Five-dimensional Gravity and the Weak Gravity Conjecture

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GEORGIOS ANTONIOU

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Advisor: Prof. Tony Gherghetta

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Abstract

The beginning of this thesis provides a brief guide to the notation we are going to use. After that, we present the Randall-Sundrum model and we outline the way it solves the hierarchy problem. To analyze the solutions, the Lagrangian is perturbed up to second order. We then examine the possibility for a massive graviton, in the context of the Randall-Sundrum models. In particular, we examine the existence of the scalar modes of the metric decomposition. We present the de-Sitter, brane-world solutions corresponding to a dS five dimensional space. Moreover, we discuss the swampland and focus on the Weak Gravity conjecture as well as on the AdS instability conjecture which follows from the former. We give the motivation and arguments supporting the Weak Gravity conjecture, derived from black hole physics. Then, we review the application of the AdS instability conjecture on Standard Model compactifications, and we retrieve recent results that support Dirac neutrinos and the normal hierarchy of the neutrino masses. We proceed by applying the same conjecture to the five-dimensional, brane-world models. For the purposes of the present thesis, we limit the analysis to relatively simple cases, involving only a small number of particles in the five-dimensional bulk. We examine the constraints set on the masses of the fermionic/bosonic degrees of freedom, as well as on the five-dimensional cosmological constant, in order to avoid AdS minima. Finally, we discuss the Scalar Weak Gravity Conjecture and a recent modification of it.

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Introduction

The physics landscape has changed significantly over the last century. Both General Relativity and Quantum Physics revolutionized the way we interpret nature. General Relativity and the Standard Model of particle physics, are both, two very successful theories, describing astrophysical processes, taking place at astronomical scales and fundamental scale interactions.

A very interesting concept arising in the context of General Relativity is that of black holes. Schwarzschild's solution of Einstein's equations provided the first example of such an object. An intereresting feature of black hole solutions is the existence of singularities, not associated with the coordinate system, but rather with the spacetime itself. Such singularities are of course a non-desirable characteristic of physical theories and indicate that General Relativity does not provide the full picture, a statement that is also supported by the fact that General Relativity is non-renormalizable. On the other hand the Standard Model of particle physics has been very accurate in describing interactions of fundamental particles up to the TeV scale, and it was basically completed with the discovery of the Higgs boson a few years ago. The Standard Model no matter how successful it has proven to be, fails to integrate gravity into the picture.

Among the many unsolved mysteries found in the high energy physics area, dark energy and dark matter are particularly important to our understanding of nature. Despite filling most of our universe, we know very little about their origin and properties. We know that dark matter interacts through gravity and that dark energy, which we describe through the term of the cosmological constant is crucial in order to retrieve the cosmological results that we have from observations. With the discovery of the accelerating expansion of our universe among others, the quest for the true nature of the cosmological constant has never been more exciting. Another crucial issue that has not yet been explained to a satisfactory -experimentally consistent- degree, is that of the *Hierarchy*. Why is the scale experienced in particle interactions so much lower than that at which gravitational effects start becoming important, namely why $M_p \sim 10^{19}$ GeV is so large when, $H_0 \sim 1$ TeV is so much smaller.

To answer that last question models have been developed during the last twenty years, which employ extra spatial dimensions with the goal of finding a naturally occurring hierarchy. One of those models is the ADD model [1, 2], proposed by Arkani-Hamed, Dimopoulos and Dvali. In that model the electroweak scale is considered to be the only fundamental scale. This should also include gravitational effects, so in order to achieve that, one consideres a number n of extra spatial dimensions. Then the gravitational scale M_p that we experience is just the effective value of $M_{p(4+n)}^{2+n} \sim m_{EW} \sim 1\,\text{TeV}$. The effective value of the Planck scale is associated with the $(4+n)$ dimensional one through the extra dimensions scale, i.e. $M_p^2 \sim R^n M_{p(4+n)}^{2+n}$. By requiring that R is smaller than the distances that could be experimentally tested at the time, it was found that $n \geq 2$. In this thesis however, we are not going to examine that model but we will focus on the Randall-Sundrum models which make use of an extra spatial dimension, that is compactified (RS1) or extends to infinity (RS2).

String theory is a candidate for a theory of quantum gravity, that will manage to unify all the fundamental interactions. The fact remains however that string theory possibilities are vast and complicated. Actually, almost any theory one can come up with in order to describe physics in the infrared (IR), is possible to arise in string theory. At the same time, string theory cannot be tested experimentally due to the high energy scale that is required in order to witness it unfold. In [12] it was argued for the first time, that not all effectively consistent theories, that arise from string theory are compatible with a quantum theory of gravity. These theories belong in the *swampland* which is defined as follows [12]:

*The semi-classical consistent effective field theories which are actually inconsistent are called the **swampland**.*

As it turns out, the swampland is much more vast than the string landscape where all the truly consistent effective field theories belong. The swampland arguments stem among others from finiteness criteria like the finiteness of the volume of scalar fields and of the rank of the gauge fields. There are many conjectures, strongly tied to gravity, that are used to separate between the consistent and the inconsistent effective field theories. To begin with, the global symmetries conjecture says that quantum gravity should not have any global symmetries. In this thesis we will focus on the Weak Gravity conjecture which states that gravity should be the weakest force experienced. Originating form the Weak Gravity conjecture the AdS Instability conjecture says that in non-supersymmetric theories, AdS space must exhibit instabilities. Additionally, the distance conjecture states that in a moduli space parametrized by the expectation values of some fields ϕ, starting from any point of the space, there exists another point in space so that the geodesic distance between the two points is infinite.

Physics	**Scale (GeV)**
$\Lambda_4^{1/4}$	10^{-12}
Neutrinos	10^{-11}
Electron	10^{-3}
Electroweak	10^{3}
Supersymmetry	10^{4}
String Theory	10^{18}

Table 1: Energy scales in physics.

1 Basics

In this first section we will include most of the differential geometry terms that we are going to use throughout this thesis. Since the models that we are examining are in many cases five dimensional, we use capital letters for the five-dimensional parts, lower case letters for the four-dimensional parts and the letters i or j when we refer to the three-dimensional part of the theory.

1.1 Notation in five dimensions

1. The signature for the metric elements we are going to use is $(-++++)$. The Minkowskian metric in five dimensions is $\eta_{MN} = \text{diag}(-1, 1, 1, 1, 1)$.

2. The line element is defined as

$$ds^2 = g_{MN}\, dx^M dx^N,$$

where $M, N = 0, 1, 2, 3, 4$ and G_{MN} is the symmetric, metric tensor satisfying $g_{MN} = g_{NM}$.

3. We define the Christoffel Symbols as follows:

$$\Gamma^C_{AB} = \frac{1}{2} g^{CD} (g_{AD,B} + g_{DB,A} - g_{AB,D}). \tag{1}$$

Notice that the Christoffel symbols are symmetric under the exchange of the two lower indices, i.e. $\Gamma^C_{AB} = \Gamma^C_{BA}$.

4. The Riemann tensor is:

$$R^D{}_{ACB} = \Gamma^D_{AB,C} + \Gamma^D_{CP}\Gamma^P_{BA} - (C \leftrightarrow B). \tag{2}$$

The Riemann tensor R_{DACB}, is antisymmetric under the exchange of the first two or the last two indices and symmetric under the exchange of the two outter or inner indices, i.e. $R_{DACB} = -R_{ADCB} = -R_{DABC} = R_{BACD} = R_{DCAB}$. It also satisfies the identity $R_{ABCD} + R_{ADBC} + R_{ACDB} = 0$.

5. The Ricci tensor is given by:

$$R_{AB} = R^C_{ACB} = \Gamma^C_{AB,C} + \Gamma^C_{CP}\Gamma^P_{BA} - \Gamma^C_{AC,B} - \Gamma^C_{BP}\Gamma^P_{CA}. \tag{3}$$

6. The Ricci scalar is given by:

$$R = g^{AB} R_{AB}, \tag{4}$$

and it is symmetric under exchange of its indices, i.e. $R_{AB} = R_{BA}$.

7. We define the covariant derivative according to the following expression:

$$\nabla_R A^{MN\ldots}_{PS\ldots} = \partial_R A^{MN\ldots}_{PS\ldots} + \Gamma^M_{RK} A^{KN\ldots}_{PS\ldots} + \Gamma^N_{RK} A^{MK\ldots}_{PS\ldots} - \Gamma^K_{RP} A^{MN\ldots}_{KS\ldots} - \Gamma^K_{RS} A^{MN\ldots}_{PK\ldots} \,. \tag{5}$$

1.2 Einstein equations

In order to derive the Einstein equations in the general case, we start from the Einstein-Hilbert action in five dimensions:

$$S = \int \left[\frac{R}{2\kappa_{(5)}} - 2\Lambda_5 + \mathcal{L}_M \right] \sqrt{-g}\, d^5x \,, \tag{6}$$

where

- R is the Ricci scalar
- Λ_5 is the cosmological constant in five dimensions.
- $\mathcal{L}_M$ is the Lagrangian describing the matter in our theory.
- G is the determinant of the metric tensor.

We then minimize the action with respect to metric variations

$$\frac{\delta S}{\delta g^{MN}} = 0.$$

If we take into account that:

$$\delta R^P_{MPN} = \nabla_P(\delta\Gamma^P_{NM}) - \nabla_N(\delta\Gamma^P_{PP}) \,,$$

$$R = g^{MN} R_{MN} \Rightarrow \delta R = R_{MN}\delta g^{MN} + g^{MN}\delta R_{MN} \,,$$

$$\nabla_P g^{MN} = 0 \,,$$

$$\frac{1}{\sqrt{-g}} \frac{\delta\sqrt{-G}}{\delta g^{MN}} = -\frac{1}{2} g_{MN} \,.$$

If we define the energy-momentum tensor to be

$$T_{MN} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g_{MN}}, \tag{7}$$

then the Einstein equations read:

$$G_{MN} - 2\, g_{MN}\Lambda_5 = k_{(5)} T_{MN}. \tag{8}$$

2 Basics of the Randall-Sundrum Model

The Randall-Sundrum model managed to solve the hierarchy problem, by introducing a spatial extra dimension compactified and extending between two 3-branes, once of which corresponds to the TeV scale universe. Later the model was modified so that the extra dimension extends to infinity and is not bound by the TeV brane.

2.1 Compactified extra dimension

Let's start by writing down the Einstein-Hilbert action for d-flat compact extra dimensions:

$$S_{EH} = M^{d+2} \int d^d y \, d^4 x \sqrt{-g_{(d+4)}} R_{(d+4)} \tag{9}$$

where:

$$M^{d+2} = \frac{1}{2k_{(d+4)}}$$

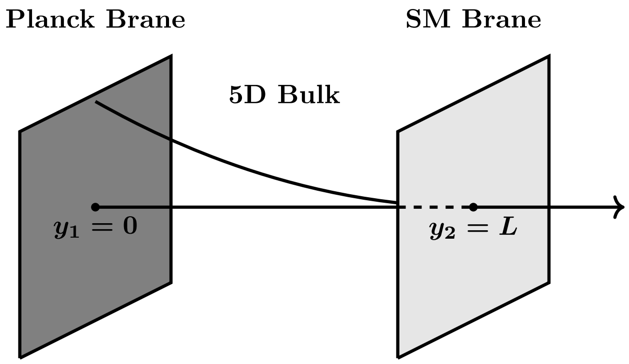

Figure 1: Depiction of the 5D Bulk in between of the Planck and the TeV branes.

We consider an extra space dimension y taking values between $y = -L$ and $y = +L$. This extra dimension is compactified on an S^1 circle possessing a $\mathbb{Z}_2$ symmetry, where $\mathbb{Z}_2$ corresponds to the multiplicative group $\{-1, 1\}$ and thus identifying points $\{x^\mu, y\}$ with the ones $\{x^\mu, -y\}$. In other words we will work in a $S^1/\mathbb{Z}^2$ orbifold. The factor $M^{(d+2)}$ ensures that the action is dimensionless. Let's actually make a quick check of the dimensions in eq. (9). Taking a look in the expressions defining the Ricci scalar we see that:

$$[R] \sim [g^{MN}] \cdot [R_{MN}] \sim 1 \cdot [\Gamma^P_{MN}]^2 \sim [g^{MN}]^2 \cdot [\partial_M]^2 \cdot [g_{MN}]^2 \sim 1 \cdot [L]^{-2} \cdot 1 = [L]^{-2}$$

Of course $[g] \sim 1$, in cartesian coordinates, because it is the determinant of the matrix (g_{MN}) and:

$$[d^d y \, d^4 x] \sim [L]^{d+3} [T]$$

which means that

$$[M]^{d+2} \sim [m] \cdot [L]^{-(d-1)} \cdot [T]^{-2}$$

Since $[S] \sim [E] \cdot [T] = [m] \cdot [L]^2 \cdot [T]^{-1}$. Effectively, in 4-D we know that:

$$k_{(4)} = \frac{8\pi G}{c^4} \Rightarrow M_{(4)}^2 = \frac{c^4}{16\pi G} \Rightarrow [M_{(4)}]^2 \sim [L]^4 \cdot [T]^{-4} \cdot [L]^{-3} \cdot [m] \cdot [T]^2 = [m] \cdot [L] \cdot [T]^{-2}$$

So we see that for $d = 0$ we get the desired 4D result.

$$S = \int d^4\, x \int_{-L}^{+L} dy \sqrt{-g} \left(M_{(5)}^3 R_{(5)} - 2\Lambda_{(5)} \right) - \sum_{i=1,2} \int d^4 x \sqrt{-\gamma_i}\, \lambda_i. \tag{10}$$

Regarding the extra dimension y, we are imposing:

- *periodicity*: $y \to y + 2L$
- *symmetry*: $y \to -y$
- *tension*: The constants λ_i determine the tension on the 3-branes 1 (Planck) and 2 (Standard Model).
- *metric determinants*: $\gamma_i = g(x^\mu, y = y_i)$

3-brane (1) is located at $y = 0$, while 3-brane (2) is located at $y = L$.

