h(X_0) = M_t^h + \int_0^t \mathcal{L}u_h(X_s)ds = M_t^h - \frac{1}{2}\int_0^t \left({}_{E^*}\langle A^2 h, X_s\rangle_E - {}_{E^*}\langle Ah, : X_s^3 :\rangle_E\right) ds,$$

where M^h is a martingale additive functional with $\langle M^h\rangle_t = t\|h\|^2_{V_0^{-1}}$.

In fact, by [AR91, Proposition 4.5],

$$\langle M^h\rangle_t = \int_0^t \langle \nabla u_h(X_s), \nabla u_h(X_s)\rangle_{V_0^{-1}} ds = t\|h\|^2_{V_0^{-1}}.$$

For $f = B^*\bar{Q}h \in U$, with $h \in V_0^{-1}$, define $W_t^f := M_t^h$ and let $D := span\{B^*Qe_k : k \in \mathbb{Z}^2 \setminus \{(0,0)\}\}$. Since $\|B^*Qh\|_U^2 = \|h\|^2_{V_0^{-1}}$, it is easy to check that $\langle W^f, W^g\rangle_t = t\langle f, g\rangle_U$ for $f, g \in D$, where $\langle W^f, W^g\rangle_t$ is the bracket process of W^f and W^g. Moreover D is dense in U and $W_t^{\cdot}$ is $\mathbb{Q}$-linear on D, since the embedding $U \to V_0^{-1-s}(\mathbb{T}^2,\mathbb{R}^2)$ is Hilbert-Schmidt for any $s > 0$. By [AR91, Theorem 6.2], there exist a map $W : \Omega \to C([0,\infty); V_0^{-1-s}(\mathbb{T}^2,\mathbb{R}^2))$, and a properly Λ-exceptional set $S \subset E$, i.e. $\nu(S) = 0$ and $\mathbb{P}^z\left(X(t) \in E \setminus S, \forall t \geq 0\right) = 1$ for $z \in E \setminus S$, such that $\forall z \in E \setminus S$, W is a U-cylindrical Wiener process on $(\Omega, \mathcal{M}_t, \mathbb{P}^z)$ such that for any $f \in D$

$${}_{V_0^{-1-s}}\langle W, f\rangle_{V_0^{1+s}} = W^f, \mathbb{P}^z - a.s.,$$

where ${}_{V_0^{-1-s}}\langle \cdot, \cdot\rangle_{V_0^{1+s}}$ is defined by (3.35). In particular,

$$\langle B^* h, W_t\rangle_{V_0^{-1}(\mathbb{T}^2,\mathbb{R}^2)} = \langle W_t, B^*Qh\rangle_U = M_t^h,$$

and $W = (W^1, W^2)$, where $W^i : \Omega \to C([0,\infty); E), i = 1, 2$ are two independent L_0^2-cylindrical Wiener processes under $\mathbb{P}^z$ for any $z \in E \setminus S$. □

3.4.2 Relation between the two solutions

In the following we discuss the relation between M constructed above and the shifted equation (1.12). In fact, by Lemma 2.1 we have that $\mathcal{C}^{-\alpha} \subset V^{-1-s_0}$ for $\alpha \in (0,1)$, $\mathcal{C}^{-\alpha} \in \mathcal{B}(V^{-1-s_0})$ and $\nu(\mathcal{C}^{-\alpha} \cap E) = 1$. For W constructed in Theorem 3.16 define $Z(t) := \int_0^t e^{-(t-s)A^2/2} B dW_s$.

Theorem 3.17. *Let $\alpha \in (0, \frac{1}{3})$, $\alpha < \beta < 2 - \alpha$. There exists a properly Λ-exceptional set $S_2 \subset E$ in the sense of Theorem 3.16 such that for every $z \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2$ under $\mathbb{P}^z$, $Y := X - Z \in C((0,T];\mathcal{C}^\beta) \cap C([0,T];\mathcal{C}^{-\alpha})$ is a solution to the following equation:*

$$Y(t) = \frac{1}{2}\int_0^t e^{-(t-s)A^2/2} A \sum_{l=0}^{3} C_3^l Y(s)^l : Z(s)^{3-l} : ds + e^{-\frac{t}{2}A^2} X(0). \tag{3.40}$$

Here $C((0,T];\mathcal{C}^\beta)$ is equipped with the norm $\sup_{t\in[0,T]} t^{\frac{\beta+\alpha}{4}} \|\cdot\|_\beta$. Moreover,

$$\mathbb{P}^z[X(t) \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2, \forall t \geq 0] = 1 \text{ for } z \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2.$$

Proof For $z \in E \setminus S$ under $\mathbb{P}^z$ we have that

$$X(t) = \frac{1}{2}\int_0^t e^{-(t-\tau)A^2/2} A : X(\tau)^3 : d\tau + Z(t) + e^{-\frac{t}{2}A^2} X(0).$$

Since ν is an invariant measure for X, by Lemma 2.1 and Lemma 3.2 we conclude that for every $T \geq 0$, $p > 1, \delta > 0$, with $2\delta - \alpha < 0$, and $p_0 > 1$ large enough

$$\begin{aligned}&\int \mathbb{E}^z \int_0^T \| : X(\tau)^3 : \|_{-\alpha}^p d\tau \nu(dz) \lesssim \int \mathbb{E}^z \int_0^T \| : X(\tau)^3 : \|_{B^{\delta-\alpha}_{p_0,p_0}}^p d\tau \nu(dz)\\ =&T \int \| : \phi^3 : \|_{B^{\delta-\alpha}_{p_0,p_0}}^p \nu(d\phi) \lesssim T \int \| : \phi^3 : \|_{2\delta-\alpha}^p \nu(d\phi) < \infty,\end{aligned}$$

which implies that there exists a properly Λ-exceptional set $S_1 \supset S$ such that for $z \in E \setminus S_1$ P^z-a.s.

$$: X(\cdot)^3 :\in L^p(0,T;\mathcal{C}^{-\alpha}), \quad \mathbb{E}^z \int_0^T \| : X(\tau)^3 : \|_{-\alpha}^p d\tau < \infty, \quad \forall p > 1.$$

Here we used Lemma 2.1 to deduce the first result. The second, however, does not imply the first directly because of (2.1). Lemma 2.2 implies that for $\alpha < \beta < 2 - \alpha$

$$\int_0^t e^{-(t-\tau)A^2/2} A : X(\tau)^3 : d\tau \in C([0,\infty);\mathcal{C}^\beta) \quad \mathbb{P}^z - a.s..$$

Now by Lemma 2.2 we conclude that for $z \in \mathcal{C}^{-\alpha} \setminus S_1$, $e^{-\frac{t}{2}A^2} X(0) \in C([0,T],\mathcal{C}^{-\alpha}) \cap C((0,T],\mathcal{C}^\beta)$. Thus,

$$X - Z \in C([0,T],\mathcal{C}^{-\alpha}) \cap C((0,T],\mathcal{C}^\beta) \quad \mathbb{P}^z - a.s..$$

Since $\mathbb{P}^\nu \circ X(t)^{-1} = \nu$, by Lemma 3.6 we conclude that under $\mathbb{P}^\nu$, by Fubini's theorem $Y := X - Z$ satisfies (3.40) and for ν-a.e. $z \in E$ under $\mathbb{P}^z$, $Y := X - Z$ satisfies (3.40).

In the following we prove that these results hold under $\mathbb{P}^z$ for z outside a properly Λ-exceptional set. First we have $Z \in C([0,\infty);\mathcal{C}^{-\alpha})$ $\mathbb{P}^\nu$-a.s., which combined with $X - Z \in C([0,T],\mathcal{C}^{-\alpha})$ implies

$$\mathbb{P}^\nu[X \in C([0,\infty),\mathcal{C}^{-\alpha})] = 1.$$

We also have

$$\begin{aligned}\bar{Y}(s,t_0) :=& X(s+t_0) - Z(s+t_0) = \frac{1}{2}\int_{t_0}^{t_0+s} e^{-(t_0+s-\tau)A^2/2} A : X(\tau)^3 : d\tau \\ &+ e^{-sA^2/2}(X(t_0) - Z(t_0)) \in C((0,\infty)^2;\mathcal{C}^\beta) \quad \mathbb{P}^\nu - a.s..\end{aligned}$$

Similar arguments as in the proof of Lemma 3.6 imply that $\forall s > 0, t_0 \geq 0$

$$\begin{aligned}\mathbb{P}^\nu(: X(s+t_0)^3 :=& \sum_{l=0}^{3} C_3^l \bar{Y}(s,t_0)^l : Z(s+t_0)^{3-l} :, \\ & X \in C([0,\infty),\mathcal{C}^{-\alpha}), \bar{Y} \in C((0,\infty)^2;\mathcal{C}^\beta)) = 1,\end{aligned}$$

In the following we use I_{t,t_0} to denote the equality

$$\begin{aligned}&\int_0^t e^{-(t-s)A^2/2} A : X(s+t_0)^3 : ds \\ =& \sum_{l=0}^{3} \int_0^t e^{-(t-s)A^2/2} A C_3^l \bar{Y}(s,t_0)^l : Z(s+t_0)^{3-l} : ds.\end{aligned}$$

Then using Fubini's theorem we know that

$$\mathbb{P}^\nu(I_{t,t_0} \text{ holds } \forall t \geq 0, a.e. t_0 \geq 0, X \in C([0,\infty);\mathcal{C}^{-\alpha}), \bar{Y} \in C((0,\infty)^2;\mathcal{C}^\beta)) = 1.$$

Here we used $X \in C([0,\infty);\mathcal{C}^{-\alpha})$ for $\alpha < \frac{1}{3}$ to make the right hand side of I_{t,t_0} meaningful. It is obvious that the right hand side of the first equality is continuous with respect to t_0. Since $\int_0^t e^{-(t-s)A^2/2} A : X(s+t_0)^3 : ds = \int_{t_0}^{t+t_0} e^{-(t-s+t_0)A^2/2} A : X(s)^3 : ds$ we know that $\int_0^t e^{-(t-s)A^2/2} A : X(s+t_0)^3 : ds$ is also continuous with respect to t_0 and we obtain that

$$\mathbb{P}^\nu(I_{t,t_0} \text{ holds } \forall t, t_0 \geq 0, X \in C([0,\infty);\mathcal{C}^{-\alpha}), \bar{Y} \in C((0,\infty)^2;\mathcal{C}^\beta)) = 1.$$

This implies that there exists a properly Λ-exceptional set $S_2 \supset S_1$ such that for $z \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2$ under $\mathbb{P}^z$

$$\mathbb{P}^z(X \in C([0,\infty);\mathcal{C}^{-\alpha}), I_{t,t_0} \text{ holds } \forall t, t_0 \geq 0) = 1.$$

Indeed, define

$$\Omega_0 := \{\omega : X \in C([0,\infty);\mathcal{C}^{-\alpha}), : Z^k :\in C((0,\infty);\mathcal{C}^{-\alpha}), k = 1,2,3, I_{t,t_0} \text{ holds } \forall t, t_0 \geq 0\},$$

and let $\Theta_t : \Omega \to \Omega, t > 0$, be the canonical shift, i.e. $\Theta_t(\omega) = \omega(\cdot + t), \omega \in \Omega$. Then it is easy to check that

$$\Theta_t^{-1}\Omega_0 \supset \Omega_0, \quad t \in \mathbb{R}^+,$$

and

$$\Omega_0 = \bigcap_{t>0, t\in\mathbb{Q}} \Theta_t^{-1}\Omega_0.$$

On the other hand, by the Markov property we know that

$$\mathbb{P}^z(\Theta_t^{-1}\Omega_0) = P_t(1_{\Omega_0})(z), \forall z \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2$$

which by [MR92, Chapter IV Theorem 3.5] is Λ-quasi-continuous in the sense of [MR92, Chapter III Definition 3.2] on E. It follows that for every $t > 0$

$$\mathbb{P}^z(\Theta_t^{-1}\Omega_0) = 1 \quad q.e. z \in E,$$

which yields that

$$\mathbb{P}^z(\Omega_0) = 1 \quad q.e. z \in E.$$

Here q.e. means that there exists a properly Λ-exceptional set such that outside this exceptional set the result holds. Now Y satisfies (3.40) $\mathbb{P}^z$-a.s. for $z \in (\mathcal{C}^{-\alpha} \cap E)\backslash S_2$. Moreover, for $z \in (\mathcal{C}^{-\alpha} \cap E)\backslash S_2$ $Y \in C([0,\infty);\mathcal{C}^{-\alpha}) \cap C([0,T],\mathcal{C}^{\beta}), Z \in C([0,\infty);\mathcal{C}^{-\alpha})$ $\mathbb{P}^z$-a.s., which implies that

$$\mathbb{P}^z[X(t) \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2, \forall t \geq 0] = 1 \text{ for } z \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2.$$

□

Corollary 3.18. *Let $\bar{X} = \bar{Y} + Z$ where $\bar{Y}$ is the unique solution to (3.19). ν is an invariant measure of $\bar{X}$.*

Proof By Theorem 3.17 and the uniqueness of the solution to (3.19) we know that $X \overset{d}{=} \bar{X}$, $\mathbb{P}^z - a.s. \forall z \in (\mathcal{C}^{-\alpha} \cap E) \setminus S_2$, which combined with $\nu(\mathcal{C}^{-\alpha} \cap E) = 1$ implies that ν is an invariant measure of $\bar{X}$. □

3.4.3 Markov uniqueness in the restricted sense

In this subsection we prove Markov uniqueness in the restricted sense and the uniqueness of the martingale (probabilistically weak) solutions to (3.1) if the solution has ν as an invariant measure.

By [MR92, Chapter 4, Section 4b] it follows that there is a point separating countable $\mathbb{Q}$-vector space $D \subset \mathcal{F}C_b^\infty$ such that $D \subset D(L(\Lambda))$. Let $\Lambda^{\text{q.r.}}$ be the set of all quasi-regular Dirichlet forms $(\tilde{\Lambda}, D(\tilde{\Lambda}))$ (cf. [MR92]) on $L^2(E;\nu)$ such that $D \subset D(L(\tilde{\Lambda}))$ and $\tilde{\Lambda} = \Lambda$ on $D \times D$. Here for a Dirichlet form $(\tilde{\Lambda}, D(\tilde{\Lambda}))$ we denote its generator by $(L(\tilde{\Lambda}), D(L(\tilde{\Lambda})))$.

In the following we consider the martingale problem in the sense of [AR94] and probabilistically weak solutions to (3.1):

Definition 3.19. *(i) A ν-special standard process $M = (\Omega, \mathcal{F}, (\mathcal{M}_t), X_t, (\mathbb{P}^z))$ in the sense of [MR92, Chapter IV] with state space E is said to solve the martingale problem for $(L(\Lambda), D)$ if for all $u \in D$, $u(X(t)) - u(X(0)) - \int_0^t L(\Lambda)u(X(s))ds$, $t \geq 0$, is an $(\mathcal{M}_t)$-martingale under $\mathbb{P}^\nu$.*

(ii) A ν-special standard process $M = (\Omega, \mathcal{F}, (\mathcal{M}_t), X_t, (P^z))$ with state space E is called a probabilistically weak solution to (3.1) if there exists two map $W^i : \Omega \to C([0,\infty); E)$ $i = 1,2$ such that for ν-a.e. z under $\mathbb{P}^z$, $W := (W^1, W^2)$ is an $L_0^2(\mathbb{T}^2, \mathbb{R}^2)$-cylindrical Wiener process with respect to $(\mathcal{M}_t)$ and the sample paths of the associated process satisfy (3.39) for all $h \in V^{3+s_0}$.

Remark 3.20. *If M is a probabilistically weak solution to (3.1), we can easily check that it also solves the martingale problem. Conversely, if M solves the martingale problem, then with the same argument in Theorem 3.16, there exists an $L^2_0(\mathbb{T}^2,\mathbb{R}^2)$-cylindrical Wiener process W such that (X,W) satisfies (3.39) for $h\in V^{3+s_0}$. That is to say, these two definitions are equivalent.*

To explain the uniqueness result below we also introduce the following concept:

Two strong Markov processes M and M' with state space E and transition semigroups $(p_t)_{t>0}$ and $(p'_t)_{t>0}$ are called *ν-equivalent* if there exists $S\in\mathcal{B}(E)$ such that (i) $\nu(E\backslash S)=0$, (ii) $\mathbb{P}^z[X(t)\in S,\forall t\geq 0]=\mathbb{P}'^z[X'(t)\in S,\forall t\geq 0]=1, z\in S$, (iii) $p_tf(z)=p'_tf(z)$ for all $f\in\mathcal{B}_b(E),t>0$ and $z\in S$.

Combining Theorem 3.9 and Theorem 3.10, we obtain Markov uniqueness in the restricted sense for $(L(\Lambda),D)$ (see part (iii)) and the uniqueness of martingale (probabilistically weak) solutions to (1.1) if the solution has ν as an invariant measure (see part (i), (ii)):

Theorem 3.21. *(i) There exists (up to ν-equivalence) exactly one probabilistically weak solution M to (3.1) satisfying $\mathbb{P}^z(X\in C([0,\infty);E))=1$ for ν-a.e. and having ν as an invariant measure, i.e. for the transition semigroup $(p_t)_{t\geq 0}$, $\int p_tfd\nu=\int fd\nu$ for $f\in L^2(E;\nu)$.*

(ii) There exists (up to ν-equivalence) exactly one ν-special standard process M with state space E solving the martingale problem for $(L(\Lambda),D)$ and satisfying $\mathbb{P}^z(X\in C([0,\infty);E))=1$ for ν-a.e. and having ν as an invariant measure.

(iii) $\sharp\Lambda^{q.r.}=1$. Moreover, there exists (up to ν-equivalence) exactly one ν-special standard process M with state space E associated with a Dirichlet form $(\Lambda,D(\Lambda))$ solving the martingale problem for $(L(\Lambda),D)$.

Proof The proof is the same as [RZZ17b, Theorem 3.12].

For (i), suppose that M^1 is a probabilistically weak solution to (3.1) and let p^1_t be the transition semigroup (of sub-probability kernels) associated with M^1. Since ν is an invariant measure and

$$p^1_tf\to^{t\to 0}f,$$

for $f\in\mathcal{F}C^\infty_b$, by [MR92, Chapter II, Subsection 4a] $(p^1_t)_{t>0}$ uniquely determines a strongly continuous contraction semigroup $(T^1_t)_{t>0}$ of operators on $L^2(E;\nu)$. By the proof of Theorem 3.17 we know that the solution to (3.39) having ν as an invariant measure minus Z also satisfies (3.40) under $\mathbb{P}^\nu$. Moreover, by the pathwise uniqueness of solutions to (3.40) we obtain that $p^1_tf(z)=P_tf(z)$ ν-a.e. for all $f\in\mathcal{B}_b(E),t>0$, which implies that p^1_t is associated with the Dirichlet form $(\Lambda,D(\Lambda))$ obtained in Section 3.2. Here P_t is the semigroup properly associated with $(\Lambda,D(\Lambda))$ obtained in Section 3.2. Since M^1 is a ν-special standard process and has continuous paths, by [MR92, Chapter 4, Theorem 1.15, Theorem 5.1] M^1 is properly associated with $(\Lambda,D(\Lambda))$. Then by [MR92, Chapter 4, Theorem 6.4] M^1 is ν-equivalent to M obtained in Section 3.2, which implies (i) easily.

(ii) follows from the first result and the above Remark.

The second result in (iii) follows from the first result and [AR95, Theorem 3.4]. We only prove the first. Since for every $(\tilde{\Lambda},D(\tilde{\Lambda}))\in\Lambda^{\text{q.r.}}$ there exists a unique Markov process $\tilde{M}$ associated with $(\tilde{\Lambda},D(\tilde{\Lambda}))$ and Theorem 3.16 holds for $\tilde{M}$, by Theorems 3.17 and 3.10 we know that for the semigroup $\tilde{p}_t$ associated with $\tilde{M}$ we have $\tilde{p}_tf=P_tf$ ν-a.e.

for $f \in \mathcal{B}_b(E)$, which implies that $\tilde{p}_t$ is a ν-version of the semigroup T_t associated with $(\Lambda, D(\Lambda))$. Then by [MR92, Chapter I] we know that $(\Lambda, D(\Lambda)) = (\tilde{\Lambda}, D(\tilde{\Lambda}))$. Now (iii) follows.

□

3.4.4 Stationary solution

Now we consider the stationary case. In this case, we can obtain a probabilistically strong solution to 3.1. Take two different stationary solutions X_1, X_2 to 3.1 with the same initial condition $\eta \in \mathcal{C}^{-\alpha} \cap E$, $\alpha > 0$, α small enough, having the distribution ν. We have

$$X_i(t) = e^{-\frac{t}{2}A^2}\eta + \frac{1}{2}\int_0^t e^{-\frac{t-\tau}{2}A^2} A : X_i(\tau)^3 : d\tau + Z(t),$$

where Z is the stochastic convolution

$$Z(t) = \int_0^t e^{-\frac{t-s}{2}A^2} B dW_s.$$

By a similar argument as in the proof of Theorem 3.17 and using Lemma 3.2 we have that for every $p > 1$

$$\mathbb{E}\int_0^T \| : X_i(\tau)^3 : \|_{-\alpha}^p d\tau = T \int \| : \phi^3 : \|_{-\alpha}^p \nu(d\phi) < \infty.$$

Then Lemma 2.2 implies that for $\alpha > 0$, $\alpha < \beta < 2 - \alpha$

$$\int_0^t e^{-\frac{t-\tau}{2}A^2} A : X_i(\tau)^3 : d\tau \in C([0,T]; \mathcal{C}^\beta) \quad \mathbb{P} - a.s..$$

Thus by Lemma 2.2 we conclude that

$$X_i - Z \in C((0,T]; \mathcal{C}^\beta) \quad \mathbb{P} - a.s.,$$

where $C((0,T]; \mathcal{C}^\beta)$ is equipped with the norm $\sup_{t\in[0,T]} t^{\frac{\beta+\alpha}{4}} \| \cdot \|_\beta$. Moreover, similar arguments as in the proof of Theorem 3.6 yield that if $\alpha > 0$ with α small enough, $X_i - Z$ is a solution to the following equation

$$Y(t) = \frac{1}{2}\int_0^t e^{-(t-s)A^2/2} A \sum_{l=0}^{3} C_3^l Y(s)^l : Z(s)^{3-l} : ds + e^{-\frac{t}{2}A^2}\eta. \qquad (3.41)$$

Here the Wick powers of Z are defined as in Lemma 3.4.

Now by [LR15, Proposition G.0.5] we know the solutions to equation (3.41) are also the solutions to (3.19) and by uniqueness of the solutions to (3.19) in Theorem 3.10, this implies that

$$X_1 - Z = X_2 - Z \text{ on } [0,T] \quad \mathbb{P} - a.s..$$

Then the pathwise uniqueness holds for the stationary solutions to (3.1). Now by the existence of the stationary martingale solution (cf. [MR99]) and the Yamada-Watanabe Theorem in [Kur07] we obtain:

Theorem 3.22. *For any initial condition $X(0) \in \mathcal{C}^{-\alpha} \cap E$ with distribution ν and $\alpha > 0$, α small enough, there exists a unique probabilistically strong solution X to (3.1) such that X is a stationary process, i.e. for every probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ with a U-Wiener process W, there exists an $\mathcal{F}_t$-adapted stationary process $X : [0,T] \times \Omega \to E$ such that for $\mathbb{P} - a.s.$ $\omega \in \Omega$, X satisfies (3.1). Moreover, for $0 < \beta < 2 - \alpha$*

$$X - Z \in C((0,T]; \mathcal{C}^\beta) \quad \mathbb{P} - a.s..$$

3.5 Ergodicity

Let $X = Y + Z$ where Y is the solution to equation (3.19). By the uniqueness of the solution Y we have that X is a Markov process. Let P_t be the semigroup of X, i.e

$$P_t\Phi(x) = \mathbb{E}\Phi\left(X(t,x)\right), \quad \forall \Phi \in C_b(V_0^{-1}).$$

We recall that the U-cylindrical Wiener process W take values in $C([0,T], V_0^{-1-s_0}(\mathbb{T}^2,\mathbb{R}^2))$, $\mathbb{P}-a.s.$, for any $s_0 > 0$. Let $\mathcal{D}$ denote the Fréchet derivative of functions on $C([0,T], V_0^{-1-s_0}(\mathbb{T}^2,\mathbb{R}^2))$ (i.e. with respect to the noise). We also denote the Cameron-Maritin space by $\mathcal{CM} := \{\omega : \partial_t\omega \in L^2([0,T], L_0^2(\mathbb{T}^2;\mathbb{R}^2)), \omega(0) = (0,0)\}$. Here we view $\partial_t\omega$ as a function on $[0,T]\times\mathbb{T}^2$ rather than lying in the tangent space of $\mathbb{T}^2$.

Proposition 3.23. *For a fixed $x \in V_0^{-1}$, let $\mathfrak{X}_t^x := X(t,x) = Z_t + Y(t,x)$ be a map from $C([0,T], V_0^{-1-s_0})$ to V_0^{-1}. For any $\omega \in \mathcal{CM}$ its directional derivative $\mathcal{D}\mathfrak{X}_t^x(\omega)$ is given in mild form as*

$$\mathcal{D}\mathfrak{X}_t^x(\omega) = \frac{1}{2}\int_0^t e^{-(t-s)A^2/2}A\sum_{l=0}^{2} 3C_2^l Y^{2-l}(s) : Z_s^l : \mathcal{D}\mathfrak{X}_s^x(\omega)ds + \int_0^t e^{-(t-s)A^2/2}Bd\omega_s. \tag{3.42}$$

The proof of Proposition 3.23 can be obtained by using approximation or the implicit function theorem (see [Dri03, Theorem 19.28], [HM18], [TW16]).

Let D denote the Fréchet derivative of functions on V_0^{-1}. We also consider the following equation:

$$\begin{cases} \partial_t J_{s,t}h = -\dfrac{1}{2}A^2 J_{s,t}h + \dfrac{1}{2}A\left(\displaystyle\sum_{l=0}^{2} 3C_2^l Y^{2-l}(t) : Z_t^l : J_{s,t}h\right) \\ J_{s,s}h = h \in V_0^{-1} \end{cases}. \tag{3.43}$$

Then $J_{0,t}h = DX(t,x)(h)$, i.e. it is the derivative of $X(t,\cdot)$ in the direction h. For $\omega \in \mathcal{CM}$, by Duhamel's principle

$$\mathcal{D}\mathfrak{X}_t^x(\omega) = \int_0^t J_{s,t}B\partial_s\omega(s)ds. \tag{3.44}$$

We define the stopping time

$$\tau^r := \inf\{t \in (0,T) : t^\rho || : Z_t^k : ||_{-\alpha} > r, k = 1,2,3\}, \tag{3.45}$$

where $\rho > 0$ is a small enough constant introduced in Lemma 3.4.

Proposition 3.24. *For any $x \in V_0^{-1}$ with $\|x\|_{H^{-1}} \le R$, there exists constants $C_1(R), C_2(R)$ such that for all $t \le \tau^r$*

$$\sup_{s\le t} ||Y_s||_{H^{-1}} \vee \int_0^t ||Y_s||_{L^4}^4 ds \vee \int_0^t ||Y_s||_{H^1}^2 ds \le C_1 \quad \textit{and} \quad \sup_{s\le t} ||J_{0,s}h||_{H^{-1}} \le C_2 ||h||_{H^{-1}}$$

Proof The first bound with constant C_1 follows from the proof of Theorem 3.7.

For the second bound, we note that $J_{0,t}h$ satisfies the following equation:

$$\begin{cases} \dfrac{du}{dt} = -\dfrac{1}{2}A^2u + \dfrac{1}{2}A\left(\displaystyle\sum_{l=0}^{2} 3C_2^l Y^{2-l}(t) : Z_t^l : u\right) \\ u(0) = h \end{cases}.$$

Taking scalar product with $(-A)^{-1}u$, we obtain that

$$\frac{d}{dt}||u||_{H^{-1}}^2 + ||u||_{H^1}^2 = -3\langle Y^2 + 2YZ + : Z^2 :, u^2\rangle,$$

that is

$$\frac{d}{dt}||u||_{H^{-1}}^2 + ||u||_{H^1}^2 \leq 6|\langle YZ, u^2\rangle| + 3|\langle : Z^2 :, u^2\rangle|.$$

Following the same argument that we used to estimate (3.31) and using the first bound, we use Grönwall's inequality to obtain the second bound.

□

Let $\chi_r \in C^\infty(\mathbb{R})$ such that $\chi_r(\zeta) \in [0,1]$ for all $\zeta \in \mathbb{R}$, and

$$\chi_r(\zeta) = \begin{cases} 1, |\zeta| \leq \dfrac{r}{2} \\ 0, |\zeta| \geq r \end{cases}.$$

Following the notation in [TW16], we set

$$C^{3,-\alpha}(0,T) := C([0,T]; \mathcal{C}^{-\alpha}) \times C((0,T]; \mathcal{C}^{-\alpha})^2, \tag{3.46}$$

and $\underline{Z} := (Z, : Z^2 :, : Z^3 :) \in C^{3,-\alpha}(0,T)$. We also define

$$|||\underline{Z}|||_t := \max_{k=1,2,3} \left\{ \sup_{0\leq s\leq t} s^\rho || : Z_s^k : ||_{-\alpha} \right\}.$$

Theorem 3.25. *(Bismut-Elworthy-Li Formula) Let $x \in V_0^{-1}$, $\Phi \in C_b^1(V_0^{-1})$ and ω be a process taking values in the Cameron-Martin space $\mathcal{CM}$ with $\partial_s\omega$ adapted. Assume that there exists a deterministic constant $C \equiv C(t)$ such that $||\partial_s\omega||_{L^2(0,t;U)} \leq C$ $\mathbb{P}-a.s.$. Then we have*

$$\begin{aligned} \mathbb{E}[D\Phi(\mathfrak{X}_t^x)(\mathcal{D}\mathfrak{X}_t^x(\omega))\chi_r(|||\underline{Z}|||_t)] = {} & \mathbb{E}\left(\Phi(\mathfrak{X}_t^x)\chi_r(|||\underline{Z}|||_t)\int_0^t \langle\partial_s\omega(s), dW_s\rangle\right), \\ & - \mathbb{E}\left(\Phi(\mathfrak{X}_t^x)\partial_+\chi_r(|||\underline{Z}|||_t)(\omega)\right) \end{aligned} \tag{3.47}$$

where

$$\begin{aligned} \partial_+\chi_r(|||\underline{Z}|||_t)(\omega) &= \partial_\zeta\chi_r(|||\underline{Z}|||_t)\partial_+|||\underline{Z}|||_t(\underline{Y}), \\ \partial_+|||\underline{Z}|||_t(\underline{Y}) &= \lim_{\delta\to 0^+} \frac{|||\underline{Z}+\delta\underline{Y}|||_t - |||\underline{Z}|||_t}{\delta}, \end{aligned} \tag{3.48}$$

$\underline{Y} = (Q_\omega(\cdot), 2ZQ_\omega(\cdot), 3 : Z^2 : Q_\omega(\cdot)) \in C^{3,-\alpha}(0,t)$ *and*

$$Q_\omega(\cdot) := \int_0^{\cdot} e^{-(\cdot-s)A^2/2}B\partial_s\omega(s)ds.$$

Proof This can be proved by the same calculation as that in the proof of [TW16, Theorem 5.4].

Let $\delta > 0$ and $u = \partial_t \omega \in L^2(0, t; U)$. For every $n \geq 1$, we define the shift $T_{\delta u}$ by

$$T_{\delta u} : Z_t^n ::= \sum_{k=0}^{n} (\delta Q_\omega(t))^{n-k} : Z_t^k :$$

and we let $T_{\delta u}\underline{Z} = \left(T_{\delta u} : Z^k :\right)_{k=1}^3$.

Let $X^\delta(t, x) = T_{\delta u} Z_t + Y^\delta$, where Y^δ solves the equation

$$\begin{cases} \partial_t Y^\delta = -\frac{1}{2} A^2 Y^\delta + \frac{1}{2} A \left(\sum_{k=0}^{3} C_3^k (Y^\delta)^{3-k} T_{\delta u} : Z^k : \right) \\ Y^\delta(0) = x \end{cases}.$$

We follow the idea in [Nor86] and [TW16] to construct a probability measure $\mathbb{P}^\delta$ such that the law of $T_{\delta u} Z$ under $\mathbb{P}^\delta$ is the same as the law of Z under $\mathbb{P}$. Then we can obtain the identity

$$\partial_{\delta+} \mathbb{E}^{\mathbb{P}^\delta} \left(\Phi(X^\delta(t, x)) \chi_r(|\!|\!| T_{\delta u} \underline{Z} |\!|\!|_t) \right) \Big|_{\delta=0} = 0. \tag{3.49}$$

To construct $\mathbb{P}^\delta$, let $v^\delta := -\delta \int_0^t \langle u(s), dW_s \rangle$ and define

$$H^\delta(r) := \exp\{v^\delta(r) - \frac{\delta^2}{2} \int_0^r \|u(s)\|_U^2 ds\}.$$

By our assumptions on ω, the Novikov's condition is satisfied, i.e.

$$\mathbb{E} \exp(\frac{\delta^2}{2} \int_0^t \|u\|_U^2 ds) < \infty,$$

thus by [DPZ14, Theorem 10.14, Proposition 10.17], $W^\delta(r) := W(r) + \delta \int_0^r u(s) ds$ is a U-cylindrical Wiener process under $\mathbb{P}^\delta$, where $d\mathbb{P}^\delta := H^\delta d\mathbb{P}$. Moreover we can obtain that

$$T_{\delta u} Z_r = \int_0^r e^{-(r-s)A^2/2} B dW_s^\delta.$$

Then (3.49) follows.

