Pseudoergodic Operators and Periodic Boundary Conditions

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Abstract

There is an increasing literature on random non self-adjoint infinite matrices with motivations ranging from condensed matter physics to neural networks. Many of these operators fall into the class of "pseudoergodic" operators, which allows the elimination of probabilistic arguments when studying spectral properties. Parallel to this is the increased awareness that spectral properties of non self-adjoint operators, in particular stability, may be better captured via the notion of pseudospectra as opposed to spectra. Although it is well known that the finite section method applied to these matrices does not converge to the spectrum, it is often found in practice that the pseudospectrum behaves better with appropriate boundary conditions. We make this precise by giving a simple proof that the finite section method with periodic boundary conditions converges to the pseudospectrum of the full operator. Our results hold in any dimension (not just for banded bi-infinite matrices) and can be considered as a generalisation of the well known classical result for banded Laurent operators and their circulant approximations. Furthermore, we numerically demonstrate a convergent algorithm for the pseudospectrum including cases where periodic boundary conditions converge faster than the method of uneven sections. Finally we show that the result carries over to pseudoergodic operators acting on l^p spaces for $p \in [1, \infty]$.

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1 Introduction

Random matrices appear in a wide number of contexts throughout the sciences, ranging from applied physics through to areas of pure mathematics such as number theory [13, 38, 64, 65]. In particular, the study of random Jacobi operators can be traced back at least as far as the famous Anderson model [2, 3]. More recently, over the past two decades there has been a considerable amount of interest in the study of random non self-adjoint (NSA) operators on separable Hilbert spaces [15, 16, 31, 35, 36]. As well as their interesting mathematical properties, motivation for studying such operators can be found in condensed matter physics: conductivity of disordered media, flux lines in superconductors and asymmetric hopping particles, and even in population biology [1, 39, 50, 53, 66]. One is often interested in how the spectrum and pseudospectrum behave under truncation of the operators to finite matrices [29, 40, 44, 60, 73, 74] and this can lead to algorithms to compute spectral properties numerically. Many of the operators studied in these papers are *pseudoergodic* (also sometimes referred to as stochastic Laurent matrices in the $l^2(\mathbb{Z})$ case), which roughly means that every possible finite pattern in the matrix elements appears somewhere up to arbitrary precision (see Definition 1.1). This allows the treatment of such random operators in a deterministic fashion, leading to simplified proofs of spectral properties which often depend only weakly on the distribution of matrix elements (e.g. on its support).

In general, spectral properties of NSA operators are much more difficult to analyse than their self-adjoint relatives due to the absence of the spectral theorem and instability of eigenvalues/spectra [35, 37]. This has led to the study of pseudospectra. If A is a bounded operator, then for $\epsilon > 0$ the ϵ-pseudospectrum is defined as

$$\mathrm{Sp}_\epsilon(A) := \mathrm{Sp}(A) \cup \{z \notin \mathrm{Sp}(A) : \|R(z, A)\|^{-1} \leq \epsilon\} \tag{1.1}$$

or equivalently via the stability condition [67]:

$$\mathrm{Sp}_\epsilon(A) = \mathrm{cl}\Big(\bigcup_{\|B\| \leq \epsilon} \mathrm{Sp}(A + B)\Big)$$

where cl denotes closure. Here we use the notation $R(z, A) = (A - zI)^{-1}$ for the resolvent. Amongst its uses, the pseudospectrum was crucial for the solution of the long-standing problem of computing the spectrum from matrix values [48] and it has a useful role in non-Hermitian quantum mechanics [56]. It has been argued that pseudospectra, as opposed to spectra, may provide more useful information when studying NSA operators, in particular when studying the notion of stability [35, 72]. We refer the reader to [74] for more applications.

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This paper studies the limit of pseudospectra of finite truncations (finite section) of pseudoergodic operators. We show that in the limit of increasing system size, pseudospectra converge to the pseudospectra of the full operator if we apply *periodic boundary conditions*. This result was conjectured in [34] for a one dimensional lattice model but has so far remained an open problem.[1] The result presented here holds for any dimension and any finite range interaction pseudoergodic operator (not just tridiagonal). In other words, the passage from finite volumes to the infinite volume case is continuous with respect to the pseudospectrum. This can be considered as a compliment to the well known corresponding result for banded Laurent operators - it is precisely the fact that pseudoergodic operators "look the same" under translational shifts that allows us to prove this result. We also convert this new result into an efficient algorithm that converges to the pseudospectrum in the Hausdorff metric, which we numerically demonstrate on a wide range of examples. Furthermore, the fact that we can approximate the pseudospectrum of the full infinite dimensional operator using square matrices allows the numerical computation of pseudospectra for $p \neq 2$ and we prove the convergence of pseudospectra in this case also.[2] This is in contrast to the method of uneven sections (see [33] and the Appendix) which uses rectangular matrices and for which no such generalisation is numerically possible. As well as being of interest from the finite section point of view, our results have practical significance. We numerically demonstrate the algorithm for general p and also compare the $p = 2$ case to the method of uneven sections (which is only available for $p = 2$) where in some cases the new algorithm converges much faster.

It should be mentioned that in sharp contrast to our results, spectra of finite sections are often very different to that of the full operator, particularly in the NSA case. The classic example of this is the NSA Anderson model introduced by Hatano and Nelson in the context of vortex pinning in type-II superconductors [49] which has been widely studied [25, 26, 35, 50, 51, 66, 71]. The model showed that an imaginary gauge field in a disordered one-dimensional lattice can induce a delocalization transition. This pseudoergodic operator acts on $l^2(\mathbb{Z})$ via

$$(Hx)_n = e^{-g}x_{n-1} + e^{g}x_{n+1} + V_n x_n \tag{1.2}$$

where $g > 0$ and V is a (real-valued) random potential. Truncating the operator to $\operatorname{span}\{e_{-n}, ..., e_n\}$ and adopting periodic boundary conditions gives a spectrum with the famous "bubble and wings". Goldsheid and Khoruzhenko have studied the convergence of the spectral measure in the periodic case as $n \to \infty$ in [43, 44]. This can be very different to the spectrum of the operator on $l^2(\mathbb{Z})$ [35], highlighting the difficulty in computing the spectrum. Applying no boundary conditions and simply taking the matrix[3] $P_n A P_n$ is even worse. In this case the matrix is similar to a real symmetric matrix, hence has completely real spectrum! Already we can see stability playing a role - as n increases, the condition number of the similarity transform increases exponentially for $g \neq 0$. There are certain cases where the obvious finite section $P_n A P_n$ behaves better and we refer the reader to [28, 29] for a thorough study of the famous "hopping sign model" (also a pseudoergodic operator) where, remarkably, this is the case.

1.1 Definitions and Main Results

Given $A \in \mathcal{B}(l^2(\mathbb{Z}^d))$ and $i, j \in \mathbb{Z}^d$, we will denote the inner product $\langle Ae_j, e_i \rangle$ with respect to the canonical basis by $A_{i,j}$.

Definition 1.1 (Pseudoergodicity)**.** *Let A be a bounded operator acting on $l^2(\mathbb{Z}^d)$. Given a collection $\underline{M} = \{M_k\}_{k \in \mathbb{Z}^d}$ of compact subsets $M_k \subset \mathbb{C}$, we say that $\underline{M}$ is permissible if only finitely many of the M_k are not $\{0\}$. Given a permissible $\underline{M}$, we say that A is pseudoergodic with respect to $\underline{M}$ if $A_{i,j} \in M_{i-j}$ and the following property holds. Given any $\epsilon > 0$, finite subsets $S_k \subset \mathbb{Z}^d$ and functions $F_k : S_k \to M_k$, there exists a translation T acting on $\mathbb{Z}^d$ such that*

$$\sup_{i \in S_k} \left| A_{T(i),T(i)-k} - F_k(i) \right| < \epsilon, \quad \forall k \in \mathbb{Z}. \tag{1.3}$$

We define $\mathcal{A}(\underline{M})$ be the class of pseudoergodic operators with respect to $\underline{M}$, and Ω^d to be the class of pseudoergodic operators acting on $l^2(\mathbb{Z}^d)$.

[1] Most results in the literature consider either special cases of tridiagonal pseudoergodic operators or use the theory of limit operators to write the pseudospectrum of pseudoergodic operators acting on $l^2(\mathbb{Z})$ as the union of pseudospectra over all possible periodic submatrices (see for example [45, 46]), which is not helpful from a numerical perspective.

[2] Although it must be said that one cannot use singular values of matrices to characterise the pseudospectrum when $p \neq 2$ and it appears the only known method of computing $p-$pseudospectra of finite matrices is via directly estimating $\|R(z, A)\|_p$ over some fine grid. The computation of matrix norms of finite square matrices for $p \neq 1, 2$ or ∞ is NP-hard [52] so this currently seems intractable for large matrices for any p other than these.

[3] Throughout P_n will denote the orthogonal projection onto $\operatorname{span}\{e_{-n}, e_{-n+1}, ..., e_n\}$ in the case of $l^2(\mathbb{Z})$.

A few remarks are in order. Note first that in the case of $d = 1$, such an operator must be banded by the assumption that only finitely many of the M_k are not $\{0\}$. Second, it is also clear that such operators must be bounded for any d. This is also true when considering these infinite matrices as operators acting on $l^p(\mathbb{Z}^d)$ (see Section 3) for which we use the same definition of pseudoergodicity. Third, the same translation T is required to work for all the diagonals simultaneously and it is clearly sufficient to only test those diagonals that are non-zero. The idea is that every possible finite "pattern" is realised up to an arbitrarily small error in each of the selected diagonal. In the case of the (one-dimensional) NSA Anderson model with i.i.d. diagonals with support M, A is clearly almost surely pseudoergodic with respect to $M_1 = \{e^g\}, M_0 = M$ and $M_{-1} = \{e^{-g}\}$ (with all other diagonals being zero). This can be extended to the hopping sign model, random tridiagonal operators and many other variants studied in the literature.

It is straightforward to show that the maps $\mathrm{Sp}(\cdot)$ and $\mathrm{Sp}_\epsilon(\cdot)$ are constant on each $\mathcal{A}(\underline{M})$ (see [36] for the case of pseudoergodic potentials, exactly the same argument can be extended to the cases in this paper). We then let A_n^{per} denote the nth order truncation of $A \in \Omega^d$ with natural periodic boundary conditions (see Section 2, in particular equation (2.7)). In the Hilbert space case of $l^2(\mathbb{Z}^d)$ our main result is the following.

Theorem 1.2. *Let $A \in \Omega^d$ and $\epsilon > 0$, then $\lim_{n\to\infty} \mathrm{Sp}_\epsilon(A_n^{per}) = \mathrm{Sp}_\epsilon(A)$ in the Hausdorff metric and $\mathrm{Sp}_\epsilon(A_n^{per}) \subset \mathrm{Sp}_\epsilon(A)$. Define the algorithm $\Gamma_n(A) = \texttt{PseudoSpecPer}(A, n, \epsilon)$, then $\lim_{n\to\infty} \Gamma_n(A) = \mathrm{Sp}_\epsilon(A)$ in the Hausdorff metric and $\Gamma_n(A) \subset \mathrm{Sp}_\epsilon(A)$.*

The routine alluded to in the above theorem is written in pseudocode as

Function `PseudoSpecPer` (A, n, ϵ)
  Input : n, A pseudoergodic, $\epsilon > 0$
  Output: $\Gamma \subset \mathbb{C}$, an approximation to $\mathrm{Sp}_\epsilon(A)$
  $G = \frac{1}{n}(\mathbb{Z} + i\mathbb{Z}) \cap B_n(0)$
  for $z \in G$ **do**
    $B = A_n^{per} - zI$
    $C = (A_n^{per})^* - \bar{z}I$
    $S = B^*B$
    $T = C^*C$
    $p = \texttt{IsPosDef}(S - \epsilon^2)$
    $q = \texttt{IsPosDef}(T - \epsilon^2)$
    $\nu(z) = \min(p, q)$
  end
  $\Gamma = \bigcup\{z \in G \,|\nu(z) = 0\}$.
end

Here $B_n(0)$ denotes the closed ball of radius n around 0 and the `IsPosDef` routine determines whether a matrix is positive definite (returns 1) or not (returns 0). This can be done by using an incomplete Cholesky decomposition [33] (chosen for stability and speed of computation). If wanted, this can be altered to use only finitely many arithmetic operations and comparisons. It is also efficient to restrict/alter the ball G to be any region of the complex plane where one is interested in computing the pseudospectrum (e.g. near a rough approximation of the pseudospectrum).

The results can also be extended to the $l^p(\mathbb{Z}^d)$ case where the definition of pseudoergodicity remains the same and we use a superscript to denote pseudospectra with respect to the corresponding operator norm.