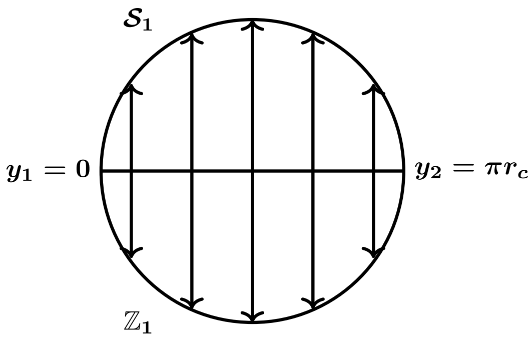

Figure 2: Depiction of the orbifold.

2.1.1 Randall-Sundrum Metric

The four-dimensional part of the five-dimensional metric should be flat and static. A way to achieve that is if we impose the following 5D metric:

$$ds^2 = F(y)\eta_{\mu\nu}dx^\mu dx^\nu + dy^2,\ F(y) = e^{-2\sigma(y)},$$

where $\mu, \nu = 0, 1, 2, 3$. The warp factor $F(y)$ becomes a constant if we fix the fifth dimension, so with a rescaling, we end up with the usual Minkowski spacetime if $y = constant$. We can get this in a slightly different form:

$$ds^2 = g_{MN} dx^M dx^N, \tag{11}$$

where

$$g_{MN} = e^{-2\sigma(y)} \eta_{\mu\nu} + \delta_M^5 \delta_N^5 \quad , \quad g^{MN} = e^{2\sigma(y)} \eta^{\mu\nu} + \delta_5^M \delta_5^N. \tag{12}$$

For this metric the Einstein Tensor components are:

$$G_{\mu\nu} = -\left(3\sigma''(y) - 6\sigma'(y)^2\right) e^{-2\sigma(y)} \eta_{\mu\nu} \quad , \quad G_{44} = 6\sigma'(y)^2.$$

Now we need to find the energy-momentum tensor corresponding to this case. In our case the S_M in the 5D bulk is:

$$S_M = \int d^5x \sqrt{-g} \, (-2\Lambda_{(5)} - \lambda_1 \delta(y) - \lambda_2 \delta(y - L)).$$

The energy momentum tensor is given by:

$$T_{MN} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{MN}} = -2 \, g_{MN} \Lambda_{(5)} - \eta_{\mu\nu} (\lambda_1 \delta(y) + \lambda_2 \delta(y - L)) \delta_M^\mu \delta_N^\nu.$$

We now see, that the (44) component of the Einstein equations gives:

$$G_{44} = \frac{1}{2M_{(5)}^3} T_{44} \Rightarrow \sigma'(y)^2 = -\frac{\Lambda_{(5)}}{6M_{(5)}^3} > 0.$$

That means that the 5D cosmological constant is negative resulting in a 5D anti-de Sitter spacetime (AdS_5). So,

$$\Lambda_{(5)} < 0 \Rightarrow \sigma'(y) = \pm k \Rightarrow \sigma(y) = k|y|,$$

where $k = \sqrt{-\Lambda_{(5)}/6M_{(5)}^3}$. In the expression above the $y \to -y$ invariance has been considered. We now need to consider the rest of the field equations. Taking into account the above relation for σ we find from all the other equations that:

$$3\sigma'' = -\frac{1}{2M_{(5)}^3} (\lambda_1 \delta(y) + \lambda_2 \delta(y - L)).$$

So we need to evaluate the second derivative of σ. That is,

$$\sigma' = k(\Theta(y) - \Theta(-y)) \Rightarrow \sigma'' = 2k(\delta(y) - \delta(y - L)).$$

In order for the equations to be satisfied we demand that:

$$\lambda_1 = -\lambda_2 = 12kM_{(5)}^3. \tag{13}$$

2.1.2 Hierarchy

In this subsection we will see how physical "lengths" in 4D, in the second brane depend on the ones in the first brane. To do that we will consider two different actions, one describing gravity and the other one describing particle masses. Let's find the effective 4-dimensional gravity action, starting from the 5D one:

$$S = \int d^5x \sqrt{-g^{(5)}} M_{(5)}^3 R_{(5)}.$$

We can write the 5D metric determinant as:

$$g_{MN}^{(5)} = e^{-2\sigma(y)} g_{\mu\nu}^{(4)} \delta_N^\mu \delta_N^\nu + \delta_M^5 \delta_N^5 \Rightarrow g^{(5)} = -8e^{\sigma(y)} g^{(4)}. \tag{14}$$

Now, we can write:

$$S = M_{(5)}^3 \int d^5x \left(e^{-4k|y|} \sqrt{-g^{(4)}} \right) R_{(5)}.$$

So, we need to think how $R_{(5)}$ is related to $R_{(4)}$. In the effective action we are interested in the term involving $R_{(4)}$, which is the 4D part of the Ricci scalar constructed only by the $g_{\mu\nu}^{(4)}$ part of the metric. In our model, $g_{\mu\nu}^{(4)} = \eta_{\mu\nu}$, so we will consider small perturbations around the 4D part of me metric:

$$\tilde{g}_{MN}^{(5)} = \overbrace{e^{-2k|y|} \underbrace{\left(g_{\mu\nu}^{(4)} + h_{\mu\nu} \right)}_{\text{Creates } R_{(4)}} \delta_M^\mu \delta_N^\nu}^{\text{4D Part}} + \delta_M^5 \delta_N^5, \quad |h| \ll 1. \tag{15}$$

In order to see how these translates to $R_{(4)}$ we need to see how a few things change under small variations, first. To do that, let's rewrite the perturbed metric given by expression (15), in a slightly different way:

$$\tilde{g}_{MN}^{(5)} = g_{MN}^{(5)} + \tilde{h}_{MN}, \; \tilde{h}_{MN} = e^{-2k|y|} h_{\mu\nu} \delta_M^\mu \delta_N^\nu. \tag{16}$$

We also need the inverse metric.

Mathematical Statement 1. $\tilde{g}_{(5)}^{MN} = g_{(5)}^{MN} - \tilde{h}^{MN}$

Proof. If $\tilde{g}_{(5)}^{MN} = g_{(5)}^{MN} - \tilde{h}^{MN}$ then

$$\tilde{g}_{(5)}^{MP} \tilde{g}_{PN}^{(5)} = \overbrace{g_{(5)}^{MP} g_{PN}^{(5)}}^{\delta_N^M} + \overbrace{g_{(5)}^{MP} \tilde{h}_{PN}}^{\tilde{h}^M{}_N} - \overbrace{g_{MP}^{(5)} \tilde{h}^{PN}}^{\tilde{h}_M{}^N} - \overbrace{\tilde{h}^{MN} \tilde{h}_{MN}}^{\mathcal{O}(h^2)}.$$

So if $\tilde{h}^M{}_N = \tilde{h}_M{}^N$, we end up with the desired result:

$$\tilde{g}_{(5)}^{MP} \tilde{g}_{PN}^{(5)} \approx \delta_N^M$$

□

Let's see what is happening with the Christoffel symbols under this perturbation. Now remember that we are interested in finding the part of $R_{(5)}$ corresponding to $R_{(4)}$ so we are interested in what the 4D part of (15) contributes, and we do not care about the terms arising from differentiation with respect to y either. So, we will calculate quantities with small indices going from 1 to 4.

Mathematical Statement 2. $\Gamma^K_{MN} \supset g^{\kappa\rho}_{(4)} \left(h_{\mu\rho,\nu} + h_{\rho\nu,\mu} - h_{\mu\nu,\rho}\right) \delta^K_\kappa \delta^\mu_M \delta^\nu_N$

Proof.

$$\Gamma^K_{MN} = \frac{1}{2} \tilde{g}^{KP} (\tilde{g}_{MP,N} + \tilde{g}_{PN,M} - \tilde{g}_{MN,P}).$$

Looking at (15) and (17), we also have that:

$$\tilde{g}^{KP} \supset e^{2k|y|} \left(g^{\kappa\rho}_{(4)} + h^{\kappa\rho} \right) \delta^K_\kappa \delta^P_\rho$$

and

$$\tilde{g}_{MP,N} \supset e^{-2k|y|} h_{\mu\rho,\nu} \delta^\mu_N \delta^\rho_P \delta^\nu_N,$$

so, the 4D part of the Christoffel symbols we are interested in derives from:

$$\Gamma^K_{MN} \supset \frac{1}{2} e^{2k|y|} \left(g^{\mu\nu}_{(4)} + h^{\mu\nu} \right) \delta^K_\mu \delta^P_\nu (e^{-2k|y|} h_{\mu\rho,\nu} + e^{-2k|y|} h_{\nu\rho,\mu} - e^{-2k|y|} h_{\mu\nu,\rho}) \delta^\mu_N \delta^\rho_P \delta^\nu_N,$$

or

$$\Gamma^K_{MN} \supset g^{\kappa\rho}_{(4)} \left(h_{\mu\rho,\nu} + h_{\rho\nu,\mu} - h_{\mu\nu,\rho}\right) \delta^K_\kappa \delta^\mu_M \delta^\nu_N + \mathcal{O}(h^2) \supset g^{\kappa\rho}_{(4)} \left(h_{\mu\rho,\nu} + h_{\rho\nu,\mu} - h_{\mu\nu,\rho}\right) \delta^K_\kappa \delta^\mu_M \delta^\nu_N.$$

□

Mathematical Statement 3. $R_{(5)} \supset e^{2k|y|} R_{(4)}$

Proof. The above result means that:

$$\Gamma^K_{MN} \supset \Gamma^\kappa_{\mu\nu} \delta^K_\kappa \delta^\mu_M \delta^\nu_N.$$

We also see that since the Ricci tensor will involve only derivatives of Γ, it will be:

$$R^{(5)}_{MN} \supset R^{(4)}_{\mu\nu} \delta^\mu_N \delta^\nu_N.$$

Since we are not interested in terms of order $\mathcal{O}(h^2)$ we can neglect terms $\mathcal{O}(\Gamma^2)$ in the expression giving the Ricci tensor. Remembering once more that:

$$\tilde{g}^{KP} \supset e^{2k|y|} \left(g^{\kappa\rho}_{(4)} + h^{\kappa\rho} \right) \delta^K_\kappa \delta^P_\rho,$$

as well as the fact that $R_{MN} \sim \mathcal{O}(h)$ we can write:

$$R_{(5)} \supset e^{2k|y|} g_{(4)}^{\kappa\rho} R_{\kappa\rho}^{(4)} = e^{2|k|y} R_{(4)}.$$

□

Thus,

$$S_{eff} \supset M_{(5)}^3 \int d^5 x e^{-2k|y|} \sqrt{-g^{(4)}} R_{(4)}.$$

So, if we perform the integration of the 5th dimension:

$$S_{eff} \supset \frac{M_{(5)}^3}{k}(1 - e^{-2kL}) \int d^4 x \sqrt{-g_{(4)}} R_{(4)}. \tag{17}$$

Therefore, we see that the suppression of the Planck mass which we identify as

$$M_{Pl}^2 = \frac{M_{(5)}^3}{k}(1 - e^{-2kL}), \tag{18}$$

is small if kL is large enough. We see, that gravity crosses "easily" from one 3-brane to the other through the fifth dimension. If, on the other hand, we consider the Higgs field living in the second 3-brane, described by the following 4-D action:

$$S_{Higgs} = \int d^4 x \sqrt{-\gamma_2} \left[\gamma_2^{\mu\nu} (D_\mu H^\dagger)(D_\nu H) - \lambda (H^\dagger H - v^2)^2 \right], \tag{19}$$

but

$$\gamma_{\mu\nu_2}^{(4)} = g_{MN}^{(5)}(y = y_2) \delta_\mu^M \delta_\nu^N,$$

which means that:

$$\gamma_2 = g_{(4)} e^{-8kL} \quad \text{and} \quad \gamma_2^{\mu\nu} = g_{(4)}^{\mu\nu} e^{-2kL}, \text{ so}$$

$$S_{Higgs} = \int d^4 x \sqrt{-g_{(4)}} e^{-4kL} \left[g_{(4)}^{\mu\nu} e^{-2kL} (D_\mu H^\dagger)(D_\nu H) - \lambda (H^\dagger H - v^2)^2 \right],$$

If now, we rescale the field as $\tilde{H} = e^{-kL} H$ and $\tilde{v} = e^{-kL} v$ we end up with:

$$S_{Higgs} = \int d^4 x \sqrt{-g_{(4)}} \left[g_{(4)}^{\mu\nu} (D_\mu \tilde{H}^\dagger)(D_\nu \tilde{H}) - \lambda (\tilde{H}^\dagger \tilde{H} - \tilde{v}^2)^2 \right]. \tag{20}$$

We see, therefore, that the effective vacuum expectation value of the Higgs field is suppressed as

$$v \to \tilde{v} = e^{-kL} v. \tag{21}$$

Summing up, we saw that even though gravity remains independent of the extra dimension provided that $kL \gg 1$, the weak interaction is exponentially suppressed.

