Applied Stochastic Modelling in Financial Economics

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Declaration of Authorship

I declare that, except where explicit reference is made to the contribution of others, that this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Printed Name:

Signature:

This thesis is dedicated to my wife, Xinxin

for her special care of me during the writing of this thesis

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Abstract

This Ph.D thesis focuses on two applied frameworks of stochastic controlling and optimisation in financial economics. The first focus (Chapter 1) is on the convergence trading with testing cross-listed stock arbitrage. The second (Chapter 2 to Chapter 4) is on the sequential studies of wealth inequality.

In Chapter 1, the convergence trading is established by dynamic programming, by setting objective at maximising trading utility function with constraint characterising the mean-reversion between price spread. Compared to past research that Liu and Timmermann (2013), the cointegrating vector has been inserted inside, meanwhile the volatility factor has been split into multiple layers attributed by the relevant information sets. A wide range of empirical tests have been conducted for the cross-listed stock trading, including both in-the-sample and out-of-sample tests for Eurozone, UK, US and China stock exchanges based on shares and CFD trading. The testing result is convincing that stochastic optimal control has the potentiality to amplify statistical arbitrages.

Chapter 2, initiates the research of wealth inequality. It first replicates the consumption-saving framework proposed by Karatzas (1991) under stochastic general equilibrium, by applying convex duality optimisation. This is to study the influence from a household's homogenous preference of consumption on the dynamical evolutions of wealth and concentration. Assuming that the household's income is exogenously given and adopting simulation. The simulation results suggest that consumption preference has no significant impact on wealth inequality but on the volatility of

wealth inequality.

Chapter 3 simplifies the endogenous price density dependent on an agent's risk aversion. Meanwhile the standard (Pareto) optimal consumption is re-modified by maximising the utility for both household's consumption and the expected saving the end of the dynasty, other than maximizing consumption only. The simulation illustrates that although the homogenous (heterogenous) risk aversion of a household's consumption could affect the progression of wealth concentration but it has no obvious association with wealth inequality. Moreover the discreteness of heterogenous risk aversion has no significant impact on wealth inequality throughout the dynastic horizon, when each household's income exogenously given.

Chapter 4 endogenizes the labour income and capital gain into the household revenue. Wage is endogenous from the technology progress (Total Factor Productivity) of the industry. While capital gain is endogenous from a completed competitive financial market with zero-profit condition for financial intermediates. Each household's income is endogenously driven by technology progress following the neoclassical economic growth framework. At each development stage, the attributions of wage and capital gain follow contingent claim analysis and the consumptions satisfy the Pareto optimal, inherited from the solution in Chapter 3 with convex duality optimisation. Our structural model not only endogenously features agent risk aversion but also the productive factor growth, human capital, TFP and labor force, which further makes it possible the analysis of the effects of all these factors on wealth inequality as a whole

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Chapter 1

Optimal convergence trading

1.1 Introduction to convergence trading

With the goal of maximizing investors' terminal utility, we construct a non-threshold based trading model within which the analytical solution of the optimal trading weights for daily rebalancing has been derived via stochastic optimal controlling. Having released the constraint that the cointegrating vector equals to one, we have proposed a more practical trading strategy that is applicable to much wider range categories of cointegrated assets. This paper is the first stochastic optimization paper that carries on extensive out-the-sample experiments on the cross-listed stock portfolios, which facilitates the comparative studies among Chinese and European, UK and US stock markets. We further test the time-delay arbitrage of the cross-listed stocks by employing two paralleled trading mechanisms, respectively equity-based contracts for difference (CFD) and real shares trading. Our empirical results illustrate that the time-delay arbitrage of the cross-listed stocks strategy based on the analytical solution of weights yields relatively stable and better performance than that of the home market index.

Convergence trading strategies have been applied for capturing arbitrage opportunities of the temporary anomalies between the relative-value of two assets. Which are proposed to converge to the long-term equilibrium based on the historical repeating. Follow the studies of Alexander (1999), Jurek and Yang (2007) and Liu and Timmermann (2013) that the individual asset price is random walking although, some sorts of their combined price show the cointegration relationship. It implies the cointegration error series (hereafter, relative-price discrepancies) between the asset pair is stationary around the cointegration equilibrium proposed by Engle and Granger (1987).

One of the simple pair trading strategy is arguably profitable based on the historical price dynamics and simple contrarian principles, tested by Gatev et al. (2006). However, they assert the risk-adjusted returns of convergence trading will converge to zero accompanying the stock market turns to be more and more efficient. Also, these arbitrageurs may suffer big losses by the unforeseen permanent anomalies of the price equilibrium, which is attributed by the significant changes of stock fundamentals. Therefore, on the bilateral developed market, the simultaneous price arbitrage[1] of cross-listed stocks attract the increasingly number of funds. It can effectively relief the unforeseen permanent anomalies of the price equilibrium because of the law of one price. Also, arbitrageurs profit from the short-lived price discrepancy attributed by the information spill-over between markets, explained by Gagnon and Karolyi (2010). However, the information spill-over has been shrunk accompanying the accelerating evolution of the information technique. It keeps squeezing the available profitability of the simultaneous price arbitrage of cross-listed stocks. On the contrary, intuitively the information technique progress seemly cannot easily erase the relative-price discrepancies between cross-listed stocks. Because the noisy trading activities during the non-overlap time between bilateral markets cannot stop intuitively[2]. Therefore, it is such worthy for this study to check whether the time-delay arbitrage of cross-listed stocks is feasible or not by utilising stochastic optimal control. Probably it may be one of feasible arbitrage patterns in the nearly strong efficient market.

[1]The simultaneous price arbitrage can be adopted in an easy way. Because numbers of large capitalized stocks express their quotation through American Depositary Receipts (ADRs) and Global Depositary Receipts (GDRs). Meanwhile the liquidity of the 24-hour over-the-counter (OTC) markets have being progressed to adequate for trading, liquidity risk to be less.

[2]In often there is no significant public information released beyond the working time of headquarter of the cross-listed corporate in the home market. The discrepancy of the closing prices between the home and foreign market is majorly attributed by the time-delay and noise information. The closing prices of cross-listed stock pair may show (logarithm) cointegration relationship in the long-run. Therefore, the stock price of home market in 24-hour Over-the-Counter (OTC) market usually fluctuate around the last closing price until the next day morning auction.

Despite that Griffin et al. (2010) show that the data-snooping is helpless to profit in the emerging markets because of the volatile fundamentals. It is also worthy to check whether the differential market microstructures and noise trading of the time-delay between emerging markets (e.g. Shanghai and Hong Kong stock exchange) can contribute the profit for their cross-listed stock arbitrage.

Liu and Timmermann (2013) argue that the simple contrarian principles[3] is failed to exploit the short-term trade-off between risk and return in the consequential and optimal way. Moreover, the equilibrium model shown by Kondor (2009) indicates that the available profit tends to be eliminated by the intensive competition among arbitrageurs. They rely on the simple contrarian thresholds, competitively seek to position them at the largest price discrepancy point. However, most of them neglect their current fleeting arbitrage opportunities. While this drawback can be released by the analytical solution of trading weight in this paper. Because it can continually and dynamically advise investors to positions where required to be rebalance to, throughout the whole trading horizon.

Without capital constraints, the relative-price arbitrageurs tend to be over aggressive when the relative-price discrepancy diverge further and further, Grossman and Miller (1988). In addition, the empirical evidence of He and Krishnamurthy (2011) shows that the weak credit or liquidity may crash the financial positions of trade brokers. Therefore, the common relative-price arbitrageur might be enforcedly wind up their positions because of lacking spare capitals for margin call. Therefore, we involve the risk aversion level in the exponential utility function to scale the capital constrain to control the liquidity risk, similarly as the works of Tourin and Yan (2013) and Liu and Timmermann (2013).

Comparing with the recent stochastic optimal convergence trading models such as the studies of Mudchanatongsuk et al. (2008) and Liu and Timmermann (2013), we extend the cointegrating vector of asset pairs from one to any figure. It theoretically

[3]One example of simple contrarian principles, is to position by setting 2 times standard deviations of the relative-price discrepancy series. The fixed threshold leads the aggregated portfolio wealth and trading weights piecewise changes.

solids the stochastic optimal convergence trading is potentially suitable for a wide range of asset categories (e.g. relative-price arbitrage between the paralleled stocks under the same industrial index). Also it is the rare study related to time-delay arbitrage of cross-listed stocks with abundant empirical analysis that covers major developed and emerging stock markets. Both in-the-sample and out-the-sample tests with their pivotal performance indicators have been presented.

The section 2.1 interprets and clarifies the assumption that stock volatility can be decomposed to three components reflecting market, index (industrial) and firm-specific information. The section 2.2 shows the stochastic differential equation on expressing cointegrated asset dynamic with corresponding discrete form as well. The section 2.3 presents the analytical solution of daily trading weight which can be adopted on the relative-price arbitrage. Section 3 carries on one complete empirical experiment of the analytical solution obtained in 2.3 on the time-delay arbitrage of cross-listed stocks. In specific, the section 3.1 express the wide range of portfolios with specific stock selection criteria. The section 3.2 and 3.3 explain the feasible mechanisms of real trading applications and the constrained capital utilization. At last but not least, the section 3.4 shows each cross-listed stock portfolio performance of their time-delay arbitrage by employing the analytical solution, also the key indicators can benchmark index have been supplied.

1.2 Volatility Decomposition

According to Efficient Market Hypothesis (EMH) Fama (1970) stock return reflects all available information related to the specific firm, index and market level. The abnormal returns of stocks can be decomposed into systematic and idiosyncratic components based on the CAPM augured by Sharpe (1964) and the five-factor asset pricing model Fama and French (2017). The idiosyncratic component of volatility is driven by the firm level information while market and index information steer the systematic volatility component. Let us define

$$\sigma_s^2 = \beta^2 \sigma_{sys}^2 + \sigma_{idi}^2 \tag{1.1}$$

where σ_{sys}^2 is the systematic risk component and σ_{idi}^2 is the idiosyncratic risk component. Extending (1.1) to be

$$\sigma_s^2 = (1 - w_x)\sigma_{sys}^2 + w_x \sigma_{sys}^2 + (\beta^2 - 1)\sigma_{sys}^2 + \sigma_{idi}^2 \tag{1.2}$$

where $(1 - w_x)\sigma_{sys}^2$ is the stock return volatility component driven by market level information, $w_x \sigma_{sys}^2$ is the volatility component driven by index level information, and $(\beta^2 - 1)\sigma_{sys}^2$ and σ_{idi}^2 are driven by firm level information. $(\beta^2 - 1)\sigma_{sys}^2$ and σ_{idi}^2 can be respectively treated as exogenous and endogenous volatility component corresponding the firm level information. Specifically, $(\beta^2 - 1)\sigma_{sys}^2$ is the systematic volatility component is transmitted to the firm level component based on the intrinsic characteristics of the firm through the adapter coefficient $\beta^2 - 1$.

It can be argued that the stock return volatility component driven by market level information is heavily influenced by the current state and the expectations of the macroeconomics factors of the economy, Engle and Rangel (2008) and Conrad and Loch (2015). This argument is represented by factor $(1 - w_x)\sigma_{sys}^2$. On the other hand, it has been shown the industry (index) level information also possesses sizeable power on stock returns $w_x \sigma_{sys}^2$, Roll (1988) and Piotroski and Roulstone (2004).

Let us assume a cross listed stock. The firm level information can be assumed to have the similar effect to the stock volatility in the two markets. However, this cannot be said for the other components. Regional cultures, the market participant characteristics (e.g. the proportions of institutional and retailing investors) and structural factors (e.g. currency fluctuation, ex-dividend-date and tax-induced heterogeneity) differ between markets. Thus, the same news can derive differential magnitudes of sentiment and volatility of stock returns Froot and Dabora (1999), Baker et al. (2012) and Corredor et al. (2013). Additionally, there is information spillover (lead-lag effects) among the indices and the markets explained by Singh et al. (2010). In case of different stocks that belong to the same index the same logic applies and there is also information spillover between these stocks King (1966), Ramnath (2002), Hou (2007) and Kelly and Ljungqvist (2012).

Consequently, we assume that the observable stock price S_t assumed compositing three components, $S(t) = f(t) + in(t) + M(t)$. $M(t)$ is the observable whole market that stock price belonging to (e.g FTSE ALL Share Index), $f(t)$ is the price part purely reflecting the endogenous (unique) information of the firm fundamentals and $in(t)$ reflects the endogenous information derived by the fundamentals of each firm in the index $f(t)$ and $in(t)$ are unobservable at time t. The cross-sectional information transmitted from other firms under the identical index is excluded in $f(t)$ but involved in $in(t)$. Similarly, the cross-sectional information transmitted from other indexes under the same stock market is not contained in $in(t)$ but in $M(t)$. If define $IN(t)$ as the observable index price (e.g. FTSE100) then this can be decompose to $IN(t) = in(t) + M(t)$.

Based on the above exposition, each stock price can be presented as

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma_f dZ_1(t) + \sigma_{in} dZ_2(t) + \sigma_M dZ_3(t) \tag{1.3}$$

where μ is drift rate of price dynamics, σ_f and σ_{in} are the volatility factors related to the endogenous information from the fundamentals of the specific firm and index respectively. σ_M is the volatility factor related to the composite information of the whole market. $\sigma_f dZ_1(t)$, $\sigma_{in} dZ_2(t)$ and $\sigma_m dZ_3(t)$ are the diffusion components corresponding to the information set reflected by series $f(t)$, $in(t)$ and $M(t)$ respectively. They mutually determine the stochastic fluctuation of each stock return, the cross-sectional information transmitted from other firms (indexes) under the identical index (market) is excluded in $f(t), (in(t))$. In addition, the information of the specific firm (index) spilling into the $in(t), (M(t))$ can be effectively diluted if there are many stocks (indexes) paralleled under the same index (market), this convinces the three diffusion components can be independent at least in a large volume market. Hence it results $Z_1(t)$, $Z_2(t)$ and $Z_3(t)$ are independent Brownian motions.

Following (1.3) the information of the specific firm (index) transmitted to $in(t), (M(t))$ can be effectively diluted if there are large number of stocks (indexes) trading under the same index (market). Then σ_f and σ_{in} can be approximated as the $\sqrt{N}$ fold volatility of the change rates of the unobservable processes $f(t)$ and $in(t)$. They can be estimated through formulas (1.4) and (1.5) (the formal proof see Apeendix)

$$\sigma_{in} = \sqrt{N \cdot \mid \text{Var}(R_{IN}) - \text{Var}(R_M) \mid} \tag{1.4}$$

$$\sigma_f = \sqrt{N \cdot \mid \text{Var}(R_S) - \text{Var}(R_{IN}) \mid} \tag{1.5}$$

where R_S, R_{IN} and R_M are the price returns of the observable series $S(t)$, $In(t)$ and $M(t)$ and N is the length of the sample estimation. The diffusion terms $\sigma_f dZ_1(t)$, $\sigma_{in} dZ_2(t)$ at each continuous time point could switch between a positive and negative. If $Var(R_S) < Var(R_{IN}) < Var(R_M)$ then these two diffusions expectation are negative. This means that the intrinsic properties (endogenous information) of the fundamentals of the specific firm and index weaken the volatility transmitted from the market level. If $Var(R_S) > Var(R_{IN}) > Var(R_M)$ then then the two diffusions expectation are positive. In that scenario, the intrinsic properties of the specific firm and index strengthen the volatility transmitted from the market level. If $Var(R_{IN}) > Var(R_M)$ and $Var(R_I N) > Var(R_S)$, then the expectation diffusion $\sigma_f dZ_1(t)$ in that specific period is negative but that of $\sigma_{in} dZ_2(t)$ is positive. This implies the intrinsic properties of the fundamentals of the index strengths the volatility transmitted from the market (e.g. the performance of the Dow Jones U.S. Financials Index during the 2008 financial crisis) while the endogenous information from the firm fundamentals of the stabilize their fluctuation.

1.3 Optimal trading on mean-reverted spread

1.3.1 Mean-reverted spread

A stock pair is probably cointegrated when both stock absorbing the aligned information from their similar fundamentals. The linear combination of cointegrated series is a stationary process as described by Engle and Granger (1987). The logarithms of stock prices $A(t)$ and $B(t)$ are cointegrated if they satisfy

$$x(t) = \ln(A(t)) - Q\ln(B(t)) \tag{1.6}$$

$$dx(t) = k(\theta(t) - x(t))dt + \eta dW(t) \tag{1.7}$$

$x(t)$ is the cointegration error (relative price changing) while $(1, -Q)$ is the cointegrating vector. The equation (1.7) ensures the spread $x(t)$ as a stationary process, it keeps fluctuating around the cointegrating equilibrium θ in the long-term horizon. The distance $\theta - x(t)$ is the equilibrium error correction term. $\theta(t) - x(t) > 0$ ($\theta(t) - x(t) < 0$) implies the price-spread between asset $A(t)$ and $B(t)$ going to be larger (smaller). k is the moving speed of price spread around the cointegration equilibrium, η is the volatility factor and $W(t)$ is a standard Brownian motion. The process (1.7) captures the feature of mean reversion with stochastic fluctuations[4]

Rewriting (1.6) as

$$A(t) = e^{x(t)}B(t)^Q \tag{1.8}$$

By applying Itô formula on Eq.(1.8), it gives

$$\begin{aligned} dA(t) = e^{x(t)}B(t)^Q dx(t) + \frac{1}{2}e^{x(t)}B(t)^Q(dx(t))^2 + Qe^{x(t)}B(t)^{Q-1}dB(t) \\ \frac{1}{2}Q(Q-1)e^{x(t)}B(t)^{Q-2}(dB(t))^2 + Qe^{x(t)}B(t)^{Q-1}dx(t)dB(t) \end{aligned} \tag{1.9}$$

Denoting the asset B satisfies expression in Eq.(1.3) as

$$\frac{dB(t)}{B(t)} = \mu dt + \sigma_f dZ_1(t) + \sigma_{in} dZ_2(t) + \sigma_M dZ_3(t) \tag{1.10}$$

The fact that the cross-sectional information is being transmitted between stocks (indexes) over the relevant index (market) is evidenced by Singh et al. (2010), King (1966), Hou (2007) and Ramnath (2002). Assuming that the relative changing rate of price (spread $x(t)$) can reflect the information transmission between cointegrated

[4]Alternatively, instead of the equation (1.3) the Cox-Ingersoll-Ross, the GARCH diffusion, Non-Gaussian Ornstein-Uhlenbeck processes can be similarly applied by Gregory et al. (2010). In this study, we follow the classic Ornstein-Uhlenbeck processes to easily maintain the analytical tractability.

stocks over firm, index and market levels. The correlations between the diffusion terms in the SDE (1.7) and every diffusion of SDE (1.10) are defined by

$$\mathrm{E}(dZ_i(t)dW(t)) = \rho_i dt, \quad i = 1, 2, 3 \tag{1.11}$$

In addition the relationship among volatility factors of the SDE (1.10) does satisfy

$$\sigma_B^2 = \sigma_f^2 + \sigma_{in}^2 + \sigma_M^2 \tag{1.12}$$

By plugging (1.7), (1.10) and (1.12) into (1.9), it yields

$$\begin{aligned}\frac{dA(t)}{A(t)} = &\left[k(\theta(t) - x(t)) + Q\mu + \frac{1}{2}\eta^2 + \frac{1}{2}Q(Q-1)\sigma_B^2\right] dt \\ &+\eta Q(\sigma_f\rho_1 + \sigma_{in}\rho_2 + \sigma_M\rho_3)dt + \eta dW(t) \\ &+Q(\sigma_f dZ_1(t) + \sigma_{in} dZ_2(t) + \sigma_M dZ_3(t))\end{aligned} \tag{1.13}$$

Suppose that the cointegration relationship between a stock pair is consistent with the volatility decomposition (1.1) and (1.2) and stock B is the lead stock (host). Each stock price change is incorporated on the fundamental information referred to the specific firm, index and market levels. The volatility factors are related to the endogenous fundamental information from the firm, index and market levels. Except σ_B^2 (the N fold of variance of the return rate of stock B) and ρ_i, the rest parameters of the SDE (1.13) are linked from the process (1.3), (1.6) and (1.7). The lag stock price A can be presented by (1.13) and their cointegration relationship with stock B. If writing the relative price dynamics of stock A in a discrete format, it can benefit us more obviously to see the relationship between stock pair based on their cointegration relationship. Reordering (1.13) and plugging (1.7), (1.10) and (1.12) into (1.9), obtain

$$\begin{aligned}\frac{dA(t)}{A(t)} = dx(t) + Q\frac{dB(t)}{B(t)} &+ Q(\sigma_f\rho_1 + \sigma_{in}\rho_2 + \sigma_M\rho_3)dt \\ &+\frac{1}{2}\eta^2 dt + \frac{1}{2}Q(Q-1)\sigma_B^2 dt\end{aligned} \tag{1.14}$$

Following the maximized log-likelihood estimation, the estimator of η in the process (1.7) can be given as

$$\eta = \sqrt{\sum_{t=0}^{T}[(x(t+1) - x(t)) - \frac{1}{T}k(\theta - x(t))]^2} \tag{1.15}$$

It can be approximated in further as

$$
\begin{aligned}
\eta &= \sqrt{\sum_{t=0}^{T}(x(t+1)-x(t))^2} - \frac{1}{\sqrt{T}}k\left(\theta - \frac{1}{T}\sum_{t=0}^{T}x(t)\right) \\
&= \sqrt{\sum_{t=0}^{T}T\cdot \mathrm{Var}(\Delta x(t))} - \frac{1}{\sqrt{T}}k\left(\theta - \frac{1}{T}\sum_{t=0}^{T}x(t)\right)
\end{aligned}
\tag{1.16}
$$

In the long horizon T, $x(t)$ fluctuates around its long-term equilibrium θ, it has

$$
\theta \approx \frac{1}{T}\sum_{t=0}^{T}x(t) \tag{1.17}
$$

Therefore, combining (1.16) and (1.17) yield us that

$$
\eta \approx \sqrt{\sum_{t=0}^{T}T\cdot \mathrm{Var}(\Delta x(t))}, \quad \Delta x(t) \sim 0 \tag{1.18}
$$

The volatility factor can be approximated as (see appendix)

$$
\sigma_B \approx \sqrt{N\cdot \mathrm{Var}(R_B^i(t))} \tag{1.19}
$$

σ_B can be estimated through a log likelihood estimator. However, the estimation will lack accuracy (diffusion ambiguity). Based on the leading-lag effect the recent short-term information can be transmitted between the cointegrated stocks and influence their price. We can use the short-term process of return rate of stock B, $R_B^i(t)$ to estimate σ_B following (1.19). The subscript i stands for the most recent short term with sampling length N. Recent data should provide computationally efficient superior estimations on the volatility factor.

The estimation of η will be based on the long-term process $\Delta x^j(t)$, subject to (1.18). The subscript j denotes the long-term process with sampling length T. In addition, the appendix shown that

$$
\sigma_M = \sqrt{T\cdot \mathrm{Var}(R_M(t))} \tag{1.20}
$$

$$
|\sigma_f| = \sqrt{T\cdot |\mathrm{Var}(R_S(t)) - \mathrm{Var}(R_{IN}(t))|} \tag{1.21}
$$

$$
|\sigma_{in}| = \sqrt{T\cdot |\mathrm{Var}(R_{IN}(t)) - \mathrm{Var}(R_M(t))|} \tag{1.22}
$$

Subject to (1.20), (1.21) and (1.22), it can be approximated that

$$\rho_1\sigma_f\eta = T\rho_1\sqrt{\mathrm{Var}(R_f^j(t))}\cdot\sqrt{\mathrm{Var}(\Delta x^j(t))} = T\cdot\mathrm{Cov}(R_f^j(t),\Delta x^j(t)) \tag{1.23}$$

$$\rho_2\sigma_{in}\eta = T\rho_2\sqrt{\mathrm{Var}(R_{in}^j(t))}\cdot\sqrt{\mathrm{Var}(\Delta x^j(t))} = T\cdot\mathrm{Cov}(R_{in}^j(t),\Delta x^j(t)) \tag{1.24}$$

$$\rho_3\sigma_M\eta = T\rho_3\sqrt{\mathrm{Var}(R_M^j(t))}\cdot\sqrt{\mathrm{Var}(\Delta x^j(t))} = T\cdot\mathrm{Cov}(R_M^j(t),\Delta x^j(t)) \tag{1.25}$$

Overall, substituting (1.18)∼(1.25) into (1.14), the discrete form of changing rate of stock A can be expressed by the return rate of stock B and the cointegration error,

$$\begin{aligned} R_A^c(t) &= (\Delta x^c(t) + QR_B^c(t)) + \frac{N}{2}Q(Q-1)\mathrm{Var}(R_B^i(t)) + \frac{T}{2}\mathrm{Var}(\Delta x^j(t)) \\ &\quad + QT\left[\mathrm{Cov}(R_f^j(t),\Delta x^j(t)) + \mathrm{Cov}(R_{in}^j(t),\Delta x^j(t)) + \mathrm{Cov}(R_M^j(t),\Delta x^j(t))\right] \\ &\quad j = \underbrace{1,2,3\cdots c}_{T} \quad i = \underbrace{t-N+1, t-N+2, t-N+3\cdots c}_{N}, \quad N \ll T \\ &\mathrm{Cov}(R_f^j(t),\Delta x^j(t)) = \rho_1\sqrt{\mathrm{Var}(\Delta X^j(t))}\cdot\sqrt{\mid \mathrm{Var}(R_B^j(t)) - \mathrm{Var}(R_{IN}^j)(t)\mid} \\ &\mathrm{Cov}(R_{in}^j(t),\Delta x^j(t)) = \rho_2\sqrt{\mathrm{Var}(\Delta X^j(t))}\cdot\sqrt{\mid \mathrm{Var}(R_{IN}^j(t)) - \mathrm{Var}(R_M^j)(t)\mid} \end{aligned} \tag{1.26}$$

where $R_A(t)$, $R_B(t)$ and $R_M(t)$ are the observable return rates of the stock A, stock B and the market index, that stock B belongs to. $\Delta x(t)$ is the changing rate of the cointegration error where $R_f(t)$ and $R_{in}(t)$ are the return rates of the unobservable processes $f(t)$ and $in(t)$, discussed in section (1.2). The subscript c denotes the current step and j indicates the long-term process up to the current step c with sampling length T. The subscript i indicates the short-term process up to current step with sampling length N.

In (1.26), there are four components dominating the current-step return rate $R_A^c(t)$. The first one is $\Delta x^c(t)+QR_B^c(t)$ implies that $\Delta x^c(t)$ and $R_B^c(t)$ mutually synchronizes $R_A^c(t)$ in the current step. If considering the long-term case where $\mathbb{E}(\Delta x^j(t)) \sim 0$, then $\mathbb{E}(\Delta x^j(t) + QR_B^j(t)) = Q\mathbb{E}(R_B^j(t))$. That reveals R_A is approximately Q times of R_B in the long run.

The second term $\mathrm{Var}(R_B^i(t))$ is the short-term lead-lag factor. It implies the current step return rate of stock A, $R_A^c(t)$ is affected by the volatility of the latest N steps return rate process of stock B, $R_B^i(t)$. This is consistent with the price lead-lag effect mentioned by Hou (2007) that it takes a few steps for the endogenous information of stock B to be transmitted to A and influencing its return rate.

The third term $\mathrm{Var}(\Delta x^j(t))$ is the long-term cointegration volatility factor. The coefficient $T/2$ illustrates the longer (more durable) is the cointegration relationship between two stocks, the more influential will be the volatility of the long-term process of the cointegration error (relative price) to the current-step return rate of stock A.

The last component $\mathrm{Cov}(R_f^j(t), \Delta x^j(t)) + \mathrm{Cov}(R_{in}^j(t), \Delta x^j(t)) + \mathrm{Cov}(R_M^j(t), \Delta x^j(t))$ is a long-term commovement factor, refeclting the long-term commovement between $R_f^j(t)$, $R_{in}^j(t)$ and $R_M^j(t)$ with $\Delta x^j(t)$ affect the current step return $R_A^c(t)$. This is consistent to the empircal evidence provided by King (1966), Ramnath (2002) and Kelly and Ljungqvist (2012). The long-term unique information of stock B (stock B hosts these information) from the fundamentals of the firm, index and market level strengthens the changes of relative price spread $\Delta x^j(t)$. The greater Q reflects the more significant influencing power.

1.3.2 Optimal trading weight

The pair trading is consistent to the self-financing, $h(t)$ and $-h(t)$ respectively refers to the trading weight of stock A and B. The wealth dynamics of each trading pair follows

$$\frac{dv(t)}{v(t)} = h(t)\left(\frac{dA(t)}{A(t)} - \frac{dB(t)}{B(t)}\right) \tag{1.27}$$

By substituting (1.10) and (1.13) into (1.27), the wealth dynamic of trading pair is specified as

$$\begin{aligned}\frac{dv(t)}{v(t)} = h(t)&\left[k(\theta(t) - x(t)) + \mu(Q-1) + \frac{1}{2}\eta^2 + \frac{1}{2}Q(Q-1)\sigma_B^2\right]dt \\ &+h(t)\eta Q(\sigma_f\rho_1 + \sigma_{in}\rho_2 + \sigma_M\rho_3)dt + h(t)\eta dW(t) \\ &+h(t)(Q-1)(\sigma_f dZ_1(t) + \sigma_{in}dZ_2(t) + \sigma_M dZ_3(t))\end{aligned} \tag{1.28}$$

In addition, the utility at the trading termination requires to be maximized

$$u(t,T) = \max_{h(t)}\ \mathrm{E}_t\left[\frac{1}{1-\gamma}V(T)^{1-\gamma}\right] \tag{1.29}$$

where γ denotes the coefficient of risk aversion.

Although the specific form of utility function is unknown at current position. It can be treated as a continuous stochastic optimisation problem. In specific, denoting the each current step as t and the forward step as τ. The spread $x(t)$ and stock price $A(t)$ and $B(t)$ reflect the whole available information adaptive to t, denoted as $\mathcal{F}_t$. Due to forthcoming information is unknown, we continuously target the utility of next step achieving to the level that equals to the maximization of current step

$$u(\tau) = \max_{h(t)} \mathrm{E}_t \left[u(t) \mid \mathcal{F}_t \right], \quad t < \tau \;\; t \in (0, T) \tag{1.30}$$

Investor's utility u at each time point should be decided by the portfolio wealth v and the spread price x, subject to Eq.(1.7) and (1.28) and following dynamic programming, it contributes the Hamilton-Jacobi-Bellman (HJB) that

$$\begin{aligned} &\frac{\partial u}{\partial t} + \max_h \mathrm{E}\{\frac{1}{2}\frac{\partial u}{\partial^2 v}\varphi_3 h^2 v^2 + \frac{\partial u}{\partial v \partial x}\varphi_1 hv + [k(\theta - x) + \varphi_2]\frac{\partial u}{\partial v}hv \\ &\qquad\qquad\qquad\qquad + k(\theta - x)\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial u}{\partial^2 x}\eta^2\} = 0 \\ &\varphi_1 = \eta^2 + (Q-1)(\sigma_f\rho_1 + \sigma_{in}\rho_2 + \sigma_M\rho_3)\eta \\ &\varphi_2 = \mu(Q-1) + \frac{1}{2}\eta^2 + \frac{1}{2}Q(Q-1)\sigma_B^2 + Q(\sigma_f\rho_1 + \sigma_{in}\rho_2 + \sigma_M\rho_3)\eta \\ &\varphi_3 = \eta^2 + (Q-1)^2\sigma_B^2 + 2(Q-1)(\sigma_f\rho_1 + \sigma_{in}\rho_2 + \sigma_M\rho_3)\eta \end{aligned} \tag{1.31}$$

Taking the first-order condition on the HJB, obtain the initial solution of optimal trading position of stock A

$$h^* = -\frac{\varphi_1 \frac{\partial u}{\partial v \partial x} + [k(\theta - x) + \varphi_2]\frac{\partial u}{\partial v}}{v\varphi_3 \frac{\partial u}{\partial^2 v}} \tag{1.32}$$

Plugging (1.32) back into the HJB

$$\begin{aligned} &\frac{1}{2}\left\{\varphi_1 \frac{\partial u}{\partial v \partial x} + [k(\theta - x) + \varphi_2]\frac{\partial u}{\partial v}\right\}^2 - \varphi_3[\frac{\partial u}{\partial t} + k(\theta - x)\frac{\partial u}{\partial x} \\ &\qquad\qquad\qquad + \frac{1}{2}\frac{\partial u}{\partial^2 x}\eta^2]\frac{\partial u}{\partial^2 v} = 0 \end{aligned} \tag{1.33}$$

The solution of $\frac{\partial u}{\partial v \partial x}$, $\frac{\partial u}{\partial v}$ and $\frac{\partial u}{\partial^2 v}$ cannot be obtained directly from (1.33). Thus, as similar as Korn and Kraft (2002), the separation of variables approach requires to be adopted.

Initially conjecturing

$$u(t,x,v) = v^{1-\gamma}\xi(t)e^{\alpha(t)x+\beta(t)x^2} \tag{1.34}$$

where $\alpha(t)$, $\beta(t)$ and $\xi(t)$ are the separated variables which at termination $\alpha(T) = \beta(T) = 0$ and $\xi(T) = (1-\gamma)^{-1}$, to recover the utility function to be identical as (1.29).