Theorem 1.3. *Let $p \in [1, \infty]$ and $A \in \Omega^d$. Then $\lim_{n\to\infty} \mathrm{Sp}_\epsilon^p(A_n^{per}) = \mathrm{Sp}_\epsilon^p(A)$ in the Hausdorff metric and $\mathrm{Sp}_\epsilon^p(A_n^{per}) \subset \mathrm{Sp}_\epsilon^p(A)$.*

This can also be used in a similar routine to `PseudoSpecPer` (see Section 3). The $2-$norm and any other $p-$norm of an $n \times n$ matrix can only differ by a factor of $\sqrt{n}$ and hence the different notions of pseudospectra are only useful for large or infinite matrices. There have been examples [54] where this difference is expected to be important. In particular, the $1-$norm and $\infty-$norm pseudospectra are relevant for probability theory [11,12] and heat flow [4].

1.2 Connections with Previous Work

There is a vast literature on the study of the finite section method and conditions under which it converges. For large classes of operators, an open question is whether suitable *boundary conditions* applied to finite

sections leads to convergence of the spectrum/pseudospectrum as the matrix size increases and the finite section method has often been viewed in connection with *Toeplitz theory*. In particular, we refer the reader to work by Böttcher [14, 24], Böttcher & Silberman [22], Lindner [59, 61], Marletta [62] and Marletta & Scheichl [63]. For an operator algebra point of view, we refer the reader to [5–7, 27]. Our results fit nicely into this framework by answering this question for a large class of well studied operators in the case of the pseudospectrum.

Definition 1.1 is similar to that of [36], where the definition of pseudoergodicity first appeared in the literature, however there are some differences. We do not consider an arbitrary group of permutations of $\mathbb{Z}^d$ but rather consider only translations which respect the structure of $\mathbb{Z}^d$. We also consider arbitrary pseudoergodic non-diagonal parts rather than the case of a sum of a pseudoergodic potential and operator that commutes with the action of the permutation group. Extensions to $\mathbb{Z}^d \times \mathcal{S}$ (equivalently vector-valued sequences) with $\mathcal{S}$ finite are also given (see Section 2.4). Note also that if we select elements of the diagonals independently from a probability distribution of support M_k, then A will be pseudoergodic almost surely. Weaker conditions ensure pseudoergodicity but we will use this context for our numerical examples.

An extreme version of pseudoergodic operators are banded Laurent operators. It is well known in this case that the l^2 spectrum and pseudospectrum of the corresponding circulant matrices converge to that of the infinite dimensional operator (the matrices are normal and one can easily prove convergence of the spectrum using Fourier analysis). Perhaps surprising is that fact that Toeplitz operators under pure finite section (truncation with no added boundary conditions) also share nice properties with regards to pseudospectra. This statement is false for the spectrum as is easily seen by considering the shift operator. The following is taken from [20, 21] where $\mathrm{Sp}^p_\epsilon(A)$ denotes the pseudospectrum with respect to the l^p norm:

Theorem 1.4. *Let b be a Laurent polynomial and $T(b)$ the corresponding Toeplitz operator with finite sections $T_n(b)$. Then for $p \in (1, \infty)$ we have the following convergence in the Hausdorff metric*

$$\lim_{n\to\infty} \mathrm{Sp}^p_\epsilon(T_n(b)) = \mathrm{Sp}^p_\epsilon(T(b)) \cup \mathrm{Sp}^p_\epsilon(T(\tilde{b})),$$

where $\tilde{b}(\zeta) = b(-\zeta)$.

In particular we recover convergence for the case $p = 2$. Amongst its extensions, this result has been shown to hold for piecewise continuous symbol [19, 23]. There are also studies considering randomly perturbed Laurent and Toeplitz operators [15–18].

With regards to pseudoergodic operators, work has focused on the case of pure finite section (no periodic boundary conditions). Remarkably this converges to the pseudospectrum of the full (doubly infinite operator) for the hopping sign model [29]. Furthermore, for tridiagonal pseudoergodic operators, it has been proven [30] that the pseudospectrum using pure finite section converges for $p = 2$ to the corresponding pseudospectrum of the operator A_+ on $l^2(\mathbb{N})$ (which is also independent of the operator sampled from the pseudoergodic class). This convergence can overestimate the pseudospectrum of the doubly infinite operator, in particular for the NSA Anderson model example we give later. However, the pseudospectrum still captures in some sense the famous bubble and wings. Finally, it was shown in [30] that for tridiagonal A, the $2-$norm pseudospectra of A_+ is the union of $\mathrm{Sp}^2_\epsilon(A)$ together with a set G that closes the gap between $\mathrm{Sp}(A_+)$ and $\mathrm{Sp}_{ess}(A_+)$, a result completely analogous to the Toeplitz/Laurent case.

Finally, it should be mentioned that our results fit into the framework of the *Solvability Complexity Index* (SCI) hierarchy. The SCI provides a classification hierarchy [9, 10, 48] of spectral problems providing classifications according to their computational difficulty. The SCI of a class of spectral problems is the least number of limits needed in order to compute the spectrum of operators in this class. In particular, denoting the algorithm in Theorem 1.2 by $\Gamma_{\epsilon,n}$, our results show that computing the pseudospectrum of operators in Ω^d requires one limit and computing the spectrum requires at most two limits via

$$\mathrm{Sp}(A) = \lim_{\epsilon\downarrow 0} \lim_{n\to\infty} \Gamma_{\epsilon,n}(A).$$

This is in contrast to the class of general bounded operators where three limits are needed to compute the spectrum [9].

1.3 Organisation of the Paper

The paper is organised as follows. In Section 2 we prove our main results for the Hilbert space case $l^2(\mathbb{Z}^d)$. These are generalised to l^p spaces in Section 3 where we recover the full pseudospectrum for $p \in [1, \infty]$. In Section 4 we present examples, which includes an explanation of the results in [33] that gave a method for

increasing the speed of convergence to the pseudospectrum of the NSA Anderson model. We also demonstrate that the speed of convergence using periodic boundary conditions can be superior to the method of "uneven sections" proposed in [33], with substantial speed up in higher dimensions ($d > 1$). Finally we conclude with a discussion of the results in Section 5.

2 The Hilbert Space Case

Throughout this section, we will use $\|\cdot\|$ to denote the standard l^2 norm and assume that $\underline{M}$ is permissible. Let $A \in \mathcal{B}(l^2(\mathbb{Z}^d))$ and define the injection modulus by

$$\sigma_1(A) := \inf\{\|Ax\| : x \in l^2(\mathbb{Z}^d), \|x\| = 1\}, \tag{2.1}$$

which is equal to the smallest singular value in the case of finite matrices. Define the function

$$\psi_A(z) := \min\{\sigma_1(A - zI), \sigma_1(A^* - \bar{z}I)\}. \tag{2.2}$$

It is well known that $\psi_A(z) = \|R(z, A)\|^{-1}$ and hence we can characterise the pseudospectrum via

$$\mathrm{Sp}_\epsilon(A) = \{z \in \mathbb{C} : \psi_A(z) \leq \epsilon\}. \tag{2.3}$$

As part of the proof of Theorem 1.2, we will show that for n larger than the bandwidth of A

$$\limsup_{l\to\infty} \psi_{A_l^{per}}(z) \leq \psi_A(z) \leq \psi_{A_n^{per}}(A),$$

where A_n^{per} denotes the finite sections of A with appropriate periodic boundary conditions (see below). We begin with the simpler case of $d = 1$ and then discuss the generalisation to $d > 1$. These are then used to prove Theorem 1.2. Finally we discuss the generalisation to vector-valued l^2 sequences (where matrix valued entries $A_{i,j}$ are considered).

2.1 The case of $d = 1$

We will first deal with the case of $d = 1$ since it presents the key ideas without additional notational complexity. Given $A \in \mathcal{B}(l^2(\mathbb{Z}))$, let $A_n^o \in \mathbb{C}^{(2n+1)\times(2n+1)}$ denote the matrix formed by P_nAP_n with P_n the orthogonal projection onto $\mathrm{span}\{e_{-n}, e_{-n+1}, ..., e_n\}$. In other words, A_n^o is the matrix formed by standard finite section with open boundary conditions. Our first lemma states that in the limit $n \to \infty$, $\psi_{A_n^o}(z) \leq \psi_A(z)$ and uses only the properties of bandedness and boundedness of $A \in \Omega^1$.

Lemma 2.1. *Let $A \in \Omega^1$ with $A \in \mathcal{A}(\underline{M})$, then for any $z \in \mathbb{C}$, $\limsup_{n\to\infty} \psi_{A_n^o}(z) \leq \psi_A(z)$.*

Proof. Let $\delta > 0$, then by definition there exists some $\tilde{x} \in l^2(\mathbb{Z})$ of norm 1 such that $\|(A - zI)\tilde{x}\| \leq \sigma_1(A - zI) + \delta$. Let $x_k = P_k\tilde{x}/\|P_k\tilde{x}\|$ then, since $P_k\tilde{x} \to \tilde{x}$ as $k \to \infty$ and A is bounded, it follows that for large enough $k \geq k_0$, $\|(A - zI)x_k\| \leq \sigma_1(A - zI) + 2\delta$. Set $x = x_{k_0}$, which has norm one by construction. Since the support of x is finite and A is banded, we must have $(A_n^o - zI)x = (A - zI)x$ for $n \geq m + k_0$ where m is the bandwidth of A given by

$$m := \max\{|k| : M_k \neq 0\}. \tag{2.4}$$

Hence $\sigma_1(A_n^o - zI) \leq \|(A_n^o - zI)x\| \leq \sigma_1(A - zI) + 2\delta$. Since $\delta > 0$ was arbitrary, it follows that

$$\limsup_{n\to\infty} \sigma_1(A_n^o - zI) \leq \sigma_1(A - zI).$$

Since the adjoint is also banded, we can prove the same inequality replacing $\sigma_1(A_n^o - zI)$ by $\sigma_1((A_n^o)^* - \bar{z}I)$ and $\sigma_1(A - zI)$ by $\sigma_1(A^* - \bar{z}I)$ in exactly the same way. The result now follows. □

Given $A \in \mathcal{A}(\underline{M})$, let $L_n^{\mathrm{b.c.}}$ be a lower diagonal matrix, with matrix values uniformly bounded in n, such that $(L_n^{\mathrm{b.c.}})_{i,j} = 0$ if $j > i + m - (2n + 1)$, where i, j are indexed in $\{-n, -n + 1, ..., n\}$ and m is defined in (2.4). Similarly let $U_n^{\mathrm{b.c.}}$ be an upper diagonal matrix, with matrix values uniformly bounded in n, such that $(U_n^{\mathrm{b.c.}})_{i,j} = 0$ if $i > j + m - (2n + 1)$. The superscript b.c. stands for the boundary conditions being imposed which are captured by these upper and lower diagonal matrices. Let $A_n^{\mathrm{b.c.}} = A_n^o + L_n^{\mathrm{b.c.}} + U_n^{\mathrm{b.c.}}$. For fixed m, letting $n \to \infty$, we can conclude in exactly the same way as the above that

$$\limsup_{n\to\infty} \psi_{A_n^{\mathrm{b.c.}}}(z) \leq \psi_A(z). \tag{2.5}$$

The point is that the boundary conditions only act *locally*. We denote periodic boundary conditions by a superscript per and in this case we *fix* the non-zero entries of L_n^{per} and U_n^{per} such that these are given by

$$(L_n^{per})_{i,j} \in M_{i-j-(2n+1)} \text{ if } j \leq i+m-(2n+1), \quad (U_n^{per})_{i,j} \in M_{i-j+(2n+1)} \text{ if } i \leq j+m-(2n+1).$$

Note that we are allowing any choice up to these constraints. The above ensure that the coupling between sites (i.e. the non-zero diagonals) is consistent if one defines the matrix

$$A_{n,N}^{per} := \underbrace{\begin{pmatrix} A_n^o & L_n^{per} & & & \\ U_n^{per} & A_n^o & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & L_n^{per} \\ & & & U_n^{per} & A_n^o \end{pmatrix}}_{N \text{ blocks}},$$

where each block is a $n \times n$ matrix. The following proposition is the key result in showing periodic boundary conditions are a good choice for calculating pseudospectra of pseudoergodic operators.

Proposition 2.2. *Consider the above set up with $A \in \mathcal{A}(\underline{M})$ (and $d = 1$). For all $n \geq m$ and all $z \in \mathbb{C}$ we have $\psi_{A_n^{per}}(z) \geq \psi_A(z)$.*

Proof. We will show that for $n \geq m$ and all $z \in \mathbb{C}$ we have $\sigma_1(A_n^{per} - zI) \geq \sigma_1(A - zI)$. Dealing with $\sigma_1(A_n^{per*} - \bar{z}I)$ is similar and together these give the result.