2.2 Graviton modes

As discussed in [4] the radius of the extra dimension can be taken to be infinite. However, it is important to examine if the effective 4-dimensional graviton modes are experimentally consistent. To determine the graviton modes we will assume perturbations in $g^{(4)}_{\mu\nu}$ of the following form:

$$g_{MN} \to e^{-2k|y|}(\eta_{\mu\nu} + h_{\mu\nu})\delta^{\mu}_{M}\delta^{\nu}_{N} + \delta^{5}_{M}\delta^{5}_{N}. \tag{22}$$

By making a simple change of variable, this metric can take a conformally flat form. To do that notice that the length element for the unperturbed metric can be written as:

$$ds^2 = e^{-2k|y|}\left(\eta_{\mu\nu} + dz^2\right). \tag{23}$$

where

$$dy^2 = e^{-2k|y|}dz^2 \Rightarrow k|z| = e^{k|y|} + constant,$$

and we choose the constant so $z = 0$ when $y = 0$, thus,

$$k|z| = e^{k|y|} - 1.$$

In our case

$$g_{MN} = e^{-2k|y|}\eta^{(5)}_{MN},$$

where $\eta^{(5)}_{MN} = \mathrm{diag}(-1, 1, 1, 1, 1)$. Or in terms of z:

$$g_{MN} = \frac{1}{(1 + k|z|)^2}\eta^{(5)}_{MN},$$

for the element given in (23). Now,

$$\sigma(z) = \ln(1 + k|z|)$$

and so

$$\begin{aligned}
\sigma'(z) =& \frac{k\,\mathrm{sign}(z)}{1 + k|z|} = \frac{k\left[(\Theta(z) - \Theta(z - L_z)) - (\Theta(-z) - \Theta(-z + L_z))\right]}{1 + k|z|} \Rightarrow \\
\sigma''(z) =& \frac{2k(\delta(z) - \delta(z - L_z))}{1 + k|z|} - \frac{k^2}{(1 + k|z|)^2} \Rightarrow \\
\\
\sigma''(z) =& 2k(\delta(z) - \delta(z - L_z))e^{-\sigma} - \sigma'^2.
\end{aligned} \tag{24}$$

In our analysis the perturbed metric elements are $\tilde{g}_{MN} = \eta_{MN} + h_{MN}$. Let us first calculate $\tilde{G}_{MN}$:

Mathematical Statement 4. $\tilde{\Gamma}^{\Sigma}_{MN} \approx \frac{1}{2}\eta^{\Sigma P}(h_{MP,N} + h_{PN,M} - h_{MN,P})$

Proof.

$$\tilde{\Gamma}^{\Sigma}_{MN} = \frac{1}{2}\tilde{g}^{\Sigma P}(\tilde{g}_{MP,N}+\tilde{g}_{PN,M} - \tilde{g}_{MN,P}) = \frac{1}{2}(\eta^{\Sigma P} - h^{\Sigma}P)(h_{MP,N} + h_{PN,M} - h_{MN,P}) \Rightarrow$$
$$\tilde{\Gamma}^{\Sigma}_{MN} = \frac{1}{2}\eta^{\Sigma P}(h_{MP,N} + h_{PN,M} - h_{MN,P})$$

□

Mathematical Statement 5. $R^{\Sigma}_{MPN} \approx \frac{1}{2}\eta^{\Sigma\Lambda}(h_{\Lambda P,MN} - h_{MP,\Lambda N} - h_{\Lambda N,MP})$

Proof. The expression giving the Riemann tensor will simplify:

$$R^{\Sigma}_{MPN} = -\Gamma^{\Sigma}_{MP,N} + \Gamma^{\Sigma}_{MN,P} + \mathcal{O}(h^2),$$

because by looking at mathematical statement 5 we can see that $\Gamma^2 \sim \mathcal{O}(h^2)$. Thus by using that result:

$$R^{\Sigma}_{MPN} = -\frac{1}{2}\eta^{\Sigma\Lambda}(h_{M\Lambda,PN}+h_{\Lambda P,MN}-h_{MP,\Lambda N}-h_{M\Lambda,NP}-h_{\Lambda N,MP}+h_{MN,\Lambda P})+\mathcal{O}(h^2),$$

or,

$$R^{\Sigma}_{MPN} = -\frac{1}{2}\eta^{\Sigma\Lambda}(h_{\Lambda P,MN} - h_{MP,\Lambda N} - h_{\Lambda N,MP}) + \mathcal{O}(h^2)$$

□

Mathematical Statement 6. $\tilde{G}_{MN} = -\frac{1}{2}(h^{\Sigma}{}_{\Sigma,MN} - h_{M\Sigma,}{}^{\Sigma}{}_{N} - h^{\Sigma}{}_{N,M\Sigma} + h_{MN,}{}^{\Sigma}{}_{\Sigma} - h^{\Sigma}{}_{\Sigma,}{}^{N}{}_{N}\eta_{MN})$

Proof. We want to find the Einstein tensor components to first order with respect to h. The Ricci tensor will be

$$\begin{aligned}\tilde{R}_{MN} = \tilde{R}^{\Sigma}_{M\Sigma N} &= -\frac{1}{2}\eta^{\Sigma P}(h_{P\Sigma,MN} - h_{M\Sigma,PN} - h_{PN,M\Sigma} + h_{MN,P\Sigma}) \\ &= -\frac{1}{2}(h^{\Sigma}{}_{\Sigma,MN} - h_{M\Sigma,}{}^{\Sigma}{}_{N} - h^{\Sigma}{}_{N,M\Sigma} + h_{MN,}{}^{\Sigma}{}_{\Sigma})\end{aligned}$$

and

$$\begin{aligned}\tilde{R} = \tilde{R}_{MN}g^{MN} &= -\frac{1}{2}(h^{\Sigma}{}_{\Sigma,MN} - h_{M\Sigma,}{}^{\Sigma}{}_{N} - h^{\Sigma}{}_{N,M\Sigma} + h_{MN,}{}^{\Sigma}{}_{\Sigma})(\eta^{MN} + h^{MN}) \\ &= -\frac{1}{2}(h^{\Sigma}{}_{\Sigma,}{}^{N}{}_{N} - h^{N}{}_{\Sigma,}{}^{\Sigma}{}_{N} - h^{\Sigma}{}_{N,}{}^{N}{}_{\Sigma} + h^{N}{}_{N,}{}^{\Sigma}{}_{\Sigma}) = -h^{\Sigma}{}_{\Sigma,}{}^{N}{}_{N}.\end{aligned}$$

So, now,

$$\tilde{G}_{MN} = \tilde{R}_{MN} - \frac{\tilde{R}}{2} g_{MN} = -\frac{1}{2}(h^{\Sigma}{}_{\Sigma,MN} - h_{M\Sigma,}{}^{\Sigma}{}_{N} - h^{\Sigma}{}_{N,M\Sigma} + h_{MN,}{}^{\Sigma}{}_{\Sigma})$$
$$+ \frac{1}{2} h^{\Sigma}{}_{\Sigma,}{}^{N}{}_{N}(\eta_{MN} + h_{MN}) =$$
$$= -\frac{1}{2}(h^{\Sigma}{}_{\Sigma,MN} - h_{M\Sigma,}{}^{\Sigma}{}_{N} - h^{\Sigma}{}_{N,M\Sigma} + h_{MN,}{}^{\Sigma}{}_{\Sigma} - h^{\Sigma}{}_{\Sigma,}{}^{N}{}_{N}\eta_{MN})$$

□

Now if these flactuations do not have a 5-D component and are transverse and traceless:

$$\begin{aligned} &h_{M4} = 0, \\ &h^{M}{}_{M} = 0 \Rightarrow h^{\mu}{}_{\mu} = 0, \\ &\partial_M h^{MN} \Rightarrow \partial_\mu h^{\mu\nu} = 0. \end{aligned} \tag{25}$$

We get a simple expression for the Einstein tensor:

$$\tilde{G}_{MN} \approx -\frac{1}{2}\partial_P \partial^P h_{\mu\nu}.$$

Regarding the degrees of freedom we see that since h_{MN} symmetric

$$d.o.f. = \frac{5 \times 5 - 5}{2} + 5 = 15$$

Eqs (25) are $5 + 1 + 4 = 10$ in total. Thus we are left with 5 d.o.f.. Let's go back to the big expression giving $G_{\mu\nu}$ in terms of $\tilde{G}_{\mu\nu}$. We want to calculate up to order $\mathcal{O}(h)$ in $D = 5$, so the product of the Christoffel symbols can be ignored, as we end up with $\mathcal{O}(\Gamma^2)$. Moreover the covariant derivative of a scalar function is simply the partial derivative:

$$\begin{aligned} G_{MN} =& -\frac{1}{2}\partial_P \partial^P h_{\mu\nu} + 3\left[\tilde{\nabla}_M \tilde{\partial}_N \sigma + \tilde{\partial}_M \sigma \tilde{\partial}_N \sigma - \tilde{g}_{MN}\left(\tilde{\nabla}_P(\partial^P \sigma) - (\tilde{\partial}\sigma)^2\right)\right] \\ =& -\frac{1}{2}\partial_P \partial^P h_{\mu\nu} + 3\left[\tilde{\partial}_M \tilde{\partial}_N \sigma - \Gamma^K_{MN}\partial_K \sigma + \tilde{\partial}_M \sigma \tilde{\partial}_N \sigma \right. \\ & \left. - (\tilde{\eta}_{MN} + h_{MN})\left(\partial_P \partial^P \sigma + \Gamma^P_{PK}\partial^K \sigma - (\tilde{\partial}\sigma)^2\right)\right]. \end{aligned}$$

Let's evaluate this for our metric. To do so, note that the derivatives give zero if the are not with respect to z, so:

$$
\begin{aligned}
G_{MN} =& -\frac{1}{2}\partial_P\partial^P h_{\mu\nu} + 3\left[\tilde{\partial}_M\tilde{\partial}_N\sigma - \Gamma^5_{MN}\partial_5\sigma + \tilde{\partial}_M\sigma\tilde{\partial}_N\sigma\right.\\
&\left. - (\tilde{\eta}_{MN} + h_{MN})\left(\partial_5\partial^5\sigma + \Gamma^P_{P5}\partial^5\sigma - (\tilde{\partial}_5\sigma)^2\right)\right]\\
=& -\frac{1}{2}\partial_P\partial^P h_{\mu\nu} + 3\left[\tilde{\partial}_M\tilde{\partial}_N\sigma - \tilde{\Gamma}^5_{MN}\sigma' + \tilde{\partial}_M\sigma\tilde{\partial}_N\right.\\
&\left. - (\tilde{\eta}_{MN} + h_{MN})\left(\sigma'' + \tilde{\Gamma}^P_{P5}\sigma' - \sigma'^2\right)\right].
\end{aligned}
$$

We are interested in the 4D part of the einstein tensor. Taking into account mathematical statement 4 and Eqs (25)

$$
\begin{aligned}
&h_{M5} = 0,\\
&\tilde{\Gamma}^5_{MN} = \frac{1}{2}h'_{\mu\nu},\\
&\tilde{\Gamma}^P_{P5} = \frac{1}{2}\tilde{\eta}^{P\Sigma}\tilde{h}_{P\Sigma,5} = \frac{1}{2}(\eta^{P\Sigma}\tilde{h}_{P\Sigma})_{,5} = \frac{1}{2}h^M{}_{M,5} = 0,
\end{aligned}
$$

$$
G_{\mu\nu} = -\frac{1}{2}\partial_P\partial^P h_{\mu\nu} + \frac{3}{2}h'_{\mu\nu}\sigma' - 3(\tilde{\eta}_{MN} + h_{MN})(\sigma'' - \sigma'^2). \tag{26}
$$

Now we need to calculate the components of the energy-momentum tensor. To do that, let's write the general expression for the energy momentum tensor which is:

$$
T_{MN} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{MN}} = -2\frac{\delta\mathcal{L}_M}{\delta g^{MN}} + g_{MN}\mathcal{L}_M, \tag{27}
$$

which gives

$$
\begin{aligned}
S &= \int\sqrt{-g}\left(\frac{1}{2k_{(5)}}R_{(5)} + \mathcal{L}_M\right),\\
G_{MN} &= k_{(5)}T_{MN}.
\end{aligned}
$$