Substituting (1.34) and corresponding derivatives $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial v}$, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial v \partial x}$, $\frac{\partial u}{\partial^2 v}$ and $\frac{\partial u}{\partial^2 x}$ into (1.33), it gives

$$\begin{aligned}
&[\varphi_3\gamma\frac{\partial \beta}{\partial t} + 2\varphi_1^2(1-\gamma)\beta^2 - 2\varphi_3\gamma k\beta + 2\varphi_3\gamma\eta^2\beta^2 - 2\varphi_1(1-\gamma)k\beta \\
&+\frac{1}{2}(1-\gamma)k^2]x^2 + [\varphi_3\gamma\frac{\partial \alpha}{\partial t} + 2\varphi_1^2(1-\gamma)\alpha\beta - \varphi_3\gamma k\alpha + 2\varphi_3\gamma\eta^2\alpha\beta \\
&+2\varphi_1(1-\gamma)k\theta\beta + 2\varphi_3\gamma k\theta\beta + 2\varphi_1\varphi_2(1-\gamma)\beta - (1-\gamma)k^2\theta \\
&-\varphi_2(1-\gamma)k - \varphi_1(1-\gamma)k\alpha]x + [\frac{1}{2}(1-\gamma)(k^2\theta^2 + \varphi_2^2 + 2k\varphi_2\theta) \\
&+(1-\gamma)\varphi_1\alpha k\theta + (1-\gamma)\varphi_1\varphi_2\alpha + \frac{\gamma}{\xi}\varphi_3\frac{\partial \xi}{\partial t} + \varphi_3\gamma k\theta\alpha + \frac{1}{2}(1-\gamma)\varphi_1^2\alpha^2 \\
&\qquad\qquad\qquad + \gamma\varphi_3\eta^2\beta + \frac{1}{2}\gamma\varphi_3\eta^2\alpha^2] = 0
\end{aligned} \tag{1.35}$$

Subjected to (1.35), the polynomial corresponding to the coefficient of x^2 can be re-arranged to be

$$\frac{\partial \beta}{\partial t} + \frac{2\varphi_1^2(1-\gamma) + 2\varphi_3\gamma\eta^2}{\varphi_3\gamma}(\beta - \beta_+)(\beta - \beta_-) = 0 \tag{1.36}$$

it is an Riccati equation and β roots satisfy an quadratic equation

$$\beta^2 - \frac{\varphi_1(1-\gamma)k + \varphi_3\gamma k}{\varphi_1^2(1-\gamma) + \varphi_3\gamma\eta^2}\beta + \frac{(1-\gamma)^2 k}{4\varphi_1^2(1-\gamma) + 4\varphi_3\gamma\eta^2} = 0 \tag{1.37}$$

The roots can be generated by

$$\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix} = \frac{\varphi_1(1-\gamma)k + \varphi_3\gamma k \pm \sqrt{\varphi_3\gamma^2(\varphi_3 - 2\varphi_1 + \eta^2) + \varphi_3\gamma(2\varphi_1 - \eta^2)}}{2\varphi_1^2(1-\gamma) + 2\varphi_3\gamma\eta^2} \tag{1.38}$$

(1.36) can be rewritten to be

$$d\ln|\frac{\beta - \beta_+}{\beta - \beta_-}| = \frac{-2\varphi_1^2(1-\gamma) - 2\varphi_3\gamma\eta^2}{\varphi_3\gamma}(\beta_+ - \beta_-)dt \tag{1.39}$$

Integrating bothside of (1.39) and exsiting $\beta(T)=0$, it gives

$$\int_0^T d\ln|\frac{\beta-\beta_+}{\beta-\beta_-}| = \int_0^T \frac{-2\varphi_1^2(1-\gamma)-2\varphi_3\gamma\eta^2}{\varphi_3\gamma}(\beta_+-\beta_-)dt \tag{1.40}$$

$$\ln|\frac{\beta-\beta_+}{\beta-\beta_-}| - \ln|\frac{\beta_+}{\beta_-}| = \frac{2\varphi_1^2(1-\gamma)+2\varphi_3\gamma\eta^2}{\varphi_3\gamma}(\beta_+-\beta_-)(T-t) \tag{1.41}$$

From (1.41), the expression of $\beta(t)$ can obtained as

$$\beta(t) = \frac{1-e^{\frac{2\varphi_1^2(1-\gamma)+2\varphi_3\gamma\eta^2}{\varphi_3\gamma}(\beta_+-\beta_-)(T-t)}}{\frac{\beta_-}{\beta_+}-e^{\frac{2\varphi_1^2(1-\gamma)+2\varphi_3\gamma\eta^2}{\varphi_3\gamma}(\beta_+-\beta_-)(T-t)}}\beta_- \tag{1.42}$$

By substituting $\beta(t)$ roots, (1.38) into (1.42), $\beta(t)$ is solved out as

$$\begin{aligned}
\beta(t) &= \frac{k(a_1-a_3)}{a_3}[1+\frac{2a_2}{a_1-a_2-(a_1+a_2)e^{2a_2a_4k(T-t)}}] \\
a_1 &= (1-\gamma)\varphi_1+\gamma\varphi_3 \\
a_2 &= \sqrt{\gamma^2\varphi_3(\varphi_3-2\varphi_1+\eta^2)+\gamma\varphi_3(2\varphi_1-\eta^2)} \\
a_3 &= 2(1-\gamma)\varphi_1^2+2\gamma\varphi_3\eta^2 \\
a_4 &= \frac{\gamma}{\varphi_3}
\end{aligned} \tag{1.43}$$

On another side of (1.35), the polynomial corresponding to the coefficient of x can be re-arranged to be

$$\begin{aligned}
&\gamma\varphi_3\frac{\partial\alpha}{\partial t}+b_1\alpha+b_2=0 \\
&b_1 = 2\varphi_1^2(1-\gamma)\beta+2\varphi_3\gamma\eta^2\beta-\varphi_3\gamma k-\varphi_1(1-\gamma)k \\
&b_2 = [2\varphi_3\gamma k\widehat{\theta}+2\varphi_1(1-\gamma)k\widehat{\theta}+2\varphi_1\varphi_2(1-\gamma)]\beta-\varphi_2(1-\gamma)k-(1-\gamma)k^2\widehat{\theta}
\end{aligned} \tag{1.44}$$

rewritting (1.44) as

$$d\alpha = (-\frac{b_1}{\gamma\varphi_3}\alpha-\frac{b_1}{\gamma\varphi_3})dt \tag{1.45}$$

multiplying $e^{\int_0^t \frac{b_1}{\gamma\varphi_3}ds}$ both side, it gives

$$e^{\int_0^t \frac{b_1}{\gamma\varphi_3}ds}d\alpha = -\frac{b_1}{\gamma\varphi_3}e^{\int_0^t \frac{b_1}{\gamma\varphi_3}ds}\alpha dt - e^{\int_0^t \frac{b_1}{\gamma\varphi_3}ds}\frac{b_2}{\gamma\varphi_3}dt \tag{1.46}$$

Thus,

$$de^{\int_0^t \frac{b_1}{\gamma\varphi_3}ds}\alpha = -e^{\int_0^t \frac{b_1}{\gamma\varphi_3}ds}\frac{b_2}{\gamma\varphi_3}dt \tag{1.47}$$

integrating bothside of (1.47) from t to T

$$\alpha(T)e^{\int_0^T \frac{b_1(s)}{\gamma\varphi_3}ds} - \alpha(t)e^{\int_0^t \frac{b_1(s)}{\gamma\varphi_3}ds} = -\int_t^T \frac{b_2(\tau)}{\gamma\varphi_3}e^{\int_0^\tau \frac{b_1(s)}{\gamma\varphi_3}ds}d\tau \tag{1.48}$$

Subjected to (1.48), recognizing $\alpha(T) = 0$, then it yields

$$\alpha(t) = \int_t^T \frac{b_2(\tau)}{\gamma\varphi_3}e^{\int_t^\tau \frac{b_1(s)}{\gamma\varphi_3}ds}d\tau, \quad t < s < \tau < T \tag{1.49}$$

As $\alpha(t)$ and $\beta(t)$ are solved out, Eq. (1.43) and (1.49), then subjected to (1.34), the specific form of partial derivatives can given by

$$\frac{\partial u}{\partial v \partial x} = (1-\gamma)(\alpha + 2\beta x)\xi e^{\alpha x + \beta x^2} v^{-\gamma} \tag{1.50}$$

$$\frac{\partial u}{\partial v} = (1-\gamma)\xi e^{\alpha x + \beta x^2} v^{-\gamma} \tag{1.51}$$

$$\frac{\partial u}{\partial^2 v} = -\gamma(1-\gamma)\xi e^{\alpha x + \beta x^2} v^{-\gamma-1} \tag{1.52}$$

Finally, taking (1.50), (1.51) and (1.52) back into (1.32), the closed-form solution of optimal trading weight of stock A is obtained

$$h^*(t) = \frac{k(\theta - x(t)) + (\alpha(t) + 2\beta(t)x(t))\varphi_1 + \varphi_2}{\gamma\varphi_3} \tag{1.53}$$

At current position, beyond the convergence speed k and error correction term $\theta - x(t)$, other factors involving in the obtained solution h^* still have no tangible economic meaning, aim for this the discrete form is to attempt.

Plugging the specific form of φ_1, φ_2 and φ_3 into (1.53), gives

$$\begin{aligned} h^*(t) = & \frac{(Q-1)(\sigma_f\rho_1 + \sigma_{in}\rho_2 + \sigma_M\rho_3)\eta + \eta^2}{\gamma[2(Q-1)\sigma_1\eta + \eta^2 + (Q-1)^2\sigma_s^2]}(\alpha + 2\beta x(t)) \\ & + \frac{k(\theta - x(t)) + \frac{1}{2}\eta^2 + \mu(Q-1) + \frac{1}{2}Q(Q-1)\sigma_B^2 + Q(\sigma_f\rho_1 + \sigma_{in}\rho_2 + \sigma_M\rho_3)\eta}{\gamma[2(Q-1)\sigma_1\eta + \eta^2 + (Q-1)^2\sigma_s^2]} \end{aligned} \tag{1.54}$$

Substituting the equation substituting (1.18)∼(1.25) into (1.54), after arrangements, the discrete expression of the optimal trading weight can be obtained

$$
\begin{aligned}
h^*(t) = \{&\gamma[N\pi_2^2\mathrm{Var}(R_B^i(t)) + T\mathrm{Var}(\Delta x^j(t)) \\
&+2T\pi_2(\mathrm{Cov}(R_f^j(t),\Delta x^j(t)) + \mathrm{Cov}(R_{in}^j(t),\Delta x^j(t)) + \mathrm{Cov}(R_M^j(t),\Delta x^j(t)))]\}^{-1} \\
&\cdot\{k(\theta - x^c(t)) + T\pi_1\mathrm{Var}(\Delta x^j(t)) + \pi_2\mu_i + N\pi_3\mathrm{Var}(R_B^i(t)) \\
&\quad +T\pi_4(\mathrm{Cov}(R_f^j(t),\Delta x^j(t)) + \mathrm{Cov}(R_{in}^j(t),\Delta x^j(t)) + \mathrm{Cov}(R_M^j(t),\Delta x^j(t)))\} \\
&\quad \pi_1 = \frac{1}{2} + \alpha(t) + 2\beta(t)x^c(t), \quad \pi_2 = Q - 1 \\
&\quad \pi_3 = \frac{1}{2}Q(Q-1), \quad \pi_4 = Q + (Q-1)(\alpha(t) + 2\beta(t)x^c(t))
\end{aligned}
\tag{1.55}
$$

(1.55) reveals the coefficient of risk aversion, γ, the long-term cointegration volatility factor $\mathrm{Var}(\Delta x^j(t))$, the short-term trend of stock B return μ^i, the short-term lead-lag factor $\mathrm{Var}(R_B^i(t))$ and the long-term commovement factor $\mathrm{Cov}(R_f^j(t),\Delta x^j(t)) + \mathrm{Cov}(R_{in}^j(t),\Delta x^j(t)) + \mathrm{Cov}(R_M^j(t),\Delta x^j(t)))$ are determinants of the optimal trading weight on stock A at each step. Although the price changing component attributed by fundamental and industrial index level $R_f^j(t)$ and $R_{in}^j(t)$ are unobservable, the long-term commovement factors can be estimated via (1.26).

Usually in the cross market arbitrage trading, the cointegrating vector $Q \sim 1$ then (1.55) can be simplified to be

$$
\begin{aligned}
h^*(t) = \frac{\pi_1}{\gamma} + \frac{1}{\gamma\mathrm{Var}(\Delta x^j(t))}[&\frac{k(\theta - x^c(t))}{T} + \mathrm{Cov}(R_f^j(t),\Delta x^j(t)) \\
&+\mathrm{Cov}(R_{in}^j(t),\Delta x^j(t)) + \mathrm{Cov}(R_M^j(t),\Delta x^j(t)))]
\end{aligned}
\tag{1.56}
$$

Comparing (1.56) with (1.55), the short-term trend of return rate of stock B and the short-term lead-lag factor no longer affect the optimal weight. This implies that cross market trading pair converge every fast. Implicitly, the spill-over effect between intraday prices related to firm-level cross-sectional information almost disappears. Q is approximately equal to one for the stock pair that have the unanimous fundamentals. This is supported by the studies such as Alsayed and McGroarty (2012) referring to cross-listed stocks in US. The literature also supports the firm-level cross-sectional information of the host stock can be transmitted to the lag stock in an extremely short time when it comes to cross-listed stocks evidenced by Chen et al. (2009), they suggest if the cross-listed stocks are observed with a delay above 30 minutes, then the lead-lag effect is not statistically significant by Granger causality test. In addition Gagnon and Karolyi (2010) estimate through panel-regression analysis to show the daily-lead-lag coefficient is almost zero. Based on these studies, it is safe to assume that for cross-listed stocks, setting $Q = 1$. Subject to (1.56) it is worth to note when the cointegration relationship $\mathrm{Var}(\Delta x^j(t))$ is more volatile, the optimal weight will be reduced. The similar condition will be taken place when investor is more risk averse (the relative lower coefficient of risk aversion γ) or when the long-term commovement factor between two stocks $\mathrm{Cov}(R^j_{in}(t), \Delta x^j(t)) + \mathrm{Cov}(R^j_M(t), \Delta x^j(t))$ is decreasing.

1.4 Trading test data

In this section an empirical application of the previous methodology is presented. The application is focused on cross-listed stock pairs between Eurozone vs. US, UK vs. US and the Shanghai vs. Hong Kong Stock Exchanges. The host stock is named as lead stock and the foreign listed stock is named lag stock. Three different portfolios consisted by the cross-listed stocks (Eurozone-US, UK-US and China A-H share) will be constructed and evaluated through the proposed methodology. The periods of trading test under study are presented below in

Table 1.1: The first period of trading test

Cross-listed trading portfolio	In-Sample	Out-of-Sample
UK-US	14/07/2009-26/07/2011	27/07/2011-12/08/2013
Eurozone-US	14/07/2009-19/07/2011	20/07/2011-29/07/2013
China A-H share	13/07/2009-31/08/2011	01/09/2011-14/10/2013

Table 1.2: The second period of trading test

Cross-listed trading portfolio	In-Sample	Out-of-Sample
UK-US	27/07/2011-12/08/2013	13/08/2013-24/08/2015
Eurozone-US	20/07/2011-29/07/2013	30/07/2013-13/08/2015
China A-H share	01/09/2011-14/10/2013	15/10/2013-16/11/2015

In each pair the same amount of capital will be dedicated. The selected cross-listed stocks satisfy two main criteria. Their historical closing price data are available for the examined period under study, their average daily trading volumes is greater than 150000, to avoid the trading crash on prices attributed by this strategy (for the historical trading volume of each stock during In-the-sample period, see the Appendix C) . The second criteria is the log-price of each cross-listed stock pair are cointegrated during the relevant in-sample periods, keep matched to process (1.6) and (1.7). The cointegration test has been done through the Engle-Granger approach at 0.05 significant level.

Concerning the Eurozone-US portfolio, the lead stocks are listed on DAX, CAC40, FTSE-MIB, AEX or ESTX50 stock indices and are denominated in Euros. The EURONEXT 100 index will represent the Eurozone host market. The lag stocks are listed either on NYSE or the NASDAQ stock indices and are traded through the ADR denoted in US dollars. Concerning the UK-US portfolio, the UK stocks are listed on FTSE 100, priced in GBP and the FTSE All-Share index represents the host market. The US stocks are similar with the Eurozone - US portfolio. In the last portfolio, the A shares listed on the Shanghai Stock Exchange 180 index (SSE180) is the lead stock denominated in CNY while the Shanghai Stock Exchange Composite Index (SSE) represents the host market. Their lag stocks are traded in the Hong Kong Stock Exchange and are priced in Hong Kong dollars, they are named as H shares. The selected cross listed stocks that constructed by the three portfolios are presented in the Table 1.3 $\sim$ 1.6

Table 1.3: Cross-listed stock trading in Eurozone-US portfolio

Stock Name	Ticker(lead)	Ticker(lag)
AEGON	AGN.AS	AEG
Alcatel-lucent	ALU.PA	ALU
Anheuser-Busch InBev SA	ABI.BR	BUD
ArcelorMittal SA	MT.AS	MT
Eni SpA	ENI.MI	E
Fresenius Medical Care	FME.DE	FMS
Orange	ORA.PA	ORAN
RELX NV	REN.AS	RENX
Sanofi	SAN.PA	SNY
SAP SE	SAP.DE	SAP
STMicroelectronics NV	STM.MI	STM
Telecom Italia S.p.A	TIT.MI	TI
Tenaris S.A	TEN.MI	TS
TOTAL S.A	FP.PA	TOT
Unilever	UNA.AS	UN

Table 1.4: Cross-listed stock trading in UK-US portfolio

Stock Name	Ticker(lead)	Ticker(lag)
ARM holdings plc	ARM.L	ARMH
AstraZeneca plc	AZN.L	AZN
Barclays plc	BARC.L	BCS
BHP Billiton plc	BLT.L	BBL
BP plc	BP.L	BP
British American Tobacco plc	BATS.L	BTI
BT Group plc	BT-A.L	BT
Carnival plc	CCL.L	CUK
Diageo plc	DGE.L	DEO
GlaxoSmithKline plc	GSK.L	GSK
HSBC Holding plc	HSBA.L	HSBC
InterContinental Hotels Group	IHG.L	IHG
Lloyds Banking Group plc	LLOY.L	LYG
National Grid plc	NG.L	NGG
Pearson plc	PSON.L	PSO
Prudential plc	PRU.L	PUK
Randgold Resources Ltd	RRS.L	GOLD
Rio Tinto plc	RIO.L	RIO
Royal Bank of Scotland Group	RBS.L	RBS
Royal Dutch Shell plc-A	RDSA.L	RDS-A
Royal Dutch Shell plc-B	RDSB.L	RDS-B
Shire plc	SHP.L	SHPG
Smith&Newphew plc	SN.L	SNN
Unilever plc	ULVR.L	UL
Vodafone Group plc	VOD.L	VOD

Table 1.5: Cross-listed stock trading in China A-H share portfolio

Stock Name	Ticker(lead)	Ticker(lag)
Air China Ltd	601111	0753
Aluminum of China Co Ltd	601600	2600
Anhui Conch Cement Co Ltd	600585	0914
Bank of China Ltd	601988	3988
Bank of Communication Co Ltd	601328	3328
China CITIC Bank Co Ltd	601998	0998
China Eastern Airlines Co Ltd	600115	0670
China Life Insurance Co Ltd	601628	2628
China Merchants Bank Co Ltd	600036	3968

Table 1.6: Cross-listed stock trading in China A-H share portfolio

Stock Name	Ticker(lead)	Ticker(lag)
China Shenhua Energy Co Ltd	601088	1088
China Southern Airlines Co Ltd	600029	1055
Industrial&Commercial Bank of China	601398	1398
Tsingtao Brewery Co Ltd	600600	0168
Zijin Ming Group Co Ltd	601899	2899

In this trading application, the stock prices of the US and H shares have been converted back to the currencies of their relevant host market based on the relevant exchange rate. After converting the currency exchange rate, the lag stock price series also need to be adjusted by the convert ratio. Convert Ratio shows how many shares of the lag stock trade as a single share of the host stock during corresponding in-the-sample trading period.

Table 1.7: Convert Ratio of trading stocks in Eurozone-US portfolio

Stock Name	Convert Ratio(1^{st} perriod)	Convert Ratio(2^{nd} period)
AEGON	0.98	0.98
Alcatel-lucent	1.04	1.04
Anheuser-Busch InBev SA	1.01	1.02
ArcelorMittal SA	0.95	0.97
Eni SpA	0.49	0.49
Fresenius Medical Care	1.96	1.96
Orange	0.98	0.98
RELX NV	0.90	0.94
Sanofi	2.00	1.99
SAP SE	1.01	0.99
STMicroelectronics NV	0.91	0.93
Telecom Italia S.p.A	0.10	0.10
Tenaris S.A	0.49	0.50
TOTAL S.A	0.97	0.97
Unilever	0.99	1.00

Table 1.8: Convert Ratio of trading stocks in UK-US portfolio

Stock Name	Convert Ratio(1^{st} perriod)	Convert Ratio(2^{nd} period)
ARM holdings plc	32.89	33.12
AstraZeneca plc	196.43	198.7
Barclays plc	24.66	24.90
BHP Billiton plc	43.41	45.25
BP plc	16.27	16.40
British American Tobacco plc	50.10	50.12
BT Group plc	19.66	19.88
Carnival plc	96.17	98.29
Diageo plc	24.57	24.67
GlaxoSmithKline plc	50.04	50.42
HSBC Holding plc	18.87	19.50
InterContinental Hotels Group	104.58	104.69
Lloyds Banking Group plc	24.61	25.17
National Grid plc	19.38	19.86
Pearson plc	98.59	99.25
Prudential plc	49.49	49.71
Randgold Resources Ltd	98.18	98.61
Rio Tinto plc	95.23	95.68
Royal Bank of Scotland Group	48.89	49.90
Royal Dutch Shell plc-A	49.33	49.39
Royal Dutch Shell plc-B	48.56	49.26
Shire plc	33.20	33.21
Smith&Newphew plc	48.38	48.53
Unilever plc	98.47	99.34
Vodafone Group plc	10.46	10.26

Table 1.9: Convert Ratio of trading stocks in China A-H share portfolio

Stock Name	Convert Ratio(1^{st} perriod)	Convert Ratio(2^{nd} period)
Air China Ltd	1.68	1.24
Aluminum of China Co Ltd	1.84	2.08
Anhui Conch Cement Co Ltd	0.91	0.82
Bank of China Ltd	1.10	1.09
Bank of Communication Co Ltd	0.94	0.98
China CITIC Bank Co Ltd	1.26	1.21
China Eastern Airlines Co Ltd	2.25	1.62
China Life Insurance Co Ltd	0.85	1.00
China Merchants Bank Co Ltd	0.84	0.92

Table 1.10: Convert Ratio of trading stocks in China A-H share portfolio

Stock Name	Convert Ratio(1^{st} perriod)	Convert Ratio(2^{nd} period)
China Shenhua Energy Co Ltd	0.94	0.91
China Southern Airlines Co Ltd	2.48	1.42
Industrial&Commercial Bank of China	0.89	0.98
Tsingtao Brewery Co Ltd	1.01	0.92
Zijin Ming Group Co Ltd	1.54	1.80

If regarding each portfolio as one two-year closed-end fund, then following the proposed methodology section the trading horizon T is set as 504 trading days. The trading weights of lead-lag stocks is suggested by the equation (1.53) and (1.56), the inside parameter φ_1, φ_2, φ_3 and π_1 are generated with (1.31) and (1.55), with detailed parameter estimation in (1.20) $\sim$ (1.26) and the Appexdix B. The short-term window N is 15 trading days and the long-term window T is 504 trading days, both of them are sliding windows, with daily rolling frequency.

1.5 Trading application

The optimal trading weights are established on the self-financing assumption, namely, the cross-listed stock pair should be kept equal weight with opposite positions throughout whole trading. Majorly our strategy can be applied by either trading shares or trading CFDs. In the comparison CFDs trading has a few advantages than trading shares.

Firstly, CFDs trading has the relatively lower initial margin requirement rate (e.g. 5%), which benefits the capital utilisation is more efficient than that of share trading. Secondly CFDs trading does not charge the stamp duty and levy (Financial Transaction Tax). This implies CFDs trading has the lower transaction cost than Share trading. The Table 1.11 and Table 1.12 completely summaries the transaction cost components charged[5].

[5]The transaction cost is based on the open sources of Interactive Broker

Table 1.11: Transaction costs summary for share trading (Basis points)

Market	Commission	Exchange fee	Clearing	Duties	Algorithm fee	Sum
US	50	0.30	0.02	0.002	-	50.32
Eurozone	8	0.16	-	10	-	18.16
UK	8	0.45	-	25	-	33.45
China A	8	0.70	6	10	-	24.70
China H	8	0.50	0.2	10.27	-	18.97

Table 1.12: Transaction costs summary for CFD trading (Basis points)

Market	Commission	Exchange fee	Clearing	Duties	Algorithm fee	Sum
US	10	-	-	-	-	10
Eurozone	5	-	-	-	10	15
UK	5	-	-	-	15	20
China A	5	-	-	10.27	10	25.27
China H	5	-	-	-	10	15

Based on the above tables it can be approximated that the transaction costs of the relevant equity CFDs for the UK-US cross-listed pairs, Eurozone-US and China A-H share is 15bps, 12.5bps and 20bps respectively. On the contrary, the transaction costs of trading shares for these three portfolios is 42.5bps, 35bps and 22.5bps.
Thirdly CFDs trading is more flexible than shares because it is traded over the OTC markets. In specific our strategy requires the cross-listed pair should be rebalanced to the optimal trading weight at the end of each day theoretically. In addition, is computed by using the closing prices of cross-listed pair. While it is unachievable in the real trading. We cannot rebalance their weight simultaneously on both market sides because the foreign stock market is closed delayed to the host market (e.g. London Stock Exchange closing time is UTC 16:30 but New York Stock Exchange closes at UTC 21:00).
In the real trading the above problem can be more easily solved by trading CFDs. OTC markets offer major institutional investors the eligibility to trade equity CFDs throughout 24 hours. Thus traders do not need to rebalance the CFD weight of the lead stock at the host market closing time but need to rebalance the lag stock CFD

weight at the closing time of foreign market in the first. After that, during the interval time from the foreign market closing to the next day pre-market auction time of the host market, traders required to gradually rebalance the weight of lead equity CFD to the optimal position h^* by executing the orders that emerging at the closing-price of the lead stock. In specific, Major information can drive the significant price movement are origin from the host market. In often the formal information are not released beyond official working time of host market countries. This ensures the price of lead stock CFD in the OTC market keeps fluctuating around the closing-price of lead stock during the interval time between foreign market closing and the next day pre-market auction time of the host market. Moreover, the stocks we selected are blue-chip equity in the representative index (e.g. FTSE100), this guarantees their trading liquidity of the equity CFDs in the OTC market is sufficient. Therefore, above two premises can ensure institutional investors to execute the orders at the closing-price of lead stock by 24-CFDs trading. On the other hand, this applying strategy also can be achieved through the similar approach (24 hours OTC market) with trading shares. But the less trading liquidity in trading share over OTC market may attribute it is harder than trading equity CFDs.

The optimal weights h^* is a closed-form solution with computing efficiency, it is feasible in applying on bar-frequency trading with synchronous prices between cross-listed stock pair. However, for the simplicity we use the daily closing price in this test. More importantly comparing with the synchronous price, the cointegration relationship between the closing prices of stock pair is the special point. Because this relationship is leaded not only by the differential market regions (structures) but also leaded by the time overlap. This section aims to check whether this trading application can profit substantially or not, by utilising this special cointegration relationship of closing price pair from the time overlap.

1.6 Constrained Capital Utilisation

This application chooses to trade both shares and equity-based CFDs without leveraging. To the shares trading approach that we proposed to split the whole initial endowment of each trading pair to be the equal half. Half is reserved for trading the lead stock and another half for the lag stock. The optimal trading weight formula (1.56) demonstrates that investor's coefficient of risk aversion γ is one of crucial determinants on the size of h^*. Therefore it is essential to locate one particular coefficient of risk aversion γ^*, to guarantee the sum of maximized capital maintained in every trading day from both sides, cannot exceed % of the capital owned by the investor, throughout the whole trading horizon of the in-sample period. After that we carry on using the γ^* in the corresponding Out-the-Sample test.

In respect with trading equity CFDs, the 5% initial margin requirement rate allows us can traded 20 times higher capitals than the capital owned. In addition, for simplicity we do not consider risk managing issues by using adequately lower capital for trading. Specifically, the maximized capital we allocated into the CFDs trading account is lower than 15% of the capitals of each trading pair (7.5% for each side of one pair). On the other hand, we reserved higher than 85% of total capital in case the margin call problem. As similar as trading shares above, we rely on this trading capital restriction to pick up the specific coefficient of risk aversion γ during the In-the-Sample period, which should guarantee the capital used in CFDs trading cannot be greater than 15% of the total capital owned. Thereby this γ^* will be carried on in the corresponding CFDs trading in Out-the-Sample test period.