Let $\delta > 0$ and choose $x \in \mathbb{C}^{2n+1}$ such that $\|x\| = 1$ and $\|(A_n^{per} - zI)x\| \leq \sigma_1(A_n^{per} - zI) + \delta$. The idea is to extend x periodically and use pseudoergodicity. Extend x and $y = (A_n^{per} - zI)x$ periodically N times to obtain $x^N, y^N \in \mathbb{C}^{N(2n+1)}$ and consider the matrix $A_{n,N}^{per}$ above. Then we have

$$(A_{n,N}^{per} - zI)x^N = \begin{pmatrix} A_n^o & L_n^{per} & & \\ U_n^{per} & A_n^o & \ddots & \\ & \ddots & \ddots & L_n^{per} \\ & & U_n^{per} & A_n^o \end{pmatrix} \begin{pmatrix} x \\ x \\ \vdots \\ x \\ x \end{pmatrix} - zx^N = \begin{pmatrix} (A_n^o + L_n^{per})x \\ A_n^{per}x \\ \vdots \\ A_n^{per}x \\ (A_n^o + U_n^{per})x \end{pmatrix} - zx^N.$$

It follows that the vector

$$(A_{n,N}^{per} - zI)x^N - y^N = -\begin{pmatrix} U_n^{per}x \\ 0 \\ \vdots \\ 0 \\ L_n^{per}x \end{pmatrix} \tag{2.6}$$

has norm bounded by some constant, $D(m, \underline{M})$, independent of N and all x of norm 1. This is because the values of the non-zero entries of $(A_{n,N}^{per} - zI)x^N - y^N$ are uniformly bounded and there are at most $2m$ of them. The constant will in general depend on m and the maximum modulus over the set $\cup_k M_k$ but this dependence is not relevant for the argument. The idea is shown visually in Figure 1.

It follows that

$$\|(A_{n,N}^{per} - zI)x^N\| \leq \|y^N\| + D(m, \underline{M}) \leq N^{\frac{1}{2}}(\sigma_1(A_n^{per} - zI) + \delta) + D(m, \underline{M}).$$

By construction, all entries of the periodic extension $A_{n,N}^{per}$ come from the set M_k of the corresponding diagonal with respect to which A is pseudoergodic. Hence by pseudoergodicity of A, for each desired accuracy $\epsilon > 0$ there is a desired $(2n+1)N \times (2n+1)N$ submatrix of A which is ϵ close to $A_{n,N}^{per}$. Hence we can shift the support of x^N and let $w^N \in l^2(\mathbb{Z})$ equal x^N on the corresponding $(2n+1)N$ entries and zero otherwise. Choosing ϵ sufficiently small we have $\|w^N\| = \|x^N\| = N^{\frac{1}{2}}$ and

$$\|(A - zI)w^N\| \leq N^{\frac{1}{2}}(\sigma_1(A_n^{per} - zI) + \delta) + \delta + D(m, \underline{M}).$$

It follows that

$$\sigma_1(A - zI) \leq \sigma_1(A_n^{per} - zI) + 2\delta + D(m, \underline{M})N^{-\frac{1}{2}}.$$

Letting $N \to \infty$ and then $\delta \downarrow 0$ gives $\sigma_1(A_n^{per} - zI) \geq \sigma_1(A - zI)$. □

$$y = \begin{pmatrix} 1 & b_1 & 0 \\ 0 & 1 & b_2 \\ b_3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + b_1 x_2 \\ x_2 + b_2 x_3 \\ x_3 + b_3 x_1 \end{pmatrix}$$

A_2^{per} A_2^o L_2^{per}

$$A_{1,2}^{per} x^2 = \left(\begin{array}{ccc|ccc} 1 & b_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & b_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & b_3 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & b_1 & 0 \\ 0 & 0 & 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + b_1 x_2 \\ x_2 + b_2 x_3 \\ x_3 + b_3 x_1 \\ x_1 + b_1 x_2 \\ x_2 + b_2 x_3 \\ \textcircled{x_3} \end{pmatrix}$$

Figure 1: Visualisation of matching across periodic extensions for $n = 1$ and $N = 2$, on $l^2(\mathbb{Z})$. For this example the diagonal takes the value 1 and the superdiagonal takes values $b_i \in M_{-1}$. The circled index corresponds to the discrepancy with y^N (in this case the missing $b_3 x_1$ term).

2.2 The case of $d > 1$

In order to deal with higher dimensional case, it is useful to introduce some notation. Given $n \in \mathbb{N}$ and $k \in \mathbb{Z}^d$ define the index sets

$$C_n := \{-n, n+1, ..., n\}^d, \quad C_{n,k} := C_n + (2n+1)k.$$

The $C_{n,k}$ partition $\mathbb{Z}^d$ and will be used to construct the relevant periodisations. Given $N \in \mathbb{N}$, we also define

$$C_N \otimes C_n := \bigcup_{k \in C_N} C_{n,k} = C_{2Nn+N+n}.$$

For $W \subset \mathbb{Z}^d$, we define the orthogonal projections $P_W, P_W^\perp : l^2(\mathbb{Z}^d) \to l^2(\mathbb{Z}^d)$ via

$$(P_W x)_j = \begin{cases} x_j, & \text{if } j \in W, \\ 0, & \text{otherwise} \end{cases}$$

and $P_W^\perp = P_{\mathbb{Z}^d \setminus W}$. We define the shift operator $S_{n,k} : l^2(\mathbb{Z}^d) \to l^2(\mathbb{Z}^d)$ via

$$(S_{n,k} x)_j = x_{j-(2n+1)k}, j \in \mathbb{Z}^d$$

which translates between the $C_{n,k}$'s. Given $A \in \Omega^d$, and with a slight abuse of notation, we can define the matrices A_n^o and A_n^{per} acting on the range of P_{C_n} (i.e. $l^2(C_n)$) via

$$A_n^o x = P_{C_n} A P_{C_n} x, \quad A_n^{per} x = \sum_{k \in C_1} P_{C_n} S_{n,-k} A P_{C_n} x. \tag{2.7}$$

Finally, the periodisation $A_{n,N}^{per}$ acting on $l^2(C_N \otimes C_n)$ is defined via

$$A_{n,N}^{per} x = P_{C_N \otimes C_n} \sum_{k \in C_N} S_{n,k} A P_{C_n} S_{n,-k} x. \tag{2.8}$$

All of these definitions also extend to the l^p case considered in Section 3. As before, we could have taken any values from the relevant M_k's in forming the above generalisations of L_n^{per} and U_n^{per}. However, the above definitions give a much cleaner presentation. The reader is referred to Figure 2 for the case of $d = 2$ which also explains the idea of the proof below. Note that the proof of Lemma 2.1 is identical for $d > 1$ and yields (2.5) for periodic boundary conditions with the general notion of bandedness given by

$$m := \max\{\|k\|_\infty : M_k \neq \{0\}\}. \tag{2.9}$$

Care is only needed for the argument in the proof of Proposition 2.2.

Proof of extension of Proposition 2.2 to $d > 1$. Again we will only show that for $n \geq m$ and all $z \in \mathbb{C}$ we have $\sigma_1(A_n^{per} - zI) \geq \sigma_1(A - zI)$. Let $\delta > 0$ and choose $x \in l^2(C_n)$ such that $\|x\| = 1$ and $\|(A_n^{per} - zI)x\| \leq \sigma_1(A_n^{per} - zI) + \delta$. Define the periodisations

$$y^N := P_{C_N \otimes C_n} \sum_{l \in C_N} S_{n,l}(A_n^{per} - zI)x, \quad x^N := P_{C_N \otimes C_n} \sum_{l \in C^N} S_{n,l}x.$$

The key step of the proof is a result analogous to (2.6). We have that

$$\begin{aligned}(A_{n,N}^{per} - zI)x^N - y^N &= \big(P_{C_N \otimes C_n} \sum_{k \in C_N} S_{n,k} A P_{C_n} S_{n,-k} x^N\big) - zx^N - \big(P_{C_N \otimes C_n} \sum_{k \in C_N} S_{n,k} A_n^{per} x\big) + zx^N \\ &= P_{C_N \otimes C_n} \sum_{k \in C_N} \big[(S_{n,k} A P_{C_n} S_{n,-k} \sum_{l \in C_N} S_{n,l} x) - S_{n,k} \sum_{l \in C_1} P_{C_n} S_{n,-l} A P_{C_n} x\big].\end{aligned}$$

Since $P_{C_n} x = x$, we can simplify the first term in the sum via

$$P_{C_n} S_{n,-k} \sum_{l \in C_N} S_{n,l} x = P_{C_n} x.$$

This yields

$$(A_{n,N}^{per} - zI)x^N - y^N = P_{C_N \otimes C_n} \sum_{k \in C_N} \big(S_{n,k} A P_{C_n} x - S_{n,k} \sum_{l \in C_1} P_{C_n} S_{n,-l} A P_{C_n} x\big). \tag{2.10}$$

Since $n \geq m$ we can write

$$A P_{C_n} x = \sum_{t \in C_1} P_{C_{n,t}} A P_{C_n} x.$$

Note that the terms corresponding to $t = 0$ cancel in (2.10). We also have the relation

$$P_{C_n} S_{n,-l} = S_{n,-l} P_{C_{n,-l}}.$$

Putting these together in (2.10), we arrive at

$$\begin{aligned}(A_{n,N}^{per} - zI)x^N - y^N &= P_{C_N \otimes C_n} \sum_{k \in C_N} S_{n,k} \big[(\sum_{t \in C_1} P_{C_{n,t}}) - \sum_{l \in C_1} S_{n,-l} P_{C_{n,-l}} \sum_{t \in C_1} P_{C_{n,t}}\big] P_{C_n}^{\perp} A P_{C_n} x \\ &= P_{C_N \otimes C_n} \sum_{k \in C_N} \sum_{t \in C_1} \big(S_{n,k} - S_{n,k} S_{n,t}\big) P_{C_{n,t}} P_{C_n}^{\perp} A P_{C_n} x \\ &= \sum_{t \in C_1 \setminus \{0\}} \sum_{k \in C_N} P_{C_N \otimes C_n} \big(S_{n,k} - S_{n,k+t}\big) P_{C_{n,t}} A P_{C_n} x.\end{aligned}$$

Given $t \in C_1 \setminus \{0\}$, the only terms remaining in

$$\sum_{k \in C_N} P_{C_N \otimes C_n} \big(S_{n,k} - S_{n,k+t}\big) P_{C_{n,t}}$$

after cancellations are

$$\sum_{k \in C_N, t-k \notin C_N} -P_{C_N \otimes C_n} S_{n,k+t} P_{C_{n,t}}.$$

We can also restrict the sum to $k \in C_N$ such that there exists $t \in C_1$ with $t - k \notin C_N$ and denote this set inclusion via $k \in \partial C_N$. Upon swapping the order of summations again, we arrive at

$$(A_{n,N}^{per} - zI)x^N - y^N = - \sum_{k \in \partial C_N} \sum_{\substack{t \in C_1 \setminus \{0\} \\ t-k \notin C_N}} S_{n,k+t} P_{C_{n,t}} A P_{C_n} x.$$

Given $k \in \partial C_N$, the vector

$$\sum_{\substack{t \in C_1 \setminus \{0\} \\ t-k \notin C_N}} S_{n,k+t} P_{C_{n,t}} A P_{C_n} x$$

$Ax =$

	$A_n^{(-1,1)}x$	$A_n^{(0,1)}x$	$A_n^{(1,1)}x$	
	$A_n^{(-1,0)}x$	$A_n^o x$	$A_n^{(1,0)}x$	
	$A_n^{(-1,-1)}x$	$A_n^{(0,-1)}x$	$A_n^{(1,-1)}x$	

$x^N =$

x	x	x	x	x
x	x	x	x	x
x	x	x	x	x
x	x	x	x	x
x	x	x	x	x

$A_{n,N}^{per}x^N =$

without $(0,-1),(1,0),(1,-1)$	without $(0,-1)$	without $(0,-1)$	without $(0,-1)$	without $(0,-1),(-1,0),(-1,-1)$
without $(1,0)$	$A_n^{per}x$	$A_n^{per}x$	$A_n^{per}x$	without $(-1,0)$
without $(1,0)$	$A_n^{per}x$	$A_n^{per}x$	$A_n^{per}x$	without $(-1,0)$
without $(1,0)$	$A_n^{per}x$	$A_n^{per}x$	$A_n^{per}x$	without $(-1,0)$
without $(0,1),(1,0),(1,1)$	without $(0,1)$	without $(0,1)$	without $(0,1)$	without $(0,1),(-1,0),(-1,1)$

$A_{n,N}^{per}x^N - y^N = -$

with $(0,-1),(1,0),(1,-1)$	with $(0,-1)$	with $(0,-1)$	with $(0,-1)$	with $(0,-1),(-1,0),(-1,-1)$
with $(1,0)$	0	0	0	with $(-1,0)$
with $(1,0)$	0	0	0	with $(-1,0)$
with $(1,0)$	0	0	0	with $(-1,0)$
with $(0,1),(1,0),(1,1)$	with $(0,1)$	with $(0,1)$	with $(0,1)$	with $(0,1),(-1,0),(-1,1)$

Figure 2: Visualisation of how periodisation works for $d = 2$ and $z = 0$. We have $A_n^{per} = A_n^o + A_n^{(-1,1)} + A_n^{(0,1)} + A_n^{(1,1)} + A_n^{(1,0)} + A_n^{(1,-1)} + A_n^{(0,-1)} + A_n^{(-1,-1)} + A_n^{(-1,0)}$. Without $(-1,1)$ refers to this sum without $A_n^{(-1,1)}$ and with $(-1,1)$ refers to $A_n^{(-1,1)}$ etc. We clearly see that $(A_{n,N}^{per} - zI)x^N - y^N$ is supported on ∂C_N with at most 3^d terms (in fact 3 in this case) in each "box".

is supported in $C_{n,-k}$ and has norm at most $3^d\|A\|$. Since these vectors have disjoint support over different k, it follows that

$$\|(A_{n,N}^{per} - zI)x^N - y^N\| \leq 3^d\|A\|\,|\partial C_N|^{\frac{1}{2}} = \mathcal{O}(N^{\frac{d-1}{2}}). \tag{2.11}$$

It follows that

$$\|(A_{n,N}^{per} - zI)x^N\| \leq \|y^N\| + \mathcal{O}(N^{\frac{d-1}{2}}) \leq |C_N|^{\frac{1}{2}}\,(\sigma_1(A_n^{per} - zI) + \delta) + \mathcal{O}(N^{\frac{d-1}{2}}), \tag{2.12}$$

since $\|y^N\| = |C_N|^{\frac{1}{2}}\,\|y\|$. The idea behind this part of the proof is shown in Figure 2 for the case of $d = 2$.