We want to calculate the Einstein tensor for the metric containing $(t, r, \theta, \varphi, z)$. Expression (10) is referring to $x^M = (t, r, \theta, \varphi, y)$. We need to be careful with γ_i as now the metric we use (23) has a different total determinant. Let me write once again that:

$$
\gamma_i^{\mu\nu} = e^{-2\sigma(y_i)}g^{\mu\nu}_{(4)}. \tag{28}
$$

In the previous section we had that

$$
\overbrace{g_{55}}^{1}\gamma_i = g_{(5)}(x^\mu, y = y_i). \tag{29}
$$

Now the metric we use is conformally flat, so

$$\overbrace{g_{55}}^{e^{-2\sigma(z)}} \ \gamma_i = g_{(5)}(x^\mu, z = z_i). \tag{30}$$

In our case the action reads:

$$S = \int d^5\, x\sqrt{-g}\left[M_{(5)}^3 R_{(5)} \underbrace{-2\Lambda_{(5)} - \sum_{i=1,2} \lambda_i e^{\sigma(z)}\delta(z - z_i)}_{\mathcal{L}_M}\right]. \tag{31}$$

So, if we use (24) we derive instantly:

$$\begin{aligned}
T_{MN} =& - \left(2\Lambda + e^{\sigma(z)}\lambda_1\delta(z) + e^{\sigma(z)}\lambda_2\delta(z - L_z)\right)g_{MN} \\
=& - \left(2\Lambda + e^{\sigma(z)}\lambda_1\delta(z) + e^{\sigma(z)}\lambda_2\delta(z - L_z)\right)e^{-2\sigma(z)}(\eta_{MN} + h_{MN}) \\
=& - \left(2\Lambda e^{-2\sigma(z)} + e^{-\sigma(z)}\lambda_1\left[\delta(z) - \delta(z - L_z)\right]\right)(\eta_{MN} + h_{MN}) \\
=& \left[12M^3\sigma'^2 - 6M^3(\sigma'' + \sigma'^2)\right](\eta_{MN} + h_{MN}) \\
=& \, 6M^3\left(\sigma'^2 - \sigma''\right)(\eta_{MN} + h_{MN}),
\end{aligned}$$

where I have used that $\sigma'^2 = k^2 e^{-2\sigma}$, $\Lambda = -6M^3 k$ and equations (13) and (24). In 4D:

$$\kappa^2 T_{\mu\nu} = 3\left(\sigma'^2 - \sigma''\right)(\eta_{\mu\nu} + h_{\mu\nu}). \tag{32}$$

So, now, the 4D equations read:

$$\frac{1}{2}\partial_P\partial^P h_{\mu\nu} - \frac{3}{2}h'_{\mu\nu}\sigma' = 0. \tag{33}$$

In order to solve the equation above we can first transform to conformal coordinates:

$$\hat{h}_{\mu\nu} = e^{-\frac{3}{2}\sigma(z)}h_{\mu\nu}, \tag{34}$$

so that the equation now reads

$$\partial^2\hat{h}_{\mu\nu} - \left(\frac{3}{2}\sigma'' - \frac{15}{4}\sigma'^2\right)\hat{h}_{\mu\nu} = 0. \tag{35}$$

Earlier we have calculated σ' and σ'' so if we make use of the expressions we derived in (24) we find:

$$\partial^2\hat{h}_{\mu\nu} - \left[\frac{3k\delta(z)}{1 + k|z|} - \frac{15k^2}{(1 + k|z|)^2}\right]\hat{h}_{\mu\nu} = 0. \tag{36}$$

Then we can perform a Kaluza-Klein reduction to four dimensions

$$\hat{h}_{\mu\nu}(x, z) = e^{ip\cdot x}\psi(z) \quad \text{where} \quad p^2 = m^2, \tag{37}$$

and solve with respect to the function $\psi(z)$, but that is something we will omitt in this thesis. There have been many reviews over the years, summarizing and expanding the results of the Randall-Sundrum models. Here, instead, we will proceed by giving a brief analysis of how one could treat a massive graviton five-dimensional theory, since this will be of interest later on.

2.3 Expanding the Lagrangian

In this subsection we will examine perturbations in the background metric Lagranigian in the presence of a graviton mass term. The goal in the end for this particular thesis, is to derive the graviton massless scalar modes. Including a bulk mass term, the action now reads:

$$S = \int d^5x\sqrt{-g}\left[M^3 R - 2\Lambda - M^3 k^2 g_{(0)}^{MN} g_{(0)}^{AB}(a\, h_{MA}h_{NB} + b\, h_{MN}h_{AB})\right]. \tag{38}$$

We will once again take variations of the background metric. Our metric choice is the same, but we will consider it here in a slightly different notation:

$$ds^2 = \alpha(y)^2(\eta_{\mu\nu}dx^\mu dx^\nu + dy^2), \tag{39}$$

thus,

$$g_{MN} = \alpha(y)^2 \tilde{g}_{MN} = e^{-A}\tilde{g}_{MN}. \tag{40}$$

In our case

$$\begin{aligned}\sqrt{-g}R = e^{-3A/2}\sqrt{-\tilde{g}}\Big[&\tilde{R} + 4\tilde{g}^{AB}\tilde{\nabla}_A\tilde{\nabla}_B A\\ &-3\tilde{g}^{AB}\tilde{\nabla}_A A\tilde{\nabla}_B A\Big],\end{aligned}$$

but also

$$\tilde{\nabla}_A\tilde{\nabla}^A\left(e^{-3A/2}\right) = \frac{9}{4}e^{-3A/2}\tilde{\nabla}_A A\tilde{\nabla}^A A - \frac{3}{2}e^{-3A/2}\tilde{\nabla}_A\tilde{\nabla}^A A,$$

thus

$$4e^{-3A/2}\tilde{\nabla}_A\tilde{\nabla}^A A = -\frac{8}{3}\tilde{\nabla}_A\tilde{\nabla}^A\left(e^{-3A/2}\right) + 6e^{-3A/2}\tilde{\nabla}_A A\tilde{\nabla}^A A.$$

So if we neglect the total derivative we are left with

$$\sqrt{-g}R = e^{-3A/2}\sqrt{-\tilde{g}}\left[\tilde{R} + 3\tilde{g}^{AB}\tilde{\nabla}_A A\tilde{\nabla}_B A\right]. \tag{41}$$

We want to find $S^{(2)}$, so we need $\sqrt{-\tilde{g}}$ to second order as well as every following term.

Mathematical Statement 7. $\tilde{g}^{MN} = \eta^{MN} - \tilde{h}^{MN} + \tilde{h}^{MP}\tilde{h}_{PN}$

Proof. Let's suppose that

$$g^{MN} = \eta^{MN} + \lambda^{MN},$$

where $\lambda^{MN} = \lambda^{MN}_{(1)} + \lambda^{MN}_{(2)}$, and $\lambda_{(1)} \sim \mathcal{O}(\tilde{h})$, $\lambda_{(2)} \sim \mathcal{O}(\tilde{h}^2)$. Let's now demand

$$\begin{aligned} \delta^M_N = \tilde{g}_{NP}\tilde{g}^{PM} \Rightarrow \delta^M_N = (\eta_{NP} + \tilde{h}_{NP})(\eta^{PM} + \lambda^{PM}_{(1)} + \lambda^{PM}_{(2)}) \Rightarrow \\ 0 = \tilde{h}^M_N + \underbrace{\tilde{h}_{NP}\lambda^{PM}_{(1)}}_{\mathcal{O}(\tilde{h}^2)} + \underbrace{\tilde{h}_{NP}\lambda^{PM}_{(2)}}_{\mathcal{O}(\tilde{h}^3)} + \lambda^M_{N\,(1)} + \lambda^M_{N\,(2)}. \end{aligned} \tag{42}$$

If we keep up to $\mathcal{O}(h)$ we have

$$0 = \tilde{h}^M_N + \lambda^M_{N\,(1)} \Rightarrow \lambda^M_{N\,(1)} = -\tilde{h}^M_N.$$

If we substitute this to (42) and keep up to $\mathcal{O}(h^2)$ we get:

$$\lambda^M_{N\,(2)} = \tilde{h}_{NP}\tilde{h}^{PM}.$$

Indeed,

$$\tilde{g}^{MN} = \eta^{MN} - \tilde{h}^{MN} + \tilde{h}^{MP}\tilde{h}_{PN}$$

□

It is true that [11]:

$$\sqrt{-\tilde{g}} = 1 + \frac{1}{2}\tilde{h} - \frac{1}{4}\left(\tilde{h}_{MN}\tilde{h}^{MN} - \frac{1}{2}\tilde{h}^2\right),$$

where $\tilde{h} = \tilde{h}^M_M$. Looking at (38) we recognize the bulk mass term:

$$\mathcal{L}_M = -\sqrt{-g}M^3k^2 g^{MN}_{(0)} g^{AB}_{(0)} (a\, h_{MA}h_{NB} + b\, h_{MN}h_{AB}). \tag{43}$$

At this point let us underline that $g^{MN}_{(0)}$ raises indices of h_{MN} and $\tilde{g}^{MN}$ of $\tilde{h}_{MN}$. So,

$$\mathcal{L}_M = -e^{-5A/2}\sqrt{-\tilde{g}}M^3k^2(a\, h_{MA}h^{MA} + b\, h^2).$$

Mathematical Statement 8. $g^{MN}_{(0)} = e^A\,\tilde{g}^{MN}_{(0)}$ *and* $h^{MN} = e^A\tilde{h}^{MN}$

Proof. To prove this, we set them to be proportional, i.e. $h^{MN} = c\,\tilde{h}^{MN}$ and $g^{MN}_{(0)} = c\,\tilde{g}^{MN}_{(0)}$. Of course in our case $\tilde{g}^{MN}_{(0)} = \eta^{MN}$. We start with:

$$\begin{aligned} h_{MN} = e^{-A}\tilde{h}_{MN} \Rightarrow h^{KP} = e^{-A} g^{KM}_{(0)} g^{PN}_{(0)} \tilde{h}_{MN} \Rightarrow \\ c\tilde{h}^{KP} = e^{-A}c^2\tilde{h}^{KP} \Rightarrow c = e^A. \end{aligned}$$

□

But this means that

$$h_{MA}h^{MN} = \tilde{h}_{MA}\tilde{h}^{MA}. \tag{44}$$

Also by using the result of mathematical statement 8, we see that:

$$h^2 = \tilde{h}^2. \tag{45}$$

2.4 Lagrangian to 1st Order

First let's find $S^{(1)}$. The $\mathcal{L}_M$ term is of second order in the first place. We also have:

$$\mathcal{L}_\Lambda^{(1)} = -\Lambda\tilde{h}. \tag{46}$$

So, the contribution to the equations of motion is:

$$\frac{\partial\mathcal{L}_\Lambda^{(1)}}{\partial\tilde{h}^{K\Lambda}} = -\Lambda\eta_{K\Lambda}.$$

And

$$\mathcal{L}_R^{(1)} = \overbrace{-\tilde{\partial}^2\tilde{h}}^{R^{(1)}} + \frac{3}{2}\tilde{h}\eta^{AB}\tilde{\partial}_A A\tilde{\partial}_B A + 3h^{AB}\tilde{\partial}_A A\tilde{\partial}_B A. \tag{47}$$

So,

$$\frac{\partial\mathcal{L}_R^{(1)}}{\partial\tilde{h}^{K\Lambda}} = \frac{3}{2}\eta_{K\Lambda}(\tilde{\partial}A)^2 + 3\tilde{\partial}_K A\tilde{\partial}_\Lambda A.$$

2.5 Lagrangian to 2nd Order

If we keep up to $\mathcal{O}(h^2)$, we find:

$$\mathcal{L}_M^{(2)} = -e^{-5A/2}M^3k^2(a\,\tilde{h}_{MA}\tilde{h}^{MA} + b\,\tilde{h}^2). \tag{48}$$

The bulk mass term will give the following contribution if we variate with respect to $\tilde{h}_{K\Sigma}$:

$$\frac{\partial\mathcal{L}_M^{(2)}}{\partial\tilde{h}^{K\Lambda}} = -2e^{-5A/2}M^3k^2(a\,\tilde{h}_{K\Lambda} + b\,\eta_{K\Lambda}\tilde{h}), \tag{49}$$

because

$$\frac{\partial}{\partial\tilde{h}^{K\Lambda}}\left(\tilde{g}^{(0)}_{MP}\tilde{g}^{(0)}_{M\Sigma}\tilde{h}^{MN}\tilde{h}^{P\Sigma}\right) = 2\tilde{h}_{K\Lambda},$$

and

$$\frac{\partial}{\partial\tilde{h}^{K\Lambda}}\left(h^2\right) = \frac{\partial}{\partial\tilde{h}^{K\Lambda}}\left(\tilde{g}^{(0)}_{NM}\tilde{g}^{(0)}_{P\Sigma}\tilde{h}^{MN}\tilde{h}^{P\Sigma}\right) = 2\eta_{K\Lambda}\tilde{h}^2.$$

Let's move on to the contribution from Λ:

$$\mathcal{L}_\Lambda = -2\Lambda\sqrt{-g} = -2\Lambda e^{-5A/2}\sqrt{-\tilde{g}}. \tag{50}$$

Keeping only second order terms we find:

$$\mathcal{L}_\Lambda^{(2)} = \frac{1}{2}\Lambda e^{-5A/2}\left(\tilde{h}_{MN}\tilde{h}^{MN} - \frac{1}{2}\tilde{h}^2\right). \tag{51}$$

