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VIX Futures Return Decomposition

A THESIS SUBMITTED TO THE FALCULTY OF THE GRADUATE SCHOOL OF THE UNIVERSITY OF MINNESOTA

By

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Abstract

VIX futures contracts have produced negative returns. I develop a method to decompose the daily returns of VIX futures contracts in to the return components of roll down and level. I show that roll down is the largest contributor to the negative returns. The return decomposition analysis is carried out across the VIX futures term structure which includes the one- to six-month VIX futures contracts. I use time series regressions to estimate the beta coefficients of the return components relative to the VIX. The results of the regression analyses are used to create a VIX curve strategy that is combined with the S&P 500 Index.

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Chapter I
Introduction

In Chapter 1 I review the Chicago Board Options Exchange's Implied Volatility Index (VIX) and VIX futures. I discuss the mechanics and pricing of each and provide context regarding the historical performance of VIX futures contracts. I also provide several theoretical explanations for the implied volatility term structure and discuss how the VIX futures term structure has negatively impacted the performance of VIX futures contracts. I conclude by discussing how this paper extends the current literature regarding the performance attribution of VIX futures contracts.

1.1 Implied Volatility Index (VIX)

The price of an options contract is derived in a model, and the required model inputs include price of the underlying, strike price, number of days until expiration, interest rate, and implied volatility. Of the option pricing model inputs, option implied volatility is the only variable that is inferred from the option's price and is not directly observed. Option implied volatility is the market's expectation for the realized volatility of the underlying asset from the current period until expiration of the options contract (Natenberg, 2014). For example, the 30-day implied volatility of an at-the-money S&P 500 Index options contract is the market's expectation for the realized volatility of the S&P 500 Index over the next 30-days.

In finance it is a stylized fact that options contracts traded on financial assets, such as equities and bonds, have implied volatilities that on average exceed the realized volatility of the underlying asset (Coval and Shumway, 2001). The difference between the implied volatility embedded in an options contract price and the subsequent realized volatility is called the volatility risk premium (i.e. realized volatility minus implied volatility). Bakshi and Kapadia (2003) find that a negative volatility risk premium exists for buyers of equity index options and the premium is the price option investors are willing to pay away to protect their equity position.

Equity market implied volatility generally increases when equity prices decline. That has been especially true during periods of sharp equity market price declines, as market participants expect higher realized volatility in the future which leads to higher implied volatilities. The inverse price relationship between the S&P 500 Index and the VIX is evident from their observed historical negative return correlation. The negative correlation of returns between the S&P 500 Index and the VIX highlights why owners of equity securities who are concerned about price declines might have a desire to own the VIX or a VIX-related derivative contract.

The Chicago Board Options Exchange (CBOE) introduced the CBOE Implied Volatility Index (VIX) in 1993 to provide a timely and consistent measure of equity market implied volatility. Today, the VIX is widely quoted in the media and is often used to gauge the market's expectation for future realized volatility.

Originally, the VIX represented the 30-day implied volatility for the S&P 100 Index and was derived from exchange-traded S&P 100 Index option contracts. In 2003, the CBOE, in conjunction with Goldman Sachs, modified the VIX calculation methodology to reflect the 30-day implied volatility for the S&P 500 Index using price information from CBOE-traded S&P 500 Index option contracts. The VIX calculation methodology was enhanced in 2014 to include price information from S&P 500 Index weekly option contracts.

The VIX price is quoted as the S&P 500 Index 30-day implied volatility (annualized). The VIX is calculated from the prices of CBOE S&P 500 Index put and call contracts that have more than 23 days but less than 37 days to expiration. To maintain a constant 30-day implied volatility, the VIX calculation proportionally weights one-month and two-month option contracts. The proportion changes each week.[1]

The option contracts used to calculate the VIX include CBOE out-of-the-money (OTM) puts and calls on the S&P 500 Index. The center strike price of the puts and calls is the strike price that sits just below the calculated forward S&P 500 Index price. Option contracts with a zero-bid price are excluded from the VIX calculation and the number of different strike prices used in the calculation is limited by the number of consecutive strike prices with a non-zero bid price. The VIX price-squared is equal to the 30-day variance swap rate (Zhang *et al.*, 2010).

A unique characteristic of the VIX relative to other indices is that the VIX is not directly investable since a VIX cash market does not exist. The reason for a nonexistent VIX cash market is due to the cost prohibitive nature of replicating the VIX, which would require buying and selling OTM puts and calls that are generally less liquid and have wide bid-ask spreads. Transacting in a market with wide bid-ask spreads usually results in outsized trading costs. For example, Buetow and Henderson (2016) find that the average bid-ask spread for OTM S&P 500 Index puts and calls traded on the CBOE are 46.2% and 50.5% (bid-ask spread as percentage of midpoint price), respectively. Their

[1] Refer to the equation in Appendix A for VIX calculation details.

research shows that buying OTM options at the offer price and then selling them at the bid price, ceteris paribus, would result in a 50% loss in value. Furthermore, since the VIX is a measure of constant 30-day implied volatility, the options used in the calculation are continually changing as time passes. This implies that several transactions would be required monthly to replicate the VIX.

In March 2004, the CBOE launched trading of VIX futures contracts. VIX futures contracts are listed and traded electronically on the CBOE Futures Exchange (the Exchange). The Exchange lists nine consecutive monthly futures contracts and six consecutive weekly futures contracts. Each contract price is quoted as the forward S&P 500 Index 30-day implied volatility (annualized). For example, the price of a VIX futures contract with three months remaining until expiration is the market expectation for the S&P 500 Index 30-day implied volatility three months from now.

The notional value of each futures contract is equal to the futures contract price multiplied by $1,000 and the minimum price interval is 0.05 points which is equal to $50. On the day of expiration, the expiring VIX futures contract is cash-settled and will settle at VIX spot based on the opening trades of the S&P 500 Index option contracts used in the VIX calculation. The expiration date for the expiring monthly VIX futures contract is 30-days prior to the third Friday of the month immediately following the contract month.[2]

Since their inception, VIX futures have experienced a dramatic increase in trading volume and open interest. The average daily volume of the one-month VIX futures contract increased over 500 times and the total open interest of all listed VIX futures contracts increased over 600 times (from March 2004 to June 2017). On December 4,

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[2] http://cfe.cboe.com/cfe-products/vx-cboe-volatility-index-vix-futures/contract-specifications

2017, the total outstanding notional amount of all VIX futures contracts listed was approximately $8.1 billion, which compares to a total notional amount of approximately $20.1 million on March 31, 2004.

A unique feature of the VIX futures contracts, relative to other markets that offer futures contracts, is that a VIX cash market does not exist. The absence of a VIX cash market diminishes the relationship between the price of the VIX and VIX futures contracts since market participants are not able to execute cash and carry trades.[3] The lack of a cash VIX market also contributes to a wider VIX basis.[4] Buetow and Henderson (2016) point out that during extreme market events the difference in prices (and returns) between the VIX and the first and second month VIX futures contracts can be explained by the lack of a VIX cash market.

The financial performance of VIX futures contracts can be evaluated from either the change in price and associated dollar profit and loss or percentage change in price (or return). For example, consider a VIX futures contract that changes in price from 10 at time t to 10.50 at time $t+1$. The change in price and profit is 0.50 and $500, respectively, and the percentage change in price is equal to 5.0% (= 0.50/10).

During December 2009-December 2017, VIX futures experienced negative average returns and the point on the VIX futures term structure with the sharpest declines came from the one-month VIX futures contract. The leading contributor to the negative returns of VIX futures contracts is the roll down. Roll down is the return that comes from

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[3] Cash and carry refers to buying (selling) a cash market instrument and simultaneously selling (buying) a derivative instrument of the same market. The ability of market participants to engage in cash and carry trades preserves the cash and theoretical derivatives pricing relationship.

[4] Basis refers to the price differential between the cash price, or spot, and a futures contract of the same market.

a change in price due to the VIX futures contract moving down the VIX term structure as it approaches expiration. For example, when the VIX futures term structure is upward sloping, the price of each futures contract declines with the passage of time as each futures contract moves down the term structure. The magnitude of the price decline is determined by the slope, or steepness, of the term structure. I define the slope as the difference in price between two points on the term structure (e.g., the one-month VIX futures price minus the two-month VIX futures price). The roll down return from an upward sloping (contango) futures curve will be negative while the roll down from an inverted (backwardated) futures curve will be positive holding all else constant.

The slope is most negative between the one-month and two-month VIX futures contracts relative to all other term structure combinations using one- to six-month contracts. For example, the daily average slope between the one- and two-month VIX futures contracts is -0.9 versus a daily average slope of -0.2 for the five- and six-month VIX futures contracts (during December 16, 2009-December 19, 2017).

The rounded cumulative compound return of the one-month VIX futures contract is -100%. Other points on the term structure lost a similar amount with the three- and five-month VIX futures contracts producing cumulative compound returns of -99.1% and -95.3%, respectively. This research will show that most, if not all, of the negative return of VIX futures contracts is accounted by the roll down.

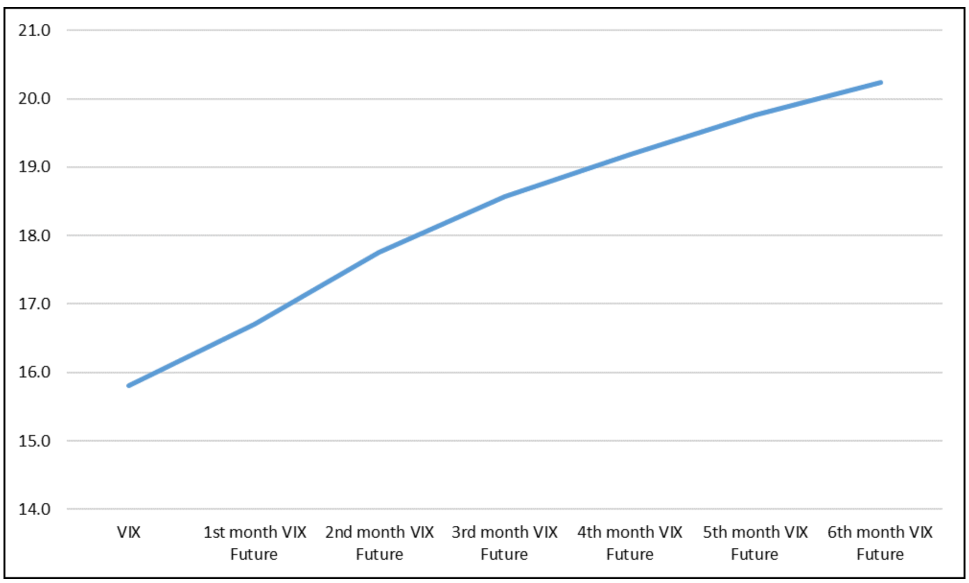
1.2 VIX Term Structure

A VIX term structure exists and can be inferred from the price of VIX futures contracts with different expirations or from S&P 500 Index option contracts using the

VIX calculation methodology and different expiration dates. Most of the time the VIX term structure is upward sloping and is described by VIX spot having the lowest price and each successive VIX futures contract with a longer time to expiration having a higher price. An upward sloping VIX term structure is analogous to an upward sloping US Treasury yield curve where the yield-to-maturity is greater for longer maturities.

VIX spot was lower than the price of the six-month VIX futures contract 92.6% of the time with an average daily price differential of -4.4 (during December 2009-December 2017). The graph below illustrates the average shape of the VIX term structure by plotting and connecting the average daily VIX price with the average daily VIX futures contract prices (during December 2009-December 2017).

Figure 1.1: Average Daily VIX and VIX Futures Closing Prices (December 2011 to December 2016)



There are various theoretical and empirical explanations for the shape of the implied volatility term structure. The main explanations include the expectations hypothesis, volatility mean reversion, and the risk premia of variance risk. The theoretical concept of the expectations hypothesis is summarized well by Campa and Chang (1995). They note that long-dated foreign currency implied volatility is equal to the average of short-dated implied volatilities spanning the same time until expiration.

The expectations hypothesis postulates that an upward sloping implied volatility term structure is explained by the market's expectation for higher implied volatility in the future. Mixon (2007) rejected the expectations hypothesis but found that the slope of the implied volatility term structure was able to forecast short-dated implied volatility. After correcting their model for the variance risk premium, the results improved but not enough to satisfy the expectation hypothesis. An important insight from the research of Mixon (2007) is that the implied volatility term structure is based on the risk neutral measure, but real world implied and realized volatility is based on the objective measure.[5] The differences in the VIX term structure between the risk neutral and objective measures is discussed by Nossman and Wilhelmsson (2009) and Simon and Campasano (2014).

Nossman and Wilhelmsson (2009) find stronger evidence of the expectations hypothesis relative to Mixon (2007). They use VIX futures and conclude that the expectations hypothesis holds for VIX futures over a 1-to-21-day forecasting horizon after adjusting for the variance risk premium. The variance risk premium is negative and is derived by estimating the risk neutral and objective parameters using a constant elasticity stochastic variance model with jumps in the variance.[6] The results show that

[5] Options are priced from the risk neutral measure which imply arbitrage-free pricing, a complete market, and discounted security values that follow a martingale process. Objective measures are probability densities based on actual price innovations.
[6] The constant elasticity stochastic variance model is a diffusion process with instantaneous volatility. The model measures the mean reversion and instantaneous volatility of the variance process.

the variance risk premium is larger for long-dated option expirations when VIX spot is high. Additionally, Nossman and Wilhelmsson (2009) find that the negative correlation between VIX futures and the S&P 500 Index is enough to incentivize investors to pay a variance risk premium for being long an instrument that will hedge S&P 500 Index losses.

Using a normal inverse Gaussian maturity dependent risk premium model, Huskaj and Nossman (2013) suggest that the variance risk premium for short-dated VIX futures contracts is negative while being positive for long-dated contracts. This is supported by showing that the beta[7] and correlation of VIX futures contracts and VIX declines as the time to expiration increases.

Johnson (2017) strongly rejects the expectations hypothesis for the VIX term structure. He proposes that changes in the term structure are due to variation in the variance risk premia embedded in the option contracts that are used to compute the VIX. Johnson (2017) shows that longer maturity VIX contracts have smaller absolute Sharpe ratios[8] compared to the absolute Sharpe ratios of short-dated VIX contracts. This indicates that variance risk is being priced differently at different maturities.

The differences in correlation between VIX futures contracts with different expirations and VIX was also noted by Zhang *et al.* (2010). However, Zhang *et al.* suggest that the VIX and VIX futures relationship can be established by modelling the instantaneous variance using a square root, mean-reverting process with a stochastic

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[7] Slope coefficient estimated from an ordinary least squares regression.
[8] Sharpe ratio is a measure of return per unit of standard deviation ($\frac{r_i - r_{rf}}{\sigma_i}$).

long-term mean level. Zhang *et al.* conclude that the term structure for VIX futures volatility is downward-sloping and is explained by mean reversion of volatility.