Now we use the pseudoergodicity property of A. Again by construction, all entries of the periodic extension $A_{n,N}^{per}$ come from the set M_k of the corresponding diagonal with respect to which A is pseudoergodic. Hence by pseudoergodicity of A, for each desired accuracy $\epsilon > 0$ there is a desired $(2(2Nn + N + n) + 1)^d \times (2(2Nn + N + n) + 1)^d$ submatrix of A which is ϵ close to $A_{n,N}^{per}$. Hence we can shift the support of x^N and let $w^N \in l^2(\mathbb{Z})$ equal x^N on the corresponding $(2(2Nn + N + n) + 1)^d$ entries and zero otherwise. Choosing ϵ sufficiently small we have $\|w^N\| = \|x^N\| = |C_N|^{\frac{1}{2}} = (2N + 1)^{\frac{d}{2}}$ and

$$\|(A - zI)w^N\| \leq (2N + 1)^{\frac{d}{2}}(\sigma_1(A_n^{per} - zI) + \delta) + \delta + \mathcal{O}(N^{\frac{d-1}{2}}).$$

It follows that

$$\sigma_1(A - zI) \leq \sigma_1(A_n^{per} - zI) + 2\delta + \mathcal{O}(N^{-\frac{1}{2}}). \tag{2.13}$$

Letting $N \to \infty$ and then $\delta \downarrow 0$ gives $\sigma_1(A_n^{per} - zI) \geq \sigma_1(A - zI)$. □

2.3 Proof of Theorem 1.2

Using these results, we can now prove Theorem 1.2.

Proof of Theorem 1.2. The inclusion $\mathrm{Sp}_\epsilon(A_n^{per}) \subset \mathrm{Sp}_\epsilon(A)$ follows from Proposition 2.2 and the characterisation in (2.3). For the convergence $\lim_{n\to\infty} \mathrm{Sp}_\epsilon(A_n^{per}) = \mathrm{Sp}_\epsilon(A)$, note that A is bounded so there exists a compact set K such that $\mathrm{Sp}_\epsilon(A) \subset K$. By Proposition 2.2 we only need to prove convergence of the sets $\mathrm{Sp}_\epsilon(A_n^{per})$ to $\mathrm{Sp}_\epsilon(A)$ restricted to K which without loss of generality we assume to be a closed ball around the origin. For any bounded operators S, T we have

$$|\sigma_1(T) - \sigma_1(S)| \leq \|S - T\|$$

and it follows that for $n \geq m$, $\psi_{A_n^{per}}(z)$ is Lipschitz over $z \in K$ with Lipschitz constant independent of n. Proposition 2.2 and Lemma 2.1 together give that

$$\psi_A(z) \leq \liminf_{n\to\infty} \psi_{A_n^{per}}(z) \leq \limsup_{n\to\infty} \psi_{A_n^{per}}(z) \leq \psi_A(z).$$

It follows that $\psi_{A_n^{per}}(z)$ converges pointwise to $\psi_A(z)$ and hence the uniform Lipschitz continuity upgrades this to uniform convergence over K. Now let $0 < \delta < \epsilon$ then the above shows that for large n we have

$$\mathrm{Sp}_{\epsilon-\delta}(A) \subset \mathrm{Sp}_\epsilon(A_n^{per}) \subset \mathrm{Sp}_\epsilon(A).$$

Finally, $\mathrm{Sp}_\eta(T)$ is continuous (w.r.t the Hausdorff metric) in η for any fixed $T \in \mathcal{B}(l^2(\mathbb{Z}^d))$. Convergence now follows since $0 < \delta < \epsilon$ was arbitrary.

To see convergence of `PseudoSpecPer` note that we have $\Gamma_n(A) \subset \mathrm{Sp}_\epsilon(A_n^{per})$ by construction. Choose a compact subset $K \subset \mathbb{C}$ with $\psi_{A_n^{per}}(z) > 2\epsilon$ for all $z \in \mathbb{C}\backslash K$ and for all n. By the uniform convergence and the Arzelá-Ascoli theorem we can choose $\delta_n \downarrow 0$ such that for all n,

$$|\psi_{A_n^{per}}(z) - \psi_{A_n^{per}}(w)| < \delta_n \text{ for all } z, w \in K \text{ with } |z - w| < 1/n. \tag{2.14}$$

Let n be large so that $K \subset [-n, n] + i[-n, n]$ and such that $\delta_n < \epsilon$. If this holds and $z_1 \in \mathrm{Sp}_{\epsilon-\delta_n}(A_n^{per})$ then there exists some $z_2 \in \frac{1}{n}(\mathbb{Z} + i\mathbb{Z}) \cap B_n(0)$ with $|z_1 - z_2| < 1/n$ and hence $|\psi_{A_n^{per}}(z_1) - \psi_{A_n^{per}}(z_2)| < \delta_n$. It follows that $z_2 \in \Gamma_n(A)$ and hence

$$\mathrm{Sp}_{\epsilon-\delta_n}(A_n^{per}) \subset \Gamma_n(A) + B_{1/n}(0) \subset \mathrm{Sp}_\epsilon(A_n^{per}) + B_{1/n}(0).$$

Let $\eta > 0$ with $\eta < \epsilon$ and choose n large such that $\epsilon - \delta_n > \eta$ then

$$\mathrm{Sp}_\eta(A_n^{per}) \subset \Gamma_n(A) + B_{1/n}(0) \subset \mathrm{Sp}_\epsilon(A_n^{per}) + B_{1/n}(0).$$

The right-hand side converges to $\mathrm{Sp}_\epsilon(A)$ and the left-hand side converges to $\mathrm{Sp}_\eta(A)$. Since $\eta < \epsilon$ was arbitrary and Sp_η is continuous in η, the desired convergence now follows. □

We have now shown why periodic boundary conditions are a natural choice for pseudoergodic operators. Although we may not have convergence of spectra (for example the 1D NSA Anderson model in the example in Section 4.1), we do obtain convergence for the pseudospectra.

2.4 Extension to vector-valued sequences

Here we briefly remark on the extension of the above arguments to vector-valued sequences. Consider the following generalisation of the standard lattice $\mathbb{Z}^d$. For some $d \in \mathbb{N}$ and finite set $\mathcal{S}$, set

$$\mathbb{X} = \mathbb{Z}^d \times \mathcal{S}.$$

We view this as the lattice $\mathbb{Z}^d$ with $|\mathcal{S}|$ sites attached to each point. In this case $l^2(\mathbb{X}, \mathbb{C}) \sim l^2(\mathbb{Z}^d, \mathbb{C}^{|\mathcal{S}|})$. Enumerating a basis of $l^2(\mathbb{X})$ (each basis vector corresponding to a site) as $\{e_{i,a} : i \in \mathbb{Z}^d, a \in \mathcal{S}\}$ allows us, for $A \in \mathcal{B}(l^2(\mathbb{X}))$, to form matrix elements

$$A_{(i,a),(j,b)} = \langle Ae_{j,b}, e_{i,a} \rangle.$$

In complete generalisation of Definition 1.1 above (where $|\mathcal{S}| = 1$), we say that a collection $\underline{M} = \{M_{k,a,b} \subset \mathbb{C} : k \in \mathbb{Z}^d, \text{ and } a, b \in \mathcal{S}\}$ is permissible if there exists $m \in \mathbb{N}$ such that $M_{k,a,b} = \{0\}$ if $\|k\|_\infty > m$. Given permissible $\underline{M}$, we say A is (translationally) pseudoergodic with respect to $\underline{M}$ if $A_{(i,a),(j,b)} \in M_{i-j,a,b}$ for all $a, b \in \mathcal{S}$ and the following property holds. Given any $\epsilon > 0$, finite subsets $S_{k,a,b} \subset \mathbb{Z}^d \times \mathcal{S}^2$ and functions $F_{k,a,b} : S_{k,a,b} \to M_{k,a,b}$ (for $k \in \mathbb{Z}^d$, and $a, b \in \mathcal{S}$), there exists a translation T acting on $\mathbb{Z}^d$ such that

$$\sup_{(i,a,b)\in S_{k,a,b}} \left| A_{(T(i),a),(T(i)-k,b)} - F_{k,a,b}(i) \right| < \epsilon, \quad k \in \mathbb{Z}^d, \text{ and } a, b \in \mathcal{S}. \tag{2.15}$$

Denote the collection of such A by $\mathcal{A}(\underline{M})$ and the union of $\mathcal{A}(\underline{M})$ over permissible $\underline{M}$ by $\Omega^{\mathbb{X}}$. Note that $A_{(i,a),(j,b)} \in M_{i-j,a,b}$ implies that

$$A_{(i,a),(j,b)} = 0, \quad \text{if } \|i - j\|_\infty > m, \tag{2.16}$$

the generalised notion of bandedness.

To treat these operators, only a slight adjustment to the definitions in Section 2.2 are needed. We now define the index sets $C_n^{\mathcal{S}} := \{-n, n+1, ..., n\}^d \times \mathcal{S}$, $C_{n,k}^{\mathcal{S}} := \big(C_n + (2n+1)k\big) \times \mathcal{S}$ and

$$C_N^{\mathcal{S}} \otimes C_n^{\mathcal{S}} := \bigcup_{k \in C_N} C_{n,k}^{\mathcal{S}} = C_{2Nn+N+n}^{\mathcal{S}}.$$

For $W \subset \mathbb{X}$, we define the orthogonal projections $P_W, P_W^{\perp} : l^2(\mathbb{X}) \to l^2(\mathbb{X})$ via

$$(P_W x)_{(j,a)} = \begin{cases} x_{(j,a)}, & \text{if } (j,a) \in W, \\ 0, & \text{otherwise} \end{cases}$$

and $P_W^{\perp} = P_{\mathbb{X} \setminus W}$ as before. The shift operator $S_{n,k}^{\mathcal{S}} : l^2(\mathbb{X}) \to l^2(\mathbb{X})$ now acts via

$$(S_{n,k}^{\mathcal{S}} x)_{(j,a)} = x_{(j-(2n+1)k,a)}, \quad j \in \mathbb{Z}^d, a \in \mathcal{S}$$

which translates between the $C_{n,k}^{\mathcal{S}}$'s. The definitions of A_n^o, A_n^{per} and $A_{n,N}^{per}$ are as before with the relevant superscripts $\mathcal{S}$ on the projections and shifts:

$$A_n^o x = P_{C_n^{\mathcal{S}}} A P_{C_n^{\mathcal{S}}} x, \quad A_n^{per} x = \sum_{k \in C_1} P_{C_n^{\mathcal{S}}} S_{n,-k}^{\mathcal{S}} A P_{C_n^{\mathcal{S}}} x, \quad A_{n,N}^{per} x = P_{C_N^{\mathcal{S}} \otimes C_n^{\mathcal{S}}} \sum_{k \in C_N} S_{n,k^{\mathcal{S}}} A P_{C_n^{\mathcal{S}}} S_{n,-k}^{\mathcal{S}} x.$$

The proof of the generalisation of Proposition 2.2 to $|\mathcal{S}| > 1$ now follows through almost verbatim with the addition of the relevant superscripts $\mathcal{S}$. For instance, the same manipulations lead to

$$(A_{n,N}^{per} - zI)x^N - y^N = - \sum_{k \in \partial C_N} \sum_{\substack{t \in C_1 \setminus \{0\} \\ t-k \notin C_N}} S_{n,k+t}^{\mathcal{S}} P_{C_{n,t}^{\mathcal{S}}} A P_{C_n^{\mathcal{S}}} x,$$

from which the rest of the argument easily follows. Lemma 2.1 also holds and together these prove the generalisation of Theorem 1.2 to $\Omega^{\mathbb{X}}$ using the same arguments as in Section 2.3.