So,

$$\frac{\partial \mathcal{L}_\Lambda^{(2)}}{\partial \tilde{h}^{K\Lambda}} = \Lambda e^{-5A/2}\left(\tilde{h}_{K\Lambda} - \frac{1}{2}\eta_{K\Lambda}\tilde{h}\right). \tag{52}$$

2.6 Modes decomposition

In this subsection we will decompose the metric into tensor, vector and scalar components.

$$g_{MN} = g^{(0)}_{MN} + h_{MN} = a^2(\eta_{MN} + \tilde{h}_{MN}). \tag{53}$$

We will follow the same process as the authors of [11] did. The metric can be expressed as

$$ds^2 = a^2\left[(1+2\phi)dy^2 + 2A_\mu\, dy\, dz^\mu + (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu\, dx^\nu\right], \tag{54}$$

where,

$$\tilde{h}_{MN} = \begin{pmatrix} h_{\mu\nu} & A_\mu \\ A_\nu & 2\phi \end{pmatrix} = 2\phi\,\delta^5_M\delta^5_N + A_\mu\delta^5_M\delta^\mu_N + A_\nu\delta^\mu_M\delta^5_\nu + h_{\mu\nu}\delta^\mu_M\delta^\nu_N. \tag{55}$$

In order to derive the equations we will need to solve eventually, we follow the technique of [11] and we make use of the null vector k_μ, taking the basis $(k^+_\mu, k^-_\mu, k^i_\mu)$, where $i = 1, 2$. These vectors are defined through the following relationships:

$$\begin{aligned} k^+_\mu &= k_\mu, \\ k^-_\mu k^{+\mu} &= 1, \\ k^-_\mu k^{-\mu} &= 0, \\ k^i_\mu k^{+\mu} &= k^i_\mu k^{-\mu} = 0, \\ k^i_\mu k^{j\mu} &= \delta^{ij}. \end{aligned} \tag{56}$$

Along a null basis, vector and tensor expansions take the following form:

$$A_\mu = k^+_\mu A^+ + k^-_\mu A^- + \sum_{i=1,2} k^i_\mu A^i, \tag{57}$$

$$\begin{aligned} T_{\mu\nu} &= k^+_\mu k^+_\nu T^{++} + 2k^+_{(\mu} k^-_{\nu)} T^{+-} + k^-_\mu k^-_\nu T^{--} \\ &\quad + \sum_{i=1,2} 2k^+_{(\mu} k^i_{\nu)} T^{+i} + \sum_{i=1,2} 2k^-_{(\mu} k^i_{\nu)} T^{-i} + \sum_{i,j=1,2} 2k^i_{(\mu} k^j_{\nu)} T^{ij}. \end{aligned} \tag{58}$$

The gauge that is fixed in [11] and which we are also fixing is the following

$$\begin{aligned} &A_\mu = A^T_\mu, \quad \partial_\mu A^\mu = 0 \\ &h_{\mu\nu} = h^{TT}_{\mu\nu} + 2\eta_{\mu\nu}\psi, \quad h^\mu{}_\mu = 8\psi, \quad \partial_\mu h^{\mu\nu} = 2\partial^\nu \psi \\ &h^{TT}_{\mu\nu} = \sum_{i,j=1,2} k^i_\mu k^j_\nu \left(h_{ij} - \frac{1}{2}\delta_{ij} h^r{}_r \right), \\ &A^T_\mu = \sum_{i=1,2} k^i_\mu A^i. \end{aligned} \tag{59}$$

Now, the $(\mu\nu)$ components of the Einstein equation read:

$$\begin{aligned} &h''_{\mu\nu} + 3\frac{a'}{a} h'_{\mu\nu} + \Box h_{\mu\nu} - 2\partial^\rho \partial_{(\mu} h_{\nu)\rho} + \partial_\mu \partial_\nu h + 2\partial_\mu \partial_\nu \phi - 2a^{-3}\left[a^3 \partial_{(\mu} A_{\nu)}\right]' \\ &\eta_{\mu\nu} \left\{ -h'' - 3\frac{a'}{a} h' - \Box h - 2\Box\phi + 6\frac{a'}{a}\phi' + 6\left[\frac{a''}{a} + 2\left(\frac{a'}{a}\right)^2\right] + 2a^{-3}(a^3 \partial_\mu A^\mu)' \right. \\ &\left. + \partial^\rho \partial^\sigma h_{\rho\sigma} \right\} - 4a^2 M^3 k^2 (\alpha h_{\mu\nu} + \beta \eta_{\mu\nu} \tilde{h}) = 0. \end{aligned} \tag{60}$$

We want to find the massless scalar modes so we will take $\Box\phi = \Box\psi = 0$. In the fixed gauge the equations take the form

$$\begin{aligned} &h^{TT''}_{\mu\nu} + 3\frac{a'}{a} h^{TT'}_{\mu\nu} + \partial_\mu \partial_\nu (2\phi + 4\psi) - 2a^{-3}\left[a^3 \partial_{(\mu} A_{\nu)}\right]' - 4a^2 M^3 k^2 \alpha h^{TT}_{\mu\nu} \\ &+ \eta_{\mu\nu} \left\{ -6\psi'' - 18\frac{a'}{a}\psi' + 6\frac{a'}{a}\phi' + 6\left[\frac{a''}{a} + 2\left(\frac{a'}{a}\right)^2\right]\phi \right. \\ &\left. -4a^2 M^3 k^2 (2\alpha\psi + \beta\tilde{h}) \right\} = 0 \end{aligned} \tag{61}$$

In the null basis, the first line from above gives

$$k^i_\mu k^j_\ni \left(h''_{ij} + 3\frac{a'}{a} h'_{ij} - 4a^2 M^3 k^2 \alpha h_{ij} \right) - 2k^i_{(\mu} k^+_{\nu)} a^{-3} (a^3 A_i)' + k^+_\mu k^+_\nu (4\psi + \phi) = 0. \tag{62}$$

Then we have the following equations:

$$h_{\mu\nu}^{TT''} + 3\frac{a'}{a}h_{\mu\nu}^{TT'} - 4a^2M^3k^2\alpha h_{\mu\nu}^{TT} = 0, \tag{63}$$

$$\left(a^3 A_\mu^T\right)' = 0, \tag{64}$$

$$\phi + 2\psi = 0, \tag{65}$$

$$3\psi'' + 9\frac{a'}{a}\psi' - 2\frac{a'}{a}\phi' - 3\left[\frac{a''}{a} + 2\left(\frac{a'}{a}\right)^2\right]\phi + 4a^2M^3k^2(2\alpha\psi + \beta\tilde{h}) = 0. \tag{66}$$

In a similar manner, from the (μy) and yy components of the Einstein equations we find:

$$\frac{a'}{a}\phi - \psi' = 0, \tag{67}$$

$$12\frac{a'}{a}\psi' - 3\left[\frac{a''}{a} + 2\left(\frac{a'}{a}\right)^2\right]\phi + 2a^2M^3k^2[2(\alpha+\beta)\phi + 8\beta\psi] = 0. \tag{68}$$

But, $a = 1/(1+ky) \Rightarrow a' = -ka^2,\ a'' = 2k^2a^3$, so if we substitute (65) and (67) in (68) we find:

$$12k^2a^2\phi - 3(2k^2a^2 + 2k^2a^2)\phi + 2a^2M^3k^2(\alpha - \beta)\phi = 0 \Rightarrow \alpha = \beta, \tag{69}$$

a results which is inconsistent with the physically meaningful Fierz-Pauli choice $\alpha = -\beta$. Therefore, the massless scalar modes are zero, $\phi = \psi = 0$.

3 de-Sitter brane world theory

In the previous section, we explored the brane world model described by Randall and Sundrum, corresponding to an AdS five-dimensional bulk. A similar model has been explored, where now $\Lambda_5 > 0$ [5, 6, 7]. in this model there exists a dS_4 brane, which can describe our universe. Considering this to correspond to our universe we can examine the accelerating expansion of our universe in the context of this five-dimensional model. That is in contrast to the Randall-Sundrum model, since there the four-dimensional brane was Minkowski. For further work done with respect to models with extra dimensions that describe our expanding universe see [8, 9].

3.1 Framework

The theory contains one brane located at $y = 0$ and the action that describes in is given by

$$S = \int d^5\, x\sqrt{-g}\left(M_{(5)}^3 R_{(5)} - 2\Lambda_{(5)}\right) - \int d^4x\sqrt{-\gamma}\,\lambda. \tag{70}$$

The metric that we are going to work with has the form

$$ds^2 = F^2(y)\left(-dt^2 + A(t)d\vec{x}^{\,2}\right) + dy^2. \tag{71}$$

The energy-momentum tensor is

$$T_{MN} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{MN}} = -2\, g_{MN}\Lambda_{(5)} - \gamma_{\mu\nu}\,\lambda\delta(y)\delta^\mu_M\delta^\nu_N. \tag{72}$$

3.2 Equations

Now, the components of the Einstein tensor read:

$$G_{00} = \frac{3}{4}\left\{\left(\frac{\dot{A}}{A}\right)^2 - 2F''\right\} \quad , \; G_{11} = G_{22} = G_{33} = \frac{1}{4}\left\{\left(\frac{\dot{A}}{A}\right)^2 - 4\ddot{A} + 6AF''\right\}$$

$$G_{44} = \frac{3}{2}\frac{AF'^2 - F\ddot{A}}{2AF^2}, \tag{73}$$

where dot and prime denote differentiation with respect to t and y respectively. The Einstein equations read:

$$\frac{\dot{A}^2}{4A^2F^2} - \frac{F''}{F} - \frac{F'^2}{F^2} = \frac{\lambda\delta(y)}{6M_5^3F} + \frac{\Lambda_5}{3M_5^3F}, \tag{74}$$

$$\frac{F''}{F} + \frac{F'^2}{F^2} - \frac{\ddot{A}}{3AF^2} + \frac{\dot{A}^2}{12A^2F^2} = -\frac{\lambda\delta(y)}{6M_5^3F} - \frac{\Lambda_5}{3M_5^3F}, \tag{75}$$

$$\frac{F'^2}{F^2} - \frac{\ddot{A}}{4AF^2} = -\frac{\Lambda_5}{6M_5^3}. \tag{76}$$

From a cosmological perspective, since our universe is expanding and the cosmological constant has been measured to be non-zero and positive, we would like to explore the scenario of an expanding 3-brane. Adding the first two equations we get

$$\ddot{A}\,A = \dot{A}^2 \Rightarrow A(t) = e^{2Ht}, \tag{77}$$

where $H > 0$. Then the Einstein equations reduce to:

$$\frac{F'^2}{F^2} - \frac{H^2}{F^2} = -\frac{\Lambda_5}{6M_5^3}. \tag{78}$$

Even though we are not going to solve this here, let us state, as it turns out, that in the case of a positive bulk cosmological constant, a solution to the hierarchy problem requires fine tuning. Despite not being able to solve the hierarchy problem, the dS five dimensional setup, might find applications in dS/CFT correspondence problems and is therefore still interesting.

4 Swampland and the Weak Gravity Conjecture

In [12] it was argued that despite the apparent consistent appearance, many effective theories of the string theory landscape of vacua, are actually inconsistent. Many low energy gravitational theories turn out to be UV incomplete belonging therefore in the swampland. A review that was used throughout this thesis is [20].

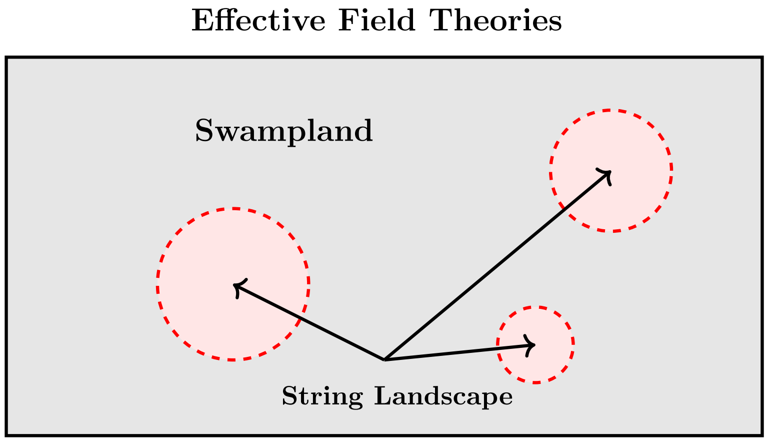

Figure 3: Depiction of the theory space, including the swampland and the string landscape.