Several previously written research papers have discussed the negative returns of VIX futures contracts, and many have suggested the negative returns are due to the upward sloping VIX futures term structure and the associated negative roll down. However, very few papers in the existing literature attribute the daily returns of VIX futures contracts to the two return components of roll down and level (changes in VIX). Also, the existing literature does not provide an analysis of the return attribution for the VIX futures term structure. The current literature focuses the return attribution on the one-month VIX futures contract.

Alexander and Korovilas (2013) emphasize that in most market regimes the VIX futures term structure is in contango and that has eroded the returns of exchange-traded funds that buy and hold VIX futures contracts. Their research focuses on the early redemption and front running issues associated with exchange-traded funds and exchange-traded notes that trade VIX futures contracts. Buetow and Henderson (2016) discuss why a cash VIX market does not exist and how that has led to a decoupling of VIX and VIX futures pricing. They show that replicating a cash VIX market is cost prohibitive due to the average illiquidity and wide bid-ask spreads of the out-of-the-money options used in the VIX calculation.

Simon and Campasano (2014) study the VIX futures basis (slope) and assess its ability to forecast changes in the VIX. Through their research they conclude that the VIX futures basis was not able to predict changes in the VIX but was able to forecast changes in the prices of VIX futures contracts. Using their research findings, they devise a profitable trading strategy that buys and sells short VIX futures contracts. However, they did not quantify how much the roll down return contributed to the negative returns of VIX futures contracts.

The research conducted by Whaley (2013) finds that VIX futures contracts are comprised of two return components, one relating to changes in VIX (level) and the other being roll down. He shows that the calculation of roll down return is deterministic and is quantified as the slope divided by the price of the constant maturity VIX futures contract measured the previous day. Whaley provides summary statistics for the slope measured at various points on the VIX futures term structure. While his work advances the literature of decomposing the VIX futures returns, it does not provide an attribution analysis for the VIX futures term structure.

1.3 Objective

Prior VIX futures research leaves open the question: how much do changes in VIX level returns and roll down returns account for the realized returns of VIX futures? My research proposes to help fill this gap by decomposing the daily returns of VIX futures and attributing the returns to the return components (roll down and level).

My first research objective is to determine what proportion the roll down return represents of the negative total return to VIX futures contracts. I plan to accomplish this using daily VIX futures prices from the CBOE and developing a methodology for decomposing and attributing the VIX futures returns. The expectation is that the roll down return component can account for a large proportion of the negative returns of VIX futures contracts.

The second objective of my research is to evaluate how returns, and return components, vary across the VIX futures term structure. Specifically, I want to answer the question: how do the total return and return components change for each contract on the term structure relative to changes in VIX? I plan to use daily returns of the VIX and the two return components (roll down and level) to conduct a regression analysis that measures the sensitivity of each return component to changes in the VIX.

The third objective of my research is to evaluate whether the regression results will allow me to construct a long-short VIX trading strategy using one-month and three-month VIX futures contracts. More specifically, I want to determine if a VIX futures curve strategy can be combined with a passive S&P 500 Index investment to create a more efficient investment compared to the passive S&P 500 Index. I plan to evaluate the dynamic nature of the estimated beta coefficients to determine the number of three-month VIX futures contracts needed to hedge a single one-month VIX futures contract in different market regimes.

1.4 Approach and Organization

In Chapter 2 I describe the data, calculations for each return component, and provide summary statistics. In Chapter 3 I describe the time series regression model I use to derive the sensitivities of each return component to the returns of the VIX. Chapter 4 is a discussion of the results from the regression and principal component models. Chapter 5 provides the results of combining the S&P 500 Index returns with the returns of a VIX futures curve strategy. In Chapter 6 I offer concluding comments and the potential for future research.

Chapter II
Data

In this chapter I describe my original data by using basic statistical measures and graphs. I then show and discuss how I decompose the total return of the VIX futures contracts into two return components, roll down and level. I provide basic statistical measures for the roll down and level returns and conclude by discussing the autoregressive properties of the various return series.

2.1 VIX Futures Data

The analysis in this chapter is based on the daily total returns for the VIX and VIX futures contracts. In addition, I analyze the daily returns of the VIX futures return components, roll down and level. The total returns are calculated from the closing prices of the VIX and monthly VIX futures contracts which are sourced from Excel data files (file) posted by the CBOE on their website. Each file represents a single monthly VIX futures contract and includes the daily open, close, high and low, and settlement prices. I calculate the total returns and derive the returns of the return components for the one- to six-month VIX futures contracts. Roll down and level returns are not calculated for the VIX given that it is a spot price. All returns are calculated for the period of December 16, 2009 to December 19, 2017. The daily returns of each monthly VIX futures contract are calculated from the settlement date of the prior contract up to and including the day before the settlement date of the current contract.

2.2 The VIX Futures Term Structure

In chapter 1 I discuss that the price of a VIX futures contract represents the expected VIX price, which is equivalent to the annualized 30-day implied volatility of the S&P 500, at expiration of the VIX futures contract. For example, if on April 8, 2018 the July VIX futures contract has a price of 19 this indicates the VIX is expected to be 19 on July 18, 2018.

The VIX term structure is constructed from the closing prices of the VIX and each VIX futures contract with expirations of one to six months. The shape of the term structure is determined by the slope, or difference in price, between two points on the term structure. In this analysis I measure the slope between each point on the term structure starting with VIX and including each successive monthly VIX futures contract. A steep term structure is characterized by large slopes between each point on the term structure, whereas a flat term structure is distinguished by small to zero slopes between each point. An inverted term structure is identified by VIX having the highest price on the term structure and each successive VIX futures contract having a lower price.

Table 2.1: Average Daily Slopes Between VIX and VIX Futures Contracts (December 15, 2009 to December 19, 2017)

	VIX minus 1-month	1-month minus 2-month	2-month minus 3-month	3-month minus 4-month	4-month minus 5-month	5-month minus 6-month	VIX Range
Average Slope	-0.9	-1.1	-0.8	-0.6	-0.5	-0.4	
Percent Negative Slope	84.2%	88.4%	91.7%	92.5%	92.6%	91.5%	
Average Slope by VIX Level							
VIX Level 1st Quintile	-1.3	-1.3	-0.9	-0.7	-0.6	-0.5	9 to 13
VIX Level 2nd Quintile	-1.2	-1.3	-0.9	-0.7	-0.5	-0.5	13 to 15
VIX Level 3rd Quintile	-0.9	-1.3	-1.0	-0.7	-0.7	-0.6	15 to 17
VIX Level 4th Quintile	-0.8	-1.4	-1.0	-0.7	-0.6	-0.5	17 to 21
VIX Level 5th Quintile	-0.1	-0.4	-0.3	-0.2	-0.3	-0.2	21 to 48

Table 2.1 includes the average slopes and average slopes based on a quintile ranking of the VIX price. I calculate the slope by subtracting from the closing price of VIX, or a VIX futures contract, the closing price of the next month VIX futures contract. For example, on January 18, 2013 the VIX closing price was 12.46 and the one-month VIX futures contract closed at 14.65 which corresponds to a slope of -2.19. The average slope is the arithmetic average of the daily slopes. I use Table 2.1 to illustrate that the VIX term structure is upward-sloping and is steepest in the first couple of months.

The average slope can be used to compare the relative steepness between two points on the term structure. In Table 2.1, observe that the average slope between the one-month and two-month VIX futures contracts is -1.1 which is the most negative among all the average slopes. The second most negative average slope (-0.9) is between VIX and the one-month VIX futures contract. They compare to the negative average slope of just -0.4 between the fifth and sixth month VIX futures contract. This highlights that on average the front part of the VIX term structure (i.e., from VIX to the three-month VIX futures contract) is steeper when compared to the rest of the term structure.

The Percent Negative Slope is the percentage of daily calculated slopes for a point on the term structure that are negative. For example, the slope of the VIX and one-month VIX futures contract is negative 84.2% of the time. Similarly, the four-month VIX futures price is lower than the five-month VIX futures price 92.6% of the time. Each highlighting that the VIX term structure is upward-sloping most of the time.

The price of VIX will influence the shape of the VIX term structure. This is shown in the bottom half of Table 2.1 where the average slopes are provided based on a quintile ranking of the VIX closing price. The average VIX closing price from the data set is 17.11. When the VIX price is above its average, the VIX term structure becomes relatively flat. Conversely, when the VIX is below its average the VIX term structure is steeper.

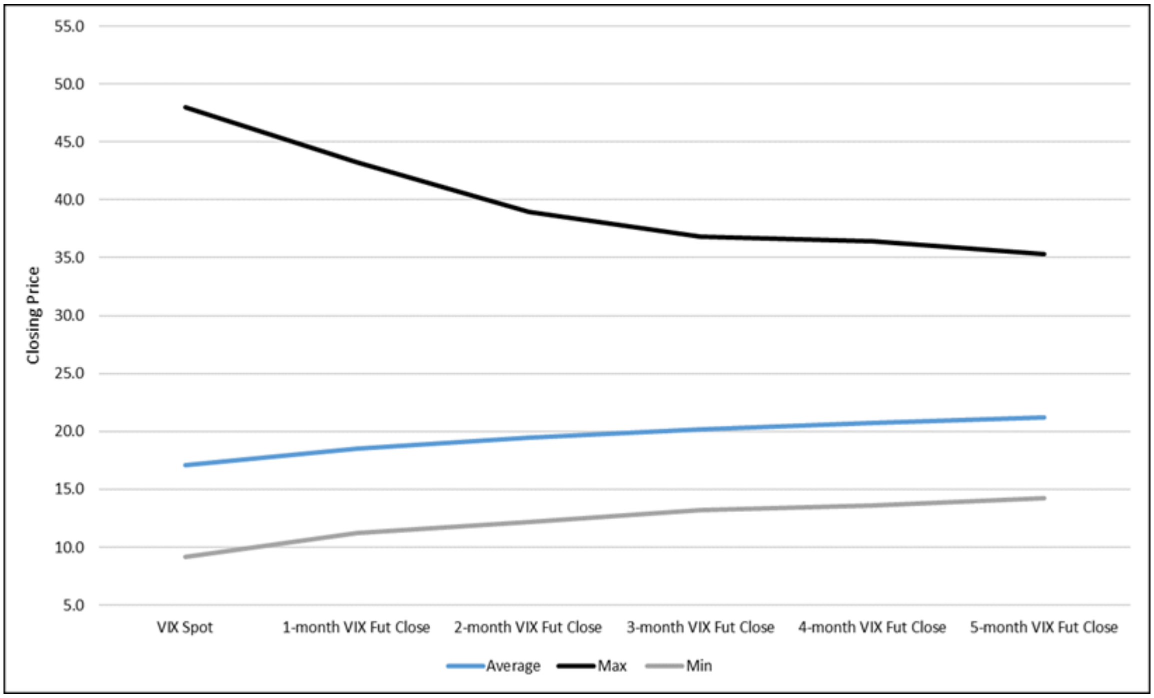
When the VIX is between 9 and 13 the average slopes between VIX and the one-month VIX futures contract (-1.3) and between the one-month and two-month VIX futures contracts (-1.3) are more negative relative to the slopes between the other VIX futures contracts. This illustrates that when the VIX closing price is well below its average the VIX term structure becomes upward sloping and relatively steep. However, when the VIX closing price is between 21 and 48 (fifth quintile) the average slope of the VIX and one-month VIX futures contract (-0.1) is less negative than the slope between the four- and five-month contracts (-0.3) highlighting that the VIX term structure becomes flat (slope near zero) with above average VIX closing prices.

Figure 2.1 portrays the three possible shapes (relatively steep, relatively flat, inverted) for the VIX term structure. These curves display the minimum, maximum, and average prices for the VIX and VIX futures contracts that have a constant number of days until expiration. I calculate the closing price of the constant n^{th} month VIX futures contract as,

$$CM_{n,t} = \frac{D_t}{T_t} * F_{n,t} + \frac{T_t - D_t}{T_t} * F_{n+1,t}, \tag{1}$$

where, $CM_{n,t}$ is the closing price of the constant n^{th} month VIX futures contract at time t, D_t is the number of days remaining until the expiration of the one-month VIX futures contract (including day t but not including the settlement date), T_t is the total number of trading days of the current one-month VIX futures contract from the settlement date of the prior one-month contract until one day prior to the current contract settlement date, and F_n is the closing price of the n^{th} month VIX futures contract.

Figure 2.1: Average, Maximum, and Minimum Daily Closing Price for VIX and Constant *nth* Month VIX Futures Contracts (December 15, 2009 to December 19, 2017)



The line representing the maximum depicts an inverted term structure which is identified by VIX having the highest closing price and the constant five-month VIX futures contract having the lowest closing price. In most cases, an inverted VIX term structure has been the result of large price declines for the S&P 500 Index. An inverted VIX term structure is uncommon and one lasting more than a few months is quite rare.

2.3 Two-Component Return Calculation

A daily total return can be calculated for VIX and each VIX futures contract. Additionally, the daily total return of each VIX futures contract can be decomposed into two return components, roll down and level. The total return of VIX futures contract over the interval [*t-1, t*] is given by,

$$r_t = \frac{F_{n,t}}{F_{n,t-1}} - 1 = \frac{P_{n,t}}{F_{n,t-1}}, \tag{2}$$

where over the interval [*t-1, t*] the terms F_n and P_n denote the *nth* month VIX futures closing price and change in price of the *nth* month VIX futures contract, respectively.

The multiperiod compounded return for *k* periods can be written as,

$$r_t[k] = \left(\prod_{j=0}^{k-1}(1 + r_{t-j})\right) - 1, \tag{3}$$

where $r_t[k]$ is the product of the *k* one-period returns.

The total return can be decomposed as,

$$r_t = Cr_t + Lr_t, \tag{4}$$

where Cr_t and Lr_t denote the roll down return and level return, respectively. More specifically, I define the roll down return at time *t* as,

$$Cr_t = \frac{C_{n,t}}{F_{n,t-1}}, \tag{5}$$

where the term $C_{n,t}$ denotes the one day roll down of the *nth* VIX futures contract at time *t*. The roll down of F_1 is calculated relative to the closing price of VIX and for all other VIX futures contracts (F_{n+1}), the roll down is calculated relative to an *n-1* constant month VIX futures closing price. I calculate the roll down at time *t* as,

$$C_{n,t} = \frac{CM_{n-1,t-1} - F_{n,t-1}}{D_t}, \tag{6}$$

and where D_t is the number of trading days remaining until the expiration of F_1 including day *t* but not including the settlement date. The constant *nth* month VIX futures closing price ($CM_{n,t}$) is calculated as the daily proportion of the *n* and *n+1* VIX futures closing prices (as in equation 1).