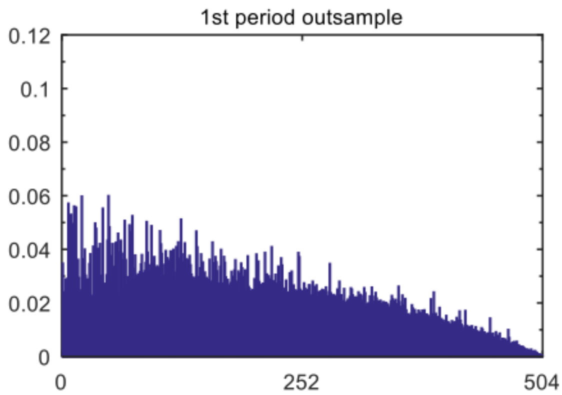
The percentages of capital utilization throughout trading tests in all periods and portfolio (CFDs&Shares) are presented below:

Table 1.13: Capital utilizing rate of UK-US portfolio in the 1^{st} period

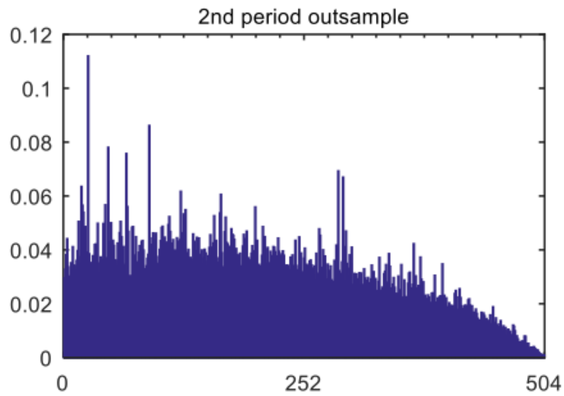
	CFDs		Shares	
Capital utilization	In-Sample	Out-of-Sample	In-Sample	Out-of-Sample
Mean	2.57%	2.41%	23.93%	21.57%
Std.	1.60%	1.18%	17.70%	14.56%
Maximum	11.11%	6.00%	85.76%	64.70%
Minimum	0.02%	0.03%	0.08%	0.14%

Table 1.14: Capital utilizing rate of UK-US portfolio in the 2^{nd} period

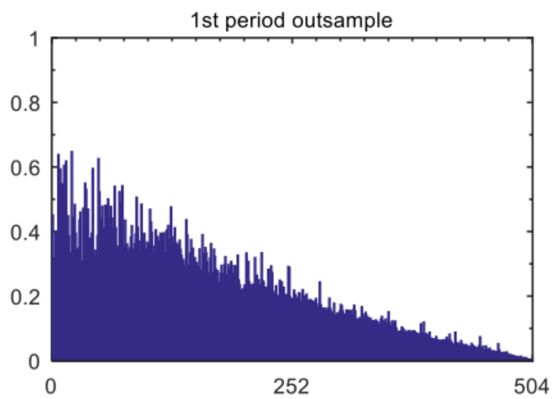
	CFDs		Shares	
Capital utilization	In-Sample	Out-of-Sample	In-Sample	Out-of-Sample
Mean	2.82%	3.14%	20.58%	22.16%
Std.	2.04%	1.35%	17.24%	13.14%
Maximum	11.03%	11.20%	82.89%	85.49%
Minimum	0.04%	0.08%	0.12%	0.30%



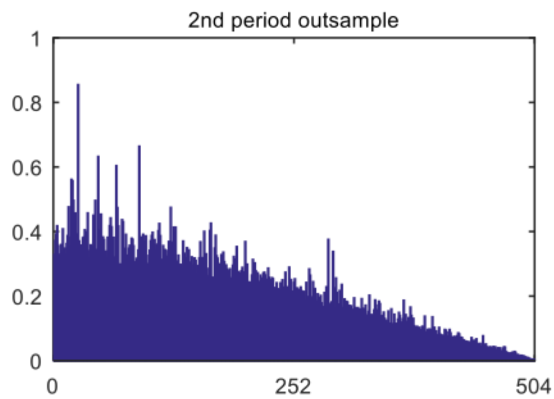
(a) CFD trading



(b) CFD trading



(c) Share trading



(d) Share trading

Figure 1.1: Capital utilizing rate of UK-US portfolio in Out-of-Sample

Table 1.15: Capital utilizing rate of Eurozone-US portfolio in the 1^{st} period

	CFDs		Shares	
Capital utilization	In-Sample	Out-of-Sample	In-Sample	Out-of-Sample
Mean	3.16%	1.17%	23.07%	13.91%
Std.	1.80%	0.70%	15.60%	9.16%
Maximum	10.20%	3.40%	82.77%	41.80%
Minimum	0.03%	0.01%	0.09%	0.11%

Table 1.16: Capital utilizing rate of Eurozone-US portfolio in the 2^{nd} period

	CFDs		Shares	
Capital utilization	In-Sample	Out-of-Sample	In-Sample	Out-of-Sample
Mean	3.04%	3.37%	23.11%	26.14%
Std.	1.72%	1.39%	16.63%	14.89%
Maximum	8.29%	6.69%	71.54%	61.77%
Minimum	0.04%	0.06%	0.15%	0.27%

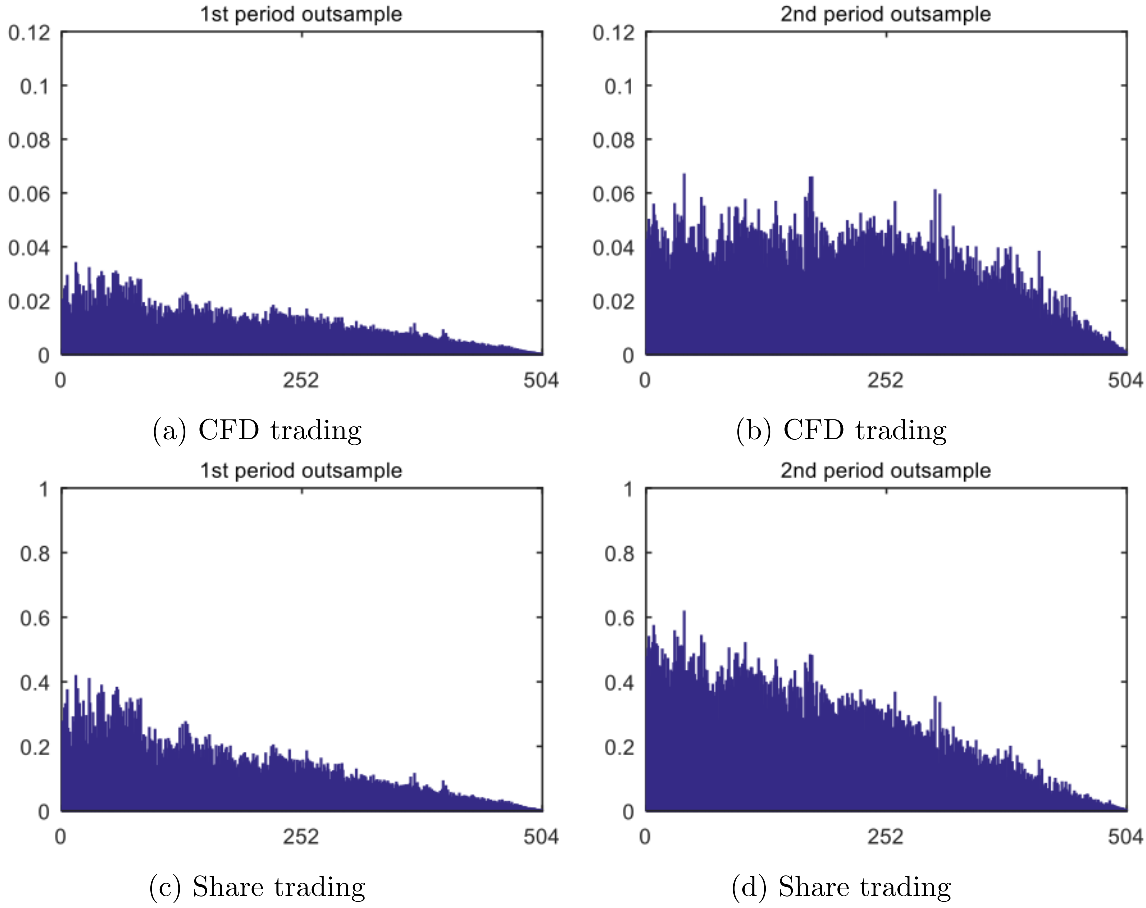


Figure 1.2: Capital utilizing rate of Eurozone-US portfolio in Out-of-Sample

The Table 1.13 ∼ Table 1.16 and Figure 1.1 and Figure 1.2 illustrate throughout the whole In-Sample and Out-of-Sample tradings, all the maximum capital percentages of UK-US portfolio and Eurozone-US portfolio are lower than the 15% (CFDs trading) and 100% (shares trading). That implies the portfolio diversification play

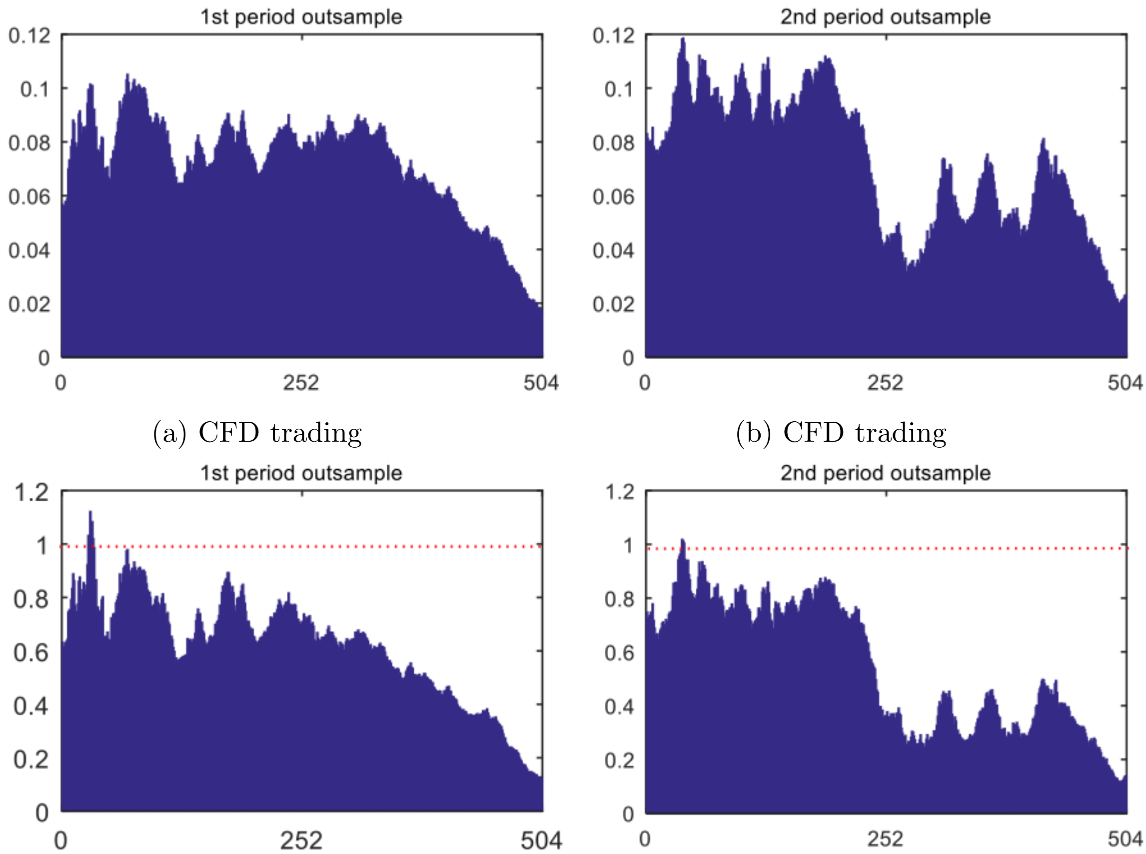
the role in enhancing capital utilisation efficiency. In process of trading UK-US and Eurozone-US portfolios with either CFDs or shares, the average trading capital percentage do not show the obvious difference between the In-sample and Out-of-Sample period trading.
The capital percentage carried on tend to gradually shrink accompanying the time to maturity. This benefits the Profit&Loss to reduce the influence from the decay of mean-reversion of price spread. More specific this kind of convergence trading default the cointegration relationship illustrated in the in-sample periods (the parameters are estimated since In-Sample horizon with rolling window) can be continually maintained in the corresponding Out-of-Sample periods. However, along the trading time passing in Out-of-Sample period, the older cointegration relationships are suffering the erosions. Overall it can be suggested that the strategy stability is acceptable on the viewpoint of portfolio trading capital for UK-US and Eurozone-US portfolio.

Table 1.17: Capital utilizing rate of China A-H share portfolio in the 1^{st} period

	CFDs		Shares	
Capital utilization	In-Sample	Out-of-Sample	In-Sample	Out-of-Sample
Mean	3.63%	7.12%	29.42%	60.96%
Std.	1.73%	1.90%	15.58%	19.78%
Maximum	11.03%	10.51%	92.96%	113.47%*
Minimum	0.71%	1.61%	4.93%	11.41%

Table 1.18: Capital utilizing rate of China A-H share portfolio in the 2^{nd} period

	CFDs		Shares	
Capital utilization	In-Sample	Out-of-Sample	In-Sample	Out-of-Sample
Mean	3.55%	7.17%	27.52%	54.04%
Std.	1.61%	2.56%	14.70%	24.35%
Maximum	9.40%	11.85%	85.01%	102.35%*
Minimum	0.89%	1.94%	5.62%	11.22%



(a) CFD trading

(b) CFD trading

(c) Share trading

(d) Share trading

Figure 1.3: Capital utilizing rate of China A-H share portfolio in Out-of-Sample

The Table 1.17, Table 1.18 and Figure 1.3 show that in the China A-H portfolio, their maximized utilization rate of capital of Share trading in the Out-of-Sample for both two periods, exceed the 100% of total owned capital, nevertheless the specific coefficient of risk aversion picked up γ^* in the In-sample period can ensuring the maximum capital required is lower than 100%. This implies during the share trading of China A-H share portfolio, there a few trading days the strategy has to borrow up to 13.47% and 2.35% over owned equity to maintain the position, it could be one nervous of risk management. Comparing with share trading for applying this strategy, the capital CFDs trading illustrates its stability, the capital utilization percentage in both two Out-of-Sample are lower enough than 15%, that is the capital using celling in CFDs trading. However it should be cautionary about both CFDs trading and shares, portfolio trading capital used during the out-sample periods are nearly twice of that used in the in-sample periods. Therefore, it can be suggested that standing on the angle of trading capital percentage, China A-H share portfolio requires the higher capital requirement and the more skilled risk management technique than that required by the UK-US and European-US portfolios.

1.7 Trading Performance

Table 1.19: UK-US CFDs portfolio performance in the 1^{st} period

Performance indicator	In-Sample	Out-of-Sample
Annualized return (ex.TC)	40.03%	25.03%
Annualized return	29.36%	14.74%
Volatility (ex.TC)	12.35%	4.09%
Volatility	12.19%	3.80%
Sharpe ratio (ex.TC)	3.08	5.63
Sharpe ratio	2.24	3.35
Information ratio (ex.TC)	1.00	0.91
Information ratio	0.49	0.35
Maximum drawdown(ex.TC)	0.80%	0.47%
Maximum drawdown	0.95%	0.61%
Maximum drawdown duration (ex.TC)	4	2
Maximum drawdown recovery duration (ex.TC)	3	2
Annualized Transaction Cost	10.67%	10.29%

Table 1.20: UK-US Shares portfolio performance in the 1^{st} period

Performance indicator	In-Sample	Out-of-Sample
Annualized return (ex.TC)	15.08%	8.38%
Annualized return	4.58%	-0.99%
Volatility (ex.TC)	5.03%	2.00%
Volatility	4.72%	1.86%
Sharpe ratio (ex.TC)	2.60	3.18
Sharpe ratio	0.55	-1.61
Information ratio (ex.TC)	-0.23	0.00
Information ratio	-0.84	-0.53
Maximum drawdown(ex.TC)	0.50%	0.43%
Maximum drawdown	1.98%	4.13%
Maximum drawdown duration (ex.TC)	4	3
Maximum drawdown recovery duration (ex.TC)	3	2
Annualized Transaction Cost	10.50%	9.37%

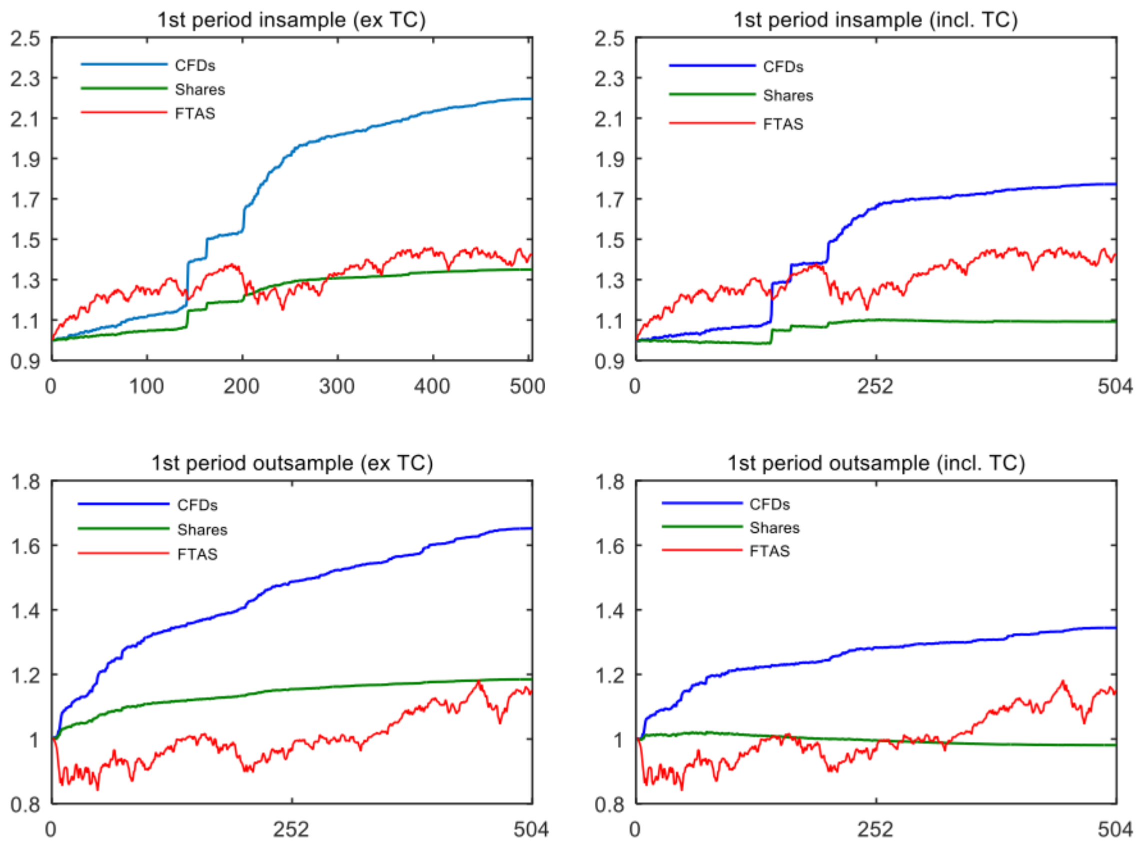


Figure 1.4: Trading performance of UK-US portfolio in the 1^{st} period

Table 1.21: UK-US CFDs portfolio performance in the 2^{nd} period

Performance indicator	In-Sample	Out-of-Sample
Annualized return (ex.TC)	47.23%	30.78%
Annualized return	30.19%	16.36%
Volatility (ex.TC)	7.29%	5.20%
Volatility	6.77%	4.74%
Sharpe ratio (ex.TC)	6.20	5.54
Sharpe ratio	4.16	3.03
Information ratio (ex.TC)	2.03	2.68
Information ratio	1.16	1.56
Maximum drawdown(ex.TC)	0.75%	0.94%
Maximum drawdown	0.95%	1.04%
Maximum drawdown duration (ex.TC)	2	3
Maximum drawdown recovery duration (ex.TC)	2	2
Annualized Transaction Cost	17.04%	14.42%

Table 1.22: UK-US Shares portfolio performance in the 2^{nd} period

Performance indicator	In-Sample	Out-of-Sample
Annualized return (ex.TC)	12.78%	7.29%
Annualized return	-0.45%	-2.72%
Volatility (ex.TC)	2.53%	1.58%
Volatility	2.24%	1.46%
Sharpe ratio (ex.TC)	4.26	3.34
Sharpe ratio	-1.09	-3.24
Information ratio (ex.TC)	0.28	0.88
Information ratio	-0.47	0.05
Maximum drawdown(ex.TC)	0.43%	0.54%
Maximum drawdown	5.31%	5.57%
Maximum drawdown duration (ex.TC)	3	3
Maximum drawdown recovery duration (ex.TC)	2	2
Annualized Transaction Cost	13.23%	10.01%

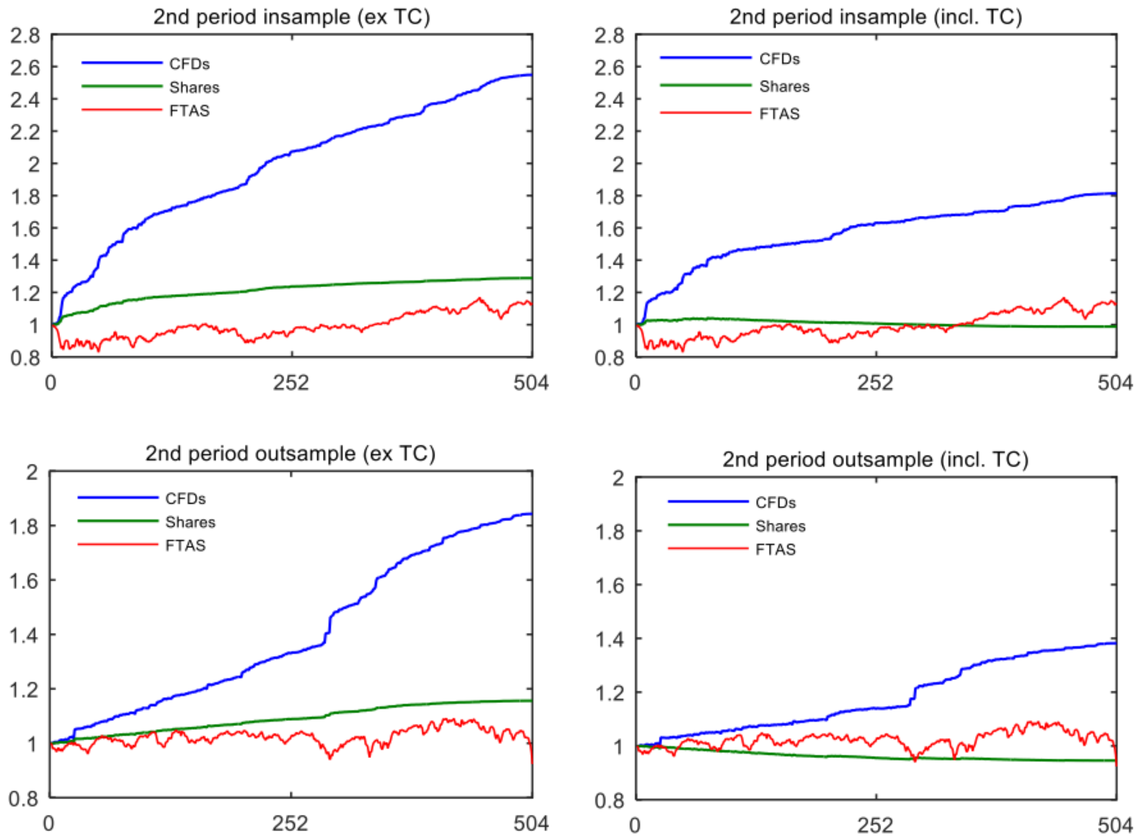


Figure 1.5: Trading performance of UK-US portfolio in the 2^{nd} period

Table 1.23: Eurozone-US CFDs portfolio performance in the 1^{st} period

Performance indicator	In-Sample	Out-of-Sample
Annualized return (ex.TC)	57.09%	17.27%
Annualized return	37.72%	12.40%
Volatility (ex.TC)	5.99%	2.40%
Volatility	5.56%	2.22%
Sharpe ratio (ex.TC)	9.20	6.36
Sharpe ratio	6.42	4.68
Information ratio (ex.TC)	2.23	0.46
Information ratio	1.25	0.24
Maximum drawdown(ex.TC)	0.49%	0.12%
Maximum drawdown	0.71%	0.14%
Maximum drawdown duration (ex.TC)	3	2
Maximum drawdown recovery duration (ex.TC)	2	2
Annualized Transaction Cost	19.37%	4.87%

Table 1.24: Eurozone-US Shares portfolio performance in the 1^{st} period

Performance indicator	In-Sample	Out-of-Sample
Annualized return (ex.TC)	13.06%	9.78%
Annualized return	0.45%	2.20%
Volatility (ex.TC)	2.00%	1.44%
Volatility	1.81%	1.19%
Sharpe ratio (ex.TC)	5.52	5.40
Sharpe ratio	-0.86	0.17
Information ratio (ex.TC)	0.00	0.12
Information ratio	-0.65	-0.23
Maximum drawdown(ex.TC)	0.33%	0.09%
Maximum drawdown	1.85%	1.18%
Maximum drawdown duration (ex.TC)	3	2
Maximum drawdown recovery duration (ex.TC)	2	2
Annualized Transaction Cost	12.61%	7.58%

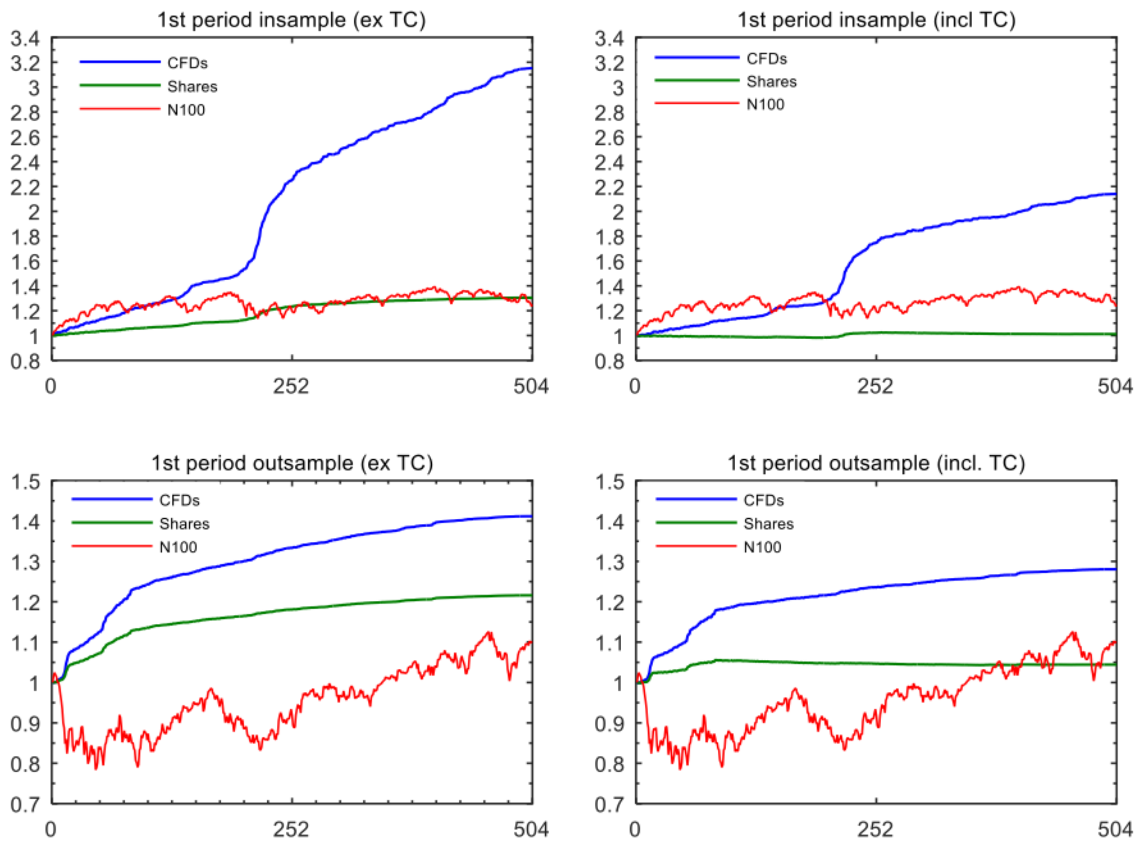


Figure 1.6: Trading performance of Eurozone-US portfolio in the 1^{st} period

Table 1.25: Eurozone-US CFDs portfolio performance in the 2^{nd} period

Performance indicator	In-Sample	Out-of-Sample
Annualized return (ex.TC)	49.25%	25.75%
Annualized return	35.25%	15.14%
Volatility (ex.TC)	6.00%	3.10%
Volatility	5.64%	2.97%
Sharpe ratio (ex.TC)	7.87	7.65
Sharpe ratio	5.89	4.42
Information ratio (ex.TC)	1.83	0.75
Information ratio	1.21	0.07
Maximum drawdown(ex.TC)	0.36%	0.47%
Maximum drawdown	0.46%	0.63%
Maximum drawdown duration (ex.TC)	3	4
Maximum drawdown recovery duration (ex.TC)	2	4
Annualized Transaction Cost	14.00%	10.61%

Table 1.26: Eurozone-US Shares portfolio performance in the 2^{nd} period

Performance indicator	In-Sample	Out-of-Sample
Annualized return (ex.TC)	15.18%	6.76%
Annualized return	3.78%	-0.91%
Volatility (ex.TC)	2.44%	1.18%
Volatility	2.16%	1.11%
Sharpe ratio (ex.TC)	5.40	4.04
Sharpe ratio	0.82	-2.62
Information ratio (ex.TC)	0.33	-0.47
Information ratio	-0.19	-0.97
Maximum drawdown(ex.TC)	0.21%	0.18%
Maximum drawdown	1.11%	2.32%
Maximum drawdown duration (ex.TC)	2	4
Maximum drawdown recovery duration (ex.TC)	2	5
Annualized Transaction Cost	11.40%	7.67%

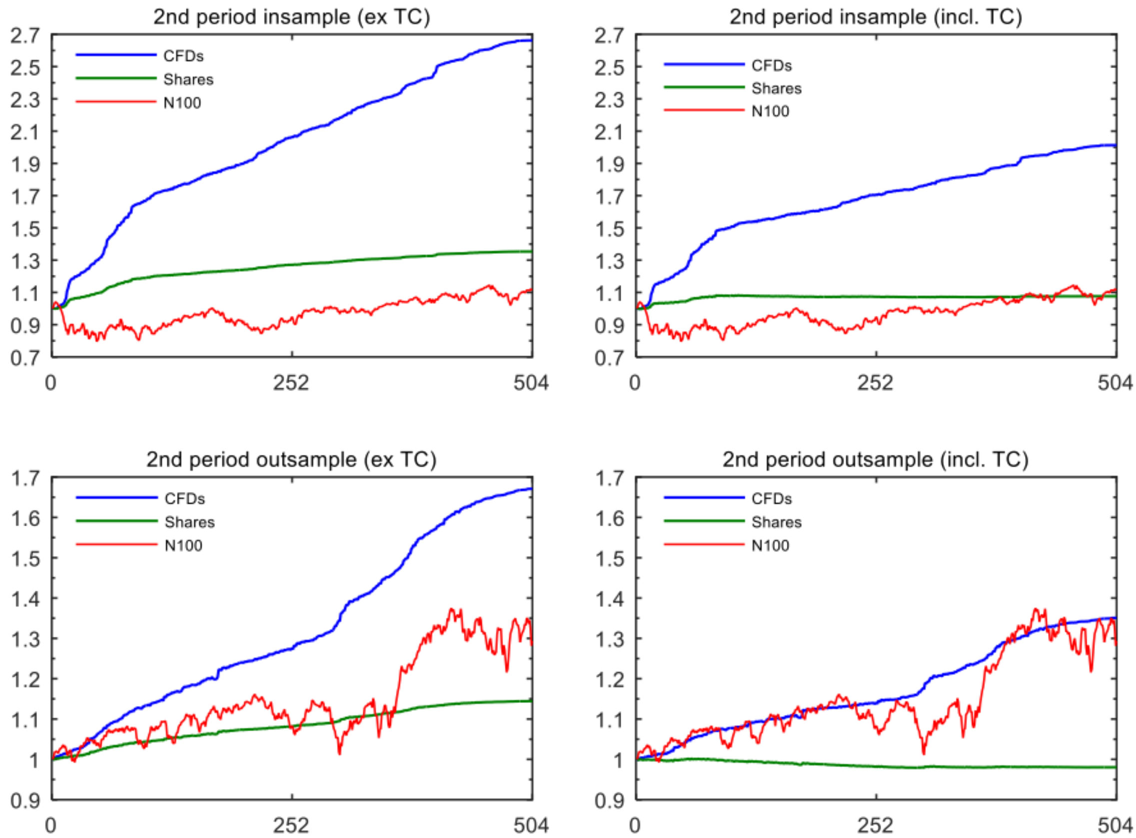


Figure 1.7: Trading performance of Eurozone-US portfolio in the 2^{nd} period

Table 1.27: China A-H share CFDs portfolio performance in the 1^{st} period

Performance indicator	In-Sample	Out-of-Sample
Annualized return (ex.TC)	34.64%	13.09%
Annualized return	27.09%	6.39%
Volatility (ex.TC)	7.04%	7.83%
Volatility	7.05%	7.78%
Sharpe ratio (ex.TC)	4.35	1.16
Sharpe ratio	3.28	0.31
Information ratio (ex.TC)	1.60	0.94
Information ratio	1.31	0.59
Maximum drawdown(ex.TC)	2.49%	6.17%
Maximum drawdown	2.84%	7.43%
Maximum drawdown duration (ex.TC)	11	20
Maximum drawdown recovery duration (ex.TC)	10	35
Annualized Transaction Cost	7.56%	6.70%

Table 1.28: China A-H share Shares portfolio performance in the 1st period

Performance indicator	In-Sample	Out-of-Sample
Annualized return (ex.TC)	14.45%	5.73%
Annualized return	10.72%	2.40%
Volatility (ex.TC)	3.20%	4.77%
Volatility	3.12%	4.65%
Sharpe ratio (ex.TC)	3.27	0.36
Sharpe ratio	2.15	-0.35
Information ratio (ex.TC)	0.85	0.58
Information ratio	0.70	0.40
Maximum drawdown(ex.TC)	1.16%	3.66%
Maximum drawdown	1.33%	4.20%
Maximum drawdown duration (ex.TC)	9	6
Maximum drawdown recovery duration (ex.TC)	10	6
Annualized Transaction Cost	3.73%	3.33%

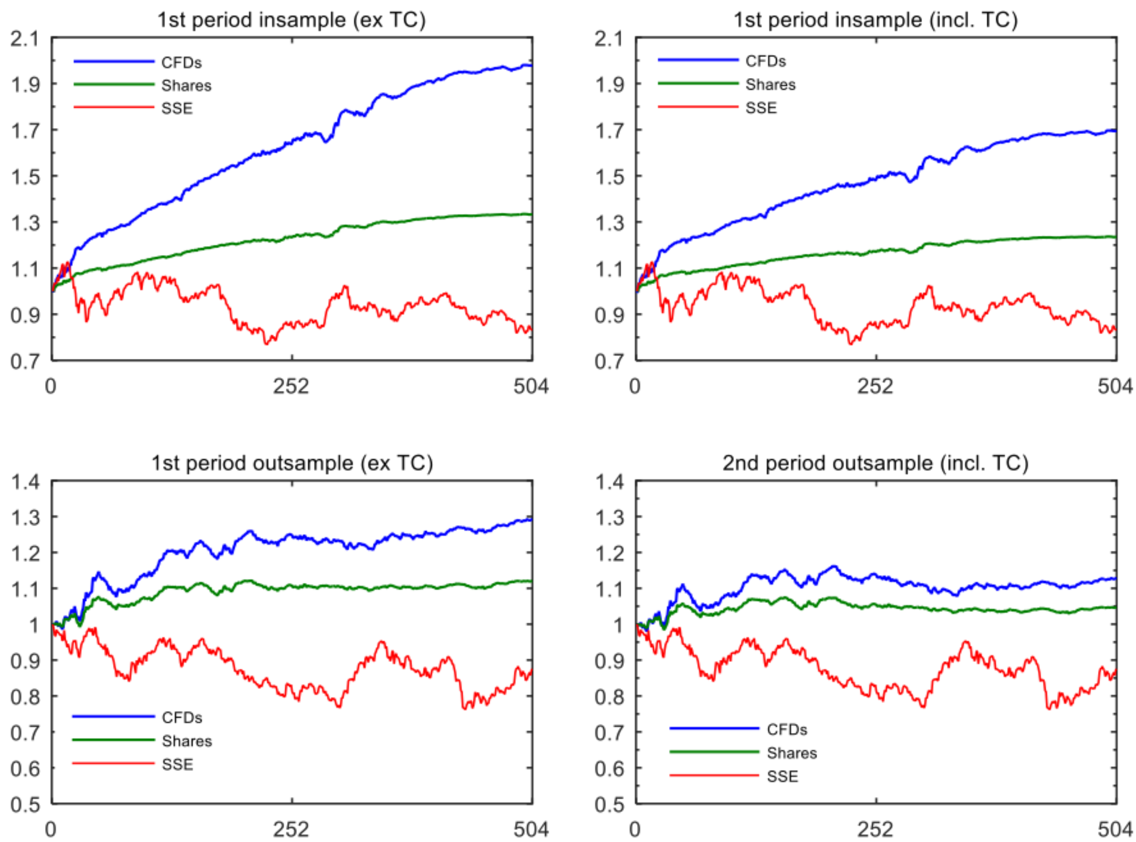


Figure 1.8: Trading performance of China A-H share portfolio in the 1st period

Table 1.29: China A-H share CFDs portfolio performance in the 2^{nd} period

Performance indicator	In-Sample	Out-of-Sample
Annualized return (ex.TC)	24.92%	15.16%
Annualized return	17.38%	8.17%
Volatility (ex.TC)	5.08%	8.22%
Volatility	4.99%	8.24%
Sharpe ratio (ex.TC)	4.12	1.36
Sharpe ratio	2.68	0.51
Information ratio (ex.TC)	1.58	-0.41
Information ratio	1.18	-0.63
Maximum drawdown(ex.TC)	3.81%	9.91%
Maximum drawdown	4.36%	10.30%
Maximum drawdown duration (ex.TC)	10	19
Maximum drawdown recovery duration (ex.TC)	7	46
Annualized Transaction Cost	7.54%	6.99%