3 The General l^p Case

In this section we will prove that the results of Section 2 can be generalised to the case of viewing the pseudoergodic operator as acting on $l^p(\mathbb{X})$, where $\mathbb{X}$ is the generalisation of $\mathbb{Z}^d$ discussed in Section 2.4. Recall that due to the definition of pseudoergodicity, the operators are banded in the generalised sense with uniformly bounded matrix values - hence their matrices can be viewed as operators acting on $l^p(\mathbb{X})$ for any $p \in [1, \infty]$. For general Banach spaces, one needs to be careful of the definition of pseudospectrum since it is possible for the resolvent norm to be constant on open sets in the resolvent [70]. This does not occur for Banach spaces which have the strong maximum property (see [42, 70] for a definition and the following theorem - the fact that $l^p(\mathbb{X})$ satisfies the required property is mentioned in [70] with results from [32, 41, 57]) and the following theorem demonstrates that we do not have to worry about this in the cases considered in this paper.

Theorem 3.1. *Suppose that X is a Banach space such that at least one of X, X^* is complex uniformly convex or such that X is finite dimensional. Then X has the strong maximum property. In particular, this holds for $l^p(\mathbb{X})$.*

This means that we shall take (1.1) as our definition of $\mathrm{Sp}_\epsilon^p(A)$ with the l^2 operator norm replaced by its l^p counterpart. Some authors differ in requiring $< \epsilon$ or the closure of such a set but in light of Theorem 3.1, we see that the closure definition and ours agree in this context. In proving results, we will find the following theorem useful (see [74]). If B is a bounded operator on the Banach space X then B^* is the adjoint operator defined on X^* (with the convention of taking anti-linear functionals following Kato [55]). In our case, this means that if $1 \leq p < \infty$ then A^* is the matrix operator defined by the usual complex conjugate defined on $l^q(\mathbb{X})$ with $1/p + 1/q = 1$.

Theorem 3.2. *Let X be a Banach space with the strong maximum property and $A \in \mathcal{B}(X)$ then $\mathrm{Sp}_\epsilon^{\mathrm{X}}(A)$ (the $\epsilon-$pseudospectrum defined using the operator norm on $A \in \mathcal{B}(X)$) is the set of $z \in \mathbb{C}$ satisfying any of the following four equivalent definitions*

I. $\|R(z,A)\|^{-1} \leq \epsilon$,

II. $z \in \mathrm{Sp}(A+E)$ *for some* $E \in \mathcal{B}(X)$ *with* $\|E\| \leq \epsilon$,

III. $z \in \mathrm{Sp}(A)$ *or there exists* $x_n \in X$ *of norm* 1 *with* $\limsup_{n\to\infty} \|(A-zI)x_n\| \leq \epsilon$,

IV. There exists $x_n \in X$ *of norm* 1 *with* $\limsup_{n\to\infty} \|(A-zI)x_n\| \leq \epsilon$ *or there exists* $y_n \in X^*$ *of norm* 1 *with* $\limsup_{n\to\infty} \|(A^*-\bar{z}I)y_n\| \leq \epsilon$.

Following [68], we define the injection and surjection modulus respectively by

$$j_X(A) = \sup\{\tau \geq 0 : \|Ax\| \geq \tau\|x\| \text{ for all } x \in X\} = \inf\{\|Ax\| : \|x\| = 1\}$$
$$q_X(A) = \sup\{\tau \geq 0 : A(B_X) \supset \tau B_X\}.$$

We then have $\|A^{-1}\|^{-1} = \min\{j(A), q(A)\}$, $j_{X^*}(A^*) = q_X(A)$ and $q_{X^*}(A^*) = j_X(A)$. Furthermore, if A is invertible then $j_X(A) = q_X(A)$. We define the functions

$$\psi^p_A(z) := \min\{j_{l^p}(A-zI), q_{l^p}(A-zI)\},$$
$$\psi^p_{A^{per}_n}(z) := \min\{j_{l^p}(A^{per}_n - zI), q_{l^p}(A^{per}_n - zI)\},$$

and note that we can characterise the pseudospectrum as $\mathrm{Sp}^p_\epsilon(A) = \{z \in \mathbb{C} : \psi^p_A(z) \leq \epsilon\}$. Assume for the remainder of this section that $\underline{M}$ is permissible. Note that we have not yet shown that $\mathrm{Sp}^p_\epsilon(A)$ is constant over all $A \in \mathcal{A}(\underline{M})$, however this follows from Theorem 4.7 (and Corollary 4.9) of [8]. Upon letting $\epsilon \downarrow 0$, this also proves that the spectrum is constant on $\mathcal{A}(\underline{M})$. Recalling the generalised bandwidth m of $A \in \mathcal{A}(\underline{M})$ in (2.16), we have the following Proposition which extends Proposition 2.2 to $p \neq 2$.

Proposition 3.3. *Let* $p \in [1,\infty]$, $d \in \mathbb{N}$ *and* $A \in \mathcal{B}(l^p(\mathbb{X}))$ *be pseudoergodic with respect to* $\underline{M}$. *For* $n \geq m$ *and all* $z \in \mathbb{C}$ *we have* $\psi^p_{A^{per}_n}(z) \geq \psi^p_A(z)$.

Proof. Assume that $A \in \mathcal{A}(\underline{M})$ and $n \geq m$. If $z \in \mathrm{Sp}^p(A)$, then $\psi^p_A(z) = 0$ and we have nothing to prove so assume that $z \notin \mathrm{Sp}^p(A)$. This implies that

$$\psi^p_A(z) = j_{l^p}(A-zI) = q_{l^p}(A-zI).$$

Since A^{per}_n acts on a finite dimensional vector space, we also have that

$$\psi^p_{A^{per}_n}(z) = j_{l^p}(A^{per}_n - zI) = q_{l^p}(A^{per}_n - zI).$$

Hence we must prove that

$$j_{l^p}(A-zI) \leq j_{l^p}(A^{per}_n - zI). \tag{3.1}$$

We begin with the case that $p < \infty$. To see that (3.1) holds in this case, we argue as in Section 2.2 (with the added notational complexity of Section 2.4 if $|\mathcal{S}| > 1$). The only real changes are that in (2.11) which now becomes

$$\|(A^{per}_{n,N} - zI)x^N - y^N\| \leq 3^d\|A\| \left|\partial C^{\mathcal{S}}_N\right|^{\frac{1}{p}} = \mathcal{O}(N^{\frac{d-1}{p}}), \tag{3.2}$$

and x^N, y^N which now have norms $\left|C^{\mathcal{S}}_N\right|^{\frac{1}{p}}$ and $\left|C^{\mathcal{S}}_N\right|^{\frac{1}{p}}\|y\|$ respectively. We now have

$$\left|C^{\mathcal{S}}_N\right|^{\frac{1}{p}} = (2N+1)^{\frac{d}{p}}\,|\mathcal{S}|^{\frac{1}{p}}.$$

The same arguments then yields

$$j_{l^p}(A^{per}_n - zI) \leq j_{l^p}(A-zI) + 2\delta + \mathcal{O}(N^{\frac{1}{p}})$$

in place of (2.13) and (3.1) then follows using exactly the same arguments.

Next we show that (3.1) holds for $p = \infty$. For this we consider the *matrix* adjoint $B = (A-zI)^*$ as an operator on $l^1(\mathbb{X})$. Note that this is not the same as the operator adjoint (which acts on a much larger space). Similarly, we consider the *matrix* adjoint $B_n = (A^{per}_n - zI)^*$ as an operator on $l^1(C^{\mathcal{S}}_n)$. B is pseudoergodic and hence

$$j_{l^1}(B) \leq j_{l^1}(B_n).$$

(Note that the periodisation commutes with taking the matrix adjoint.) But we then must have that $q_{l^\infty}(B^*) = j_{l^1}(B)$ and $q_{l^\infty}(B^*_n) = j_{l^1}(B_n)$. Hence

$$j_{l^\infty}(A-zI) = q_{l^\infty}(A-zI) = q_{l^\infty}(B^*) \leq q_{l^\infty}(B^*_n) = q_{l^\infty}(A^{per}_n - zI) = j_{l^\infty}(A^{per}_n - zI),$$

which proves (3.1) for $p = \infty$. □

Theorem 3.4. *Let $p \in [1,\infty]$ and $A \in \mathcal{B}(l^p(\mathbb{X}))$ be pseudoergodic with respect to $\underline{M}$. Then $\psi^p_{A_n^{per}}(z)$ converges uniformly to $\psi^p_A(z)$ from above on compact subsets of $\mathbb{C}$. Hence $\lim_{n\to\infty} \mathrm{Sp}^p_\epsilon(A_n^{per}) = \mathrm{Sp}^p_\epsilon(A)$ in the Hausdorff metric and $\mathrm{Sp}^p_\epsilon(A_n^{per}) \subset \mathrm{Sp}^p_\epsilon(A)$, i.e. Theorem 1.3 and its extension to $l^p(\mathbb{X})$ hold.*

Proof. Suppose that we can prove pointwise convergence. Uniform convergence follows by a similar argument as Theorem 1.2 where we have uniform Lipschitz continuity from the definition of injection modulus (and hence the surjection modulus by considering the operator dual if $p < \infty$ or the matrix adoint if $p = \infty$). By Proposition 3.3, convergence is from above and hence $\mathrm{Sp}^p_\epsilon(A_n^{per}) \subset \mathrm{Sp}^p_\epsilon(A)$. Using Theorem 3.1 and a straightforward compactness argument, it is easy to see that $\mathrm{Sp}^p_\epsilon(A)$ is continuous in ϵ. The uniform convergence of $\psi^p_{A_n^{per}}(z)$ now implies $\lim_{n\to\infty} \mathrm{Sp}^p_\epsilon(A_n^{per}) = \mathrm{Sp}^p_\epsilon(A)$ as in the proof of Theorem 1.2.

Hence we are left with proving pointwise convergence. By Proposition 3.3, it is enough to show that

$$\limsup_{n\to\infty} \psi^p_{A_n^{per}}(z) \leq \psi^p_A(z). \tag{3.3}$$

The truncation argument in the proof of Lemma 2.1 works for $p \in (1,\infty)$ (p and its dual must be finite) and hence we only have to consider the $p \in \{1,\infty\}$ cases. Note that the truncation argument shows that

$$\limsup_{n\to\infty} j_{l^1}(A_n^{per} - zI) \leq j_{l^1}(A - zI). \tag{3.4}$$

Applying this to the matrix adjoints and using the same argument as in the proof of Proposition 3.3 shows that

$$\limsup_{n\to\infty} q_{l^\infty}(A_n^{per} - zI) \leq q_{l^\infty}(A - zI). \tag{3.5}$$

Suppose that we can also show that

$$\limsup_{n\to\infty} j_{l^\infty}(A_n^{per} - zI) \leq j_{l^\infty}(A - zI). \tag{3.6}$$

Then the duality $(l^1)^* = l^\infty$ yields (again by matrix adjoints) that

$$\limsup_{n\to\infty} q_{l^1}(A_n^{per} - zI) \leq q_{l^1}(A - zI), \tag{3.7}$$

which finishes the proof of (3.3) and hence of the theorem.