I define the level return at time *t* as,

$$Lr_t = r_t - Cr_t. \tag{7}$$

2.4 Characteristics of Return Series

In the following sections I provide and discuss the return characteristics of the VIX and the one-, three-, and five-month VIX futures contracts. I leave out the results of the two-, four-, and six-month VIX futures contracts to keep the results compact but without detracting from the analysis.[9] In Section 2.4.1 I highlight the characteristics of the entire return series and then in Section 2.4.2 I segment the returns and return characteristics based on a quintile ranking of the VIX price. In Section 2.5 I provide the output of testing the various returns series for stationarity and autocorrelation.

[9] Results for the two-, four-, and six-month VIX futures contracts are provided in Appendix B.

Table 2.2: Statistical Characteristics of Daily Returns for VIX and VIX Futures Contracts (December 16, 2009 to December 19, 2017)[ab]

Return Component	Mean	Median	Cumulative	Std Dev	Min	Max	t Value
One-Month Roll Down Return	-0.7%	-0.6%	-	1.3%	-20.1%	12.3%	-26.3
One-Month Level Return	0.5%	0.1%	-	5.1%	-22.0%	35.7%	4.1
One-Month Total Return	-0.3%	-0.8%	-100.0%	5.1%	-20.9%	35.9%	-2.5
Three-Month Roll Down Return	-0.2%	-0.3%	-	0.2%	-1.3%	1.1%	-57.1
Three-Month Level Return	0.1%	-0.1%	-	2.8%	-13.2%	19.6%	0.9
Three-Month Total Return	-0.2%	-0.4%	-99.1%	2.8%	-12.9%	19.5%	-3.1
Five-Month Roll Down Return	-0.1%	-0.1%	-	0.1%	-1.1%	0.4%	-57.8
Five-Month Level Return	0.0%	-0.1%	-	2.1%	-9.0%	13.6%	0.4
Five-Month Total Return	-0.1%	-0.2%	-95.3%	2.1%	-9.0%	13.5%	-2.8
VIX	0.2%	-0.5%	-53.3%	7.7%	-29.6%	50.0%	1.4

[a] The data in Table 2.2 has been rounded.

[b] The t-Value in Table 2.2 is used to test the significance of the sample mean relative to zero at the significance level of 0.05. Specifically, $H_o: \bar{x} = 0$ against $H_a: \bar{x} \neq 0$. The critical value for the one-sided test is equal to 1.645.

2.4.1 Basic Statistics

In Table 2.2 I display basic statistical characteristics for the daily total returns of the VIX and VIX futures contracts as well as the two VIX futures return components. VIX futures contracts with expirations of one, three, and five months are shown to illustrate the differences in return statistics across the term structure.

In Table 2.2, the mean and median roll down returns are negative for each monthly VIX futures contract. The average roll down returns of the one-month VIX futures contract (-0.7%) are more negative than the average roll down returns of the three- (-0.2%) and five-month (-0.1%) contracts, which supports the view that the VIX term structure is on average upward sloping and steeper in the front-end. A futures contract has a finite life and as the time to expiration approaches, the contract loses value if an upward-sloping term persists. Hence, the negative return on an upward-sloping term structure.

Unlike the average roll down returns, the average level returns are positive for each monthly VIX futures contract. Although comparing the mean and median level returns shows that the average level returns are pulled higher by the larger relative changes in the price for VIX. For example, from December 16, 2009 to December 19, 2017 the average VIX price was 17.11 with a maximum price of 48 (180% from average) and minimum price of 9.14 (47% from average). During the same period, the maximum and minimum closing price, and average closing prices for the one-month VIX futures contract were 45, 9.88 and 17.93, respectively.

The average level return of the one-month VIX futures contract (0.5%) compared to the three- and five-month VIX futures contracts (0.1% and 0.0%) shows that the one-month VIX futures contract has the greatest sensitivity to changes in the VIX price. This is also supported by the large difference between the mean and median level returns of the one-month VIX futures contract (0.5% versus 0.1%) relative to the difference in mean and median level returns of the other two contracts.

The standard deviation of returns in Table 2.2 highlight that the total return standard deviation for each monthly VIX futures contract is primarily accounted for by the standard deviation of level returns. In fact, 96.6% of the total standard deviation of the one-month VIX futures contract is accounted for by the level return component. I calculate the percentage contribution of level return standard deviation to the total return standard deviation as the covariance of level and total returns divided by the variance of total returns. The reason that the level return standard deviation accounts for so much of the total return standard deviation is due to the strong positive correlation between the level and total returns (0.97) and the similar return standard deviations between the two return components.

Since VIX is a spot price, it does not have a roll down return component and therefore, it's standard deviation of returns can be considered both level return and total return standard deviation. The standard deviation of the level return declines starting with VIX and as the time to expiration increases (i.e. each successive VIX futures contract). For example, the VIX standard deviation of returns (7.7%) is 1.5 times greater than the standard deviation of the one-month VIX futures contract level return (5.1%) and more than 3.5 times the standard deviation of the five-month VIX futures contract level return (2.1%). This shows that the term structure of realized volatility for the VIX and VIX futures contracts is inverted. Zhang *et al* (2010) find similar results and conclude that the downward sloping realized volatility term structure of VIX futures contracts is explained by the volatility mean-reversion process.

The t-values shown in Table 2.2 are used to test the significance of the mean return for each return component relative to zero. The one-tailed critical value at the 0.05 level is 1.645. All but the VIX, three-month level, and five-month level returns are statistically different from zero at the 0.05 level. The t-values for the return components of the one-, three-, and five-month VIX futures contracts highlight the relative magnitude of loss from the roll down component and standard deviation of the level component.

2.4.2 Returns by VIX Ranking

I use a quintile ranking of the VIX price to gain perspective for the various return components under different implied volatility regimes by inspecting the average return, minimum and maximum returns, and standard deviation of each return component for the various VIX futures contracts. The quintile ranking uses the daily closing prices of VIX published by the CBOE.

Table 2.3: Daily Returns and Standard Deviation of the One-Month, Three-Month and Five-Month VIX Futures Contracts Ranked by VIX Quintile

(December 16, 2009 to December 19, 2017)[a]

		Roll Down Returns			Level Returns			Total Returns			
		1-month Roll Down Return	3-month Roll Down Return	5-month Roll Down Return	1-month Level Return	3-month Level Return	5-month Level Return	1-month Total Return	3-month Total Return	5-month Total Return	VIX
First Quintile	Avg Return	-1.4%	-0.3%	-0.2%	0.2%	-0.3%	-0.2%	-1.2%	-0.6%	-0.4%	
	Std Dev	1.5%	0.2%	0.1%	2.6%	1.6%	1.1%	2.7%	1.5%	1.1%	
	Min	-20.1%	-1.2%	-0.7%	-10.1%	-11.1%	-7.6%	-19.3%	-11.3%	-7.6%	9.1
	Max	0.3%	1.1%	0.0%	8.5%	4.0%	2.6%	6.1%	3.5%	2.3%	12.7
Second Quintile	Avg Return	-1.0%	-0.3%	-0.2%	0.0%	-0.3%	-0.3%	-0.9%	-0.6%	-0.4%	
	Std Dev	1.1%	0.1%	0.1%	3.6%	1.8%	1.4%	3.4%	1.8%	1.4%	
	Min	-10.2%	-0.9%	-1.1%	-19.9%	-7.9%	-6.7%	-16.5%	-8.1%	-6.7%	12.7
	Max	3.4%	0.0%	0.0%	14.0%	4.1%	3.5%	14.1%	3.8%	3.1%	14.5
Third Quintile	Avg Return	-0.7%	-0.3%	-0.2%	0.1%	-0.2%	-0.1%	-0.6%	-0.4%	-0.3%	
	Std Dev	1.2%	0.2%	0.1%	4.7%	2.6%	1.8%	4.8%	2.6%	1.8%	
	Min	-10.7%	-1.3%	-0.8%	-14.1%	-7.6%	-6.6%	-13.1%	-7.8%	-6.7%	14.5
	Max	12.3%	0.0%	0.0%	20.1%	10.5%	6.9%	32.4%	10.1%	6.8%	16.9
Fourth Quintile	Avg Return	-0.5%	-0.3%	-0.1%	0.6%	0.3%	0.1%	0.1%	0.0%	0.0%	
	Std Dev	0.9%	0.2%	0.1%	5.9%	3.1%	2.3%	5.8%	3.1%	2.2%	
	Min	-8.9%	-0.9%	-0.5%	-21.0%	-12.8%	-9.0%	-20.9%	-12.8%	-9.0%	16.9
	Max	3.8%	0.2%	0.0%	33.7%	10.6%	8.1%	31.0%	9.9%	7.8%	20.6
Fifth Quintile	Avg Return	-0.2%	-0.1%	-0.1%	1.3%	0.7%	0.6%	1.2%	0.7%	0.5%	
	Std Dev	1.2%	0.2%	0.1%	7.1%	4.0%	3.0%	7.1%	4.0%	3.0%	
	Min	-7.3%	-1.2%	-0.8%	-22.0%	-13.2%	-8.7%	-16.8%	-12.9%	-8.7%	20.6
	Max	5.2%	0.6%	0.4%	35.7%	19.6%	13.6%	35.9%	19.5%	13.5%	48.0

[a] The data in Table 2.3 has been rounded.

The data in Table 2.3 offers some interesting insight regarding the characteristics of the VIX term structure. For example, the large negative roll down returns of the one-month VIX futures contract (-1.4%) compared with the negative roll down returns of the three- (-0.3%) and five-month (-0.2%) VIX futures contracts highlight the fact that the front-end of the VIX term structure is generally steeper when the VIX is in the first quintile (below 12.7) of its price history. However, when the VIX price is in the fifth quintile (greater than 20.6) the average roll down returns of the one-month VIX futures contract (-0.2%) is similar to the average roll down returns of the three- (-0.1%) and five-month (-0.1%) VIX futures contracts. This illustrates the fact that the average VIX term structure is flat when the VIX price is high.

The standard deviation and the range of roll down and level returns vary greatly depending on the level of the VIX closing price. For example, the standard deviation of the one-month level returns in the first quintile (2.6%) is less than half of the standard deviation in the fifth quintile (7.1%). This demonstrates that the VIX is more volatile when the VIX is above its average price.

Similarly, the range of level returns for the one-month contract significantly expand between the first (18.6%) and fifth (57.7%) quintiles. However, the standard deviation of the one-month roll down returns in the first quintile (1.5%) is slightly greater than the fifth quintile (1.2%). And the range of roll down returns for the one-month contract decline between the first (20.4%) and fifth (12.5%) quintiles. These results support the view that the VIX term structure becomes flat when the VIX price is above its average price of 17.11.

2.4.3 VIX Futures Profit and Loss

The returns shown previously are helpful in drawing conclusions about the total return distribution of VIX futures contracts and return distributions for the roll down and level return components. The cumulative profit and loss (P&L) from buying and holding a VIX futures contract aid in illustrating the differences between the roll down and level components.

I use Table 2.4 to show the cumulative profit and loss (P&L) from buying and holding a one-, three-, and five-month VIX futures contract. Like the returns calculated earlier, a buy-and-hold VIX futures strategy involves buying and holding the current month contract up to and including one-day prior to the settlement date. On the day prior to the settlement date, the buy-and-hold strategy sells the current contract at the closing price and simultaneously buys the new month contract at the closing price.

The total, roll down, and level P&Ls are derived from equations 2, 6, and 7 and are indicative of a buy-and-hold strategy with one VIX futures contract. For example, during December 16, 2009 to December 19, 2017 the one-month VIX futures contract lost 127.89 points or -$127,885 (-127.89*1,000). During the same period the roll down component lost 228.753 points (-$228,753) and the level component gained 100.868 points ($100,868). The P&Ls shown in Table 2.4 are the summed daily P&Ls for each component during December 16, 2009 to December 19, 2017 and for each component from quintile one and five. The quintiles were determined by a quintile ranking of the VIX price.

Table 2.4: VIX Futures Cumulative P&L (December 16, 2009 to December 19, 2017)

	Cumulative P&L		
VIX Futures Contract	**Roll Down**	**Level**	**Total**
One-Month	-$228,753	$100,868	-$127,885
Three-Month	-$93,732	$7,687	-$86,045
Five-Month	-$59,712	-$878	-$60,590
Quintile One (VIX: 9.1-12.7)			
One-Month	-$71,323	$4,588	-$66,735
Three-Month	-$20,705	-$17,905	-$38,610
Five-Month	-$13,263	-$13,407	-$26,670
Quintile Five (VIX: 20.6-48.0)			
One-Month	-$14,106	$90,506	$76,400
Three-Month	-$7,722	$59,232	$51,510
Five-Month	-$5,768	$48,358	$42,590

The P&L data in Table 2.4 provide several interesting insights regarding buying and holding VIX futures contracts. For example, the negative roll down P&L of the one- and three-month VIX futures contracts (-$228,753, -$93,732) is entirely responsible for the negative total P&L of each contract. The negative roll down P&L of the five-month VIX futures contract (-$59,712) accounts for 98.6% of the negative total P&L. Additionally, the five-month contract has a negative level P&L (-$878) whereas the one- and three-month contracts have a positive level P&L ($100,868, $7,687).

In Table 2.4, I provide the P&L of each component from quintiles one and five to show their differences based on the VIX price. As discussed earlier, when the S&P 500 Index declines the VIX price increases and the VIX term structure becomes flatter and in extreme cases, the VIX term structure will invert. For example, the roll down P&Ls of quintile one are more negative than the roll down P&Ls of quintile five highlighting a steeper VIX term structure in quintile one. Additionally, the differences in roll down

P&L between the monthly VIX futures contracts show that when the VIX price is low (quintile one) the VIX term structure is steeper as illustrated by the greater roll down losses.

The difference in level P&Ls between quintiles one and five highlight that the level component produces the largest P&L when the VIX price is high. Additionally, the level P&L is greater than the roll down P&L for each VIX futures contract in quintile five. This finding implies that buying and holding a VIX futures contract when the VIX price is at or above 20.6 results in a positive total P&L.

2.5 Stationarity and Autocorrelation of Returns

To evaluate the characteristics of the return series I assess the stability of the mean and variance as well as correlation of returns with respect to time. A return series that exhibits a mean and variance that is invariant with time and where the covariances between x_t and x_{t+h} only depend on the distance between them is considered covariance stationary. Covariance stationary time series offer a greater ability to make inferences regarding future observations.