Table 1.30: China A-H share Shares portfolio performance in the 2^{nd} period

Performance indicator	In-Sample	Out-of-Sample
Annualized return (ex.TC)	10.31%	6.48%
Annualized return	6.95%	3.48%
Volatility (ex.TC)	2.50%	3.05%
Volatility	2.44%	3.03%
Sharpe ratio (ex.TC)	2.53	0.81
Sharpe ratio	1.21	-0.17
Information ratio (ex.TC)	0.84	-0.73
Information ratio	0.66	-0.83
Maximum drawdown(ex.TC)	1.50%	3.04%
Maximum drawdown	1.71%	3.21%
Maximum drawdown duration (ex.TC)	8	23
Maximum drawdown recovery duration (ex.TC)	5	25
Annualized Transaction Cost	3.36%	3.00%

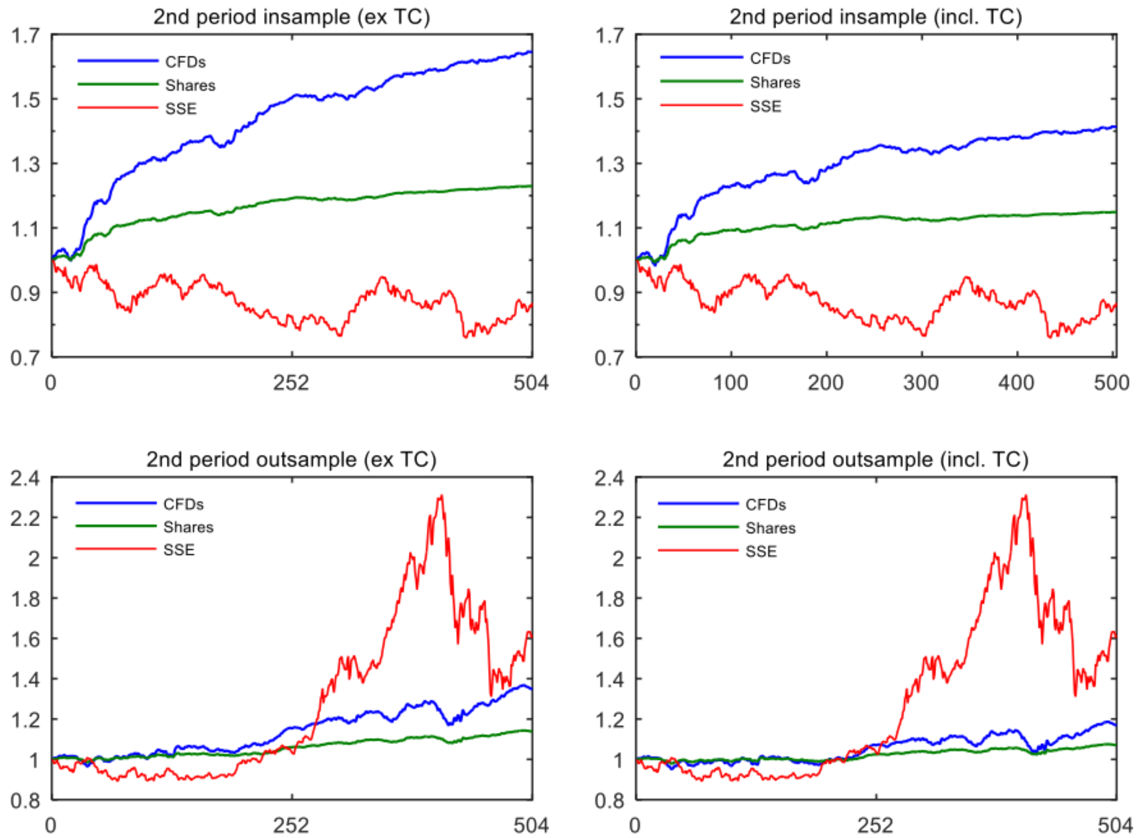


Figure 1.9: Trading performance of China A-H share portfolio in the 2^{nd} period

The trading performances of each cross-listed trading portfolio and period including (excluding) transaction cost (TC) are given by the Table 1.19 ∼ Table 1.30 and the Figure 1.4 ∼ Figure 1.9. CFDs trading with our strategy for UK-US, Eurozone-US and China A-H share portfolio all possess earning ability throughout out-samples of both two periods. It roughly spends 13.91%-26.14% of total capital to maintain the CFDs trading for UK-US and European-US portfolios, which are much lower than that spent by China A-H share from 54.04% to 60.9% of the total capital. Nevertheless, the following table shows that during the out-sample periods the CFDs trading of UK-US and Eurozone-US portfolios yield them the annualized profits from the 12.40% to 16.36% (incl. TC). Those are approximately twice higher than the CFDs trading profitability 6.39% to 8.17% (incl. TC) for China A-H share portfolio. Therefore, it can be argued the CFDs trading of UK-US or Eurozone-US portfolios have the much higher capital utilisation efficiency than that of China A share-H share portfolio to obtain the equal profits.

Without considering transaction cost, the shares trading of the whole portfolios has the earning abilities namely annualized return from 5.73% to 9.78% (ex.TC) in the out-sample trading periods. However, three out of the four return rates after deducting the transaction costs for both UK-US and Eurozone-US portfolios show the losses -0.99%, -2.72% and -0.91%. The above result indirectly demonstrates the transaction costs charged in share trading of UK-US and Eurozone-US portfolio are higher than that of China share portfolio. Their profits have been offset to be the loss because of the higher transaction cost. Moreover, the annualized transaction cost figures of share trading directly show UK-US and Eurozone-US portfolio spend around 7.58%-10.01% of total capitals to pay the transaction cost. On the contrary China A-H share just pay 3.00%-3.33% of total capitals as transaction cost. Overall we proposed it is better to adopt CFDs trading for UK-US and Eurozone-US portfolios in the real trading.

The riskless interest rate is assumed as 2% (2% is higher than the whole historical 12 month-LIBOR rate in the past 6 years) and 4% (4% is higher than the whole historical 12month-SHIBOR rate in the past 6 years) to compute the Sharpe ratios for UK-US, Eurozone-US portfolios and China A share -H share portfolio respectively. All the Sharpe ratios of their CFDs trading are positive and much higher than zero, from1.21-4.68 (incl.TC). However, their share trading shows the negative Sharpe ratios from -0.17 to -3.24 (incl.TC). Above result illustrates the share trading approach for the whole three portfolios are failed to lead the earning ability to beat up the LIBOR or SHIBOR rates. However, CFDs approach can guarantee that.

From results shown that in all cases the proposed methodology is profitable. The only exemption is for the UK-US and the Eurozone-US portfolios based on the share approach. The transaction costs reduce but do not diminish the profitability of the algorithm. It is interesting to note that the CFD approach is far more profitable than the shares trading. This can be explained by the lower transaction cost and more effective use of capital. The maximum drawdown figures are considerably low in all cases under study. In the next graphs, we benchmark the performance of our models against the relevant home market indices before and after transaction costs. More specifically for the UK-US portfolio we benchmark the results against the FTAS-ALL SHARE (FTAS) index, for the Eurozone-US portfolio against the EURONEXT100 (N100) and for the China portfolio against the Shanghai Stock Exchange composite index (SSE).

In addition, the composite index of FTAS-ALL SHARE (FTAS) index, EURONEXT100 (N100) and Shanghai Stock Exchange composite index (SSE) are respectively set as the benchmark market indices to compute the information ratio of the UK-US, Eurozone-US and China A-H share portfolios. Excluding the second period (15/10/2013 to 16/11/2015) of the China A-H share portfolio, all the Out-the-Sample information ratios of three portfolios show positive figures. That means all these three portfolios after deducting out the transaction costs still perform better than the index of their located stock markets, by shown directly in the Figure 1.4 $\sim$ Figure 1.9. To the Out-the-Samples of second period of the China A-H share portfolio, its negative information ratio -0.83 (incl. TC) does not implies it unprofitable. Which could be attributed by the policy bull market performed in the Shanghai stock market.

This paper proposes a novel trading approach for relative-price arbitrage. Through setting the objective of maximising investor utility with the constraint that relative-price discrepancy series is mean reverted by the cointegrated stocks, we have derived the optimal trading weights which is to be adaptively and continually advised for practical purpose. The proposed method can effectively liberate arbitrageurs from the fierce competition of current traditional statistically arbitrage. This research illustrates the particular form of investor utility function during the consequential trading of relative-price arbitrage, which not only contributes to relax the restriction that cointegrating vector is near one to be any figure but also to get an analytical form solution of daily trading weights. Intuitively this implies the stochastic optimal control could be potentially applied to relative-price trading with much wider categories of assets.

We have conducted extensive empirical analysis on cross listed stocks in US, UK, Eurozone and Chinese markets based on the analytical solution of daily trading weight.

It contains both in-sample and out-sample tests from July 2009 to November 2015. Our results illustrate the effectiveness of our proposed approach, especially for CFDs trading. Our method performs better than that of the home market index without significant bullishness, which further provides evidence that the time-delay arbitrage of cross-listed stocks could be profitable. The superiority of the proposed approach simply comes from avoiding the permanent anomalies of relative-price discrepancy of and the shrinking spill- over impact on more and efficient global financial markets.
Our model has further implications on bilateral trading over the newly established Shanghai-London Stock Connect and the forthcoming China's Nasdaq as well. The proposed trading strategy could be easily applied to the bilateral markets for cross market arbitrage. It would be interesting and worthwhile to test the effectiveness of our method on these bilateral market, which we leave for future research.

Chapter 2

Implementing the Karatzas (1997) stochastic general equilibrium model to study wealth inequality

2.1 Introduction

Despite many researches have been done about wealth inequality such as wolff (1992), Aiyagari (1994), Hugget (1996), Castaneda et al. (2003), De Nardi (2004, 2009) and Piketty and Saez (2013, 2014) on the perspectives of dynastic models and overleaping generation models (will be discussed later). Their researches mostly focus on calibrating the inequality trajectories over time passing. Beyond current studies, on perspective of theoretical, our research put more concentration on the evolution process of wealth concentration, that is the key driver of inequality.

The wealth inequality was reversed to decline during the middle of the last century due to a few particular causes, such as the two world wars, the Great Depression and debt-fuel recession, which destroyed much wealth accumulated by the elite class, thus reducing the inequality level at that time.

The world today is returning toward "patrimonial capitalism", in which the inherited wealth has a dominant portion of the economy, meanwhile the progressive income tax and inheritance tax are not high enough to rebalance the social redistribution. Piketty proposes that a progressive income tax reaching as high as 80%, would reduce the wealth inequality, although this is "politically impossible", thereby study on inheritance issue and income taxation are not the main focus in this thesis. Compared to the infeasibility of greater high taxation, a few other potential determinants of wealth inequality, which have relative modest influences on the elite's income and inheritance, have been investigated by quantitative modelling. These potentially feasible determinants that consumption risk aversion, the financial structure between equity-financing and debt-financing, labor force's welfare, they may light a way forward, to relieve the extreme concentration of wealth.

Among these possible determinants, if it can be confirmed that consumption preference has influential power on wealth concentration, then government could advocate better consuming habits for households, to prevent wealth gap from the continuous extensions. While whether the consumption preference has impact on inequality (distribution of wealth) or not? It is still an controversial topic. Cagetti (2001) shows that wealth concentration should be sensitive to the variation of the preference parameter in consuming utility function, conversely Krusell and Smith (1998) evidence that risk averse coefficient in consumption does not affect the distribution of wealth. Therefore it is essential to have a systematic investigation on the theoretical viewpoint, to confirm whether (homogenous) heterogenous preferences of consumption have impacts on wealth inequality, it would be one determinant of that.

Beyond the consumption preference, there may be existing other determinants influencing the evolution of wealth inequality, such as the labor-capital ratio, corporate capital structure, technology-specific investment (human capital), technological progress (Total Factor Productivity) and labour force's welfare. As an example, intuitively the labor-capital ratio, capital structure respectively determines the attribution level to capital suppliers (especially elite class) from the value-added. Thus it is useful to check their influential power on the evolutions of wealth inequality from a theoretical viewpoint. These tasks will be carried on step by step in the following three chapters.

2.2 Trend of wealth inequality

The US wealth inequality declined significantly from 1920 to 1970s, as evidenced by Wolff and Marley (1989), Davies and Shorrocks (2000) and Piketty and Saez (2014). Wolff (1992) indicates the share of total wealth held by the top 1% richest dropped from 38% in 1922 to 19% in 1976. In the United States, the wealth inequality decrease was likely led by the Great Depression and also the increased tax burden prevented the richest from recovering after World War II, Kopczuk and Saez (2004). A similar decrease of wealth inequality has been observed in the UK, Sweden and France, as observed by Davies and Shorrocks (2000) and Piketty et al. (2006). However, their inequality declined for years longer than the US, until the 1980s.

Since 1979, US income and wealth have been concentrating accompanying market boom, the rise of internet fortunes and the financial crisis, based on the investigation by Piketty and Saez (2003), Piketty and Saez (2014) and Wolff (2016). Generally, the top 1% richest households in the United States hold around 40% of the total wealth, and the top 5% hold more than half. At the other extreme that more than 10% household have little or no assets. The share of total net private wealth owned by the top 10% of wealth holders has been roughly rising from 67% to 75% since 1980 in the United States. On the other hand, the top decile's income share has been rising from 36% to 50% of the total. The Gini coefficient of net wealth [1]rose from 0.80 to 0.83 during 1979-1989. It was virtually stable at 0.83 around from 1989 to 2007. Conversely it has been sharply elevating to 0.87 from 2007 to now. Although the Gini coefficient of income has two falling periods respectively 2001-2004 and 2009-2010, it has been rising from 0.48 to 0.58 roughly since 1979. Although delayed in years than that of the US, the income and wealth in Europe have been starting to concentrate again since the 1980s.

In the past century, the inequality of income and wealth fluctuated up and down in the international perspectives. While in the recent 40 years, the top decile's share

[1]Household's net wealth (wealth per capita) after deducting liability.

and Gini coefficient both imply the inequality of income and wealth has been extending. The wealth share owned by the top decile and Gini coefficient of wealth are much greater than their corresponding income. This characterises that wealth is more extremely concentrated than income; the deterioration of wealth inequality has been more intractable than that of income throughout these 40years and seems set to continue.

2.3 Existing models in wealth inequality studies

It is a stylized fact that wealth is more extremely concentrated than income, has been highlighted by numerous studies, such as Wolff (1992), Castaneda et al. (2003), Piketty and Saez (2014) and others. Economists attempt various sorts of model trying to better understand the wealth inequality and its determinants as well. Quite a few fruitful economic models exist to uncover the myth of wealth inequality, however, most of which have limited empirical implications with regard to real observed income inequality around the world. These models can be categorised as dynastic models, life-cycle models, overlapping-generation models among others, which will be discussed in details below.

2.3.1 Dynastic models

Here we first consider an infinitely-lived agent model with the objective

$$V(x) = \underset{(c,a')}{max} \{u(c) + H_0 \mathbb{E}[v(a', z') \mid (a, z)]\} \tag{2.1}$$

where c is agent's consumption, a is the agent's asset holdings carried, that is the only asset choice that agents can use to self-insure against earning risk. a' is the saving for next period and $u(c)$ is consumption utility. H_0 is discounter, z is the labor income shock generated exogenously. $\mathbb{E}[v(a', z') \mid (a, z)]$ is the agent's expected wealth conditional on the state (a, z).

The objective of maximising consumption and wealth is subjected to the following constraint:

$$c + a' = (1 + r)a + zw \tag{2.2}$$

where r is interest rate and w is the exogenous labor income.

The main framework of dynastic model is established by recursively maximising agent's consumption and expected discounted wealth with the consuming constraint inclusive of labor income after the endowment shock, referring to Bewley (1977), Aiyagari (1994) and Huggett (1996) among others for specific descriptions of this framework. Moreover, it follows stationary equilibrium in that the aggregate capital equals the aggregate labor. The aggregate capital and aggregate labor respectively stands for the total saving and total labor supplied by all of the households of the economy. This equilibrium assumes all households' saving is accumulated by labor income, but does not consider market clearing between aggregate output and consumption.

Dynastic models have one drawback in that the wealth is significantly less concentrated than the observed data, as stated by Carroll (1997) that the intrinsic of this model is the agent creating a buffer stock of asset to self-insure against income fluctuations. However this model demonstrates that once the richer households have reached high levels of wealth (save to buffer stock), they do not save in higher (even zero) rates than the poorer. Following the argument by Dynan et al. (2004), it is inconsistent with the empirical studies on saving rates and incapable of explaining the creation of large fortunes.

Krusell and Smith (1998) add heterogeneity in discounter and consuming preference to the general dynastic model. They found it is feasible to use specific stochastic process for the discounter to replicate the variance of the cross-sectional distribution of wealth, as risk aversion cannot. However Cagetti (2001) shows that distribution of wealth should be associated with the risk averse coefficient in the utility function[2].

[2]Lawrance (1991) and Cagetti (2003) empirically evidence that heterogenous preference has effect on wealth inequality but the discounter is more significant.

2.3.2 Overlapping generations models

As the cornerstone of overlapping generations models (OLG), the life-cycle theory of consumption was developed by Modigliani and Brumberg (1954). Their framework exposes that the finitely-living agent chooses consumption by maximising the accumulating discounted utility of consumption (homogenous utility function for consumption), subject to the remaining earnings (resources) up to the life end, while their settings are deterministic, hereafter named as consumption constraint

$$a_t + \sum_{\tau=t}^{N} \frac{y_\tau}{(1+r)^{\tau+1-t}} = \frac{a_{L+1}}{(1+r)^{L+1-t}} + \sum_{\tau=t}^{L} \frac{c_\tau}{(1+r)^{\tau+1-t}} \tag{2.3}$$

where c_t and y_t respective denote the current consumption and income of the agent, while c_τ and y_τ, for $\tau > t$, denote expected income and planned consumption at the τ^{th} stage. a_t is assets at beginning of stage t. L is the finitely life span and N is the agent's earning span, $N < T$.

Assuming $a_t = 0$, $a_{L+1} = 0$, implies that agents have no inherited assets at the beginning of its life, also do not receive any inheritance (gift) at any stage point of life and that whole asset are only accumulated by saving. In addition, the current and future planned consumption should be a homogenous function of current and expected income plus initial assets. This framework has intuitions for wealth inequality, households keep on saving accompanying working, reach the maximum wealth level just at the beginning of retirement, and then accumulate their savings to the life end. Comparing with dynastic models, OLG models add intergeneration transfer on the life-cycle theory of consumption with constant probability distribution of births and deceases and labor income shock. That inheritance could be one significant cause of wealth inequality was empirically checked by Davies (1982). After that, the OLG model was updated about inequality of wealth. Its objective function is similar to that of the dynastic (2.1)

$$V(x,t) = \underset{(c,a')}{max} \{u(c) + H_0 \mathbb{E}[v(a', z', t+1) \mid (a, z, t)]\} \tag{2.4}$$

taking the life-cycle into the considering is one difference, subject to

$$c + a' = (1+r)a + e(z,t)w + h + b \tag{2.5}$$

it is consistent to the consumption constraint (2.3). Firstly, h denotes the accidental bequests which are assumed to be redistributed by planner to the alive, and b is social security benefits to the retirees. Secondly, the fraction of population at different age range are constant. Thirdly, the $e(z,t)$ implies wage after labor income shock is described by a probability distribution associated with age. The above three points are the differences in the framework between dynastic models and OLG modes.
Comparing with dynastic models, OLG has an extra constraint on the stationary equilibrium for the economy. Namely, the total lump-sum transfers to the alive households equal accidental bequest by the deceased, under an balanced planner's budget constraint during each stage.
The benchmark OLG models refer being to the ones shown by Huggett (1996) and De Nardi (2004). Despite the population being inadequate of concentrating wealth in the fat tails of wealth distribution by the OLS model. Its calibration results succeeds in matching the observed Gini coefficient of wealth.

2.3.3 Other models

Under the main framework of dynastic, OLG models and beyond bequest, a few other determinants of wealth inequality have been studied. Portfolio investment choice, one of the most important determinants, has been treated heterogeneously by earning, preference and age. Heaton and Lucas (2000) evidence that the value of a household's business asset has effects on investment portfolio choice and asset pricing. In addition, Yao and Zhang (2004) find that housing investment plays a significant role in influencing the household's investment portfolio. The various portfolio choices linked with heterogenous wealth levels should lead recursively wealth to concentrate over investment earnings not only labor incomes. Lusardi et al. (2017) show that financial knowledge influences on portfolio choice, consequently being a key determinant of wealth inequality in a stochastic life-cycle model with endogenous financial knowledge accumulation, where financial knowledge enables individuals to make wiser investment choices when facing uncertainty and imperfect insurance market.

It is useful to study the wealth concentration by considering human capital accumulation, to disentangle the whole sets of transitory sources of inequality. Becker and Tomes (1986) modelled that parents are altruistic in transfering to their children, where both human capitals and bequests are characterised as the transferring resources across generations. They show human capital transferring is relatively more important than monetary (physical) assets transferring, the bequest takes place only after an optimal threshold of human capital has been reached. Huggett et al. (2006) modelled that human capital investment in presence of heterogeneous learning capabilities and exogenous shocks, which could lead to determine the earning and wealth inequality through life-cycle.

Income inequality is one directive cause of wealth inequality. Each agent's revenue contains earnings on business (financial) assets and labor incomes. In the income portal, the taxation policy has dramatical effects on income inequality. In addition, taxation boundary on inheritances could lead to wealth concentration. Using wealth inequality data from China and India Piketty and Qian (2009) find that the progressive taxation is one of the distortionary policy tools in rising inequality. Cagetti and De Nardi (2009) show that estate taxation does distort investment decision of larger firms, thereby reducing aggregate output and savings. Removing the estate tax by raising other taxation on households to reestablish fiscal balance, can benefit top wealth quantile in a large welfare gain, but most of the population would suffer loss, reflecting the wealth concentration. In addition, if inheritance has been considered as one source of incomes, the inheritance taxation can also determine the wealth inequality. Piketty and Saez (2013) show the existence of optimal inheritance taxation under maximising the long-run steady state social welfare; they assert the optimal inheritance tax rate increase with the concentration of bequest received to balance the wealth concentration and welfare.

2.4 Motivation and structure

In this chapter, similar as the work done by Karatzas and Shreve (1998), an optimal consumption will be solved out under general equilibrium with continuous stochastic form. We will extend their work to the differing risk aversion owned by each household's consumption. We obtain the solution of heterogenous individual's consumption level for simultaneously maximizing each household's utility in both consumptions and their final savings. Each household's endowment is completely exogenous in a market clearing economy with no financial investment activity setting directly. How does risk aversion of consumption has impact on the level of wealth inequality will be studied.

2.5 Modelling setting

This chapter is to investigate whether household's consumption preference influences the wealth concentration and its volatility or not, during a finite dynasty where an agent's risk aversion is homogenous. We apply the framework introduced by Karatzas et al. (1997) that one general equilibrium is established within a market clearing condition between aggregated income and aggregated consumption[3]. Each household's income is heterogenous and exogenously given by stochastic processes assumed to be progressively measurable with respect to the augmentation of the natural filtration generated by an $\mathbb{R}^d$ valued Brownian motion $\mathbb{W} = (W_1, ..., W_d)'$ with a filtration as $(\mathcal{F}_t)$. The main result of Karatzas et al. (1997) is that the closed-form solution of optimal consumption is existing to implement a Pareto allocating rule (maximizing household's consumption utility simultaneously) in an completed market, [4] under the market clearing condition in goods. In this framework, each household's consumption through the whole dynasty is consistent with the consumption constraint set by

[3]They also implicitly assume market clearing between aggregated investment and perishable commodity.

[4]Under the market completeness of the first welfare theorem that economic planner can implement a rule to satisfy the weak Pareto efficiency of resource allocation. Where the planner is able to improve upon a decentralised market outcome, because all information are available to them and market participant's information are symmetric, no moral hazard or adverse selection possibility.

Modigliani and Brumberg (1954).

2.5.1 Exogenous household's income and the aggregation

There are K households, each of their income level[5] is $\epsilon_k(t), k = 1, 2, 3, \cdots, K$, which is assumed to follow the dynamics

$$d\epsilon_k(t) = \epsilon_k(t)\left(\mu_k(t)dt + \sum_{i=1}^{d}\sigma_{kj}(t)dW_j(t)\right). \tag{2.6}$$

These processes, alongside the utility functions $U_k(x)$ and consumptions of good are the principal building blocks of the economy. Everything else is derived from these. To proceed it is useful to introduce the aggregated income level process

$$\epsilon(t) = \sum_{i=1}^{K}\epsilon_k(t). \tag{2.7}$$

It follows the form, Eq. (2.6) that the dynamics of the aggregated income level is given by

$$d\epsilon(t) = \epsilon(t)\left(\nu(t)dt + \rho(t)'d\mathbb{W}(t)\right), \tag{2.8}$$

with

$$\nu(t) = \sum_{k=1}^{K}\frac{\epsilon_k(t)}{\epsilon(t)}\mu_k(t) \tag{2.9}$$

$$\rho(t)' = \left(\sum_{k=1}^{K}\frac{\epsilon_k(t)}{\epsilon(t)}\sigma_{k,1}(t), \cdots, \sum_{k=1}^{K}\frac{\epsilon_k(t)}{\epsilon(t)}\sigma_{k,d}(t)\right). \tag{2.10}$$

For the numerical tractability it is helpful to assume that the income changes are independent on the past, i.e.

$$\mathbb{E}\left(\frac{\epsilon_i(t)}{\epsilon_i(s)}\middle|\mathcal{F}_s\right) = \mathbb{E}\left(\frac{\epsilon_i(t)}{\epsilon_i(s)}\right). \tag{2.11}$$

Otherwise the solution of the model will require more sophisticated methods of Monte Carlo simulation.

[5]The salary level $\epsilon_k(t)$ is defined as the expected total salary of the whole life $[0, T]$ at the filtration $\mathcal{F}_t$. At each time stage, based on the adaptive salary level $\epsilon_k(t)$, the individual receives salary can be computed as $\epsilon_k(t)dt$.

2.5.2 Market clearing constraints

Karatzas et.al.(1997) consider the three market clearing constraints ensure the financial market equilibrium.

Clearing between aggregated consumptions and aggregated incomes:

$$\sum_{k=1}^{K} c_k(t) = \epsilon(t), \tag{2.12}$$

Clearing of households' investment weights:

$$\sum_{k=1}^{K} \pi_{kj}(t) = 0, \tag{2.13}$$

Clearing of households' wealths:

$$\sum_{k=1}^{K} X_k(t) = 0. \tag{2.14}$$

2.6 Problem solving under CRRA Utility

It is difficult that using general constant relative risk aversion (CRRA) utility to analytically solve out the optimal consumption. The particular form of CRRA utility is helpful in solving this problem:

$$U_k(x) = \left(\frac{\gamma^{1-\alpha}}{\alpha}\right) x^{\alpha}, \quad \gamma > 0 \tag{2.15}$$

where α plays the role of different risk aversion coefficient of household's, $\alpha \in \{-\infty, 1\}\backslash\{0\}$. Defining $I(\cdot)$ as the inverse function of the first derivative of the k-th household's utility:

$$I_k(x) = (U_k'(x))^{-1} = (\gamma^{1-\alpha} x^{\alpha-1})^{-1} \tag{2.16}$$

$$= (\gamma^{\alpha-1} x)^{\frac{1}{\alpha-1}} \tag{2.17}$$

then it exists

$$I_k\left(\frac{\epsilon(t)}{\lambda_k}\right) = \left(\gamma^{\alpha-1}\frac{\epsilon(t)}{\lambda_k}\right)^{\frac{1}{\alpha-1}} \tag{2.18}$$

where λ_k is a "Lagrange multiplier" in the problem that maximizing utility from consumption, which is used in the 2.2.1 Proposition, 2.2.2 Theorem and 2.3.1 Theorem of Karatzas (1997).

Subject to the three market clearing conditions (2.12)-(2.14), Karatzas (1997) shown in 3.4.1 Proposition that:

$$\epsilon(t) = I(H_0(t)e^{\int_0^t \beta(s)ds}; \Lambda) \tag{2.19}$$

where setting

$$I(x; \Lambda) = \sum_{k=1}^{K} I_k\left(\frac{x}{\lambda_k}\right) \tag{2.20}$$

for any given $\Lambda = (\lambda_1, ..., \lambda_k) \in (0, \infty)^K$. The function $I(\cdot; \Lambda)$ is continuous and strictly decreasing, and maps $(0, \infty)$ onto itself with $I(0+; \Lambda) = \infty$ and $I(\infty; \Lambda) = 0$; therefore, it has a continuous, strictly decreasing inverse $\mathcal{H}(\cdot; \Lambda)$ projecting $(0, \infty)$ onto $(\infty, 0)$, with $\mathcal{H}(0+; \Lambda) = \infty$ and $\mathcal{H}(\infty; \Lambda) = 0$, namely

$$I(\epsilon(t); \Lambda) = \mathcal{H}(\epsilon(t); \Lambda)^{-1} \tag{2.21}$$

and exists

$$I(\mathcal{H}(\epsilon(t); \Lambda); \Lambda) = \epsilon(t) \tag{2.22}$$

Substituting (2.18) into (2.20), it gives

$$I(\epsilon(t); \Lambda) = \sum_{k=1}^{K} \left(\gamma^{\alpha-1}\frac{\epsilon(t)}{\lambda_k}\right)^{\frac{1}{\alpha-1}} \tag{2.23}$$

$$= \epsilon(t)^{\frac{1}{\alpha-1}} \sum_{k=1}^{K} \gamma\lambda_k^{\frac{1}{1-\alpha}} \tag{2.24}$$

Subject to (2.21) and (2.24) yields

$$\mathcal{H}(\epsilon(t); \Lambda) = \left(\frac{\epsilon(t)}{\sum_{k=1}^{K} \gamma\lambda_k^{\frac{1}{1-\alpha}}}\right)^{\alpha-1}. \tag{2.25}$$

In addition, the (2.19) and (2.22) gives

$$H_0(t) = e^{-\int_0^t \beta(s)ds}\mathcal{H}(\epsilon(t); \Lambda) \tag{2.26}$$

Aligned with Theorem 2.3.1 of Karatzas et al. (1997) and (2.26) the following equation existing

$$\begin{aligned} &\mathbb{E}\int_0^T e^{-\int_0^t \beta(s)ds}\mathcal{H}(\epsilon(t); \Lambda)I_k(\frac{\mathcal{H}(\epsilon(t); \Lambda)}{\lambda_k})dt \\ &= \mathbb{E}\int_0^T e^{-\int_0^t \beta(s)ds}\mathcal{H}(\epsilon(t); \Lambda)\epsilon_k(t)dt \end{aligned} \tag{2.27}$$

Overall, by plugging (2.17) and (2.25) into (2.27), after a few arrangements, we obtain each household's optimal consumption level:

$$c_k^*(t) = (\lambda_k^*)^{\frac{1}{1-\alpha}} \gamma \left(\frac{\epsilon(t)}{\epsilon(0)} \right), \tag{2.28}$$

where

$$(\lambda_k^*)^{\frac{1}{1-\alpha}} = \frac{\epsilon(0)}{\gamma} \frac{\mathbb{E} \int_0^T e^{-\int_0^t \beta(s)ds} (\epsilon(t))^{\alpha-1} \epsilon_k(t) dt}{\mathbb{E} \int_0^T e^{-\int_0^t \beta(s)ds} (\epsilon(t))^{\alpha} dt}. \tag{2.29}$$

Each household's wealth at the start and end of the dynasty is zero, the $X_k(0)^+ = 0$ and has no capital accumulated after this dynasty $\mathcal{G}_k = 0$. To be matched to Karataz's work in this, subject to the generalised expression of household's wealth Appendix. E, (E.14) the wealth framework of each household's in this section satisfies

$$X_k(t) = \frac{1}{H_0(t)} \mathbb{E}_t \left[\int_t^T H_0(s) \left[c_k(s) - \epsilon_k(s)) \right] ds \right] \tag{2.30}$$

where $\epsilon_k(t)$ is exogenously generated by process (2.6), each household's income is completely from labor income, in absence of capital investment. The state price density $H_o(t)$ is given by (2.29), (2.25) and (2.26), with the exogenously given discounter of money time value $\beta(t)$.