Using the chain rule, $\partial_\delta \Phi\left(X^\delta(t, x)\right) = D\Phi\left(X^\delta(t, x)\right)\left(\partial_\delta X^\delta(t, x)\right)$ and

$$\partial_\delta H^\delta(t) = -H^\delta(t) \left(\int_0^t \langle u(s), dW_s \rangle_U + \delta \int_0^t \|u(s)\|_U^2 ds \right).$$

For $\partial_{\delta+}\chi(|\!|\!| T_{\delta u}\underline{Z} |\!|\!|_t)$, note that $T_{\delta u}\underline{Z} - (\underline{Z} + \delta \underline{Y}) = (0, \delta^2 Q_\omega^2, 3\delta^2 Z Q_\omega^2 + \delta^3 Q_\omega^3)$, then we get

$$\lim_{\delta \to 0^+} \frac{|\!|\!| T_{\delta u}\underline{Z} |\!|\!|_t - |\!|\!| \underline{Z} |\!|\!|_t}{\delta} = \lim_{\delta \to 0^+} \frac{|\!|\!| \underline{Z} + \delta \underline{Y} |\!|\!|_t - |\!|\!| \underline{Z} |\!|\!|_t}{\delta},$$

that is

$$\partial_{\delta+} \chi_r(|\!|\!| T_{\delta u}\underline{Z} |\!|\!|_t) \Big|_{\delta=0} = \partial_+ \chi_r(|\!|\!| \underline{Z} |\!|\!|_t)(\omega). \tag{3.50}$$

Using the bounds in Proposition 3.24, by the dominated convergence theorem we can pass the derivative inside the expectation in (3.49) and integrate by parts to obtain

$$\begin{aligned}&\mathbb{E}\left(D\Phi\left(X^\delta(t,x)\right)\left(\partial_\delta X^\delta(t,x)\right)\chi_r(\|\!|T_{\delta u}\underline{Z}|\!\|_t)H^\delta(t)\right)\Big|_{\delta=0}=\\&-\mathbb{E}\left(\Phi\left(X^\delta(t,x)\right)\chi_r(\|\!|T_{\delta u}\underline{Z}|\!\|_t)\partial_\delta H^\delta(t)\right)\Big|_{\delta=0}\\&-\mathbb{E}\left(\Phi\left(X^\delta(t,x)\right)\partial_{\delta+}\chi_r(\|\!|T_{\delta u}\underline{Z}|\!\|_t)(\omega)H^\delta(t)\right)\Big|_{\delta=0}\end{aligned}.$$

Since $\partial_\delta X^\delta(t,x)\Big|_{\delta=0}=\mathcal{D}\mathfrak{X}_t^x(\omega)$ and $\partial_\delta H^\delta(t)\Big|_{\delta=0}=-\int_0^t\langle u(s),dW_s\rangle$, combining with (3.50) we get (3.47) which completes the proof.

□

We use (3.47) to prove the following proposition.

Proposition 3.26. *There exists universal constants $\theta_1>0$ such that for every $T>0$, $x\in V_0^{-1}$ with $\|x\|_{H^{-1}}\le R$, there exists a constant $C\equiv C(T,R)>0$ satisfying*

$$|P_t\Phi(x)-P_t\Phi(y)|\le C(T,R)\frac{1}{t^{\theta_1}}\|\Phi\|_\infty\|x-y\|_{H^{-1}}+2\|\Phi\|_\infty\mathbb{P}(t\ge\tau^{\frac{r}{2}})\qquad(3.51)$$

for every $y\in V_0^{-1}$, $\|x-y\|_{H^{-1}}\le 1$, $\Phi\in C_b^1(V_0^{-1})$ and $t\in[0,T]$.

Proof Let $\Phi\in C_b^1(V_0^{-1})$. Then

$$|P_t\Phi(x)-P_t\Phi(y)|=|\mathbb{E}\left[\Phi\left(X(t,x)\right)-\Phi\left(X(t,y)\right)\right]|\le I_1+I_2,$$

where

$$\begin{aligned}I_1&:=|\mathbb{E}\left[\Phi\left(X(t,x)\right)-\Phi\left(X(t,y)\right)\chi_r(\|\!|\underline{Z}|\!\|_t)\right]|\\I_2&:=|\mathbb{E}\left[\Phi\left(X(t,x)\right)-\Phi\left(X(t,y)\right)(1-\chi_r(\|\!|\underline{Z}|\!\|_t))\right]|.\end{aligned}$$

For the second term we have that $I_2\le 2\|\Phi\|_\infty\mathbb{P}(t\ge\tau^{\frac{r}{2}})$. By the mean value theorem we get that

$$\begin{aligned}I_1&=\left|\mathbb{E}\left(\int_0^1 D\Phi\left(\mathfrak{X}_t^{z_\lambda}(y-x)\right)d\lambda\cdot\chi_r(\|\!|\underline{Z}|\!\|_t)\right)\right|\\&=\left|\int_0^1\mathbb{E}\left(D\Phi\left(\mathfrak{X}_t^{z_\lambda}\right)(y-x)\chi_r(\|\!|\underline{Z}|\!\|_t)\right)d\lambda\right|,\end{aligned}$$

where $z_\lambda:=x+\lambda(y-x)$. For any $h\in V_0^{-1}$, let ω be such that $B\partial_s\omega(s)=J_{0,s}h$ for $s\le\tau^r$ and 0 otherwise. Then $\partial_s\omega(s)$ satisfies the condition in Theorem 3.25. Furthermore, by (3.44) and $J_{0,s}J_{s,t}=J_{0,t}$ we have $\mathcal{D}\mathfrak{X}_t^{z_\lambda}(\omega)=tD\mathfrak{X}_t^{z_\lambda}(h)$. Then we can use (3.47) to obtain that

$$\begin{aligned}\mathbb{E}\left(D\left[\Phi(\mathfrak{X}_t^{z_\lambda})\right](h)\chi_r(\|\!|\underline{Z}|\!\|_t)\right)&=\frac{1}{t}\mathbb{E}\left(\Phi(\mathfrak{X}_t^{z_\lambda})\int_0^t\langle\partial_s\omega(s),dW_s\rangle\chi_r(\|\!|\underline{Z}|\!\|_t)\right)\\&-\frac{1}{t}\mathbb{E}\left(\Phi(\mathfrak{X}_t^{z_\lambda})\partial_+\chi_r(\|\!|\underline{Z}|\!\|_t)(\omega)\right).\end{aligned}$$

Then we have

$$I_1\le\frac{1}{t}\|\Phi\|_\infty\int_0^1\mathbb{E}\Big|\int_0^t\langle\partial_s\omega(s),dW_s\rangle\chi(\|\!|\underline{Z}|\!\|_t)\Big|d\lambda+\frac{1}{t}\|\Phi\|_\infty\int_0^1\mathbb{E}\Big|\partial_+\chi_r(\|\!|\underline{Z}|\!\|_t)(\omega)\Big|d\lambda.$$

For the first term we have

$$\begin{aligned}\mathbb{E}\Big|\int_0^t \langle \partial_s\omega(s), dW_s\rangle \chi(|||\underline{Z}|||_t)\Big| &\leq \mathbb{E}\Big|\int_0^{t\wedge\tau^r} \langle \partial_s\omega(s), dW_s\rangle\Big| \\ &\leq \left(\int_0^{t\wedge\tau^r} \|\partial_s\omega(s)\|_U^2 ds\right)^{\frac{1}{2}} \\ &\lesssim \left(\int_0^{t\wedge\tau^r} \|J_{0,s}h\|_{H^{-1}}^2 ds\right)^{\frac{1}{2}} \\ &\leq C_2 t\|h\|_{H^{-1}},\end{aligned}$$

where we used the Cauchy-Schwartz inequality and Itô's isometry in the second step and Proposition 3.24 in the last step.

By the definition of $\partial_+\chi_r(|||\underline{Z}|||_t)(\omega)$, we have that for any $\beta > \alpha > 0$,

$$\Big|\partial_+\chi_r(|||\underline{Z}|||_t)(\omega)\Big| \leq \partial_+|||\underline{Z}|||_t(\underline{Y}) \leq |||\underline{Y}|||_t \lesssim |||\underline{Z}|||_t\|Q_\omega(t)\|_\beta,$$

where $\underline{Y}$ is as introduced in Theorem 3.25 and we used Lemma 2.3 in the last inequality. Moreover, we use Lemma 2.2 and Lemma 2.5 to obtain

$$\|Q_\omega(t)\|_\beta \lesssim \int_0^t (t-s)^{-\frac{\beta+2}{4}}\|J_{0,s}h\|_{-2}ds \lesssim \int_0^t (t-s)^{-\frac{\beta+2}{4}}\|J_{0,s}h\|_{H^{-1}}ds \lesssim C_2 t^{\frac{2-\beta}{4}}\|h\|_{H^{-1}}.$$

Choosing β small enough, we deduce that there exists a constant $\theta_1 \in (0, \frac{1}{2})$, such that

$$I_1 \lesssim C_2\frac{1}{t^{\theta_1}}\|\Phi\|_\infty\|h\|_{H^{-1}}.$$

Letting $h = y - x$ we finish the proof.

□

We denote by $\|\mu_1 - \mu_2\|_{TV}$ the total variation distance of two probability measures μ_1, μ_2 on V_0^{-1} given by

$$\|\mu_1 - \mu_2\|_{TV} := \sup_{\|\Phi\|_{L^\infty}\leq 1}\Big|\int \Phi d\mu_1 - \int \Phi d\mu_2\Big|.$$

Theorem 3.27. *There exists $\theta \in (0,1)$ such that for any $x, y \in V_0^{-1}$ with $\|x\|_{H^{-1}} \leq R$ and $\|x-y\|_{H^{-1}} \leq 1$ there exists a constant $C(R) > 0$ satisfying*

$$\|P_t(x,\cdot) - P_t(y,\cdot)\|_{TV} \leq C(R)\|x-y\|_{H^{-1}}^\theta,$$

for every $t \geq 1$.

Proof The proof is similar as that of [TW16, Theorem 5.8].

First we fix $T > 0$, thus the constant in Proposition 3.26 does not rely on T. Without loss of generality, we assume $T \leq 1$.

By [DPZ96, Section 7.1], (3.51) is equivalent to

$$\|P_t(x) - P_t(y)\|_{\mathrm{TV}} \leq C\frac{1}{t^{\theta_1}}\|x-y\|_{H^{-1}} + 2\mathbb{P}\left(t \geq \tau^{\frac{r}{2}}\right),$$

for every $t \leq T$ and $\|x-y\|_{H^{-1}} \leq 1$.

Recall the proof of Lemma 3.3 and 3.4, we can obtain that for any $\alpha \in (0,1)$, $p>2$, there exists a small $\theta_2 \equiv \theta_2(\alpha) \in (0,1)$ and a constant $C \equiv C(T,\alpha,p) \equiv C(\alpha,p)$ (T is fixed now), such that for any $\rho \in (0,1)$, and any $n \in \mathbb{N}$

$$\mathbb{E} \sup_{s,t\in[0,T]} \frac{\| : Z^n(t) : - : Z^n(s) : \|_{-\alpha}^p}{t^{p\rho} s^{p\rho} |t-s|^{p\theta_2}} \leq C.$$

Hence we obtain that

$$\mathbb{P}\left(t > \tau^{\frac{\tau}{2}}\right) \leq \mathbb{P}\left(\|\underline{Z}\|_t > \frac{r}{2}\right) \lesssim \frac{t^{\theta_2}}{r}.$$

By the semigroup property and contractivity, we have that

$$\|P_1(x) - P_1(y)\|_{\mathrm{TV}} \leq \|P_T(x) - P_T(y)\|_{\mathrm{TV}},$$

where

$$\|P_T(x) - P_T(y)\|_{\mathrm{TV}} \leq \inf_{t\leq T} \left\{ C_1 \frac{1}{t^{\theta_1}} \|x-y\|_{H^{-1}} + C_2 \frac{1}{r} t^{\theta_2} \right\}.$$

Let $g(t) := C_1 \frac{1}{t^{\theta_1}} \|x-y\|_{H^{-1}} + C_2 \frac{1}{r} t^{\theta_2}, t>0$ and note that for $t_0 = \left(\frac{\theta_1 C_1}{\theta_2 C_2}\right)^{\frac{1}{\theta_1+\theta_2}}$, $g(t_0) = \inf_{t>0} g(t)$. If $t_0 \leq T$, then there exists a constant $C \equiv C(\theta_1,\theta_2,r)$ such that

$$\|P_T(x) - P_T(y)\|_{\mathrm{TV}} \leq g(t_0) = C\|x-y\|_{H^{-1}}^{\theta_2}.$$

Otherwise $t_0 > T$, which implies that

$$\begin{aligned}
\|P_T(x) - P_T(y)\|_{\mathrm{TV}} &\leq C_1 \frac{1}{T^{\theta_1}} \|x-y\|_{H^{-1}} + C_2 \frac{1}{r} T^{\theta_2} \\
&\leq C_1 \frac{1}{T^{\theta_1}} \|x-y\|_{H^{-1}} + C_2 \frac{1}{r} t_0^{\theta_2} \\
&= C_1 \frac{1}{T^{\theta_1}} \|x-y\|_{H^{-1}} + \tilde{C}_2 \frac{1}{r} \|x-y\|_{H^{-1}}^{\frac{\theta_2}{\theta_1+\theta_2}} \\
&\leq C(T,R,\theta_1,\theta_2,r) \|x-y\|_{H^{-1}}^{\frac{\theta_2}{\theta_1+\theta_2}}
\end{aligned}$$

for a constant $C \equiv C(T,R,\theta_1,\theta_2,r) \equiv C(R,\theta_1,\theta_2,r)$. Combining all the estimates above we deduce that

$$\|P_1(x) - P_1(y)\|_{\mathrm{TV}} \leq C(R) \|x-y\|_{H^{-1}}^{\frac{\theta_2}{\theta_1+\theta_2}},$$

which completes the proof.

□

In order to use Krylov-Bogoliubov method to prove the existence of an invariant measure, the H^{-1} uniform estimate is not enough. We need to find a space compactly embedded in H^{-1} where the solution is bounded in probability. We make use of the integrability on a smaller space, which is compactly embedded in H^{-1}. Thus we have

Theorem 3.28. *For every $x \in V_0^{-1}$, there exists a probability Borel measure ν_x on V_0^{-1} such that ν_x is an invariant measure for the semigroup $\{P_t, t \geq 0\}$ on V_0^{-1}.*

Proof By (3.23) and a similar argument as in the proof of [TW16, Corollary 3.10] we have that

$$\sup_{x\in V_0^{-1}}\sup_{t>0}(t\wedge 1)\mathbb{E}\|X(t,x)\|_{H^{-1}}^2<\infty. \tag{3.52}$$

By the uniqueness of the solution, we know $X(t,x)=Z_{t-1,t}+Y_{t-1,t}$, where $Z_{s,t}:=\int_s^t e^{-(t-r)A^2/2}BdW_r$ and $Y_{s,r}$, $r\geq t-1$ solves the equation

$$\begin{cases}\dfrac{dY_{s,r}}{dr}=-\dfrac{1}{2}A^2Y_{s,r}+\dfrac{1}{2}A\displaystyle\sum_{k=0}^{3}C_3^kY_{s,r}^{3-k}:Z_{s,r}{}^k:,\\ \quad Y_{s,s}=X(s,x).\end{cases} \tag{3.53}$$

Applying Theorem 3.7 with $Y_{t,r}$ replacing Y_r we have

$$\mathbb{E}\int_t^{t+1}\|Y_{t,r}\|_{H^1}^2dr\lesssim 1+\mathbb{E}\|Y_{t,t}\|_{H^{-1}}^2=1+\mathbb{E}\|X(t,x)\|_{H^{-1}}^2.$$

Combining this with (3.52) we deduce that for $\alpha\in(0,1)$,

$$\mathbb{E}\int_t^{t+1}\|X(s,x)\|_{\mathcal{C}^{-\alpha}}^2ds\leq\mathbb{E}\int_t^{t+1}\|Y_{t,s}\|_{H^1}^2ds+\mathbb{E}\int_t^{t+1}\|Z_{t,s}\|_{\mathcal{C}^{-\alpha}}^2ds\lesssim 1+\frac{1}{1\wedge t},$$

where we used a similar argument as in the proof of [TW16, Theorem 2.1] in the last inequality. Then we obtain that for $t\geq 1$

$$\mathbb{E}\int_1^t\|X(s,x)\|_{\mathcal{C}^{-\alpha}}^2ds\lesssim t.$$

Moreover, by (3.22) we have

$$\mathbb{E}\int_0^1\|Y_s\|_{H^1}^2ds\lesssim 1+\|x\|_{H^{-1}}^2.$$

Thus for $t\geq 1$

$$\int_0^t\mathbb{E}\|X(s,x)\|_{\mathcal{C}^{-\alpha}}^2ds\leq\int_0^1\mathbb{E}\|X(s,x)\|_{\mathcal{C}^{-\alpha}}^2ds+\int_1^t\mathbb{E}\|X(s,x)\|_{\mathcal{C}^{-\alpha}}^2ds\lesssim 1+\|x\|_{H^{-1}}^2+t.$$

By Chebyshev's inequality, for any $K>0$

$$\mathbb{P}(\|X(t,x)\|_{\mathcal{C}^{-\alpha}}>K)\leq\frac{1}{K^2}\mathbb{E}\|X(t,x)\|_{\mathcal{C}^{-\alpha}}.$$

Thus there exists a constant $C>0$, such that

$$\begin{aligned}\int_0^t\mathbb{P}(\|X(s,x)\|_{\mathcal{C}^{-\alpha}}>K)ds&\leq\frac{C}{K^2}\int_0^t\mathbb{E}\|X(s,x)\|_{\mathcal{C}^{-\alpha}}^2ds\\&\leq\frac{C}{K^2}(1+\|x\|_{H^{-1}}^2+t).\end{aligned}$$

Letting $R_t(x,\cdot)=\frac{1}{t}\int_0^tP_s(x,\cdot)ds$, for $K_\varepsilon^2:=\frac{C}{\varepsilon}$ we get

$$R_t(f\in\mathcal{C}^{-\alpha}\cap V_0^1:\|f\|_{\mathcal{C}^{-\alpha}}>K_\varepsilon)\leq R_t(f\in V_0^1:\|f\|_{\mathcal{C}^{-\alpha}}>K_\varepsilon)\leq(1+\frac{1+\|x\|_{H^{-1}}}{t})\varepsilon.$$

By [Tri06, Proposition 4.6] we know that $\{f \in \mathcal{C}^{-\alpha} \cap V_0^1 : \|f\|_{\mathcal{C}^{-\alpha}} > K_\varepsilon\}$ is a compact subset of V_0^{-1} since the embedding $\mathcal{C}^{-\alpha} \subset V^{-1}$ is compact. This implies the tightness of $\{R_t\}_{t\geq 0}$ in V_0^{-1}. By the Krylov-Bogoliubov existence theorem (see [DPZ96, Corollary 3.1.2]), there exists a sequence $t_k \nearrow \infty$ and a measure ν_x such that $R_{t_k} \to \nu_x$ weakly in V_0^{-1} and ν_x is an invariant measure for the semigroup $\{P_t\}_{t\geq 0}$.

□

To prove the exponential mixing property, we make use of the irreducibility of $\underline{Z}$ and a uniform estimate, which is slightly different from that in the proof of [TW16, Theorem 6.3].

Theorem 3.29. *There exists a constant $\lambda \in (0,1)$ and $T_0 \geq 0$ such that*

$$\|P_t(x) - P_t(y)\|_{TV} \leq 1 - \lambda,$$

for every $x, y \in V_0^{-1}$, $t \geq T_0 + 1$.

Proof From (3.23) we know that for any fixed $r > 0$, there exist $T_0, M > 0$ which are independent of ω, x, such that for any initial value $x \in V_0^{-1}$, we have that $\{\omega : |||\underline{Z}|||_{T_0} \leq M\} \subset \{\|Y(T_0)\|_{V_0^{-1}} < \frac{r}{2}\} \cap \{\|Z(T_0)\|_{V_0^{-1}} < \frac{r}{2}\}$.

By Theorem 3.27 for every $a \in (0,1)$ there exists $r \equiv r(a) > 0$ such that for every $x, y \in \bar{B}_r(0)$ and $t \geq 1$

$$\|P_t(x) - P_t(y)\|_{TV} \leq 1 - a, \tag{3.54}$$

where $B_r(u) := \{x \in V_0^{-1} : \|x-u\|_{V_0^{-1}} < r\}$. Then by (3.23) for any initial value $x \in V_0^{-1}$, there exists $b \equiv b(r) \in (0,1)$ such that

$$\begin{aligned} \mathbb{P}(\|X(T_0)\|_{V_0^{-1}} \leq r) &\geq \mathbb{P}\left(\{\|Y(T_0)\|_{V_0^{-1}} \leq \frac{r}{2}\} \cap \{\|Z(T_0)\|_{V_0^{-1}} \leq \frac{r}{2}\}\right) \\ &\geq \mathbb{P}(|||\underline{Z}|||_{T_0} \leq M) \\ &\geq b, \end{aligned} \tag{3.55}$$

where in the last step we used the irreducibility of the law of $\underline{Z}$. Here we omit the proof of the irreducibility of $\underline{Z}$, since it is the same as that of [TW16, Theorem 6.3]. Moreover, by (3.55) for any $r > 0$

$$\inf_{x \in V_0^{-1}} P_{T_0}(x, \bar{B}_r(0)) \geq b. \tag{3.56}$$

By Markov property, for any $\Phi \in C_b(V_0^{-1})$, $t \geq T_0 + 1$, and $x, y \in \bar{B}_r(0)$ we have that

$$\begin{aligned} |P_t\Phi(x) - P_t\Phi(y)| =& |\mathbb{E}\left[P_{t-T_0}\Phi(X(T_0;x)) - P_{t-T_0}\Phi(X(T_0;y))\right]| \\ =& \left| \int \left[P_{t-T_0}\Phi(\tilde{x}) - P_{t-T_0}\Phi(\tilde{y})\right] P_{T_0}(x, d\tilde{x}) P_{T_0}(y, d\tilde{y}) \right| \\ \leq& \|\Phi\|_{L^\infty} P_{T_0}(x) \otimes P_{T_0}(y) \left(\left(\bar{B}_r(0) \times \bar{B}_r(0)\right)^c \right) \\ &+ \|\Phi\|_{L^\infty} \int_{\bar{B}_r(0)\times\bar{B}_r(0)} \|P_{t-T_0}(\tilde{x}) - P_{t-T_0}(\tilde{y})\|_{TV} P_{T_0}(x, d\tilde{x}) P_{T_0}(y, d\tilde{y}). \end{aligned}$$

This implies that

$$\begin{aligned} \|P_t(x) - P_t(y)\|_{\mathrm{TV}} &\leq P_{T_0}(x) \otimes P_{T_0}(y) \left(\left(\bar{B}_r(0) \times \bar{B}_r(0)\right)^c \right) + (1-a) P_{T_0}(x) \otimes P_{T_0}(y) \left(\bar{B}_r(0) \times \bar{B}_r(0)\right) \\ &\leq 1 - a P_{T_0}(x, \bar{B}_r(0)) P_{T_0}(y, \bar{B}_r(0)) \\ &\leq 1 - ab^2, \end{aligned}$$

where we used (3.54) in the first inequality and (3.56) in the last inequality. Thus we can complete the proof by setting $\lambda = ab^2$.

□

The following corollary gives the exponential convergence to a unique invariant measure.

Corollary 3.30. *There exists a unique invariant measure $\bar{\nu}$ for the semigroup $\{P_t\}_{t\geq 0}$ such that*

$$\|P_t - \bar{\nu}\|_{TV} \leq (1-\lambda)^{\lfloor \frac{t}{T_0+1} \rfloor} \|\delta_x - \bar{\nu}\|_{TV}.,$$

for every $x \in V_0^{-1}$, $t \geq T_0 + 1$. Moreover, $\bar{\nu} = \nu$.

Proof The first result follows from the proof of [TW16, Corollary 6.6]. In fact, for any probability measures μ_1, μ_2 on V_0^{-1}, denote $M(dx, dy) := \mu_1(dx)\mu_2(dy)$. Note that

$$\|\mu_1 P_t - \mu_2 P_t\|_{\mathrm{TV}} \leq \frac{1}{2} \sup_{\|\Phi\|_{L^\infty} \leq 1} \iint |P_t\Phi(x) - P_t\Phi(y)| \, M(\mathrm{d}x, \mathrm{d}y),$$

where $\mu P_t(dx) := \int P_t(y, dx)\mu(dy)$. Thus by Theorem 3.29, for $t \geq T_0 + 1$,

$$\|\mu_1 P_t - \mu_2 P_t\|_{\mathrm{TV}} \leq (1-\lambda) \left(1 - M\left(\left\{(x,x) : x \in V_0^{-1}\right\}\right)\right).$$

By using the characterization of the total variation distance in the transportation theory (cf. [Vil09, Section 1])

$$\|\mu_1 - \mu_2\|_{\mathrm{TV}} = 2 \inf_{\mu_1, \mu_2} \left\{1 - M\left(\left\{(x,x) : x \in V_0^{-1}\right\}\right) : M(dx, dy) := \mu_1(dx)\mu_2(dy)\right\}.$$

We obtain that

$$\|\mu_1 P_t - \mu_2 P_t\|_{\mathrm{TV}} \leq (1-\lambda) \|\mu_1 - \mu_2\|_{\mathrm{TV}}.$$

By Theorem 3.28, for $x \in V_0^{-1}$, ν_x is an invariant measure. Thus we have

$$\|\nu_x - \nu_y\|_{\mathrm{TV}} = \|\nu_x P_t - \nu_y P_t\|_{\mathrm{TV}} \leq (1-\lambda)\|\nu_x - \nu_y\|_{\mathrm{TV}}.$$

This implies that $\nu_x = \nu_y$ for any $x, y \in V_0^{-1}$ i.e. $\{P_t : t \geq 0\}$ has a unique invariant measure $\bar{\nu}$. Moreover, for $t \geq T_0 + 1$,

$$\|P_t(x) - \bar{\nu}\|_{\mathrm{TV}} \leq (1-\lambda) \|P_{t-T_0-1}(x) - \bar{\nu}\|_{\mathrm{TV}},$$

which implies the first assertion.

For the second assertion, by Corollary 3.18, ν is an invariant measure of X. Hence $\bar{\nu} = \nu$. □

Remark 3.31. *In the following we give a simple and short proof for exponential convergence by the theory of Dirichlet forms.*

Similarly to [DPDT04], by comparing the two Dirichlet forms for Cahn-Hilliard equation and the dynamical Φ^4_2 model, we can obtain the spectral gap of equation (3.1). Indeed, by the same arguments in [RZZ17b] and [TW16] we know that ν is also the invariant measure for the solution to the dynamical Φ^4_2 model. We denote the Dirichlet form associated with the dynamical Φ^4_2 model by $(\bar{\Lambda}, D(\bar{\Lambda}))$, i.e.

$$\bar{\Lambda}(f,g) = \frac{1}{2} \int_E \langle Df, Dg \rangle_{L^2} d\nu, f, g \in D(\bar{\Lambda}),$$

where D denotes the gradient operator in $L^2(\mathbb{T}^2)$ (see [RZZ17b]). In [TW16] the exponential convergence for the dynamical Φ_2^4 model in total variation is proved. This implies the exponential convergence in $L^2(E,\nu)$-norm. By [Wan06, Theory 1.1, Example 1.1.2] this is equivalent to the Poincaré inequality

$$\int f^2 d\nu - (\int f d\nu)^2 \leq C\bar{\Lambda}(f,f), f \in D(\bar{\Lambda}).$$

From the proof of Theorem 3.15 we know that

$$\Lambda(f,f) = \frac{1}{2}\sum_k \int |\frac{\partial f}{\partial h_k}|^2 d\nu = \frac{1}{2}\sum_k \lambda_k \int |\frac{\partial f}{\partial e_k}|^2 d\nu \geq \frac{1}{2}\sum_k \int |\frac{\partial f}{\partial e_k}|^2 d\nu = \bar{\Lambda}(f,f),$$

where $h_k = \sqrt{\lambda_k} e_k$, $\{h_k\}_{k\in\mathbb{Z}^2\setminus\{(0,0)\}}$ is an orthonormal basis of V_0^{-1}. Then by [Wan06, Theory 1.1, Example 1.1.2] we have

$$\|P_t f - \int f d\nu\|_{L^2(E,\nu)} \leq e^{-\frac{t}{C}} \|f - \int f d\nu\|_{L^2(E,\nu)}.$$

Chapter 4

Sharp interface limit of stochastic Cahn-Hilliard equation with singular noise

In chapter 4 we obtain the convergence results arising in the study of the sharp interface limit, as $\varepsilon \searrow 0$, of the solutions to the stochastic Cahn-Hilliard equation on $\mathcal{D} := (0,1)^2$,

$$\begin{cases} \partial_t u^\varepsilon = \Delta v^\varepsilon + \varepsilon^\sigma \dot{\mathcal{W}}_t, \\ \quad v^\varepsilon = -\varepsilon \Delta u^\varepsilon + \frac{1}{\varepsilon} f(u^\varepsilon), \\ u^\varepsilon(0) = z, \end{cases} \tag{4.1}$$

with Neumann boundary conditions,

$$\frac{\partial u^\varepsilon}{\partial n} = \frac{\partial v^\varepsilon}{\partial n} = 0 \text{ on } \partial\mathcal{D}. \tag{4.2}$$

Here $f(u) = F'(u)$ and $F(u) = \frac{1}{4}(u^2-1)^2$ is the double-well potential , $\sigma > 0$ is a constant, and $\dot{\mathcal{W}}$ is a singular noise which represents the space-time white noise in Section 4.2 and the conservative noise in Section 4.4.

In the case of conservative noise, similarly as in Chapter 3, the nonlinear term is ill-defined since the solutions are expected to be distributions. Thus we consider the following renormalized equation

$$\begin{cases} du^\varepsilon = \Delta(-\varepsilon\Delta u^\varepsilon + \frac{1}{\varepsilon} : f(u^\varepsilon) :)dt + \varepsilon^\sigma B dW, \\ u^\varepsilon(0) = z \in H^{-1}, \end{cases} \tag{4.3}$$

with Neumann boundary conditions,

$$\frac{\partial u^\varepsilon}{\partial n} = \frac{\partial \Delta u^\varepsilon}{\partial n} = 0 \text{ on } \partial\mathcal{D}, \tag{4.4}$$

where $: f(u^\varepsilon) :=: f(\varphi^\varepsilon + \bar{Z}^\varepsilon) :$ is the Wick power defined in (4.46).

Equation (4.3) is also the limit of the following approximate equation:

$$du^{\varepsilon,h} = \Delta\left(-\varepsilon\Delta u^{\varepsilon,h} + \frac{1}{\varepsilon}\left(f(u^{\varepsilon,h}) - 3c^\varepsilon_{h,t} u^{\varepsilon,h}\right)\right) dt + \varepsilon^\sigma \nabla \cdot dW^h_t, \tag{4.5}$$

where $3c^\varepsilon_{h,t} u^{\varepsilon,h}$ is the renormalization term (see (4.42)-(4.44)). As $h \to 0$, $u^{\varepsilon,h}$ converges to u^ε, which is the unique solution to equation (4.3).

4.1 Notations and preliminaries

Let $\mathcal{D} := (0,1)^2$, $\mathcal{D}_T := (0,T)\times\mathcal{D}$. In this chapter, we always use $\langle\cdot,\cdot\rangle$ to denote the $L^2(\mathcal{D})$-inner product. For any $E\subset\mathcal{D}$, we denote by $\mathbb{1}_E$ the characteristic function of E, i.e.

$$\mathbb{1}_E(x) = \begin{cases} 1 \text{ if } x\in E, \\ 0 \text{ if } x\notin E. \end{cases}$$

We consider the Neumann Laplacian operator Δ on $L^2(\mathcal{D})$ with domain

$$D(\Delta) = \{u\in H^2(\mathcal{D}) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\mathcal{D}\}.$$

The operator $-\Delta$ is self-adjoint positive and has compact resolvent. It possesses a basis of eigenvectors $\{e_k\}_{k\in\mathbb{Z}^2}$ which is orthonormal in $L^2(\mathcal{D})$. In fact for $k=(k_1,k_2)\in\mathbb{Z}^2$, $e_k(x)$ is given by

$$\begin{aligned} &e_0(x) := 1, e_{(k_1,0)}(x) = \sqrt{2}\cos\pi k_1x_1, e_{(0,k_2)}(x) = \sqrt{2}\cos\pi k_2x_2, , \\ &e_k(x) := 2\cos\pi k_1x_1\cdot\cos\pi k_2x_2, k_1k_2\neq 0. \end{aligned} \tag{4.6}$$

It is associated with the eigenvalues $\{\lambda_k\}$, where $\lambda_k\simeq|k|^2$.

We also introduce a notation for the average of $g\in L^2(\mathcal{D})$:

$$m(g) := \langle g, e_0\rangle.$$

For any $\alpha\in\mathbb{R}$, we define V^α as the closure of $C^\infty(\mathcal{D})$ under the norm

$$\|g\|_{V^\alpha}^2 := m(g)^2 + \sum_k \lambda_k^\alpha\langle g, e_k\rangle^2.$$

It is easy to see that $(V^\alpha, \|\cdot\|_{V^\alpha})$ is a Hilbert space and $V^\alpha\simeq H^\alpha$, where H^α is the classical Sobolev space on domain $\mathcal{D}$ which can be defined as the closure of $C^\infty(\mathcal{D})$ under the norm

$$\|g\|_{H^\alpha}^2 = \sum_{k\in\mathbb{Z}^2}(1+\lambda_k)^\alpha\langle g, e_k\rangle^2.$$

In the rest of this chapter, we use the notation H^α to represent V^α for simplicity.

Moreover for any $s,\alpha\in\mathbb{R}$, we can define a bounded operator $(-\Delta)^s : H^\alpha\to H^{\alpha-2s}$ by:

$$(-\Delta)^s u = \sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}}\lambda_k^s u_k e_k,$$

where $u=\sum_k u_ke_k\in H^\alpha$.

We also set

$$H_0^\alpha := \{g\in H^\alpha : \langle g, e_0\rangle_{H^\alpha} = 0\},$$

where $\langle\cdot,\cdot\rangle_{H^\alpha}$ denote the inner product in H^α. Moreover we denote $L_0^2 := H_0^0$.

Finally, as what we mentioned in Introduction, the method in this chapter is heavily relied on Theorem 4.2, which holds under the assumption that the smooth solution to (1.17) exists. We assume $\Gamma_{00}\in\mathcal{C}^{3+\alpha}$ for some $\alpha\in(0,1)$, then

Theorem 4.1. *([CHY96, Theorem 1.1]). For any $\Gamma_{00}\in\mathcal{C}^{3+\alpha}$ for some $\alpha\in(0,1)$, there exists a $T>0$, such that (1.17) has a unique local solution $\{(v,\Gamma)\}_{t\in[0,T]}$, where $\Gamma\in C^{\frac{3+\alpha}{3}}([0,T];\mathcal{C}^{3+\alpha})$*

Now we fix Γ_{00} and T in the following of this chapter. Then by [ABC94, Theorem 2.1], we have that

Theorem 4.2. *Let (v, Γ_t) be a classical smooth solution to (1.17) in Theorem 4.1. For any $K > 0$ there exists a pair $(u_A^\varepsilon, v_A^\varepsilon)$ of solutions to (1.16), such that*

$$\|r_A^\varepsilon\|_{C(\mathcal{D}_T)} \lesssim \varepsilon^{K-2}.$$

Moreover, it holds that

$$\|v_A^\varepsilon - v\|_{C(\mathcal{D}_T)} \lesssim \varepsilon,$$

where v is the solution to (1.17) below. In particular, u_A^ε and v_A^ε are uniformly bounded.