Time series regression analysis relies on a covariance stationary process in the residuals. If the data series are not covariance stationary, or adjusted to be covariance stationary, then the estimated variance of the regression coefficient(s) can be biased leading to invalid t-statistics. For example, an independent variable that has a positive autocorrelated series can result in an underestimation of the beta coefficient variance, leading to overstated t-statistics. In finance it is common to assume that return series are covariance stationary since the derived returns are a form of detrending transformation.

However, using return series does not jettison the need to check and possibly correct the time series regression model for autocorrelated or heteroskedastic residuals.

In this section I discuss the results of the Augmented Dicky Fuller (ADF) unit root test for stationarity in the daily returns of the VIX, and the daily total returns and returns of each return component (roll down and level) for the one-month, three-month, and five-month VIX futures contracts. Additionally, I present the autocorrelation test results for the returns of the VIX and one-month VIX futures contract. The partial autocorrelation test results are provided in Appendix C.

The ADF test is conducted for the VIX returns and the various returns (total, roll down, level) of the VIX futures contracts. The ADF test includes a test of the null hypothesis of a unit root ($H_0: \gamma = 0$) versus the alternative hypothesis ($H_a: \gamma < 0$). The equation for the ADF-test statistic can be written as,

$$ADF\ test = \frac{\hat{\gamma}-1}{std(\hat{\gamma})}, \tag{8}$$

where $\hat{\gamma}$ is estimated in a least-squares regression (Tsay, 2010, pp. 77). The estimating equation is written as,

$$\Delta y_t = \mu + \beta t + \gamma * y_{t-1} + \sum_{i=1}^{p-1} \emptyset_i \, \Delta y_{t-i} + \varepsilon_t, \tag{9}$$

where $\emptyset_i = -\sum_{k=i+1}^{p} \gamma_k$, $\gamma *= (\sum_{j=1}^{p} \gamma_i) - 1$, and Δy is the daily change in y (the total return or return component).

Since each time series is comprised of daily derived returns, stationarity is not an issue as most return series are covariance stationary. In fact, the ADF test for the returns of the VIX and VIX futures contracts confirm this with each test result rejecting the null hypothesis (non-stationary) for the alternative hypothesis (stationary) at all levels of significance.

I now test the linear dependence of returns (*r*) at time *t* with returns at time *t*-*i* by estimating the correlation coefficient. The correlation coefficients estimated from ρ_1 to ρ_i are called the sample autocorrelation function. The equation for the sample autocorrelation function can be written as,

$$\rho_i = \frac{Cov(r_t, r_{t-i})}{\sqrt{Var(r_t)Var(r_{t-i})}}, \qquad (10)$$

where ρ_i is the correlation coefficient between r_t and r_{t-i} (Tsay, 2010, pp. 31). This is known as the lag-i autocorrelation of r_t.

Figure 2.2: Plot of the Autocorrelations for the VIX Daily Return (December 16, 2009 to December 19, 2017)

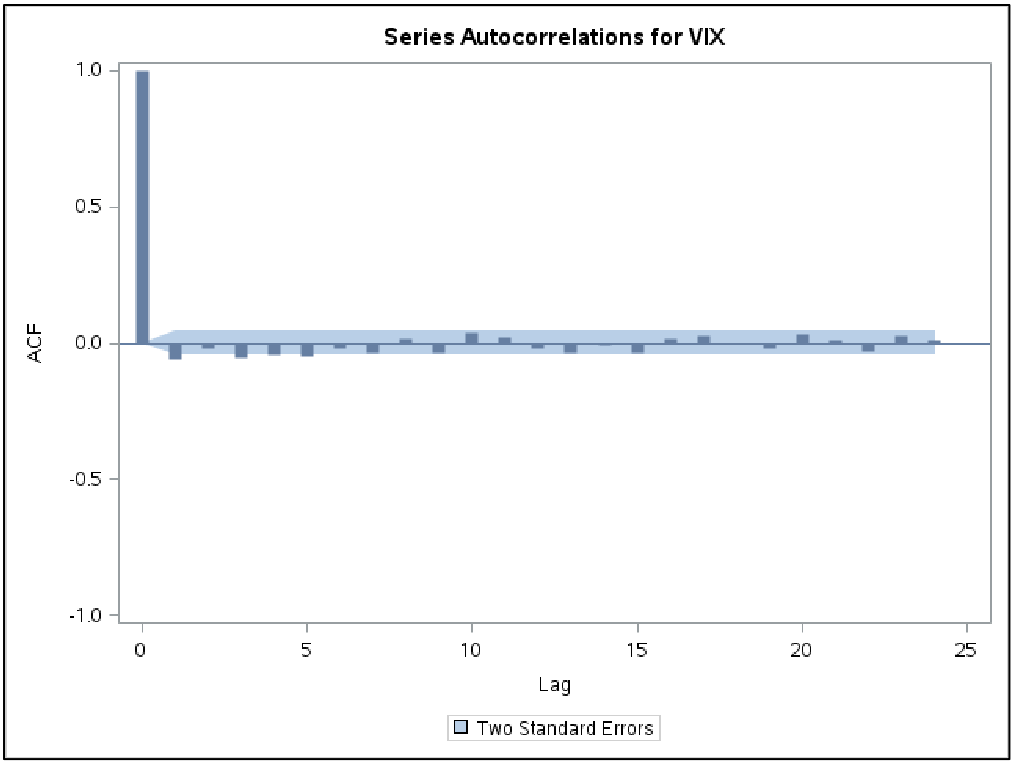


Figure 2.2 is a graphical representation of the autocorrelations for the VIX return series to the 24th lag. Beginning at lag 1, each bar represents the correlation coefficient between the *nth* lagged return and the return at time t. The light blue upper and lower bounds represent the 5% confidence interval for autocorrelation. The VIX return series appears to have some mild negative autocorrelation at the 1st and 3rd lags. Regressions that use the VIX return series will most likely need to correct for autocorrelated residuals and inferences regarding the standard deviation of VIX returns may need to correct for the autocorrelated return series.

Figure 2.3: Plot of the Autocorrelations for the Daily One-Month VIX Futures Roll Down Return

(December 16, 2009 to December 19, 2017)

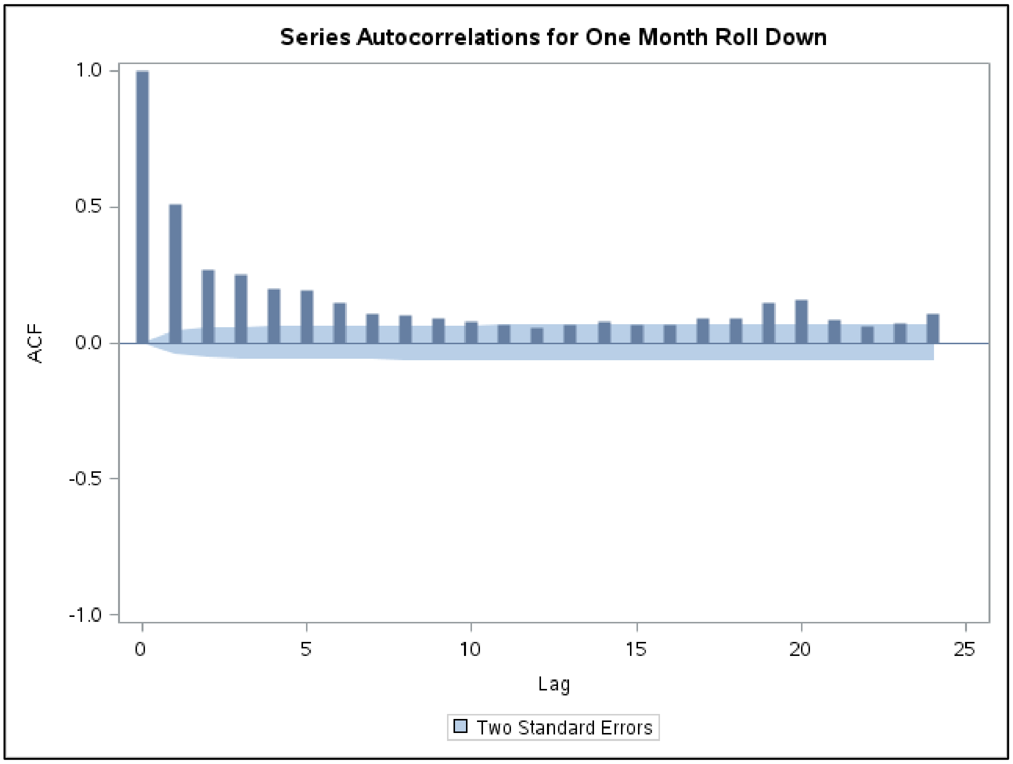


Figure 2.3 illustrates the autocorrelations for the roll down returns of the one-month VIX futures contract. These returns are highly autocorrelated and the return series follows an autoregressive process with lags 1 through 10 positive and significant at the 5% level. The partial autocorrelations for lags 1, 3, and 5 are significant at the 5% level indicating that the autoregressive process is driven by the first few lags. The residuals of any time series regression analysis using the one-month VIX futures roll down returns will need to be carefully examined and corrected for autocorrelation. Otherwise, the variance of the estimated beta coefficient will be misstated leading to invalid t-statistics. Additionally, the one-month VIX futures roll down return series should be corrected for autocorrelation before inferences regarding the standard deviation are made.

Figure 2.4: Plot of the Autocorrelations for the Daily One-Month VIX Futures Level Return

(December 16, 2009 to December 19, 2017)

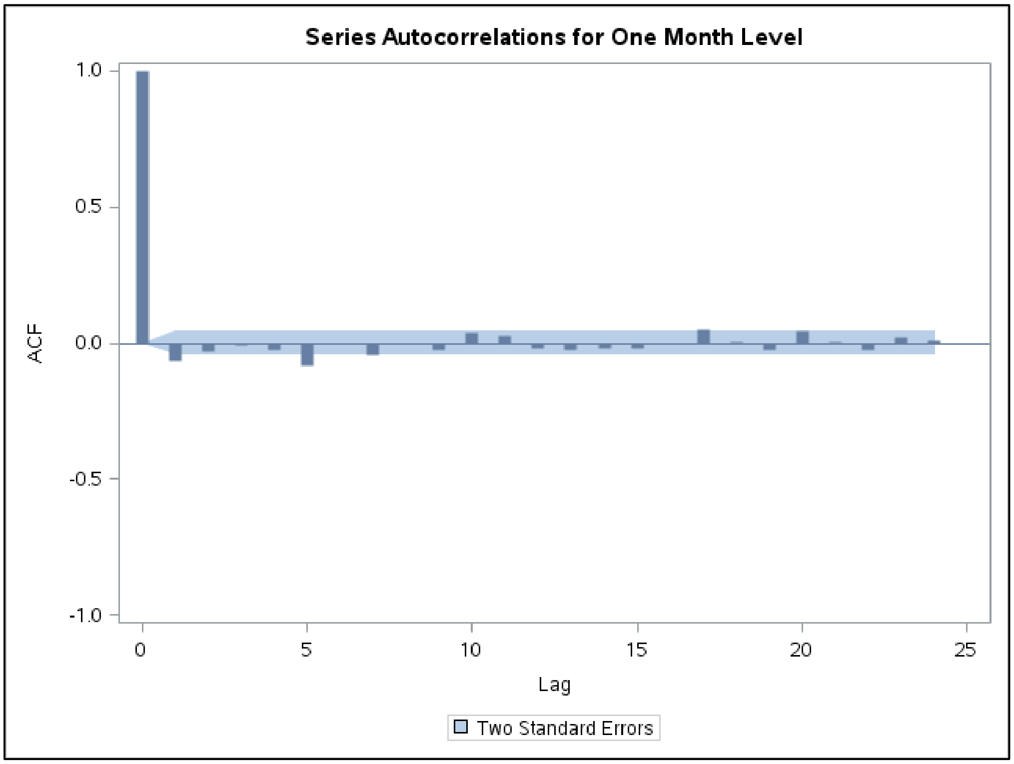


Figure 2.4 illustrates the autocorrelations for the one-month VIX futures level return series. The return series appears to have mild negative autocorrelation at the 1st and 5th lag that is significant at the 5% confidence level. Any inference the standard deviation of returns or regression results for the one-month VIX futures level return may need to be corrected for autocorrelation.

Figure 2.5: Plot of the Autocorrelations for the Daily One-Month VIX Futures Total Return

(December 16, 2009 to December 19, 2017)

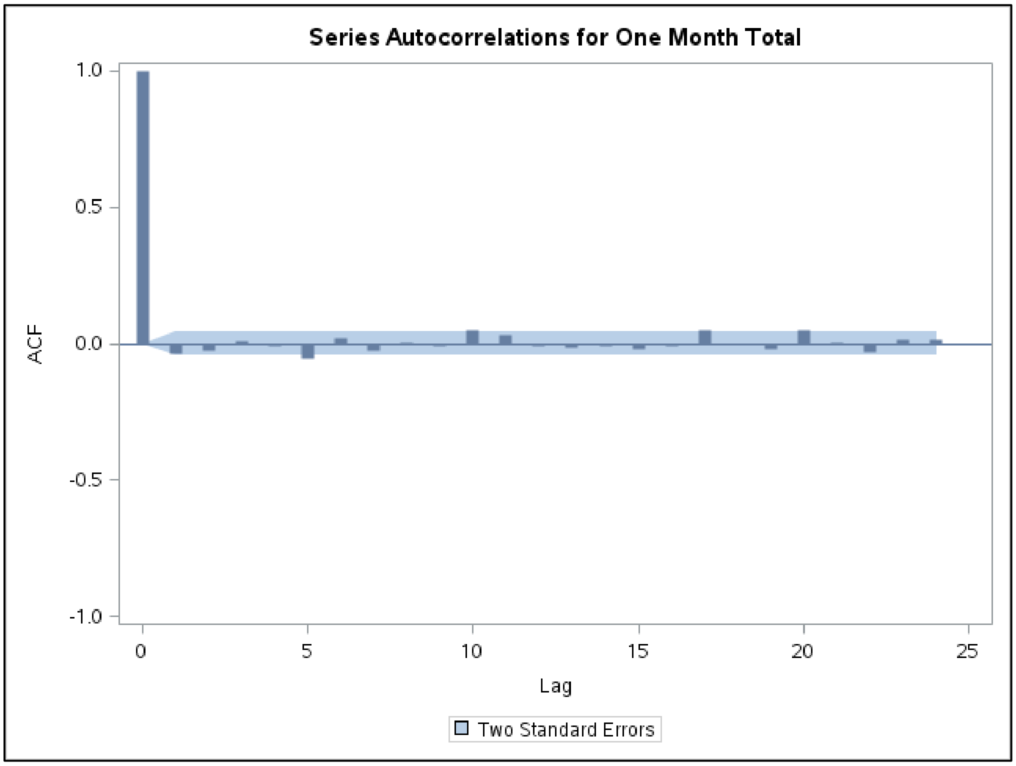


Figure 2.5 illustrates the autocorrelations for the total returns of the one-month VIX futures contract. The total return series of the one-month VIX futures contract does not appear to be autocorrelated given that none of the correlation coefficients for the lags are statistically different from zero. Therefore, the one-month VIX futures contract total return series is considered covariance stationary. Although it is interesting to note that the roll down and level returns of the one-month VIX futures contract were not covariance stationary as shown in Figures 2.3 and 2.4.