We are thus left with proving (3.6) so assume for the remainder of the proof that $p = \infty$. Given $\delta > 0$, there exists $x \in l^\infty(\mathbb{X})$ of norm 1 such that $\|(A - zI)x\| \leq j_{l^\infty}(A - zI) + \delta$. Fix any $N \in \mathbb{N}$ and define

$$(x_N)_{(i,a)} = x_{(i,a)} \max\big\{0, 1 - \frac{\|i\|_\infty}{N}\big\}, \quad i \in \mathbb{Z}^d, a \in \mathcal{S}.$$

It is clear that x_N has finite support and $P_{C^{\mathcal{S}}_n} x_N = x_N$ for large n. Now we use the fact that if $A_{(i,a),(j,b)} \neq 0$ then $\|i - j\|_\infty \leq m$ for some $m \in \mathbb{N}$. Consider the entry $((A_n^{per} - zI)x_N)_{(i,a)}$ where we assume that n is large so that this is equal to $((A - zI)x_N)_{(i,a)}$ for all (i,a). Since the operator is banded in the generalised sense, we must have

$$\big|\big((A - zI)x_N - \lambda_i(N)(A - zI)x\big)_{(i,a)}\big| \leq \frac{C(A,z)}{N}, \tag{3.8}$$

for some constant $C(A,z)$ independent of N and (i,a) where $\lambda_i(N)$ is the local factor

$$\lambda_i(N) = \max\big\{0, 1 - \frac{\|i\|_\infty}{N}\big\},$$

which converges to 1 as $N \to \infty$ for any i. Let y_N be defined by

$$(y_N)_{(i,a)} = \lambda_i(N)\big((A - zI)x\big)_{(i,a)}$$

then we have that

$$\limsup_{n\to\infty} j_{l^\infty}(A_n^{per} - zI) \leq \frac{\|(A - zI)x_N\|}{\|x_N\|} \leq \frac{\frac{C(A,z)}{N} + \|y_N\|}{\|x_N\|}.$$

But $\lim_{N\to\infty} \|x_N\| = \|x\| = 1$ and

$$\lim_{N\to\infty} \|y_N\| = \|(A - zI)x\| \leq j_{l^\infty}(A - zI) + \delta.$$

Hence

$$\limsup_{n\to\infty} j_{l^\infty}(A_n^{per} - zI) \leq j_{l^\infty}(A - zI) + \delta.$$

Since $\delta > 0$ was arbitrary this proves (3.6) and hence the Theorem. □

Remark 3.5 Bandedness was crucial in the above proof to obtain (3.8). One can in fact study $\|B\|$ and $\|B^{-1}\|$ for much more general operators B on l^∞ by looking at $\|B_0\|$ and $\|B_0^{-1}\|$ where B_0 is the restriction of B to the space of null sequences (see [47] Lemma 3.8), hence allowing similar arguments for B_0 as in the case of $p < \infty$. See also [69] for further discussion of these so-called $\mathcal{P}$-techniques.

4 Numerical Examples

We now seek to provide numerical examples of Theorems 1.2 and 1.3. We start with the Hilbert space case and then finish with an example for the $l^p(\mathbb{X})$ case. When dealing with the l^2 case, we will sometimes use p to denote the parameter of certain random variables (in agreement with the literature) but the notation will always be clear from context. Note that the inclusion $\mathrm{Sp}_\epsilon(A_n^{per}) \subset \mathrm{Sp}_\epsilon(A)$ implies that the eigenvalues of the matrix A_n^{per} lie in the spectrum of A.

4.1 The Variation of Distributions for NSA Anderson model

Here we revisit the operator defined by (1.2) in the introduction. In [43, 44] Goldsheid and Khoruzhenko presented a method of describing the limiting curves of the spectra of the periodic matrices using a potential dependent on the density of states. Throughout, parameter values are $g = 1/2$ and $M = \{\pm 1\}$ where V_n are i.i.d Bernoulli (with range $\{\pm 1\}$) of parameter p. We let A_n denote the matrix of size $2n+1$ obtained by truncating the operator H to acting on $\mathrm{span}\{e_{-n}, ..., e_n\}$ with periodic boundary conditions and the corresponding operator with non periodic (open) boundary conditions by B_n:

$$A_n = \begin{pmatrix} V_{-n} & e^{-g} & & & e^{g} \\ e^{g} & V_{-n+1} & e^{-g} & & \\ & e^{g} & V_{-n+2} & \ddots & \\ & & \ddots & \ddots & e^{-g} \\ e^{-g} & & & e^{g} & V_n \end{pmatrix}, B_n = \begin{pmatrix} V_{-n} & e^{-g} & & & \\ e^{g} & V_{-n+1} & e^{-g} & & \\ & e^{g} & V_{-n+2} & \ddots & \\ & & \ddots & \ddots & e^{-g} \\ & & & e^{g} & V_n \end{pmatrix}.$$

In the notation in Section 2, $A_n = A_n^{per}$ and $B_n = A_n^o$. B_n can be transformed to a self-adjoint operator, hence we can define, for real λ, the eigenvalue counting function

$$N_n(\lambda) = \frac{1}{n}\#\{\text{eigenvalues of } B_n \text{ in } (-\infty, \lambda)\}.$$

It is well known that, with probability 1, N_n converges to a continuous non-decreasing function N, known as the integrated density of states (dN is known as the density of states). We then define the potential

$$\Phi(z) = \int_{-\infty}^{\infty} \ln(|\lambda - z|) dN(\lambda),$$

and denote the support of dN by Σ. It is key to note that, whilst the support Σ does not change as we vary $p \in (0, 1)$, the distribution dN does! The limit distribution of the complex eigenvalues of A_n is supported on the level set $\{\Phi(z) = g\}$, a curve $\mathcal{L}$. More precisely, if we let dA_n denote the measure on the complex plane assigning mass $1/n$ to each eigenvalue of A_n then the following holds [44].

Theorem 4.1. *For the operator H defined above (where the $\{V_i\}$ are i.i.d.) the following holds almost surely.*

1. *For every compact set K in $\mathbb{C}\backslash\mathcal{L} \cup \mathbb{R}$, there exists n_1 such that for all $n > n_1$, there are no eigenvalues of A_n in K.*
2. *For any bounded continuous function f on $\mathbb{C}$,*

$$\lim_{n\to\infty} \int_{\mathbb{C}} f(z) dA_n(z) = \int_{\tilde{\Sigma}} f(\lambda) dN(\lambda) + \int_{\mathcal{L}} f(z(l))\rho(z(l)) dl,$$

where

$$\rho(z) = \frac{1}{2\pi}\left|\int_{-\infty}^{\infty} \frac{dN(\lambda)}{\lambda - z}\right|,$$

dl the arc-length element along $\mathcal{L}$ and

$$\tilde{\Sigma} = \{\lambda \in \mathrm{supp}(dN) : \Phi(\lambda + 0i) > g\}.$$

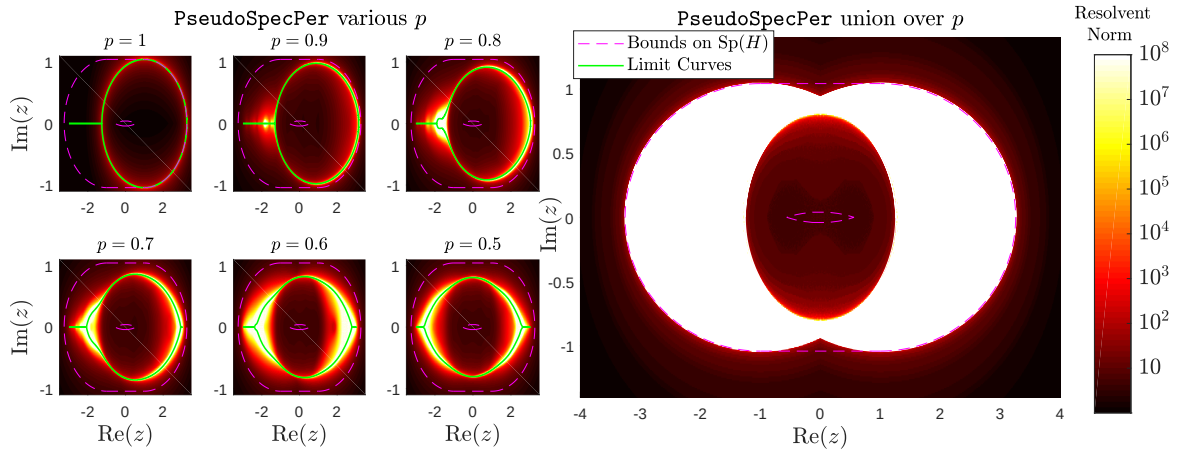

Figure 3: Left: Pseudospectra plots for `PseudoSpecPer` at $n = 10^5$ and various p. Limit curves of the finite section with periodic boundary conditions are also shown in green. Note that this explains the areas of fast vs. slow convergence of the algorithm. Right: Union of pseudospectra over various p.

We have computed the support of the limiting distribution by calculating the density of states numerically (for $n = 10^5$) and using the fact that we have convergence for the integral of any continuous function against this measure to accurately find the potential Φ. Figure 3 shows the output of `PseudoSpecPer` for $n = 10^5$ and various p along with the (approximate) limit curves $\mathcal{L} \cup \tilde{\Sigma}$ (which depend on p). The transformation $p \to 1 - p$ induces a reflection in the imaginary axis so we only show for $p \geq 0.5$. Note that these curves appear to fill up the apparent converged region in the right of Figure 3 which shows the union over p. Although `PseudoSpecPer` is guaranteed to converge for any selected p, it requires extremely large, and indeed computationally infeasible, n in order to gain convergence in some regions of the complex plane. This is simply due to the very small probability of the sequence of diagonal values needed to obtain convergence in the truncation window (infinite monkey theorem). Taking unions over different values of p causes regions near the limit curves to converge very fast and hence n can be taken much smaller. Exactly the same phenomenon occurs when the support of the diagonal potential is not discrete (for instance a uniform random variable). The set $E + M$ is contained in the spectrum, with E the ellipse $\{e^{g+i\theta} + e^{-g-i\theta} : \theta \in [0, 2\pi)\}$, and we have shown bounds on the spectrum (which lies between the two magenta curves obtained via a series argument and consideration of numerical ranges). Note in this case that the limit of the pure finite section eigenvalues is the interval $[-3, 3]$ which is neither contained in, nor contains the true spectrum.

Remark 4.2 Given this example, it is reasonable to ask whether taking the union of *spectra* of all possible periodic finite sections converges to the true spectrum. This does not hold in general. For instance, consider an operator A acting on $l^2(\mathbb{Z})$ and pseudoergodic with respect to $M_{-1} = \{0, 1\}$ in the notation of Definition 1.1. It can be shown (see [58] for a generalisation for bi-diagonal operators) that the spectrum is the closed unit disk. Periodic finite sections are of the form

$$\begin{pmatrix} 0 & \alpha_1 & & & \\ & 0 & \alpha_2 & & \\ & & 0 & \ddots & \\ & & & \ddots & \alpha_{n-1} \\ \alpha_n & & & & 0 \end{pmatrix}$$

with $\alpha_j \in \{0, 1\}$. These either have spectrum consisting of the n-th roots of unity if all the α_j are 1 or $\{0\}$ otherwise. Hence whilst the spectra of periodic finite sections lie in the true spectrum, we not recover all of the spectrum by taking their union and closure. We refer the reader to [45] for positive results and for conditions that ensure convergence to the spectrum.

4.2 A Quidiagonal Operator

Next we consider the quidiagonal operator acting on $l^2(\mathbb{Z})$ by

$$(Tx)_n = V_n x_{n-2} + W_n x_{n+1},$$

where V_n, W_n are i.i.d. random variables whose distribution has support $\{\pm 1\}$. We realise this by setting V_n and W_n to be independent Bernoulli random variables of parameters p and q respectively. The reason for selecting this operator is that it is not tridiagonal (not previously considered in the literature), hence it is in the more general setting of that considered by Theorem 1.2. Figure 4 shows the union of outputs of `PseudoSpecPer` over different p, q for $n = 10,000$ (so that the true matrix sizes are $20,001$). It is straightforward to show that the curve γ and $-\gamma$ are contained in the spectrum, where γ is the trifolium

$$\gamma(\theta) = \exp(\theta 2\pi i/6) + \exp(-\theta\pi i/6), \theta \in [0, 2\pi),$$

and these are shown in green. Note that if there were no randomness present, say $V_n = W_n = 1$ for all n, then the spectrum of T would be γ (the range of the symbol). The operator would be normal and hence the $\mathrm{Sp}_\epsilon(T)$ would consist of the spectrum together with points at most ϵ distance from the spectrum. The effect of adding randomness is most pronounced between the petals of $\gamma \cup -\gamma$. It is also straightforward to show that the pseudospectrum (and spectrum) possess many symmetries. In particular, they are invariant under complex conjugation and have rotational symmetry of order six. Note that whilst applying periodic boundary conditions ensures the pseudospectrum converges, the eigenvalue distribution seems to depend on the distribution of V_n, W_n in a non trivial manner. This is entirely analogous to the NSA Anderson model.

We also show the time taken for `PseudoSpecPer` to estimate the resolvent norm (for a single sample of the operators) on a 100×100 grid for both T and the NSA Anderson model H for $\epsilon \in \{10^{-k} : k = -8, -8.05, ..., 0\}$. We can use an interval bisection search routine to make this efficient. In order to make the algorithms as fast as possible we used an approximate minimal degree ordering of the matrices[4] to aid computing Cholesky decompositions which are the basis of our positive definiteness test. The times are shown in Figure 4, computed on a standard desktop computer with four cores. The algorithm is easily parallelisable - we simply split the searches amongst workers, and hence much faster speeds can be implemented if one has access to more cores and very little communication is needed between nodes. Note that as expected, the tridiagonal case was quicker, especially for large system sizes ($2n + 1$ in the above notation). Figure 4 also shows the time when using the method of uneven sections presented in [33]. For completeness, this is described in the Appendix and we label the algorithm `PseudoSpec`. We have not shown the outputs when using `PseudoSpec` for H and T since the difference with `PseudoSpecPer` is negligible (and cannot be detected by eye). The key difference is that `PseudoSpec` uses matrices $P_m A P_n$ with $m > n$ and does not apply boundary conditions. Hence in this case, it reduces to the computation of the smallest singular value of banded rectangular matrices. `PseudoSpec` is slightly faster than `PseudoSpecPer` as expected since the periodic matrices used in `PseudoSpecPer` are no longer banded. We will compare computation time and speed of convergence in two dimensions in the next example.