4.1 Weak Gravity Conjecture

One of the Swampland constraints is the Weak Gravity Conjecture and it states that gravity is the weakest force. Using basic black hole physics this corresponds to the statement that the mass of a charged particle should be less than or equal to the charge it would carry if it was an extremal black hole, namely $m = Q$. According to the conjecture for a theory coupled to gravity, there exists an electrically charged object of charge Q and tension T so that:

$$\frac{T}{M_P^2} \leq Q^2, \tag{79}$$

where that object could correspond to a brane. In the simplest form, for a particle, the inequality above reduces to the following:

$$\left(\frac{m}{M_P}\right)^2 \leq Q^2. \tag{80}$$

For a $U(1)$ charged particle $Q^2 = 2gq$. It is believed that a theory of quantum gravity should not obey any global symmetries. Even though, arguments in favor of that stem among others from string theory and holography, we will focus only on the black hole argument. According to that, if we feed a black hole by throwing $U(1)$ globally charged particles in it, then the mass of the black hole increases. Due to the fact that there is no field arising from the symmetry, there is no way to realize the charge of the black hole. Therefore, there is an infinite number of charged states for a black hole with a particular mass. This is problematic with respect to calculations of the black hole entropy. On the other hand, if we have a gauge $U(1)$ theory, we can write the metric for the charged black hole geometry as follows:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r} + \frac{Q^2 G}{4\pi\epsilon_0 c^4 r^2}\right) dt^2 + \left(1 - \frac{2GM}{c^2 r} + \frac{Q^2 G}{4\pi\epsilon_0 c^4 r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \quad (81)$$

For this geometry two horizons arise at radii

$$r_\pm = M \pm \sqrt{M^2 - Q^2}, \quad (82)$$

where we set $[G] = [c] = 1$ and we absorbed some factors inside Q. In order to satisfy the cosmic censorship conjecture which states that no naked singularities are allowed, the mass and the charge of the black hole must satisfy:

$$M \geq Q. \quad (83)$$

For black holes the attractive force of gravity wins over their electromagnetic repulsion and gravity is stronger. For that to be true despite us throwing $U(1)$ particles in the black hole, the latter must discharge from the extremality limit. Since there is a gauge field it is possible for the black hole to emit charged particles with $m < q$, whereas for the remaining black hole we have $M > Q$, as shown in figure 4.

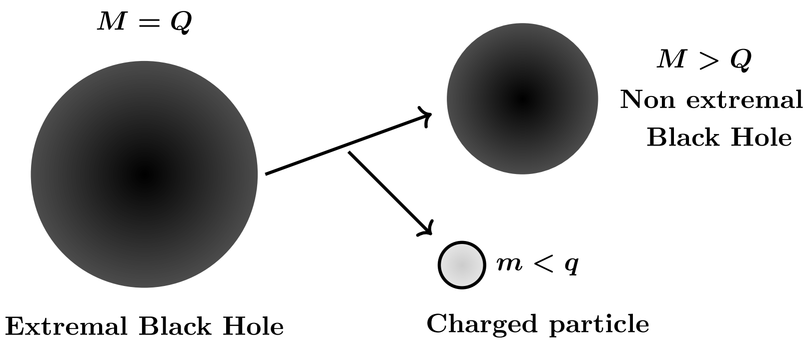

Figure 4: Black hole discharging. Extremal black holes emit charged particles that satisfy the Weak Gravity Conjecture.

In work done more recently by Ooguri and Vafa [16] the conjecture was sharpened, and it was conjectured that the equality in the above inequality is satisfied in the case of a supersymmetric theory. This modification of the Weak Gravity Conjecture provides an interesting result for an AdS geometry. Considering an AdS$_{p+2}$ geometry which is supported by a $(p+2)$-flux there exists, a p-brane solution which is nucleating outwards to the AdS boundary. This contributes to what is called AdS fragmentation [21], as it leaves behind a p-brane and an AdS geometry with units of flux reduced by one. The tension of the brane tries to contract it, in contradiction to the charge that wants to expand it. According to the non-supersymmetric Weak Gravity Conjecture howerver, charge will exceed the tension and subsequently, there always exists a nucleating brane. In this section we will make use of this conjecture which can be stated as follows:

AdS Instability Conjecture:

Anti-de Sitter space must exhibit instabilities in the case of non-supersymmetric setups.

A very interesting application has to do with compactifying the Standard model on a circle, as we review in the following subsection.

4.2 Standard Model compactification

In this subsection we will look through the constraints on the mass of the lightest neutrino, upon compactification of the SM on a circle of radius R. The Casimir energy density is given by:

$$\rho(R) = \sum_{n=1}^{\infty} \frac{2m^4}{(2\pi)^2} \frac{K_2(2\pi R\, m\, n)}{(2\pi R\, m\, n)^2}. \tag{84}$$

This can be expanded in the case of $mR \ll 1$. The potential reads:

$$V(R) = (2\pi R) \left(\frac{r}{R}\right)^3 \left\{ \Lambda_4 \mp \sum_i n_i \rho_i(R) \right\}, \tag{85}$$

where $-(+)$ is used for bosons (fermions). We can now check for the existence or not of vacua, which would indicate a problem with our theory. Whenever neutrino masses come into play we shall consider both normal and inverted hierarchy:

Normal Hierarchy: $m_1 \ll m_2,\ m_2 \approx 8.6 \times 10^{-3}\text{eV}$ and $m_3 \approx 4.9 \times 10^{-2}\text{eV}$ (86)

Inverted Hierarchy: $m_1 \approx m_2 \approx 4.9 \times 10^{-2}\text{eV}$ (87)

4.2.1 Non-zero cosmological constant $\Lambda_4 \neq 0$

By varying the mass of the lightest neutrino while keeping the value of the c.c. fixed at $\Lambda_4 = 2.6 \times 10^{-47}\,\text{GeV}^4$, we see that those vacuua exist both for normal and inverted hierarchy (figures 5 and 6). This applies to the case of both Dirac and Majorana neutrinos. From (85) we get:

$$V(R) = \frac{(\text{GeV}^{-3})}{(2\pi R)^2} \left\{ \Lambda_4 + n_\nu \rho_\nu(R) - \frac{2+2}{1440\pi^2 R^4} \right\}, \tag{88}$$

where in the last term we include 2+2 degrees of freedom for the photon and the graviton.

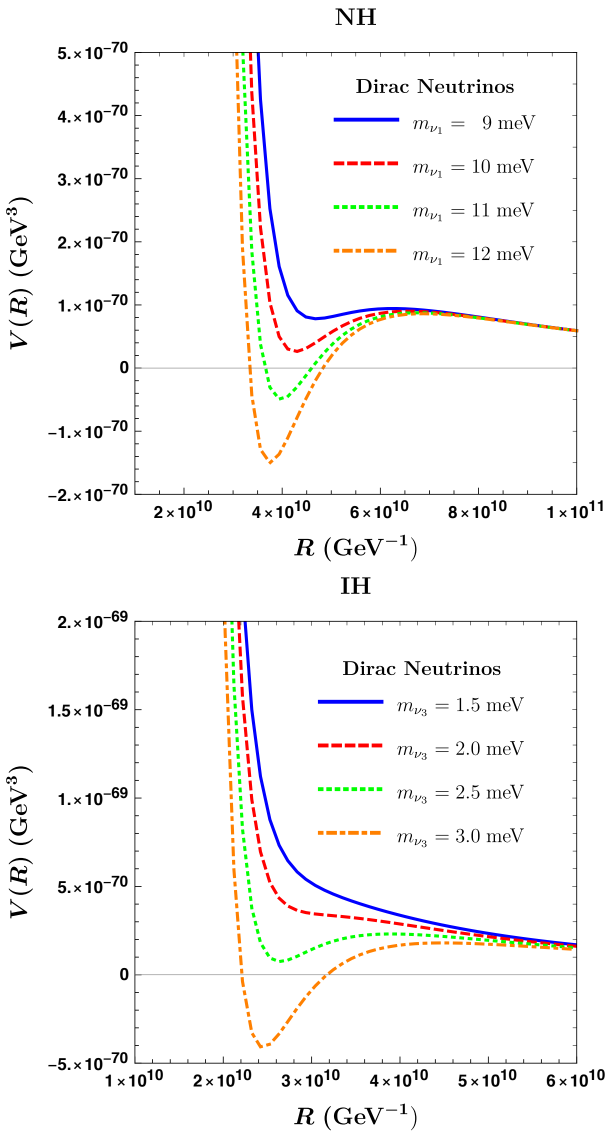

Figure 5: Potential as a function of R in the case of Dirac neutrinos for normal and inverted hierarchy.

We see that in the case of the Dirac neutrinos an AdS minimum might appear as the mass of the lightest neutrinos increases, while it is also possible to have dS minima. On the other hand, if we consider the neutrinos to be Majorana, even when the lightest neutrino is massless, an AdS minimum always appears. This could stand as an indication in favor of Dirac neutrinos.

Examining figures 5 and 6 we see that only Dirac neutrinos are in agreement with the observable value for Λ_4. In the case of normal hierarchy the mass of the lightest neutrino should satisfy the constraint $m_{\nu_1} < 1.26 \times 10^{-2}$eV, while for inverted hierarchy we find $m_{\nu_3} < 3.16 \times 10^{-3}$eV.

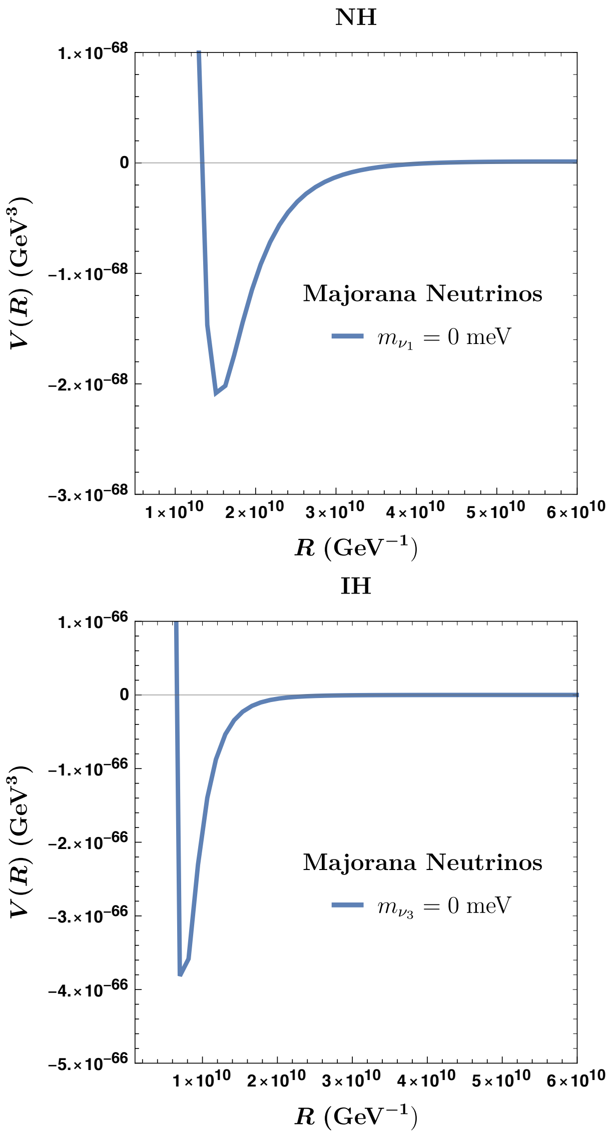

Figure 6: Potential as a function of R in the case of Majorana neutrinos for normal and inverted hierarchy.

4.2.2 Limit where the cosmological constant is zero $\Lambda_4 \to 0$

As a first scenario we will examine the assumptions under which the potential exhibits AdS vacua, if $\Lambda_4 = 0$. The massless degrees of freedom that contribute are 2+2=4 (graviton and photon), therefore:

$$V(R) = \frac{(\text{GeV}^{-3})}{(2\pi R)^2} \{2 \cdot \rho_b(R) + 2 \cdot \rho_g(R) - n_f \rho_f(R)\}\,. \tag{89}$$

We find that *AdS*-minima are always present as you can see in figure 9:

The same applies to the inverted hierarchy scenario.

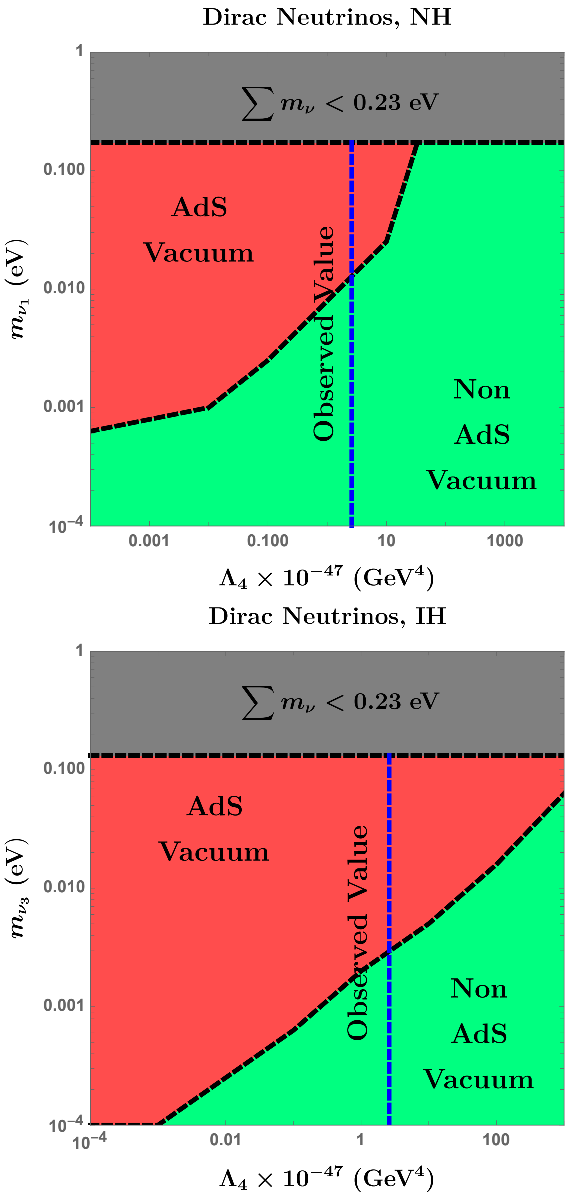

Figure 7: Bounds on the value of Λ_4 with respect to the lightest neutrino mass, in the case of Dirac neutrinos for normal and inverted hierarchy.