Chapter III
Methods

I begin Chapter 3 by discussing the assumptions of a time series regression model. I discuss covariance stationarity and highlight its importance with the asymptotic properties of a least squares regression. Next, I review and provide details for the univariate regression models that will be estimated. I conclude Chapter 3 by providing my expectations for the output of the regression models.

3.1 Time Series Regression Model

Time series regression models are used extensively in economics and finance to identify and explain the relationship between variables that are indexed by time. An example of a time series regression model in finance is one that uses the returns of the S&P 500 Index to explain the returns of a long/short equity hedge fund. Here, the returns of the long/short equity hedge fund represent the dependent variable and the returns of the S&P 500 Index represent the explanatory variable. Using this simple example, the time series least squares regression model can be written as,

$$y_t = \alpha + \beta x_t + u_t,$$

where y is the return of the long/short equity hedge fund at time t, x is the return of the S&P 500 Index at time t, β is the estimated beta coefficient, and u is the error term.

There are five assumptions that must hold for the estimated beta coefficient to be the best linear unbiased estimator (β) of the regression. The term "best" has the meaning minimum variance and unbiased means that the estimates are unbiased across all time periods (Wooldridge, 2013).

The first assumption of a time series regression model is that the model parameters are linear. The second assumption necessitates that the independent variables are not constant and are not perfectly correlated. The third assumption of a time series regression model requires that the expected value of the error term, u, is zero for all time periods. Assumption four imposes that the variance of error term is constant across all time periods and assumption five requires that the error term be serially uncorrelated. If only assumptions one, two, and three hold then the time series regression estimator, β, is considered unbiased but is not considered the best linear unbiased estimator (Wooldridge, 2013).

The law of large numbers and central limit theorem applied to cross-sectional data can be applied to time series data through sample averages and covariance stationarity. As discussed in 2.4.3, a return series (r_t) is said to be covariance stationary if both the mean and variance of r_t are time invariant and the covariance between r_t and r_{t-h}, where h is an arbitrary integer, only depends on h. Covariance stationarity requires that the correlation between r_t and r_{t-h} goes to zero sufficiently quickly as h increases. As we will see, covariance stationarity is required for asymptotically valid regression results (Tsay, 2010).

As the size of the return series (T) goes to infinity, asymptotic distribution theory can be used to describe the first and second moments (parameter estimates) of the dependent and explanatory variables. The probability limit (e.g., 95% probability) that the sample parameter estimates of the dependent and explanatory variables converge to their true parameter estimates can be achieved with a sufficiently large *T*. When the probability limit is reached, the parameter estimates are considered consistent. Although a return series can be serially correlated, the sample parameter estimates can converge to the true parameter estimates based on the probability limit provided the return series is covariance stationary (Hamilton, 1994).

The first assumption of a time series regression model, linearity, is enhanced by requiring that the independent and dependent variables be covariance stationary. The third assumption of the time series regression model is relaxed by requiring that the independent variable is contemporaneously exogenous. The fourth assumption is relaxed so that the requirement is for contemporaneous homoskedastic errors conditioned on the independent variable at time *t*. The assumptions of no perfect collinearity among the independent variables and zero correlation in the errors across time remain. Provided the time series regression model can meet these assumptions, the beta coefficient estimates are consistent, and the various inferences (e.g., t-statistics, F-statistics) are asymptotically valid (Wooldridge, 2013).

3.2 Model Estimation

I use univariate time series regressions to estimate the daily total returns and return components, roll down and level, for the one-, three-, and five-month VIX futures contracts based on contemporaneous daily returns of the VIX. Additionally, I use univariate time series regressions to estimate the daily returns of VIX from contemporaneous daily total returns of the S&P 500 Index. Three equations are estimated for each VIX futures contract and one is estimated for the VIX, which yields ten regression models estimated in total. The first set of equations are written as,

$$OVF_{i,t} = \alpha_i + \beta O_i VIX_t + \epsilon_{i,t}, \tag{11}$$

where *t* is the counter for time and *i=1,2,3* represents daily total return (*1*), roll down return (*2*), and level return (*3*) of the one-month VIX futures contract (*OVF*), α_i is the intercept of the regression equation, BO_i is the beta coefficient, and VIX_t is the daily return of the VIX. The second set of equations are written as,

$$TVF_{j,t} = \gamma_j + \beta T_j VIX_t + \epsilon_{j,t}, \tag{12}$$

where *t* is a counter for time and *j=1,2,3* represents daily total return (*1*), roll down return (*2*), and level return (*3*) of the three-month VIX futures contract (*TVF*), γ_j is the intercept of the regression equation, BT_j is the beta coefficient, and VIX_t is the daily return of the VIX. The third set of equations are written as,

$$FVF_{k,t} = \delta_k + \beta F_k VIX_t + \epsilon_{k,t}, \tag{13}$$

where *t* is a counter for time and *k=1,2,3* represents daily total return (*1*), roll down return (*2*), and level return (*3*) of the five-month VIX futures contract (*FVF*), δ_k is the intercept of the regression equation, BF_k is the beta coefficient, and VIX_t is the daily return of the VIX. The final set of equations are written as,

$$VIX_t = \theta_v + \beta_v SPX_t + \epsilon_{v,t}, \tag{14}$$

where VIX_t is the daily return of the VIX and SPX_t is the daily total return of the S&P 500 Index.

I expect the estimated beta coefficients for the total returns of the VIX futures contracts to be positive. Additionally, I anticipate the beta coefficient for the total returns of the one-month VIX futures contract being the largest of the VIX futures contracts and then declining for each successive VIX futures contract. My expectation is based on the inverted term structure for the realized volatility of VIX futures contracts which I discussed earlier.

In addition, I anticipate that the level returns will have a positive and larger estimated beta coefficient compared to the roll down returns given their contribution to the volatility of total returns of VIX futures contracts. The estimated beta coefficient for level returns of the one-month VIX futures contract should be larger than the level return beta coefficients of the other VIX futures contracts given the inverted realized volatility term structure of VIX futures contracts.

My expectations are supported by the findings of Huskaj and Nossman (2013) and Zhang (2010) where they find that the realized volatility of the VIX futures term structure is inverted and the estimated correlation coefficient between VIX and the VIX futures contracts declines as the time to expiration increases.

3.3 Regression Model Testing and Calibration

The residuals of each regression are tested for autocorrelation and heteroskedasticity. I discuss the testing methodologies and techniques used to correct for significant autocorrelation and heteroskedasticity.

3.3.1 Testing for Autocorrelated Residuals

Each time series regression model is tested for significant (two standard deviations) autocorrelation in the regression residuals. Autocorrelated residuals violate assumption five of the time series regression model assumptions and cause issues with the regression estimates and interpretation of the regression results. For example, autocorrelated residuals result in biased variance estimates of the beta coefficient leading to invalid t-statistics. The estimated correlation coefficient of each residual is written as,

$$\rho_h = \frac{Cov(e_t, e_{t-h})}{\sqrt{Var(e_t)Var(e_{t-h})}} = \frac{Cov(e_t, e_{t-h})}{Var(e_t)}, \tag{15}$$

where ρ_{h} is the correlation coefficient of the error term for the time series regression between e_t and e_h and where h is an arbitrary integer (Tsay, 2010, pp. 31).

The result from equation 19 can be used to test the null hypothesis that ρ_h is equal to 0 versus the alternative hypothesis that it is not equal to 0. The test statistic can be written as,

$$t\ ratio = \frac{\rho_h}{\sqrt{(1+2\sum_{i=1}^{h-1}\rho_i^2)/T}}, \tag{16}$$

where T is the total number of observations (Tsay, 2010, pp. 32). The null hypothesis would be rejected if the absolute value of the *t ratio*, |*t ratio*|, is greater than $Z_{\alpha/2}$ (two-tailed test critical value).

Each residual autocorrelation test is evaluated using the partial autocorrelations of the regression residuals. The partial autocorrelations remove the indirect correlations of the autocorrelation test. Meaning, the partial autocorrelation of the lag 2 residual ($\emptyset_{22}$) is the autocorrelation coefficient of e_t and e_{t-2} removing the autocorrelation of e_t and e_{t-1}. The partial autocorrelations are used in conjunction with the autocorrelations to identify the autoregressive process. The partial autocorrelation can be written as,

$$\emptyset_{11} = \rho_1, \tag{17}$$

$$\emptyset_{22} = \frac{\rho_2 - \rho_1^2}{1-\rho_1^2}, \tag{18}$$

for lags 1 and 2, and for additional lags,

$$\emptyset_{ss} = \frac{\rho_s - \sum_{j=1}^{s-1} \emptyset_{s-1},j\rho_{s-j}}{1 - \sum_{j=1}^{s-1} \emptyset_{s-1},j\rho_j}, s = 3,4,5 \ldots. \tag{19}$$

where $\emptyset_{ss}$ is the partial autocorrelation of e_t and e_{t-s} (Enders, 2015, pp. 65).

The statistical software SAS is needed to run and analyze each regression model. For each regression SAS calculates the autocorrelation function (ACF) and partial autocorrelation function (PACF). SAS produces separate ACF and PACF plots that contain the autocorrelations and partial autocorrelations up to the 25th lag. Each plot includes two standard error upper and lower bounds, which makes it easy to identify the presence of significant (5% level) autocorrelation. Visually inspecting the ACF and PACF plots serve as a reasonable substitute to calculating the Ljung-Box Q statistic.

3.3.2 Correcting for Autocorrelated Residuals

To correct for autocorrelated residuals, I use an autoregressive error model in SAS, which relies on maximum likelihood estimates. The autoregressive error model corrects for residual autocorrelation up to the specified lag. I determine the specified lag from the ACF and PACF of the least squares regression and base it on the last residual that is significantly autocorrelated at the 5% level. The corrected residuals from the autoregressive error model are saved and used to test for heteroskedasticity.

3.3.3 Testing for Heteroskedastic Residuals

Tests for residual heteroskedasticity are conducted using the Ljung-Box Q(*m*) and Lagrange Multiplier tests after the residuals have been corrected for significant autocorrelation. I test for a null hypothesis of no heteroskedasticity against an alternative hypothesis that heteroskedasticity exists at the 5% level. The null hypothesis is rejected if the p-value is less than or equal to 5%. The Ljung-Box Q(*m*) statistic can be written as,

$$Q(m) = \frac{T(T+2)\sum_{k=1}^{m} a_k^2}{T-k}, \tag{20}$$

where T is the sample size, a_k^2 is the squared residual of the *kth* lag, and m is the number of lags being tested (Tsay, 2010, pp. 32).

The Lagrange Multiplier statistics for heteroskedasticity can be written as,

$$\text{LM statistic} = n * R_{u^2}^2, \tag{21}$$

where n is the sample size, and $R_{u^2}^2$ is the r-squared from a regression of the OLS squared residuals on the explanatory variables (Woolridge, 2013, pp. 277).

3.3.4 Correcting for Heteroskedastic Errors

To correct for significant heteroskedasticity I use a GARCH model. In the presence of autocorrelated residuals I estimate the models as a combined autoregressive error model and the GARCH model in SAS. The GARCH model is estimated as a GARCH (q=1, p=1) model.

The SAS AR(*m*)-GARCH(*q*,*p*) regression model can be written as,

$y_t = x_t'\beta + v_t,$

where $v_t = \varepsilon_t - \varphi_1 v_{t-1} - \cdots - \varphi_m v_{t-m},$

and $\varepsilon_t = \sqrt{h_t} e_t,$

where $h_t = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \gamma_j h_{t-j},$

where v_t is the corrected residual, φ_m is the autoregressive parameter at lag m, and h_t is the conditional variance.[13]

3.4 Principal Component Analysis of Returns

Principal component analysis (PCA) reduces the dimensionality of the covariance matrix by uniquely combining the return vectors of the covariance matrix into a smaller set of components that explain most of the variance of the covariance matrix. More specifically, a covariance matrix ($\sum_r$) of k-dimensional return vectors, where $r = (r_1, \ldots., r_k)'$, is reduced to a smaller set y_i through a unique linear combination of r_i and w_i, where $w_i = (w_{i1}, \ldots., w_{ik})'$.

PCA is an orthogonal component analysis which requires that each y_i component be completely uncorrelated with all other y_i components. The first principal component will explain the greatest amount of variance of the covariance matrix compared to the

[13] http://support.sas.com/documentation/cdl/en/etsug/63939/HTML/default/viewer.htm#etsug_autoreg_sect024.htm

other components. The second component will explain the second greatest amount of variance followed by the third component and so on. The variance and covariance of the *ith* principal component (y_i) can be written as follows,

$$Var(y_i) = w_i' \textstyle\sum_r w_i, \tag{22}$$

for *i* = 1,…,*k*, and

$$Cov(y_i, y_j) = w_i' \textstyle\sum_r w_i, \tag{23}$$

for *i*,*j* = 1,….,*k* (Tsay, 2010, pp. 484).

If we let the eigenvalues of $\sum_r$ be represented by λ_i, where $\lambda_i = (\lambda_1, \ldots., \lambda_k)'$, then we can show that the proportion of variance explained by the *ith* principal component is written as,

$$\frac{Var(y_i)}{\sum_{i=1}^{k} Var(r_i)} = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_k} \text{ (Tsay, 2010, pp. 484).} \tag{24}$$

I conduct the principal component analysis using the total returns of the VIX futures contracts and VIX returns to identify latent market variables that exist among the return series of the VIX term structure. For the VIX futures contracts I use the total returns of the one- to six-month contracts. My hypothesis is that the PCA results will show that most of the variance of the VIX term structure can be explained by three principal components (level, slope, and curvature). This expectation is backed by the findings of Litterman and Scheinkman (1991), who find that three common factors (level, slope, curvature) explain more than 90% of the variance of US government bond returns.

Chapter IV
Results

I briefly summarize the objectives of the regression analyses. Then I provide and discuss the regression results highlighting the beta coefficients and goodness-of-fit measures for the roll down, level, and total returns of the VIX futures contracts. Next, I reference results from Chapters 2 and 4 to show that the realized volatility of the VIX term structure is downward sloping. I conclude by discussing the results of the principal components analysis.