Finally for x away from Γ_t, i.e. $d(x, \Gamma_t) > C\varepsilon$, where $d(x, \Gamma_t)$ is the distance of x to Γ_t and C is some constant which is independent to ε,

$$|u_A^\varepsilon(t,x) - 1| \lesssim \varepsilon \quad or \quad |u_A^\varepsilon(t,x) + 1| \lesssim \varepsilon.$$

4.2 The sharp interface limit for space-time white noise

Let $\mathcal{W} = W$ be an $L_0^2(\mathcal{D})$-cylindrical Wiener process on a fixed stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$.

Theorem 4.3. *([DPD96, Theorem 2.1])For $\mathbb{P} - a.s.\ \omega$, there exists a unique solution u^ε to equation (4.1) in $C([0,T]; H^{-1})$.*

We rewrite the equation (4.1) as

$$\begin{cases} du^\varepsilon = \Delta v^\varepsilon dt + \varepsilon^\sigma dW \text{ in } \mathcal{D}_T, \\ \ \ v^\varepsilon = \dfrac{1}{\varepsilon} f(u^\varepsilon) - \varepsilon \Delta u^\varepsilon \text{ in } \mathcal{D}_T. \end{cases} \tag{4.7}$$

We assume that the interface has been formed initially. That is, there exists a smooth closed curve $\Gamma_{00} \subset\subset \mathcal{D}$ such that $u^\varepsilon(0) \approx -1$ in $\mathcal{D}^-$, the region enclosed by Γ_{00}, and $u^\varepsilon(0) \approx 1$ in $\mathcal{D}^+ := \mathcal{D} \setminus (\Gamma_{00} \cup \mathcal{D}^-)$.

Our main theorem will show that as $\varepsilon \to 0$, v^ε tends to v, which, together with a free boundary $\Gamma \equiv \cup_{0 \le t \le T}(\Gamma_t \times \{t\})$, satisfies the deterministic Hele-Shaw problem (1.17).

We present now the following spectral estimate which is useful in our proof.

Proposition 4.4. *([ABC94, Proposition 3.1]) Let u_A^ε be the approximation given in Theorem 4.2. Then for all $w \in H^1$ satisfying Neumann boundary conditions such that $\int_{\mathcal{D}} w = 0$, the following estimate is valid*

$$\varepsilon \|w\|_{H^1}^2 + \frac{1}{\varepsilon} \int f'(u_A^\varepsilon) w^2 \ge -C_0 \|w\|_{H^{-1}}^2.$$

We consider the residual

$$R^\varepsilon := u^\varepsilon - u_A^\varepsilon, \tag{4.8}$$

where u^ε is the unique solution to (4.7). We show bounds for this error R^ε in our main theorem below.

Theorem 4.5. *(**Main Theorem**) Let u^ε_A be defined in Theorem 4.2 with large enough K and let u^ε be the unique solution to (4.1) with initial value $u^\varepsilon(0) = u^\varepsilon_A(0)$. For any $\sigma^* > \delta > 0$,*

$$\begin{cases} \gamma > 13, \\ \sigma^* > \dfrac{1}{3}\gamma + \dfrac{13}{3} + 2\delta, \end{cases}$$

where $\sigma^* = \sigma - \frac{1}{4}$ is introduced in Lemma 4.8, there exist a generic constant $C > 0$ and a constant $C_\delta > 0$ for all $\delta > 0$ such that the following estimates hold

$$\mathbb{P}\left[\|R^\varepsilon\|_{L^3(\mathcal{D}_T)} \le C\varepsilon^{\frac{\gamma}{3}}\right] \ge 1 - C_\delta\varepsilon^\delta,$$

$$\mathbb{P}\left[\|R^\varepsilon\|^2_{L^\infty(0,T;H^{-1})} \le C\left(\varepsilon^{\gamma-1} + \varepsilon^{\sigma^*-1-2\delta+\frac{\gamma}{3}}\right)\right] \ge 1 - C_\delta\varepsilon^\delta,$$

$$\mathbb{P}\left[\|v^\varepsilon - v^\varepsilon_A\|^2_{L^1(0,T;H^{-2})} \le C\varepsilon^{\frac{\gamma}{3}-1}\right] \ge 1 - C_\delta\varepsilon^\delta.$$

Remark 4.6. *Since δ can be as small as enough, the best choice is $\sigma > \frac{107}{12}$.*

Corollary 4.7. *There exists a subsequence $\{\varepsilon_k\}_{k=1}^\infty$ such that for $\mathbb{P}-a.s.\ \omega \in \Omega$*

$$\lim_{k\to\infty} u^{\varepsilon_k} = -1 + 2\mathbb{1}_{E_t} \ \ in\ \ L^3(\mathcal{D}_{T_0}),$$

where E_t is the region enclosed by Γ_t.

Proof The local uniqueness of (1.17) can be obtained directly by [CHY96, Theorem 1.1]. Then by the construction of u^ε_A in [ABC94], for uniformly $t \in [0, T_0]$

$$\lim_{\varepsilon\to 0} u^\varepsilon_A = -1 + 2\mathbb{1}_{E_t} \ \text{uniformly on compact subsets.}$$

Moreover all the results in Theorem 4.5 hold if replacing T by T_0.

For any $\eta > 0$, choosing ε small enough such that $C\varepsilon^{\frac{\gamma}{3}} < \eta$, then we have

$$\mathbb{P}\left[\|R^\varepsilon\|_{L^3(\mathcal{D}_T)} > \eta\right] \le \mathbb{P}\left[\|R^\varepsilon\|_{L^3(\mathcal{D}_T)} > C\varepsilon^{\frac{\gamma}{3}}\right] \le C_\delta\varepsilon^\delta,$$

which implies that $\|R^\varepsilon\|_{L^3}$ converge in probability to 0. Thus there exists a subsequence (still denoted as ε), such that

$$\lim_{\varepsilon\to 0}\|R^\varepsilon\|_{L^3(\mathcal{D}_T)} = 0 \ \ \mathbb{P}-a.s..$$

Since $R^\varepsilon = u^\varepsilon - u^\varepsilon_A$, we obtain the assertion.

□

4.3 The proof of the Main Theorem

4.3.1 The decomposition of the equation for the error

Combining (4.7), (1.16) and (4.8) we know that R^ε satisfies the following equation:

$$\begin{cases} dR^\varepsilon = -\varepsilon\Delta^2 R^\varepsilon dt + \dfrac{1}{\varepsilon}\Delta\left(f(u^\varepsilon_A + R^\varepsilon) - f(u^\varepsilon_A)\right)dt + \Delta r^\varepsilon_A dt + \varepsilon^\sigma dW, \\ \dfrac{\partial R^\varepsilon}{\partial n} = \dfrac{\partial \Delta R^\varepsilon}{\partial n} = 0 \ \text{ on } \partial\mathcal{D}. \end{cases} \tag{4.9}$$

Let $Z_t^\varepsilon := \varepsilon^\sigma \int_0^t e^{-(t-s)\varepsilon\Delta^2} dW_s$, which is the mild solution to the linear equation:

$$\begin{cases} dZ^\varepsilon = -\varepsilon\Delta^2 Z^\varepsilon dt + \varepsilon^\sigma dW, \\ \dfrac{\partial Z^\varepsilon}{\partial n} = \dfrac{\partial \Delta Z^\varepsilon}{\partial n} = 0 \ \text{ on } \ \partial\mathcal{D}. \end{cases} \tag{4.10}$$

Then $Y^\varepsilon := R^\varepsilon - Z^\varepsilon$ satisfies:

$$\begin{cases} dY^\varepsilon = -\varepsilon\Delta^2 Y^\varepsilon dt + \dfrac{1}{\varepsilon}\Delta\left(f'(u_A^\varepsilon)(Y^\varepsilon + Z^\varepsilon) + \mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon)\right) dt + \Delta r_A^\varepsilon dt, \\ \dfrac{\partial Y^\varepsilon}{\partial n} = \dfrac{\partial \Delta Y^\varepsilon}{\partial n} = 0 \ \text{ on } \ \partial\mathcal{D}. \end{cases} \tag{4.11}$$

where $\mathcal{N}(u, v) := f(u + v) - f(u) - f'(u)v$.

Moreover, we define a stopping time T_ε by:

$$T_\varepsilon := T \wedge \inf\{t > 0 : \int_0^t \|Y_s^\varepsilon\|_{L^3}^3 ds > \varepsilon^\gamma\}, \tag{4.12}$$

for some $\gamma > 1$.

4.3.2 Estimate for Z^ε

Lemma 4.8. *For any $\delta > 0$, there exists a constant $C_\delta > 0$, such that*

$$\mathbb{P}[\Omega_\delta] > 1 - C_\delta \varepsilon^\delta,$$

where $C_1 > 0$ is a universal constant, $\Omega_\delta := \{\|Z\|_{C(\mathcal{D}_T)} \le C_1 \varepsilon^{\sigma^* - 2\delta}\}$, and $\sigma^* := \sigma - \frac{1}{4}$.

Proof By the factorization method in [DP04] we have that for $\kappa \in (0, 1)$

$$Z^\varepsilon(t) = \varepsilon^\sigma \frac{\sin(\pi\kappa)}{\pi} \int_0^t (t - s)^{\kappa - 1} \langle M(\varepsilon(t - s), x, \cdot), U(s)\rangle ds,$$

where $M(\varepsilon t, x, y)$ is the kernel of the semigroup $\{e^{-\varepsilon t\Delta^2}\}$ and

$$U^\varepsilon(s, \cdot) = \int_0^s (s - r)^{-\kappa} e^{-\varepsilon(s-r)\Delta^2} dW_r.$$

Similarly to the proof of Lemma 2.12 in [DP04], we have that

$$\mathbb{E}\left[\|Z^\varepsilon(t)\|_{\mathcal{C}(\mathcal{D}_T)}\right] \lesssim_T \varepsilon^\sigma \mathbb{E}\left[\|U^\varepsilon\|_{L^{2p}(\mathcal{D}_T)}\right]. \tag{4.13}$$

It suffices to estimate $\mathbb{E}\left[\|U^\varepsilon\|_{L^{2p}(\mathcal{D}_T)}\right]$ for $p > \frac{1}{2\kappa}$.

In fact, we have that

$$\begin{aligned} \mathbb{E}\left[\|U^\varepsilon(s)\|_{L^{2p}(\mathcal{D}_T)}^{2p}\right] &\lesssim \int_{\mathcal{D}_T} \mathbb{E}\left[\left|\int_0^s (s - r)^{-\kappa} e^{-\varepsilon(s-r)\Delta^2} dW_r\right|^{2p}\right] dsdx \\ &\lesssim \int_{\mathcal{D}_T} \left(\mathbb{E}\left[\left|\int_0^s (s - r)^{-\kappa} e^{-\varepsilon(s-r)\Delta^2} dW_r\right|^2\right]\right)^p dsdx. \end{aligned} \tag{4.14}$$

Here we used that $U^\varepsilon(x)$ belongs to the first order Wiener-chaos and Gaussian hypercontractivity (cf. [Nua13, Section 1.4.3] and [Nel73]) in the second inequality. Moreover, we obtain that

$$\mathbb{E}\left[\left|\int_0^s (s-r)^{-\kappa} e^{-\varepsilon(s-r)\Delta^2} dW_r\right|^2\right] \lesssim \int_0^s \int_{\mathcal{D}} (s-r)^{-2\kappa} M(\varepsilon(s-r), x, y)^2 dy ds. \quad (4.15)$$

Since $M(t, x, y)$ is the kernel of $e^{-t\Delta^2}$, we have that for any $g \in L^2$

$$\int_{\mathcal{D}} M(t, x, y) g(y) dy = e^{-t\Delta^2} g(x) \simeq \sum_k \langle g, e^{-t|k|^4} e_k \rangle e_k(x).$$

Hence

$$M(t, x, y) \simeq \sum_k e^{-t|k|^4} e_k(x) e_k(y). \quad (4.16)$$

where e_k is defined in (4.6). Note that $e_k(x)e_k(y) = \frac{1}{2}(e_k(x-y) + e_k(x+y)$. Thus we obtain

$$M(t, x, y) \simeq \sum_k e^{-t|k|^4} (e_k(x-y) + e_k(x+y)) := P(t, x-y) + P(t, x+y), \quad (4.17)$$

Then (4.15) becomes

$$\mathbb{E}\left[\left|\int_0^s (s-r)^{\kappa} e^{-\varepsilon(s-r)\Delta^2} dW_r\right|^2\right] \lesssim \int_0^s \int_{\mathcal{D}} (s-r)^{-2\kappa} \left(P(\varepsilon(s-r), x-y)^2 + P(\varepsilon(s-r), x+y)^2\right) dy ds. \quad (4.18)$$

By [SW72, p282, (c)], we have that

$$|P(t, x)| \lesssim |x|^{-2} e^{-\frac{t}{|x|^4}} \lesssim t^{-\frac{\eta}{4}} |x|^{-2+\eta}, \ \forall \eta \in [0, 2]. \quad (4.19)$$

Then taking (4.18) into (4.19), we deduce that

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^s (s-r)^{\kappa} e^{-\varepsilon(s-r)\Delta^2} dW_r\right|^2\right] &\lesssim \varepsilon^{-\frac{\eta}{2}} \int_0^s \int_{\mathcal{D}} (s-r)^{-2\kappa-\frac{\eta}{2}} \left(|x+y|^{-4+2\eta} + |x-y|^{-4+2\eta}\right) dy ds \\ &\lesssim \varepsilon^{-\frac{\eta}{2}} s^{1-2\kappa-\frac{\eta}{2}} |x|^{-2+2\eta}.\end{aligned} \quad (4.20)$$

Here we require that

$$1 - 2\kappa - \frac{\eta}{2} > 0, \quad -2 + 2\eta > 0,$$

that is

$$1 < \eta < 2 - 4\kappa, \quad (4.21)$$

which can be obtained by choosing small enough $\kappa > 0$. Hence by (4.13) and (4.14), we obtain that for any $p \geq 1$

$$\mathbb{E}\left[\|U^\varepsilon\|_{L^{2p}(\mathcal{D}_T)}\right] \lesssim \varepsilon^{\sigma - \frac{\eta}{4}}.$$

This implies that for any $2 > \eta > 1$,

$$\mathbb{E}\left[(\|Z^\varepsilon\|_{\mathcal{C}(\mathcal{D}_T)})\right] \lesssim \varepsilon^{\sigma - \frac{\eta}{4}}. \quad (4.22)$$

Hence we can obtain our results by Chebyshev's inequality.

□

4.3.3 Local-in-time estimate for Y^ε up to T_ε on the set Ω_δ

Now we fix an $\omega \in \Omega_\delta$, thus by Lemma 4.8, $\|Z^\varepsilon(\omega)\|_{C(\mathcal{D}_T)} \lesssim \varepsilon^{\sigma^*-2\delta}$. All estimates in this section in on Ω_δ.

By taking inner product with $(-\Delta)^{-1}Y^\varepsilon$ in both side of equation (4.11) we have that

$$\frac{1}{2}\frac{d\|Y^\varepsilon\|_{H^{-1}}^2}{dt} + \varepsilon\|Y_t^\varepsilon\|_{H^1}^2 = -\frac{1}{\varepsilon}\langle f'(u_A^\varepsilon)(Y^\varepsilon + Z^\varepsilon) + \mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon), Y^\varepsilon\rangle - \langle r_A^\varepsilon, Y^\varepsilon\rangle. \quad (4.23)$$

In the following, we estimate the right hand side of (4.23) separately:

Using Proposition 4.4 we have that

$$-\frac{1}{\varepsilon}\langle f'(u_A^\varepsilon)Y^\varepsilon, Y^\varepsilon\rangle \leq \varepsilon\|Y^\varepsilon\|_{H^1}^2 + C_0\|Y^\varepsilon\|_{H^{-1}}^2. \quad (4.24)$$

For $-\frac{1}{\varepsilon}\langle f''(u_A^\varepsilon)(Y^\varepsilon, Z^\varepsilon\rangle$ by Theorem 4.2 we know that u_A^ε is uniformly bounded in $\mathcal{D}_T$. Thus we have that

$$\frac{1}{\varepsilon}|\langle f''(u_A^\varepsilon)Y^\varepsilon, Z^\varepsilon\rangle| \lesssim \frac{1}{\varepsilon}\|Y^\varepsilon\|_{L^3}\|Z^\varepsilon\|_{L^{\frac{3}{2}}} \lesssim \varepsilon^{\sigma^*-1-2\delta}\|Y^\varepsilon\|_{L^3}, \quad (4.25)$$

where we used Hölder's inequality in the first inequality and Lemma 4.8 in the last inequality.

By [ABC94, Lemma 2.2], we have that $v\mathcal{N}(u,v) \geq -C|v|^3$. Then

$$\begin{aligned}-\frac{1}{\varepsilon}\langle \mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon), Y^\varepsilon\rangle &= -\frac{1}{\varepsilon}\langle \mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon), Y^\varepsilon + Z^\varepsilon\rangle + \frac{1}{\varepsilon}\langle \mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon), Z^\varepsilon\rangle \\ &\lesssim \frac{1}{\varepsilon}\|Y^\varepsilon + Z^\varepsilon\|_{L^3}^3 + \frac{1}{\varepsilon}|\langle \mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon), Z^\varepsilon\rangle| \\ &\lesssim \frac{1}{\varepsilon}\|Y\|_{L^3}^3 + \varepsilon^{3(\sigma^*-2\delta)-1} + \frac{1}{\varepsilon}|\langle \mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon), Z^\varepsilon\rangle|,\end{aligned} \quad (4.26)$$

where we used Lemma 4.8 in the last inequality.

For $|\langle \mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon), Z^\varepsilon\rangle|$, by the Taylor expansion, $\mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon) = f''(u_A^\varepsilon + \theta(Y^\varepsilon + Z^\varepsilon))(Y^\varepsilon + Z^\varepsilon)^2 = 6(u_A^\varepsilon + \theta(Y^\varepsilon + Z^\varepsilon))(Y^\varepsilon + Z^\varepsilon)^2$, where $\theta \in (0,1)$. Then we have

$$\begin{aligned}|\langle \mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon), Z^\varepsilon\rangle| &\lesssim \varepsilon^{\sigma^*-2\delta}\|\mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon)\|_{L^1} \\ &\lesssim \varepsilon^{\sigma^*-2\delta}(\|Y^\varepsilon + Z^\varepsilon\|_{L^3}^3 + \|Y^\varepsilon + Z^\varepsilon\|_{L^2}^2) \\ &\lesssim \varepsilon^{3(\sigma^*-2\delta)} + \varepsilon^{4(\sigma^*-2\delta)} + \varepsilon^{\sigma^*-2\delta}\|Y^\varepsilon\|_{L^3}^2 + \varepsilon^{\sigma^*-2\delta}\|Y^\varepsilon\|_{L^3}^3,\end{aligned} \quad (4.27)$$

where we used the uniform boundness of u_A^ε in the second inequality and Lemma 4.8 in the first and the last inequality.

For $|\langle r_A^\varepsilon, Y^\varepsilon\rangle|$, by Theorem 4.2 we have

$$|\langle r_A^\varepsilon, Y^\varepsilon\rangle| \lesssim \varepsilon^{K-2}\|Y^\varepsilon\|_{L^1} \lesssim \varepsilon^{K-2}\|Y^\varepsilon\|_{L^3}. \quad (4.28)$$

Let $\sigma^* > \delta$, $\varepsilon < 1$, δ be small enough and K large enough. Collecting (4.23)-(4.28) together, by using Hölder's inequality we have

$$\frac{d\|Y^\varepsilon(t)\|_{H^{-1}}^2}{dt} \lesssim \|Y^\varepsilon\|_{H^{-1}}^2 + \frac{1}{\varepsilon}\|Y^\varepsilon\|_{L^3}^3 + \varepsilon^{\sigma^*-1-2\delta}(\|Y^\varepsilon\|_{L^3} + \|Y^\varepsilon\|_{L^3}^2 + \|Y^\varepsilon\|_{L^3}^3) + \varepsilon^{3(\sigma^*-2\delta)-1}.$$

Then for any $t \le T_\varepsilon$ we have

$$\begin{aligned}\|Y^\varepsilon(t)\|_{H^{-1}}^2 &\lesssim \int_0^t e^{t-s}\left(\frac{1}{\varepsilon}\|Y^\varepsilon\|_{L^3}^3 + \varepsilon^{\sigma^*-1-2\delta}\|Y^\varepsilon\|_{L^3} + \varepsilon^{3(\sigma^*-2\delta-)-1}\right)ds \\ &\lesssim_T \frac{1}{\varepsilon}\int_0^t \|Y^\varepsilon\|_{L^3}^3 d\tau + \varepsilon^{\sigma^*-1-2\delta}\left(\int_0^t \|Y^\varepsilon\|_{L^3}^3 d\tau\right)^{\frac{1}{3}} + \varepsilon^{3(\sigma^*-2\delta)-1} \\ &\lesssim \varepsilon^{\gamma-1} + \varepsilon^{\sigma^*-1-2\delta+\frac{\gamma}{3}} + \varepsilon^{3(\sigma^*-2\delta)-1}.\end{aligned} \tag{4.29}$$

To estimate $L^2(0,T_\varepsilon;H^1)$ norm of Y^ε, we use the estimate presented in [ABC94, p.171]

$$-\frac{1}{\varepsilon}\int_0^t \int_{\mathcal{D}} f''(u_A^\varepsilon)g^2 dx ds \lesssim \varepsilon^{-\frac{2}{3}}(\int_0^t \|g\|_{L^3}^3 ds)^{\frac{2}{3}}, \quad \forall g \in L^3.$$

Then

$$-\frac{1}{\varepsilon}\int_0^t \langle f''(u_A^\varepsilon)Y^\varepsilon, Y^\varepsilon\rangle ds \le \varepsilon^{-\frac{2}{3}}(\int_0^t \|Y^\varepsilon\|_{L^3}^3 ds)^{\frac{2}{3}} \lesssim \varepsilon^{\frac{2}{3}(\gamma-1)}. \tag{4.30}$$

Combining (4.23), (4.25)-(4.28) and (4.30) we have for any $t \le T_\varepsilon$

$$\int_0^t \|Y^\varepsilon\|_{H^1}^2 ds \lesssim \varepsilon^{\frac{2}{3}\gamma-\frac{5}{3}} + \varepsilon^{\sigma^*-2-2\delta+\frac{\gamma}{3}} + \varepsilon^{3(\sigma^*-2\delta)-2} + \varepsilon^{\gamma-2}. \tag{4.31}$$

4.3.4 Final step: Globalization $T_\varepsilon \equiv T$

Let

$$\gamma_1 := (\gamma-1)\wedge(3(\sigma^*-2\delta)-1)\wedge(\sigma^*-1-2\delta+\frac{\gamma}{3}),$$

$$\gamma_2 := (\frac{2}{3}\gamma-\frac{5}{3})\wedge(3(\sigma^*-2\delta)-2)\wedge(\sigma^*-2-2\delta+\frac{\gamma}{3})\wedge(\gamma-2) = (\frac{2}{3}\gamma-\frac{5}{3})\wedge(\gamma_1-1),$$

then we have for any $t \le T_\varepsilon$

$$\sup_{s\in[0,t]}\|Y^\varepsilon\|_{H^{-1}}^2 \lesssim \varepsilon^{\gamma_1}, \quad \int_0^t \|Y^\varepsilon\|_{H^1}^2 ds \lesssim \varepsilon^{\gamma_2}. \tag{4.32}$$

We use the Sobolev's embedding of H^β into L^p with $\beta := 2(\frac{1}{2}-\frac{1}{p}) = \frac{p-2}{p}$. Then by the interpolation we have

$$\|Y^\varepsilon\|_{L^3} \lesssim \|Y^\varepsilon\|_{H^{\frac{1}{3}}} \lesssim \|Y^\varepsilon\|_{H^1}^{\frac{2}{3}}\|Y^\varepsilon\|_{H^{-1}}^{\frac{1}{3}}.$$

For any $t \le T_\varepsilon$ by (4.32) we have

$$\begin{aligned}\int_0^t \|Y^\varepsilon\|_{L^3}^3 ds &\lesssim \sup_{t\in[0,t]}\|Y^\varepsilon\|_{H^{-1}} \int_0^t \|Y^\varepsilon\|_{H^1}^2 ds \\ &\lesssim \varepsilon^{\frac{\gamma_1}{2}+\gamma_2}.\end{aligned} \tag{4.33}$$

Then we have that for ε small enough, $T_\varepsilon = T$, if $\gamma < \frac{\gamma_1}{2}+\gamma_2$.
Let $\gamma_1 > \frac{2}{3}\gamma - \frac{2}{3}$ such that $\gamma_2 = \frac{2}{3}\gamma-\frac{5}{3}$, then we only need

$$\gamma_1 > \frac{2}{3}\gamma + \frac{10}{3}.$$

i.e.

$$\begin{cases} \gamma - 1 > \frac{2}{3}\gamma + \frac{10}{3} \\ 3(\sigma^* - 2\delta) - 1 > \frac{2}{3}\gamma + \frac{10}{3} \\ \sigma^* - 1 - 2\delta + \frac{\gamma}{3} > \frac{2}{3}\gamma + \frac{10}{3}. \end{cases}$$

A direct calculation yields that

$$\begin{cases} \gamma > 13 \\ \sigma^* > \frac{1}{3}\gamma + \frac{13}{3} + 2\delta, \end{cases} \tag{4.34}$$

which also implies $\gamma_1 = (\gamma - 1) \wedge (\sigma^* - 1 - 2\delta + \frac{\gamma}{3})$.

Since $R^\varepsilon = Y^\varepsilon + Z^\varepsilon$, by Lemma 4.8 we have for any $\omega \in \Omega_\delta$

$$\begin{aligned} \|R^\varepsilon(\omega)\|_{L^3(\mathcal{D}_T)} &\lesssim \varepsilon^{\frac{\gamma}{3}} + \varepsilon^{\sigma^*-2\delta} \lesssim \varepsilon^{\frac{\gamma}{3}}, \\ \|R^\varepsilon(\omega)\|^2_{L^\infty(0,T;H^{-1})} &\lesssim \varepsilon^{\gamma-1} + \varepsilon^{\sigma^*-1-2\delta+\frac{\gamma}{3}}. \end{aligned} \tag{4.35}$$

Finally, note that

$$v^\varepsilon - v^\varepsilon_A = \varepsilon\Delta(Y^\varepsilon + Z^\varepsilon) - \frac{1}{\varepsilon}\left(f(u_\varepsilon) - f(u^\varepsilon_A)\right).$$

Therefore, by using the embedding $C(\mathcal{D}) \subset L^2$

$$\|\Delta(Y^\varepsilon + Z^\varepsilon)\|_{L^1(0,T;H^{-2})} \lesssim \|Y^\varepsilon\|_{L^2(0,T;L^2)} + \|Z^\varepsilon\|_{C(\mathcal{D}_T)} \lesssim \varepsilon^{\frac{\gamma}{3}} + \varepsilon^{\sigma^*-2\delta}.$$

Moreover, similarly to what we do above

$$\begin{aligned} f(u^\varepsilon) - f(u^\varepsilon_A) &= f'(u^\varepsilon_A)R^\varepsilon + \mathcal{N}(u^\varepsilon_A, R^\varepsilon) \\ &= f'(u^\varepsilon_A)R^\varepsilon + f''(u^\varepsilon_A + \theta(R^\varepsilon))(R^\varepsilon)^2 \\ &= f'(u^\varepsilon_A)R^\varepsilon + 6(u^\varepsilon_A + \theta(R^\varepsilon))(R^\varepsilon)^2. \end{aligned}$$

Since $\{u^\varepsilon_A\}$ are uniformly bounded in ε and $\theta \in [0,1]$, we have that

$$\begin{aligned} \|f(u^\varepsilon) - f(u^\varepsilon_A)\|_{L^1(0,T;H^{-2})} &\lesssim \|f(u^\varepsilon) - f(u^\varepsilon_A)\|_{L^1(\mathcal{D}_T)} \\ &\lesssim \|(R^\varepsilon)^3\|_{L^1(\mathcal{D}_T)} + \|(R^\varepsilon)^2\|_{L^1(\mathcal{D}_T)} + \|R^\varepsilon\|_{L^1(\mathcal{D}_T)} \\ &\lesssim \|R^\varepsilon\|_{L^3(\mathcal{D}_T)} + \|R^\varepsilon\|^2_{L^3(\mathcal{D}_T)} + \|R^\varepsilon\|^3_{L^3(\mathcal{D}_T)} \\ &\lesssim \varepsilon^{\frac{\gamma}{3}} + \varepsilon^{\sigma^*-2\delta} \\ &\lesssim \varepsilon^{\frac{\gamma}{3}}, \end{aligned}$$

where we use the Sobolev embedding $L^1 \subset H^{-2}$ in the first inequality.

Hence we deduce that

$$\begin{aligned} \|v^\varepsilon - v^\varepsilon_A\|_{L^1(0,T;H^{-2})} &\lesssim \varepsilon^{\frac{\gamma}{3}+1} + \varepsilon^{\frac{\gamma}{3}-1} \\ &\lesssim \varepsilon^{\frac{\gamma}{3}-1}. \end{aligned} \tag{4.36}$$

Combining it with (4.34), we obtain our results stated in the Theorem 4.5.

4.4 Sharp interface limit for conservative noise

In this section, we will consider the case that $\mathcal{W} = \nabla \cdot W$, where W is an $L^2_0(\mathcal{D}, \mathbb{R}^2)$-cylindrical Wiener process on stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$. For $g \in L^2_0(\mathcal{D}, \mathbb{R}^2)$, we denote its component functions by $g_1, g_2 \in L^2_0(\mathcal{D})$, i.e. $g(x) = (g_1(x), g_2(x)), \forall x \in \mathcal{D}$. There exist two independent $L^2_0(\mathcal{D})$-cylindrical Wiener processes W^1 and W^2 such that $W = (W^1, W^2)$.

Following a similar argument as in [RZZ17b, RYZ18], in this case, the solution to (4.3) is distribution-valued. Thus we consider the approximate equation (4.5) instead.