2.7 Simulation and Analysis

The objective in this research is to investigate whether the homogenous preference of household's consumptions could influence wealth inequality or not in the economy through out the whole finite-lived dynasty. In this section, the corresponding simulation result for the model established in the section (2.5) and (2.6) is going to be demonstrated. For the simplicity of the particular CRRA utility, the consumption preference is driven by the risk aversion α but the γ in consumption utility keeps constant, and the money discounter β is also time-invariant. By reordering the coefficient of risk aversion α range from -6 to -0.1 with the incremental -0.1, it allows us to see whether the simulated averaging figure and the standard deviation of the wealth inequality (Gini coefficient process) is varing or not. The specific parameters employed in this simulation test are given in the Table (2.1),

Table 2.1: Summary of key parameters in case 1

Parameter	Symbol	Value
Household number	K	100
Terminal stage	T	80
Number of income risk resources	d	6
Initial income level	$\epsilon_k(0)$	1
Income growth rate	μ	1
Income volatility averaging	$\bar{\sigma}_{kj}$	0.12
std.of income volatility	σ_ρ	0.02
Money discounter	β	0.03
Invariant coefficient of risk aversion	γ	2
Nonhomothetic coefficients of risk aversion	α	$[-6.0 : -0.1 : -0.1]$

Note: $[a : b : c]$ denote the series on the closed interval from a to c with increment b.

Considering the form of net wealth could be negative, this issue can be solved by employing the measurement of normalized Gini coefficient proposed by Chen et al. (1982) that

$$G(t,\cdot) = 1 - \frac{\frac{1}{K}\sum_{\nu=\omega(t)+1}^{K}[1+2(K-\nu)]x_\nu(t)}{1+\frac{2}{K}\sum_{\nu=1}^{\omega(t)}\nu x_\nu(t)} \tag{2.31}$$

$$x_\nu = \frac{X_\nu(t)}{\sum_{\nu=1}^{K} X_\nu(t)} \tag{2.32}$$

$\{X_\nu(t), \nu = 1, \cdots, K^x\}$ is the ascending order series of household's wealth $\{X_k(t), k = 1, \cdots, K\}$ generated by (2.30) at each stage. To the first $\omega(t)^{th}$ [6] lower $x_\nu(t)$ satisfies $\sum_{\nu=1}^{\omega(t)} x_\nu(t) = 0$ and $\sum_{\nu=1}^{\omega(t)+1} x_\nu(t) > 0$, the $\omega(t)^{th}$ is time-variant. The Normalised Gini coefficient is also the measurement of wealth inequality in the following chapters. There are 100 households assumed living throughout the dynasty. Each household's labor income is following the geometric Brownian motion process subject to Eq. (2.6). Every household's initial income level and income growth rate are assumed to be in homogeneity, namely $\epsilon_k(0) = 1$ and $\mu_k = \mu = 1$. Consequentially, their labor incomes turn to be in heterogeneity because their income volatility factor are different. In

[6] $\omega(t)$ is to identify each household's wealth ranking at each stage t, it is time-variant.

simplicity, their various income volatility factors in all satisfy the distribution that

$$\sigma_{kj} \sim N(\bar{\sigma}_{kj}, \sigma_\rho), \tag{2.33}$$

although it is strengthen that every household's income shouldering the identical risk resources under the differing magnitude dependent on the different volatilities. These households' incomes are given in the Figure 2.1[7]:

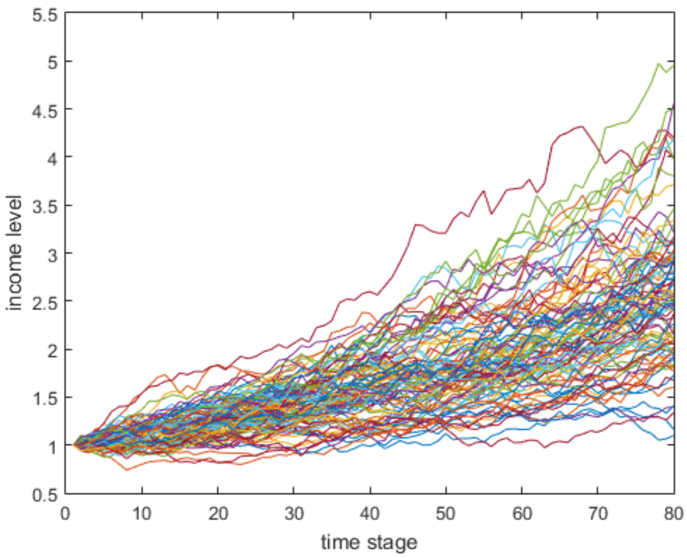


Figure 2.1: Household's labor income process throughout the whole dynasty, they are generated following Eq. (2.6), parameters used in simulation are given in Table 2.1

[7]These exogenously given labor incomes also will be used in the simulation analysis of the next chapter.

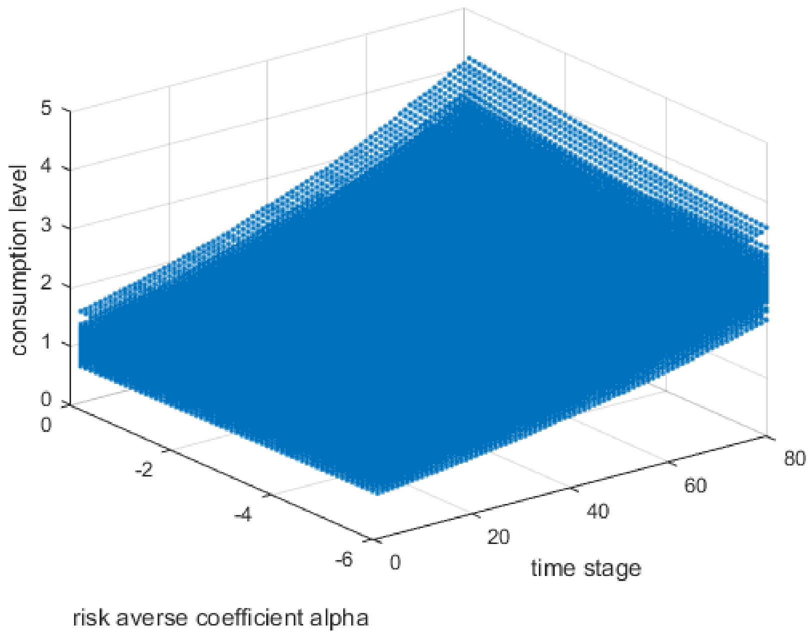
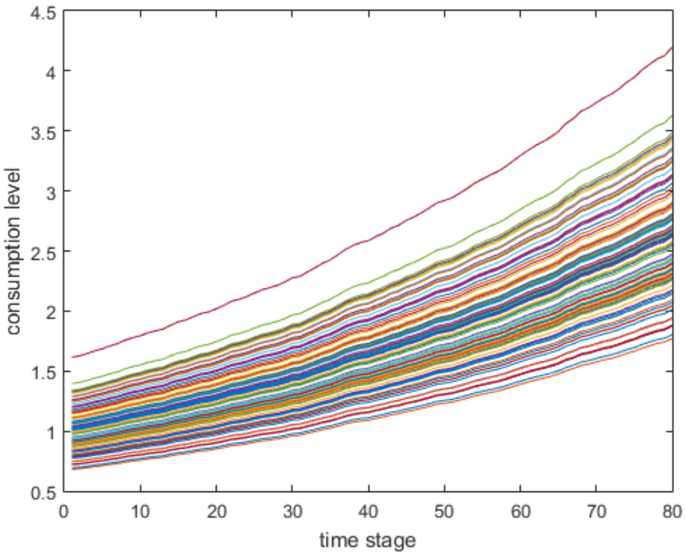
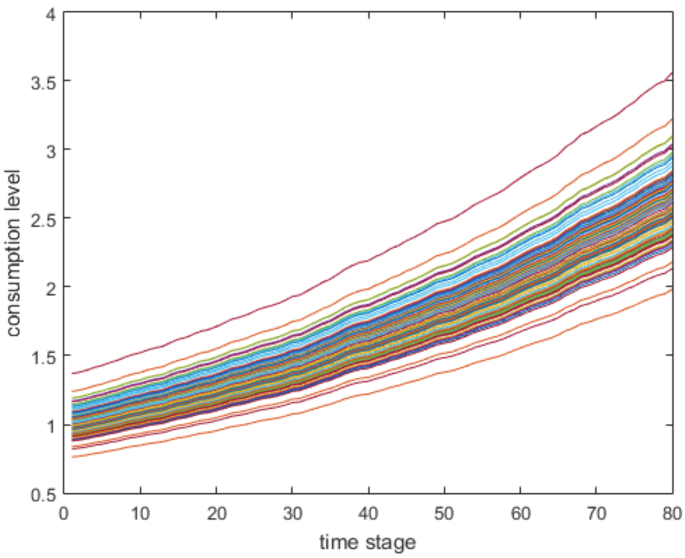


Figure 2.2: consumption scatter accompanying the varying coefficients of risk aversion and time passing, each consumption scatter is generated following Eq. (2.28) with Eq. (2.29), parameters used in the simulation is shown in Table 2.1.



(a) α = -6.0



(b) α = -0.1

Figure 2.3: consumption spectrum with the differing coefficients of risk aversion, α

The Figure 2.2 and Figure 2.3 together illustrate that under the homogenous consuming preference, the heterogenous income levels lead each household's consumption to be different at each stage. Their consumptions [8] have similar trends accompanying

[8]Subject to Eq. (2.28), the smooth of consumption spectrum is attributed by the comparative smooth of the aggregated income $\epsilon(t) = \sum_{k=1}^{K} \epsilon_k(t)$, nevertheless each household's income process is stochastic.

the increase of their incomes. The coefficient of risk aversion α has influence on the concentration of households' consumptions at each stage. The higher α implies the more concentrated consumptions among households.

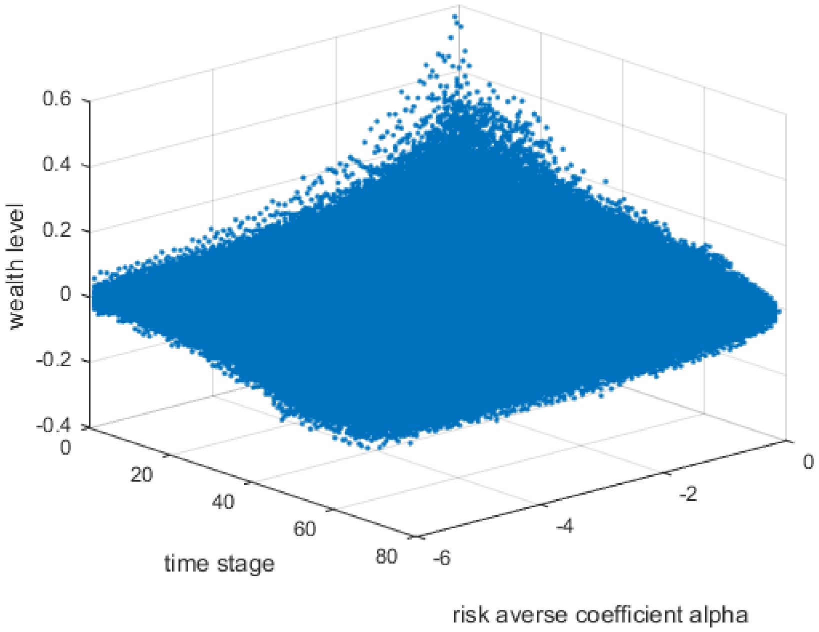
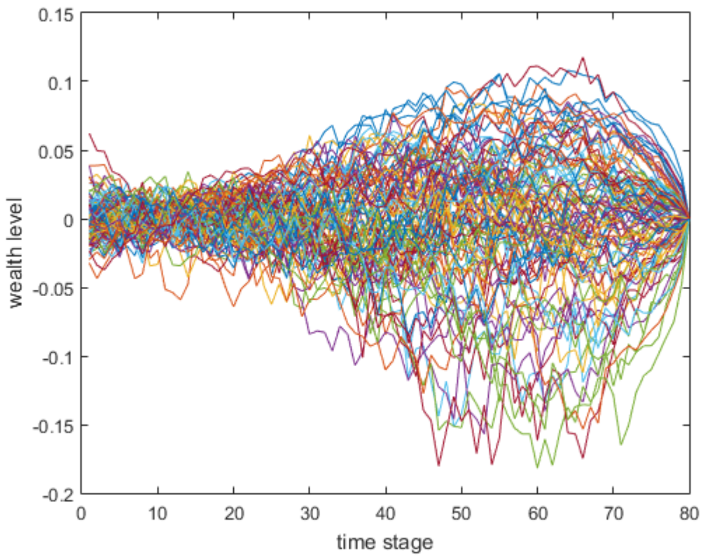
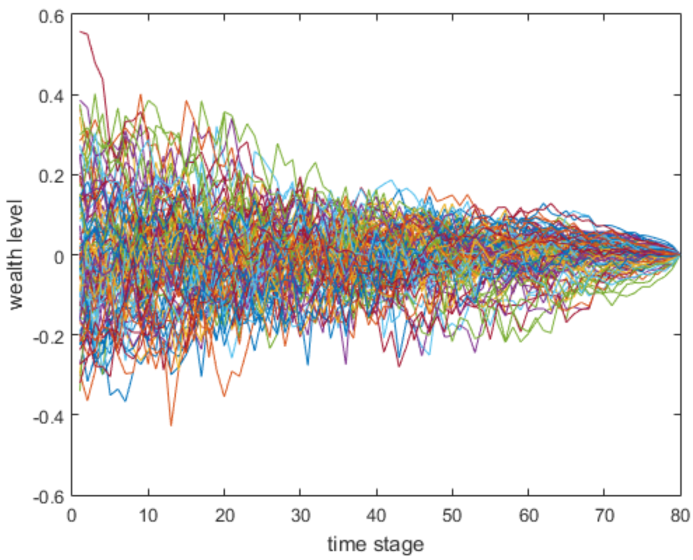


Figure 2.4: Wealth scatter along the varying coefficients of risk aversion and time passing



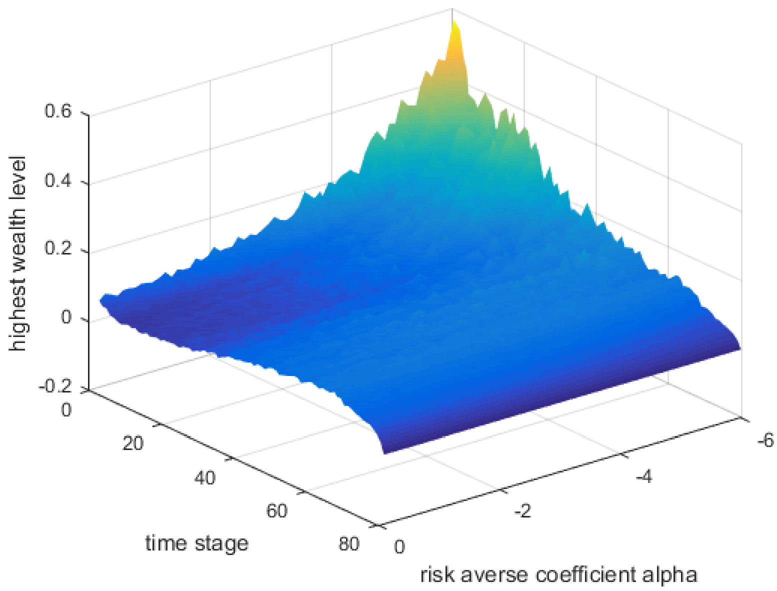
(a) α = -6.0



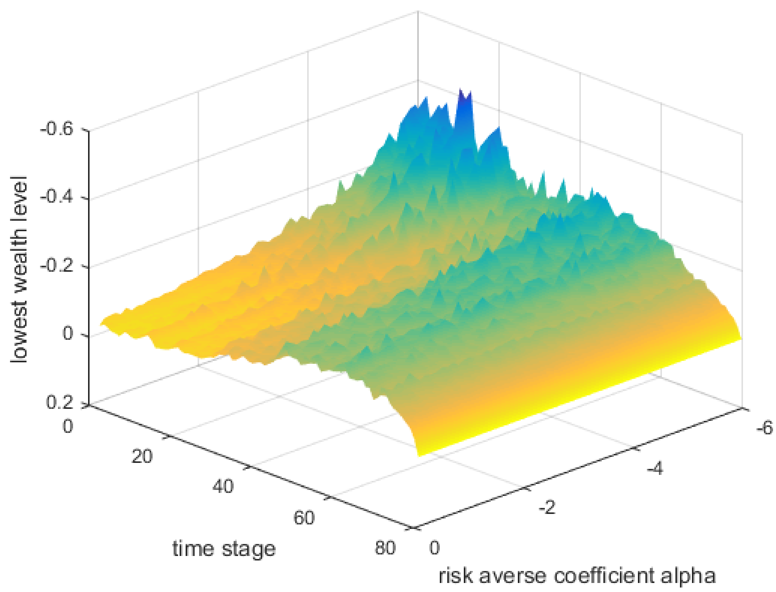
(b) α = -0.1

Figure 2.5: wealth trajectories through time passing with different coefficients of risk aversion, α

The Figure 2.4 and Figure 2.5 illustrate that risk aversions in consumption across households could have impacts on the wealth trajectories. Specifically, the majority of wealth series demonstrates the regime-switching from a converging trend to diverging, then recover back to be converging till household's wealth near to be zero. However this regime-switching status tend to disappear accompanying the coefficieints of risk aversion α decreasing from -0.1 to -6.0.



(a) highest wealth surface



(b) lowest wealth surface

Figure 2.6: highest & lowest wealth surface corresponding differing risk aversion and time stage

The Figure 2.6 shows the highest wealth and lowest wealth under the same risk aversion α in consumption are around symmetric. At a comparatively high α from -0.1 to -6, the evolution trajectories for the highest (lowest) wealth by time passing is regime-switching, it firstly diverges from the averaging wealth level [9] and then converge back to the average wealth again. Other other hand, if the coefficient of risk aversion α is a comparatively low such as -6, the initial wealth gap between top rich (poor) with wealth averaging could be very large, this gap is converging back to the averaging level to the end of dynasty. In addition, by observing the dimension that highest (lowest) wealth level vs. risk averse coefficient alpha, the lower α implies that the highest (lowest) wealth level deviates further from the averaging level (roughly zero) at the initial stage.

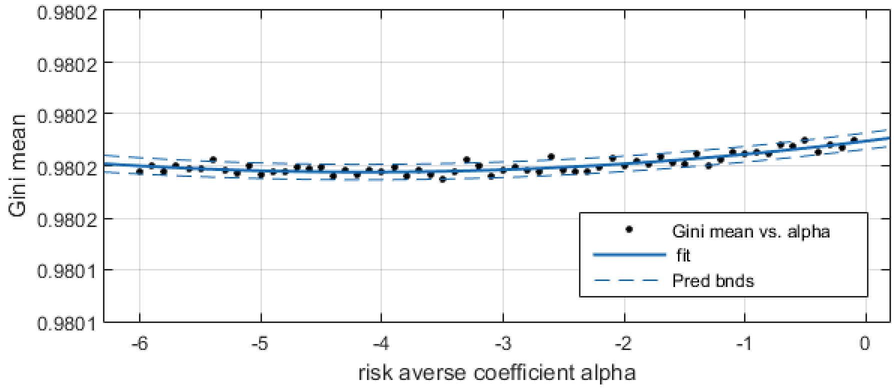


Figure 2.7: Gini coefficient averaging vs. alpha

[9]subject to Figure 2.6, the averaging wealth is around zero during the whole dynastic life.

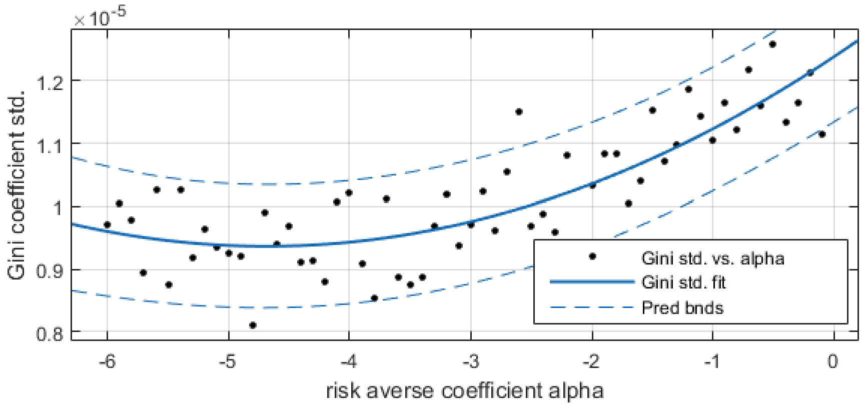


Figure 2.8: Gini coefficient std. vs. alpha

Figure 2.7 illustrates that there is no significant effect of the risk aversion in consumption on wealth inequality of the dynasty. However Figure 2.8 shows the risk aversion could have impacts on the volatility of wealth inequality, these impacts roughly follow the U-shaped relationship shown by the Gini std. fit line. This simulation of the specific scenario may suggest that the coefficient of risk aversion α ranging from -4 to -5 could be an optimal interval to reduce the volatility of the dynastic inequality on wealth.

2.8 Conclusion and future research

This chapter explores the consumption-saving framework explained by Karatzas (1997) which is under stochastic general equilibrium, by applying convex duality optimisation. Each household's income is exogenously given with implicated financial market exchanging economy. There is a pair of comparative studies has been implemented, the simulations illustrate the risk aversion level in homogenous consumption preferences have no obvious impacts on the evolutions of wealth. However, the theoretically modelling suggests the homogenous risk aversion could influence the evolution patterns of wealth concentration, the simulations shown the less risk averse, the evolutions of wealth more probably show regime-switch during a dynasty, this implies a weaker stratification of the wealth inequality.

In respect to the future robustness check, subject to the Table 2.1, the risk invariant coefficient of risk aversion γ could be adjusted to other varying level, to check whether the regime-switch of wealth concentration trend through the dynasty is still existing or not. More further, the real calibration could be applied, by estimating the income growth rate, money discounter with classifying the different risk stems.

Chapter 3

Optimal consumption under quadratic utility

3.1 Introduction

Comparing with Karatzas (1997), this chapter we worked out an equilibria state price density, that is endogenously determined by risk aversion, deposit rate and household?s revenue level. This also simplify the whole modelling in describing the wealth concentration progress. In addition, each household's optimal consumption is set as maximising the utility for their accumulative consumptions as simultaneous as their utilities in terminal wealth at the end of dynasty. Also, we release the constraint on the clearing that aggregated investment weights are zero. We are adopting quadratic utility function benefits us to extend the problem from homogenous consumption preference to the heterogeneous, although the whole optimisation framework is still under convex duality approach. The simulation is also carried out on the premise of household?s income being exogenously given and implicating no financial capital gain flow from business sectors. Meanwhile we release the constraint on the equilibrium where aggregated wealth and investment weight are both zero. Also the optimal consumption solution is to maximises household's utilities in consumption as well as their terminal wealth utilities of the dynasty. This chapter illustrates the homogenous (heterogeneous) risk aversion can affect the progressing of wealth concentration, remind us that just employing wealth inequality measure could let us neglect the social stratification problem as worse than we think.

3.2 Modelling setting

3.2.1 State Price Density and wealth framework

For convenience in using the state price density of wealth hereafter, one sort of its generalised form has been denoted. When evaluating each individual's wealth, it is essential to adjust the extra risk and money time value. The extra risk adjustor is according to

$$dZ(t) = -Z(t)\theta(t)'dW(t), \quad Z(0) = 1, \tag{3.1}$$

where $\theta(t)$ is the price of market risk and $W(t) = (W_1(t) \cdots W_z(t))'$ is a z-dimensional independent Brownian motion vector, standing for z risk sources on asset prices. Denoting the state-price density of individual's wealth as $H_0(t)$ involves a money time value discounter and the extra risk adjustor. The generalised form of $H_0(t)$ can be expressed as

$$\begin{aligned} H_0(t) &= e^{-rt}Z(t) \\ &= e^{-\left(rt+\int_0^t \theta(\tau)'dW(\tau)+\frac{1}{2}\int_0^t \|\theta(\tau)\|^2 d\tau\right)}, \quad 0 < \tau < t, \end{aligned} \tag{3.2}$$

and its dynamics follows

$$dH_0(t) = -H_0(t)\Bigg(rdt + \theta(t)'dW(t)\Bigg) \tag{3.3}$$

Each individual's wealth satisfies the dynamics

$$dX_k(t) = (rX_k(t) + \epsilon_k(t) - c_k(t))\, dt, \tag{3.4}$$

where $c_k(t)$ is the individual's consumption level at the filtration $\mathcal{F}_t$, $c_k(t)dt$ is individual's consumption at t. Each individual wealth is consistent with the measure of net worth, it includes all assets (real estate, financial wealth and vehicles) and liabilities (mortgages and other debts) held by the household, all of these taking place through an exchange economy.

In this section, it is highlighted that each household's investment has still not been considered. Therefore, each household's income source is just their labor incomes.

Financial investment activity condition imposing the issue of the price of market risk does not need to be considered, this is a particular case of Eq. (3.2) in absence of stochastic terms. Then, the state price density for individual's wealth in absence of investing, is equivalent to a money time value discounter

$$H_0^1(t) = e^{-rt}. \tag{3.5}$$

Subject to the generalized expression of wealth, Appendix E. (E.14) and Eq. (3.4), Eq. (3.5), also assuming that each household's wealth just at the start of the initial stage is zero $X_k(0)^+ = 0$ and no capital accumulated throughout the whole dynasty $\mathcal{G}_k = 0$, then each individual's wealth is

$$X_k(t) = \frac{1}{H_0^1(t)} \mathbb{E}_t \int_t^T H_0^1(s) \left(c_k(s) - \epsilon_k(s)\right) ds \tag{3.6}$$

It is stressed that the state price density is exogenously dependent on the interest rate r at current position as shown in Eq. (3.5), but it turns to be endogenously dependent on the risk averse coefficient γ and household's income level after below modelling, illustrated by Eq. (3.31) forthcoming.

3.3 Optimal consumption under quadratic utility

For the tractability in solving the heterogenous optimal consumption level, we choose a quadratic form of consumption utility

$$U_k(c_k(t)) = c_k(t) - \frac{1}{2}\gamma_k c_k^2(t), \quad \gamma_k > 0, c_k(t) \in \left(0, \gamma_k^{-1}\right), \tag{3.7}$$

where γ_k is heterogenous risk aversion in consumption, which varies among different households.

Its derivative $U_k'(c_k(t)) = 1 - \gamma_k c_k(t)$ and the inverse of the derivative, denoted as $I_k(c_k(t)) = \left[U_k'(c_k(t))\right]^{-1} = (1 - c_k(t))/\gamma_k$ are both strictly decreasing with $c_k(t) \in \left(0, \gamma_k^{-1}\right)$. In a general form

$$I_k(x) = \frac{1-x}{\gamma_k}. \tag{3.8}$$

Based on the Legendre-Fenchel transform (Rockafellar (1970) Karatzas and Shreve, 1998),

$$\widetilde{U}_k(\lambda_k) \triangleq \max_{c_k(t)\in\left(0,\gamma_k^{-1}\right)} \{U_k(c_k(t)) - \lambda_k c_k(t)\} = U_k(I_k(\lambda_k)) - \lambda_k I_k(\lambda_k), \tag{3.9}$$

the domain of λ_k should be identical to that of $c_k(t)$, namely$\lambda_k \in \left(0, \gamma_k^{-1}\right)$. $\widetilde{U}_k(\lambda_k)$ is a convex decreasing function and satisfies

$$\widetilde{U}_k'(\lambda_k) = (\lambda_k - 1)/\gamma_k, \lambda_k \in \left(0, \gamma_k^{-1}\right), \tag{3.10}$$

$$U_k(c_k(t)) = \min_{\lambda_k\in\left(0,\gamma_k^{-1}\right)} \left\{\widetilde{U}(\lambda_k) + c_k(t)\lambda_k\right\}, c_k(t) \in \left(0, \gamma_k^{-1}\right) \tag{3.11}$$

$$= \widetilde{U}\left((\lambda_k - 1)/\gamma_k\right) + c_k(t)(\lambda_k - 1)/\gamma_k \qquad , \tag{3.12}$$

$$\widetilde{U}_k(\gamma_k^{-1}) = U(0), \quad \widetilde{U}_k(0) = U(\gamma_k^{-1}). \tag{3.13}$$

Solving the optimal consumption level $c_k^*(t)$, this problem can be treated as one stochastic controlling with,
the objective function[1] :

$$\begin{aligned} G = \quad & \underset{c_k(t)}{argmax}\, \mathbb{E}\left[\int_0^T U_k(c_k(t))dt + U_k(X_k(T))\right] \\ = & \underset{c_k(t)}{argmax}\, \mathbb{E}\left[\int_0^T (c_k(t) - \frac{1}{2}\gamma_k c_k^2(t))d\tau + X_k(T) - \frac{1}{2}\gamma_k X_k^2(T)\right] \end{aligned} \tag{3.14}$$

s.t.

Constraint 1:

$$\mathbb{E}\left[\int_0^T H_0(t)c_k(t)dt + H_0(T)X_k(T)\right] \leq X_k(0), \tag{3.15}$$

Intuitively, it is one budget constraint that the expected discounted terminal wealth plus the expected discounted total consumptions cannot exceed each individual's initial endowment. The discounting is accomplished by the state price density $H_0(t)$. Also it almost surely exists by the proof in Appendix A.

Constraint 2:

$$dX_k(t) = \left[rX_k(t) + \mathcal{E}_k(t,\cdot) - c_k(t)\right]dt + \sigma(t, X_k, \cdot)dW(t), \tag{3.16}$$

[1]The optimal consumption should benefit the individual achieving the highest expectation of total consumption utility throughout the whole life long. If the household's revenue exceeds the rational consumption, then they saves (invests) the retained revenue (to earn more) for the future consumption. Conversely, if they are confident for their future earning ability, nonetheless current revenue is less than their rational consumption. They may choose an overdraft consumption.

where $\mathcal{E}_k(t,\cdot)$ is each individual's whole revenue level at the filtration $\mathcal{F}_t$ from all feasible revenue resources. $\sigma(t,\mathcal{E}_k,X_k,\cdot))$ is an diffusion factor, $W(t)$ is the z-dimensional independent Brownian motion vector as that of Eq. (3.1)

Subjected to Eq. (3.1), Eq. (3.2) and Eq. (3.16) by applying Ito's formula on $Z(t)e^{-rt}X_k(t)$, it can offer one generalized expression of adjusted individual's wealth dynamics

$$dH_0(t)X_k(t) = H_0(t)\left(\mathcal{E}_k(t,\cdot) - c_k(t)\right)dt + H_0(t)\Omega(t,\sigma,X_k,\theta,\cdot)dW(t), \tag{3.17}$$

where the diffusion term Ω has different form up to particular scenarios.

Constraint 3:

$$\sum_{k=1}^{K} c_k^*(t) = \mathcal{E}(t,\cdot), \tag{3.18}$$

It is market equilibrium between aggregated production level and consumption level. where $\mathcal{E}(t,\cdot)$ is the produce level of total goods at filtration $\mathcal{F}_t$ in the whole economy.