4.1 Regression Results

The objective of the regression analysis is to study the relationship between the return components (roll down, level, total) of the VIX futures contracts at different points on the VIX futures term structure and the returns of VIX. I am interested to learn if the linear relationship between VIX and the VIX futures contracts decreases as the time to expiration for the VIX futures contracts increases. I use the regression models to estimate the daily roll down, level, and total returns of the one-, three-, and five-month VIX futures contracts by regressing the returns of the VIX futures contracts on the returns of VIX. Additionally, I use the regression model to estimate the returns of VIX by regressing the VIX returns on the total returns of the S&P 500 Index.

4.1.1 VIX and VIX Futures

In estimating each regression, I identified significant (1%) autocorrelated and heteroskedastic residuals. The results shown in Table 4.1 are from models that combine an autoregressive error model and GARCH model to correct for autocorrelated and heteroskedastic residuals. The autoregressive error model corrects for the autocorrelated residuals using the maximum likelihood method. The GARCH model is a (q=1, p=1) model.

In Table 4.1 I report the results of the ten regressions including the estimated beta coefficients and associated p-values. I also include the results for several goodness-of-fit measures including the R-square, root mean square error (RMSE), Akaike information criterion (AIC), and the Schwartz Bayesian information criterion (SBC).

The AIC and SBC measures are useful for evaluating and selecting a parsimonious model. The AIC and SBC become smaller and approach $-\infty$ as the model fit improves. The AIC and SBC can be written as,

$$AIC = -\frac{2\ln(L)}{T} + \frac{2n}{T}, \tag{25}$$

$$SBC = -\frac{2\ln(L)}{T} + \frac{n\ln(T)}{T}, \tag{26}$$

where *L* is the maximized value of the log of the likelihood function, *T* is the number of usable observations, and *n* is the number of parameters estimated (Enders, 2015, pp. 70).

Table 4.1: Regression Results for VIX and VIX Futures (December 16, 2009 to December 19, 2017)

	VIX	One-Month VIX Futures	Three-Month VIX Futures	Five-Month VIX Futures
Equation	**14**	**11**	**12**	**13**
Roll Down				
α		-0.006	-0.003	-0.002
prob > |t|		<.0001	<.0001	<.0001
β		-0.025	-0.002	-0.001
prob > |t|		<.0001	<.0001	<.0001
r-square		0.24	0.68	0.74
RMSE		0.01	0.00	0.00
AIC		-15021	-23855	-26013
SBC		-14965	-23810	-25968
Level				
α		0.002	0.000	-0.001
prob > |t|		<.0001	0.1377	0.0146
β		0.594	0.289	0.197
prob > |t|		<.0001	<.0001	<.0001
r-square		0.74	0.67	0.61
RMSE		0.03	0.02	0.01
AIC		-9641	-11159	-12048
SBC		-9602	-11120	-12020
Total				
α	0.006	-0.005	-0.003	-0.002
prob > |t|	<.0001	<.0001	<.0001	<.0001
β	-6.690	0.585	0.284	0.196
prob > |t|	<.0001	<.0001	<.0001	<.0001
r-square	0.65	0.70	0.65	0.61
RMSE	0.05	0.03	0.02	0.01
AIC	-6778	-9448	-11101	-12019
SBC	-6728	-9391	-11068	-11963

The regression results in Table 4.1 offer insight regarding the relationship between the returns of VIX futures contracts and the returns of VIX. For example, the estimated beta coefficients for the total returns of the one-, three, and five-month VIX futures contracts decrease as the time to expiration increases from one to five months. The beta coefficient for the total return of the one-month VIX futures contract is 0.585, which is more than twice the beta coefficient of the three-month VIX futures contract

(0.284) and almost three times greater than the total return beta coefficient for the five-month VIX futures contract (0.196). Each of the estimated beta coefficients is significant at the 1% level. The estimated beta coefficients for the total returns of the one-, three-, and five-month VIX futures contracts indicate that the contracts will change by approximately 0.58%, 0.28%, and 0.20% for every 1.0% change in VIX, respectively.

The goodness-of-fit measures improve for each model going from the one-month to the three- and five-month VIX futures contracts. The improvement can be seen by comparing the RMSE, AIC, and SBC, which are all lower for the total returns of the five-month VIX futures contract compared to the one- or three-month VIX futures contracts. For example, the RMSE for the total returns of the five-month VIX futures contract is 0.01 whereas the RMSE for the one-month VIX futures contract is 0.03. However, the differences among the goodness-of-fit measures is the result of a lower standard deviation of total returns for the five-month VIX futures contract compared to either the one- or three-month VIX futures contract.

The regression results for the level returns are like the regression results of the total returns. For example, the level return beta coefficient of the one-month VIX futures contract is 0.594 which compares to 0.289 and 0.197 for the beta coefficients of the three- and five-month VIX futures contracts, respectively. The beta coefficient estimates for each VIX futures contract is significant at the 1% level.

Like the total returns, the level return goodness-of-fit measures show slight improvement in the three- and five-month VIX futures contracts versus the one-month VIX futures contract. Additionally, the value of the goodness-of-fit measures for the level returns are close to those of the total returns. For example, the RMSE of both the one-month level and one-month total returns is 0.03. The similarity in regression results between the level and total returns of each VIX futures contract is a result of the strong positive correlation between the two return series and the return series having very similar standard deviations. As noted earlier, the level returns of the one-month VIX futures contract account for 96.6% of the one-month total returns.

The regression results for the roll down returns are quite distinct from the level and total returns. The roll down beta coefficients are negative for the one-, three-, and five-month VIX futures contracts. The moderately negative beta coefficients are to be expected given the negative covariance between the roll down and VIX returns. This can be inferred from the Pearson correlation matrix in the appendix (Table D.1). In addition, the negative correlation between the roll down and VIX returns can be explained by the average negative returns of the roll down component and the positive average returns of VIX. As shown earlier, the average slope between each of the VIX futures contracts is negative (i.e., upward sloping term structure) more than 84% of the time, which explicates the negative roll down returns.

The regression results for VIX in Table 4.1 show that the VIX and S&P 500 Index are negatively correlated and the VIX is much more volatile than the S&P 500 Index. With an estimated beta coefficient of -6.690, the regression results indicate that the VIX returns are more than six times as volatile as the S&P 500 Index returns. The RMSE of the regression is approximately 5%, which indicates that the differences between the actual and predicted values can be +/- 10% at the 95% level.

4.1.2 VIX Realized Volatility Term Structure

The results from Table 2.2 and Table 4.1 illustrate that the realized volatility term structure for VIX (i.e., VIX and VIX futures contracts) is downward sloping. For example, in Table 2.2 I find that the standard deviation of returns is greatest for VIX followed in order by the total returns of the one-, three-, and five-month VIX futures contracts. The total return beta coefficients in Table 4.1 are greatest for the one-month VIX futures contract and decline for the three- and five-month VIX futures contracts. The declining beta coefficients for the total returns of the VIX futures contracts is illustrative of a downward sloping realized volatility term structure.

4.2 PCA Results

I provide and discuss the results of the principal component analysis. In the discussion I draw conclusions regarding the eigenvector results and the roll down and level return components.

4.2.1 VIX Futures

The PCA was conducted from the VIX returns and the total returns of the one- to six-month VIX futures contracts. Using those seven return series provides a return for each point on the VIX term structure from the spot (VIX) to the six-month point. The eigenvalues and eigenvectors of each principal component are evaluated to identify latent variables (i.e. hidden variables) that exist across the term structure of returns and explain a large amount of the variance of returns. The relative size of each eigenvalue is used to determine the relative importance as a principal component. I include all seven principal components in the graph and table that follow but limit the analysis to the first three principal components given the low amount of variance explained by the fourth to seventh principal components.

Figure 4.1: Proportion of Variance Explained by Each Principal Component

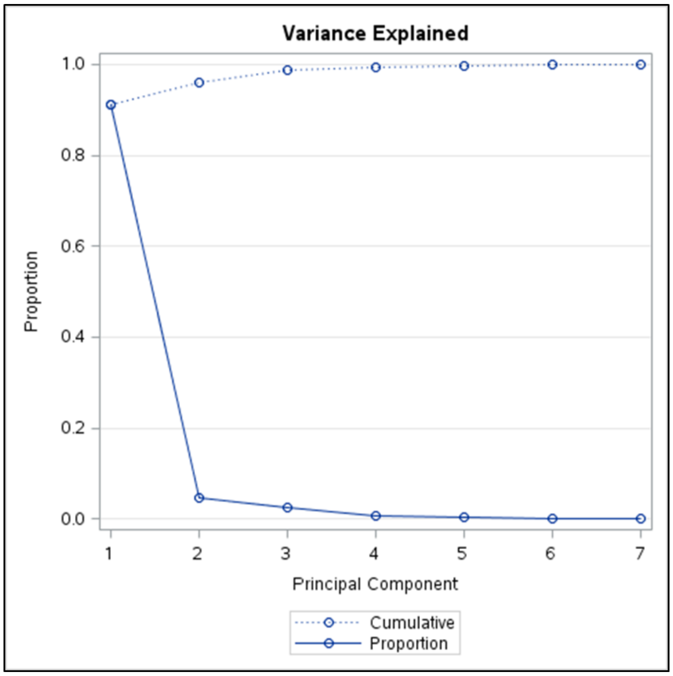


Figure 4.1 is a graph showing the amount of variance explained by the principal components. The x-axis represents each principal component (1 to 7). The y-axis is representative of the proportion of variance explained by the *ith* principal component and is calculated as the eigenvalue of the *ith* principal component divided by the total eigenvalue of the PCA.

The first principal component explains 91.2% of the total variance of the return series. The second and third principal components explain 4.8% and 2.6%, respectively.

In sum, the first three principal components explain 98.6% of the total variance of all the return series.

Each principal component is derived through a unique linear weighting of the return series for VIX and the VIX futures contracts. The unique weighting scheme of each principal component is called the eigenvector. The eigenvectors in Table 4.2 are used to identify latent variables that exist in the VIX and VIX futures return series. Given the low amount of variance explained by principal components 4 to 7 (1.4%), I focus on interpreting the eigenvectors of the first three principal components.

Table 4.2: Eigenvectors and Percent of Variance Explained from PCA with VIX Returns and Total Returns of One- to Six-Month VIX Futures Contracts

	Eigenvectors						
Variable	**PC1**	**PC2**	**PC3**	**PC4**	**PC5**	**PC6**	**PC7**
VIX	0.341	0.801	0.486	-0.062	0.041	-0.010	-0.015
One-Month Total Return	0.373	0.279	-0.622	0.535	-0.324	0.062	0.043
Two-Month Total Return	0.386	0.029	-0.426	-0.311	0.716	-0.220	-0.104
Three-Month Total Return	0.390	-0.134	-0.076	-0.548	-0.271	0.491	0.457
Four-Month Total Return	0.390	-0.226	0.077	-0.242	-0.425	-0.195	-0.717
Five-Month Total Return	0.385	-0.305	0.266	0.188	-0.079	-0.637	0.490
Six-Month Total Return	0.380	-0.344	0.335	0.468	0.347	0.512	-0.152
Percent of Variance Explained	91.3%	4.8%	2.6%	0.7%	0.3%	0.2%	0.1%

The eigenvector of the first principal component (PC1) shows that the first principal component is approximately an equally-weighted linear combination of all the return series. The equal-weighted eigenvector values and large eigenvalue of PC1 (91.2%) results in PC1 being interpreted as a variable representing level returns. From the PCA results I conclude that the level returns account for most of the variance in the

VIX and VIX futures return series. My conclusion is consistent with the analysis provided in section 2.4.1.

The second principal component (PC2) has positive eigenvector values for the return series representing the front-end of the VIX term structure (VIX to two-month) and negative eigenvector values for the return series representing the back-end of the VIX term structure. In addition, the absolute eigenvector values are largest at the ends of the VIX term structure (VIX and six-month). From that I interpret PC2 as a variable for the slope of the VIX term structure. The slope variance accounts for 4.8% of the total variance of the VIX return series.

Although the third principal component (PC3) only accounts for 2.6% of the total variance of the return series, the eigenvector values indicate that it is a variable representing the curvature of the VIX term structure.

Chapter V
Application

I review the Portfolio Strategy objectives which include constructing a portfolio using the S&P 500 Index and a VIX futures curve strategy to earn a return that is equal to or better than the S&P 500 Index while producing a lower standard deviation of returns. I discuss the methodology for the Portfolio Strategy and highlight the weighting scheme used for the S&P 500 Index and the VIX Strategy. I conclude by discussing the regression and summary risk and return statistics.

5.1 Portfolio Strategy

The objective of the Portfolio Strategy is to construct an efficient portfolio that consists of both a passive investment in the S&P 500 Index and a VIX futures curve strategy that achieves a lower risk with the same or better return as the S&P 500 Index. I define risk as the standard deviation of returns and drawdown. Drawdown is the cumulative percentage return of the Portfolio Strategy conditioned on periods of sharp declines for the S&P 500 Index. Having a lower risk implies that the Portfolio Strategy will have a lower standard deviation of returns and will lose less when the S&P 500 Index experiences a sharp decline. Achieving a lower standard deviation of returns while earning the same or better return as the S&P 500 Index will result in a higher risk-adjusted return (i.e., Sharpe Ratio) compared to the S&P 500 Index. The Sharpe Ratio is a measure of the return per unit of risk and is calculated as the return in excess of the risk-free rate divided by the return standard deviation.

5.2 Portfolio Strategy Methodology

The Portfolio Strategy is constructed as a passive investment in the S&P 500 Index combined with an investment in the VIX futures curve strategy (VIX Strategy). The VIX Strategy consists of a long position in the one-month VIX futures contract and two short positions in the 3-month VIX futures contracts. The ratio of VIX futures contracts (i.e., 1:-2) is held constant when deriving the VIX Strategy returns. VIX Strategy returns are calculated as the total dollar profit or loss divided by the total absolute notional amount of the VIX futures contracts.

The Portfolio Strategy returns are produced using a beginning of month weighting scheme that weights the VIX Strategy returns by 60% and the passive S&P 500 Index returns by 40%. The Portfolio Strategy weights are rebalanced back to 60% VIX Strategy and 40% S&P 500 Index at the beginning of each new month. The 60%/40% weighting scheme and monthly rebalancing is used because of its favorable risk-return characteristics compared to other weighting schemes and rebalancing frequencies. Multiple weighting combinations of the VIX Strategy and S&P 500 Index were tested ranging from 10% VIX Strategy and 90% S&P 500 Index to 90% VIX Strategy and 10% S&P 500 Index. Each combination involved changing the weights in 10% increments while ensuring the total weight summed to 100%. Evaluating the different weighting combinations included comparing the annualized returns, annualized standard deviations, Sharpe Ratios, and drawdown returns.