4.4.1 Existence and uniqueness of solutions to equation (4.3)

In order to consider the convolution of the noise with an approximate delta function, we need to extend the noise to the whole space $\mathbb{R}^2$. Considering the Neumann boundary condition, it is reasonable to extend it evenly to $[-1, 1]^2$ first, then do a periodical extension to the whole space. That is, for any function g on $\mathcal{D}$ which satisfies the Neumann boundary condition, we view it as a function $\bar{g}$ on $\mathbb{R}^2$ by

$$\bar{g}(x) := g(|x_1+k_1|, |x_2+k_2|), \ \ \forall x = (x_1, x_2) \in \mathbb{R}^2, \ \ \forall k = (k_1, k_2) \in \mathbb{Z}^2 \ \text{ when } \ x+k \in [-1, 1]^2.$$

Moreover, for $x \in \mathbb{R}^2$ and $t > 0$, define

$$\bar{M}(t, x) = -\mathcal{F}^{-1}(e^{-\frac{t}{2}|\pi\cdot|^4})(x),$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transformation on $\mathbb{R}^2$. By Poisson summation formula, for any $(x, y) \in \mathcal{D}^2$

$$M(t, x, y) := \sum_{k\in\mathbb{Z}^2} \left(\bar{M}(t, x + y + 2k) + \bar{M}(t, x - y + 2k)\right)$$

is the kernel of $e^{-t\Delta^2}$ on $\mathcal{D}$, where Δ is the Neumann Laplacian operator on $\mathcal{D}$. A direct calculation yields that for any $g \in L^2(\mathcal{D})$

$$\int_{\mathcal{D}} M(t, x, y)g(y)dy = \int_{\mathbb{R}^2} \bar{M}(t, x - y)\bar{g}(y)dy. \tag{4.37}$$

Define

$$K(t, x, y) := -\nabla_y M(t, x, y) = \sum_{k\in\mathbb{Z}^2} \left(\bar{K}(t, x + y + 2k) - \bar{K}(t, x - y + 2k)\right),$$

where $\bar{K}(t, x) = (\bar{K}^1(t, x), \bar{K}^2(t, x)) := -\nabla\bar{M}(t, x)$, thus for any $t > 0$, $\bar{K}^j(t, \cdot)$ is the inverse Fourier transformation of the function $\eta \to -\pi i\eta_j e^{-\frac{t}{2}|\pi\eta|^4}$, i.e.

$$\bar{K}^j(t, x) := -\mathcal{F}^{-1}(\pi i\eta_j e^{-\frac{t}{2}|\pi\eta|^4})(x).$$

We use $\mathcal{S}(\mathbb{R}^2)$ to denote the Schwartz function on $\mathbb{R}^2$, $\mathcal{S}'(\mathbb{R}^2)$ to denote the Schwartz distribution on $\mathbb{R}^2$ and ${}_{\mathcal{S}'(\mathbb{R}^2)}\langle\cdot, \cdot\rangle_{\mathcal{S}(\mathbb{R}^2)}$ to denote the dual between $\mathcal{S}(\mathbb{R}^2)$ and $\mathcal{S}'(\mathbb{R}^2)$. Then we know that $\bar{K}^j(t, \cdot) \in \mathcal{S}(\mathbb{R}^2)$ for any $t > 0$. Moreover we define Z^ε by

$$Z^\varepsilon(t, x) := \varepsilon^\sigma \int_0^t \langle K(t - s, x, \cdot), dW_s\rangle_{L^2(\mathcal{D},\mathbb{R}^2)} = \varepsilon^\sigma \sum_{j=1}^{2} \int_0^t {}_{\mathcal{S}'(\mathbb{R}^2)}\langle \bar{K}^j(t - s, x - \cdot), d\bar{W}^j_s\rangle_{\mathcal{S}(\mathbb{R}^2)}. \tag{4.38}$$

Here $\bar{W} = (\bar{W}^1, \bar{W}^2)$, $\bar{W}^j,\ j = 1, 2$ is two i.i.d Wiener processes defined by

$$_{\mathcal{S}'(\mathbb{R}^2)}\langle \bar{W}^j, g\rangle_{\mathcal{S}(\mathbb{R}^2)} = \langle W, \tilde{g}\rangle_{L^2(\mathcal{D})},$$

for any $g \in \mathcal{S}(\mathbb{R}^2)$ and $\tilde{g} \in L^2(\mathcal{D})$ is defined as

$$\tilde{g}(x) := \sum_{k\in\mathbb{Z}^2} \left(g(x + 2k) - g(-x + 2k)\right), \quad x \in \mathcal{D}.$$

For simplicity we write

$$Z^\varepsilon(t, x) = \varepsilon^\sigma \sum_{j=1}^{2} \int_0^t {}_{\mathcal{S}'(\mathbb{R}^2)}\langle \bar{K}^j(t - s, x - \cdot), d\bar{W}^j_s\rangle_{\mathcal{S}(\mathbb{R}^2)} := \varepsilon^\sigma \int_0^t {}_{\mathcal{S}'}\langle \bar{K}(t - s, x - \cdot), d\bar{W}_s\rangle_{\mathcal{S}}.$$

We also denote

$$\bar{Z}^\varepsilon := Z^\varepsilon + e^{-\varepsilon t\Delta^2} m(z), \tag{4.39}$$

where $z \in H^{-1}$, $m(z)$ is defined in Section 4.1. Then $\bar{Z}^\varepsilon$ is the mild solution to the linear equation

$$\begin{cases} d\bar{Z}^\varepsilon = -\varepsilon\Delta^2\bar{Z}^\varepsilon + \varepsilon^\sigma BdW, \\ \bar{Z}^\varepsilon(0) \equiv m(z) \in \mathbb{R}, \end{cases}$$

with Neumann boundary conditions,

$$\frac{\partial \bar{Z}^\varepsilon}{\partial n} = \frac{\partial \Delta\bar{Z}^\varepsilon}{\partial n} = 0 \text{ on } \partial\mathcal{D},$$

where

$$D(B) = H^1(\mathcal{D}, \mathbb{R}^2), B = \operatorname{div}, D(B^*) = H^1(\mathcal{D}), B^* = -\nabla. \tag{4.40}$$

Let ρ_h be an approximate delta function on $\mathbb{R}^2$ given by

$$\rho_h(x) = h^{-2}\rho(\frac{x}{h}), \qquad \int \rho = 1.$$

Define for any $(t, x) \in \mathcal{D}_T$

$$\begin{aligned} Z^{\varepsilon,h}(t, x) :&= \varepsilon^\sigma \int_0^t {}_{\mathcal{S}'}\langle \bar{K}(\varepsilon(t - r), x - \cdot), d\bar{W}^h_s\rangle_{\mathcal{S}} \\ &= \varepsilon^\sigma \int_0^t {}_{\mathcal{S}'}\langle \bar{K}_h(\varepsilon(t - r), x - \cdot), d\bar{W}_s\rangle_{\mathcal{S}}, \end{aligned} \tag{4.41}$$

where $\bar{W}^h = \bar{W} * \rho_h$, and $\bar{K}_h(t, x) = (\bar{K}^1_h(t, x), \bar{K}^2_h(t, x))$,

$$\bar{K}^j_h(t, x) = \int_{\mathbb{R}^2} \bar{K}^j(t, x - y)\rho_h(y)dy.$$

For fixed $\varepsilon, h > 0$, let $\varphi^{\varepsilon,h}$ be a solution to the following equation on $\mathcal{D}$

$$\begin{cases} \dfrac{d\varphi^{\varepsilon,h}}{dt} = \Delta(-\varepsilon\Delta\varphi^{\varepsilon,h} + \dfrac{1}{\varepsilon} : f(\varphi^{\varepsilon,h} + \bar{Z}^{\varepsilon,h}) :) \\ \varphi^{\varepsilon,h}(0) = (z - m(z)) * \rho_h, \end{cases} \tag{4.42}$$

with Δ the Neumann Laplacian operator on $\mathcal{D}$. Here $: f(\varphi^{\varepsilon,h} + \bar{Z}^{\varepsilon,h}) :$ is the Wick power defined by

$$: f(\varphi^{\varepsilon,h} + \bar{Z}^{\varepsilon,h}) ::= \sum_{k=0}^{3} C_3^k : \left(\bar{Z}^{\varepsilon,h}\right)^{3-k} : \left(\varphi^{\varepsilon,h}\right)^k \tag{4.43}$$

where for any $k = 0, 1, 2, 3$

$$: \left(\bar{Z}^{\varepsilon,h}\right)^k ::= \sum_{l=0}^{3} C_3^k : \left(Z^{\varepsilon,h}\right)^{k-l} : \left(e^{-\varepsilon t \Delta^2} m(z)\right)^k,$$

$$\begin{aligned} &: \left(\bar{Z}^{\varepsilon,h}\right)^0 : := 1, \ : \left(\bar{Z}^{\varepsilon,h}\right) ::= \left(\bar{Z}^{\varepsilon,h}\right), \ : \left(\bar{Z}^{\varepsilon,h}\right)^2 := \left(\bar{Z}^{\varepsilon,h}\right)^2 - c^{\varepsilon}_{h,t}(x), \\ &: \left(\bar{Z}^{\varepsilon,h}\right)^3 : := \left(\bar{Z}^{\varepsilon,h}\right)^3 - 3c^{\varepsilon}_{h,t}(x) \left(\bar{Z}^{\varepsilon,h}\right). \end{aligned}$$

and

$$c^{\varepsilon}_{h,t}(x) = \mathbb{E}\left[\bar{Z}^{\varepsilon,h}(t,x)^2\right]. \tag{4.44}$$

Lemma 4.9. *([LR15, Example 5.2.27]) For any $\varepsilon, h > 0$, there exists a unique solution $\varphi^{\varepsilon,h} \in C([0,T]; L^2(\mathcal{D}))$ to equation (4.42).*

Since $m(z) \in \mathbb{R}$, similar as in the proof in Lemma 3.3 or [MW17, RZZ17b, RYZ18], for any $k = 1, 2, 3$, as $h \to 0$, $: \left(\bar{Z}^{\varepsilon,h}\right)^k :$ converges in $C([0,T], \mathcal{C}^\alpha)$ for any $\alpha < 0$ whose limit is denoted as $: \left(\bar{Z}^{\varepsilon}\right)^k :$. Here $\mathcal{C}^\alpha$ is defined as the Besov space $B^\alpha_{\infty,\infty}$, see Section 2.1 and the reference therein for details.

Then we denote

$$\begin{cases} \dfrac{d\varphi^\varepsilon}{dt} = \Delta(-\varepsilon\Delta\varphi^\varepsilon + \dfrac{1}{\varepsilon} : f(\varphi^\varepsilon + \bar{Z}^\varepsilon) :), \\ \varphi^\varepsilon(0) = z - m(z) \in H_0^{-1}, \end{cases} \tag{4.45}$$

where

$$: f(\varphi^\varepsilon + \bar{Z}^\varepsilon) ::= \sum_{k=0}^{3} C_3^k : \left(\bar{Z}^\varepsilon\right)^{3-k} : (\varphi^\varepsilon)^k. \tag{4.46}$$

Theorem 4.10. *([RYZ18, Theorem 4.4]) For $\mathbb{P} - a.s.\ \omega$, there exists a unique solution φ^ε to equation (4.45) in $C([0,T]; H_0^{-1})$ for any fixed $\varepsilon > 0$.*

Remark 4.11. *We note that in Chapter 3 and [RYZ18], we consider the periodical boundary condition, which is different from the Neumann boundary condition. But by our extension method as we explained before, a similar proof follows.*

In fact, $\varphi^\varepsilon = \lim_{h\to 0} \varphi^{\varepsilon,h}$ in $C([0,T]; H_0^{-1})$. Let $u^{\varepsilon,h} := \varphi^{\varepsilon,h} + Z^{\varepsilon,h}$, $u^{\varepsilon,h}$ also converges to u^ε in $C([0,T]; H^{-1})$, which is the unique solution to (4.3).

4.4.2 The sharp interface limit of equation (4.3)

Similarly as in the proof of Theorem 4.5 we prove that for a suitable choice $h(\varepsilon)$, the solutions to (4.3) will converge to the solution to deterministic Hele-Shaw model (1.17).

The method is a modification of the one in Section 4.3. We consider the residual

$$R^{\varepsilon,h} := u^{\varepsilon,h} - u^\varepsilon_A. \tag{4.47}$$

Let $Y^{\varepsilon,h} = R^{\varepsilon,h} - Z^{\varepsilon,h}$, which satisfies

$$\begin{aligned} dY^{\varepsilon,h} =& -\varepsilon\Delta^2 Y^{\varepsilon,h} dt + \frac{1}{\varepsilon}\Delta\left(f'(u_A^\varepsilon)(Y^{\varepsilon,h}+Z^{\varepsilon,h}) + \mathcal{N}(u_A^\varepsilon, Y^{\varepsilon,h}+Z^{\varepsilon,h})\right) dt \\ & - \frac{c_{h,t}^\varepsilon}{\varepsilon}\Delta(u_A^\varepsilon + Z^{\varepsilon,h} + Y^{\varepsilon,h}) + \Delta r_A^\varepsilon dt, \end{aligned} \tag{4.48}$$

where $c_{h,t}^\varepsilon$ is defined in (4.44). For $Y^{\varepsilon,h}$ we also have the energy estimate:

$$\begin{aligned} \frac{1}{2}\frac{d\|Y^{\varepsilon,h}\|_{H^{-1}}^2}{dt} + \varepsilon\|Y^{\varepsilon,h}(t)\|_{H^1}^2 =& -\frac{1}{\varepsilon}\langle f'(u_A^\varepsilon)(Y^{\varepsilon,h}+Z^{\varepsilon,h}) + \mathcal{N}(u_A^\varepsilon, Y^{\varepsilon,h}+Z^{\varepsilon,h}), Y^{\varepsilon,h}\rangle \\ & - \langle r_A^\varepsilon, Y^{\varepsilon,h}\rangle + \frac{c_{h,t}^\varepsilon}{\varepsilon}\langle u_A^\varepsilon + Y^{\varepsilon,h} + Z^{\varepsilon,h}, Y^{\varepsilon,h}\rangle. \end{aligned} \tag{4.49}$$

In order estimate Y^ε, we still need the estimation of $Z^{\varepsilon,h}$ and $c_{h,t}^\varepsilon$. Analogously to Lemma 4.8 we have

Lemma 4.12. *There exists a constant $C_2 > 0$ such that for any $0 < \beta \le 1$,*

$$\mathbb{E}\left[\|Z^{\varepsilon,h}\|_{C(\mathcal{D}_T)}\right] \le C_2\varepsilon^{\sigma_*}h^{-2},$$

where $\sigma_* = \sigma - \frac{\beta}{4}$. Then for any $\delta > 0$, there exists a constant $C_\delta > 0$, such that

$$\mathbb{P}\left[\Omega'_\delta\right] > 1 - C_\delta\varepsilon^\delta,$$

where $\Omega'_\delta = \{\|Z^{\varepsilon,h}\|_{C(\mathcal{D}_T)} \le C_2\varepsilon^{\sigma_*-2\delta}h^{-2}\}$.

Proof We follow a similar proof as in Lemma 4.8. A factorization formula implies that

$$Z^{\varepsilon,h}(t,x) = \varepsilon^\sigma\frac{\sin\pi\kappa}{\pi}\int_0^t (t-s)^{\kappa-1}\langle M(\varepsilon(t-s), x-\cdot), U^{\varepsilon,h}(s)\rangle ds,$$

where $M(t,x,y)$ is the kernel of $e^{-t\Delta^2}$ and

$$U^{\varepsilon,h}(s,x) = \int_0^t \langle (t-r)^{-\kappa}K_h(\varepsilon(t-r), x, \cdot), dW_s\rangle_{L^2(\mathcal{D},\mathbb{R}^2)},$$

where K_h is defined in (4.55). Combined with (4.56), we have that

$$|K_h(\varepsilon t, x, y)| \lesssim (\varepsilon t)^{-\frac{\beta}{4}}h^{-\eta}\left(|x-y|^{-\zeta} + |x+y|^{-\zeta}\right),$$

where $\beta, \zeta, \eta \ge 0$ and $\beta + \zeta + \eta = 3$. Similarly to (4.15)-(4.20) we have that

$$\begin{aligned} \mathbb{E}\left[\left|U^{\varepsilon,h}(s,x)\right|^2\right] &\lesssim \varepsilon^{-\frac{\beta}{2}}h^{-2\eta}\int_0^s\int_{\mathcal{D}}(s-r)^{-2\kappa-\frac{\beta}{2}}\left(|x+y|^{-2\zeta} + |x-y|^{-2\zeta}\right) dyds \\ &\lesssim \varepsilon^{-\frac{\beta}{2}}h^{-2\eta}s^{1-2\kappa-\frac{\beta}{2}}|x|^{2-2\zeta}, \end{aligned} \tag{4.50}$$

where we require that

$$1 - 2\kappa - \frac{\beta}{2} > 0, \quad \zeta < 1.$$

Similarly to (4.14), we have that

$$\mathbb{E}\left[\|U^{\varepsilon,h}\|_{L^{2p}(\mathcal{D}_T)}\right] \lesssim \varepsilon^{\sigma-\frac{\beta}{4}}h^{-\eta}.$$

Let $\eta = 2$ and $\kappa > 0$ be small enough such that $\beta < 1-\zeta < 2-4\kappa$, $\zeta < 1$. Similarly as in the proof of Lemma 2.7 in [DP04], we have that

$$\begin{aligned}\mathbb{E}\left[\|Z^{\varepsilon,h}(t)\|_{\mathcal{C}(\mathcal{D}_T)}\right] &\lesssim_T \varepsilon^{\sigma}\mathbb{E}\left[\|U^{\varepsilon,h}\|_{L^{2p}(\mathcal{D}_T)}\right] \\ &\lesssim \varepsilon^{\sigma-\frac{\beta}{4}}h^{-2}.\end{aligned} \tag{4.51}$$

Then by Chebyshev's inequality, we finish the proof.

□

For $c^{\varepsilon}_{h,t}$, we have the following estimate:

Lemma 4.13. *There exists a constant $C > 0$ such that for any $(t,x) \in \mathcal{D}_T$ and any $\varepsilon, h \in (0,1)$,*

$$|c^{\varepsilon}_{h,t}(x)| \leq -C\varepsilon^{2\sigma-1}\log h.$$

Proof Following a similar argument as in (4.16), (4.17) and (4.19), we obtain that for all $g \in (g_1, g_2) \in L^2(\mathcal{D}, \mathbb{R}^2)$

$$\begin{aligned}\int_{\mathcal{D}} K(t,x,y)g(y)dy &= \int_{\mathcal{D}} K^1(t,x,y)g_1(y)dy + \int_{\mathcal{D}} K^2(t,x,y)g_2(y)dy \\ &\simeq \sum_k \left(\langle g_1, |k_1|e_k\rangle + \langle g_2, |k_2|e_k\rangle\right) e^{-t|k|^4} e_k(x).\end{aligned}$$

Hence

$$K(t,x,y) \simeq \sum_k |k|e^{-t|k|^4} e_k(x)e_k(y). \tag{4.52}$$

where e_k is defined in (4.6). Note that $e_k(x)e_k(y) = \frac{1}{2}\left(e_k(x-y) + e_k(x+y)\right)$. Thus we obtain

$$K(t,x,y) \simeq \sum_k |k|e^{-t|k|^4}(e_k(x-y) + e_k(x+y)) := P_2(t,x-y) + P_2(t,x+y). \tag{4.53}$$

By [SW72, p282, (c)], we have that for any $(t,x) \in \mathcal{D}_T$,

$$|P_2(t,x)| \lesssim |x|^{-3}e^{-\frac{t}{|x|^4}} \lesssim \left(t^{\frac{1}{3}} + |x|\right)^{-3}. \tag{4.54}$$

Thus we obtain for any $t \in [0,T]$, $x, y \in \mathcal{D}$,

$$|K(\varepsilon t, x, y)| \lesssim \left((\varepsilon t)^{\frac{1}{4}} + |x-y|\right)^{-3} + \left((\varepsilon t)^{\frac{1}{4}} + |x+y|\right)^{-3}.$$

We can extend the definition of $K(t,x,y)$ for $x, y \in \mathbb{R}^2$ with the same form as in (4.52), and denote

$$K_h(t,x,y) := \int_{\mathbb{R}^2} \rho_h(z)K(t,x,y-z)dz. \tag{4.55}$$

Therefore (4.41) becomes

$$Z^{\varepsilon,h}(t,x) = \varepsilon^{\sigma}\int_0^t \langle K_h(\varepsilon(t-r), x-\cdot), dW_s\rangle_{L^2(\mathcal{D},\mathbb{R}^2)}$$

Then by [Hai14, Lemma 10.17] we have that

$$|K_h(\varepsilon t,x,y)| \lesssim \Big((\varepsilon t)^{\frac14}+|x-y|+h\Big)^{-3}+\Big((\varepsilon t)^{\frac14}+|x+y|+h\Big)^{-3}. \tag{4.56}$$

Then we have that for any $(t,x)\in\mathcal{D}_T$.

$$\begin{aligned}|c^\varepsilon_{h,t}(x)| &\le \varepsilon^{2\sigma}\int_0^t\int_{\mathcal{D}}|K^\varepsilon_h(t-r,x,y)|^2drdy\\ &\lesssim \varepsilon^{2\sigma-1}\int_0^{t\varepsilon}\int_{\mathcal{D}}\Big(r^{\frac14}+|x-y|+h\Big)^{-6}drdy+\varepsilon^{2\sigma-1}\int_0^{t\varepsilon}\int_{\mathcal{D}}\Big(r^{\frac14}+|x+y|+h\Big)^{-6}drdy\\ &\lesssim -\varepsilon^{2\sigma-1}\log h.\end{aligned} \tag{4.57}$$

□

Now we have the following main result in this section:

Theorem 4.14. *Let $u^{\varepsilon,h}$ be the unique solution to (4.3) and u^ε_A be defined in Theorem 4.2 with large enough $K>0$. For some $\theta>0$ such that $\varepsilon^\theta\lesssim h^2$, we assume that*

$$\begin{cases}\gamma>13,\\ \sigma>\dfrac13\gamma+\dfrac{13}{3}+\theta.\end{cases} \tag{4.58}$$

Then there exist a generic constant $C>0$ and a constant $C_\delta>0$ for all $0<\delta<\frac{\sigma}{2}-\frac16\gamma-\frac{13}{6}-\frac{\theta}{2}$ such that the following estimates hold

$$\begin{aligned}\mathbb{P}\Big[\|R^{\varepsilon,h}\|_{L^3(\mathcal{D}_T)}\le C\varepsilon^{\frac{\gamma}{3}}\Big]&\ge 1-C_\delta\varepsilon^\delta,\\ \mathbb{P}\Big[\|R^{\varepsilon,h}\|^2_{L^\infty(0,T;H^{-1})}\le C\left(\varepsilon^{\gamma-1}+\varepsilon^{\sigma_*-1-2\delta-\theta}\right)\Big]&\ge 1-C_\delta\varepsilon^\delta,\\ \mathbb{P}\Big[\|v^{\varepsilon,h}-v^\varepsilon_A\|^2_{L^1(0,T;H^{-2})}\le C\varepsilon^{\frac{\gamma}{3}-1}\Big]&\ge 1-C_\delta\varepsilon^\delta.\end{aligned} \tag{4.59}$$

Proof The proof is similar to Section 4.3.

Again we define a stopping time

$$T^{\varepsilon,h}:=T\wedge\inf\{t>0:\int_0^t\|Y^{\varepsilon,h}(\tau)\|^3_{L^3}d\tau>\varepsilon^\gamma\}. \tag{4.60}$$

Then let $t<T^{\varepsilon,h}$ and fix an $\omega\in\Omega'_\delta$. Since

$$h^{-2}\lesssim\varepsilon^{-\theta} \tag{4.61}$$

for some $\theta>0$. We have that

$$-\log h\lesssim-\frac{\theta}{2}\log\varepsilon\lesssim\varepsilon^{-\delta},\quad |c^\varepsilon_{h,t}|\lesssim\varepsilon^{2\sigma-1-\delta}. \tag{4.62}$$

For $\frac{c^\varepsilon_{h,t}}{\varepsilon}\langle u^\varepsilon_A+Y^{\varepsilon,h}+Z^{\varepsilon,h},Y^{\varepsilon,h}\rangle$ we have that for small enough ε

$$\int_0^t\frac{c^\varepsilon_{h,t}}{\varepsilon}|\langle u^\varepsilon_A+Y^{\varepsilon,h}+Z^{\varepsilon,h},Y^{\varepsilon,h}\rangle|d\tau\lesssim\varepsilon^{\frac{\gamma}{3}-1}c^\varepsilon_{h,t}\lesssim\varepsilon^{2\sigma+\frac{\gamma}{3}-2-\delta}.$$

For the rest terms on the right hand side of (4.49), we follow the proof in Section 4.3 by replacing the estimate for Z^ε with the estimate of $Z^{\varepsilon,h}$ in Lemma 4.12. Thus we have that for small enough ε and $t \leq T^{\varepsilon,h}$

$$\sup_{\tau\in[0,t]} \|Y^{\varepsilon,h}(\tau)\|^2_{H^{-1}} d\tau \lesssim \varepsilon^{2\sigma+\frac{\gamma}{3}-2-\delta} + \varepsilon^{\gamma-1} + \varepsilon^{\sigma_*-1-2\delta+\frac{\gamma}{3}-\theta} + \varepsilon^{3(\sigma_*-2\delta-\theta)-1}.$$

Also,

$$\int_0^t \|Y^{\varepsilon,h}(\tau)\|^2_{H^1} d\tau \lesssim \varepsilon^{2\sigma+\frac{\gamma}{3}-3-\delta} + \varepsilon^{\frac{2}{3}(\gamma-1)-1} + \varepsilon^{\sigma_*-2-2\delta+\frac{\gamma}{3}-\theta} + \varepsilon^{3(\sigma_*-2\delta-\theta)-2} + \varepsilon^{\gamma-2}.$$

Hence we have

$$\sup_{\tau\in[0,t]} \|Y^{\varepsilon,h}(\tau)\|^2_{H^{-1}} d\tau \lesssim \varepsilon^{\gamma_1}, \quad \int_0^t \|Y^{\varepsilon,h}(\tau)\|^2_{H^1} d\tau \lesssim \varepsilon^{\gamma_2}, \tag{4.63}$$

where

$$\gamma_1 := (2\sigma + \frac{\gamma}{3} - 2 - \delta) \wedge (\sigma_* - 1 - 2\delta - \theta + \frac{\gamma}{3}) \wedge (3(\sigma_* - 2\delta - \theta) - 1) \wedge (\gamma - 1),$$

$$\gamma_2 := (\gamma_1 - 1) \wedge (\frac{2}{3}\gamma - \frac{5}{3}).$$

Similarly to (4.33), we have

$$\int_0^t \|Y^{\varepsilon,h}\|^3_{L^3} d\tau \lesssim \varepsilon^{\frac{\gamma_1}{2}+\gamma_2}.$$

In order to prove $T^{\varepsilon,N} = T$ for small enough ε, we need to prove $\gamma < \frac{1}{2}\gamma_1 + \gamma_2$. First we assume that $\gamma_2 = \frac{2}{3}\gamma - \frac{5}{3}$, i.e.

$$\gamma_1 > \frac{2}{3}\gamma - \frac{2}{3}.$$

Then $\frac{1}{2}\gamma_1 + \gamma_2 > \gamma$ yields

$$\gamma_1 > \frac{2}{3}\gamma + \frac{10}{3}.$$

A direct calculation yields that

$$\begin{cases} \gamma > 13, \\ \sigma > \frac{1}{3}\gamma + \frac{13}{3} + 2\delta + \theta + \frac{\beta}{4}, \end{cases}$$

which implies that

$$\gamma_1 = (\sigma_* - 1 - 2\delta - \theta + \frac{\gamma}{3}) \wedge (\gamma - 1).$$

Since $\delta, \beta > 0$ can be as small as enough, we can only assume that (4.58) hold and let $0 < 2\delta < \sigma - \frac{1}{3}\gamma - \frac{13}{3} - \theta$.

Since $R^{\varepsilon,h} = Y^{\varepsilon,h} + Z^{\varepsilon,h}$, and $H^1 \subset L^3$, we can obtain the estimate of $R^{\varepsilon,h}$ which is similar to (4.35). Moreover let

$$v^{\varepsilon,h} := -\varepsilon \Delta u^{\varepsilon,h} + \frac{1}{\varepsilon}\left(f(u^{\varepsilon,h}) - 3c^\varepsilon_{h,t} u^{\varepsilon,h}\right),$$

similarly to (4.36), we obtain that

$$\begin{aligned}\|v^{\varepsilon,h}-v_A^{\varepsilon}\|^2_{L^1(0,T;H^{-2})} &\lesssim \varepsilon\|R^{\varepsilon,h}\|_{L^2(\mathcal{D}_T)}+\frac{1}{\varepsilon}\|R^{\varepsilon,h}\|_{L^1(\mathcal{D}_T)}+\frac{1}{\varepsilon}\|c^{\varepsilon}_{h,t}u^{\varepsilon,h}\|^2_{L^1(0,T;H^{-2})}\\ &\lesssim \varepsilon\|R^{\varepsilon,h}\|_{L^3(\mathcal{D}_T)}+\frac{1}{\varepsilon}\|R^{\varepsilon,h}\|_{L^3(\mathcal{D}_T)}+\frac{1}{\varepsilon}\|c^{\varepsilon}_{h,t}\|_{L^\infty}\|R^{\varepsilon,h}+u^{\varepsilon}_A\|_{L^3(\mathcal{D}_T)}\\ &\lesssim \varepsilon^{\frac{\gamma}{3}-1}-\varepsilon^{2\sigma-2}\log h\lesssim \varepsilon^{\frac{\gamma}{3}-1}.\end{aligned}$$

□

Remark 4.15. *It is easy to see that (4.58) implies $\sigma > \frac{26}{3}+\theta$. This implies that the faster that h converges to 0 than ε, the smaller σ could be. Since θ can be small enough, the lower bound for σ is $\frac{26}{3}$.*

Note that the lower bound for σ is smaller than the case of space-time white noise (see Remark 4.6). In fact we can also consider the mollified space-time white noise in Section 4.2 just as in this section. By comparing ε with the converging speed of the noise, the lower bound for σ could be much smaller.

Corollary 4.16. *There exist subsequences $\{\varepsilon_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ with $\varepsilon_k^{\theta}\lesssim h_k^2$ such that for $\mathbb{P}-a.s.\ \omega\in\Omega$*

$$\lim_{k\to\infty} u^{\varepsilon_k,h_k} = -1+2\mathbb{1}_{E_t} \ \ in \ \ L^3(\mathcal{D}_{T_0}),$$

where E_t is the region enclosed by Γ_t.

Proof The proof is the same as Corollary 4.7, we ignore it here for simplicity.

□

Chapter 5

Weak solutions to the sharp interface limit of stochastic Cahn-Hilliard equations

In this chapter, we consider stochastic Cahn-Hilliard equations driven by two types of noise.

First, we consider the sharp interface limit of the following stochastic Cahn-Hilliard equation on a bounded smooth open domain $\mathcal{D} \subset \mathbb{R}^d$ ($d = 2, 3$):

$$\begin{cases} du^\varepsilon = \Delta v^\varepsilon dt + \varepsilon^\sigma dW_t, \quad (t,x) \in (0,T] \times \mathcal{D}, \\ v^\varepsilon = -\varepsilon \Delta u^\varepsilon(t) + \dfrac{1}{\varepsilon} f(u^\varepsilon(t)), \quad (t,x) \in [0,T] \times \mathcal{D}, \\ \dfrac{\partial u^\varepsilon}{\partial n} = \dfrac{\partial v^\varepsilon}{\partial n} = 0, \quad (t,x) \in [0,T] \times \partial\mathcal{D}, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x), \quad x \in \mathcal{D}. \end{cases} \tag{5.1}$$

Here W is a Q-Wiener process where Q satisfies (5.4) and (5.5). $f(u) = F'(u)$ where $F(u) = \frac{1}{4}(u^2-1)^2$ is the double well potential and the initial data u_0^ε satisfies

$$\begin{cases} \sup\limits_{0<\varepsilon\le 1} \displaystyle\int_{\mathcal{D}} \left(\frac{\varepsilon}{2} |\nabla u_0^\varepsilon(x)|^2 + \frac{1}{\varepsilon} F(u_0^\varepsilon(x)) \right) dx \le \mathcal{E}_0 < \infty, \\ \dfrac{1}{|\mathcal{D}|} \displaystyle\int_{\mathcal{D}} u_0^\varepsilon(x) dx = m_0 \in (-1,1) \quad \forall \varepsilon \in (0,1]. \end{cases} \tag{5.2}$$

In the last section, we consider the equation driven by a “smeared” noise:

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \Delta v^\varepsilon + \varepsilon^\sigma \xi_t^\varepsilon, \quad (t,x) \in (0,T] \times \mathcal{D}, \\ v^\varepsilon = -\varepsilon \Delta u^\varepsilon(t) + \dfrac{1}{\varepsilon} f(u^\varepsilon(t)), \quad (t,x) \in [0,T] \times \mathcal{D}, \\ \dfrac{\partial u^\varepsilon}{\partial n} = \dfrac{\partial v^\varepsilon}{\partial n} = 0, \quad (t,x) \in [0,T] \times \partial\mathcal{D}, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x), \quad x \in \mathcal{D}, \end{cases} \tag{5.3}$$

where $\xi_t^\varepsilon = \frac{dW^\varepsilon}{dt}$, $W_t^\varepsilon := \int_{-\infty}^{\infty} \rho_\varepsilon(t-s) W_s ds$ and ρ_ε is an approximate delta function on $\mathbb{R}$. Formally as $\varepsilon \to 0$, $\xi^\varepsilon \to \frac{dW}{dt}$.

5.1 Preliminary

5.1.1 Basic notations and assumptions

In the following, we denote by S^{d-1} the unit sphere in $\mathbb{R}^d$ and $\vec{\nu}$ a generic element in S^{d-1}.

We assume that $\mathcal{D}$ is a smooth bounded open domain in $\mathbb{R}^d$ $(d = 2, 3)$. Let Q be an linear operator on $L^2(\mathcal{D})$, which is commuted with Δ and satisfies

$$Qe_0 = 0, \tag{5.4}$$

where $e_0(x) \equiv 1$ for any $x \in \mathcal{D}$ and

$$\mathrm{Tr}((-\Delta)Q) < +\infty. \tag{5.5}$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a stochastic basis and defined on it a Q-Wiener process W on $L^2(\mathcal{D})$.

According to [DPD96, Remark 2.2], we have that

Theorem 5.1. *Assume that Q satisfies (5.4), (5.5), then for $\mathbb{P} - a.s.$ ω, the equation (5.1) has a unique analytic weak solution $u^\varepsilon \in C([0,T]; H^1 \cap L^4)$.*

Let u^ε be the solution to equation (5.1), we set

$$\mathcal{E}^\varepsilon(t) := \mathcal{E}^\varepsilon(u^\varepsilon)(t) = \int_{\mathcal{D}} e^\varepsilon(u^\varepsilon(t,x))dx, \quad e^\varepsilon(u^\varepsilon) := \frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon}F(u^\varepsilon). \tag{5.6}$$

5.1.2 Definition of a weak solution to the limit of equation (5.1)

Now we recall the following definition of weak solutions to the limit of equation (5.1) introduced in [Che96, Definition 2.1]:

Definition 5.2. *A triple (E, v, V) is called a weak solution to the limit of equation (5.1) if the following holds:*

(i) $E = \cup_{t\in[0,T]}(\{t\} \times E_t)$ is a subset of $\mathcal{D}_T$ and $\mathbb{1}_E \in C([0,T]; L^1) \cap L^\infty(0,T; BV)$;

(ii) $v \in L^2(0,T; H^1)$;

(iii) $V = V(t,x,p)$ is Radon measure on $\mathcal{D}_T \times P$ and for almost every $t \in [0,T]$, $V^t := V(t,\cdot,\cdot)$ is a varifold on $\mathcal{D}$, and there exist Radon measure μ^t on $\bar{\mathcal{D}}$, μ^t-measurable functions $c_1^t, \cdots, c_d^t$, and μ^t-measurable P-valued functions $p_1^t, \cdots, p_d^t$ such that

$$0 \le c_i^t \le 1 \ \ (i = 1, \cdots, d), \quad \sum_{i=1}^d c_i^t \ge 1, \quad \sum_{i=1}^d p_i^t \otimes p_i^t = \mathrm{I} \ \ \mu^t - a.e., \tag{5.7}$$

$$2S|D\mathbb{1}_{E_t}|(x)dx \le d\mu^t(x) \quad \left(S = \int_{-1}^1 \sqrt{\frac{F(s)}{2}}ds = \frac{2}{3}\right), \tag{5.8}$$

$$\int\int_{\mathcal{D}\times P} \psi(x,p)dV^t(x,p) = \sum_{i=1}^d \int_{\mathcal{D}} c_i^t(x)\psi(x, p_i^t(x))d\mu^t(x) \quad \forall \psi \in C_c(\mathcal{D} \times P); \tag{5.9}$$

(iv) For any $t \in (0,T]$ and for almost every $\tau \in (0,t)$,

$$\int_0^t \int_{\mathcal{D}} (-2\mathbb{1}_{E_\tau}\partial_t\psi + \nabla v \cdot \nabla\psi)\, dxd\tau = \int_{\mathcal{D}} 2\mathbb{1}_{E_0}\psi(0,\cdot) \quad \forall \psi \in C_c^1([0,t) \times \bar{\mathcal{D}}), \tag{5.10}$$

$$-\langle D\mathbb{1}_{E_t}, v\vec{Y}\rangle := \langle \mathbb{1}_{E_t}, \mathrm{div}(v\vec{Y})\rangle = \frac{1}{2}\langle \delta V^t, \vec{Y}\rangle \quad \forall \vec{Y} \in C_c^1(\mathcal{D}; \mathbb{R}^d), \tag{5.11}$$

$$\mu^t(\bar{\mathcal{D}}) + \int_\tau^t \int_{\mathcal{D}} |\nabla v|^2 \le \mu^\tau(\bar{\mathcal{D}}). \tag{5.12}$$

5.1.3 Main results for Q-Wiener noise

Theorem 5.3. *Assume that $\sigma \geq \frac{1}{2}$ and (5.2) hold. Let Q satisfy (5.4) and (5.5). Let $(u^\varepsilon, v^\varepsilon)$ be the solution to (5.1). Then there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}, \tilde{\mathbb{P}})$, $(\tilde{u}^\varepsilon, \tilde{v}^\varepsilon) \in C([0,T];L^2) \times L^2(0,T;H^1)$ with $\tilde{\mathbb{P}} \circ (\tilde{u}^\varepsilon, \tilde{v}^\varepsilon)^{-1} = \mathbb{P} \circ (u^\varepsilon, v^\varepsilon)^{-1}$ on $C([0,T];L^2) \times L^2(0,T;H^1)$. There also exists a subsequence ε_k such that as $\varepsilon_k \searrow 0$ the following holds:*

(i) There exists a measurable set $E \subset \tilde{\Omega} \times \mathcal{D}_T$, such that $\mathbb{1}_E$ is $\{\tilde{\mathcal{F}}_t\}$-adapted in $L^2(\mathcal{D})$ and for $\tilde{\mathbb{P}} - a.s.\ \omega$

$$\tilde{u}^{\varepsilon_k}(\omega) \to -1 + 2\mathbb{1}_{E(\omega)}, \quad a.e.\ in\ \mathcal{D}_T\ and\ in\ C^\beta([0,T];L^2)$$

for any $\beta < \frac{1}{12}$ where $E(\omega) := \{(t,x) \in \mathcal{D}_T : (\omega,t,x) \in E\}$;

(ii) There exists v which is weakly measurable in $L^2(0,T;H^1)$, such that for $\tilde{\mathbb{P}} - a.s.\ \omega$

$$\tilde{v}^{\varepsilon_k}(\omega) \to v(\omega) \quad weakly\ in\ L^2(0,T;H^1);$$

(iii) There exist random variables $\mu \in \mathfrak{M}_R$ and $\{\mu_{ij}\}_{i,j=1}^d \in \mathfrak{M}^{d\times d}$ such that for $\tilde{\mathbb{P}} - a.s.\ \omega$

$$\begin{aligned} e^{\varepsilon_k(\omega)}(\tilde{u}^{\varepsilon_k(\omega)})dxdt \to d\mu(\omega,t,x) \quad & weakly\ in\ \mathfrak{M}_R, \\ \varepsilon_k \partial_{x_i}\tilde{u}^{\varepsilon_k}(\omega)\partial_{x_j}\tilde{u}^{\varepsilon_k}(\omega)dxdt \to d\mu_{ij}(\omega,t,x) \quad & weakly\ in\ \mathfrak{M},\ \ \forall i,j = 1,\cdots,d. \end{aligned} \tag{5.13}$$

(iv) For $\tilde{\mathbb{P}} - a.s.\ \omega$, there exists Radon measure $V(\omega)$ on $\mathcal{D}_T \times P$, and $\mu^t(\omega,x)dt = d\mu(\omega,t,x)$ such that for any $t \in (0,T]$ and $\vec{Y} \in C_c^1(\mathcal{D}_t;\mathbb{R}^d)$

$$\int_0^t \langle \delta V^s, \vec{Y} \rangle ds = \int_0^t \int_{\mathcal{D}} \nabla \vec{Y} : \left(I d\mu(s,x) - (\mu_{ij}(s,x))_{d\times d} \right). \tag{5.14}$$

For $\tilde{\mathbb{P}} - a.s.\ \omega$, $(E(\omega), v(\omega), V(\omega))$ satisfies all the properties in Definition 5.2 except (5.12). In particular, if $\sigma > \frac{1}{2}$, (5.12) holds, thus $(E(\omega), v(\omega), V(\omega))$ is a weak solution in the sense of Definition 5.2.