Subject to Eq. (3.12) and imposing constraint 1, Eq. (3.15) to the objective, Eq. (3.14) with optimization multiplier $\lambda_k \in \left(0,\gamma_k^{-1}\right)$, yields

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T U_k(c_k(t))dt + U_k(X_k(T))\right] \\
&+\lambda_k\left\{X_k(0) - \mathbb{E}\left[\int_0^T H_0(t)c_k(t)dt + H_0(T)X_k(T)\right]\right\} \\
&= \mathbb{E}\int_0^T \left[U_k(c_k(t)) - \lambda_k H_0(t)c_k(t)\right]dt \\
&\qquad +\mathbb{E}\left[U_k(X_k(T)) - \lambda_k H_0(T)X_k(T)\right] + \lambda_k X_k(0) \\
&\leq \mathbb{E}\int_0^T \widetilde{U}_k(\lambda_k H_0(t))dt + \mathbb{E}\widetilde{U}_k(\lambda_k H_0(T)) + \lambda_k X_k(0).
\end{aligned} \tag{3.19}$$

Following Eq. (3.12), to achieve the objective, Eq. (3.14), namely leading Eq. (3.19) to equality, the following conditions should be satisfied:

$$c_k(t) = I_k\left(\lambda_k H_0(t)\right) = \frac{1-\lambda_k H_0(t)}{\gamma_k}, \tag{3.20}$$

$$X_k^*(t) = I_k\left(\lambda_k H_0(T)\right) = \frac{1-\lambda_k H_0(T)}{\gamma_k}, \tag{3.21}$$

thus it can be primarily confirmed that

$$c_k^*(t) \in \left\{ c_k(t) = \frac{1 - \lambda_k H_0(t)}{\gamma_k} \right\}. \tag{3.22}$$

Further, subject to constraint 2, Eq. (3.16) and Eq. (3.2), temporally setting $Z(t) = 1$ and integrating $dZ(t)e^{-rt}X_k(t)$ from 0 to T and taking expectation, obtain

$$\mathbb{E}\int_0^T H_0(t)c_k(t)dt = \mathbb{E}\int_0^T H_0(t)\mathcal{E}_k(t,\cdot)dt + X_k(0) - H_0(T)\mathbb{E}(X_k(T)). \tag{3.23}$$

Substituting Eq. (3.22) into Eq. (3.23) gives

$$\begin{aligned} &\frac{1}{\gamma_k}\mathbb{E}\int_0^T H_0(t)dt - \frac{\lambda_k}{\gamma_k}\mathbb{E}\int_0^T H_0^2(t)dt = \\ &\qquad \mathbb{E}\int_0^T H_0(t)\mathcal{E}_k(t,\cdot)dt + X_k(0) - H_0(T)\mathbb{E}(X_k(T)), \end{aligned} \tag{3.24}$$

this switches the problem from solving $c_k^*(t)$ to solving $\lambda_k^{c,*} \in \{\lambda_k\}$ which satisfies Eq. (3.24).

Plugging Eq. (3.20) into the constraint 3, Eq. (3.18), gives

$$\mathcal{E}(t,\cdot) = \sum_{k=1}^{K} I_k\left(\lambda_k^{c,*} H_0(t)\right), \tag{3.25}$$

additionally, assuming state price density is dependent on every household's income level and their risk aversion in consumption, satisfies the intermediated form that

$$H_0(t) = \mathcal{H}(\mathcal{E}_1(t,\cdot), ..., \mathcal{E}_K(t,\cdot); \lambda_1^{c,*}, ..., \lambda_K^{c,*}), \tag{3.26}$$

Then plugging Eq. (3.26) back into Eq. (3.25) the aggregated income (output) can be temporarily written with the intermediated form $\mathcal{H}$ as

$$\mathcal{E}(t,\cdot) = \sum_{k=1}^{K} I_k(\lambda_k^{c,*}\mathcal{H}), \tag{3.27}$$

Simultaneously Eq. (3.8) and Eq. (3.27) lead

$$\mathcal{E}(t,\cdot) = \sum_{k=1}^{K} \frac{1}{\gamma_k}(1 - \lambda_k^{c,*}\mathcal{H}) \tag{3.28}$$

$$\mathcal{H}(\mathcal{E}_1(t,\cdot), ..., \mathcal{E}_K(t,\cdot); \lambda_1^{c,*}, ..., \lambda_K^{c,*}) = \frac{\sum_{k=1}^{K}\frac{1}{\gamma_k} - \mathcal{E}(t,\cdot)}{\sum_{k=1}^{K}\frac{\lambda_k^{c,*}}{\gamma_k}}. \tag{3.29}$$

Subjected to Eq. (3.26) and Eq. (3.29) and $H_0(0) = 1$, they contribute us

$$\sum_{k=1}^{K} \frac{\lambda_k^{c,*}}{\gamma_k} = \sum_{k=1}^{K} \frac{1}{\gamma_k} - \mathcal{E}(0, \cdot), \tag{3.30}$$

$$H_0(t) = \frac{\sum_{k=1}^{K} \frac{1}{\gamma_k} - \mathcal{E}(t, \cdot)}{\sum_{k=1}^{K} \frac{1}{\gamma_k} - \mathcal{E}(0, \cdot)}. \tag{3.31}$$

Substituting Eq. (3.31) into Eq. (3.24)

$$\begin{aligned} &\frac{1}{\gamma_k}\mathbb{E}\int_0^T \frac{\sum_{k=1}^{K} \frac{1}{\gamma_k} - \mathcal{E}(t, \cdot)}{\sum_{k=1}^{K} \frac{1}{\gamma_k} - \mathcal{E}(0, \cdot)} dt - \frac{\lambda_k^{c,*}}{\gamma_k}\mathbb{E}\int_0^T \left(\frac{\sum_{k=1}^{K} \frac{1}{\gamma_k} - \mathcal{E}(t, \cdot)}{\sum_{k=1}^{K} \frac{1}{\gamma_k} - \mathcal{E}(0, \cdot)}\right)^2 dt \\ &\quad = \mathbb{E}\int_0^T \frac{\sum_{k=1}^{K} \frac{1}{\gamma_k} - \mathcal{E}(t, \cdot)}{\sum_{k=1}^{K} \frac{1}{\gamma_k} - \mathcal{E}(0, \cdot)} \mathcal{E}_k(t, \cdot) dt + X_k(0) - H_0(T)\mathbb{E}(X_k(T)), \end{aligned} \tag{3.32}$$

by re-ordering, we obtain the solution of optimal multiplier $\lambda_k^{c,*}$ for maximising each household's utility in consuming as

$$\begin{aligned} \lambda_k^{c,*} = \frac{\Gamma - \mathcal{E}(0, \cdot)}{\mathbb{E}\int_0^T [\Gamma - \mathcal{E}(t, \cdot)]^2 dt} \{ \mathbb{E}\int_0^T & [\Gamma - \mathcal{E}(t, \cdot)][1 - \gamma_k \mathcal{E}_k(t, \cdot)] dt \\ & + \gamma_k \mathcal{G}_k [\Gamma - \mathcal{E}(0, \cdot)] \}, \end{aligned} \tag{3.33}$$

where $\Gamma = \sum_{k=1}^{K} \frac{1}{\gamma_k}$ and $\mathcal{G}_k = \mathbb{E}(H_0(T)X_k(T)) - X_k(0)$ is the desired growth rate of wealth by each household, up to stage T.

Subject to Eq. (3.22), Eq. (3.31) and Eq.(3.33), it yields us an generalised solution of each household's optimal consumption level

$$\begin{aligned} c_k^*(t) = \frac{1}{\gamma_k} & - \frac{\Gamma - \mathcal{E}(t, \cdot)}{\gamma_k \mathbb{E}\left\{\int_0^T [\Gamma - \mathcal{E}(t, \cdot)]^2 dt\right\}} \mathbb{E}\left\{\int_0^T [\Gamma - \mathcal{E}(t, \cdot)][1 - \gamma_k \mathcal{E}_k(t, \cdot)] dt\right\} \\ & - \mathcal{G}_k X_k(0) \frac{[\Gamma - \mathcal{E}(0, \cdot)][\Gamma - \mathcal{E}(t, \cdot)]}{\mathbb{E}\left\{\int_0^T [\Gamma - \mathcal{E}(t, \cdot)]^2 dt\right\}} \end{aligned} \tag{3.34}$$

Each household's revenue purely stems from the salary generated exogenously by Eq. (2.6), because there is no financial investment activity in this economy, the consequence is that no extra market risk to be borne in evaluating wealth. Then isolated exogenous incomes and aggregation are matched to the element of generalised form solution of optimal consumption:

$$\begin{aligned} \mathcal{E}_k(t, \cdot) &= \epsilon_k(t), \\ \mathcal{E}(t, \cdot) &= \epsilon(t), \end{aligned} \tag{3.35}$$

Each household's rational consumption level can simultaneously maximise the utilities of consumptions and terminal wealth saving satisfying

$$c_k^*(t) = \frac{1}{\gamma_k} - \frac{\Gamma - \epsilon(t)}{\gamma_k \mathbb{E}\left\{\int_0^T [\Gamma - \epsilon(t)]^2 dt\right\}} \mathbb{E}\left\{\int_0^T [\Gamma - \epsilon(t)][1 - \gamma_k \epsilon_k(t)] dt\right\}$$
$$-\mathcal{G}_k X_k(0) \frac{[\Gamma - \epsilon(0)][\Gamma - \epsilon(t)]}{\mathbb{E}\left\{\int_0^T [\Gamma - \epsilon(t)]^2 dt\right\}} \tag{3.36}$$

Considering each household's wealth just at the start and the termination of the dynasty to be both zero, $X_k(T) = X_k(0) = 0$, then is subject to $\mathcal{G}_k = \mathbb{E}(H_0(T)X_k(T)) - X_k(0)$, it gives us the expected growth rate of wealth $\mathcal{G}_k = 0$. In addition, subject to $\Gamma = \sum_{k=1}^{K} \frac{1}{\gamma_k}$ and $\epsilon(t) = \sum_{k=1}^{K} \epsilon_k(t)$, setting $\frac{1}{\gamma_k} < \epsilon_k(t)$, can guarantee $\Gamma - \epsilon(t) < 0$ also equivalents $\frac{1}{\epsilon_k(t)} < \gamma_k$ ensures $1 - \gamma_k \epsilon_k(t) < 0$. Therefore, based on the overall condition, $c_k^*(t)$ will be positive. Otherwise, the sign of $c_k^*(t)$ cannot be analysed unless by using numerical experiment.

3.4 Simulation and Analysis

Following the solution of optimal consumption under the quadratic utility function Eq. (3.36), there are a few simulated tests going to be taken, namely whether the homogenous coefficient of risk aversion (γ) can influence the wealth concentration (both averaging Gini and its volatility) along the dynasty or not. The rest respect of the heterogenous risk averse in consumption to determine whether or not the magnitude of difference among household's risk aversions (different standard deviations on the heterogenous risk averse coefficients, γ_k) have impacts on wealth concentration.

3.4.1 The effect of homogenous consuming preference on wealth inequality

Each household's income process still employs the identical one in the last chapter, as shown in Figure 2.1. The coefficient of risk aversion is γ under the quadratic utility function described in the modelling section (3.3) but it is homogenous in this simulation. In oder to check whether or not this homogenous risk averse γ could determine wealth inequality, it ranges from 0.05 to 0.35 with the incremental 0.005.

Table 3.1: Summary of key parameters in case 2

Parameter	Symbol	Value
Household number	K	100
Terminal stage	T	80
Number of income risk resources	d	6
Initial income level	$\epsilon_k(0)$	1
Income growth rate	μ	1
Income volatility averaging	$\bar{\sigma}_{kj}$	0.12
std. of income volatility	σ_ρ	0.02
Expected growth rate on wealth	$\mathcal{G}_k$	0
Variant coefficients of risk aversion	γ	[0.05 : 0.005 : 0.35]

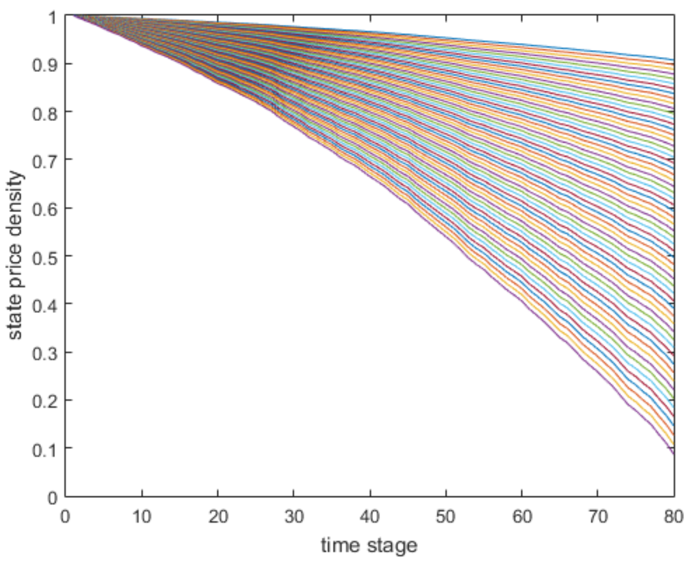


Figure 3.1: spectrum of state price density with gamma domain 0.05 to 0.35

The Figure 3.1 graphs the spectrum of state price endogenously generated by the household's incomes and their corresponding homogenous risk averse in consumption, by following Eq. (3.31). This spectrum shows that state price density is increasing by a higher risk averse γ with reasonable characteristics. Eq. (3.5) and Eq. (3.31) contribute $\frac{\sum_{k=1}^{K} \frac{1}{\gamma_k} - \mathcal{E}(t,\cdot)}{\sum_{k=1}^{K} \frac{1}{\gamma_k} - \mathcal{E}(0,\cdot)} = e^{-rt}$, this implicates that the interest rate r is endogenously driven by aggregated income level $\mathcal{E}(t, \cdot)$ and the set of risk averse γ_k.

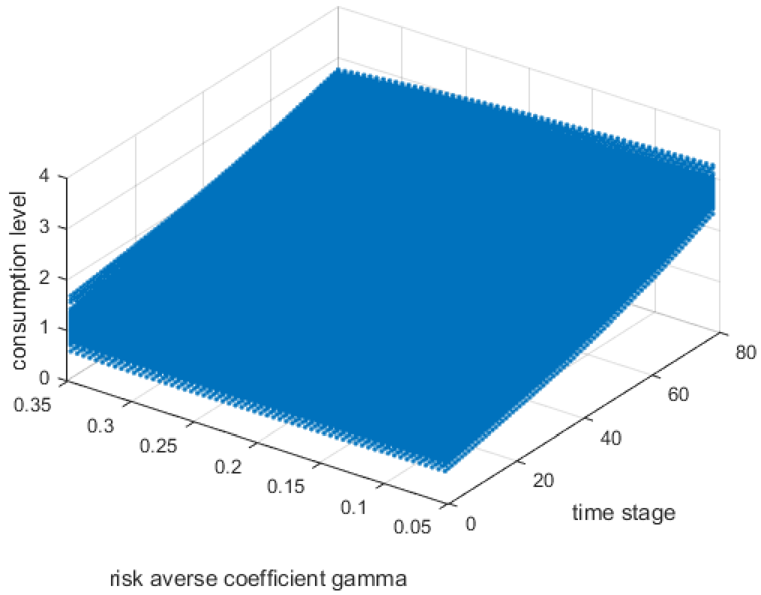
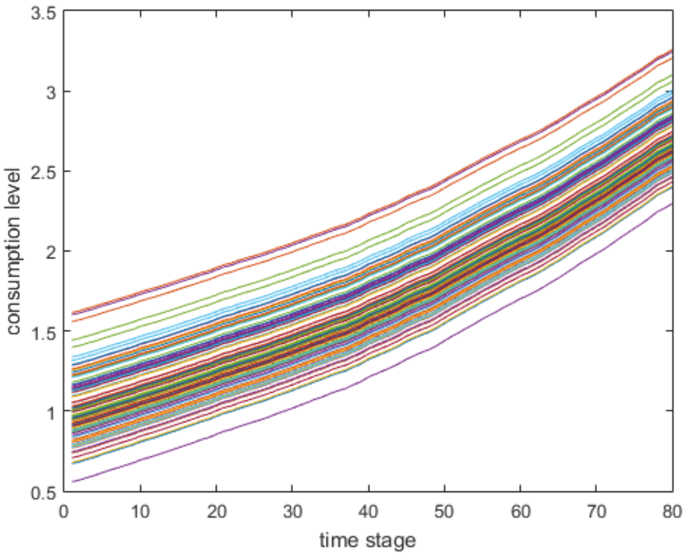
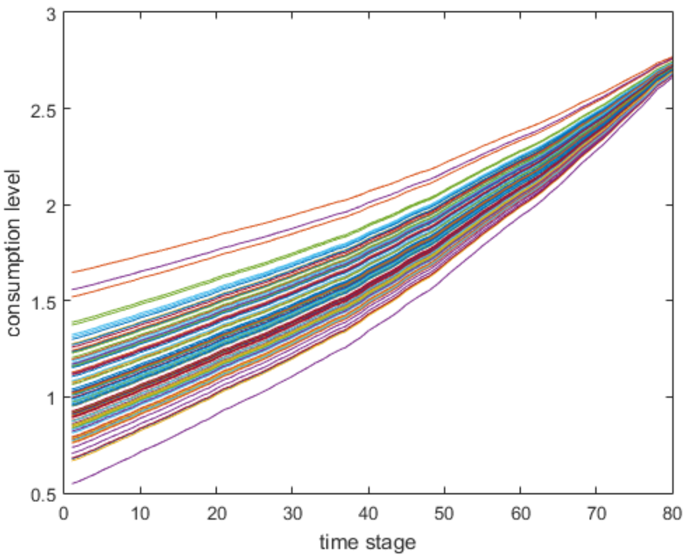


Figure 3.2: Consumption scatter along the varying of risk averse coefficient and time stage



(a) $\gamma = 0.05$



(b) $\gamma = 0.35$

Figure 3.3: consumption spectrum with differing gamma

From Figure 2.1, Figure 3.2 and Figure 3.3 demonstrate that each household's consumption level is roughly positive to their income level under the optimal consumption under quadratic utility. In addition, accompanying the risk averse γ increasing and dynasty life passing, households' consumptions tend to be more concentrated.

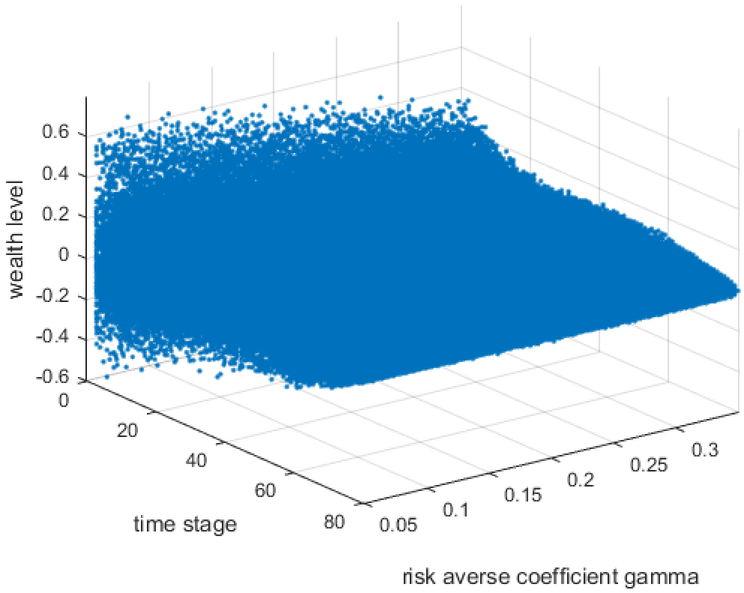
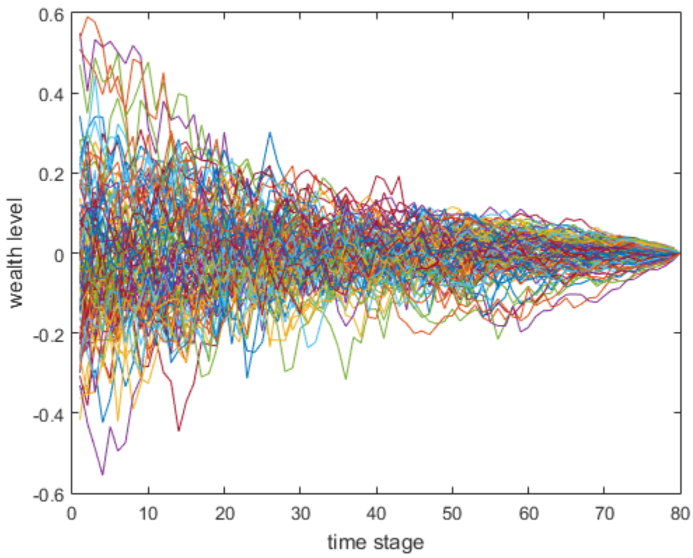
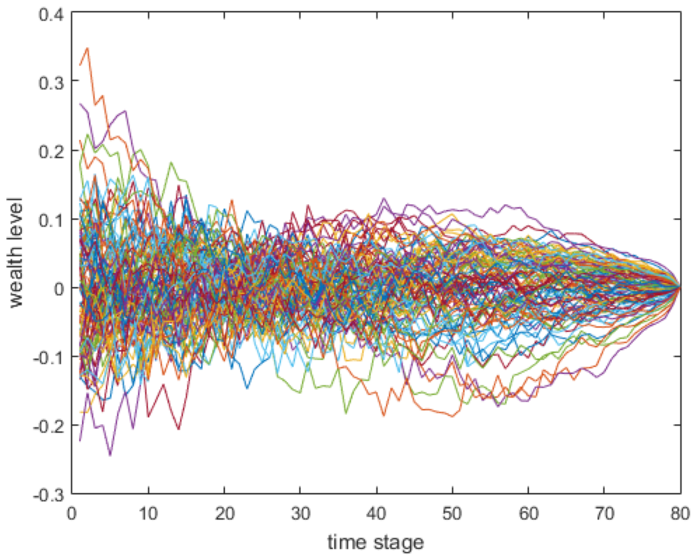


Figure 3.4: Wealth scatter along the varying of risk averse coefficient and time stage



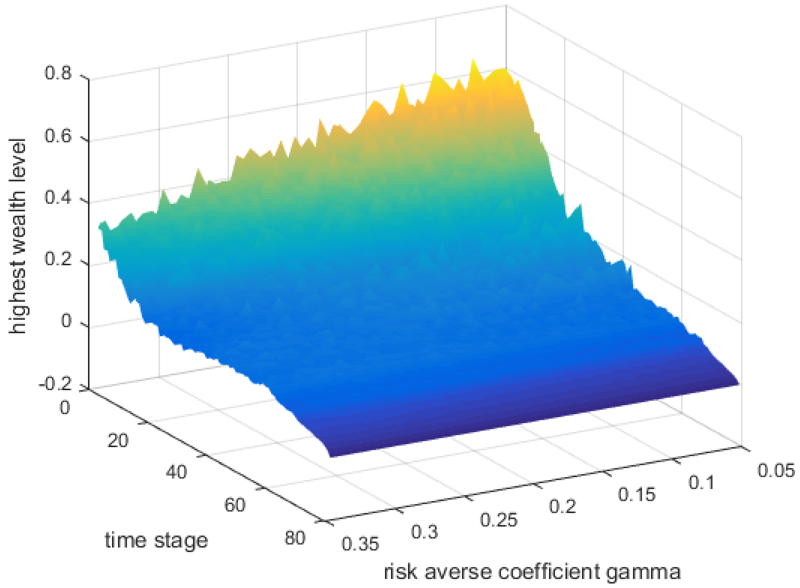
(a) $\gamma = 0.05$



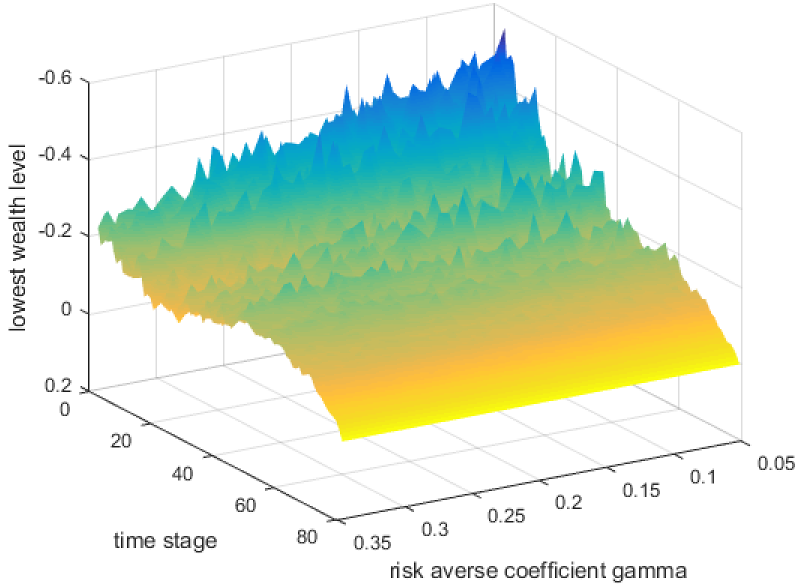
(b) $\gamma = 0.35$

Figure 3.5: wealth trajectory through time with differing gamma

Figure 3.4 and Figure 3.5 show under optimal consumption with quadratic utility, the risk aversion γ has determine power on wealth concentration. The lower γ attributes households' wealths converging more steeply to the averaging wealth level (zero) among households. Additionally, results illustrate that the discrete magnitude among households' wealth is highest at the just starting point of the dynasty.



(a) highest wealth surface



(b) lowest wealth surface

Figure 3.6: highest & lowest wealth surface corresponding to differing gammas and stage

Figure 3.6 illustrate the highest and lowest wealth at each dynastic stage symmetrically converging to zero along time passing. By rising the risk averse γ, both the highest and lowest converging speeds turn to be faster. In addition, the higher risk averse γ implies the deviation of highest (lowest) household's wealth is greater from the common wealth level (zero in this case, because it default no capital accumulated throughout the whole dynasty).

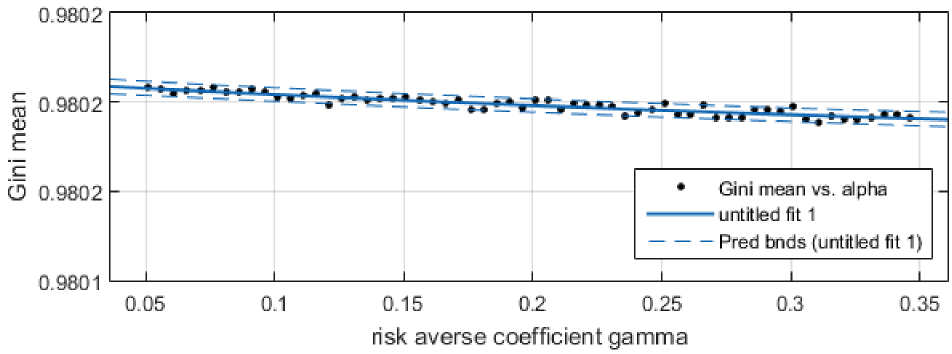


Figure 3.7: Gini coefficient averaging vs. γ

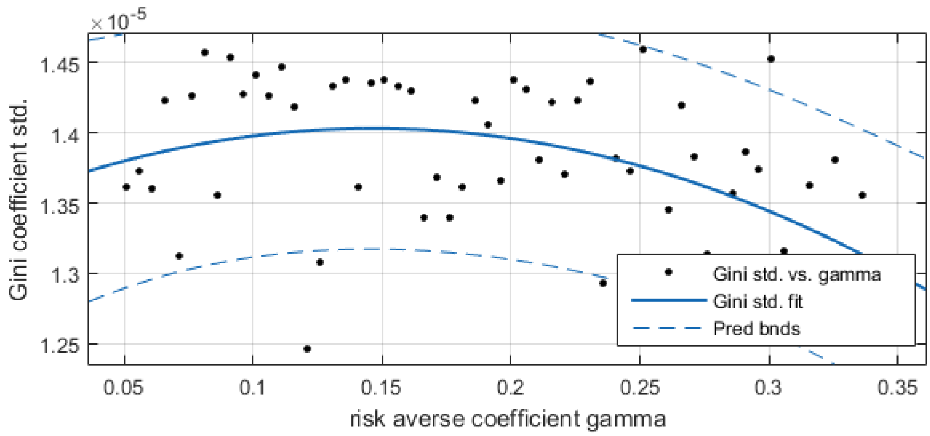


Figure 3.8: Gini coefficient std. vs. γ

Although Figure 3.7 shows a negative correlation between risk averse γ and Gini mean of the dynasty, it is not observable to suggest the risk averse in household's consumption has a significant influencing power on wealth inequality. Figure 3.8 cannot confirm a clear relationship between risk aversion in consumption and the volatility of wealth inequality throughout the dynasty. However, under the second-order Taylor polynomial fitting, there is one U-shaped relationship existing between them.

3.4.2 Effect of heterogenous consuming preference on wealth inequality

In this section, the risk averse coefficient in consumption utility will be switched from the homogenous to be heterogenous among households. The differing risk averse coefficient is set as γ_k satisfies the normal distribution that

$$\gamma_k \sim N(\gamma, \sigma_{\gamma k}^2), \tag{3.37}$$

the risk averse coefficient mean is located as $\gamma = 0.2$ in this simulation. $\sigma_{\gamma k}$ is the standard deviation, it stands for the discreteness of the different risk averse coefficient in consumption. This simulation is to investigate whether or not the discreteness of the differing risk averse coefficient γ_k in consumption is one of determinants of the wealth concentration of households.

Household income is still exogenously given and identically matched to the result generated in the simulation section (2.7, Figure 2.1) by the process, Eq. (2.6). All the other parameters are the same as the previous two simulations, only the heterogenous risk averse coefficient set. The standard deviation of risk averse coefficient ranges from 0.0105 to 0.04 with the incremental 0.0005.

Table 3.2: Summary of key parameters in case 3

Parameter	Symbol	Value
Household number	K	100
Terminal stage	T	80
Number of income risk resources	d	6
Initial income level	$\epsilon_k(0)$	1
Income growth rate	μ	1
Income volatility averaging	$\bar{\sigma}_{kj}$	0.12
std. of income volatility	σ_ρ	0.02
Expected growth rate on wealth	$\mathcal{G}_k$	0
Risk averse coefficient averaging	γ	0.2
Std. of risk averse coefficient	$\sigma_{\gamma k}$	[0.0105 : 0.0005 : 0.04]

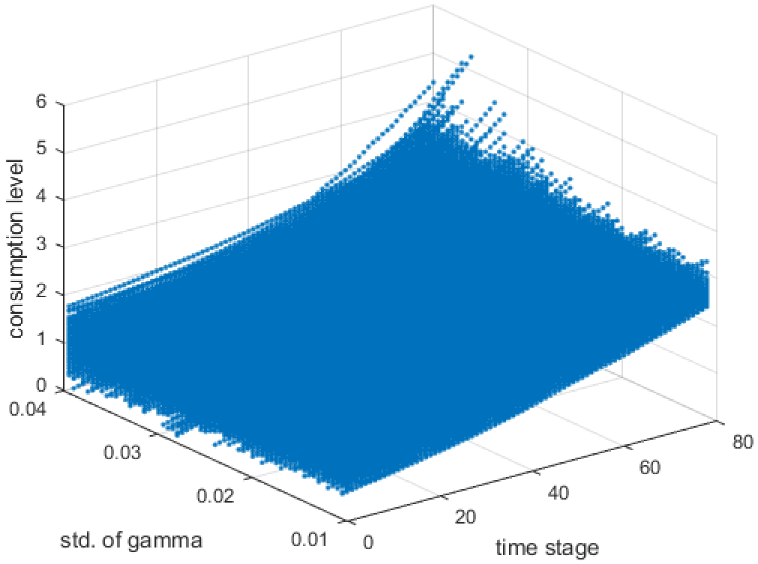
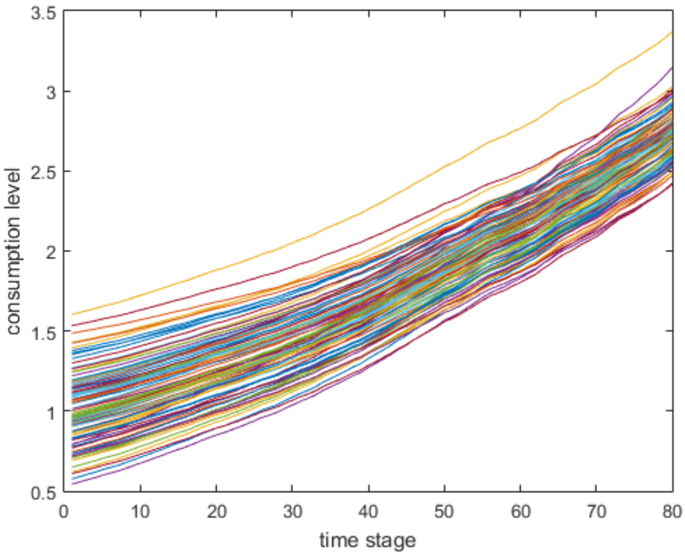
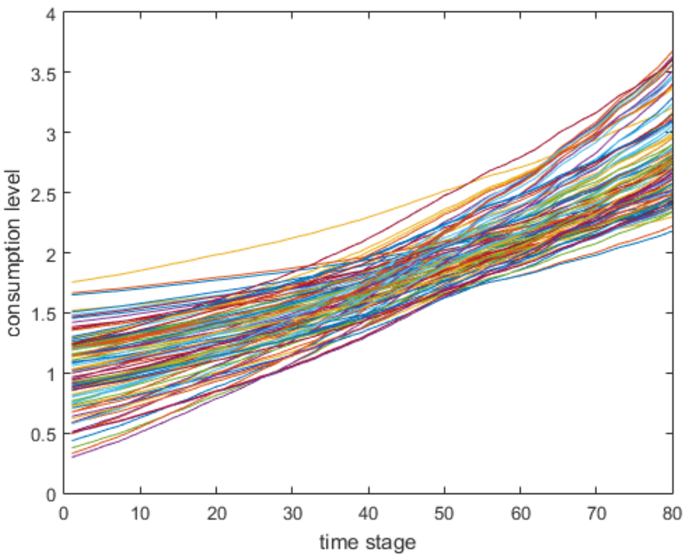


Figure 3.9: Consumption scatter along the varying std. of risk averse coefficient and stage



(a) gamma std. = 0.01



(b) gamma std. = 0.04

Figure 3.10: consumption spectrum with differing std. of averaging gamma 0.20

Figure 3.9 and Figure 3.10 reflect that the consumption level of households' are changing from "parallelized" consumptions with their income level, to the switching-styled among consumptions, accompanying the widening (diverging) discreteness of risk aversion in consumption (the standard deviation of gamma moving from 0.01 to 0.04).