I construct the VIX Strategy using the one- and three-month VIX futures contracts based on the results from Table 2.3, Table 4.1, and Table 5.1. The level return component is the most desirable VIX return component to have exposure to when investing in the S&P 500 Index as the level return component will increase when the S&P 500 Index decreases. Unlike the level return component, the roll down return component is the least desirable return component to have exposure to as it produces most of the negative returns of a VIX futures contract and will continue providing negative returns when the VIX term structure is upward sloping.

The results provided in Table 2.3 show how the average returns and risk characteristics for the one-, three-, and five-month VIX futures contracts vary conditional on the VIX price. For example, the average roll down returns of the one-month VIX futures contract change for each quintile and are most negative in the first quintile (-1.4%) and least negative in the fifth quintile (-0.2%). This compares to the average roll down returns of the three-month VIX futures contract, which is the same for quintiles 1 to 4 and only change by 0.2% in the fifth quintile. Comparing the conditional average roll down returns of the one-, three-, and five-month VIX futures contracts across the different quintiles illustrate that changes in the VIX futures term structure are most pronounced at the one-month contract. The results also indicate that using a static ratio (e.g., 1:-2) of one-month to three-month VIX futures contracts derived from unconditional returns will result in the roll down returns of the VIX Strategy being over-hedged or under-hedged depending on the VIX price.

The conditional average level returns of the one-month VIX futures contract demonstrate similar return behavior as the one-month roll down returns. For example, the one-month average level returns change in every quintile and are the smallest in the first quintile (0.2%) and largest in the fifth quintile (1.3%). The results compare to the average three-month level returns which are the same for quintiles one and two, negative in quintiles 1 to 3, and only increase by a multiple of 2.3 between the first and fifth quintiles. The results illustrate that being short the three- or five-month VIX futures contract when the VIX price is below 16.9 will result in positive average returns.

The regression results from Table 4.1 show that the beta coefficient for the level returns of the one-month VIX futures contract are the largest compared to the other level return beta coefficients. In fact, the beta coefficient for the one-month level returns is more than two times the beta coefficient of the three-month level returns and slightly more than three times the beta coefficient for the five-month level returns. The regression results from Table. 4.1 support the view that the one-month VIX futures contract is the most responsive futures contract on the VIX futures term structure to changes in the VIX price. Therefore, the one-month VIX futures contract provides the greatest exposure to the level return.

The hedge ratio of one-month to three-month VIX futures contracts is determined by regressing the returns of the three-month VIX futures contract on the one-month VIX futures contract. I provide the regression results in Table 5.1. The regression residuals of the roll down return component have been corrected for autocorrelation using the autoregressive error model discussed in earlier. The regression residuals for the level and total return components did not appear to be autocorrelated or heteroskedastic.

Table 5.1: Results from Regressing the Daily Returns of the Three-Month VIX Futures Contract on the One-Month VIX Futures Contract (December 16, 2009 to December 19, 2017)

	α (prob > |t|)	β (prob > |t|)	r-square	RMSE	AIC	SBC
Roll Down Return	-0.002 (<.0001)	0.014 (<.0001)	0.02	0.00	-21801	-21773
Level Return	-0.002 (<.0001)	0.503 (<.0001)	0.83	0.01	-12329	-12317
Total Return	-0.001 (0.0446)	0.501 (<.0001)	0.83	0.01	-12309	-12298

In Table 5.1 I show the regression results including the beta coefficients and goodness-of-fit measures for the roll down, level, and total return components. The beta coefficients for level and total returns are approximately 0.50 and are significant at 1%. They indicate that the ratio of one-month VIX futures to three-month VIX futures is 1:2. Therefore, I calculate the VIX Strategy returns as a static combination of two short positions in the three-month VIX futures contract for every long position in the one-month VIX futures contract.

5.3 Portfolio Strategy Results

In Table 5.2 I provide summary risk and return statistics for the long S&P 500 Index, VIX Strategy, and Portfolio Strategy. I evaluate the effectiveness of the Portfolio Strategy relative to the long S&P 500 Index by comparing the annualized return, annualized standard deviation, Sharpe ratio, and performance during S&P 500 Index drawdowns. Additionally, I compare the skewness and kurtosis as a means of evaluating the distributional characteristics of each return series. The annualized return is calculated as the geometric return annualized based on 252 trading days per year. The annualized standard deviation is calculated as the standard deviation of daily returns multiplied by the square root of 252.

The returns presented in the S&P 500 Index Drawdowns section of Table 5.2 are calculated as the cumulative compound returns during the period shown. The specific dates were chosen based on the peak-to-trough decline for the S&P 500 Index.

Table 5.2: Risk and Return Statistics for the S&P 500 Index, VIX Strategy, and Portfolio Strategy

(December 16, 2009 to December 19, 2017)

	S&P 500 Index	VIX Strategy	Portfolio Strategy
Annualized Return	11.7%	11.6%	11.8%
Annualized Standard Deviation	14.7%	12.4%	10.6%
Skewness	-0.4	0.0	-0.1
Kurtosis	4.6	17.7	6.1
Sharpe Ratio	0.8	0.9	1.1
Means *t*-Test		-0.3	0.4
***F*-Test**		0.7	0.5
S&P 500 Index Drawdowns			
May 2, 2011-Oct. 3, 2011	-19.4%	1.0%	-7.6%
July 20, 2015-Aug. 25, 2015	-13.1%	1.9%	-4.3%
Nov. 3, 2015-Feb. 11, 2016	-12.2%	0.2%	-4.8%

The Portfolio Strategy outperforms the S&P 500 Index with approximately 28% less standard deviation. The combination of a slightly better annualized return and lower standard deviation results in the Portfolio Strategy having a higher Sharpe Ratio (1.1 versus 0.8) compared to the S&P 500 Index. Additionally, the skewness of the Portfolio Strategy (-0.1) compared to the skewness of the S&P 500 Index (-0.4) indicates that the Portfolio Strategy has less of a left tailed distribution compared to the S&P 500 Index.

The results for kurtosis show that the Portfolio Strategy has slightly more extreme observations compared to the S&P 500 Index. However, the skewness results for the VIX Strategy show that the extreme observations fall on both sides of the mean return.

During the periods identified as S&P 500 Index drawdowns, the Portfolio Strategy greatly outperformed the S&P 500 Index. For example, during the period of May 2, 2011-October 3, 2011 the S&P 500 Index declined 19.4% while the Portfolio Strategy declined 7.6%. The outperformance during each drawdown period was a result of the strong performance of the VIX and one-month VIX futures contract. For example, the VIX increased in price from 16 to 45.45 from May 2, 2011 to October 3, 2011.

The Means *t*-Test values shown in Table 5.2 are the test statistics for comparing the annualized returns of the Portfolio Strategy and VIX Strategy to the annualized returns of the S&P 500 Index. The null hypothesis is equal annualized returns (i.e., $H_0\colon r_{portfolio\ strategy} = r_{S\&P\ 500}\ and\ r_{VIX\ strategy} = r_{S\&P\ 500}$) against an alternative hypothesis that they are not equal. The absolute critical value for a two-tailed test is 1.96. The test statistics for the Portfolio Strategy (0.4) and VIX Strategy (-0.3) do not exceed 1.96 or -1.96 and therefore indicate that their annualized returns are not statistically different from the annualized returns of the S&P 500 Index.

Table 5.2 includes the *F*-Test statistics for comparing the variances of the Portfolio Strategy and VIX Strategy to the variance of the S&P 500 Index. The null hypothesis is equal variances (i.e., $H_0\colon \sigma^2_{portfolio\ strategy} = \sigma^2_{S\&P\ 500}\ and\ \sigma^2_{VIX\ strategy} = \sigma^2_{S\&P\ 500}$) against a two-tailed alternative hypothesis of unequal variances. The critical value at the 0.01 significance level for a two-tailed *F*-Test is 1.000. The *F*-Test statistics for both the Portfolio Strategy and VIX Strategy are less than 1.000 indicating that their variances are statistically different from the variance of the S&P 500 Index.

The results shown in Table 2.3 and Table 5.2 compare to the results of Berkowitz and DeLisle (2018) shown in Exhibit 5. Although the two analyses are conducted during different dates and using different proportions of S&P 500 Index and VIX futures contracts, the results from Table 2.3 and Table 5.2 can be used to compare against their methodology and findings. For example, Berkowitz and DeLisle (2018) buy the new month VIX futures contract only when the prior month-end VIX is below 20.2.

The results from Table 2.3 imply that Berkowitz and DeLisle (2018) should have bought the new month VIX futures contract when the month-end VIX was equal to or greater than 20.2. In Table 2.3 I show that the average total returns for the one-, three-, and five-month VIX futures contracts are the greatest when VIX is equal to or greater than 20.6 (fifth quintile). The results from Table 2.3 indicate that Berkowitz and DeLisle (2018) could have produced more appealing results if they bought the new month VIX futures contract when the prior month-end VIX was equal to or greater than 20.2.

The Portfolio Strategy results in Table 5.2 compare favorably to the results of Berkowitz and DeLisle (2018) because of the VIX Strategy. The VIX Strategy mitigates the negative roll down return of the one-month VIX futures contract by shorting three-month VIX futures contracts while retaining an average positive level return. While they don't explicitly quantify the negative roll down returns of the VIX futures contracts, the results from Table 2.4 show that the roll down component is accountable for the entire negative total P&L (and return) of the one- and three-month VIX futures contracts.

Chapter VI
Summary and Conclusion

6.1 Summary of Findings

VIX futures began trading on the CBOE Futures Exchange in March 2004 and have since provided a means of trading forward S&P 500 Index implied volatility. The popularity of trading VIX futures contracts is measured by the increase in volume and open interest of the one-month contract, which increased 500 times and 600 times, respectively from March 2004 to June 2017. The VIX term structure is upward sloping most of the time, although the realized volatility of the VIX term structure is inverted.

The performance of VIX futures contracts has garnered considerable attention in the finance literature. However, most of the finance literature has focused on the total returns with little attention directed at the decomposition of the returns. More specifically, the finance literature has noted the negative performance of VIX futures contracts but only a select number have tried to quantify how much of the negative return is due to roll down. Therefore, my first research objective is to: *determine what proportion the roll down return represents of the negative total return to VIX futures contracts*.

I develop a methodology to decompose the total returns of VIX futures contracts into the two return components, roll down and level. I find that roll down accounts for most, if not all, of the negative returns for the one-, three-, and five-month VIX futures contracts. Additionally, I derive the profit and loss (P&L) from roll down and level, which is unique in the context of the existing finance literature.

The methodology extends to each point on the VIX futures term structure allowing me to achieve the second research objective: *to evaluate how returns, and return components, vary across the VIX futures term structure*. I present the time series regressions I use to analyze the linear relationships between VIX and the return components of the one-, three-, and five-month VIX futures contracts. Additionally, I apply and discuss the principal component model to identify the latent factor structure of the VIX term structure.

I discuss the results from my time series regression models. Specifically, I find that the total return and level return beta coefficients decrease as the time to expiration increases, which is indicative of an inverted realized volatility term structure. Using the results of the principal component model, I find that level returns account for most of the VIX futures variance.

I apply a Portfolio Strategy and compare the results to the return and risk characteristics of the S&P 500 Index. The Portfolio Strategy is a combination of the VIX Strategy and passive S&P 500 Index. The VIX Strategy is constructed using the time series regression results. This analysis relates to my third objective: *to evaluate whether the regression results will allow me to construct a long-short VIX trading strategy using one-month and three-month VIX futures contracts*. I find that the Portfolio Strategy outperforms the S&P 500 Index and has less risk.

6.2 Conclusions

These results indicate that the negative total return of the one- and three-month VIX futures contracts is entirely due to the negative roll down returns. The negative roll down return accounts for approximately 98.6% of the negative total return of the five-month VIX futures contract. The results of the return and P&L decomposition for multiple points on the VIX term structure are unique and add to the existing VIX term structure literature. For example, the existing literature focuses on total return calculations for the VIX futures contracts (Alexander and Korovilas (2013), Bekowitz and DeLisle (2018), Simon and Campasano (2014), Whaley (2013)). Additionally, literature showing P&L by roll down and level has not yet been identified.

Using these results, I conclude that the level return component is responsible for more than 90% of the standard deviation of total returns for each VIX futures contract. Additionally, the level return and total return standard deviations each decrease as the time to expiration increases. These results are indicative of an inverted realized volatility term structure and they are consistent with prior findings in the literature.

The Portfolio Strategy outperforms the S&P 500 Index with less risk. The favorable results are attributable to the VIX Strategy. Specifically, the short three-month VIX futures contracts offset the negative roll down of the one-month VIX futures contract while the level return component of the one-month VIX futures contract provides a hedge to a long position in the S&P 500 Index. These results highlight the significance of disaggregating the total returns and P&L of the VIX futures contracts into the components of roll down and level. Insight regarding the roll down and level components from across the VIX term structure offer the ability to construct a strategy that maximizes either component.

6.3 Future Areas of Research

The returns of the roll down and level return components vary greatly based on the level of VIX. I find that the average roll down and level returns each increase as the VIX level rises. Additionally, the standard deviation of the level return component becomes greater as the VIX level increases. These results indicate that estimating the time series regressions conditional on the VIX level would produce beta coefficient estimates that materially change given different levels of VIX. For example, the estimated beta coefficient of the level return component may become larger (smaller) for higher (lower) levels of VIX. Additionally, the change in beta coefficient estimates for different levels of VIX may not be linear, indicating that an element of convexity exists in the return components.

A natural extension of this research is to conduct quantile regressions to estimate the beta coefficients of the total return and return components (roll down and level) based on quantile rankings of the VIX. The results from quantile regressions could be used to conditionally adjust the ratio of one- and three-month VIX futures contracts used in the VIX Strategy. A conditionally adjusted ratio should reduce the amount of under-hedge or over-hedge during different market regimes.

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Appendix A
VIX calculation

The VIX calculation is $\sigma^2 = \frac{2}{T}\sum\frac{\Delta K_i}{K_i^2}e^{RT}Q(K_i) - \frac{1}{T}[\frac{F}{K_0} - 1]^2$ where $\sigma = \frac{VIX}{100}$, *T* is the time to expiration where each calendar day is divided into minutes, *F* is the forward S&P 500 Index level identified by the strike price where the absolute difference between call and put price is the smallest, K_0 is the at-the-money strike price and is the strike price that is immediately below *F*, K_i is the *ith* out-of-the-money strike price for the calls and puts, $\Delta K_i = \frac{K_{i+1} - K_{i-1}}{2}$ is the interval between strike prices, *R* is the risk-free interest rate, and $Q(K_i)$ is the midpoint of the bid-ask spread for each option K_i.