Theorem 5.4. *Let $\sigma \geq \frac{1}{2}$, with the same notations as in Theorem 5.3, and suppose that the assumptions in Theorem 5.3 hold. Then in radially symmetric case, that is $\mathcal{D} = B_1$, where B_1 is the unit ball in $\mathbb{R}^d$ and that u_0^ε is radially symmetric, we have that*

$$d\mu = 2S|D\mathbb{1}_{E_t}|dxdt\ \ as\ Radon\ measure\ on\ \mathcal{D}_T.$$

In particular, for a.e.$t \in [0,T]$, V^t is a $(d-1)$-rectifiable varifold (see [Sim83, Section 11, Section 38] for definition), i.e.

$$dV(t,x,p) = 2S|D\mathbb{1}_{E_t}|dxdt\delta_{\vec{\nu}_{E_t}(t,x)}(dp)\ \ as\ Radon\ measure\ on\ \mathcal{D}_T \times P,$$

where $\vec{\nu}_{E_t}$ is defined in (2.3). Then we have that

$$\begin{cases} (d\mu_{ij})_{d\times d} = \vec{\nu}_{E_t} \otimes \vec{\nu}_{E_t} d\mu\ \ as\ Radon\ measure\ on\ \bar{\mathcal{D}}_T, \\ \quad v(t,x) = S\vec{\nu}_{E_t}(x) \cdot \vec{H}_{V^t}(x)\ \ on\ \operatorname{supp}(|D\mathbb{1}_{E_t}|)\ \ for\ a.e.\ t \in [0,T], \end{cases} \tag{5.15}$$

$\vec{H}_{V^t}$ is the mean curvature vector of V^t defined in Definition 2.22 and $\delta_{\vec{\nu}}$ is the Dirac measure concentrated at $\vec{\nu} \in P$.

Remark 5.5. *Since E_t is a BV set for a.e. $t \in [0,T]$, by Theorem 2.16, $\partial^* E_t$ is a $(d-1)$-rectifiable set and*

$$|D\mathbb{1}_{E_t}| = \mathcal{H}^{d-1}\lfloor\partial^* E_t.$$

Then in radial symmetric case, for a.e. $t \in [0,T]$

$$\mu^t = 2S|D\mathbb{1}_{E_t}| = 2S\mathcal{H}^{d-1}\lfloor\partial^* E_t.$$

By Remark 2.8, when E_t is a smooth domain, $\vec{H}_{V^t}$ is just the classical mean curvature vector of ∂E_t and $\vec{\nu}_{E_t}$ is the inward normal vector of ∂E_t. Thus the last equation in (5.15) gives a weak formula of the third equation in (1.5).

5.1.4 Remarks on the definition of weak solutions

Suppose that (E, v, V) is a weak solution of Definition 5.2. In the following, we show how Definition 5.2 is connected with (1.5). This has been obtained in [Che96, Subsection 2.4]. We give more details for complete results.

Observe that in distribution sense, $\partial_t \mathbb{1}_E$ is defined for any $\psi \in C_c^1([0,t)\times\bar{\mathcal{D}})$

$$\int_0^t\int_{\mathcal{D}}(\partial_t\mathbb{1}_E)\psi = \int_0^t\int_{\mathcal{D}}\partial_t(\mathbb{1}_E\psi) - \int_0^t\int_{\mathcal{D}}\mathbb{1}_E\partial_t\psi = -\int_{\mathcal{D}}\mathbb{1}_{E_0}\psi(0,x)dx - \int_0^t\int_{\mathcal{D}}\mathbb{1}_E\partial_t\psi dxds,$$

Thus (5.10) implies that in distribution sense

$$2\partial_t\mathbb{1}_E = \Delta v, \quad \text{in } [0,T]\times\mathcal{D}.$$

Since $v \in L^2(0,T;H^1)$, Δv and $\frac{\partial v}{\partial n}$ are ill-defined in (1.5). They have to be understood in distribution sense. We suppose that (v,Γ) is smooth enough such that Δv and $\frac{\partial v}{\partial n}$ are well-defined.

Suppose that $\overline{\mathcal{D}\setminus E} \subset \mathcal{D}$. Denote $\Gamma_t := \partial E_t \setminus \partial\mathcal{D}$ and let $\mathcal{D}^+ = E_t^o \cap \mathcal{D}$ be the interior of E_t in $\mathcal{D}$ and $\mathcal{D}^- = \mathcal{D}\setminus\bar{E}_t$.

For the first equation in (1.5): For any $x \in \mathcal{D}\setminus\Gamma$, $\Delta v(x) = 0$ since $\mathbb{1}_E(x)$ is a constant in time. More precisely, let $\psi \in C_c^1([0,t)\times\bar{\mathcal{D}})$ and $\text{supp}\psi(s,\cdot) \subset \mathcal{D}\setminus\Gamma_s$ for any $s \in [0,t)$, we have that

$$\int_0^t\int_{\mathcal{D}}\mathbb{1}_{E_t}\partial_t\psi dxds = \int_0^t\int_{\mathcal{D}}\partial_t\psi dxds = -\int_{\mathcal{D}}\psi(0,\cdot)dx = -\int_{\mathcal{D}}\mathbb{1}_{E_0}\psi(0,\cdot)dx.$$

Then (5.10) implies that

$$\int_0^t\int_{\mathcal{D}}\nabla v\cdot\nabla\psi dxds = 0,$$

which is the weak formula of the first equation in (1.5).

For the second equation in (1.5): Since $\overline{\mathcal{D}\setminus E} \subset \mathcal{D}$, $\partial\mathcal{D}^+ = \partial\mathcal{D}\cup\Gamma$. For any

$\psi \in C_c^1([0,t)\times \bar{\mathcal{D}^+})$ and $\text{supp}\psi(s,\cdot)\subset \mathcal{D}^+$ for any $s\in[0,t)$,

$$
\begin{aligned}
\int_0^t\int_{\partial\mathcal{D}}\frac{\partial v}{\partial n}\psi d\mathcal{H}^{d-1}ds =& \int_0^t\int_{\mathcal{D}^+}\text{div}(\nabla v\psi)dxds\\
=& \int_0^t\int_{\mathcal{D}^+}\nabla v\cdot\nabla\psi dxds + \int_0^t\int_{\mathcal{D}^+}\Delta v\psi dxds\\
=& \int_0^t\int_{\mathcal{D}}\nabla v\cdot\nabla\psi dxds + 2\int_0^t\int_{\mathcal{D}}(\partial_t\mathbb{1}_E)\psi dxds\\
=& \int_0^t\int_{\mathcal{D}}\nabla v\cdot\nabla\psi dxds - 2\int_{\mathcal{D}}\mathbb{1}_{E_0}\psi(0,x)dx - 2\int_0^t\int_{\mathcal{D}}\mathbb{1}_E\partial_t\psi dxds\\
=&0,
\end{aligned}
\tag{5.16}
$$

where we used (5.10) in the last equality. Thus we obtain in distribution sense the following holds.

$$
\frac{\partial v}{\partial n}=0, \quad \text{on } [0,T]\times\partial\mathcal{D}.
$$

For the last equation in (1.5): For any $\psi\in C_c^1(\bar{\mathcal{D}}_t)$

$$
\begin{aligned}
\int_{\mathcal{D}}\partial_t\mathbb{1}_{E_t}\psi d\mathcal{H}^d =& -\frac{1}{2}\int_{\mathcal{D}}\nabla v\nabla\psi d\mathcal{H}^d\\
=& -\frac{1}{2}\int_{\mathcal{D}^+}\nabla v\nabla\psi d\mathcal{H}^d - \frac{1}{2}\int_{\mathcal{D}^-}\nabla v\nabla\psi d\mathcal{H}^d\\
=& \frac{1}{2}\int_{\mathcal{D}^+}\text{div}(\nabla v\psi)d\mathcal{H}^d + \frac{1}{2}\int_{\mathcal{D}^-}\text{div}(\nabla v\psi)d\mathcal{H}^d\\
=& \frac{1}{2}\int_{\Gamma_t}(\partial_n v^+ - \partial_n v^-)\psi d\mathcal{H}^{d-1}.
\end{aligned}
\tag{5.17}
$$

By using the weak formula of normal velocity in [Ton19, (2.5)], we have that

$$
\partial_t\int_{\Gamma_t}gd\mathcal{H}^{d-1} - \int_{\Gamma_t}\partial_t g d\mathcal{H}^{d-1} = \int_{\Gamma_t}\mathcal{V}\nabla g\cdot\vec{n} - gH\mathcal{V}d\mathcal{H}^{d-1}.
$$

Let φ satisfy

$$
\begin{aligned}
&\Delta\varphi=\psi,\\
&\frac{\partial\varphi}{\partial n}=0 \ \text{ on } \ \partial\mathcal{D},
\end{aligned}
$$

and take $g=\frac{\partial\varphi}{\partial n}$, we obtain that

$$
\begin{aligned}
\int_{\mathcal{D}}\partial_t\mathbb{1}_{E_t}\psi d\mathcal{H}^d =& \partial_t\int_{\mathcal{D}}\mathbb{1}_{E_t}\psi d\mathcal{H}^d - \int_{\mathcal{D}}\mathbb{1}_{E_t}\partial_t\psi d\mathcal{H}^d\\
=& \partial_t\int_{\mathcal{D}^+}\Delta\varphi d\mathcal{H}^d - \int_{\mathcal{D}^+}\Delta\partial_t\varphi d\mathcal{H}^d\\
=& \partial_t\int_{\Gamma_t}\frac{\partial\varphi}{\partial n}d\mathcal{H}^{d-1} - \int_{\Gamma_t}\partial_t\frac{\partial\varphi}{\partial n}d\mathcal{H}^{d-1}\\
=& \int_{\Gamma_t}\mathcal{V}\nabla\frac{\partial\varphi}{\partial n}\cdot\vec{n} - \frac{\partial\varphi}{\partial n}H\mathcal{V}d\mathcal{H}^{d-1}
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\Gamma_t} \Delta\varphi \mathcal{V} d\mathcal{H}^{d-1} + \int_{\Gamma_t} \frac{\partial \varphi}{\partial n} \mathcal{V} (\operatorname{div} \vec{n} - H)\, d\mathcal{H}^{d-1} \\
&= \int_{\Gamma_t} \psi \mathcal{V} d\mathcal{H}^{d-1} + \int_{\Gamma_t} \frac{\partial \varphi}{\partial n} \mathcal{V} (\operatorname{div} \vec{n} - H)\, d\mathcal{H}^{d-1}.
\end{aligned}
$$

By our assumption, $\mathcal{D}^+$ is the exterior of Γ. Thus $\vec{n}$ is the inward normal vector of the interior of Γ. Then if $\Gamma_t = \{(x, y) \in \mathcal{D} : y = \phi(x) \in \mathbb{R}\}$, it is well-known that

$$
H = \operatorname{div}\left(\frac{\nabla\phi}{\sqrt{1 + |\nabla\phi|^2}}\right), \vec{n} = -\frac{1}{\sqrt{1 + |\nabla\phi|^2}}(-\nabla\phi, 1)^T,
$$

which implies that $\operatorname{div} \vec{n} = H$ on Γ_t. Hence we obtain that

$$
\int_{\Gamma_t} \psi \mathcal{V} d\mathcal{H}^{d-1} = \frac{1}{2} \int_{\Gamma_t} (\partial_n v^+ - \partial_n v^-) \psi d\mathcal{H}^{d-1},
$$

which yields that in distribution sense

$$
\mathcal{V} = \frac{1}{2}(\partial_n v^+ - \partial_n v^-).
$$

Therefore we know that (5.10) is a weak formulation of all the equations in (1.5) except the third equation.

For the third equation in (1.5): following the argument in [Che96, Subsection 2.4], we can only prove the third equation in weak sense in the radial symmetric case as in Theorem 5.4 and Remark 5.5.

In general case, it was shown in [RT08, Theorem 3.1, Theorem 3.2], under the assumption that for *a.e.* $t \in [0, T]$, $v^\varepsilon(t, \cdot) \to v(t, \cdot)$ weakly in $W^{1,p}$ for $p > d$, the authors proved that

$$
v(t, x) = S\vec{H}_{V^t} \cdot \vec{\nu}_{E_t}, \ \ \mathcal{H}^{d-1} - a.e.\ x \in \partial^* E_t. \tag{5.18}
$$

But the assumption that $v^\varepsilon \to v$ weakly in $W^{1,p}$ for $p > d$ has not been obtained until now since we can only obtain the convergence in $H^1 = W^{1,2}$.

In fact, identifying the value of v on the interface Γ_t is the main task of varifold approach to study the sharp interface limit of both Cahn-Hilliard equation and Allen-Cahn equation (cf. [HT00, Ton02, Ton05, RS06, Le08, RT08]). Until now, a fully rigorous proof for the (deterministic) Cahn-Hilliard equation is still missing.

5.2 Convergence

5.2.1 Lyapunov functional $\mathcal{E}^\varepsilon$ and basic estimates

In the deterministic case, where no forcing terms are present, the function $\mathcal{E}^\varepsilon$ defined in (5.6) decreases in time. In stochastic case, the authors in [DPD96] showed a similar property when $\varepsilon = 1$ and (5.4) is satisfied. Using the same trick we can prove a similar result.

Lemma 5.6. *There exists a constant which only depends on T and $0 < \varepsilon_0 < 1$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and any $p \geq 1$,*

$$
\mathbb{E} \sup_{t \in [0,T]} \mathcal{E}^\varepsilon(t)^p \leq C_T(\varepsilon^{2\sigma - 1} + \mathcal{E}_0)^p, \tag{5.19}
$$

and

$$\mathbb{E}\left(\int_0^T \|\nabla v^\varepsilon\|_{L^2}^2 dt\right)^p \leq C_T(\varepsilon^{2\sigma-1} + \mathcal{E}_0)^p. \tag{5.20}$$

Proof We will not give all the details of the proof since it is the same as [DPD96, Section 2.3], we only calculate the order of ε for every term in the following.

Applying Itô's formula on $\mathcal{E}^\varepsilon$, we have that

$$\begin{aligned} d\mathcal{E}^\varepsilon(u^\varepsilon) &= \langle D\mathcal{E}^\varepsilon(u^\varepsilon), du^\varepsilon\rangle + \frac{\varepsilon^{2\sigma}}{2}\mathrm{Tr}(QD^2\mathcal{E}^\varepsilon(u^\varepsilon))dt \\ &= -\langle \nabla v^\varepsilon, \nabla v^\varepsilon\rangle dt + \frac{\varepsilon^{2\sigma+1}}{2}\mathrm{Tr}(-\Delta Q)dt + \frac{\varepsilon^{2\sigma-1}}{2}\mathrm{Tr}(f'(u^\varepsilon)Q)dt + \varepsilon^\sigma\langle v^\varepsilon, dW_t\rangle. \end{aligned} \tag{5.21}$$

By using the same trick as in [DPD96, Section 2.3] we have that

$$\mathrm{Tr}(f'(u^\varepsilon)Q) \lesssim 1 + \varepsilon\mathcal{E}^\varepsilon(u^\varepsilon).$$

Hence we deduce from (5.21) that for any $p \geq 1$,

$$\mathbb{E}\left(\sup_{t\in[0,T]}\mathcal{E}^\varepsilon(t) + \int_0^T \|\nabla v^\varepsilon\|_{L^2}^2 ds\right)^p \lesssim \mathbb{E}\left(\mathcal{E}_0 + \varepsilon^{2\sigma-1} + \varepsilon^{2\sigma+1} + \varepsilon^{2\sigma}\sup_{t\in[0,T]}\mathcal{E}^\varepsilon(t) + \varepsilon^\sigma\sup_{t\in[0,T]}|M^\varepsilon(t)|\right)^p$$

where $M^\varepsilon(t) := \int_0^t\langle v^\varepsilon, dW_s\rangle$. Let ε be small enough, we have that

$$\mathbb{E}\left(\sup_{t\in[0,T]}\mathcal{E}^\varepsilon(t)\right)^p + \mathbb{E}\left(\int_0^T \|\nabla v^\varepsilon\|_{L^2}^2 ds\right)^p \lesssim \varepsilon^{p(2\sigma-1)} + \mathcal{E}_0^p + \mathbb{E}\sup_{t\in[0,T]}|M^\varepsilon(t)|^p,$$

By Burkholder-Davis-Gundy's inequality

$$\begin{aligned} \mathbb{E}\sup_{t\in[0,T]}|M^\varepsilon(t)|^p &\lesssim \mathbb{E}\left(\langle M^\varepsilon\rangle_T\right)^{\frac{p}{2}} = \mathbb{E}\left(\int_0^T \|\sqrt{Q}v^\varepsilon(t)\|_{L^2}^2 dt\right)^{\frac{p}{2}} \\ &\lesssim \mathbb{E}\left(\int_0^T \|\nabla v^\varepsilon\|_{L^2}^2 dt\right)^{\frac{p}{2}}. \end{aligned}$$

Then by Young's inequality, for any $\kappa > 0$, there exists a constant $C_1 \equiv C_1(T)$ such that

$$\mathbb{E}\sup_{t\in[0,T]}|M^\varepsilon(t)|^p \leq C_1 + \kappa\mathbb{E}\left(\int_0^T \|\nabla v^\varepsilon\|_{L^2}^2 ds\right)^p.$$

Thus for a small enough $\kappa > 0$, there exists a constant $C_T > 0$ such that

$$\mathbb{E}\sup_{t\in[0,T]}\mathcal{E}^\varepsilon(t)^p + (1-\kappa)\mathbb{E}\left(\int_0^T \|\nabla v^\varepsilon\|_{L^2}^2 ds\right)^p \leq C_T(\varepsilon^{p(2\sigma-1)} + \mathcal{E}_0^p),$$

which implies our results. □

Corollary 5.7. *There exists a constant $C_T > 0$, such that for any $p \geq 1$*

$$\mathbb{E}\sup_{t\in[0,T]}\left(\int_{\mathcal{D}} F(u^\varepsilon(t,x))dx\right)^p \leq C_T\varepsilon^p(\mathcal{E}_0^p + \varepsilon^{p(2\sigma-1)}) \tag{5.22}$$

and

$$\mathbb{E}\sup_{t\in[0,T]}\|u^\varepsilon(t)\|_{L^4}^{4p} \leq C_T(1 + \mathcal{E}_0^p + \varepsilon^{p(2\sigma-1)}). \tag{5.23}$$

In the rest of this section, we always assume $\sigma \geq \frac{1}{2}$.

5.2.2 Estimates for $\{u^\varepsilon\}$

We introduce a function $g^\varepsilon(t,x)$ defined by

$$g^\varepsilon(t,x) := G(u^\varepsilon(t,x)), \tag{5.24}$$

where

$$G(u) := \int_{-1}^{u} \sqrt{2F(x)}dx, \quad \forall u \in \mathbb{R}.$$

Observe that

$$\int_{\mathcal{D}} |\nabla g^\varepsilon(t,\cdot)| = \int_{\mathcal{D}} \sqrt{2F(u^\varepsilon)}|\nabla u^\varepsilon|dx \le \int_{\mathcal{D}} e^\varepsilon(u^\varepsilon)(t)dx = \mathcal{E}^\varepsilon(t), \tag{5.25}$$

and there are positive constants c_1, c_2 such that

$$c_1|u_1-u_2|^2 \le |G(u_1)-G(u_2)| \le c_2|u_1-u_2|(1+|u_1|+|u_2|), \quad \forall u_1,u_2 \in \mathbb{R}. \tag{5.26}$$

Lemma 5.8. *There exists constant $C_T>0$ which only depends on T, such that for any $\beta \in (0,\frac{1}{12})$,*

$$\mathbb{E}\left(\|g^\varepsilon\|_{L^\infty(0,T;W^{1,1})} + \|g^\varepsilon\|_{C^\beta([0,T];L^1)} + \|u^\varepsilon\|_{C^\beta([0,T];L^2)}\right) \le C_T$$

Proof Similarly to the proof of [Che96, Lemma 3.2], let ρ be any fixed mollifier satisfying

$$\rho \in C^\infty(\mathbb{R}^d), \quad 0 \le \rho \le 1, \quad \mathrm{supp}\rho \subset B_1(0), \quad \int_{\mathbb{R}^d} \rho = 1,$$

where B_1 is the unit ball in $\mathbb{R}^d$ centered at 0. For any small $\eta>0$, we define

$$u^\varepsilon_\eta(t,x) = \int_{B_1} \rho(y)u^\varepsilon(t,x-\eta y)dy.$$

Here we assume that u^ε is extended to $\{x\in\mathbb{R}^d : d(x,\mathcal{D}) \le \eta_0\}$ by

$$u^\varepsilon(t,y+\eta n(y)) = u^\varepsilon(t,y-\eta n(y)), \quad y \in \partial\mathcal{D}, \eta \in [0,\eta_0],$$

where η_0 is a small positive number and $n(y)$ is the unit outward normal vector to $\partial\mathcal{D}$ at $y \in \partial\mathcal{D}$.

Then by (5.23), we have that for any $p>1$, $\eta \in (0,\eta_0)$,

$$\mathbb{E}\sup_{t\in[0,T]} \|\nabla u^\varepsilon_\eta(t)\|^p_{L^2} \lesssim \eta^{-p}\mathbb{E}\sup_{t\in[0,T]} \|u^\varepsilon(t)\|^p_{L^2} \lesssim \eta^{-p}, \tag{5.27}$$

and

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}\left(\int_{\mathcal{D}} |u^\varepsilon_\eta - u^\varepsilon|^2dx\right)^p &\le \mathbb{E}\left(\sup_{t\in[0,T]}\int_{\mathcal{D}}\int_{B_1} \rho(y)|u^\varepsilon(t,x-\eta y)-u^\varepsilon(t,x)|^2dydx\right)^p \\
&\lesssim \mathbb{E}\left(\sup_{t\in[0,T]}\int_{\mathcal{D}}\int_{B_1} \rho(y)|g^\varepsilon(t,x-\eta y)-g^\varepsilon(t,x)|dydx\right)^p \\
&\lesssim \eta^p\mathbb{E}\sup_{t\in[0,T]}\|\nabla g^\varepsilon(t)\|^p_{L^1} \\
&\le \eta^p\mathbb{E}\sup_{t\in[0,T]}\mathcal{E}^\varepsilon(t)^p \lesssim \eta^p,
\end{aligned} \tag{5.28}$$

where we use (5.26) in the second inequality and (5.25), (5.19) in the last line.

For any $0 \le \tau < t \le T$, by using $u^\varepsilon(t) - u^\varepsilon(\tau) = \int_\tau^t \Delta v^\varepsilon(s)ds + \varepsilon^\sigma(W_t - W_\tau)$ (in weak sense), we have that

$$\begin{aligned}
&\mathbb{E}\left(\int_{\mathcal{D}} |\left(u_\eta^\varepsilon(t,x) - u_\eta^\varepsilon(\tau,x)\right)\left(u^\varepsilon(t,x) - u^\varepsilon(\tau,x)\right)|dx\right)^p \\
\le&\mathbb{E}\left(\int_\tau^t \int_{\mathcal{D}} |\nabla v^\varepsilon(s,x)\left(\nabla u_\eta^\varepsilon(t,x) - \nabla u_\eta^\varepsilon(\tau,x)\right)|dxds\right)^p \\
&+ \varepsilon^{p\sigma}\mathbb{E}\left(\int_{\mathcal{D}} |\left(u_\eta^\varepsilon(t,x) - u_\eta^\varepsilon(\tau,x)\right)(W_t - W_\tau)|dx\right)^p \\
\lesssim&\mathbb{E}\left(\int_\tau^t \int_{\mathcal{D}} |\nabla v^\varepsilon|^2\right)^{\frac{p}{2}} (t-\tau)^{\frac{p}{2}} \left(\mathbb{E}\sup_{s\in[0,T]} \|\nabla u_\eta^\varepsilon(s)\|_{L^2}^p\right) \\
&+ \varepsilon^{p\sigma}\mathbb{E}\sup_{s\in[0,T]} \|u_\eta^\varepsilon(s)\|_{L^2}^p \left(\mathbb{E}\|W_t - W_\tau\|_{L^2}^{2p}\right)^{\frac{1}{2}} \\
\lesssim&(t-\tau)^{\frac{p}{2}}\eta^{-p} + (t-\tau)^{\frac{p}{2}}\varepsilon^{p\sigma} \\
\lesssim&\eta^{-p}(t-\tau)^{\frac{p}{2}},
\end{aligned} \tag{5.29}$$

where in the third inequality we use (5.20), (5.23), (5.27) and the fact that

$$\mathbb{E}\|W_t - W_\tau\|_{L^2}^{2p} \lesssim |t-\tau|^p.$$

Then we have that

$$\begin{aligned}
\mathbb{E}\|u^\varepsilon(t) - u^\varepsilon(\tau)\|_{L^2}^{2p} \lesssim&\mathbb{E}\left(\int_{\mathcal{D}} |\left(u_\eta^\varepsilon(t,x) - u_\eta^\varepsilon(\tau,x)\right)\left(u^\varepsilon(t,x) - u^\varepsilon(\tau,x)\right)|dx\right)^p \\
&+ \mathbb{E}\left(\int_{\mathcal{D}} |\left(u^\varepsilon(t,x) - u_\eta^\varepsilon(t,x)\right)\left(u^\varepsilon(t,x) - u^\varepsilon(\tau,x)\right)|dx\right)^p \\
&+ \mathbb{E}\left(\int_{\mathcal{D}} |\left(u^\varepsilon(\tau,x) - u_\eta^\varepsilon(\tau,x)\right)\left(u^\varepsilon(t,x) - u^\varepsilon(\tau,x)\right)|dx\right)^p \\
\lesssim&\eta^{-p}(t-\tau)^{\frac{p}{2}} + \left(\mathbb{E}\left(\sup_{t\in[0,T]} \|u_\eta^\varepsilon(t) - u^\varepsilon(t)\|_{L^2}\right)^{2p}\right)^{\frac{1}{2}} \left(\mathbb{E}\sup_{t\in[0,T]} \|u^\varepsilon\|_{L^2}^{2p}\right)^{\frac{1}{2}} \\
\lesssim&\eta^{-p}(t-\tau)^{\frac{p}{2}} + \eta^{\frac{p}{2}},
\end{aligned}$$

where we use (5.29) in the second inequality and (5.23), (5.28) in the last inequality. If we take $\eta = \eta_0 \wedge (t-\tau)^{\frac{1}{3}}$, we have that

$$\mathbb{E}\|u^\varepsilon(t) - u^\varepsilon(\tau)\|_{L^2}^{2p} \lesssim \eta^{-p}(t-\tau)^{\frac{p}{2}} + \eta^{\frac{p}{2}} \le (t-\tau)^{\frac{p}{6}}. \tag{5.30}$$

Moreover, using (5.26) we have that

$$\begin{aligned}
\mathbb{E}\|g^\varepsilon(t) - g^\varepsilon(\tau)\|_{L^1}^p \lesssim& \mathbb{E}\left(\int_{\mathcal{D}} |u^\varepsilon(t,x) - u^\varepsilon(\tau,x)|\left(1 + |u^\varepsilon(t,x)| + |u^\varepsilon(\tau,x)|\right)dx\right)^p \\
\lesssim& \mathbb{E}\|u^\varepsilon(t) - u^\varepsilon(\tau)\|_{L^2}^p \left(1 + \mathbb{E}\sup_{t\in[0,T]} \|u^\varepsilon\|_{L^2}^p\right) \\
\lesssim& (t-\tau)^{\frac{p}{12}},
\end{aligned} \tag{5.31}$$

where we use (5.23) and (5.30) in the last inequality.

Finally by Kolmogorov's criteria (see e.g. [DPZ14, Theorem 3.3]), for any $0 < \beta < \frac{1}{12}$,

$$\mathbb{E}\left(\|g^\varepsilon\|_{C^\beta([0,T];L^1)} + \|u^\varepsilon\|_{C^\beta([0,T];L^2)}\right) \lesssim 1.$$

Moreover by (5.25)

$$\mathbb{E} \sup_{t\in[0,T]} \|\nabla g^\varepsilon(t)\|_{L^1} \lesssim 1.$$

Thus

$$\mathbb{E}\|g\|_{L^\infty(0,T;W^{1,1})} \lesssim \mathbb{E} \sup_{t\in[0,T]} \|\nabla g^\varepsilon(t)\|_{L^1} + \mathbb{E} \sup_{t\in[0,T]} \|g^\varepsilon(t)\|_{L^1} \lesssim 1$$

□

5.2.3 Estimates for $\{v^\varepsilon\}$

We want to obtain the estimate of v^ε in the space H^1. By (5.20) and Poincaré-Wirtinger inequality, it is enough to estimate $\bar{v^\varepsilon} := \frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} v^\varepsilon(x)dx$.

Lemma 5.9. *For any $\delta > 0$, there exists a constant $C \equiv C(\delta, T) > 0$, such that*

$$\mathbb{P}\left(\int_0^T \|v^\varepsilon(t)\|_{H^1}^2 dt \le C\right) \ge 1 - \delta.$$

Proof For any $R > 0$, set

$$A_R := \left\{\omega \in \Omega : \|u^\varepsilon(\omega)\|_{C([0,T];L^2)} + \sup_{t\in[0,T]} \mathcal{E}^\varepsilon(t)(\omega)^p + \int_0^T \|\nabla v^\varepsilon(\omega,t)\|_{L^2}^2 dt \le R\right\}.$$

By the same argument as in [Che96, Lemma 3.4] and using an integration by parts formula, we have that

$$\bar{v}^\varepsilon = \frac{\int_{\mathcal{D}} \left(D^2\psi : (e(u^\varepsilon)\mathrm{I} - \varepsilon\nabla u^\varepsilon \otimes \nabla u^\varepsilon) - u^\varepsilon \nabla\psi \cdot \nabla v^\varepsilon - u^\varepsilon \Delta\psi(v^\varepsilon - \bar{v}^\varepsilon)\right)}{\int_{\mathcal{D}} \Delta\psi u^\varepsilon}$$

where $D^2\psi$ is the Hessen matrix of ψ, ψ is the unique solution to

$$\begin{cases} -\Delta\psi = u_\eta^\varepsilon - \bar{u}_\eta^\varepsilon \ \text{ in } \ \mathcal{D}, \\ \dfrac{\partial\psi}{\partial n} = 0 \ \text{ on } \ \partial\mathcal{D}. \end{cases}$$

Here u_η^ε is defined in the same way as in the proof of Lemma 5.8.

Then for a fixed $\omega \in A_R$, all the estimates in the proof of [Che96, Lemma 3.4] hold. Thus we have that there exists a constant C_R such that for any $\omega \in A_R$, $t \in [0,T]$ and any $\eta, \varepsilon \in (0,1)$

$$|\bar{v}^\varepsilon(\omega,t)| \le C_R \frac{\eta^{-1}(1 + \varepsilon^{1/2}\eta^{-d/2})(\mathcal{E}^\varepsilon(t)(\omega) + \|\nabla v^\varepsilon(\omega,t)\|_{L^2(\mathcal{D})})}{1 - m_0^2 - \sqrt{\varepsilon} - \sqrt{\eta}},$$

where $m_0 = \bar{u_0^\varepsilon} \in (-1,1)$ is as in (5.2). Taking η small and independent of ε, we obtain that there exists constant $\tilde{C}_R > 0$ such that for any $\omega \in A_R$, $t \in [0,T]$,

$$\int_0^T \bar{v}^\varepsilon(t)^2 dt \le \tilde{C}_R.$$

Hence we have

$$A_R \subset \left\{ \int_0^T \bar{v}^\varepsilon(t)^2 dt \le \tilde{C}_R \right\}.$$

Moreover, by Poincaré-Wirtinger inequality

$$\|v^\varepsilon\|_{H^1} \lesssim |\bar{v^\varepsilon}| + \|\nabla v^\varepsilon\|_{L^2},$$

thus for any $R > 0$ there exists a constant $C_R > 0$, such that

$$\mathbb{P}\left(\int_0^T \|v^\varepsilon(t)\|_{H^1}^2 dt \le C_R \right) \ge \mathbb{P}\left(\int_0^T \bar{v}^\varepsilon(t)^2 dt \le \tilde{C}_R, \|\nabla v^\varepsilon\|_{L^2(\mathcal{D}_T)}^2 \le R \right) \ge \mathbb{P}(A_R).$$

By Lemma 5.6 and Lemma 5.8, using Chebyshev's inequality, we have that for any $\delta > 0$, there exists a constant $R \equiv R(\delta) > 0$, such that

$$\mathbb{P}(A_R) \ge 1 - \delta.$$

Then we obtain the assertion of the lemma. □

5.2.4 Tightness

For any $\beta < \frac{1}{12}$, we denote

$$\mathcal{X}^1 := \mathbb{R} \times L^\infty_{w^*}(0,T) \times C^\beta([0,T]; L^2_w) \times C^\beta([0,T]; L^1) \times L^2_w(0,T; H^1), \tag{5.32}$$

where $L^2_w(0,T;H^1)$ is the space $L^2(0,T;H^1)$ equipped with the weak topology, L^2_w is the space L^2 equipped with the weak topology and $L^\infty_{w^*}(0,T)$ is the space $L^\infty(0,T)$ equipped with the weak-* topology. We also denote

$$\mathcal{X}^2 := \mathfrak{M}^{d\times d} \times \mathfrak{M}_R, \tag{5.33}$$

where $\mathfrak{M}$ is the space of all finite signed measure on $\mathcal{D}_T$ and $\mathfrak{M}_R \subset \mathfrak{M}$ is the space of all Radon measure on $\mathcal{D}_T$. $\mathfrak{M}_R$ and $\mathfrak{M}$ are equipped with the total variation norm $\|\cdot\|_{TV}$ and weak topology, respectively. Here an element in $\mathfrak{M}^{d\times d}$ is a $d \times d$ $\mathfrak{M}$-valued matrix $\{\mu_{ij}\}_{i,j=1}^d$ where $\mu_{ij} \in \mathfrak{M}$.