4.3 2D NSA Anderson Model

Our aim here is to demonstrate that, whilst using periodic boundary conditions can be slightly slower to implement than the algorithm presented in [33] for fixed n, it can perform faster convergence of pseudospectra as $n \to \infty$, particularly in higher dimensions. We will consider two cases of the NSA Anderson model in two dimensions given by

$$(Ax)_{m,n} = e^{-g} x_{m-1,n} + e^{g} x_{m+1,n} + e^{-h} x_{m,n-1} + e^{h} x_{m,n+1} + V_{m,n} x_{m,n}, \quad m, n \in \mathbb{Z}$$

where, as usual, $V_{m,n}$ are i.i.d. random variables. We found that varying the distribution of the potential V whilst keeping its support constant caused the algorithms to converge at different rates in different regions of the complex plane, in a completely analogous fashion to the one dimensional case discussed above. Hence we shall only consider one distribution in each of our cases but stress that the same conclusions hold for other distributions.

Our first example is the case of $g = h = 0.3$ and $V_{m,n} = \pm 3$ with equal probability $1/2$. One can show that the sum of ellipses $E_g + E_h$, with $E_\alpha = \{e^{\alpha + i\theta} + e^{-\alpha - i\theta} : \theta \in [0, 2\pi)\}$, is in the spectrum and similarly

[4] Obviously these are different for each algorithm. We shall use this ordering without further comment and note that it can be quickly computed from one test point using standard routines in `MATLAB`.

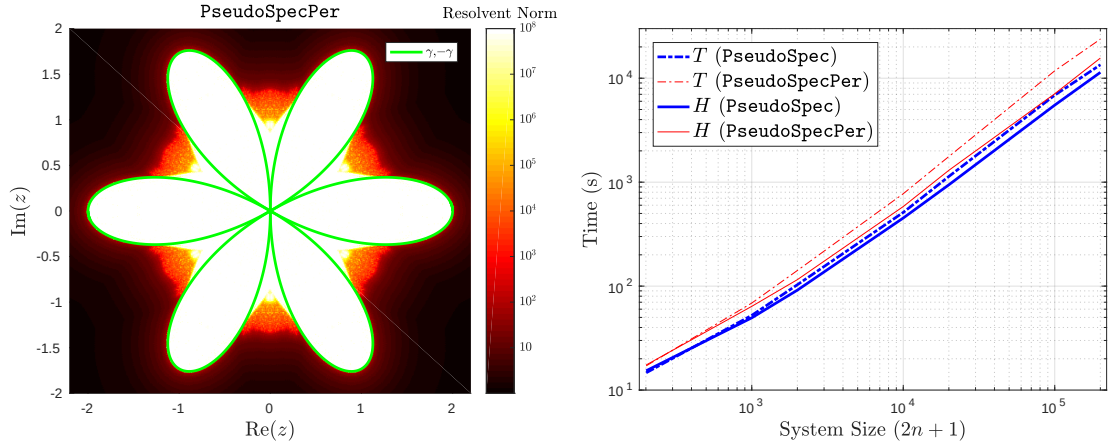

Figure 4: Left: Output of `PseudoSpecPer` for $n = 10^4$. Right: Comparison of times for both methods for the tridiagonal operator H and the quidiagonal operator T. Note that `PseudoSpec` is slightly faster.

bound the spectrum with convex hulls. We tested `PseudoSpec` and `PseudoSpecPer` on sites in a square of length L, corresponding to system sizes L^2. The result for $L = 200$ is shown in Figure 5. We clearly see that employing periodic boundary conditions gives a much better approximation of the resolvent norm. The faster convergence can be quantified by studying specific z. We choose the points $z_1 = 0, z_2 = 4\cosh(0.3)+3+0.1i$ and $z_1 = 4\sinh(0.3)i$. These points lie in the spectrum and for our choice of distribution we expect that convergence will be fastest for z_1 and slowest for z_3 (changing the distribution can reverse this). This is confirmed in Figure 5 were we have also shown the estimates of $\|R(z, A)\|^{-1}$ at these points over a mean of 100 runs for different L. Note also that periodic boundary conditions consistently approximates $\|R(z, A)\|^{-1}$ better, even in the regions of slow convergence.

Our second example considers the case when the support of the potential is not discrete. We let $g = \sinh^{-1}(0.1)$ and $h = 0$ with $V_{m,n}$ having uniform distribution in $[-1/5, 1/5]$. For these parameters, the spectrum is known to be $E_g + E_h$ [36]. Figure 6 shows the analogous plots for this case. We choose the test points $z_1 = 0, z_2 = 2(1 + \cosh(\sinh^{-1}(0.1))) + 0.2$ and $z_1 = 0.2i$. Exactly the same conclusions can be drawn. We also tested for complex potentials which produced similar results.

The fact that `PseudsoSpecPer` performs better can be understood as follows. `PesuoSpec` only uses one sample ($L \times L$ square) of the potential. However, as soon as we adopt periodic boundary conditions, we effectively gain a larger number of samples due to translations modulo $L \times L$. This larger sample isn't independent but we are able to use more information due to the larger number of possible couplings (non-zero diagonals) between sites. For instance, periodic boundary conditions take advantage of the fact that sites along the edge of the square can couple to those on the opposite edge. This effect becomes more increased in higher dimensions. In Figure 7 we have shown results over z_i but now after taking the minimum over 11 distributions, each sampled 20 times. Note that `PseudoSpecPer` generally produces an estimate several orders of magnitude better than `PseudoSpec`. Finally, we compare the time taken for both methods. As before, a 100×100 grid was chosen in each case with $\epsilon \in \{10^k : k = 8, 8.05, ..., 0\}$. The results are also shown in Figure 7. Again, there is no difference between these methods in terms of how the time scales with system size with both scaling less than quadratically with the system size, but `PseudoSpec` is slightly faster.

Remark 4.3 It must be said that the above explanation is purely heuristic and does not always hold. In the general $\mathbb{Z}^d$ case, one has for $p < \infty$ and $x \in l^p(C_n)$ that

$$\|(A_n^{per} - zI)x\|_p = \left\| \sum_{k \in C_1} P_{C_n} S_{n,-k}(A - zI)P_{C_n} x \right\|_p \leq \sum_{k \in C_1} \|P_{C_n} S_{n,-k}(A - zI)P_{C_n} x\|_p$$
$$\leq 3^{\frac{d}{q}} \Big(\sum_{k \in C_1} \|P_{C_n} S_{n,-k}(A - zI)P_{C_n} x\|_p^p \Big)^{\frac{1}{p}} = 3^{\frac{d}{q}} \|(A - zI)P_{C_n} x\|_p ,$$

where we have used Hölder's inequlity and $1/p + 1/q = 1$ (this extends also to $p = \infty$). This gives $j_{l^1}(A_n^{per} - zI) \leq j_{l^1}((A - zI)P_{C_n})$. However, this can be false for $p > 1$. For instance consider the shift operator $(Ax)_j = x_{j+1}$ (on $l^p(\mathbb{Z})$) and $z = 1 \in \mathrm{Sp}(A)$. We can of course consider $n \times n$ periodic approximations (as

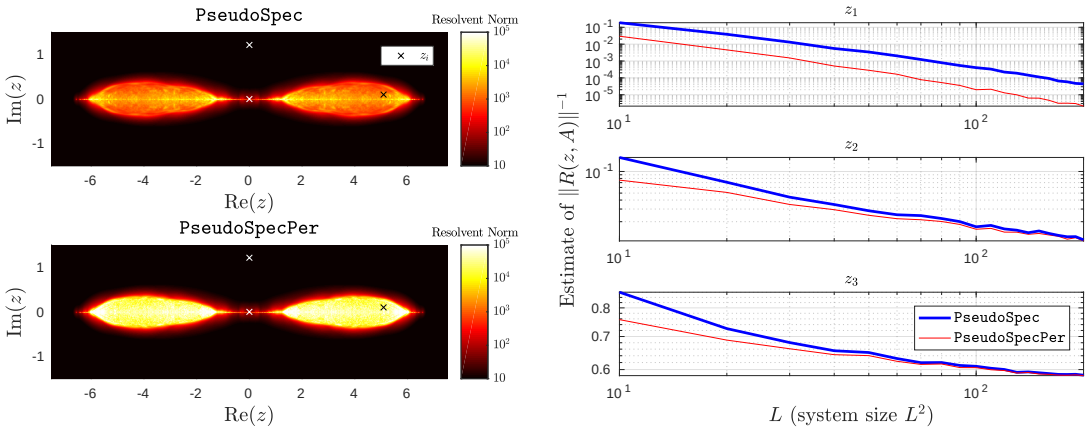

Figure 5: Left: Case one, output for $L = 200$. Note `PseudoSpecPer` performs better. The test points are shown as blak/white crosses. Right: The estimate of the resolvent norm for different system sizes at the different points z_i. The point z_3 converges very slowly for this distribution.

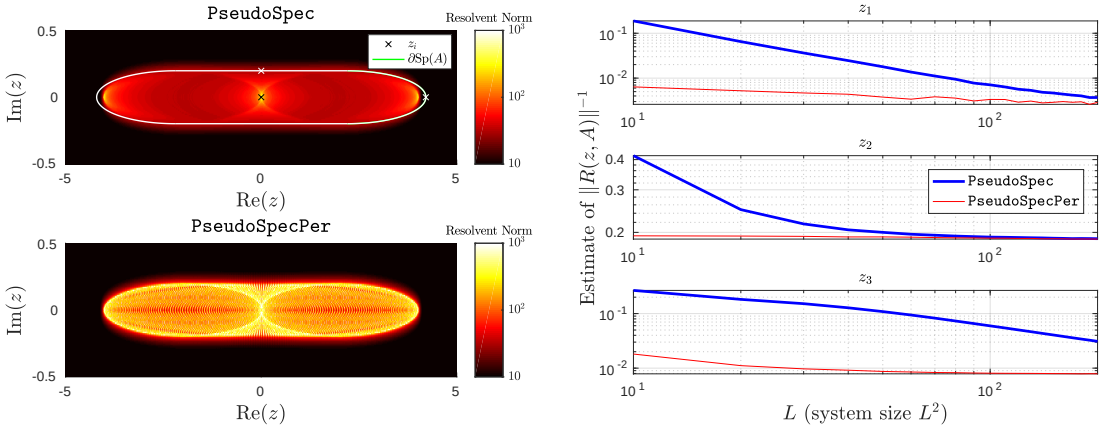

Figure 6: Analogous figure for the second case considered. We have also shown the boundary of the spectrum in the left plot.

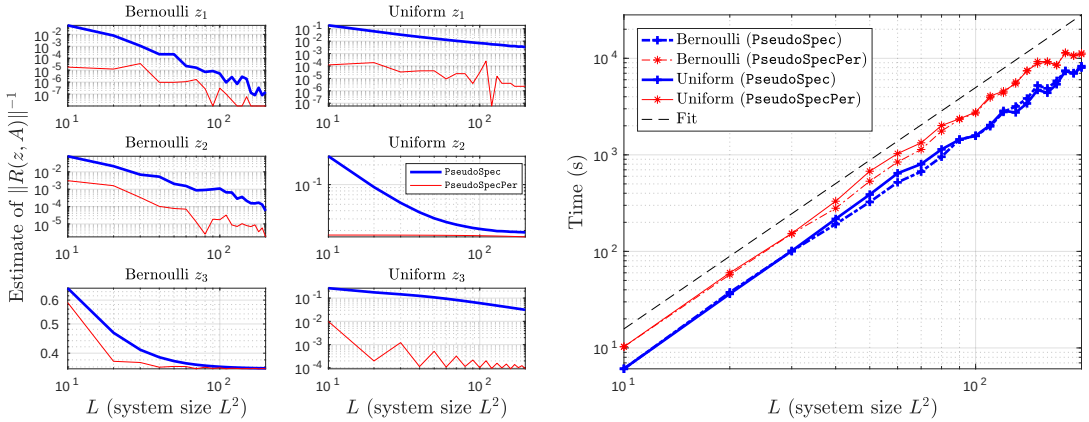

Figure 7: Left: Minimum over 11 distributions with 20 samples each. Note that in some cases periodic boundary conditions ensure an estimate several orders of magnitude better. Right: Time taken in both cases for `PseudoSpec` and `PseudoSpecPer`. The time appears to grow roughly as $\sim L^{2.5}$ (shown), less than quadratically with the system size.