4.3 Weak Gravity Conjecture in 5D brane world theories

In this subsection, we will consider compactification on a circle of the brane world theories that were presented in the previous sections. As we saw earlier, Randall-Sundrum 1 and 2 models describes a theory with an AdS bulk and therefore belongs already in the Swampland. We will see that regardless of the positive contributions to the potential, it never rises above zero and can only exhibit AdS minima if any. In the compactified potential we will include the contributions of the Casimir energy in five dimensions, in addition to the bulk cosmological constant term. We will employ the following metric

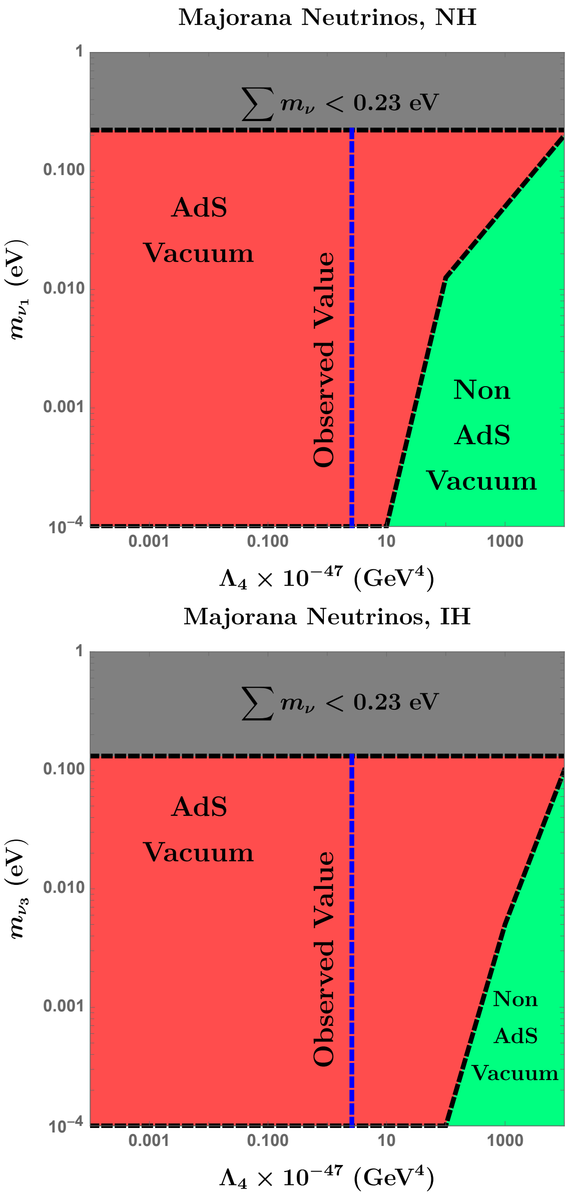

Figure 8: Bounds on the value of Λ_4 with respect to the lightest neutrino mass, in the case of Majorana neutrinos for normal and inverted hierarchy

decomposition

$$ds^2_{(5)} = \frac{r^2}{R^2} ds^2_{(4)} + F(y)A(t)R^2 d\varphi^2 + \text{graviphoton}, \tag{90}$$

where, $ds^2_{(4)} = F(y)\left\{-dt^2 + A(t)d\vec{x}^2_{2D}\right\} + dy^2$. The effective potential now, reads:

$$V(R) = (FA)^{1/2}(2\pi R)\left(\frac{r}{R}\right)^4 \{\Lambda_5 + n_f\rho_f(R) - n_b\rho_b(R)\} \tag{91}$$

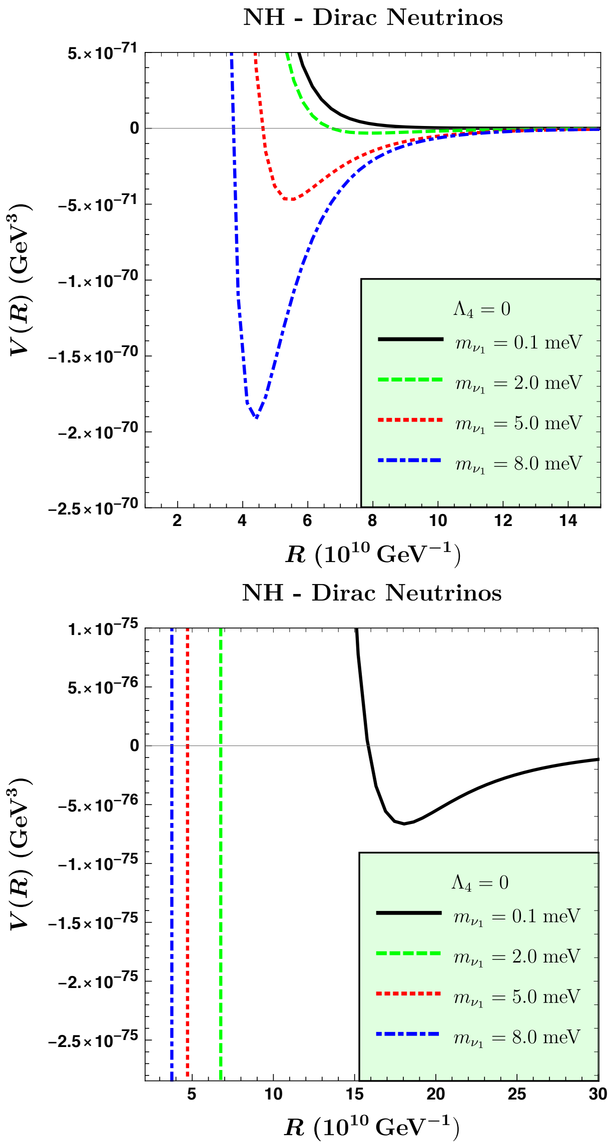

Figure 9: Potential as a function of R in the case of Dirac neutrinos for normal hierarchy.

For simplicity we define

$$\mathcal{V}(R) \equiv \frac{2\pi}{R^3}\left\{\Lambda_5 + n_f \rho_f(R) - n_b \rho_b(R)\right\} \quad \text{and} \quad \alpha \equiv (FA)^{1/2}, \tag{92}$$

so that

$$V(R) = \alpha\, \mathcal{V}(R). \tag{93}$$

The existence or not of minima is determined by $\mathcal{V}(R)$, since the factor α contributes merely as a scale factor that blows up at $R = 0$, which is way far from the position we expect the minima to appear at. We see that the existence or not of AdS minima depends on the factor in the brackets and is therefore irrelevant to the value employed by y.

4.3.1 Casimir energy in five dimensions

As reviewed in [17] upon considering the contributions to the vacuum Casimir energy, the Casimir energy density in the general case and in d-dimensions reads:

$$\rho(R) = \sum_{n=1}^{\infty} \frac{2m^d}{(2\pi)^{d/2}} \frac{K_{d/2}(2\pi R\, m\, n)}{(2\pi R\, m\, n)^{d/2}}. \tag{94}$$

This can be expanded in the case of small mR. In the 4D case we find:

$$\begin{aligned}\rho(R) =& \sum_{n=1}^{\infty} \left\{ \frac{1}{16\pi^6 n^4 R^4} - \frac{(mR)^2}{16\left(\pi^4 n^2\right) R^4} + \mathcal{O}\left(mR\right)^6 \right\} \\ =& \frac{1}{1440\pi^2 R^4} - \frac{1}{96\pi^2 R^4}\left(mR\right)^2 + \mathcal{O}\left(mR\right)^4 ,\end{aligned} \tag{95}$$

where $\gamma \approx 0.577$ is the Euler gamma. We see that $[\rho]_{4D} = [m]^4$. In 5D

$$\begin{aligned}\rho(R) =& \sum_{n=1}^{\infty} \left\{ \frac{3}{128\pi^7 n^5 R^5} - \frac{(mR)^2}{64\left(\pi^5 n^3\right) R^5} + \frac{(mR)^4}{64\pi^3 n R^5} - \frac{(mR)^5}{60\pi^2 R^5} + \mathcal{O}\left(mR\right)^6 \right\} \\ =& \frac{3\,\zeta(5)}{128\pi^7 R^5} - \frac{\zeta(3)}{64\pi^5 R^5}\left(mR\right)^2 + \mathcal{O}(mR)^4,\end{aligned} \tag{96}$$

and $[\rho]_{5D} = [m]^5$.

4.3.2 Small mR approximation

We will now examine the case where except for the massless photon, we include (at least at first) massive bosonic degrees of freedom as well.

$$\begin{aligned}\mathcal{V}(R) =& \frac{2\pi r^4}{R^3}\left\{\Lambda_5 + n_f \rho_f(R) - n_b \rho_b(R)\right\} \\ =& \frac{2\pi r^4}{R^3}\left\{\Lambda_5 - (3)\frac{3\,\zeta(5)}{128\pi^7 R^5} + \sum_i n_b^i \rho_b^i(R) + \sum_i n_f^i \rho_f^i(R)\right\}.\end{aligned} \tag{97}$$

The expansion for small mR yields

$$
\begin{aligned}
\mathcal{V}(R) \approx & \frac{2\pi r^4}{R^3}\left\{\Lambda_5 - (3)\frac{3\,\zeta(5)}{128\pi^7 R^5} - \sum_i n_i^b\left[\frac{3\,\zeta(5)}{128\pi^7 R^5} - \frac{\zeta(3)}{64\pi^5 R^5}\,(m_i^b R)^2\right]\right.\\
& \left. + \sum_i n_i^f\left[\frac{3\,\zeta(5)}{128\pi^7 R^5} - \frac{\zeta(3)}{64\pi^5 R^5}\,(m_i^f R)^2\right]\right\}\\
= & \frac{2\pi r^4\Lambda_5}{R^3} + (n_f - n_b - 3)\,\frac{3\,\zeta(5)\,r^4}{64\pi^6 R^8} + \frac{\zeta(3)\,r^4}{32\pi^4 R^8}\left(M_b^2 - M_f^2\right)R^2,
\end{aligned}
\tag{98}
$$

where $n_{b,f} \equiv \sum_i n_i^{b,f}$ and $M_{b,f}^2 = \sum_i n_i^{b,f} m_i^2$. Then at the minimum we should have:

$$
\begin{aligned}
& -\frac{6\pi r^4\Lambda_5}{R_0^4} - (n_f - n_b - 3)\,\frac{3\,\zeta(5)\,r^4}{8\pi^6 R_0^9} - \frac{3\zeta(3)\,r^4}{16\pi^4 R_0^9}\left(M_b^2 - M_f^2\right)R_0^2 = 0\\
\Rightarrow & \left(M_b^2 - M_f^2\right)R_0^2 = -\,(n_f - n_b - 3)\,\frac{2\,\zeta(5)}{\pi^2\zeta(3)} - \frac{32\pi^5\Lambda_5}{\zeta(3)}R_0^5
\end{aligned}
\tag{99}
$$

In order for this to be a minimum and not a maximum we demand:

$$
\begin{aligned}
& +\frac{24\pi r^4\Lambda_5}{R_0^4} + (n_f - n_b - 3)\,\frac{27\,\zeta(5)\,r^4}{8\pi^6 R_0^{10}} + \frac{21\zeta(3)\,r^4}{16\pi^4 R_0^{10}}\left(M_b^2 - M_f^2\right)R_0^2 > 0\\
\Rightarrow & -\frac{18\pi\Lambda_5}{R_0^5} + (n_f - n_b - 3)\,\frac{3\,\zeta(5)}{4\pi^7 R_0^7} > 0.
\end{aligned}
\tag{100}
$$

At that point the potential employs the value:

$$
\mathcal{V}(R_0) \approx \frac{\pi r^4\Lambda_5}{R_0^3} - (n_f - n_b - 3)\,\frac{\zeta(5)\,r^4}{64\pi^6 R_0^8}.
\tag{101}
$$

4.3.3 The dS-bulk brane world scenario

We are interested in examining the presence of AdS vacua in the case of $\Lambda_5 > 0$. Following the analysis of the previous subsection, and using eq. (101), we see that provided the existence of a minimum, we should check the value of Λ_5 in order to determine whether or not we end up with an AdS vacuum. Firstly, let's ignore fermionic contributions to the potential, and consider only the contributions from the massless photon and graviton. The potential now reads

$$
V(R) = (FA)^{1/2}(2\pi R)\left(\frac{r}{R}\right)^4\left\{\Lambda_5 - (3)\frac{3\,\zeta(5)}{128\pi^7 R^5}\right\}.
\tag{102}
$$

At small R the $\gamma + g$ term is dominant and goes to minus infinity, whereas at large R the potential has a runaway form. A maximum appears at R_{max} which is given by

$$
R_{max} = \left[\frac{71\zeta(5)}{384\pi^7\Lambda_5}\right]^{\frac{1}{5}}.
\tag{103}
$$

The value that R acquires at the maximum describes a mass scale that is therefore given by:

$$m_s = \frac{1}{R_{max}} = \left[\frac{71\zeta(5)}{384\pi^7\Lambda_5}\right]^{-\frac{1}{5}}. \tag{104}$$

We will now vary the value of the cosmological constant in five dimensions, and examine the possible constraints on the femionic masses and degrees of freedom that will not allow for the appearance of an AdS vacuum. We will take the graviton to be massless and therefore $n_b = 8$. We will scan the parameter space for a very large set of Λ_5. We will then move on to discuss about the potential phenomenological constraints. In the previous section we derived eq. (100), which tells us that if $n_f < n_b + 3$ then we can not have a minimum. If we look at eq. (99) we can further investigate the possible cases. In the massless graviton scenario we have $M_b = 0$ and $n_b = 5$. If the contributing fermionic degrees of freedom are massless, then we just have $n_f < n_b + 3$ for the equation to be satisfied. But this contradicts the condition above and therefore we can not have an AdS minimum in this case.