Let $\hat{\mathbb{P}}^\varepsilon$ be the probability measure on $\mathcal{X}^1 \times \mathcal{X}^2$ defined by

$$\hat{\mathbb{P}}^\varepsilon := \mathbb{P} \circ \left(\varepsilon^{-1} \sup_{t\in[0,T]} \|F(u^\varepsilon)\|_{L^1}, \mathcal{E}^\varepsilon(u^\varepsilon), u^\varepsilon, G(u^\varepsilon), v^\varepsilon, e^\varepsilon(u^\varepsilon)dxdt, \{\varepsilon \partial_{x_i} u^\varepsilon \partial_{x_j} u^\varepsilon dxdt\}_{ij} \right)^{-1}. \tag{5.34}$$

In the following we will prove that $\{\hat{\mathbb{P}}^\varepsilon\}_\varepsilon$ is tight on $\mathcal{X}^1 \times \mathcal{X}^2$. This is equivalent to prove the tightness of every component.

For $\sup_{t\in[0,T]} \|F(u^\varepsilon)\|_{L^1}$, by (5.22) and Chebyshev's inequality, we know that

$$\mathbb{E}\varepsilon^{-1} \sup_{t\in[0,T]} \int_{\mathcal{D}} F(u^\varepsilon) dx \lesssim 1.$$

Then we have that for any $\delta > 0$, there exists a constant $K_1 > 0$ such that

$$\mathbb{P}\left(\varepsilon^{-1} \sup_{t\in[0,T]} \|F(u^\varepsilon)\|_{L^1} \le K_1 \right) \ge 1 - \delta.$$

For $\mathcal{E}^\varepsilon$, by (5.19) and Chebyshev's inequality, we have that for any $\delta > 0$, there exists a constant $K_2 > 0$ such that

$$\mathbb{P}\left(\sup_t \mathcal{E}_t^\varepsilon \leq K_2\right) \geq 1 - \delta.$$

By the Banach-Alaoglu theorem, any bounded set in $L^\infty(0,T)$ is precompact in $L^\infty_{w^*}(0,T)$, thus $\mathbb{P} \circ (\mathcal{E}^\varepsilon(u^\varepsilon))^{-1}$ is tight on $L^\infty_{w^*}(0,T)$.

For u^ε, by the Banach-Alaoglu theorem, any bounded set in L^2 is precompact in L^2_w. Then by a generalized Arzelà-Ascoli theorem, any bounded set in $C^\beta([0,T];L^2)$ is precompact in $C^\gamma([0,T];L^2_w)$ for any $0 < \gamma < \beta$. Hence we obtain the tightness of $\mathbb{P} \circ (u^\varepsilon)^{-1}$ on $C^\gamma([0,T];L^2_w)$ by using Chebyshev' inequality and Lemma 5.8.

For $G(u^\varepsilon)$, by Lemma 5.8 we have that for any $\delta > 0$, there exists a constant $K_3 > 0$ such that

$$\mathbb{P}\left(\|G(u^\varepsilon)\|_{L^\infty(0,T;W^{1,1})} + \|G(u^\varepsilon)\|_{C^\beta([0,T];L^1)} \leq K_3\right) \geq 1 - \delta.$$

Since $W^{1,1}$ is compactly embedded into L^q for any $q \in [1, \frac{d}{d-1}]$, then by a generalized Arzelà-Ascoli theorem for any $0 < \gamma < \beta$, the set

$$\{g \in C^\gamma([0,T];L^1) : \|g\|_{L^\infty(0,T;W^{1,1})} + \|g\|_{C^\beta([0,T];L^1)} \leq K\}$$

is compact in $C^\gamma([0,T];L^1)$, which implies the tightness of $\mathbb{P} \circ (G(u^\varepsilon))^{-1}$ in $C^\gamma([0,T];L^1)$ for any $\gamma < \frac{1}{12}$.

For v^ε, the tightness of $\mathbb{P} \circ (v^\varepsilon)^{-1}$ in $L^2_w(0,T;H^1)$ is followed by Lemma 5.9 and the Banach-Alaoglu theorem.

For $\varepsilon\partial_{x_i}u^\varepsilon\partial_{x_j}u^\varepsilon$ and $e^\varepsilon(u^\varepsilon)$, since $L^1(\mathcal{D}_T)$ is embedded into $\mathfrak{M}$. Moreover for any $f \in L^1(\mathcal{D}_T)$, we have that

$$f(t,x)dxdt = f^+dxdt - f^-dxdt.$$

Since $\mathcal{D}_T$ is a compact set, we have that $f^+dxdt, f^-dxdt \in \mathfrak{M}_R$. By Theorem 2.11, any bounded set in $\mathfrak{M}_R$ w.r.t. total variation norm is precompact in $\mathfrak{M}_R$ w.r.t weak topology, which implies that any bounded set in $\mathfrak{M}$ w.r.t. total variation norm is precompact in $\mathfrak{M}$ w.r.t weak topology. Thus by (5.19) and

$$\|\varepsilon\partial_{x_i}u^\varepsilon\partial_{x_j}u^\varepsilon\|_{L^1(\mathcal{D}_T)} \lesssim \varepsilon\|\nabla u^\varepsilon\|_{L^1(\mathcal{D}_T)} \lesssim \sup_{t\in[0,T]} \mathcal{E}_t^\varepsilon,$$

$$\|e^\varepsilon(u^\varepsilon)\|_{L^1(\mathcal{D}_T)} \lesssim \sup_{t\in[0,T]} \mathcal{E}_t^\varepsilon,$$

we obtain the tightness of $\mathbb{P} \circ \left(e^\varepsilon(u^\varepsilon)dxdt, \{\varepsilon\partial_{x_i}u^\varepsilon\partial_{x_j}u^\varepsilon dxdt\}_{ij}\right)^{-1}$ in $\mathcal{X}^2$.

Hence we proved the tightness of $\{\hat{\mathbb{P}}^\varepsilon\}_\varepsilon$ in $\mathcal{X}^1 \times \mathcal{X}^2$. Then by using a Jakubowski's version of the Skorokhod Theorem in the form given by [BO13, Theorem A.1], which was proved in [Jak98]:

Theorem 5.10. *Let $\mathcal{X}$ be a topological space such that there exists a sequence $\{f_n\}_{n\geq 1}$ of continuous functions $f_n : \mathcal{X} \to \mathbb{R}$ that separate points of $\mathcal{X}$. Let us denote by $\mathscr{S}$ the σ-algebra generated by the maps $\{f_n\}$. Then:*

(j1) *every compact subset of $\mathcal{X}$ is metrizable;*

(j2) *every Borel subset of a σ-compact set in $\mathcal{X}$ belongs to $\mathscr{S}$;*

(j3) every probability measure supported by a σ-compact set in $\mathcal{X}$ has a unique Radon extension to the Borel σ-algebra on $\mathcal{X}$;

(j4) if (μ_n) is a tight sequence of probability measures on $(\mathcal{X}, \mathscr{S})$, then there exists a subsequence $(n_k)_{k\geq 1}$, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{X}$-valued Borel measurable random variables X_k, X such that μ_{n_k} is the law of X_k and X_k converge almost surely to X. Moreover, the law of X is a Radon measure.

We obtain that

Theorem 5.11. *Assume $\sigma \geq \frac{1}{2}$. There exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}\}_{t\in[0,T]}, \tilde{\mathbb{P}})$ on $\mathcal{X}^1 \times \mathcal{X}^2$, a subsequence ε_k (we still denote it as ε for simplicity) and*

$$\left\{\left(\varepsilon^{-1} \sup_{t\in[0,T]} \|F(\tilde{u}^\varepsilon)\|_{L^1}, \mathcal{E}^\varepsilon(\tilde{u}^\varepsilon), \tilde{u}^\varepsilon, G(\tilde{u}^\varepsilon), \tilde{v}^\varepsilon, e^\varepsilon(\tilde{u}^\varepsilon)dxdt, \{\varepsilon\partial_{x_i}\tilde{u}^\varepsilon\partial_{x_j}\tilde{u}^\varepsilon dxdt\}_{ij}\right)\right\} \subset \mathcal{X}^1\times\mathcal{X}^2$$

and

$$(a, \mathcal{E}, u, g, v, \mu, \{\mu_{ij}\}_{ij}) \in \mathcal{X}^1 \times \mathcal{X}^2,$$

such that

(i) $\tilde{\mathbb{P}}\circ\big(\varepsilon^{-1}\sup_{t\in[0,T]} \|F(\tilde{u}^\varepsilon)\|_{L^1}, \mathcal{E}^\varepsilon(\tilde{u}^\varepsilon), \tilde{u}^\varepsilon, G(\tilde{u}^\varepsilon), \tilde{v}^\varepsilon, e^\varepsilon(\tilde{u}^\varepsilon)dxdt, \{\varepsilon\partial_{x_i}\tilde{u}^\varepsilon\partial_{x_j}\tilde{u}^\varepsilon dxdt\}_{ij}\big)^{-1} = \hat{\mathbb{P}}^\varepsilon$ on $\mathcal{X}^1 \times \mathcal{X}^2$,

(ii) $\big(\varepsilon^{-1}\sup_{t\in[0,T]} \|F(\tilde{u}^\varepsilon)\|_{L^1}, \mathcal{E}^\varepsilon(\tilde{u}^\varepsilon), \tilde{u}^\varepsilon, G(\tilde{u}^\varepsilon), \tilde{v}^\varepsilon, e^\varepsilon(\tilde{u}^\varepsilon)dxdt, \{\varepsilon\partial_{x_i}\tilde{u}^\varepsilon\partial_{x_j}\tilde{u}^\varepsilon dxdt\}_{ij}\big)$ converges to $(0, \mathcal{E}, u, g, v, \mu, \{\mu_{ij}\}_{ij})$ in $\mathcal{X}^1 \times \mathcal{X}^2$, $\tilde{\mathbb{P}} - a.s$, as $\varepsilon \searrow 0$.

In particular, for $\tilde{\mathbb{P}} - a.s.\omega$, there exists a Borel set $E(\omega) \subset \tilde{\Omega} \times \mathcal{D}_T$, such that as $\varepsilon \searrow 0$

(iii) $u^\varepsilon \to u$ in $C^{\frac{\beta}{2}}([0,T]; L^2)$, $g = G(u) = 2S\mathbb{1}_E$ a.e. in $\mathcal{D}_T$ and in $C^\beta([0,T]; L^1)$, $u = -1 + 2\mathbb{1}_E$ a.e. in $\mathcal{D}_T$ and in $C^\beta([0,T]; L^2)$.

Moreover, denote $E = \{(\omega, t, x) \in \Omega\times D_T : (t, x) \in E(\omega)\}$, $E_t := \{(\omega, x) : (\omega, t, x) \in E\}$, then $\mathbb{1}_{E_t}$ is $\{\tilde{\mathcal{F}}\}_{t\in[0,T]}$-adapted in $L^2(\mathcal{D})$ and satisfies the following:

(iv) For all $\beta < \frac{1}{12}$, $\tilde{\mathbb{P}}\left(\mathbb{1}_E \in C^\beta([0,T]; L^1)\right) = 1$,

(v) $\tilde{\mathbb{P}}\left(|E_t| = |E_0| = \frac{1+m_0}{2}|\mathcal{D}|, \forall t \in [0,T]\right) = 1$,

(vi) $\tilde{\mathbb{P}}\left(\mathbb{1}_E \in L^\infty(0,T; BV)\right) = 1$.

Proof Since $\mathcal{X}^1 \times \mathcal{X}^2$ is locally convex space and its dual space is separable, by [Rud73, Theorem 3.4], the condition in Theorem 5.10 holds. Thus the Skorokhod theorem Theorem 5.10 yields the first assertion and the existence of convergence subsequence to

$$(a, \mathcal{E}, u, g, v, \mu, \{\mu_{ij}\}_{ij}) \ \text{ in } \ \mathcal{X}^1 \times \mathcal{X}^2.$$

Since $\tilde{\mathbb{P}} \circ (\tilde{u}^\varepsilon, \tilde{v}^\varepsilon)^{-1} = \mathbb{P} \circ (u^\varepsilon, v^\varepsilon)^{-1}$, we have that for any $h \in H^1$,

$$\varepsilon^{-\sigma}\left(\int_{\mathcal{D}} (\tilde{u}^\varepsilon(t) - \tilde{u}^\varepsilon(0))hdx + \int_0^t \nabla\tilde{v}^\varepsilon \cdot \nabla hdx\right)$$

is a Wiener process on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with covariance $\|Q^{\frac{1}{2}}h\|^2_{L^2}$. Thus there exists a Q-Wiener process $\tilde{W}$ on L^2 which is defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Then we have that for any $h \in H^1$

$$\int_{\mathcal{D}} (\tilde{u}^\varepsilon(t) - \tilde{u}^\varepsilon(0))hdx + \int_0^t \int_{\mathcal{D}} \nabla\tilde{v}^\varepsilon \cdot \nabla hdx = \varepsilon^\sigma \int_0^t \langle h, d\tilde{W}_s\rangle. \tag{5.35}$$

Moreover, we denote $\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}$ be completion under $\tilde{\mathbb{P}}$ of the natural filtration generated by $\{\tilde{W}_t\}_{t\in[0,T]}$, thus $\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}$ is a normal filtration. By [DPD96], we know that for any $\varepsilon > 0$, $\tilde{u}^\varepsilon$ is the unique solution, thus by Yamada–Watanabe theorem (see e.g. [LR15, Theorem E.0.8]) $\{\tilde{u}^\varepsilon\}_t$ is $\{\tilde{\mathcal{F}}_t\}_t$-adapted in $L^2(\mathcal{D})$. Since $\tilde{u}^\varepsilon \to u$ in $C([0,T]; L^2_w)$, we know that u is $\{\tilde{\mathcal{F}}_t\}_t$-adapted in $L^2(\mathcal{D})$.

In the rest of this proof, we ignore the notation $\tilde{}$ if there is no confusion.

By (5.26), we know that for any $t, \tau \in [0,T]$, any $\varepsilon > 0$

$$|u^\varepsilon(t) - u^\varepsilon(\tau)|^2 \lesssim |G(u^\varepsilon(t)) - G(u^\varepsilon(\tau))|,$$

thus we have that for $\tilde{P} - a.s.\omega$

$$\|u^\varepsilon(t) - u^\varepsilon(\tau)\|^2_{L^2} \lesssim \|G(u^\varepsilon(t)) - G(u^\varepsilon(\tau))\|_{L^1}.$$

Since $G(u^\varepsilon) \to g$ in $C^\beta([0,T], L^1)$ for any $\beta < \frac{1}{12}$, let $\varepsilon \to 0$ we have that

$$\limsup_{\varepsilon\to 0} \|u^\varepsilon(t) - u^\varepsilon(\tau)\|^2_{L^2} \lesssim \|g(t) - g(\tau)\|_{L^1} \lesssim |t-\tau|^\beta.$$

Since $u^\varepsilon(t) \to u(t)$ in L^2_w, by the weakly lower-semicontinuity, we have that

$$\|u(t) - u(s)\|^2_{L^2} \le \liminf_{\varepsilon\to 0} \|u^\varepsilon(t) - u^\varepsilon(\tau)\|_{L^2} \lesssim |t-\tau|^\beta.$$

Hence we obtain that $u \in C^{\frac{\beta}{2}}([0,T]; L^2)$ $\mathbb{P} - a, s,$. Similarly we have for any $\varepsilon, h > 0$

$$\|u^\varepsilon - u^h\|^2_{L^2} \lesssim \|G(u^\varepsilon) - G(u^h)\|_{L^1}, \quad \tilde{\mathbb{P}} - a.s..$$

Let $h \to 0$, we obtain

$$\|u^\varepsilon - u\|^2_{L^2} \lesssim \|G(u^\varepsilon) - g\|_{L^1}, \quad \tilde{\mathbb{P}} - a.s.,$$

which implies that $u^\varepsilon \to u$ in $C^{\frac{\beta}{2}}([0,T]; L^2)$ $\tilde{\mathbb{P}} - a.s.$.

On the other hand, by (5.22) we know that

$$\mathbb{E} \sup_{t\in[0,T]} \int_{\mathcal{D}} (|u^\varepsilon| - 1)^2 dx \lesssim \mathbb{E} \sup_{t\in[0,T]} \|F(u^\varepsilon)\|_{L^1} \lesssim \varepsilon.$$

As $\varepsilon \to 0$, we have that for $\tilde{\mathbb{P}} - a.s.$ $|u| \equiv 1$ in L^2, $\forall t \in [0,T]$, which implies that for $\tilde{\mathbb{P}} - a.s.$ there exists a measurable set $E(\omega)$ in $\mathcal{D}_T$, such that

$$u = -1 + 2\mathbb{1}_E, \quad \tilde{\mathbb{P}} - a.s..$$

Since u is $\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}$-adapted in L^2, we know $\mathbb{1}_E$ is also $\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}$-adapted in L^2.

Moreover by the right hand side of (5.26), we obtain that for $\tilde{\mathbb{P}} - a.s.\omega$,

$$\|g - G(u)\|_{L^1} = \lim_{\varepsilon\to 0} \|G(u^\varepsilon) - G(u)\|_{L^1} \lesssim \lim_{\varepsilon\to 0} \|u^\varepsilon - u\| = 0$$

which implies that $g = G(u) = 2S\mathbb{1}_E$. Hence we proved the assertion (iii).

Using the estimate (5.30), we have that for any $t, \tau \in [0,T]$,

$$\mathbb{E}\|\mathbb{1}_{E_t} - \mathbb{1}_{E_\tau}\|^{2p}_{L^1} \lesssim \mathbb{E}\|\mathbb{1}_{E_t} - \mathbb{1}_{E_\tau}\|^{2p}_{L^2} \lesssim \lim_{\varepsilon\to 0} \mathbb{E}\|u^\varepsilon(t) - u^\varepsilon(\tau)\|^{2p}_{L^2} \lesssim |t-\tau|^{\frac{p}{6}}.$$

Then the assertion (iv) followed by the Kolmogorov's criteria.

Note that the equation (5.1) is conserved, i.e. for any $t \in [0,T]$,

$$\int_{\mathcal{D}} u^\varepsilon(t,x)dx \equiv \int_{\mathcal{D}} u_0^\varepsilon(x)dx = |\mathcal{D}|m_0.$$

Since $u^\varepsilon \to u = -1 + 2\mathbb{1}_E$, we have that $|E_t| = \frac{1+m_0}{2}|\mathcal{D}|$. This proved the assertion (v).

Finally set $g^\varepsilon := G(u^\varepsilon)$, by (5.25) we know that

$$|Dg^\varepsilon(t,\cdot)|(\mathcal{D}) = \int_{\mathcal{D}} |\nabla g^\varepsilon(t,x)|dx \leq \mathcal{E}^\varepsilon(t).$$

As $\varepsilon \searrow 0$, since $g^\varepsilon \to g = 2S\mathbb{1}_E$ in $C([0,T];L^1)$ and $\mathcal{E}^\varepsilon \to \mathcal{E}$ in $L^\infty_{w*}(0,T)$, by [AFP00, Proposition 3.13], we obtain that $Dg^\varepsilon \to Dg$ in $L^\infty(0,T;BV)$. Then by the lower semi-continuity of the BV norm we obtain that $|D\mathbb{1}_{E_t}|(\mathcal{D}) = \frac{1}{2S}|Dg(t,\cdot)| \leq \frac{1}{2S}\mathcal{E}(t)$. This completes the proof of the theorem.

□

5.2.5 Proof of Theorem 5.3

Now we are in a position to prove Theorem 5.3. Before we begin the proof, we need to first recall some crucial lemmas to estimate the following "discrepancy" measure $\zeta^\varepsilon(u^\varepsilon)dx$

$$\zeta^\varepsilon(u^\varepsilon)dx := \left(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 - \frac{1}{\varepsilon}F(u^\varepsilon)\right)dx. \tag{5.36}$$

Lemma 5.12. *([Che96, Lemma 4.4, Theorem 3.6]). Let*

$$\mathcal{K}^\varepsilon := \left\{(u,v) \in H^2(\mathcal{D}) \times L^2(\mathcal{D}) : v = -\varepsilon\Delta u + \frac{1}{\varepsilon}f(u) \ \ in \ \mathcal{D}, \ \ \frac{\partial u}{\partial n} = 0 \ \ on \ \partial\mathcal{D}\right\}.$$

There exist positive constants C_0 and $\eta_0 \in (0,1]$ such that for every $\eta \in (0,\eta_0]$, every $\varepsilon \in (0,1]$, and every $(u^\varepsilon, v^\varepsilon) \in \mathcal{K}^\varepsilon$,

$$\int_{\{x\in\mathcal{D};|u^\varepsilon|\geq 1-\eta\}} \left[e^\varepsilon(u^\varepsilon) + \varepsilon^{-1}f^2(u^\varepsilon)\right] \leq C_0\eta \int_{\{x\in\mathcal{D};|u^\varepsilon|\leq 1-\eta\}} \varepsilon|\nabla u^\varepsilon|^2 + C_0\varepsilon\int_{\mathcal{D}}(v^\varepsilon)^2. \tag{5.37}$$

Moreover there exist continuous, non-increasing, and positive functions $M_1(\eta)$ and $M_2(\eta)$ defined on $(0,\eta_0]$ such that for every $\eta \in (0,\eta_0]$, every $\varepsilon \in (0, \frac{1}{M_1(\eta_0)}]$, and every $(u^\varepsilon, v^\varepsilon) \in \mathcal{K}^\varepsilon$, we have that

$$\int_{\mathcal{D}} (\zeta^\varepsilon(u^\varepsilon))^+ dx \leq \eta\int_{\mathcal{D}} e^\varepsilon(u^\varepsilon)dx + \varepsilon M_2(\eta)\int_{\mathcal{D}}(v^\varepsilon(x))^2dx, \tag{5.38}$$

where $(\zeta^\varepsilon(u^\varepsilon))^+$ is the positive part of $\zeta^\varepsilon(u^\varepsilon)$.

Proof of Theorem 5.3

Let $\{u_0^\varepsilon(\cdot)\}_\varepsilon$ be a family of initial data satisfying (5.2). Let $(u^\varepsilon, v^\varepsilon)$ be the solution of (5.1) with initial value u_0^ε. The first three assertions can be obtained directly by Theorem 5.11.

In the following we fixed ω such that all the assertions in Theorem 5.11 hold. For simplicity of notation, we also denote ε_k by ε and omit the notation tilde $\tilde{}$ in the Theorem 5.11.

Since $G(u^\varepsilon) \to 2S\mathbb{1}_E$ and $|DG(u^\varepsilon)| \leq e^\varepsilon(u^\varepsilon)$ for every ε and every $(t,x) \in \mathcal{D}_T$, by the lower semicontinuity of the BV norms, we have that

$$2S|D\mathbb{1}_{E_t}|dtdx \leq d\mu,$$

which is the inequality (5.8).

For any $\psi \in C_c^1([0,t) \times \bar{\mathcal{D}})$, denote $h(t,u) := \int_{\mathcal{D}}(1+u(x))\psi(t,x)dx$. Since $(u^\varepsilon, v^\varepsilon)$ is a solution to equation (5.1), by Itô's formula we have that for any $\tau \in (0,t)$

$$h(t,u^\varepsilon(t)) - h(0,u^\varepsilon(0)) = \int_0^t \int_{\mathcal{D}} \partial_t\psi(\tau,x)(1+u^\varepsilon(\tau,x))dxd\tau + \int_0^t \langle \psi(\tau,\cdot), du^\varepsilon(\tau)\rangle,$$

combined with $\psi(t) \equiv 0$, which yields that

$$\begin{aligned}-\int_{\mathcal{D}}(1+u^\varepsilon(0,x))\psi(0,x)dx = &\int_0^t \int_{\mathcal{D}} \partial_t\psi(\tau,x)(1+u^\varepsilon(\tau,x))dxd\tau - \int_0^t\int_{\mathcal{D}} \nabla v^\varepsilon \nabla\psi \\ &+ \varepsilon^\sigma \int_0^t \langle \psi(\tau,\cdot), dW_\tau\rangle.\end{aligned}$$

Let $\varepsilon \searrow 0$, we obtain that the identity (5.10).

In addition, for any $t \in (0,T]$, $\vec{Y} \in C_c^1(\mathcal{D}_t;\mathbb{R}^d)$, a direct calculation by integration by parts yields that

$$\begin{aligned}\int_{\mathcal{D}} \vec{Y}\cdot\nabla u^\varepsilon v^\varepsilon &= \int_{\mathcal{D}} \vec{Y}\cdot\nabla u^\varepsilon\left(-\varepsilon\Delta u^\varepsilon + \frac{1}{\varepsilon}f(u^\varepsilon)\right) \\ &= -\int_{\mathcal{D}} D\vec{Y} : (e^\varepsilon(u^\varepsilon) - \varepsilon\nabla u^\varepsilon \otimes \nabla u^\varepsilon) + \int_{\partial\mathcal{D}} e^\varepsilon(u^\varepsilon)\vec{Y}\cdot\vec{n}_{\partial\mathcal{D}} \\ &= -\int_{\mathcal{D}} D\vec{Y} : (e^\varepsilon(u^\varepsilon) - \varepsilon\nabla u^\varepsilon \otimes \nabla u^\varepsilon).\end{aligned}$$

The last equality holds because $\mathcal{D}$ is an open domain thus $\vec{Y} \equiv 0$ on $\partial\mathcal{D}$. Then taking integration from $s=0$ to $s=t$ and letting $\varepsilon \searrow 0$, we obtain

$$\int_0^t 2\mathbb{1}_E \mathrm{div}(v\vec{Y})dxds = \int_0^t \int_{\mathcal{D}} D\vec{Y} : (\mathrm{Id}\mu - (d\mu_{ij})_{d\times d}). \tag{5.39}$$

It remains to construct V to finish the proof. Note that for any $0 < \tau < t < T$,

$$\int_\tau^t \int_{\bar{\mathcal{D}}} d\mu(s,x) = \lim_{\varepsilon\searrow 0}\int_\tau^t\int_{\mathcal{D}} e^\varepsilon(u^\varepsilon)dsdx = \int_\tau^t \mathcal{E}(s)ds. \tag{5.40}$$

Therefore, in the sense of Radon measure,

$$d\mu(t,x) = d\mu^t(x)dt.$$

By (5.40) we have $\mu^t(\bar{\mathcal{D}}) = \mathcal{E}(t)$ for *a.e.* $t \in (0,T]$. Consequently, for *a.e.* $t \in (0,T]$ and *a.e.* $\tau \in (0,t)$, by (5.21), we have that

$$\mu^t(\bar{\mathcal{D}}) = \mathcal{E}(t) = \lim_{\varepsilon\searrow 0}\mathcal{E}^\varepsilon(t) = \lim_{\varepsilon\searrow 0}\left(\mathcal{E}^\varepsilon(u^\varepsilon)(\tau) - \int_\tau^t\int_{\mathcal{D}}|\nabla v^\varepsilon|^2\right)$$

$$+\lim_{\varepsilon\searrow 0}\left(\varepsilon^{2\sigma+1}\int_\tau^t \mathrm{Tr}(-\Delta Q)ds+\frac{\varepsilon^{2\sigma-1}}{2}\int_\tau^t \mathrm{Tr}(f'(u^\varepsilon)Q)ds+\varepsilon^\sigma\int_\tau^t\langle v^\varepsilon, dW_\tau\rangle\right).$$

Similar as in the proof of Lemma 5.6, we have that for $\sigma > \frac{1}{2}$,

$$\lim_{\varepsilon\searrow 0}\left(\varepsilon^{2\sigma+1}\int_\tau^t \mathrm{Tr}(-\Delta Q)ds+\frac{\varepsilon^{2\sigma-1}}{2}\int_\tau^t \mathrm{Tr}(f'(u^\varepsilon)Q)ds+\varepsilon^\sigma\int_\tau^t\langle v^\varepsilon, dW_\tau\rangle\right)=0.$$

Hence we deduce that

$$\mu^t(\bar{\mathcal{D}})\le \mathcal{E}(\tau)-\int_\tau^t\int_{\mathcal{D}}|\nabla v|^2dxds=\mu^\tau(\bar{\mathcal{D}})-\int_\tau^t\int_{\mathcal{D}}|\nabla v|^2dxds,$$

which is the inequality (5.12).

Next, we study the relation between μ_{ij} and μ. Observe that for any $t\in(0,T]$, and $\vec{Y},\vec{Z}\in C(\bar{\mathcal{D}}_t;\mathbb{R}^d)$,

$$\begin{aligned}\varepsilon\int_0^t\int_{\mathcal{D}}\vec{Y}^T\left(\nabla u^\varepsilon\otimes\nabla u^\varepsilon\right)\vec{Z}=&\varepsilon\int_0^t\int_{\mathcal{D}}\sum_{i,j}Y^iZ^j\partial_{x_i}u^\varepsilon\partial_{x_j}u^\varepsilon dxdt\\ \le&\varepsilon\int_0^t\int_{\mathcal{D}}|\vec{Y}||\vec{Z}||\nabla u^\varepsilon|^2dxdt\\ \le&\int_0^t\int_{\mathcal{D}}|\vec{Y}||\vec{Z}|e^\varepsilon(u^\varepsilon)+\int_0^t\int_{\mathcal{D}}|\vec{Y}||\vec{Z}|\zeta^\varepsilon(u^\varepsilon),\end{aligned}\tag{5.41}$$

where $\vec{Y}^T$ is the transpose of vector $\vec{Y}$. Here in the last inequality we use the definition of $\zeta^\varepsilon(u^\varepsilon)$ in (5.36) then $e^\varepsilon(u^\varepsilon)+\zeta^\varepsilon(u^\varepsilon)=\varepsilon|\nabla u^\varepsilon|^2$.

By taking η as small as enough in (5.38), we have that

$$\lim_{\varepsilon\searrow 0}\int_0^t\int_{\mathcal{D}}|\vec{Y}||\vec{Z}|\zeta^\varepsilon(u^\varepsilon)\le 0.$$

Thus letting $\varepsilon\searrow 0$ in (5.41), we obtain that

$$\int_0^t\int_{\bar{\mathcal{D}}}\vec{Y}^T\left(d\mu_{ij}\right)_{d\times d}\vec{Z}\le\int_0^t\int_{\mathcal{D}}|\vec{Y}||\vec{Z}|d\mu.\tag{5.42}$$

Therefore, in the sense of measure $|d\mu_{ij}(t,x)|\le d\mu(t,x)$. Consequently, there exists μ-measurable functions $\nu_{ij}(t,x)$ such that

$$d\mu_{ij}(t,x)=\nu_{ij}(t,x)d\mu(t,x),\quad \mu-a.e.\ (t,x)\in\bar{\mathcal{D}}_T.$$

By the definition of μ_{ij} and (5.42), we have that

$$0\le(\nu_{ij})_{d\times d}=(\nu_{ij}(t,x))_{d\times d}\le \mathrm{I},\quad \mu-a.e.\ (t,x)\in\bar{\mathcal{D}}_T.$$

Therefore we have that

$$(\nu_{ij})_{d\times d}=\sum_{i=1}^d\lambda_i\vec{\nu}_i\otimes\vec{\nu}_i,\quad \mu-a.e.,$$

where $\vec{\nu}_i$, $i = 1, \cdots, d$ are μ-measurable unit vectors and λ_i, $i = 1 \cdots, d$ are μ-measurable functions, which satisfy

$$0 \le \lambda_i \le 1 \ (i = 1, \cdots, d), \quad \sum_{i=1}^{d} \lambda_i \le 1, \quad \sum_{i=1}^{d} \vec{\nu}_i \otimes \vec{\nu}_i = \mathrm{I}, \quad \mu - a.e.. \tag{5.43}$$

It then follows from (5.39) that for *a.e.* $t \in (0, T]$ and for every $\vec{Y} \in C_c^1(\mathcal{D}, \mathbb{R}^d)$,

$$\begin{aligned} 2\int_{\mathcal{D}} \mathbb{1}_{E_t} \mathrm{div}\left(v(t,x)\vec{Y}(x)\right) dx &= \int_{\mathcal{D}} \nabla\vec{Y}(x) : \left(\mathrm{I} - \sum_{i=1}^{d} \lambda_i(t,x)\vec{\nu}_i(t,x) \otimes \vec{\nu}_i(t,x)\right) d\mu^t(x) \\ &= \int_{\mathcal{D}} \nabla\vec{Y}(x) : \sum_{i=1}^{d} c_i^t(x) \left(\mathrm{I} - \vec{\nu}_i(t,x) \otimes \vec{\nu}_i(t,x)\right) d\mu^t(x), \end{aligned}$$

where

$$c_i^t(x) = \lambda_i(t,x) + \frac{1}{d-1}\left(1 - \sum_{i=1}^{d} \lambda_i(t,x)\right).$$

Clearly, for *a.e.* $t \in (0, T]$, $0 \le c_i^t \le 1$ and $\sum_{i=1}^{d} c_i^t \ge 1$ for $\mu^t - a.e.$. Define $p_i^t = \{\vec{\nu}_i(t,x), -\vec{\nu}_i(t,x)\} \in P$ and V^t as in (5.9), then V is defined by $dV(t,x,p) = dV^t(x,p)dt$, i.e.

$$dV(t,x,p) = \sum_{i=1}^{d} c_i^t(x) \delta_{p_i^t(x)}(p) d\mu^t(x) dp dt,$$

satisfying (iii) of Definition 2.22.

Then by (2.6),

$$\int_{\mathcal{D}} \mathbb{1}_{E_t}(x) \mathrm{div}\left(v(t,x)\vec{Y}(x)\right) dx = \frac{1}{2}\langle \delta V^t, \vec{Y} \rangle.$$

Thus we obtain (5.11). Hence we proved (iv) of Theorem 5.3. This completes the proof of Theorem 5.3.