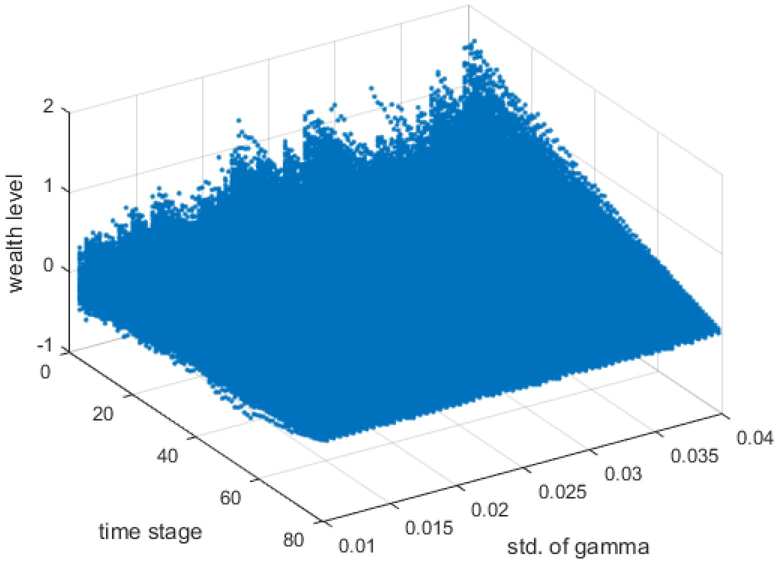
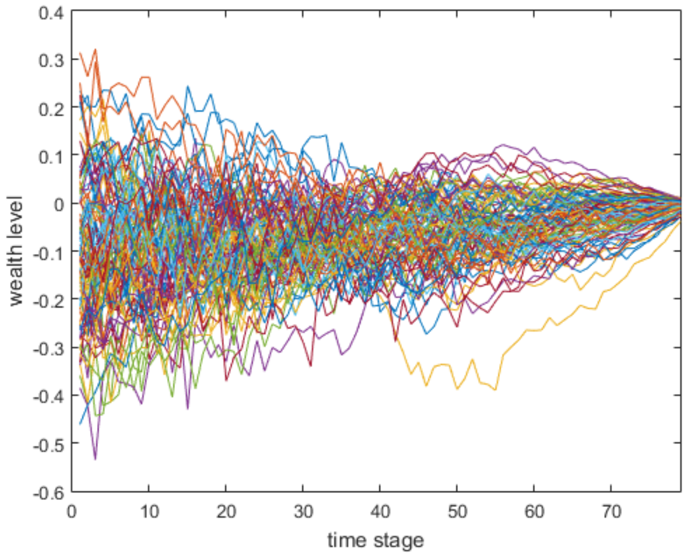
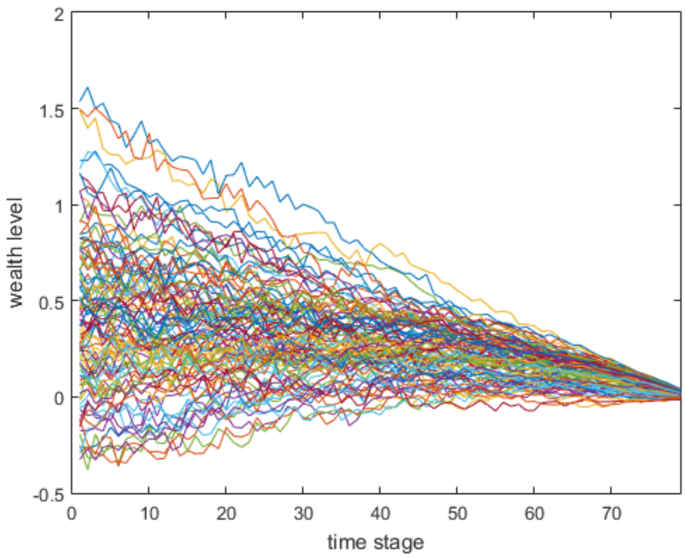


Figure 3.11: simulated wealth scatter along the varying std. of constant gamma 0.2 and time stage

Figure 3.11 and Figure 3.12 shown it could not prevent household's wealth from converging to zero at the end, with adjusting the differentiation magnitude of heterogenous consumption preference (the discreteness of differing risk aversion coefficient, γ_k). However, if risk aversions are more discrete among households, then the smoother their wealth converging to the same level along time passing. In addition, the results show that the averaging level (common level) of wealth distribution at each time stage is not roughly zero anymore as similar in previous simulations. It could be argued that the discreteness of risk aversion in consumptions may have effects on the averaging level of households' wealth.

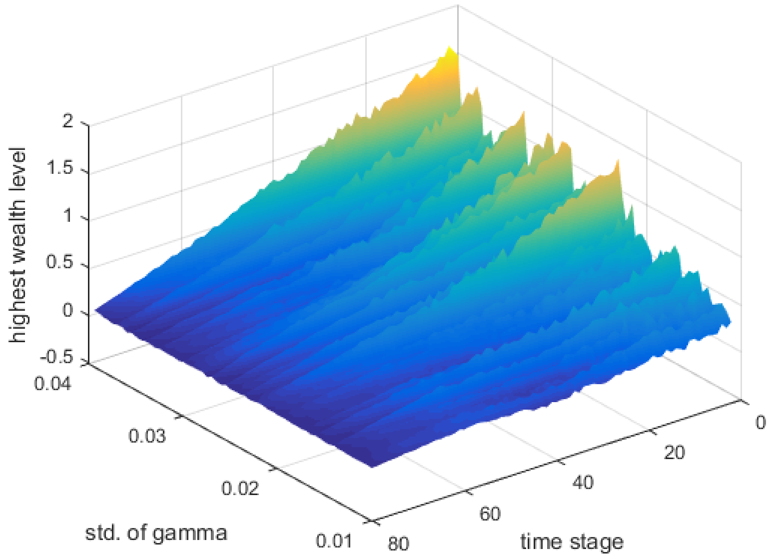


(a) gamma std. = 0.01

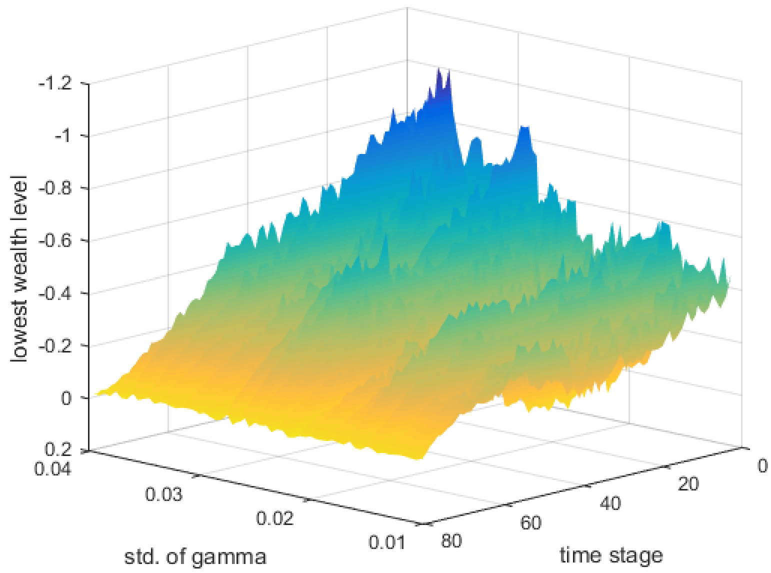


(b) gamma std. = 0.04

Figure 3.12: wealth trajectory through time with differing std. of fixed gamma 0.20



(a) highest wealth surface



(b) lowest wealth surface

Figure 3.13: highest & lowest wealth surface corresponding differing std. of constant gamma 0.20

Figure 3.13 illustrates that the higher discreteness of risk aversion of household's consumption leads the higher highest-wealth (lowest-wealth) level. The highest and lowest wealth are symmetrically converging to the zero value of capital accumulated at the end of the dynasty.

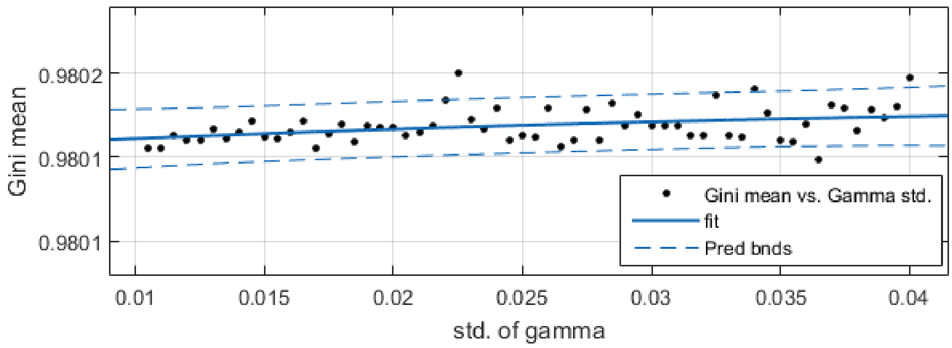


Figure 3.14: Gini coefficient averaging vs. std. of gamma

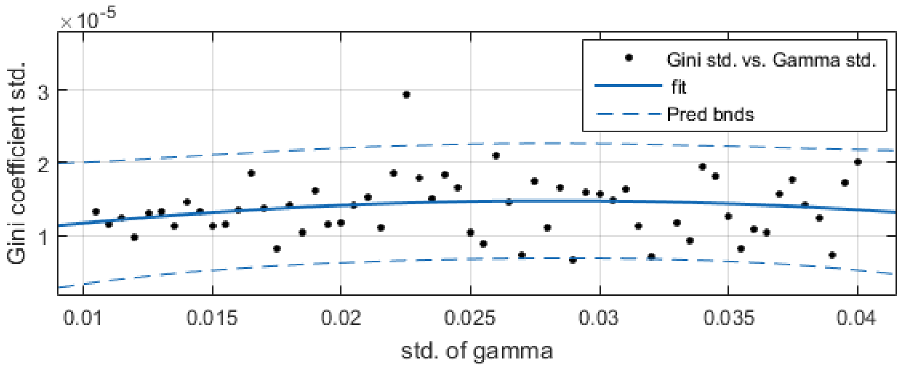


Figure 3.15: Gini coefficient std. vs. std. of gamma

The wealth inequality roughly is positive to discreteness of risk aversion in consumption, however it cannot significantly influence the inequality. Similarly, there is no obvious impact of discreteness of the differing risk aversion in consumption on the volatility of wealth inequality.

3.5 Conclusion and future research

This chapter we improved it is more consistent to reality that the state price density satisfies the equilibrium level endogenously driven by risk aversion, deposit rate and household's revenue levels jointly. This also simplify the whole modelling in describing the wealth concentration progress. In addition, each household?s optimal consumption is set as maximising the utility for their accumulative consumptions and their utility in terminal wealth at dynasty end. We are adopting quadratic utility function benefits us to extend the problem from homogenous consumption preference to the heterogeneous. This chapter illustrates the homogenous (heterogeneous) risk aversion can affect the progressing of wealth concentration, however cannot affect wealth inequality obviously. On the other hand, this result is equivalent to previous chapter conclusion, as one of robustness comparing. Moreover, the deviation magnitude among differing risk aversion has no significant impact on wealth inequality, however can influence the evolution of wealth concentrations.

For the robustness check planned in the future, subject to the Table 3.1, the expected growth rate on wealth could be set as varying in different, it could be comparing calibrations among multilateral nations, they have differentiated GDP growth rate, treating them as equivalent as the wealth growth rate. On the other hand, the income growth rate should be set as differentiated and consistent to its real trend. Thereby, subject to the Figure 3.5, we may check whether the economic growth (income progress level) is one important determinant causing the regime-switch of wealth concentration trend through the dynasty is still existing or not.

Chapter 4

Determinants of wealth inequality with endogenous income

4.1 Introduction

This chapter is still consistent to general equilibrium framework with maximizing social welfare, achieve the Pareto optimality of household's consumption, productive factors and structures at each stage, simultaneously maximizing terminal wealth. Hesitating from previous chapter, the market clearing condition just keep the aggregated incomes equal to the aggregated consumption.

In the previous chapters, household's income is given exogenously and only stems from labor incomes. However, the real household's income should constitute labor wage and financial capital gain, because households contributing their labor forces and savings to different kinds of business sectors in an exchanging economy with financial market. Thus, it could be essential to see whether dynamics of factors and structures during business sector's development have influences or not, on the evolutions of wealth inequality, initiated from an economic complex system.

Moreover, comparing with the previous two chapters and Karatzas (1997), we endogenote the wage and capital gain under a complex economic system which is not only compatible with neoclassical economic growth framework. Also, the complex economic system built in this chapter, it let us glance on if the latecomer's advantage advocated have powerful effect on inequality on the multilateral country sides. The latecomer's advantage is advocated by Lin (2011) to the global technological frontier, can be adoptable for the developing counties in the globalised open trade environment.

The real household's income should constitute labor wage and financial capital gain, because households contributing their labor forces and savings to different kinds of business sectors in an exchanging economy with financial market. Contingent claim analysis is crucial for our system, it enables the allocation of output (industrial value-added) to capital suppliers and labor suppliers are dynamically varying, adaptive to the technological gap shrinking (catching up) between developing and developed countries.

4.2 Modelling

This chapter builds a theoretical model to see whether some endogenously productive factors and structures could influence the wealth inequality or not? Comparing with the previous chapters, this chapter will embed the wealth inequality study into a neo-classical economic growth model, whose labor incomes and financial capital gain are heterogeneously and endogenously driven by business sector's progress in labor (technological) productivity. Moreover, each household's consumption still satisfies the Pareto optimality as shown as previous chapter, meanwhile household dynamically invest their savings (after consumings) into an incomplete financial market following general strategies.

By utilising the approach of contingent claim analysis, the closed-form solution for the labor and financial capital have been obtained, and several dimensions have been involves inside, such as total factor productivity (TFP), productive factors and output. Furthermore, household income stems from wage and financial capital gain, compensating them for supplying labor and capital to business sectors. Moreover, at each stage, the financial asset available for each household's investment is determined after their optimal consumption, which is consistent with the work in the Chapter 3. Based on these premises , we check whether a few factors have influential power on the evolution of wealth inequality or not, by using simulating experiments, it obtains a few innovative answers.

4.2.1 Individual's wealth within investment gain

Each individual act as a productive factors (capital and labor) supplier. They contribute labor and financial capital (equity, lending etc., physical capital is capitalised to be financial capital) to various kinds of business sectors. Each household may supply either one or simultaneously supply both two kind of productive factors. In return they receive revenue from the business sector working for. The revenue is constituted by the wage (basic salary and bonus) by supplying the labor force to sectors, also include financial capital gain (equity gain and interest) by supplying financial capital. Households reply on their revenue to maintain the daily consumption in priority. For consuming more in the future, they reinvest the retained savings on different business sectors via a completed financial market at each period end, following general investing strategies.

There are $n+1$ securities trading continuously in the market, corresponding to the n business entities[1] and one money market $B(t)$, such as the treasury bill. For simplicity, the riskless interest rate is constant r, money price evolves according to

$$dB(t) = rB(t)dt \tag{4.1}$$

The evolutions of the price-per-share of the security $P_i(t)$ of the i^{th} business entity at stage t satisfy:

$$dP_i(t) = P_i(t)dt\left(b_i(t) + \sum_{j=1}^{z} \sigma_{ij}(t)dW_j(t)\right), \quad i = 1, ..., n. \tag{4.2}$$

where $b_i(t)$ is the drift of the i^{th} risky security, in vector $\{b(t) = (b_1(t), ..., b_n(t))'\}$ and $\sigma_{ij}dW_j(t)$ are the risk sources driven by the independent z-dimensional Brownian motions $W(t) = (W_1(t), \cdots, W_z(t))'$. With this interpretation, the volatility coefficient

[1] A few business sectors have subordinate or joint relationships. Nonetheless juxtaposing them as completely securitised

$\sigma_{ij}(t)$ reflects the intensity of the j^{th} source of uncertainty influences the price of the i^{th} risky security, in vector process $\sigma(t) = \{\sigma_{ij}(t)\}_{n\times z}$.

In absence of labor income from business sectors, financial capital gain from their investment (putting their savings into the correspond physical capital in different business sectors) could be the only source for households to maintain their consumptions. Under this circumstance every household's wealth $X_k(t)$ at time t can be presented as:

$$\begin{aligned} dX_k(t) &= \sum_{i=1}^{n} \pi_{k,i}(t)\frac{dP_i(t)}{P_i(t)} + \left(X_k(t) - \sum_{i=1}^{n}\pi_{k,i}(t)\right)\frac{dB(t)}{B(t)} - c_k(t)dt \\ &= \sum_{i=1}^{n}\pi_{k.i}(t)\left(b_i(t) + \sum_{j=1}^{z}\sigma_{ij}(t)dW_j(t)\right) + \left(X_k(t) - \sum_{i=1}^{n}\pi_{k,i}(t)\right)rdt - c_k(t)dt \\ &= \pi_k'(t)[(b(t) - r\mathbf{1}_n)dt + \sigma(t)dW(t)] - c_k(t)dt + rX_k(t)dt \end{aligned} \tag{4.3}$$

where $\mathbf{1}_n = (1, ..., 1)'$, $\pi_{k,i}(t)$ denotes how much money the k^{th} household invests in each of the available securities, $\pi_k(t) = (\pi_{k,1}(t), \cdots, \pi_{k,n}(t))'$ and the term $X_k(t) - \sum_{i=1}^{n}\pi_{k,i}(t)$ is each household's investment in money market.

After inserting the wage part $\epsilon_k(t)$ into the revenue and rewriting Eq.(4.3) with matrix forms. The household's wealth dynamic satisfies

$$dX_k(t) = [\epsilon_k(t) + rX_k(t) - c_k(t)]dt + \pi_k(t)'[(b(t) - r\mathbf{1}_n)dt + \sigma(t)dW(t)]. \tag{4.4}$$

Following portfolio theory, Markowitz (1952), the specific form of the price of market risk $\theta(t)$, can be presented as

$$\begin{aligned} \theta(t) &= \sigma(t)'(\sigma(t)\sigma(t)')^{-1}(b(t) - r\mathbf{1}_n). \\ b(t) &= \begin{pmatrix} b_1(t)\cdots b_i(t)\cdots b_n(t)\end{pmatrix}_{1\times n} \\ \sigma(t) &= \begin{pmatrix} \sigma_{11}(t) & \cdots & \sigma_{1j}(t) & \cdots & \sigma_{1z}(t) \\ \vdots & \ddots & & & \vdots \\ \sigma_{i1}(t) & & \sigma_{ij}(t) & & \sigma_{iz}(t) \\ \vdots & & & \ddots & \vdots \\ \sigma_{n1}(t) & \cdots & \sigma_{nj}(t) & \cdots & \sigma_{nz}(t) \end{pmatrix}_{n\times z} \end{aligned} \tag{4.5}$$

$b(t)$ is the matrix of capital return rate covering the whole categories of business sectors. $\sigma(t)$ is a market risk matrix fully covering the risk sources from all financings for the whole physical capitals in the business sectors[2].

4.2.2 Household's revenue

Each household's salary level $\epsilon_k(t)$ can be regarded as the varying labour value $L_i(t, \cdot)$ accumulative throughout the whole dynasty. For simplicity, the salary payment follows equalitarianism inside each business sector i, where $m_{i,L}$ is the constant number of employees working for it:

$$\epsilon_k(t)dt \triangleq dL_i(t)/m_{i,L}, \tag{4.6}$$

On the other side, the earning rate $b_i(t)$ for each unit financing security of sector i is denoted as

$$b_i(t)dt \triangleq dF_i(t)/\big(n_{i,F}P_i(t)\big), \tag{4.7}$$

where P_i is the endogenous security price and $n_{i,F}$ is fixed number of securities issued by the sector i. $L_i(t, \cdot)$ and $F_i(t, \cdot)$ are determined by the changes of different specific factors will be shown by Eq. (4.49) and Eq. (4.50) later.

Subject to Eq, (4.6) and Eq. (4.9), their discrete form can be written as

$$\epsilon_k(t)\Delta t \triangleq \Delta L_i(t)/m_{i,L} \tag{4.8}$$

$$b_i(t)\Delta t \triangleq \Delta F_i(t)/\big(n_{i,F}P_i(t)\big), \tag{4.9}$$

It is crucial to solve out the set containing every household's investment portfolio on each security at time stage, $\{\pi_{k,i}(t)\}_{n\times K}$. It is strength that $\pi_{k,i}(t), t = 0, 1, 2, \cdots, T$ should meet the below constraints simultaneously.

The constraint of the sum of each row, corresponds the market value of each business sector at stage t:

$$\sum_{k=1}^{K} \pi_{k,i}(t + \Delta t) = n_{i,F}P_i(t), \tag{4.10}$$

[2]In this research, the physical capital is assumed equivalent to the financial capital.

intuitively, among the K households, the aggregation of their investment weights on the business sector i during stage $t+\Delta t$ is roughly equal to the unit security price of the sector i multiplied by its security amount $n_{i,F}$.
The constraint of the sum of each column, corresponds to each household's wealth figure at stage t:

subject to Eq. (4.4) in discrete form,

if the k^{th} order household's net wealth is positive, $X_k(t) + [\epsilon_k(t) + rX_k(t) - c_k(t)]\Delta t + \sum_{i=1}^{n} \pi_{k,i}(t)[b_i(t)\Delta t + \sum_{j=1}^{z} \sigma_{ij}\big(W_j(t) - W_j(t-\Delta t)\big)] > 0$, then at the start of stage $t+\Delta t$, the sum of his investments on the whole range of n business sectors, $\{\pi_{k,i}(t+\Delta t), i = 1, \cdots, n\}$ should equal his total wealth available at the end of stage t:

$$\sum_{i=1}^{n} \pi_{k,i}(t+\Delta t) = X_k(t) + [\epsilon_k(t) + rX_k(t) - c_k(t)]\Delta t \\ + \sum_{i=1}^{n} \pi_{k,i}(t)[b_i(t) + \sum_{j=1}^{z} \sigma_{ij}\big(W_j(t) - W_j(t-\Delta t)\big)],^{3} \tag{4.11}$$

if the k^{th} order household's net wealth is non-positive at the end of stage t, $X_k(t) + [\epsilon_k(t) + rX_k(t) - c_k(t)]\Delta t + \sum_{i=1}^{n} \pi_{k,i}(t)[b_i(t)\Delta t + \sum_{j=1}^{z} \sigma_{ij}\big(W_j(t) - W_j(t-\Delta t)\big)] \leq 0$, then he cannot participate investing on anyone of business sectors:

$$\pi_{k,i}(t+\Delta t) = 0, i =, 1, 2, \cdots, n. \tag{4.12}$$

where $P_i(t)$ and $X_k(t)$ can be recursively solved out according to Eq. (4.2) and Eq. (4.4) respectively,

$$P_i(t+\Delta t) = P_i(t)[1 + b_i(t)\Delta t + \sum_{j=1}^{z} \sigma_{ij}\big(W_j(t+\Delta t) - W_j(t)\big], \quad P_i(0) = \frac{F_i(0)}{n_{i,F}} \tag{4.13}$$

$$X_k(t+\Delta t) = X_k(t) + [\epsilon_k(t) + rX_k(t) - c_k(t)]\Delta t + \pi_k(t)'[b(t)\Delta t + \sigma\big(W(t+\Delta t) - W(t)\big)], \tag{4.14}$$

with the initial conditions that $X_k(0) = \sum_{i=1}^{n} F_i(0)/K$, $\epsilon_k(0) = 0$, $b_i(0) = 0$, $\pi_{k,i}(0) = F_i(0)/K$, $\pi_k(0)' = (\pi_{k,1}(0), \cdots, \pi_{k,i}(0), \cdots, \pi_{k,n}(0))$. The investment portfolio of each household's stands on the same starting line, the capitalised value of the whole business sectors, equally belong to each household at the initial condition of the dynasty.

4.2.2.1 Economic system with general investment strategy

This chapter is to investigate the evolution of wealth inequality under an economic system where households make investment decisions at each time stage following General Strategies (GS) in a completed market. The GS satisfies:

$$\{\pi_{k,i}(t)\} = \begin{pmatrix} \pi_{1,1}(t) & \cdots & \pi_{k,1}(t) & \cdots & \pi_{K,1}(t) \\ \vdots & \ddots & & & \vdots \\ \pi_{1,i}(t) & & \pi_{k,i}(t) & & \pi_{K,i}(t) \\ \vdots & & & \ddots & \vdots \\ \pi_{1,n}(t) & \cdots & \pi_{k,n}(t) & \cdots & \sigma_{K,n}(t) \end{pmatrix}_{n\times K} \tag{4.15}$$

Be consistent to Eq. (4.10), the sum of each GS's elements in row meets

$$\sum_{k=1}^{K} \pi_{k,i}(t) = n_{i,F} P_i(t-\Delta t), \tag{4.16}$$

it presents the sum of all household's investments at just start of time t equal to the capitalized value of the i^{th} business sector at just end of time $t-\Delta t$, namely its unit stock price $P_i(t-\Delta t)$ multiplying the time-invariant number of stocks $n_{i,F}$.

Be consistent to Eq. (4.11), the sum of each GS's elements in column satisfies

$$\begin{aligned} \sum_{i=1}^{n} \pi_{k,i}(t) = {} & X_k(t-\Delta t) + [\epsilon_k(t-\Delta t) + rX_k(t-\Delta t) - c_k(t-\Delta t)]\Delta t \\ & + \sum_{i=1}^{n} \pi_{k,i}(t-1)[b_i(t-\Delta t) + \sum_{j=1}^{z} \sigma_{ij}\big(W_j(t-\Delta t) - W_j(t-2\Delta t)\big)], \end{aligned} \tag{4.17}$$

it presents the sum of the k^{th} household's investments on the full range of business sectors at just start of time t, equal to his available wealth at the just end of time $t-\Delta t$. His available wealth at end of $t-\Delta t$ are constituted by the just start wealth at $t-\Delta t$, $X_k(t-\Delta t)$ plus his after-consumption saving $[\epsilon_k(t-\Delta t)+rX_k(t-\Delta t)-c_k(t-\Delta t)]\Delta t$, then plus his financial capital again during $t-\Delta t$, by investing on business sectors, $\sum_{i=1}^{n} \pi_{k,i}(t-\Delta t)[b_i(t-\Delta t) + \sum_{j=1}^{z} \sigma_{ij}\big(W_j(t-\Delta t) - W_j(t-2\Delta t)\big)]$.

Further more, this household's labor income level and financial capital gain level at $t-\Delta t$, $\epsilon_k(t-\Delta t)$ and $b_i(t-\Delta t)$ respectively follow Eq. (4.6) and Eq. (4.9), namely,

$$\epsilon_k(t-\Delta t) \triangleq \frac{T[L_i(t-\Delta t,\cdot) - L_i(t-2\Delta t,\cdot)]}{m_{i,L}} \tag{4.18}$$

$$b_i(t-\Delta t) \triangleq \frac{T[F_i(t-\Delta t) - F_i(t-2\Delta t)]}{n_{i,F} P_i(t-2\Delta t)} \tag{4.19}$$

where aggregated labor value and financial capital value L_i and F_i are endogenously driven by this business sector's Total Factor Productivity (TFP), $A_{i,t}$ and a few other factors, these will be explained in Section (4.2.2.2). The analytical formulas to compute L_i and F_i at each time stage will be shown in Eq. (4.49) and Eq. (4.50) later. Each business sector's stock price $P_i(t-2\Delta t)$ can be computed following Eq. (4.13) as that

$$P_i(t-2\Delta t) = P_i(t-3\Delta t)[1+b_i(t-3\Delta t)\Delta t+\sum_{j=1}^{z}\sigma_{ij}\big(W_j(t-2\Delta t)-W_j(t-3\Delta t)], \tag{4.20}$$

subject to Eq. (4.19) and Eq. (4.20), it can be known that each business sector's earning rate of unit capital investment, b_i and its stock price P_i can be recursively iterated out, namely $b_i(t-\Delta t)$ is computed by the known $P_i(t-3\Delta t)$ and $b_i(t-3\Delta t)$. Moving back to the Eq. (4.17), the k^{th}household's consumption $c_k(t-\Delta t)$ is the optimal consumption level obtained from the previous chapter, which is solved out under general equilibrium framework. The specific solution of the $c_k(t)$ will be shown later in Eq. (4.57) with the aggregated incomes from both labor income and capital accumulation across the whole economy, $\mathcal{E}_k$, Eq. (4.55) and Eq. (4.56).

Subject to the GS matrix, Eq. (4.15), it is a problem solving a few general strategies chosen by household at the starting time of each stage t. It can be known that there could be $n \times K - (n + K) + 1$ groups of general strategies can be solved out for each stage GS, with the n constraints Eq. (4.16) and the K constraints Eq. (4.17). In addition, our target is to check the inequality evolution accompanying different kind of factor changes, therefore, it had better to make expectation for the whole $n \times K - (n + K) + 1$ groups of general strategies, it means we find out the most possible investment decision made by each household in the economy in expectation, and their expected general strategies comprehensively leading the wealth inequality evolution throughout the dynastic development.

In addition, be consistent to Eq. (4.14), the $X_k(t-\Delta t)$ required by Eq. (4.17) can be computed by

$$\begin{aligned} X_k(t-\Delta t) = X_k(t-2\Delta t) + [\epsilon_k(t-2\Delta t) + rX_k(t-2\Delta t) - c_k(t-2\Delta t)]\Delta t \\ +\pi_k(t-2\Delta t)'[b(t-2\Delta t)\Delta t + \sigma\big(W(t-\Delta t) - W(t-2\Delta t)\big)], \end{aligned} \tag{4.21}$$

Thus, the Eq. (4.21), Eq. (4.17) and Eq. (4.15) together illustrate that the process of each household's wealth X_k and their general investment strategies $\pi_{k,i}(t)$ can be recursively simulated out step by step.

4.2.2.2 Endogenous Labor income and Financial Capital

Subject to the Solow-Swan model with human capital improved by Mankiw et al. (1992) that

$$Y_{i,t} = F_{i,t}^{\alpha} H_{i,t}^{\beta} \big(A_{i,t} L_{i,t}\big)^{1-\alpha-\beta} \tag{4.22}$$

where Y_t is the business sector's output at stage t (hereafter, omit i for all of parameters), F_t is physical capital (financial capital), H_t is human capital, A_t stands for technique level (Total Factor Productivity, TFP among business sectors), L_t is labor and α and β respectively are the elasticity of output with respect to physical capital and labor. Subject to Eq. (4.22)

$$\begin{aligned} \ln\big(Y_t/Y_{t-1}\big) =& \alpha \ln\big(F_t/F_{t-1}\big) + \beta \ln\big(H_t/H_{t-1}\big) \\ &+ \big(1-\alpha-\beta\big)\ln\big(A_t/A_{t-1}\big) + \big(1-\alpha-\beta\big)\ln\big(L_t/L_{t-1}\big) \end{aligned} \tag{4.23}$$

with an implicit assumption that the sector's TFP progresses uniformly through each stage $\mathbb{E}\ln\big(A_{t+1}/A_t\big) = \mathbb{E}\ln\big(A_t/A_{t-1}\big)$. Reordering Eq. (4.23) to be

$$\begin{aligned} \ln\big(A_t/A_{t-1}\big) =& \frac{\ln\big(Y_t/Y_{t-1}\big)}{1-\alpha-\beta} - \ln\big(L_t/L_{t-1}\big) \\ &- \frac{\alpha}{1-\alpha-\beta}\ln\big(F_t/F_{t-1}\big) - \frac{\beta}{1-\alpha-\beta}\ln\big(H_t/H_{t-1}\big) \end{aligned} \tag{4.24}$$

after release the implicit assumption, by adding a noise term

$$\begin{aligned} \ln\big(A_t\big) = \ln\big(A_{t-1}\big) + \frac{\ln\big(Y_{t-1}/Y_{t-2}\big)}{1-\alpha-\beta} + \frac{1}{2}\sigma^2 - \frac{1}{2}\sigma^2 - \ln\big(L_t/L_{t-1}\big) \\ -\frac{\alpha}{1-\alpha-\beta}\ln\big(F_t/F_{t-1}\big) - \frac{\beta}{1-\alpha-\beta}\ln\big(H_t/H_{t-1}\big) + \sigma Z_t \end{aligned} \tag{4.25}$$

Subject to Eq. (4.25), the dynamics of TFP can be presented by

$$\frac{dA_t}{A_t} = \big[\mu_t - l_t - (\omega_t + d_t - r) - \delta_t\big]dt + \sigma dZ_t \tag{4.26}$$

with

$$\mu_t \triangleq \tfrac{\ln\left(Y_{t-1}/Y_{t-2}\right)}{1-\alpha-\beta} - \tfrac{1}{2}\sigma^2 \tag{4.27}$$

$$l_t \triangleq \ln\left(L_{t-1}/L_{t-2}\right) \tag{4.28}$$

$$\omega_t + d_t - r \triangleq \tfrac{\alpha}{1-\alpha-\beta}\ln\left(F_{t-1}/F_{t-2}\right) \tag{4.29}$$

$$\delta_t \triangleq \tfrac{\beta}{1-\alpha-\beta}\ln\left(H_{t-1}/H_{t-2}\right) \tag{4.30}$$

where σdZ_1 is innovation risk of TFP dynamics, Z is Brownian motion. The drift component of TFP dynamics is linearly composited by the changes of endowment structure, μ_t, l_t, $w_t + d_t - r$ and δ_t respectively correspond to the changes of output, labor, capital and human capital. w_t and d_t separately imply the proportion of A_t dynamics attributed by equity-financing and debt-financing.