Appendix B
Statistical Characteristics of VIX and VIX Futures

Table B.1: Statistical Characteristics of Daily Returns for VIX and VIX Futures Contracts (December 16, 2009 to December 19, 2017)[ab]

Return Component	Mean	Median	Cumulative	Std Dev	Min	Max	t Value
Two-Month Roll Down Return	-0.3%	-0.4%	-	0.3%	-1.8%	2.0%	-46.2
Two-Month Level Return	0.1%	-0.1%	-	3.6%	-17.2%	25.8%	1.2
Two-Month Total Return	-0.2%	-0.5%	-99.8%	3.6%	-15.9%	25.6%	-3.1
Four-Month Roll Down Return	-0.2%	-0.2%	-	0.1%	-0.9%	0.5%	-57.0
Four-Month Level Return	0.0%	-0.1%	-	2.4%	-10.7%	16.1%	0.3
Four-Month Total Return	-0.2%	-0.3%	-97.6%	2.4%	-10.4%	15.9%	-3.0
Six-Month Roll Down Return	-0.1%	-0.1%	-	0.1%	-0.7%	0.2%	-55.5
Six-Month Level Return	0.0%	-0.1%	-	1.9%	-8.4%	11.7%	0.1
Six-Month Total Return	-0.1%	-0.2%	-93.7%	1.9%	-8.4%	11.7%	-2.8
VIX	0.2%	-0.5%	-53.3%	7.7%	-29.6%	50.0%	1.4

[a] The data in Table B.1 has been rounded.

[b] The t-Value in Table B.1 is used to test the significance of the sample mean relative to zero at the significance level of 0.05. Specifically, $H_o\colon \bar{x} = 0$ against $H_a\colon \bar{x} \neq 0$. The critical value for the one-sided test is equal to 1.645.

Table B.2: VIX Futures Cumulative P&L (December 16, 2009 to December 19, 2017)

	Cumulative P&L		
VIX Futures Contract	**Roll Down**	**Level**	**Total**
Two-Month	-$122,322	$9,677	-$112,645
Four-Month	-$67,014	-$5,426	-$72,440
Six-Month	-$50,164	-$5,786	-$55,950
Quintile One (VIX: 9.1-12.7)			
Two-Month	-$27,715	-$20,005	-$47,720
Four-Month	-$16,316	-$14,844	-$31,160
Six-Month	-$11,553	-$13,137	-$24,690
Quintile Five (VIX: 20.6-48.0)			
Two-Month	-$7,686	$62,516	$54,830
Four-Month	-$4,445	$49,845	$45,400
Six-Month	-$4,620	$44,920	$40,300

Appendix C
Partial Autocorrelations of VIX and VIX Futures Return

Figure C.1: Plot of the Partial Autocorrelations for the Daily VIX Return (December 16, 2009 to December 19, 2017)

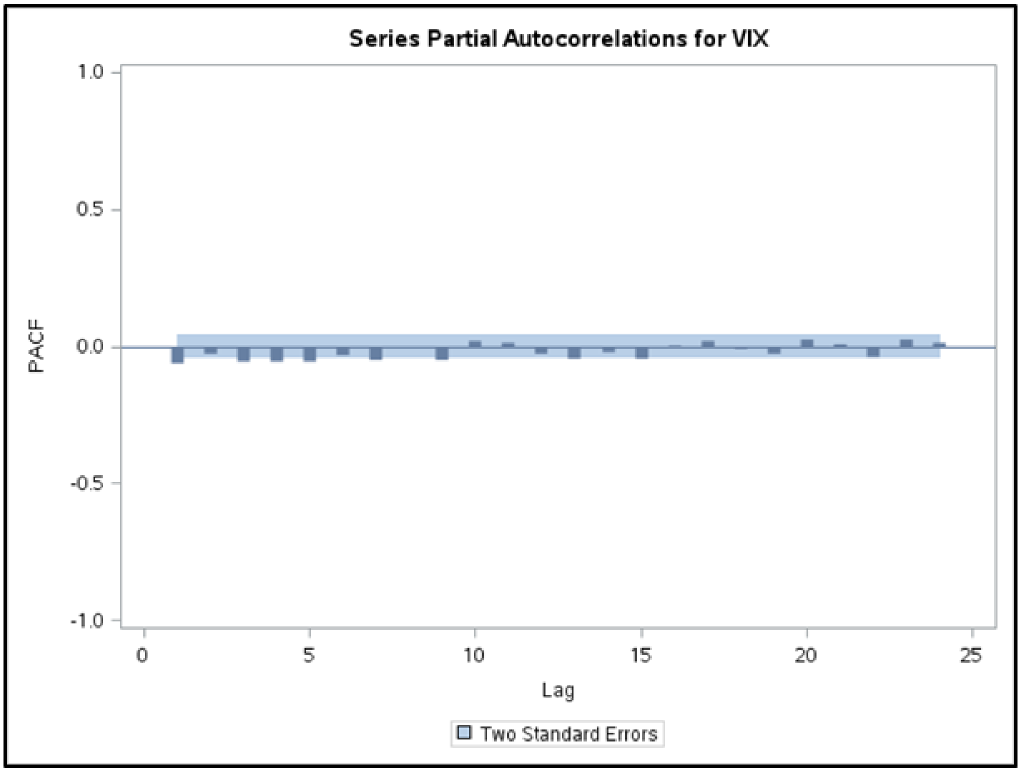


Figure C.2: Plot of the Partial Autocorrelations for the Daily One-Month VIX Futures Roll Down Return (December 16, 2009 to December 19, 2017)

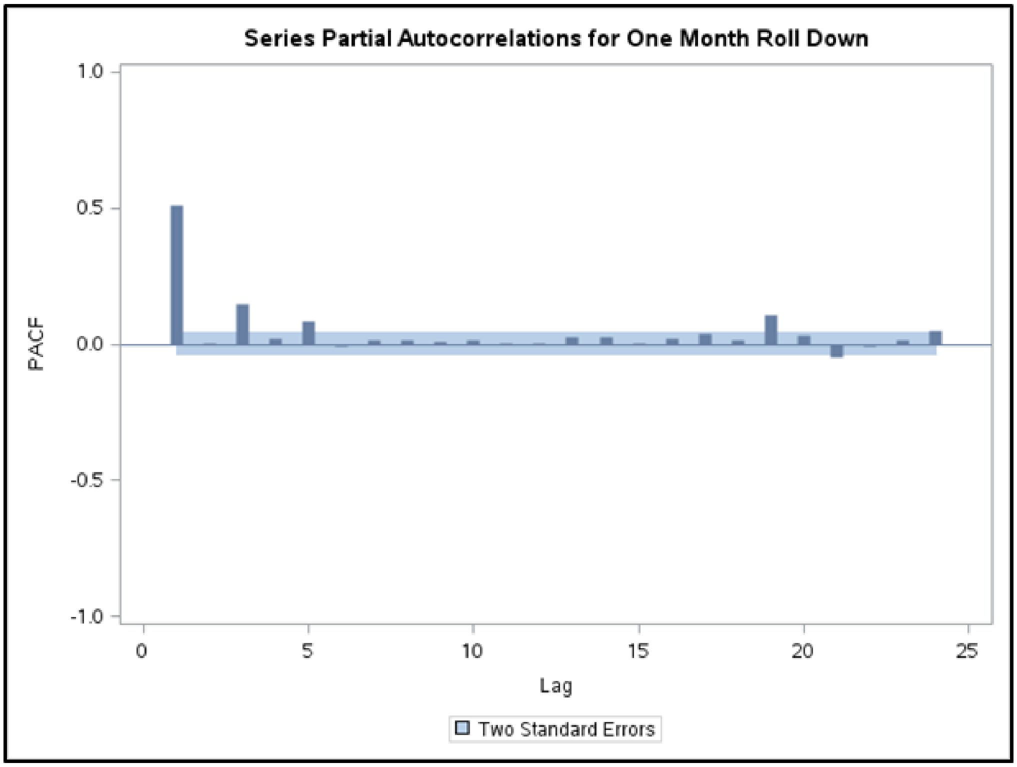


Figure C.3: Plot of the Partial Autocorrelations for the Daily One-Month VIX Futures Level Return

(December 16, 2009 to December 19, 2017)

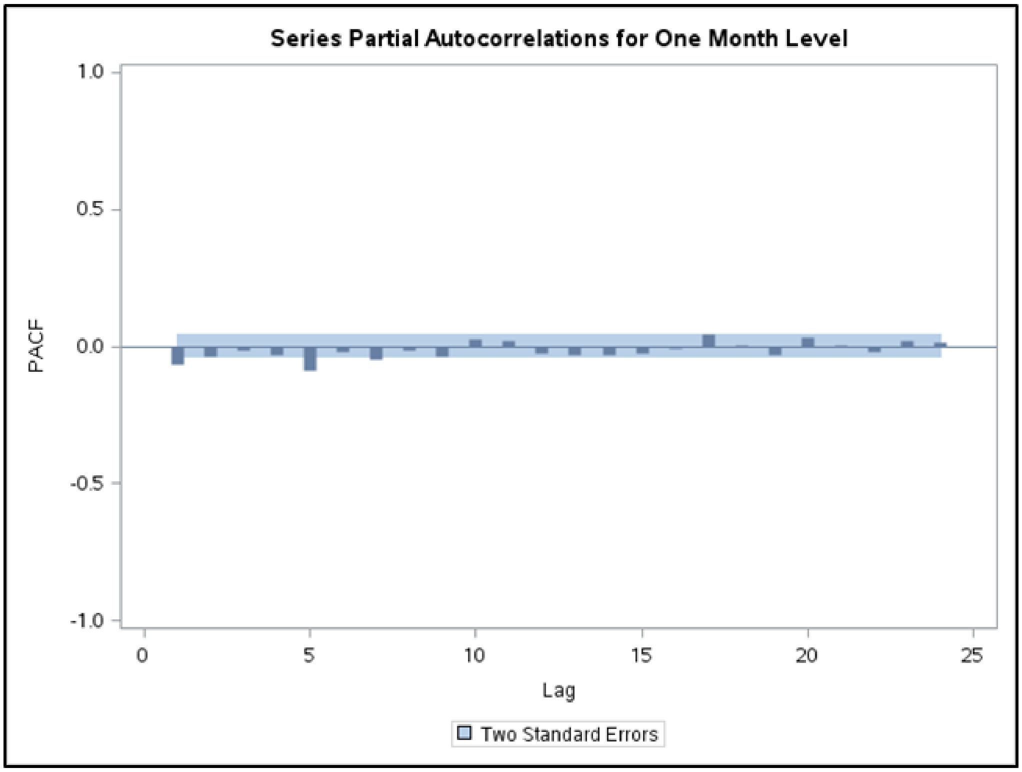
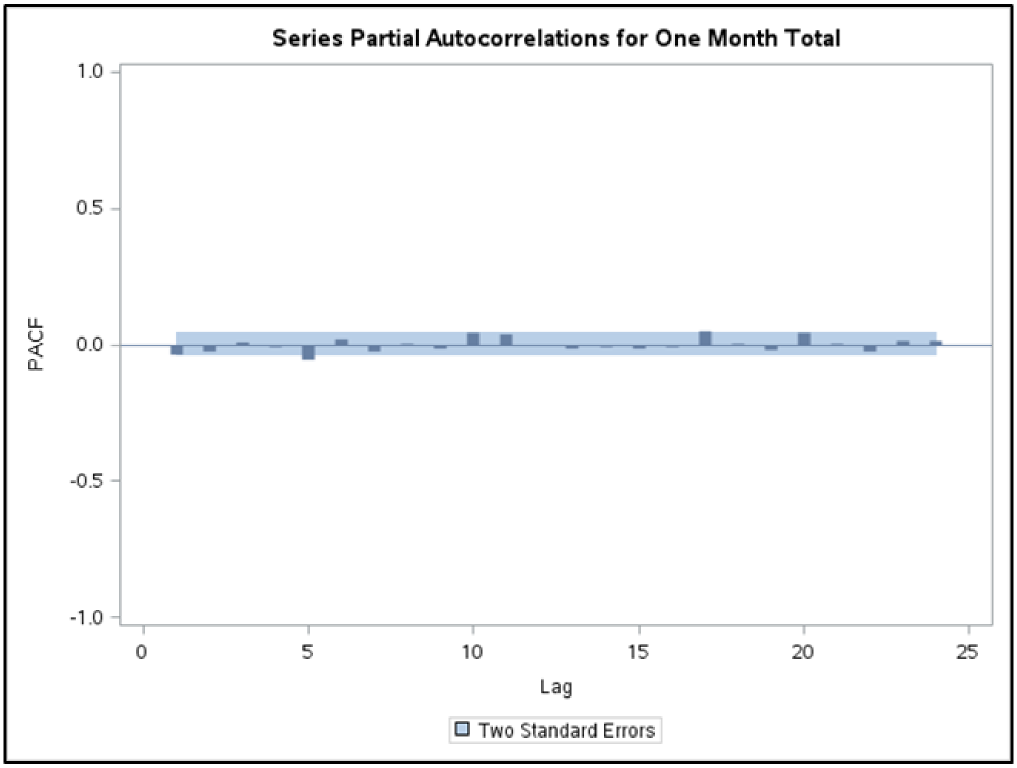


Figure C.4: Plot of the Partial Autocorrelations for the Daily One-Month VIX Futures Total Return

(December 16, 2009 to December 19, 2017)



Appendix D
Return Correlation Matrix

Table D.1: Correlation Matrix Derived from Daily Returns (December 16, 2009 to December 19, 2017)

Pearson Correlation Coefficients										
	VIX	One Month Roll Down	One Month Level	One Month Total	Three Month Roll Down	Three Month Level	Three Month Total	Five Month Roll Down	Five Month Level	Five Month Total
VIX	**1.00**									
One Month Roll Down	-0.10	**1.00**								
One Month Level	0.86	-0.12	**1.00**							
One Month Total	0.83	0.13	0.97	**1.00**						
Three Month Roll Down	-0.09	0.36	-0.13	-0.04	**1.00**					
Three Month Level	0.81	0.01	0.91	0.91	-0.05	**1.00**				
Three Month Total	0.81	0.04	0.90	0.91	0.02	1.00	**1.00**			
Five Month Roll Down	-0.08	0.35	-0.12	-0.03	0.61	-0.05	-0.01	**1.00**		
Five Month Level	0.78	0.00	0.87	0.86	-0.02	0.96	0.96	-0.05	**1.00**	
Five Month Total	0.78	0.02	0.86	0.86	0.01	0.96	0.96	0.01	1.00	**1.00**