5.2.6 The case that $\sigma = \frac{1}{2}$

As what is shown in the last subsection, for $\sigma = \frac{1}{2}$, the limit of solution to equation (5.1) satisfies all the definition in Definition 5.2 except (5.12). Instead we have

Proposition 5.13. *Let μ^t be as in Theorem 5.3, then*

$$\mu^t(\bar{\mathcal{D}}) + \int_\tau^t \int_{\mathcal{D}} |\nabla v|^2 \le \mu^\tau(\bar{\mathcal{D}}) + C_Q(t - \tau), \quad \tilde{P} - a.s., \tag{5.44}$$

where $C_Q := Tr(Q)$.

Proof By using the method as in subsection 5.2.5, we have that for $\sigma = \frac{1}{2}$

$$\begin{aligned} \mu^t(\bar{\mathcal{D}}) =& \mathcal{E}(t) = \lim_{\varepsilon \searrow 0}\left(\mathcal{E}^\varepsilon(u^\varepsilon)(\tau) - \int_\tau^t \int_{\mathcal{D}} |\nabla v^\varepsilon|^2\right) \\ &+ \lim_{\varepsilon \searrow 0}\left(\varepsilon^2 \int_\tau^t \mathrm{Tr}(-\Delta Q) ds + \frac{1}{2}\int_\tau^t \mathrm{Tr}(f'(u^\varepsilon)Q) ds + \varepsilon^{\frac{1}{2}} \langle \int_\tau^t v^\varepsilon, dW_\tau \rangle\right) \\ &\le \mathcal{E}(\tau) - \int_\tau^t \int_{\mathcal{D}} |\nabla v|^2 dx ds + (t-\tau)\mathrm{Tr}(Q). \end{aligned}$$

The last inequality holds because $|u^\varepsilon| \to 1$ and $f'(u^\varepsilon) = 3(u^\varepsilon)^2 - 1$. Thus we obtain (5.44).

□

Remark 5.14. *By Proposition 5.13 and the analysis in the proof of Theorem 5.3 in Subsection 5.2.5. In the case that $\sigma = \frac{1}{2}$, the energy μ^t may grow a little faster than that in deterministic case. But as what we will show in the next section, at least in radial symmetric case, the perturbation by the noise $\varepsilon^{\frac{1}{2}} dW$ is not strong enough, such that the limit of equation (5.1) also converges to deterministic Hele-Shaw model (in a weak sense). Thus we conjecture that in general for $\mathbb{P}-a.s.$ ω, the sharp interface limit of (5.1) satisfies the deterministic Hele-Shaw model (1.5):*

$$\begin{cases} \Delta v = 0 \text{ in } \mathcal{D} \setminus \Gamma_t, \ t > 0, \\ \dfrac{\partial v}{\partial n} = 0 \text{ on } \partial \mathcal{D}, \\ \quad v = SH \text{ on } \Gamma_t, \\ \quad \mathcal{V} = \dfrac{1}{2}(\partial_n v^+ - \partial_n v^-) \text{ on } \Gamma_t. \end{cases}$$

5.3 Case of radial symmetry for $\sigma \geq \frac{1}{2}$

In this section we are going to prove Theorem 5.4. In this case of radial symmetry, we assume $\mathcal{D} = B_1$.

Any function u in this section of the form $u(x) \equiv u(|x|)$. For convenience, we do not distinguish functions of $x \in B_1$ from functions of $r \in [0, 1)$. We only distinguish the integrals of dx from that of dr, due to consideration of singularities at the origin.

Denote $r = |x|$, then the equation (5.1) should be changed as

$$\begin{cases} du^\varepsilon = \partial_{rr} v^\varepsilon dt + \dfrac{d-1}{r} \partial_r v^\varepsilon dt + \varepsilon^\sigma dW_t, \quad (t, r) \in (0, T) \times [0, 1), \\ v^\varepsilon = -\varepsilon \partial_{rr} u^\varepsilon(t) - \dfrac{d-1}{r} \partial_r u^\varepsilon + \dfrac{1}{\varepsilon} f(u^\varepsilon(t)), \quad (t, x) \in (0, T) \times [0, 1), \\ \partial_r u^\varepsilon(t, 1) = \partial_r v^\varepsilon(t, 1) = 0, \quad t \in [0, T], \\ u^\varepsilon(0, r) = u_0^\varepsilon(r), \quad x \in \mathcal{D}. \end{cases} \tag{5.45}$$

Here W_t is given by

$$W_t = \sum_{k \in \mathbb{Z}^d} \alpha_k b_k(r) \beta_k(t), \tag{5.46}$$

where $\{\beta_k\}_{k \in \mathbb{Z}^d}$ is a sequence of independent Brownian motions and $\{\alpha_k\}$ satisfies

$$\sum_{k \in \mathbb{Z}^d} \lambda_k \alpha_k^2 < \infty. \tag{5.47}$$

$\{b_k\}_{k \in \mathbb{Z}^d}$ is an orthogonal basis in $L_0^2(0, 1)$, which is defined as $\{f \in L^2(0, 1) : \int_0^1 f(r) r^{d-1} dr = 0\}$, i.e.

$$\int_0^1 b_k(r) r^{d-1} dr = 0, \quad \forall k \in \mathbb{Z}^d. \tag{5.48}$$

Note that (5.46)-(5.48) is just the radially symmetric version of condition (5.4) and (5.5). Moreover, all the results we obtained in the previous section also hold for this case. In particular, there exists a Borel set $\tilde{E} \subset [0,T] \times [0,1)$ such that $E = \{(t,x) \in \mathcal{D}_T : (t,|x|) \in \tilde{E}\}$ and $\tilde{E}_t := \{r \in [0,1) : (t,r) \in [0,T] \times [0,1)\}$ is a BV set in $[0,1)$ for any $t \in [0,T]$.

Remark 5.15. *For the existence of radial symmetric solution to (5.1) under the assumption in this section, we only need to check that any solution u^ε to (5.1) is invariant under the rotation transformation. Then by the uniqueness, we can obtain that u^ε is radial symmetric.*

In fact, any rotation transformation in $\mathbb{R}^d$ can be identified as an orthogonal matrix with determinant 1, i.e. an element in $SO(d)$. For any $A \in SO(d)$, a direct calculation yields that

$$\nabla(v \circ A) = (\nabla v) \circ A \quad (\Delta v) \circ A = \Delta(v \circ A).$$

Then we have that for any solution $(u^\varepsilon, v^\varepsilon)$ to (5.1), $(u^\varepsilon \circ A, v^\varepsilon \circ A)$ is also a solution to (5.1). By the uniqueness of solutions to (5.1), if the initial value of $(u^\varepsilon, v^\varepsilon)$ is radial symmetric, $(u^\varepsilon, v^\varepsilon)$ is also radial symmetric. In this case, equation (5.1) is equivalent to (5.45).

Moreover, a direct calculation yields that

$$\int_{B_1} |\nabla u(|x|)|^2 dx = \int_0^t r^{d-1} |\partial_r u(r)|^2 dr.$$

Since in 1-dimensional case, $H^1([0,1])$ is embedded in $C([0,1])$, we have that in radial symmetric case, for a.e. $t \in [0,T]$, $u^\varepsilon(t,\cdot), v^\varepsilon(t,\cdot) \in C(B_1 \setminus B_\delta)$ for any $\delta \in (0,1)$.

We also mention that all the results in [Che96, Section 5] only depend on the second equation in (5.45) and the estimate of $(u^\varepsilon, v^\varepsilon)$. Thus with a similar proof, we obtain the following theorems.

Theorem 5.16. *Assume that $\{(\tilde{u}^\varepsilon, \tilde{v}^\varepsilon)\}$ is obtained in Theorem 5.11. Then*

$$\lim_{\varepsilon \searrow 0} \int_0^T \int_{\mathcal{D}} \left|\zeta^\varepsilon(\tilde{u}^\varepsilon)\right| dx dt = 0, \quad \tilde{\mathbb{P}} - a.s.,$$

where $\zeta^\varepsilon(\tilde{u}^\varepsilon)$ is the discrepancy measure defined in (5.36).

Proof In this proof, we ignore the notation tilde ˜ for simplicity.

For a fixed ω such that all the assertions in Theorem 5.11 hold. By the same proof as [Che96, Theorem 5.1], we have that there exists a constant $C > 0$ he following estimates

$$\int_{B_\delta} e^\varepsilon(u^\varepsilon) dx \le C\delta M^\varepsilon(t), \quad \forall \delta \in (0,1), \tag{5.49}$$

where $M^\varepsilon(t) := 1 + \mathcal{E}^\varepsilon(u^\varepsilon)(t) + \|v^\varepsilon\|_{H^1}^2 \in L^1(0,T)$, and

$$\sup_{0<r<1} \left| r^{d-1} \left(\zeta^\varepsilon(u^\varepsilon) + v^\varepsilon u^\varepsilon\right)\right| \le C M^\varepsilon(t). \tag{5.50}$$

Hence for any small δ and η,

$$\int_{\mathcal{D}} |\zeta^\varepsilon(u^\varepsilon)| dx \le \int_{B_\delta \cup \{|u^\varepsilon| \ge 1-\eta\}} |\zeta^\varepsilon(u^\varepsilon)| dx + \int_{\mathcal{D} \cap \{r > \delta, |u^\epsilon| \le 1-\eta\}} |\zeta^\varepsilon(u^\varepsilon)| dx$$

$$\leq \int_{B_\delta \cup \{|u^\varepsilon| \geq 1-\eta\}} e^\varepsilon (u^\varepsilon)\, dx + \int_{\mathcal{D} \cap \{r>\delta, |u^\epsilon| \leq 1-\eta\}} \left[|v^\varepsilon| \, |u^\varepsilon| + r^{1-d} C M^\varepsilon(t) \right] dx$$
$$\leq \int_{B_\delta \cup \{|u^\varepsilon| \geq 1-\eta\}} e^\varepsilon (u^\varepsilon)\, dx + \int_{\mathcal{D} \cap \{r>\delta, |u^\epsilon| \leq 1-\eta\}} \left[|v^\varepsilon| \, (1-\eta) + r^{1-d} C M^\varepsilon(t) \right] dx,$$

where we used the definition of ζ^ε and $e^\varepsilon(u^\varepsilon)$ and (5.49) in the second inequality.

For the first integral above, we have that

$$\int_{B_\delta \cup \{|u^\varepsilon| \geq 1-\eta\}} e^\varepsilon (u^\varepsilon)\, dx \leq \int_{B_\delta} e^\varepsilon (u^\varepsilon)\, dx + \int_{\{|u^\varepsilon| \geq 1-\eta\}} e^\varepsilon (u^\varepsilon)\, dx$$
$$\leq C\delta M^\varepsilon(t) + C_0 \eta M^\varepsilon(t) + C_0 \varepsilon M^\varepsilon(t),$$

where we used (5.49) and (5.37) in the second inequality.

For the second integral, we have that

$$\int_{\mathcal{D} \cap \{r>\delta, |u^\epsilon| \leq 1-\eta\}} \left[|v^\varepsilon| \, (1-\eta) + r^{1-d} C M^\varepsilon(t) \right] dx \leq \int_{\{|u^\epsilon| \leq 1-\eta\}} \left[|v^\varepsilon| + \delta^{1-d} C M^\varepsilon(t) \right] dx$$
$$\leq \mathcal{H}^d(\{|u^\epsilon| \leq 1-\eta\}) \left(M^\varepsilon(t)^{\frac{1}{2}} + \delta^{1-d} C M^\varepsilon(t) \right).$$

By Theorem 5.11 we know that $\varepsilon^{-1} F(u^\varepsilon)$ is uniformly bounded in $L^\infty(0,T;L^1)$. Thus there exists a constant $C_1 > 0$ such that

$$\mathcal{H}^d(\{|u^\epsilon| \leq 1-\eta\}) \leq \mathcal{H}^d(\{||u^\epsilon| - 1| \geq \eta\}) \leq \eta^{-2} \int_{\mathcal{D}} F(u^\varepsilon) dx \leq C_1 \eta^{-2} \varepsilon.$$

Combining all the estimates above, we have that for any $\eta, \delta > 0$, there exists a constant $C(\delta,\eta) > 0$, such that

$$\int_{\mathcal{D}} \left| \zeta^\varepsilon(\tilde{u}^\varepsilon) \right| dx \leq C_2 \left(\delta + \eta + \varepsilon + C(\delta,\eta)\varepsilon \right) M^\varepsilon(t),$$

C_2 is independent of $\varepsilon, \eta, \delta$. Integrating the last inequality in $(0,T)$ and letting first $\varepsilon \to 0$ and then δ, η to 0, we can obtain the theorem.

□

In the following, we are going to prove

$$d\mu = 2S|D\mathbb{1}_E| dxdt.$$

To prove this, we need a technical lemma:

Lemma 5.17. *([Che96, Lemma 5.4]) For every small positive constant $\delta > 0$, there exists a small positive constant $\varepsilon_0(\delta)$ and a large positive constant $C(\delta) > 0$, such that for every $\varepsilon \in (0, \varepsilon_0(\delta)]$, if $(u^\varepsilon, v^\varepsilon)$ is a pair satisfying the second equation in (5.45) and*

$$\|v^\varepsilon\|_{H^1(B_1)} \leq \delta^{-1}, \quad \int_{B_1} e^\varepsilon(u^\varepsilon) dx \leq \mathcal{E}_0,$$

then the following hold:

(i). If $(a,b) \subset (\delta, 1]$ is an open interval where $|u^\varepsilon| < 1 - |\ln\varepsilon|^{-\frac{1}{2}}$, then for a.e. $t \in [0,T]$, u^ε is strictly monotonic in (a,b) and $|b-a| \leq C(\delta)\varepsilon|\ln\varepsilon|$.

(ii). Denote $A^\varepsilon := \{r \in [2\delta, 1-2\delta] : u^\varepsilon(r) = 0\}$*, then*

$$\int_{2\delta}^{1-2\delta} r^{d-1} e^\varepsilon(u^\varepsilon) dr - C(\delta)\sqrt{\varepsilon} \le 2S \sum_{r \in A^\varepsilon} r^{d-1} \le \int_{2\delta - C(\delta)\varepsilon|\ln\varepsilon|}^{1-2\delta+C(\delta)\varepsilon|\ln\varepsilon|} r^{d-1} e^\varepsilon(u^\varepsilon) dr + C(\delta)\sqrt{\varepsilon}.$$

(iii). For any $r \in A^\varepsilon$*,*

$$\left| v^\varepsilon(r) + \operatorname{sgn}(u_r^\varepsilon(r)) \frac{S(d-1)}{r} \right| \le C(\delta)\varepsilon^{1/8}.$$

(iv). If $r_1 \neq r_2$ *in* A^ε*, then*

$$|r_1 - r_2| \ge \frac{1}{C(\delta)}.$$

Theorem 5.18. *Let* $\{(\tilde{u}^{\varepsilon_k}, \tilde{v}^{\varepsilon_k})\}_k$ *are radially symmetric solutions of (5.45) which satisfy all the assertions in Theorem 5.11. Then for any* $t \in (0,T]$*,* $\psi \in C_c(\mathcal{D}_t)$*,*

$$\int_0^t \int_{\mathcal{D}} \psi(t,x) d\mu(t,x) = 2S \int_0^t \int_{\mathcal{D}} \psi |D\mathbb{1}_{E_t}| dx dt, \quad \tilde{\mathbb{P}} - a.s..$$

Proof The proof is a modification of the proof of [Che96, Theorem 5.3]. The only difference is that in stochastic case, by (5.21), Theorem 5.4 and Proposition 5.13, we know that for all $\sigma \ge \frac{1}{2}$, there exists a $h^\varepsilon \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0,T))$ such that

$$\tilde{\mu}_\varepsilon^t(\mathcal{D}) := \int_{\mathcal{D}} e^\varepsilon(\tilde{u}^\varepsilon(t,x)) dx = \tilde{\mathcal{E}}^\varepsilon(t) \le \mathcal{E}_0 + h^\varepsilon(t) \ \ \mathbb{P} - a.s.,$$

where for $\mathbb{P} - a.s.\ \omega$, $\{h^\varepsilon(\omega, \cdot)\}_\varepsilon$ is bounded in $L^2(0,T)$, while in deterministic case as in [Che96], h^ε is just 0. Then the rest proof just follows the proof of [Che96, Theorem 5.3] for a fixed ω.

We ignore the notation tilde ˜ in Theorem 5.11 for simplicity.

By (5.21), Theorem 5.4 and Proposition 5.13, we know that for all $\sigma \ge \frac{1}{2}$, there exists a $h^\varepsilon \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0,T))$ such that

$$\mu_\varepsilon^t(\mathcal{D}) = \mathcal{E}^\varepsilon(t) \le \mathcal{E}_0 + h^\varepsilon(t) \ \ \mathbb{P} - a.s., \tag{5.51}$$

where $d\mu_\varepsilon^t := e^\varepsilon(u^\varepsilon) dx$ and for $\mathbb{P} - a.s.\ \omega$, $\{h^\varepsilon(\omega, \cdot)\}_\varepsilon$ is bounded in $L^2(0,T)$.

In the following we fix ω such that all the assertions in Theorem 5.11 hold, such that (5.51) holds, and such that $\{h^\varepsilon(\omega, \cdot)\}_\varepsilon$ is bounded in $L^2(0,T)$.

The following proof is a modification of the proof of [Che96, Theorem 5.3]. We use a contradiction argument. Since $2S|D\mathbb{1}_{E_t}|dx \le d\mu^t$, we assume that there exists $T_0 \in (0,T]$, such that

$$\int_0^{T_0} \int_{\mathcal{D}} d\mu(t,x) = \int_0^{T_0} \int_{\mathcal{D}} d\mu^t dt > 2S \int_0^{T_0} \int_{\mathcal{D}} |D\mathbb{1}_{E_t}| dx dt.$$

Since $d\mu = d\mu^t dt$ is a Radon measure on $\mathcal{D}_T$, we know that $\lim_{\delta \searrow 0} \int_0^T \mu^t(B_\delta) dt = 0$ and $\lim_{\delta \searrow 0} \int_0^T \mu^t(B_1 \setminus \bar{B}_{1-\delta}) dt = 0$. Thus there exists $\delta > 0$ such that

$$\int_0^{T_0} \int_{B_{1-2\delta} \setminus \bar{B}_{2\delta}} d\mu \ge 2S \int_0^{T_0} \int_{\mathcal{D}} |D\mathbb{1}_{E_t}(x)| \, dx dt + \delta \left(T_0 + 2\sqrt{T_0 \mathcal{E}_0 + C_h(\omega)} + 1 \right), \tag{5.52}$$

where $C_h(\omega) := \sup_{\varepsilon \in [0,1)} \|h^\varepsilon(\omega,\cdot)\|^2_{L^2(0,T_0)}$. For simplicity we denote $C_{T_0} := \sqrt{T_0\mathcal{E}_0 + C_h(\omega)}$. Since $d\mu^\varepsilon := d\mu^t_\varepsilon dt := e^\varepsilon(u^\varepsilon)dxdt \to d\mu$, there exists a large positive integer $J \equiv J(\delta)$ such for all $j \geq J$,

$$\int_0^T \int_{B_{1-2\delta}\setminus \bar{B}_{2\delta}} d\mu_t^{\varepsilon_j}(x)dt \geq 2\sigma \int_0^T \int_{\mathcal{D}} |D\mathbb{1}_{E_t}|\, dxdt + \delta(T_0 + 2C_{T_0}).$$

Denote

$$\varphi_\delta(t) := \mu_t^{\varepsilon_j}(B_{1-2\delta} \setminus \bar{B}_{2\delta}), \quad \phi(t) := 2S|D\mathbb{1}_{E_t}|(\mathcal{D}).$$

we have that

$$\begin{aligned}\int_0^{T_0} \varphi_\delta(t) - \phi(t)dt \leq & \int_{\{\varphi_\delta - \phi \geq \delta\}} \varphi_\delta(t) - \phi(t)dt + \int_{\{\varphi_\delta - \phi < \delta\}} \varphi_\delta(t) - \phi(t)dt \\ \leq & \delta T_0 + \int_{\{\varphi_\delta - \phi \geq \delta\}} \varphi_\delta(t) - \phi(t)dt \\ \leq & \delta T_0 + \int_{\{\varphi_\delta - \phi \geq \delta\}} \varphi_\delta(t)dt \\ \leq & \delta T_0 + \|\varphi_\delta\|_{L^2(0,T_0)}\sqrt{\mathcal{H}^1(\{t \in [0,T_0] : \varphi_\delta(t) - \phi(t) \geq \delta\})} \\ \leq & \delta T_0 + C_{T_0}\sqrt{\mathcal{H}^1(\{t \in [0,T_0] : \varphi_\delta(t) - \phi(t) \geq \delta\})}.\end{aligned}$$

In the last inequality we used (5.51) and that

$$\varphi_\delta(t) \leq \mu_t^{\varepsilon_j}(\mathcal{D}) \leq \mathcal{E}_0 + h^{\varepsilon_j}.$$

By (5.52), we obtain that

$$\mathcal{H}^1(\{t \in [0,T_0] : \varphi_\delta(t) - \phi(t) \geq \delta\}) \geq 4\delta^2 > 0.$$

Moreover, since v^ε converges in $L^2_w(0,T;H^1)$ thus is uniformly bounded in $L^2(0,T;H^1)$, we have that

$$\mathcal{H}^1(\{t \in [0,T_0] : \|v^\varepsilon\|_{H^1} \leq \delta^{-1}\} = 1 - \mathcal{H}^1(\{t \in [0,T_0] : \|v^\varepsilon\|_{H^1} > \delta^{-1}\} > 1 - \delta^2\|v^\varepsilon\|^2_{L^2(0,T;H^1)} > 0.$$

Hence, for each $j > J$, there exists $t_j \in [0,T_0]$ such that

$$\|v^{\varepsilon_j}(t_j,\cdot)\|_{H^1} \leq \delta^{-1}, \quad \mu_{t_j}^{\varepsilon_j}(B_{1-2\delta} \setminus \bar{B}_{2\delta}) \geq 2S|D\mathbb{1}_{E_{tj}}|(\mathcal{D}) + \delta. \tag{5.53}$$

Now we show that (5.53) is wrong for j large enough.

For each $j \geq J$, we define

$$\begin{aligned} A^j &:= \left\{r \in [\delta, 1-\delta]; r \in \operatorname{supp}\left(\left|D\mathbb{1}_{\tilde{E}_{t_j}}\right|\right)\right\}, \\ A^{\varepsilon_j} &:= \left\{r \in [2\delta, 1-2\delta]; u^{\varepsilon_j}(r,t_j) = 0\right\}, \end{aligned}$$

where $\tilde{E} \subset [0,T] \times [0,1)$ such that $E = \{(t,x) \in \mathcal{D}_T : (t,|x|) \in \tilde{E}\}$ and $\tilde{E}_t := \{r \in [0,1) : (t,r) \in [0,T] \times [0,1)\}$ is a BV set in $[0,1)$ for any $t \in [0,T]$. By Theorem 2.16,

$$|D\mathbb{1}_{\tilde{E}}| = \mathcal{H}^0 \lfloor \operatorname{supp}(|D\mathbb{1}_{\tilde{E}}|), \quad |D\mathbb{1}_E| = \mathcal{H}^{d-1}\lfloor \operatorname{supp}(|D\mathbb{1}_E|). \tag{5.54}$$

Moreover, $\operatorname{supp}(|D\mathbb{1}_E|)$ is a $(d-1)$-rectifiable set and $\operatorname{supp}(|D\mathbb{1}_{\tilde{E}}|)$ is a 0-rectifiable set. Since $\mathcal{H}^0$ is just the counting measure of points in $[0,1)$, we have that $\operatorname{supp}(|D\mathbb{1}_{\tilde{E}}|) =$

$\{r \in [0,1) : |D\mathbb{1}_{\tilde{E}}|(\{r\}) \neq 0\}$ which is at most countable. Denote $R(x) := |x|$ is a Lipshitz function on $\mathcal{D}$, thus $R(E) = \tilde{E}$ and $R\left(\mathrm{supp}\left(|D\mathbb{1}_E|\right)\right) = \mathrm{supp}\left(|D\mathbb{1}_{\tilde{E}}|\right)$ which implies that $(|D\mathbb{1}_E|)$ consists of countable $(d-1)$-spheres. By (5.54) and Fubini's theorem, we have that

$$|D\mathbb{1}_{E_t}|(\mathcal{D}) = \int_0^1 \mathcal{H}^{d-1}(\mathcal{D}\cap R^{-1}(y))dy = \int_0^1 \omega_d y^{d-1}\mathbb{1}_{\mathrm{supp}\left(|D\mathbb{1}_{\tilde{E}}|\right)}(y)dy = \sum_{y\in \mathrm{supp}\left(|D\mathbb{1}_{\tilde{E}}|\right)} \omega_d y^{d-1}.$$

Then we obtain that

$$|D\mathbb{1}_{E_{t_j}}|(\mathcal{D}) \geq \sum_{r\in A^j} \omega_d r^{d-1} \geq \delta^{d-1}\omega_d(\sharp A^j),$$

where $\sharp A^j$ is the number of elements in A^j which is finite since $|D\mathbb{1}_{E_{t_j}}|(\mathcal{D})$ is finite. By the second estimate in Lemma 5.17, A^{ε_j} is also a finite set.

Moreover, by the first inequality in (5.53) and Lemma 5.17, we have that

$$\mu_t^{\varepsilon_j}\left(B_{1-2\delta}\setminus \bar{B}_{2\delta}\right) = \int_{2\delta}^{1-2\delta} r^{d-1}e^{\varepsilon}(u^{\varepsilon})dr \leq 2S\sum_{r\in A^{\varepsilon}} r^{d-1} + C(\delta)\sqrt{\varepsilon}.$$

Thus since $\omega_d > 1$, there exists a large integer $J_1 \geq J$ such that

$$\mu_{t_j}^{\varepsilon_j}\left(B_{1-2\delta}\setminus \bar{B}_{2\delta}\right) \leq 2S\sum_{r\in A^{\varepsilon_j}} \omega_d r^{d-1} + \frac{\delta}{2} \quad \forall j \geq J_1.$$

Hence by the second inequality in (5.52),

$$\sum_{r\in A^{\epsilon_j}} \omega_d r^{d-1} \geq \sum_{r\in A^j} \omega_d r^{d-1} + \frac{\delta}{4S}, \quad \forall j \geq J_1 \tag{5.55}$$

Denote

$$l_j := \sqrt{\varepsilon_j} + \sup_{t\in[0,T_0]}\int_\delta^1 |u^{\varepsilon_j}(t,r) + 1 - 2\mathbb{1}_{\tilde{E}_t}(r)|r^{d-1}dr.$$

Since $u^{\varepsilon_j} \to -1 + 2\mathbb{1}_E$ in $C^\beta([0,T];L^1)$,

$$\sup_{t\in[0,T_0]}\int_{\mathcal{D}} |u^{\varepsilon_j}(t,x) + 1 - 2\mathbb{1}_{E_t}(x)|dx = \sup_{t\in[0,T_0]}\int_0^1 r^{d-1}|u^{\varepsilon_j}(t,r) + 1 - 2\mathbb{1}_{\tilde{E}_t}(r)|dr \to 0,$$

for a fixed $\delta > 0$, we have that $\lim_{j\to\infty} l_j = 0$.

We claim that the definition of l_j and (5.26) imply the existence of $J_2 \geq J_1$ such that

$$\min_{r_1,r_2\in A^i, r_1\neq r_2} |r_1 - r_2| \leq 4l_j, \quad \forall j \geq J_2, \tag{5.56}$$

which is a contradiction to Lemma 5.17. We prove the (5.56) in the following two steps.

First, if $A^{\varepsilon_j} \subset \cup_{r\in A^j}(r - 2l_j, r + 2l_j)$, we claim that for some $r \in A^j$, there exist at least two elements of A^{ε_j} in $(r - 2l_j, r + 2l_j)$, which concludes (5.56).

If this claim is not true, that is, for any $r \in A^j$, there exists at most one $r_0 \in A^{\varepsilon_j} \cap (r - 2l_j, r + 2l_j)$. Denote

$$\underline{A}^j := \{r \in A^j : \exists r_0 \in A^{\varepsilon_j}, r_0 \in (r - 2l_j, r + 2l_j)\}.$$

Then $\sharp \underline{A}^j = \sharp A^{\varepsilon_j} \leq \sharp A^j$. Note that the number of elements in A^j in bounded in j since

$$\sharp A^j \leq \delta^{1-d} \omega_\delta^{-1} |D\mathbb{1}_{E_{t_j}}|(\mathcal{D}) = \delta^{1-d} \omega_\delta^{-1} \mathcal{E}^{\varepsilon_j}(t_j)$$

and $\mathcal{E}^\varepsilon(t)$ is uniformly in ε bounded in $L^\infty(0,T)$. By (5.55) we have that

$$\begin{aligned}
\frac{\delta}{4S} &= \sum_{r \in A^{\varepsilon_j}} \omega_d r^{d-1} - \sum_{r \in A^j} \omega_d r^{d-1} \\
&= \sum_{r \in A^{\varepsilon_j}} \omega_d r^{d-1} - \sum_{r \in \underline{A}^j} \omega_d r^{d-1} - \sum_{r \in A^j \setminus \underline{A}^j} \omega_d r^{d-1} \\
&\leq \sum_{r \in \underline{A}^j} \omega_d \left((r + 2l_j)^{d-1} - r^{d-1} \right) - \sum_{r \in A^j \setminus \underline{A}^j} \omega_d r^{d-1} \\
&\leq 2l_j \# \underline{A}^j - \sum_{r \in A^j \setminus \underline{A}^j} \omega_d r^{d-1} \\
&\leq C(\delta) l_j - \sum_{r \in A^j \setminus \underline{A}^j} \omega_d r^{d-1},
\end{aligned}$$

which is impossible for big j since $\lim_{j \to \infty} l_j = 0$.

Then if $A^{\varepsilon_j} \subset \cup_{r \in A^j} (r - 2l_j, r + 2l_j)$ does not hold, there exists $r_1 \in A^{\varepsilon_j}$ such that $r_1 \notin \cup_{r \in A^j} (r - 2h_j, r + 2h_j)$, i.e. $(r_1 - 2l_j, r_2 + 2l_j) \cap A^j = \varnothing$. Therefore, $\lim_{\varepsilon \to 0} u^\varepsilon \equiv 1$ or $\lim_{\varepsilon \to 0} u^\varepsilon \equiv -1$ on $(r_1 - 2l_j, r_2 + 2l_j)$. Without loss of generality, we assume $\lim_{\varepsilon \to 0} u^\varepsilon \equiv -1$ on $(r_1 - 2l_j, r_2 + 2l_j)$. Thus there exists $(a_1, b_1) \subset (\delta, 1)$ such that $r_1 \in (a_1, b_1)$ and $|u^{\varepsilon_j}| < 1 - |\ln \varepsilon|^{-\frac{1}{2}}$ on (a_1, b_1). By the first assertion of Lemma 5.17, u^{ε_j} is monotonic on (a_1, b_1) and $|b_1 - a_1| \leq C(\delta) \varepsilon_j |\ln \varepsilon_j|$. Let ε_j be small enough such that $(a_1, b_1) \subset (r_1 - 2l_j, r_2 + 2l_j)$.

We assume $\partial_r u^{\varepsilon_j}(r_0) > 0$, i.e. u^{ε_j} is monotone increasing on (a_1, b_1). Since $\lim_{\varepsilon \to 0} u^\varepsilon \equiv -1$ on $(r_1 - 2l_j, r_2 + 2l_j)$, we have that

$$\mathcal{H}^d(u^\varepsilon > 0, |u^\varepsilon| > \delta) \leq \int_{B_1 \setminus \bar{B}_\delta} |u^{\varepsilon_j}(t,x) + 1 - 2\mathbb{1}_{E_t}(x)| dx \leq l_j.$$

Since u^ε is continuous, there must be a $r_2 \in (r_1, r_1 + 2l_j) \cap A^{\varepsilon_j}$.

In the case that $\partial_r u^{\varepsilon_j}(r_0) < 0$, a similar argument yields that there exists $r_2 \in (r_1 - 2l_j, r_1) \cap A^{\varepsilon_j}$. Anyway, we obtain $r_2 \in A^{\varepsilon_j}$, such that $|r_1 - r_2| \leq 4l_j$.

Thus we proved (5.56), which is a contradiction to Lemma 4.2. Then we finish the proof of the Theorem.

□

5.4 Proof of Theorem 5.4

The definition of V in (5.9) can be written as

$$dV^t(x,p) = \sum_{i=1}^{d} c_i^t(x) \delta_{p_i^t(x)} d\mu^t(x) dp.$$

From (5.7) we know that

$$d\|V^t\|(x) = \sum_{i=1}^{d} c_i^t(x) d\mu^t(x) \geq d\mu^t(x).$$

First, we will show for *a.e.* $t \in [0,T]$, V^t is a $(d-1)$-rectifiable varifold.

As what we mentioned in Remark 5.5, for *a.e.* $t \in [0,T]$, E_t is a BV set, thus for *a.e.* $t \in [0,T]$

$$\mu^t = 2S|D\mathbb{1}_{E_t}| = 2S\mathcal{H}^{d-1}\lfloor \partial^* E_t.$$

Here $\mathcal{H}^{d-1}\lfloor \partial^* E_t$ is the $(d-1)$-dimensional Hausdorff measure on $\partial^* E_t$. $\partial^* E_t$ is the reduced boundary of E_t (see (2.5))

$$\partial^* E_t = \{x \in \mathcal{D} : |\vec{\nu}_{E_t}(x)| = 1\} = \text{supp}(|D\mathbb{1}_{E_t}|).$$

Moreover, by Theorem 2.16, $\partial^* E_t$ is a countably $(d-1)$-rectifiable set and

$$\lim_{\rho \searrow 0} \frac{|D\mathbb{1}_{E_t}|(B_\rho(x))}{\rho^{d-1}} = \omega_{d-1}, \quad \mathcal{H}^{d-1} - a.e.\ x \in \partial^* E_t, \tag{5.57}$$

where $B_\rho(x)$ is the ball in $\mathbb{R}^d$ with radius ρ and centered at x and ω_{d-1} is the area of unite sphere in $\mathbb{R}^{d-1}$. Since $\|V^t\| \geq \mu^t = 2S|\mathbb{1}_{E_t}|$, by Theorem 2.21, to show V^t is rectifiable, we need to show V^t has locally bounded first variation. In fact, by (5.54), $\text{supp}(D\mathbb{1}_{\tilde{E}_t})$ is a countable set, thus

$$d|D\mathbb{1}_{E_t}(x)| = \omega_d \sum_{r \in \text{supp}(D\mathbb{1}_{\tilde{E}_t})} \delta_r(|x|)dx,$$

where δ_r is the Dirac measure on $\mathbb{R}$. Thus $\partial^* E_t = \partial E_t$ which consists of countable $(d-1)$-spheres. Then by trace theorem

$$\int_{\partial E_t} |v|^2 d\mathcal{H}^{d-1} \lesssim \|v\|^2_{H^1}.$$

By (5.11),

$$\begin{aligned}|\langle \delta V^t, \vec{Y} \rangle| &\lesssim |\int_{\partial E_t} v\vec{Y} \cdot \vec{n}_{E_t} d\mathcal{H}^{d-1}| \\ &\lesssim \|Y\|_{L^\infty} \int_{\partial E_t} |v| d\mathcal{H}^{d-1} \\ &\lesssim \|v\|_{H^1}\|Y\|_{L^\infty},\end{aligned}$$

which implies V^t has locally bounded first variation.