	$n=3$	$n=5$	$n=11$	$n=51$	$n=101$	$n=501$
$j_{l^2}(A_n^{per}-I)$	1.00	6.18×10^{-1}	2.85×10^{-1}	6.16×10^{-2}	3.11×10^{-2}	6.27×10^{-3}
$j_{l^2}((A-zI)P_n)$	7.65×10^{-1}	5.18×10^{-1}	2.61×10^{-1}	6.04×10^{-2}	3.08×10^{-2}	6.26×10^{-3}

Table 1: Comparisons of periodic boundary conditions and uneven sections for the shift operator. Both $j_{l^2}(A_n^{per}-I)$ and $j_{l^2}((A-zI)P_n)$ converge to zero as $n\to\infty$. P_n denotes the projection onto the n-dimensional subspace.

opposed to $(2n+1)\times(2n+1)$) for which $j_{l^p}(A_n^{per}-I)=0$ if n is even. Table 1 shows some results for $p=2$. Whilst $j_{l^2}(A_n^{per}-I)\geq j_{l^2}((A-zI)P_n)$ for odd n, it appears their ratio converges to 1 as $n\to\infty$.

4.4 Coupled Anderson Model (example with $|\mathcal{S}|>1$)

Our final example for the Hilbert space case demonstrates the algorithm when $|\mathcal{S}|>1$. We consider two coupled one dimensional NSA Anderson models, where the couplings between different sites are now random variables. We match up each site of both models at each lattice point, giving rise to the finite matrices

$$A_n^{per}=\begin{pmatrix} V_1 & a_1 & e^{-g} & 0 & & & e^{g} & 0\\ b_1 & W_1 & 0 & e^{-h} & & & 0 & e^{h}\\ e^{g} & 0 & V_2 & a_2 & e^{-g} & 0 & & \\ 0 & e^{h} & b_2 & W_2 & 0 & e^{-h} & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots\\ e^{-g} & 0 & & & e^{g} & 0 & V_n & a_n\\ 0 & e^{-h} & & & 0 & e^{h} & b_n & W_n \end{pmatrix}.$$

V_i and W_i describe the potentials in each model which have parameters g and h respectively. The a_i and b_i describe the interactions between the models at each point of the lattice $\mathbb{Z}$. We take the example of $g=1$, $h=1/10$ with V_i and W_i taking values in $\{\pm1/2\}$ (Bernoulli independent random variables with parameter q). For the interactions we set $b_i=1$ and a_i taking values in $\{\pm1\}$ (Bernoulli independent random variables with parameter p). Figure 8 shows the output over $p\in\{0,0.05,0.1,...,1\}, q\in\{0,1/10,...,1\}$ for $n=10^4$. Note that this example also demonstrates how we can naturally treat *periodically pseudoergodic* operators.

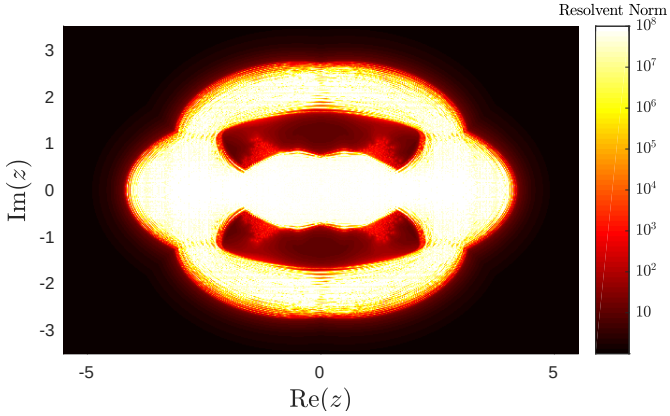

Figure 8: Estimate of resolvent norm for the coupled Anderson models. We have taken the maximum over outputs with parameters $p \in \{0, 0.05, 0.1, ..., 1\}, q \in \{0, 1/10, ..., 1\}$.

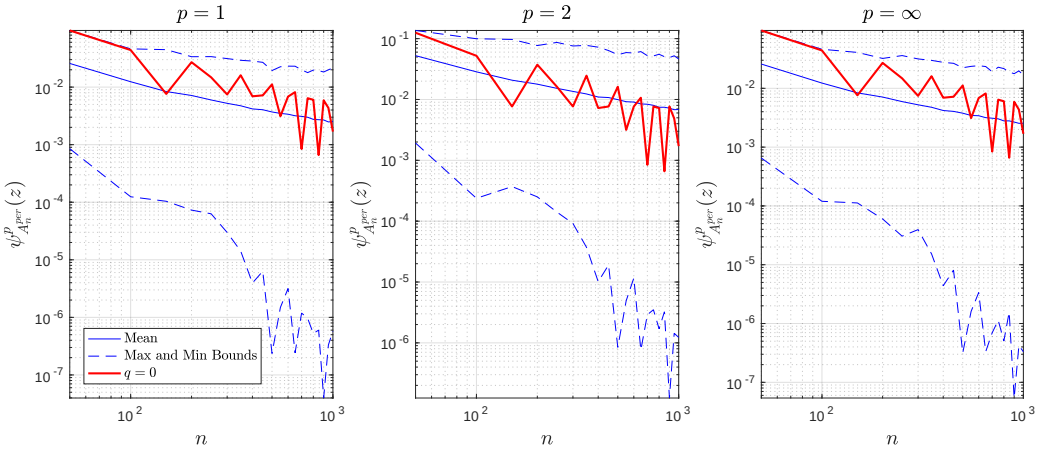

Figure 9: Convergence of $\psi^p_{A_n^{per}}(z)$, where z is the nearest point on the curve $T(\theta)$ to the origin. We have shown $q = 0$ (the Laurent operator), as well as the maximum, minimum and mean over 1000 Bernoulli parameters when we alter the first super-diagonal to have entries in $\{\pm 1\}$. This last case corresponds to the entries being independent and having value -1 with probability $1 - q$ for $q = 0, 1/999, 2/999, ..., 1$.

4.5 Example for $l^p(\mathbb{X})$

As an example for l^p spaces we consider an alteration of the Laurent operator with symbol $T(\theta) = -(\exp(-i\theta) + i\exp(i\theta) + \exp(2i\theta))$. For this operator, the spectrum is simply the curve traversed by the symbol and it is straightforward to show that $\psi^1_{A_n^{per}}(z) = \psi^\infty_{A_n^{per}}(z)$ ($A_n^{per} - zI$ are circulant so have circulant inverses with the same l^1 and l^∞ operator norms). We alter the first super-diagonal to have entries in $\{\pm 1\}$ independently with the value 1 being chosen with probability q. Figure 9 shows the convergence of $\psi^p_{A_n^{per}}(z)$, where z is the point on the symbol curve closest to the origin (so that the limit is zero), for a range of 1000 values of q. Note that this demonstrates we do not always obtain monotone convergence (in contrast to the method of uneven sections). Figure 10 shows typical pseudospectral plots for one realisation.

As mentioned in the introduction, one cannot use singular values of matrices to characterise the pseudospectrum and it appears the only known method of computing $p-$pseudospectra is via directly estimating $\|R(z, A)\|_p$ over some fine grid. The computation of matrix norms of finite square matrices for $p \neq 1, 2$ or ∞ is NP-hard [52] so this currently seems intractable for large matrices for any p other than these.

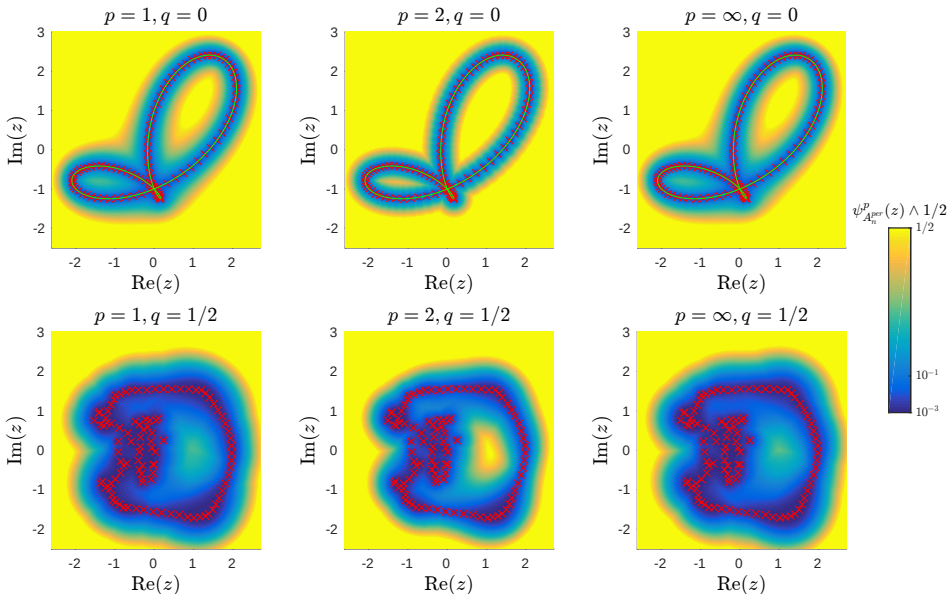

Figure 10: Typical pseudospectral plots for one realisation at $n = 100$. Eigenvalues of the periodic matrices are shown in red.

5 Conclusion

There is now a vast literature on spectral studies of random NSA operators. Most of these fall into Definition 1.1 (and more generally that in Section 2.4) of pseudoergodicity. We have shown that under this general condition, finite sections with periodic boundary conditions are very well behaved in terms of pseudospectra. Given that the spectrum need not be recovered in the infinite system size limit, it is perhaps remarkable that the pseudospectrum is. This example demonstrates why pseudospectra may be the correct spectral property to study for non normal operators, a viewpoint backed up by many important examples [74].

As well as our main theorem, we have demonstrated that periodic boundary conditions provide a useful numerical tool, particularly in higher dimensions. For many pseudoergodic operators, it outperforms the algorithm `PseudoSpec` (method of uneven sections) proposed in [33], though it is slightly slower. Future work will aim at studying this trade-off and exactly when periodic boundary conditions outperform `PseudoSpec`.[5] Finally, the simple proof method used for our main theorem has been adapted to the general l^p setting (where `PseudoSpec` is not applicable) using a different argument for the $p = \infty$ case. The results presented here hold for more general operators such as those which are periodically pseudoergodic. Future work will aim to extend this type of result to semi-infinite matrices (stochastic Toeplitz matrices).

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Appendix: Method of Uneven Sections

Here we briefly recall the method of uneven sections described in [33] to compute pseudospectra, and we denote the algorithm by `PseudoSpec`. By an effective enumeration of our basis sites, we can realise our operators as matrices acting on $l^2(\mathbb{N})$ with respect to the canonical basis $\{e_1, e_2, ...\}$. All the operators studied in this paper then have the property that there is a function $f : \mathbb{N} \to \mathbb{N}$ (which we can explicitly evaluate) such that

$$A_{i,j} = A_{j,i} = 0,$$

[5] Although `PseudoSpec` does converge in many cases where periodic boundary conditions do not.

if $i > f(j)$. This simply captures the finite range interaction of our operators (and the adjoint). For instance, if our original operator is tridiagonal when viewed in the original basis of $l^2(\mathbb{Z})$, then with the enumeration $\{e_0, e_{-1}, e_1, e_{-2}, e_2, ...\}$ giving $l^2(\mathbb{Z}) \cong l^2(\mathbb{N})$ we can take $f(n) = n + 2$. The algorithm can then be written as:

Function `PseudoSpec` (A, n, ϵ, f)
  Input : n, $A \in \mathcal{B}(l^2(\mathbb{N}))$, $\epsilon > 0$, $f : \mathbb{N} \to \mathbb{N}$
  Output: $\Gamma \subset \mathbb{C}$, an approximation to $\mathrm{Sp}_\epsilon(A)$
  $G = \frac{1}{n}(\mathbb{Z} + i\mathbb{Z}) \cap B_n(0)$
  for $z \in G$ **do**
    $B = (A - zI)(1 : f(n), 1 : n)$
    $C = (A - zI)^*(1 : f(n), 1 : n)$
    $S = B^*B$
    $T = C^*C$
    $p = \mathtt{IsPosDef}(S - \epsilon^2)$
    $q = \mathtt{IsPosDef}(T - \epsilon^2)$
    $\nu(z) = \min(p, q)$
  end
  $\Gamma = \bigcup\{z \in G \,|\nu(z) = 0\}$
end

The basic idea is that the function $\psi_{A,n}(z) := \min\{\sigma_1((A - zI)P_n), \sigma_1((A^* - zI)P_n)\}$ is non-increasing in n and converges uniformly to $\|R(z, A)\|^{-1}$ from above on compact subsets of $\mathbb{C}$.

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