- Two or less fermions: In this case $n_f \leq 8$ and therefore eq. (100) is not satisfied. No AdS minimum is found in this scenario.
- Three fermions: In this case it is possible to end up with AdS minima. What follows is a plot, presented in figure 10, revealing the regions that allow for the existence of minima and the ones that do not. We variate the cosmological constant from a value of 10^{-90} GeV to 10^{90} GeV.
- More than three fermions: Further increasing the fermionic degrees of freedom will increase the mass limit at which AdS minima start appearing. To see that we can make use of eq. (99):

$$M_f^2 = (n_f - 8)\frac{2\zeta(5)}{\pi^2\zeta(3)R_0^2} + \frac{32\pi^5\Lambda_5}{\zeta(3)}R_0^3. \tag{105}$$

  We can also see that in figure 10.
- If the graviton is massive, then in five dimensions it will have nine degrees of freedom. In this scenario for three fermions, we will have $n_f = n_b$. Then from (99), the condition for AdS minima to appear is:

$$(M_b^2 - M_f^2)R_0^2 = -\frac{32\pi^5\Lambda_5}{\zeta(3)}R_0^5 \tag{106}$$

  For a positive cosmological constant, this requires

$$M_b > M_f \Rightarrow 3m_\gamma^2 + 9m_g^2 > \sum_i n_i^f m_i^2. \tag{107}$$

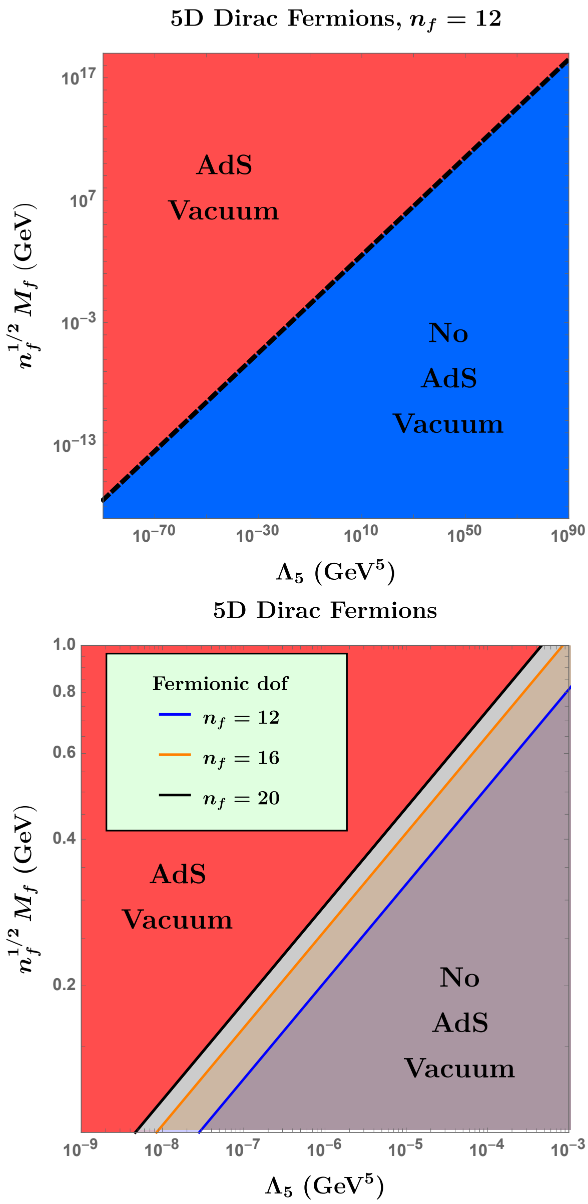

Figure 10: (Top) Bounds on the values of Λ_5 with respect to the fermionic masses contributing to the potential. For this figure we consider twelve fermionic degrees of freedom, which translates to three Dirac fermions. (Bottom) Bounds on the values of Λ_5 with respect to the fermionic masses contributing to the potential for $n_f \geq 8$.

But even in this scenario, only a maximum can be retrieved and no minimum.

To summarize:

Massless graviton	1. $M_f^2 = (n_f - 8)\frac{2\zeta(5)}{\pi^2\zeta(3)R_0^2} + \frac{32\pi^5\Lambda_5}{\zeta(3)}R_0^3$ 2. $(n_f - 12)\,\frac{\zeta(5)}{24\pi^8 R_0^2} > \Lambda_5$
Massive graviton	No AdS minimum

Table 2: Swampland conditions for massless and massive graviton and $n_f/4$ massive fermions in the dS brane world scenario.

4.3.4 Randall-Sundrum Models

As we saw on previous sections, the five-dimensional cosmological constant in these models is negative, creating an AdS five-dimensional bulk. If we look at eq. (99), and we take the scenario of a massless graviton, then we can rewrite it as

$$M_f^2 R_0^2 = (n_f - 8)\frac{2\zeta(5)}{\pi^2\zeta(3)} + \frac{32\pi^5\Lambda_5}{\zeta(3)}R_0^5 > 0. \tag{108}$$

Provided that the second term is always negative, we realize that for $n_f < 8$ and thus less than two fermions, no AdS minima occur. However, the potential goes to minus infinity and therefore we will not examine it in the context of the AdS-instability conjecture. If now, $n_f \geq 8$ then AdS minima always occur because there is no way to uplift the potential by picking a more positive cosmological constant for an AdS bulk. So, once again we will not examine constraints set by the AdS-instability conjecture.

5 Scalar Weak Gravity Conjecture in 5D gravity models

As we saw, the Weak Gravity Conjecture demands the existence of particles who experience gravity as the weakest force. In our discussion so far the forces were the gravitational and the elctromagnetic force. The conjecture, however should apply even in the scenario of scalar interactions, mediated by a scalar particle ϕ.

5.1 Scalar Weak Gravity Conjecture

In that case the conjecture needs to be modified in order to take into account the extra scalar force. To write the new relationship, we need first to figure out what is the the strength of the interaction. We consider a three point interaction given by the Lagrangian

$$\mathcal{L}_I = \frac{1}{2}\underbrace{\left(2m_H\lambda\phi + m_H^2\right)}_{m^2}|H|^2. \tag{109}$$

The constant λ is the coupling constant of the scalar interactions. We see now that for the new effective mass m, we can write

$$\frac{\partial m^2}{\partial \phi} = 2m_H\lambda \Rightarrow \lambda = \frac{m}{m_H}\partial_\phi m, \tag{110}$$

so if we take the vacuum expectation value of the scalar field ϕ to be $\langle\phi\rangle = 0$, then $\langle m\rangle = m_H$, and therefore

$$\lambda = \langle\partial_\phi m\rangle. \tag{111}$$

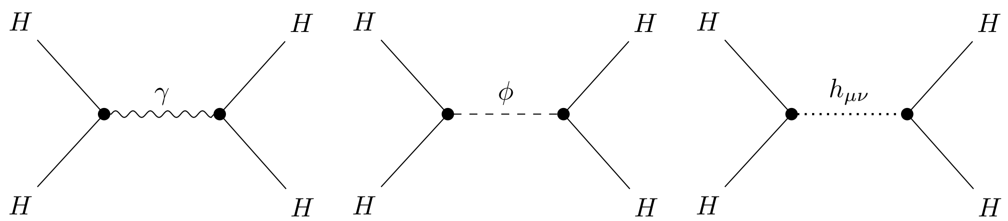

Figure 11: Feynman diagrams of the the tree level scalar interactions between the particles in our theory.

If we look at figure 11, the Feynman diagram in the middle represents the interaction mediated by the scalar field. This interaction gives rise to an attractive potential $\sim \frac{\mu^2}{r}$. As in the electric Weak Gravity Conjecture that we saw before, the Strong Scalar Weak Gravity Conjecture can be stated as follows:

$$m \leq \lambda M_p. \tag{112}$$

5.2 Strong Scalar Weak Gravity Conjecture

Palti's formulation, however, does not apply to any scalar field. It applies to WGC scalars that experience scalar interactions mediated by ϕ, and the Weak Gravity Conjecture does not apply to ϕ itself. In [22], the conjecture is generalized so that it includes any scalars in the theory. Now, we also need to consider the four-point interaction between the scalars. In figure 12 we see the scalar interactions we must take into account:

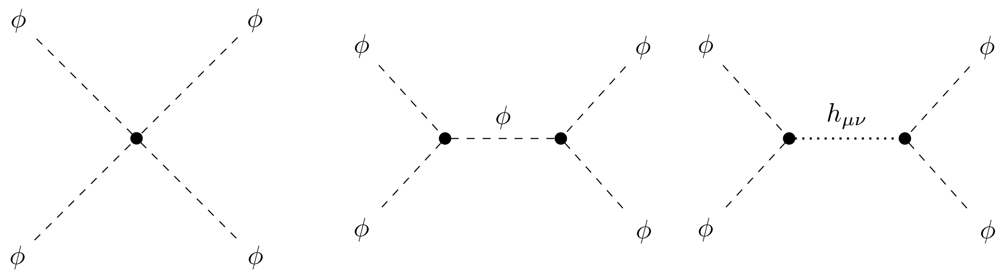

Figure 12: Feynman diagram of the four point interaction between scalars ϕ.

The Strong Weak Gravity Conjecture is expressed through the following inequality:

$$\frac{(V'')^2}{M_p^2} \leq 2(V''')^2 - V''V'''', \tag{113}$$

or we can define a new variable according to [22]

$$\tilde{\chi} = M_p^2 \left[2 \left(\frac{V'''}{V''} \right) - \frac{V''''}{V''} \right], \tag{114}$$

so then the condition that needs to be satisfied is $\tilde{\chi} \geq 1$. The first term on the right, represents the attractive force coming from the three-point interaction diagram, whereas the second term on the right, represents a repulsive force. The factors are chosen so that nice results in the case of an axion potential are retrieved. In the case of compactification of one of the dimensions, the radius R of the compactification itself corresponds to a scalar field ϕ through the relationship

$$R = r \exp \left[\frac{\phi}{M_{(4-d)}} \right] \Rightarrow \phi = M_{(4-d)} \ln \left(\frac{R}{r} \right), \tag{115}$$

where we will set $r = 1\,\text{GeV}$, as we did earlier. Also, the $(4-d)$-dimensional Planck mass is related to the 4-dimensional Planck mass through the compactification relation $M_{(4-d)} = (2\pi R) M_p^2$. In [22] the strong Scalar Weak Gravity Conjecture is used in order to examine again the Standard Model compactification. According to this conjecture, the inverse hierarchy scenario belongs in the swampland. It will be interesting to check the results that this conjecture will give in the five-dimensional models but this is beyond the scope of this thesis.

Conclusions

In this thesis we discussed the basics of the Randall-Sundrum models. We saw how they solve the hierarchy problem and we examined metric perturbations. Additionally, we examined the scenario of massive gravity in five dimensions, by including a graviton mass term in the action and we considered perturbations of the metric to second order. Our goal was to examine the massless scalar modes, using the method given in [11] to examine the results retreived in [10]. We found that for the Fierz-Pauli choice of the mass term parameters, the massless scalar modes are zero.

We then discussed a seemingly different topic. That of swampland, and some conjectures, which distinguish between consistent and inconsistent with quantum gravity, effective field theories. We used the AdS-Instability Conjecture to see how it constraints neutrino masses after compactifying the Standard Model on a circle, like it was done in [19] and we verified the results derived there. Those results are in favor of Dirac instead of Majorana neutrinos, and normal instead of inverted neutrino mass hierarchy.

Consequently, we discussed the constraints on fermionic masses and degrees of freedom, set due to the AdS-Instability Conjecture, on de-Sitter brane-world models. We showed that for massless graviton and photon, unless the fermionic degrees of freedom are $n \geq 8$, no AdS minima occur and we graphically represented the allowed region in terms of cosmological constant and fermionic masses for different choices of n_f.

Finally, we briefly talked about the Scalar Weake Gravity Conjecture and a recent modification of it, namely the Strong Scalar Weak Gravity Conjecture. Deriving constraints on brane-world models, by using this conjecture is a project that we plan to work on. Apart from that, more scenarios need to be tested that are more interesting phenomenologically and that is something we intend to examine.

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