Thus by the definition of rectifiability and the expression of V^t, we have that

$$dV^t(x,p) = 2S|D\mathbb{1}_{E_t}|dx\delta_{\vec{\nu}_{E_t}(t,x)}(dp) \text{ as Radon measure on } \mathcal{D} \times P,$$

i.e.

$$dV(t,x.p) = dV^t(x.p)dt = d\mu^t(t,x)\delta_{\vec{\nu}_{E_t}(t,x)}(dp).$$

Hence we conclude that $c_1^t = 1$, $c_2^t = \cdots = c_d^t = 0$ and $p_1^t = \vec{\nu}_{E_t}$. Then by the construction of V in subsection 5.2.5, we have that $\lambda_1 = 1$, $\lambda_2 = \cdots = \lambda_d = 0$ and

$$(d\mu_{ij})_{d\times d} = \vec{\nu}_{E_t} \otimes \vec{\nu}_{E_t} d\mu.$$

Then by (5.11), for any $\vec{Y} \in C_c^1(\mathcal{D}, \mathbb{R}^d)$, we have that

$$-\langle \delta V^t, \vec{Y} \rangle = 2\langle D\mathbb{1}_{E_t}, v\vec{Y} \rangle = 2\langle \vec{\nu}_{E_t}|D\mathbb{1}_{E_t}|, v\vec{Y} \rangle = \frac{1}{S}\langle \|V^t\|, v\vec{Y} \cdot \vec{\nu}_{E_t} \rangle.$$

Hence by the Definition 2.22, we obtain that

$$S\vec{H}_{V^t} = v\vec{\nu}_{E_t} \;\; \|V^t\| - a.e.,$$

where $\vec{H}_{V^t}$ is the mean curvature vector of V^t in Definition 2.22. This also implies that for $\|V^t\| - a.e.\ x \in \mathcal{D} \setminus \mathrm{supp}(|D\mathbb{1}_{E_t}|)$, $\vec{H}_{V^t} = 0$. Thus we have that

$$v = S\vec{H}_{V^t} \cdot \vec{\nu}_{E_t} \;\text{ on }\; \mathrm{supp}(|D\mathbb{1}_{E_t}|).$$

5.5 The case for "smeared" noise

We observe that the requirement $\sigma \geq \frac{1}{2}$ only comes from the second variation term $\frac{\varepsilon^{2\sigma-1}}{2}\mathrm{Tr}(f'(u^\varepsilon)Q)$ in (5.21) when we apply Itô's formula on $\mathcal{E}^\varepsilon(u^\varepsilon)$. If there were no such term $\frac{\varepsilon^{2\sigma-1}}{2}\mathrm{Tr}(f'(u^\varepsilon)Q)$, Theorem 5.11 would hold for $\sigma \geq 0$.

This motivates us to consider the following equation:

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \Delta v^\varepsilon + \varepsilon^\sigma \xi_t^\varepsilon, \quad (t,x) \in [0,T] \times \mathcal{D}, \\ v^\varepsilon = -\varepsilon \Delta u^\varepsilon(t) + \dfrac{1}{\varepsilon} f(u^\varepsilon(t)), \quad (t,x) \in [0,T] \times \mathcal{D}, \\ \dfrac{\partial u^\varepsilon}{\partial n} = \dfrac{\partial v^\varepsilon}{\partial n} = 0, \quad (t,x) \in [0,T] \times \partial\mathcal{D}, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x), \quad x \in \mathcal{D}, \end{cases} \tag{5.58}$$

where u_0^ε satisfies (5.2) and ξ_t^ε is formally defined by $\xi_t^\varepsilon = \int_{-\infty}^{\infty} \rho_\varepsilon(t-s)dW_s$. In fact, let $(W_t, t \geq 0)$ be a Q-Wiener process on $L_0^2(\mathcal{D})$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Q satisfies (5.4) and (5.5). We extend the definition of $(W_t, t \geq 0)$ to negative time by considering an i.i.d Q-Wiener process $(\hat{W}_t, t \geq 0)$ and setting $W_t = \hat{W}_{-t}$ for $t < 0$. Then $(W_t, t \in \mathbb{R})$ is a two-sided Q-Wiener process on L_0^2. Let ρ be a mollifying kernel i.e.

$$\rho \in C^\infty(\mathbb{R}), \quad 0 \leq \rho \leq 1, \quad \mathrm{supp}\rho \subset [-1,1], \quad \int_{\mathbb{R}} \rho = 1, \quad \rho(t) = \rho(-t).$$

For $\gamma > 0$ we set $\rho_\varepsilon(t) = \varepsilon^{-\gamma}\rho(\frac{t}{\varepsilon^\gamma})$. Then the approximate Wiener process W_t^ε is defined as

$$W_t^\varepsilon := \int_{-\infty}^{\infty} \rho_\varepsilon(t-s) W_s ds, \tag{5.59}$$

Its derivative is defined as

$$\xi_t^\varepsilon := \frac{dW_t^\varepsilon}{dt} = \int_{-\infty}^{\infty} \rho_\varepsilon(t-s) dW_s. \tag{5.60}$$

Since ρ_ε is supported on $[-\varepsilon^\gamma, \varepsilon^\gamma]$, only the definition on negative time $[-\varepsilon^\gamma, 0)$ of W_t is used. Thus we have that for any $g \in L^2(\mathcal{D})$

$$\begin{aligned} \int_0^T \langle g(t), \xi_t^\varepsilon \rangle dt &= \int_0^T \langle g(t), \int_{-\varepsilon^\gamma}^{t+\varepsilon^\gamma} \rho_\varepsilon(t-s) dW_s \rangle dt \\ &= \int_{-\varepsilon^\gamma}^{T+\varepsilon^\gamma} \langle \int_0^T \rho_\varepsilon(s-t) g(t) dt, dW_s \rangle. \end{aligned} \tag{5.61}$$

Lemma 5.19. *There exists a constant which only depends on T such that for any $\varepsilon \in (0,1]$ and any $p \geq 1$, any $\sigma \geq 0$*

$$\mathbb{E} \sup_{t\in[0,T]} \mathcal{E}^\varepsilon(t)^p \leq C_T(\varepsilon^\sigma + \mathcal{E}_0)^p, \tag{5.62}$$

and

$$\mathbb{E}\left(\int_0^T \|\nabla v^\varepsilon\|_{L^2}^2 dt\right)^p \leq C_T(\varepsilon^\sigma + \mathcal{E}_0)^p. \tag{5.63}$$

Proof The proof is a modification of Lemma 5.6.

Note that the noise in equation 5.58 is smooth in time, which enable us to apply Newton-Leibniz formula on $\mathcal{E}^\varepsilon$ to avoid the second variation term in (5.21). We have that

$$\frac{d\mathcal{E}^\varepsilon(u^\varepsilon)}{dt} = \langle D\mathcal{E}^\varepsilon(u^\varepsilon), \partial_t u^\varepsilon\rangle = -\langle \nabla v^\varepsilon, \nabla v^\varepsilon\rangle + \varepsilon^\sigma\langle v^\varepsilon, \xi_t^\varepsilon\rangle. \tag{5.64}$$

By (5.61) we know that

$$\int_0^T \langle v^\varepsilon(t), \xi_t^\varepsilon\rangle dt = \int_{-\varepsilon^\gamma}^{T+\varepsilon^\gamma} \langle \rho_\varepsilon * v^\varepsilon(t), dW_t\rangle,$$

where we simply denote

$$\rho_\varepsilon * v^\varepsilon(t) := \int_0^T \rho_\varepsilon(t-s)v^\varepsilon(s)ds.$$

Similarly as the proof in Lemma 5.6. by Burkholder-Davis-Gundy type inequality, we have that

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]} \left|\int_0^T \langle v^\varepsilon(t), \xi_t^\varepsilon\rangle dt\right| &\lesssim \mathbb{E}\sup_{t\in[0,T]}\left|\int_0^T \langle \rho_\varepsilon * v^\varepsilon(t), dW_t\rangle\right| \\
&\lesssim \left(\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^T \langle \rho_\varepsilon * v^\varepsilon(t), dW_t\rangle\right|^2\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_0^T \mathbb{E}\|\sqrt{Q}(\rho_\varepsilon * v^\varepsilon(t))\|_{L^2}^2 dt\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_0^T \left(\rho_\varepsilon * \mathbb{E}\|\nabla v^\varepsilon\|_{L^2}\right)^2 dt\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_0^T \mathbb{E}\|\nabla v^\varepsilon\|_{L^2}^2 dt\right)^{\frac{1}{2}},
\end{aligned}$$

where we used the Young's inequality in the last inequality. The rest is the same as in the proof of Lemma 5.6. We omit it here for simplicity.

□

With the same notation and proof as in Theorem 5.11, we can obtain a tightness result for any $\sigma \geq 0$.

Theorem 5.20. *Assume $\sigma \geq 0$, Q satisfies (5.4) and (5.5). There exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ on $\mathcal{X}^1 \times \mathcal{X}^2$, a subsequence ε_k (we still denote it as ε for simplicity) and*

$$\left\{\left(\varepsilon^{-1}\sup_{t\in[0,T]}\|F(\tilde{u}^\varepsilon)\|_{L^1}, \mathcal{E}^\varepsilon(\tilde{u}^\varepsilon), \tilde{u}^\varepsilon, G(\tilde{u}^\varepsilon), \tilde{v}^\varepsilon, e^\varepsilon(\tilde{u}^\varepsilon)dxdt, \{\varepsilon\partial_{x_i}\tilde{u}^\varepsilon\partial_{x_j}\tilde{u}^\varepsilon dxdt\}_{ij}\right)\right\} \subset \mathcal{X}^1\times\mathcal{X}^2$$

and

$$(a, \mathcal{E}, u, g, v, \mu, \{\mu_{ij}\}_{ij}) \in \mathcal{X}^1 \times \mathcal{X}^2,$$

such that

(i) $\tilde{P}\circ\left(\varepsilon^{-1}\sup_{t\in[0,T]}\|F(\tilde{u}^\varepsilon)\|_{L^1}, \mathcal{E}^\varepsilon(\tilde{u}^\varepsilon), \tilde{u}^\varepsilon, G(\tilde{u}^\varepsilon), \tilde{v}^\varepsilon, e^\varepsilon(\tilde{u}^\varepsilon)dxdt, \{\varepsilon\partial_{x_i}\tilde{u}^\varepsilon\partial_{x_j}\tilde{u}^\varepsilon dxdt\}_{ij}\right)^{-1} = \hat{\mathbb{P}}^\varepsilon$ *on* $\mathcal{X}^1 \times \mathcal{X}^2$,

(ii) $\left(\varepsilon^{-1}\sup_{t\in[0,T]}\|F(\tilde{u}^\varepsilon)\|_{L^1}, \mathcal{E}^\varepsilon(\tilde{u}^\varepsilon), \tilde{u}^\varepsilon, G(\tilde{u}^\varepsilon), \tilde{v}^\varepsilon, e^\varepsilon(\tilde{u}^\varepsilon)dxdt, \{\varepsilon\partial_{x_i}\tilde{u}^\varepsilon\partial_{x_j}\tilde{u}^\varepsilon dxdt\}_{ij}\right)$ *converges to* $(0, \mathcal{E}, u, g, v, \mu, \{\mu_{ij}\}_{ij})$ *in* $\mathcal{X}^1 \times \mathcal{X}^2$, $\tilde{\mathbb{P}} - a.s$, *as* $\varepsilon \searrow 0$.

In particular, for $\tilde{\mathbb{P}} - a.s.\omega$, *there exists a Borel set* $E \in \mathcal{D}_T$, *such that as* $\varepsilon \searrow 0$

(iii) $u^\varepsilon \to u$ *in* u *in* $C^\beta([0,T]; L^2)$, $g = G(u) = 2S\mathbb{1}_E$ *a.e. in* $\mathcal{D}_T$ *and in* $C^\beta([0,T]; L^1)$, $u = -1 + 2\mathbb{1}_E$ *a.e. in* $\mathcal{D}_T$ *and in* $C^\beta([0,T]; L^2)$.

Moreover, denote $E_t := \{x : (t,x) \in E\}$. *Then*

(iv) For all $\beta < \frac{1}{12}$, $\tilde{\mathbb{P}}\left(\mathbb{1}_E \in C^\beta([0,T]; L^1)\right) = 1$,

(v) $\tilde{\mathbb{P}}\left(|E_t| = |E_0| = \frac{1+m_0}{2}|\mathcal{D}|, \forall t \in [0,T]\right) = 1$,

(vi) $\tilde{\mathbb{P}}(\mathbb{1}_E \in L^\infty(0,T; BV)) = 1$.

Proof For all $\sigma \geq 0$, one can check that with Lemma 5.19 true, all the estimate in Subsection 5.2.2 and 5.2.3 hold for the solution $(u^\varepsilon, v^\varepsilon)$ to equation (5.58). Then the same proof as Theorem 5.11 follows.

□

Moreover, for $\sigma \geq 0$, the same argument as in Subsection 5.2.5 yields that

Theorem 5.21. *Assume that $\sigma \geq 0$ and (5.2) hold. Let $(u^\varepsilon, v^\varepsilon)$ be the solution to (5.58). Then there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, $(\tilde{u}^\varepsilon, \tilde{v}^\varepsilon) \in C([0,T]; L^2) \times L^2(0,T; H^1)$ with $\tilde{\mathbb{P}}\circ(\tilde{u}^\varepsilon, \tilde{v}^\varepsilon)^{-1} = \mathbb{P}\circ(u^\varepsilon, v^\varepsilon)^{-1}$ on $C([0,T]; L^2)\times L^2(0,T; H^1)$. There also exists a subsequence ε_k such that as $\varepsilon_k \searrow 0$ the following holds:*

(i) There exists a measurable set $E \subset \tilde{\Omega} \times \mathcal{D}_T$, *such that for* $\tilde{\mathbb{P}} - a.s.\ \omega$

$$\tilde{u}^{\varepsilon_k}(\omega) \to -1 + 2\mathbb{1}_{E(\omega)}, \quad a.e.\ in\ \mathcal{D}_T\ and\ in\ C^\beta([0,T]; L^2)$$

for any $\beta < \frac{1}{12}$ *where* $E(\omega) := \{(t,x) \in \mathcal{D}_T : (\omega, t, x) \in E\}$;

(ii) There exists a random variable $v \in L^2_w(0,T; H^1)$ *(v is weakly measurable in* $L^2(0,T; H^1)$*) such that for* $\tilde{\mathbb{P}} - a.s.\ \omega$

$$\tilde{v}^{\varepsilon_k}(\omega) \to v(\omega) \quad weakly\ in\ L^2(0,T; H^1);$$

(iii) There exist random variables $\mu \in \mathfrak{M}_R$ *and* $\{\mu_{ij}\}_{i,j=1}^d \in \mathfrak{M}^{d\times d}$ *such that for* $\tilde{\mathbb{P}} - a.s.\ \omega$

$$\begin{aligned} e^{\varepsilon_k(\omega)}(\tilde{u}^{\varepsilon_k(\omega)})dxdt &\to d\mu(\omega,t,x) \quad weakly\ in\ \mathfrak{M}_R, \\ \varepsilon_k\partial_{x_i}\tilde{u}^{\varepsilon_k}(\omega)\partial_{x_j}\tilde{u}^{\varepsilon_k}(\omega)dxdt &\to d\mu_{ij}(\omega,t,x) \quad weakly\ in\ \mathfrak{M},\ \forall i,j = 1,\cdots,d. \end{aligned} \tag{5.65}$$

(iv) For $\tilde{\mathbb{P}}-a.s.\ \omega$, there exists a Radon measure $V(\omega)$ on $\mathcal{D}_T\times P$, and $\mu^t(\omega,x)dt = d\mu(\omega,t,x)$ such that for any $t\in(0,T]$ and $\vec{Y}\in C_0^1(\mathcal{D}_t;\mathbb{R}^d)$

$$\int_0^t\langle\delta V^s,\vec{Y}\rangle ds=\int_0^t\int_{\mathcal{D}}\nabla\vec{Y}:\left(Id\mu(s,x)-(\mu_{ij}(s,x))_{d\times d}\right). \tag{5.66}$$

In particular, for $\tilde{\mathbb{P}}-a.s.\ \omega$, $(E(\omega),v(\omega),V(\omega))$ satisfies all the properties in Definition 5.2 except (5.12). If $\sigma>0$, (5.12) holds, thus $(E(\omega),v(\omega),V(\omega))$ is a weak solution in the sense of Definition 5.2.

Proof The proof is almost the same as in Subsection 5.2.5. The only difference is in the proof of that the existence of a Q-Wiener process on L^2 cannot be obtained directly such that for any $\varepsilon>0$, (5.35) holds. We use the original equation (5.58) to prove (5.10) directly.

In fact, for any $\psi\in C_c^1([0,t)\times\mathcal{D})$,

$$-\int_{\mathcal{D}}(1+u^\varepsilon(0,x))\psi(0,x)dx=\int_0^t\int_{\mathcal{D}}\partial_t\psi(\tau,x)(1+u^\varepsilon(\tau,x))dxd\tau-\int_0^t\int_{\mathcal{D}}\nabla v^\varepsilon\nabla\psi dxd\tau$$
$$+\varepsilon^\sigma\int_0^t\int_{\mathcal{D}}\psi(\tau,x)\xi^\varepsilon(\tau,x)dxd\tau.$$

Thus for $\mathbb{P}-a.s.\omega\in\Omega$,

$$\lim_{\varepsilon\to0}\left(\int_0^t\int_{\mathcal{D}}\partial_t\psi(\tau,x)(1+u^\varepsilon(\tau,x))dxd\tau+\int_{\mathcal{D}}(1+u^\varepsilon(0,x))\psi(0,x)dx-\int_0^t\int_{\mathcal{D}}\nabla v^\varepsilon\nabla\psi dxd\tau\right)=0.$$

Since $\tilde{\mathbb{P}}\circ(\tilde{u}^\varepsilon,\tilde{v}^\varepsilon)^{-1}=\mathbb{P}\circ(u^\varepsilon,v^\varepsilon)^{-1}$, we have that for $\tilde{P}-a.s.\omega\in\tilde{\Omega}$ and any $\sigma>0$,

$$\lim_{\varepsilon\to0}\left(\int_0^t\int_{\mathcal{D}}\partial_t\psi(\tau,x)(1+\tilde{u}^\varepsilon(\tau,x))dxd\tau+\int_{\mathcal{D}}(1+\tilde{u}^\varepsilon(0,x))\psi(0,x)dx-\int_0^t\int_{\mathcal{D}}\nabla\tilde{v}^\varepsilon\nabla\psi dxd\tau\right)=0,$$

which yields that

$$\int_0^t\int_{\mathcal{D}}\partial_t\psi(\tau,x)(1+\tilde{u}(\tau,x))dxd\tau+\int_{\mathcal{D}}(1+\tilde{u}(0,x))\psi(0,x)dx-\int_0^t\int_{\mathcal{D}}\nabla\tilde{v}\nabla\psi dxd\tau=0.$$

Thus we obtain (5.10). The rest proof is the same as the proof of Theorem 5.3 in Subsection 5.2.5.

□

Moreover in radial symmetric case,

Theorem 5.22. *Let $\sigma\geq0$, with the same notations as in Theorem 5.21, and suppose that the assumptions in Theorem 5.21 hold. Then in radially symmetric case, that is $\mathcal{D}=B_1$, where B_1 is the unit ball in $\mathbb{R}^d$ and that u_0^ε is radially symmetric, we have that*

$$d\mu=2S|D\mathbb{1}_{E_t}|dxdt\ \text{ as Radon measure on }\ \mathcal{D}_T.$$

In particular, for a.e.$t\in[0,T]$, V^t is a $(d-1)$-rectifiable varifold, i.e.

$$dV(t,x,p)=2S|D\mathbb{1}_{E_t}|dxdt\delta_{\vec{\nu}_{E_t}(t,x)}(dp)\ \text{ as Radon measure on }\ \mathcal{D}_T\times P.$$

Then we have that

$$\begin{cases} (d\mu_{ij})_{d\times d} = \vec{\nu}_{E_t} \otimes \vec{\nu}_{E_t} d\mu \ \ \textit{as Radon measure on } \bar{\mathcal{D}}_T, \\ \quad v(t,x) = S\vec{\nu}_{E_t}(x) \cdot \vec{H}_{V^t}(x) \ \ \textit{on } \operatorname{supp}(|D\mathbb{1}_{E_t}|) \ \textit{for a.e. } t \in [0,T], \end{cases} \tag{5.67}$$

$\vec{H}_{V^t}$ *is the mean curvature vector of* V^t *defined in Definition 2.22 and* $\delta_{\vec{\nu}}$ *is the Dirac measure concentrated at* $\vec{\nu} \in P$.

Proof It suffice to prove

$$d\mu = 2S|D\mathbb{1}_{E_t}|dxdt \ \text{ as Radon measure on } \mathcal{D}_T,$$

then the following is the same as the proof of Theorem 5.4 in Section 5.3.

In fact, by taking $h^\varepsilon(t,x) = \int_0^t \int_{\mathcal{D}} v^\varepsilon(s,x)\xi^\varepsilon(s,x)dsdx$ in (5.51), then all the proof followed as in the proof of Theorem 5.18. Thus we can finish the proof.

□

Remark 5.23. *The same as in Remark 5.5, in radial symmetric case,* $v = SH$ *on* Γ_t *in a weak sense. Thus in radial symmetric case, for all* $\sigma > 0$ *the sharp interface limit of equation 5.58 satisfies the deterministic Hele-Shaw model (1.5) in a weak sense. In general we also conjecture that the sharp interface limit of equation 5.58 satisfies the deterministic Hele-Shaw model (1.5)*

$$\begin{cases} \Delta v = 0 \ \textit{in } \mathcal{D} \setminus \Gamma_t, \ t > 0, \\ \dfrac{\partial v}{\partial n} = 0 \ \textit{on } \partial\mathcal{D}, \\ \quad v = SH \ \textit{on } \Gamma_t, \\ \quad \mathcal{V} = \dfrac{1}{2}(\partial_n v^+ - \partial_n v^-) \ \textit{on } \Gamma_t. \end{cases}$$

Now we will focus on the case that $\sigma = 0$. Note that the triple (E, v, V) obtained in Theorem 5.21 satisfies all the definition in Definition 5.2 except (5.10) and (5.12). Let $\mathcal{D}^+ = E_t^o \cap \mathcal{D}$ be the interior of E_t in $\mathcal{D}$ and $\mathcal{D}^- = \mathcal{D} \setminus \bar{E}_t$.

Theorem 5.24. *Let* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, E, v *be as in Theorem 5.21 and* Q *be an operator satisfying (5.4),(5.5). Then there exists a Q-Wiener process* $\tilde{W}$ *on* $L^2(\mathcal{D})$, *which is defined on* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, *such that*

$$2d\mathbb{1}_{E_t} = \Delta v dt + d\tilde{W}_t,$$

in the sense that for any $t \in [0,T]$ *and* $\psi \in C_c^1([0,t) \times \bar{\mathcal{D}})$,

$$\int_0^t \int_{\mathcal{D}} (-2\mathbb{1}_{E_\tau}\partial_t\psi + \nabla v \cdot \nabla\psi)\, dxd\tau = \int_{\mathcal{D}} 2\mathbb{1}_{E_0}\psi(0,\cdot) + \int_0^t \langle \psi(\tau,\cdot), d\tilde{W}_\tau \rangle. \tag{5.68}$$

Proof For any $h \in H^1$, denote

$$M_h^\varepsilon := \int_{\mathcal{D}} (u^\varepsilon(t) - u^\varepsilon(0))hdx + \int_0^t \nabla v^\varepsilon \cdot \nabla h dx$$

and

$$\tilde{M}_h^\varepsilon := \int_{\mathcal{D}} (\tilde{u}^\varepsilon(t) - \tilde{u}^\varepsilon(0))hdx + \int_0^t \nabla \tilde{v}^\varepsilon \cdot \nabla h dx.$$

Clearly,

$$M_h^\varepsilon = \int_{\mathcal{D}} h(x) W_t^\varepsilon(x) dx$$

and as $\varepsilon \to 0$, M_h^ε converges to a Wiener process with covariance $\|Q^{\frac{1}{2}} h\|_{L^2}^2$. Since $\mathbb{P} \circ (M_h^\varepsilon)^{-1} = \tilde{P} \circ (\tilde{M}_h^\varepsilon)^{-1}$, we know that the law of $\tilde{M}_h^\varepsilon$ converges to a Wiener process with covariance $\|Q^{\frac{1}{2}} h\|_{L^2}^2$. Moreover

$$\lim_{\varepsilon \to 0} \tilde{M}_h^\varepsilon = \int_{\mathcal{D}} (u(t) - u(0)) h dx + \int_0^t \nabla v \cdot \nabla h dx, \quad \tilde{P} - a.s..$$

Thus we obtain that

$$\int_{\mathcal{D}} (u(t) - u(0)) h dx + \int_0^t \nabla v \cdot \nabla h dx$$

is a Wiener process with covariance $\|Q^{\frac{1}{2}} h\|_{L^2}^2$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Then there exists a Q-Wiener process $\tilde{W}$ on L^2, which is defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, such that

$$\langle W_t, h \rangle = \int_{\mathcal{D}} (u(t) - u(0)) h dx + \int_0^t \int_{\mathcal{D}} \nabla v \cdot \nabla h dx ds.$$

Thus we obtain the following equation for u:

$$du = \Delta v dt + dW_t$$

Similar to the proof in Subsection 5.2.5, the Itô's formula yields that for any $\psi \in C_c^1([0,t) \times \bar{\mathcal{D}})$

$$\begin{aligned} -\int_{\mathcal{D}} (1 + u(0,x)) \psi(0,x) dx = & \int_0^t \int_{\mathcal{D}} \partial_t \psi(\tau, x)(1 + u(\tau, x)) dx d\tau - \int_0^t \int_{\mathcal{D}} \nabla v \nabla \psi dx d\tau \\ & + \int_0^t \langle \psi(\tau, \cdot) dW_\tau \rangle, \end{aligned}$$

i.e.

$$\int_0^t \int_{\mathcal{D}} (-2 \mathbb{1}_{E_\tau} \partial_t \psi + \nabla v \cdot \nabla \psi) \, dx d\tau = \int_{\mathcal{D}} 2 \mathbb{1}_{E_0} \psi(0, \cdot) + \int_0^t \langle \psi(\tau, \cdot), d\tilde{W}_\tau \rangle.$$

Similarly to the discussion in Subsection 5.1.4, (5.68) is a weak formula for

$$2 d \mathbb{1}_{E_t} = \Delta v dt + d\tilde{W}_t.$$

□

Corollary 5.25. *For any $\psi \in C_c^1([0,t) \times \bar{\mathcal{D}})$, with* $\mathrm{supp}\psi \subset \mathcal{D} \setminus \Gamma_t$,

$$\int_0^t \int_{\mathcal{D}} \nabla v \cdot \nabla \psi d\mathcal{H}^d ds = \int_0^t \langle \psi, d\tilde{W}_s \rangle.$$

This is in fact a weak formula for

$$\Delta v = -\frac{dW_t}{dt} \quad in \;\; \mathcal{D} \setminus \Gamma_t.$$

Proof Since $\psi \in C_c^1([0,t) \times \bar{\mathcal{D}})$ and $\text{supp}\psi \subset \mathcal{D} \setminus \Gamma_t$, we know that $\mathbb{1}_E \psi = \psi$ and $\mathbb{1}_E \partial_t \psi = \partial_t \psi$. Thus

$$\int_0^t \int_{\mathcal{D}} \mathbb{1}_{E_t} \partial_t \psi dx = \int_0^t \int_{\mathcal{D}} \partial_t \psi dx = -\int_{\mathcal{D}} \psi(0,\cdot) dx = -\int_{\mathcal{D}} \mathbb{1}_{E_0} \psi(0,\cdot) dx.$$

Then by (5.68), we can finish the proof.

□

Remark 5.26. *Similar to the deterministic case, Δv and $\frac{\partial v}{\partial n}$ are ill-defined. The equation of (v, Γ) should be understood in distribution sense. We suppose that (v, Γ) is smooth enough such that Δv and $\frac{\partial v}{\partial n}$ are well-defined.*

We also suppose that $\overline{\mathcal{D} \setminus E} \subset \mathcal{D}$. Denote $\Gamma_t := \partial E_t \setminus \partial \mathcal{D}$ and let $\mathcal{D}^+ = E_t^o \cap \mathcal{D}$ be the interior of E_t in $\mathcal{D}$ and $\mathcal{D}^- = \mathcal{D} \setminus \bar{E}_t$.

For any $\psi \in C_c^1([0,t) \times \bar{\mathcal{D}})$, with $\text{supp}\,\psi(t,\cdot) \cap \Gamma_t = \emptyset$,

$$\begin{aligned}
\int_0^t \int_{\partial\mathcal{D}} \frac{\partial v}{\partial n} \psi d\mathcal{H}^{d-1} ds &= \int_0^t \int_{\partial\mathcal{D}^+} \frac{\partial v}{\partial n} \psi d\mathcal{H}^{d-1} ds - \int_0^t \int_{\Gamma_t} \frac{\partial v}{\partial n} \psi d\mathcal{H}^{d-1} ds \\
&= \int_0^t \int_{\partial\mathcal{D}^+} \frac{\partial v}{\partial n} \psi d\mathcal{H}^{d-1} ds \\
&= \int_0^t \int_{\mathcal{D}^+} div(\nabla v \psi) d\mathcal{H}^d ds \\
&= \int_0^t \int_{\mathcal{D}^+} \nabla v \cdot \nabla \psi d\mathcal{H}^d ds + \int_0^t \int_{\mathcal{D}^-} \Delta v \psi d\mathcal{H}^d ds \\
&= \int_0^t \langle \psi, d\tilde{W}_s \rangle + \int_0^t \int_{\mathcal{D}^+} \Delta v \psi d\mathcal{H}^d ds \\
&= \int_0^t 2\langle \psi, d\mathbb{1}_{E_s} \rangle = 0,
\end{aligned} \tag{5.69}$$

where we used Corollary 5.25 in the fifth equality. The last equality holds because $\text{supp}\,\psi(t,\cdot) \cap E_t = \emptyset$.

Formally we have that in distribution sense

$$\frac{\partial v}{\partial n} = 0 \quad in \quad [0,T] \times \partial\mathcal{D}.$$

To calculate the velocity of Γ_t, formally we denote $\hat{v} = v + \Delta^{-1}\dot{\tilde{W}}$, where $\dot{\tilde{W}}$ is the formal derivative $\frac{d\tilde{W}}{dt}$. Then we have

$$2\partial_t \mathbb{1}_{E_t} = \Delta \hat{v},$$

and $\hat{v} = 0$ in $[0,T] \times (\mathcal{D} \setminus \Gamma_t)$. For any $\psi \in C_c^1(\bar{\mathcal{D}}_t)$

$$\begin{aligned}
\int_0^t \int_{\mathcal{D}} \partial_t \mathbb{1}_{E_t} \psi d\mathcal{H}^d ds &= -\frac{1}{2} \int_0^t \int_{\mathcal{D}} \nabla \hat{v} \nabla \psi d\mathcal{H}^d ds \\
&= -\frac{1}{2} \int_0^t \int_{\mathcal{D}^+} \nabla \hat{v} \nabla \psi d\mathcal{H}^d ds - \frac{1}{2} \int_0^t \int_{\mathcal{D}^-} \nabla \hat{v} \nabla \psi d\mathcal{H}^d ds \\
&= \frac{1}{2} \int_0^t \int_{\mathcal{D}^+} \text{div}(\nabla \hat{v} \psi) d\mathcal{H}^d ds + \frac{1}{2} \int_0^t \int_{\mathcal{D}^-} \text{div}(\nabla \hat{v} \psi) d\mathcal{H}^d ds \\
&= \frac{1}{2} \int_0^t \int_{\Gamma_t} (\partial_n \hat{v}^+ - \partial_n \hat{v}^-) \psi d\mathcal{H}^{d-1} ds.
\end{aligned} \tag{5.70}$$

Then following the same calculation as in Section 5.1.4, we obtain that in distribution sense

$$\mathcal{V} = \frac{1}{2}(\partial_n \hat{v}^+ - \partial_n \hat{v}^-).$$

Thus formally we have

$$\mathcal{V}dt = \frac{1}{2}\left[\frac{\partial}{\partial n}\right]_{\Gamma_t} (vdt + \Delta^{-1} d\tilde{W}_t), \tag{5.71}$$

Here $\left[\frac{\partial}{\partial n}\right]_{\Gamma_t}$ *is defined by*

$$\left[\frac{\partial}{\partial n}\right]_{\Gamma_t} f = \partial_n f^+ - \partial_n f^-,$$

where f^+, f^- *is the restriction of* f *on* $\mathcal{D}^+$, $\mathcal{D}^-$, *respectively.*

Remark 5.27. *For the value of* v *on* Γ_t, *since Theorem 5.22 holds for all* $\sigma \geq 0$. *Combining it with Corollary 5.25, Remark 5.26 and (5.71), we prove that in radial case, when* $\sigma = 0$, *the sharp interface limit of equation (5.58) is the formally the stochastic Hele-Shaw model (1.29). For general case, we conjecture that the sharp interface limit also satisfies (1.29):*

$$\begin{cases} \Delta v dt = -dW_t \ in\, \mathcal{D} \setminus \Gamma_t,\ t > 0, \\ \dfrac{\partial v}{\partial n} = 0 \ on\, \partial\mathcal{D}, \\ v = SH \ on\, \Gamma_t, \\ \mathcal{V}dt = \dfrac{1}{2}\left[\dfrac{\partial}{\partial n}\right]_{\Gamma_t} (vdt + \Delta^{-1} dW_t). \end{cases}$$

Remark 5.28. *The idea of "smeared noise" in this section can be also applied to the case of space-time white noise by considering the mollified space-time white noise. That is, considering the convolution of space-time white noise with a mollifier both in space and in time. Then a similar result can be obtained.*

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