Intuitively, Eq.(4.24) exposits TFP A_t as the driver to create residual profit deducting physical capital, labor and human capital from output, for each business sector during producing. A few studies represented by Griliches (1979) clarifies the TFP A_t evolution associated to technological research and development risk (R&D), expressed by the diffusion term σdZ_t of the Eq. (4.24). On the other hand, some studies represented by Lin (2011) finds developing economies may have a latecomer advantage in productive technology, because their technology has not achieved the global technological frontier. Thereby the latecomer advantage benefits the business sectors in emerging nations almost shouldering minor R&D risk than that of nations whose industrial sectors on the global technological frontier. Accordingly, the Eq.(4.26) can be treated equivalently as the business sector's profitability dynamics of Developing Economies, comparing with Eq. (4.24), the R&D risk can be neglected, the business sector's profitability (or TFP) $A^{\mathbb{D}}$ could be much stable and smoothy than the profitability on technological frontier, and satisfies

$$dA_t^{\mathbb{D}} = (\mu_t + r - l_t - \omega_t - d_t - \delta_t)A_t^{\mathbb{D}}dt, \tag{4.31}$$

In addition, an theoretical background should be existing in this research that, the markets among among economies are open no trade barrier between developed and developing countries, the industrial location (division) of each kind of business sector is freely determined by the competitive equilibrium for the open trade, similar as the explanation by Hertel (1997) .

Subject to Eq. (4.26) and Eq. (4.31), conventional economic analysis expresses the expected gross salary rate (wage rate and labor welfare) according to

$$[l_t + k(\mu_t + r - l_t - \omega_t - d_t - \delta_t)]dt. \tag{4.32}$$

However, Eq. (4.32) ignores the reserved part of residual profit under a budget management policy for the incremental activities of the business sector, such as a forthcoming project. Therefore, being consistent with the budget management, the gross salary (or financial capital gain)[4] available at each time stage, $\varphi(t, \cdot)$ should be treated as the difference between the adjacent contingent claims adaptive to the prospect of the sector development

$$\varphi_{t+1} = \mathcal{V}_{t+1} - \mathcal{V}_t, \tag{4.33}$$

$$\mathcal{V}_t = \mathbb{E}_t \int_t^T H_{t,\tau}\mathcal{G}_\tau d\tau + H_{t,T}\mathcal{V}_T, \tag{4.34}$$

the $\mathcal{G}$ is the continuously periodical payment flow expected to be attributed to either the labor force or financial capital suppliers, conditional on the business sector progress adaptive to each time sage t. In the conventional economic aspect, under the single closed economy, the retaining profit and the periodical payment attributed to labor force or financial capital suppliers is linear relationship in simplicity, denoted by

$$\mathcal{G}\big(A, t\big) = \xi(\cdot)A \tag{4.35}$$

where, ξ is the linear adaptor.

Moreover, between the bilateral economies with open trade condition, under the

[4]When $\varphi(t, \cdot)$ corresponds the business sector's gross salary at each period, it could be denoted by L_t, alternatively when $\varphi(t, \cdot)$ corresponds financial capital gains (no matter direct financing or indirect financing, it could be denoted by F_t .

competitive equilibrium between the identical industrial sector in developed economy (technological frontier) and developing economy (own latecomer advantage in producing technology), the periodical payment attributed to labor force or financial capitals suppliers should be linear to the differential level of technology of the identical business sector:

$$\mathcal{G}\big(A, A^D, t\big) = \xi_1(\cdot)A + \xi_2(\cdot)A^D \tag{4.36}$$

where ξ_1 and ξ_2 are the linear adaptors respective for A and A^D. However, the emerging economic perspective from new economic thinking begin regarding its could be nonlinear between them, also the attribution relationship for the value added of each business sector to its labor and financial capital suppliers could be endogenously driven by relative difference between the technology in developed economy and the in developing economy, $\frac{A^D}{A}$:

$$\mathcal{G}\big(A, A^D, t\big) = \xi_1\big(\frac{A^D}{A}, \cdot\big)A + \xi_2\big(\frac{A^D}{A}, \cdot\big)A^D + \Omega\big(A, A^D, t\big) \tag{4.37}$$

where $\Omega\big(A, A^D, t\big)$ is the nonlinear component, its architecture is still unsure and it will be solved out in the end of derivation as shown as Eq. (4.49) and Eq. (4.50).

Lets calling back Eq. (4.33), subject to the nonlinearity in Eq. (4.37), the adaptive payoff $\mathcal{V}_t$ should be endogenously driven by A, A^D and t in nonlinearity. Moreover $H_{t,\tau}$ is the time discounter[5]. The $H_{t,T}\mathcal{V}_T$ presents the terminal payment by the business sector to its stakeholders, for labor force, it is the dismissal compensation fee for employment, on the other side for financial capital suppliers, it is the asset collateral priority for loan lenders and then residuals for shareholders. The $\mathcal{V}_t$ is the expected payoff accumulative from t to T conditional on each development stage t by this business sector, equivalently, $\mathcal{V}_t$ can be treated as the expected total outputs from t to T by this business sector, and nothing left at the end of liquidation stage.

Similar to employee payment, investor payment also satisfies Eq. (4.33) and Eq. (4.34). We use L_t and F_t to express $\mathcal{V}_t$ respectively for the labor and capital for the

[5]Following Eq. (3.2), the state price density at τ conditional to t, $\tau < c < t < T$, can be written as $H_{t,\tau} = e^{-\left[r(\tau-t)+\int_t^\tau \theta(c)'dW(c)+\frac{1}{2}\int_t^\tau \|\theta(c)\|^2dc\right]}$

business sector at t.

Under the competitive equilibrium among open economies, each specific business sector's salary and capital payoff should be variant upon on time stage, technological frontier TFP A_t and latecomer advantage TFP $A^{\mathbb{D}}$, namely $\mathcal{V}(t, A, A^{\mathbb{D}})$, for simplicity written as $\mathcal{V}_t$, subject to Eq. (4.26) and Eq. (4.31) applying Ito's formula on $H_{0,\tau}\mathcal{V}$ yields

$$\begin{aligned} dH_{0,t}\mathcal{V}_\tau = & -H_{0,t}[rdt + \theta(t)'dW(t)]\mathcal{V}_\tau + H_{0,t}\{(\mu + r - l - \omega - d - \delta) \\ & \cdot A^{\mathbb{D}}\frac{\partial \mathcal{V}_\tau}{\partial A^{\mathbb{D}}}dt + (\mu + r - l - \omega - d)A\frac{\partial \mathcal{V}_\tau}{\partial A}dt + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 \mathcal{V}_\tau}{\partial A^2}dt \\ & + A\frac{\partial \mathcal{V}_\tau}{\partial A}\sigma dZ\ \}. \end{aligned} \tag{4.38}$$

Integrating on both sides with the conditional expectation up to t

$$\begin{aligned} \mathbb{E}_t[H_0\mathcal{V}_T - \mathcal{V}_\tau] = & -r\mathbb{E}_t\int_t^T H_{0,\tau}\mathcal{V}_\tau d\tau + \mathbb{E}_t\int_t^T H_{0,\tau}(\mu + r - l - \omega - d - \delta)A^{\mathbb{D}}\frac{\partial \mathcal{V}_\tau}{\partial A^{\mathbb{D}}} \\ & + \mathbb{E}_t\int_t^T H_{0,\tau}\left[(\mu + r - l - \omega - d)A\frac{\partial \mathcal{V}_\tau}{\partial A} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 \mathcal{V}_\tau}{\partial S^2}\right]d\tau, \end{aligned} \tag{4.39}$$

and plugging Eq. (4.34) into Eq. (4.39), the below PDE can be obtained

$$\begin{aligned} 0 = & -r\mathcal{V}_\tau + (\mu + r - l - \omega - d)A\frac{\partial \mathcal{V}_\tau}{\partial A} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 \mathcal{V}_\tau}{\partial A^2} \\ & + (\mu + r - l - \omega - d - \delta)A^{\mathbb{D}}\frac{\partial \mathcal{V}_\tau}{\partial A^{\mathbb{D}}} + \mathcal{G}. \end{aligned} \tag{4.40}$$

The $\mathcal{V}_\tau$ can be assumed as homogenous of degree one in A and $A^{\mathbb{D}}$, the solution could be structured as $\mathcal{V} = A^{\mathbb{D}}Q(x)$, where x is a nonlinear adapter between the business sector's value attributable to labor (or financial capital supplier) and the sector's residual profitability contributed by the latecomer productive advantage in emerging economy, namely $xA^{\mathbb{D}} = A$. Comparing with Eq. (4.32), the $Q(x)$ is dynamical fraction of the aggregated payments occupied in $A^{\mathbb{D}}$. $Q(x)$ involves the paying change term attributed by the differential prospects adaptive to the adjacent time stages.

Substituting $\mathcal{V} = A^{\mathbb{D}}Q(x)$ and the relevant derivatives $\partial\mathcal{V}/\partial A^{\mathbb{D}} = Q$, $\partial\mathcal{V}/\partial A = \partial Q/\partial x$ and $\partial^2\mathcal{V}/\partial A^2 = (A^{\mathbb{D}})^{-1}\partial^2 Q/\partial x^2$ into (4.40), the below ODE can be obtained

$$(\mu - l - \omega - \delta)Q + (\delta + d)x\frac{\partial Q}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 Q}{\partial x^2} + \theta x = 0, \tag{4.41}$$

for the employees' gross salary case, existing $\theta = l + kA^{\mathbb{D}}/A$ contains the linearity percentage of both basic salary and welfare benefit on the entity value. In respect to the business sector's investor side, existing $\theta = d + \omega$ is the linearity proportion of the payment to them. Conjecturing the specific form of $Q(x)$ in (4.41) as

$$Q(x) = \frac{\theta x}{l + \omega + k\frac{A^{\mathbb{Q}}(t)}{A} + d - \mu} + B_1 x^{y_1} + B_2 x^{y_2} \tag{4.42}$$

where the fraction is the linear component of the $Q(x)$. It indicates the linear part of the payments to employees or investors occupied in the whole payment flows of the entity. The $B_1 x^{y_1}$ and $B_2 x^{y_2}$ are the curvature parts of the $Q(x)$, they correspond the nonlinear various parts of payments inferenced by the contingent prospects. y_1 and y_2 are the real root pair follow the quadratic equation

$$\frac{1}{2}\sigma^2 y(y-1) + (\delta + d)y + (\mu - l - \omega - \mu) = 0 \tag{4.43}$$

imposing the condition that the rigid paying (basic salary and equity withdraw) proportion $\mu < \frac{(\delta + d - \frac{1}{2}\sigma^2)^2}{2\sigma^2} + l + \omega + \delta$ to ensure existence of two real roots. Then the roots can be sorted as

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{(\frac{1}{2}\sigma^2 - \delta - d) \pm \sqrt{(\delta + d - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(l + \omega + \delta - \mu)}}{\sigma^2} \tag{4.44}$$

In order to solve out the specific form of B_1 and B_2 in Eq.(4.42), two boundary conditions are required. One boundary condition is that the labor requirement converges to zero at the end of development T, accompanying the TFP A_T progress to be very high. This tendency is consistent with that explained by Mankiw et al. (1992) Jones (1995), the TFP (aggregated labour productivity) growth is driven by human capital accumulation, accompanying the share of human capital participating larger and larger share in economic growth, then the intensity of labor and physical capital tend to shrink. The physical (financial) capital relative to technique reaches an upper limit D at T:

$$\begin{aligned} Q_{L,T}(x_T) &= L_T(A_T, A_T^{\mathbb{D}})/A_T^{\mathbb{D}} = 0 \\ Q_{F,T}(x_T) &= F_T(A_T, A_T^{\mathbb{D}})/A_T^{\mathbb{D}} = D \end{aligned} \tag{4.45}$$

the subscript L and F of Q respectively identifies that they correspond to $\mathcal{V}_t^L$ or $\mathcal{V}_t^F$, because in previously, we simply denoted both $\mathcal{V}_t^L$ and $\mathcal{V}_t^F$ sharing the homogenous form as $\mathcal{V}(t, A_t)$.

On the other side, at the just start stage t_0 of economic development, the intensity of labor initiates from the whole, there is no R&D component involved in TFP, then there is no technological risk at start stage, $x_{t_0} = A_{t_0}/A_{t_0}^{\mathbb{D}} = 1$. At just starting t_0, there is no demand for labor and capital, namely $L_{t_0} = 0$ and $F_{t_0} = 0$. Thereby, the incremental Q'_{L,t_0} and Q'_{F,t_0} stand for the actual requirements for labor and capital at the just end of t_0. By considering the labor welfare has not yet been given, then it existing

$$\begin{aligned} Q_{L,t_0}(x_{t_0} = 1) = (1+k)Q'_{L,t_0}(x_{t_0} = 1) - k \\ Q_{F,t_0}(x_{t_0} = 1) = (1+k)Q'_{F,t_0}(x_{t_0} = 1) \end{aligned} \tag{4.46}$$

based on the two boundaries of the Eq. (4.45) and Eq. (4.46), the simultaneous equations sets can be respectively established:

$$\begin{cases} \frac{l+k\frac{A^{\mathbb{D}}(t)}{A}}{l+k\frac{A^{\mathbb{D}}(t)}{A}+d+\omega-\mu} D + B_{1,L} \cdot D^{y_1} + B_{2,L} \cdot D^{y_2} = 0 \\ B_{1,L} + B_{2,L} = k\left[\frac{l+k\frac{A^{\mathbb{D}}(t)}{A}}{l+k\frac{A^{\mathbb{D}}(t)}{A}+d+\omega-\mu} + y_1 B_{1,L} + y_2 B_{2,L} - 1\right] \end{cases} \tag{4.47}$$

and

$$\begin{cases} \frac{\omega+d}{l+k\frac{A^{\mathbb{D}}(t)}{A}+d+\omega-\mu} D + B_{1,F} \cdot D^{y_1} + B_{2,F} \cdot D^{y_2} = D \\ B_{1,F} + B_{2,F} = k\left[\frac{\omega+d}{l+k\frac{A^{\mathbb{D}}(t)}{A}+d+\omega-\mu} + y_1 B_{1,F} + y_2 B_{2,F}\right] \end{cases} \tag{4.48}$$

where $B_{1,L}$ and $B_{2,L}$, $B_{1,F}$ and $B_{2,F}$ separately correspond to the coefficients B_1, B_2 of Q_L and Q_F. Their specific forms can be solved out from the simultaneous equations, Eq. (4.47) and Eq. (4.48). After that plugging them back into Eq. (4.42), based on the homogeneous expression $\mathcal{V} = A^{\mathbb{D}}Q(x)$, the analytical solutions of the endogenous labor and financial capital $L_t(A_t,\cdot)$, $F_t(A_t,\cdot)$ at t satisfy

$$L_t(A_t,\cdot) = \frac{lA_t + kA_t^{\mathbb{D}}}{l+\omega+k\frac{A_t^{\mathbb{D}}}{A_t}+d-\mu} - \frac{\left\{k(\omega+d-\mu)D^{y_2-y_1}+[y_2(1+k)-1][l+k\frac{A_t^{\mathbb{D}}(t)}{A_t}]D^{1-y_1}\right\}A_t^{\mathbb{D}}(t)^{1-y_1}A_t^{y_1}}{(l+\omega+k\frac{A_t^{\mathbb{D}}(t)}{A_t}+d-\mu)\left\{y_2(1+k)-1-D^{y_2-y_1}[y_1(1+k)-1]\right\}} + \frac{\left\{k(\omega+d-\mu)+[y_1(1+k)-1](l+k\frac{A_t^{\mathbb{D}}(t)}{A_t})D^{1-y_1}\right\}A_t^{\mathbb{D}}(t)^{1-y_2}A_t^{y_2}}{(l+\omega+k\frac{A_t^{\mathbb{D}}(t)}{A_t}+d-\mu)\left\{y_2(1+k)-1-D^{y_2-y_1}[y_1(1+k)-1]\right\}}, \tag{4.49}$$

$$F_t(A_t,\cdot) = \frac{(\omega+d)A_t}{l+\omega+k\frac{A_t^{\mathbb{D}}}{A_t}+d-\mu} + \frac{\left\{k(\omega+d)D^{y_2-y_1}+[y_2(1+k)-1](l+k\frac{A_t^{\mathbb{D}}(t)}{A_t}-\mu)D^{1-y_1}\right\}A_t^{\mathbb{D}}(t)^{1-y_1}A_t^{y_1}}{(l+\omega+k\frac{A_t^{\mathbb{D}}(t)}{A_t}+d-\mu)\left\{y_2(1+k)-1-D^{y_2-y_1}[y_1(1+k)-1]\right\}} - \frac{\left\{k(\omega+d)+[y_1(1+k)-1](l+k\frac{A_t^{\mathbb{D}}(t)}{A_t}-\mu)D^{1-y_1}\right\}A_t^{\mathbb{D}}(t)^{1-y_2}A_t^{y_2}}{(l+\omega+k\frac{A_t^{\mathbb{D}}(t)}{A_t}+d-\mu)\left\{y_2(1+k)-1-D^{y_2-y_1}[y_1(1+k)-1]\right\}}, \tag{4.50}$$

Subject to Eq. (4.27) $\sim$ (4.30), the solutions Eq. (4.49) and Eq. (4.50) release that at stage t, each business sector's demands on labor L_t and physical (financial) capital F_t are endogenously driven by the technological progress A_t, the adaptively growth rate of output and productive factors, $\ln\left(Y_{t-1}/Y_{t-2}\right)$, $\ln\left(L_{t-1}/L_{t-2}\right)$, $\ln\left(F_{t-1}/F_{t-2}\right)$ and $\ln\left(H_{t-1}/H_{t-2}\right)$, the labor welfare k and the technical innovation risk σ.

4.2.3 Optimal consumption with financial investment

Eq.(4.4) shows each household's revenue contains the gross salary and financial capital gain from different business sectors. Follows the optimal consumption level under quadratic utility function, matching Eq. (4.4) with Eq. (3.16), gives

$$\mathcal{E}_k(t,\cdot) = \epsilon_k(t) + \pi_k'(t)(b(t) - r\mathbf{1}_n), \quad t = 0,1,2,\cdots,T.^6 \tag{4.51}$$

[6] subject to $\epsilon_k(0)=0$, $b_i(0)=0$, $\pi_k(0)' = (\pi_{k,1}(0),\cdots,\pi_{k,i}(0),\cdots,\pi_{k,n}(0))$, $\pi_{k,i}(0) = F_i(0)/K$, leads that $\mathcal{E}_k(0) = -r\frac{\sum_{i=1}^n F_i(0)}{K}$

Originally, the aggregate revenue level of the whole economy is

$$\mathcal{E}(t,\cdot)=\sum_{k=1}^{K}\epsilon_k(t,\cdot)+\sum_{k=1}^{K}\pi_k'(t)(b(t)-r\mathbf{1}_n)+r\sum_{k=1}^{K}X_k(t),\quad t=0,1,2,\cdots,T,\tag{4.52}$$

the assumption no individual saving for interest for (4.11) and (4.14) shows

$$\sum_{i=1}^{n}\pi_{k,i}(t)=X_k(t)\tag{4.53}$$

substituting Eq. (4.53) into Eq. (4.52) yields

$$\mathcal{E}(t,\cdot)=\sum_{k=1}^{K}\epsilon_k(t,\cdot)+\sum_{k=1}^{K}\pi_k'(t)b(t),\quad t=0,1,2,\cdots,T,\tag{4.54}$$

furthermore, based on $K=\sum_{i=1}^{n}m_{i,L}$, Eq. (4.6), Eq. (4.9) and Eq. (4.10) results $\mathcal{E}(0,\cdot)=0$ and Eq. (4.54) can be rewritten as

$$\begin{aligned}\mathcal{E}(t,\cdot)&=\sum_{i=1}^{n}m_{i,L}\epsilon_k(t,\cdot)+\sum_{i=1}^{n}\sum_{k=1}^{K}\pi_{k,i}(t)b_i(t)\\&=\sum_{i=1}^{n}T[L_{i,t}(A_t,\cdot)-L_{i,t-1}(A_{t-1},\cdot)]+\sum_{i=1}^{n}T[F_{i,t}(A_t,\cdot)-F_{i,t-1}(A_t,\cdot)]\\&=T\sum_{i=1}^{n}[\widehat{A}_{i,t}(A_t,\cdot)-\widehat{A}_{i,t-1}(A_{t-1},\cdot)],t=1,2,\cdots,T\end{aligned}\tag{4.55}$$

where $\widehat{A}_{i,t}(A_t,\cdot)=L_{i,t}(A_t,\cdot)+F_{i,t}(A_t,\cdot)$, in respect to Eq. (4.49) and Eq. (4.50), it can be presented as

$$\begin{aligned}\widehat{A}_{i,t}(A_t,\cdot)=&\frac{l+\omega+k\frac{A^{\mathbb{D}}(t)}{A(t)}+d}{l+\omega+k\frac{A^{\mathbb{D}}(t)}{A(t)}+d-\mu}A_{i,t}(A_t,\cdot)\\&+\frac{\mu\left\{kD^{y_2-y_1}-[y_2(1+k)-1]D^{1-y_1}\right\}A^{\mathbb{D}}(t)^{1-y_1}A(t)^{y_1}}{(l+\omega+k\frac{A^{\mathbb{D}}(t)}{A(t)}+d-\mu)\left\{y_2(1+k)-1-D^{y_2-y_1}[y_1(1+k)-1]\right\}}\\&-\frac{\mu\left\{k-[y_1(1+k)-1]D^{1-y_1}\right\}A^{\mathbb{D}}(t)^{1-y_2}A(t)^{y_2}}{(l+\omega+k\frac{A^{\mathbb{D}}(t)}{A(t)}+d-\mu)\left\{y_2(1+k)-1-D^{y_2-y_1}[y_1(1+k)-1]\right\}}\end{aligned}\tag{4.56}$$

Along each stage of economic development, heterogenous households' optimal consumptions satisfy the result we obtained in the last chapter, Eq. (3.34), which maximizing the social welfare (household's consumptions) and their heritage wealth at the end of dynasty, recalling Eq. (3.34) that

$$\begin{aligned} c_k^*(t) = \frac{1}{\gamma_k} - \frac{\Gamma - \mathcal{E}(t,\cdot)}{\gamma_k \mathbb{E}\left\{\int_0^T [\Gamma - \mathcal{E}(t,\cdot)]^2 dt\right\}} \mathbb{E}\{\int_0^T [\Gamma - \mathcal{E}(t,\cdot)][1 - \gamma_k \epsilon_k(t) \\ + \gamma_k \pi_k'(t)(b(t) - r\mathbf{1}_n)]dt\} - \mathcal{G}_k X_k(0) \frac{[\Gamma - \mathcal{E}(0,\cdot)][\Gamma - \mathcal{E}(t,\cdot)]}{\mathbb{E}\left\{\int_0^T [\Gamma - \mathcal{E}(t,\cdot)]^2 dt\right\}} \end{aligned} \tag{4.57}$$

In this chapter, household's revenue has been endogenously given by Eq. (4.55) and Eq. (4.56). At each stage, the optimal consumption level $c_k^*(t)$ will be substituted to the Eq. (4.11), recursively yielding the solution of household's investment weight $\{\pi_{k,i}(t+1)\}$ rebalancing to the next stage. After that, it permits recursively solving out the trajectories of households' net wealth along the whole dynasty and investigating the key determinants embedded in the endogenous revenue flow impacting on inequality.

4.3 Simulation and Analysis

In this simulation, we attempt to investigate whether changes of productive factors and structures can influence wealth inequality or not, such as the changes of sector's output, labor, financial capital, human capital, the capital structure (between equity and debt-financing fraction), worker's welfare from residual profit. Moreover, the capital gain has been regarded as one of income sources for household's saving, which is endogenously generated by the operating business sector to contribute output. Under this circumstance that labor and investment earning are endogenously generated, it is worthy to re-check if the homogenous risk averse in household's consumption can determine the wealth inequality or not.

The analytical solutions have been obtained. However the simulation load is still astonished attributed by the endogenous structures for both revenue resources and heterogenous consumption among households. In specific revenue resources contains the labor income distributed by their affiliated sector and the capital gain from their investment in kinds of sectors. Each business sector makes differently particular decision in allocating the wage and capital financing return to their labor and capital-supplier, at each stage. Simultaneously each household's investment decision is endogenously driven by their differing consumptions at each stage. Moreover each household's optimal consumption level is adjusted by their revenue status. These endogenous structures are dynamically and recursively carried on, stage by stage throughout the whole dynasty.

Fortunately, the high performance computing[7] can realise this simulation experiment.

In respect to the fixed parameter adopted in the simulation, they are shown in Table 4.1 and the parameter corresponds the potential determinants of wealth inequality, is to be stated in Table 4.2:

Table 4.1: Summary of fixed parameters

Parameter	Symbol	Value
Household number	K	30
Terminal stage	T	80
Initial technique	A_{i,t_0}	1.1
riskless rate	r	0.05
Upper limit of physical capital over technique at T	D	6
Change of labor	l	0.3
Technological progress volatility factor	σ	0.05
Correlation between technique and product innovation	ρ	0.1

Table 4.2: Potential determinants of wealth inequality

Parameter	Symbol	Value
debt-financing share of output	ω	0.1
equity-financing share of output	d	[0.05: 0.35]
change of human capital	δ	0.1
change of sector output	μ	0.02
worker's welfare in residual profit	k	0.2
household's risk aversion	γ	0.25

[7]The simulations are achieved by employing Alibaba Cloud High Performance Computing Server.

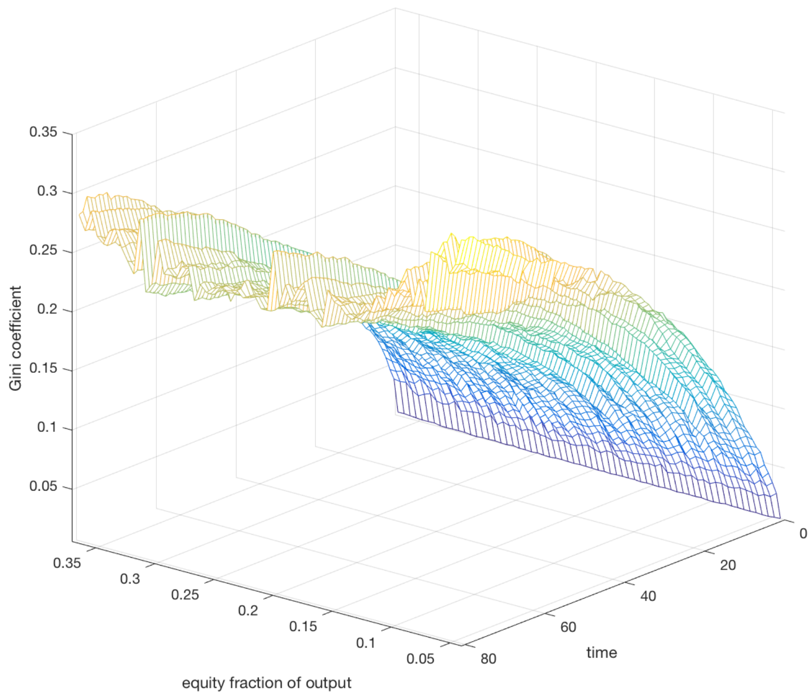


Figure 4.1: equity fraction vs. Gini coefficient along time

Figure 4.1 illustrates this model is capable to describe the inequality deteriorating along time passing, this is consistent with the fact that the wealth keeps on concentrating. There is approximately " smile " relationship between equity-financing fraction and the wealth inequality at each stage. This simulation suggests that the share of equity-financing ω_t is one determinant in wealth concentrating. The overloading or shortage of equity-financing could further accelerate the wealth inequality.

Table 4.3: Potential determinants of wealth inequality

Parameter	Symbol	Value
debt-financing share of output	ω	[0: 0.5]
equity-financing share of output	d	0.2
change of human capital	δ	0.1
change of sector output	μ	0.02
labor's welfare in residual profit	k	0.2
household's risk aversion	γ	0.25

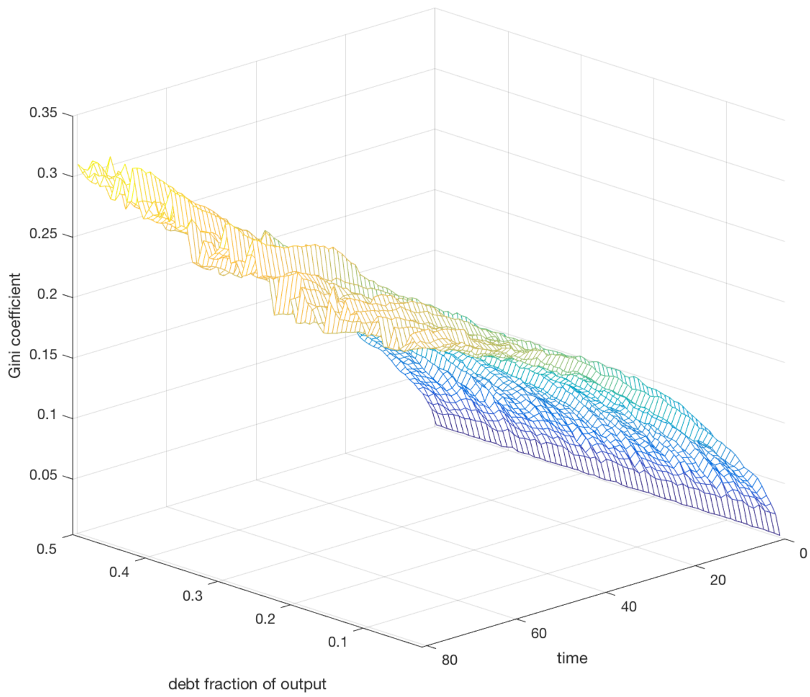


Figure 4.2: debt-financing fraction vs. Gini coefficient along time

The Figure 4.2 illustrates the inequality of wealth could be negatively correlated to the debt-financing share. Theoretically, in absence of thinking the efficiency of equity/debt capital financing and financial market frictions, if the economy is more bank-orientated, then the inequality of wealth expanses more fiercely.

Table 4.4: Potential determinants of wealth inequality

Parameter	Symbol	Value
debt-financing share of output	ω	0.2
equity-financing share of output	d	0.2
change of human capital	δ	0.1
change of sector output	μ	[0 : 0.20]
worker's welfare in residual profit	k	0.2
household's risk aversion	γ	0.25

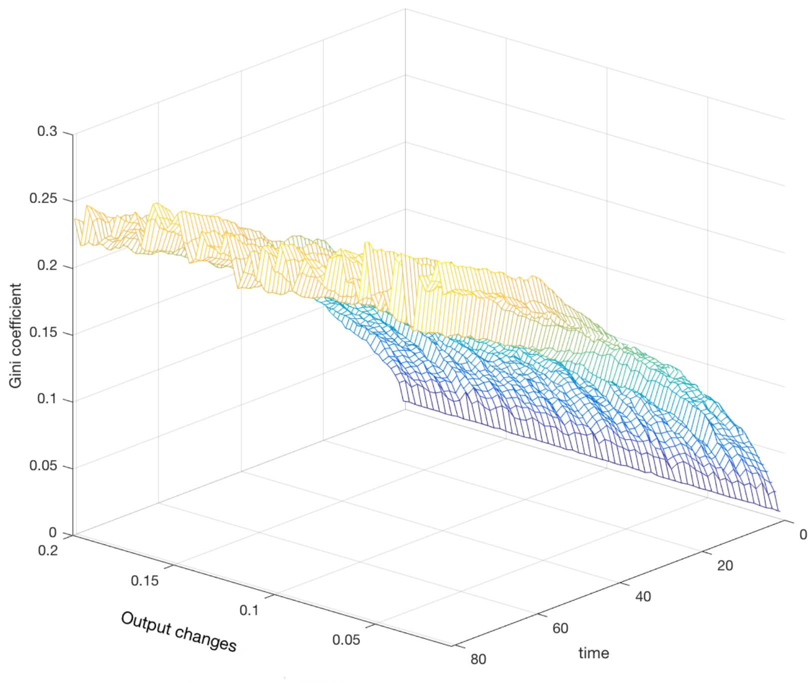


Figure 4.3: Output changes vs. Gini coefficient along time

Figure 4.3 exposits that business sector's output growth is seemly not a cause for the expansion of wealth inequality, by controlling other factors or economic structures as invariant.

Table 4.5: Potential determinants of wealth inequality

Parameter	Symbol	Value
debt-financing share of output	ω	0.2
equity-financing share of output	d	0.2
change of human capital	δ	0.1
change of sector output	μ	0.02
worker's welfare in residual profit	k	[0.1:0.8]
household's risk aversion	γ	0.25

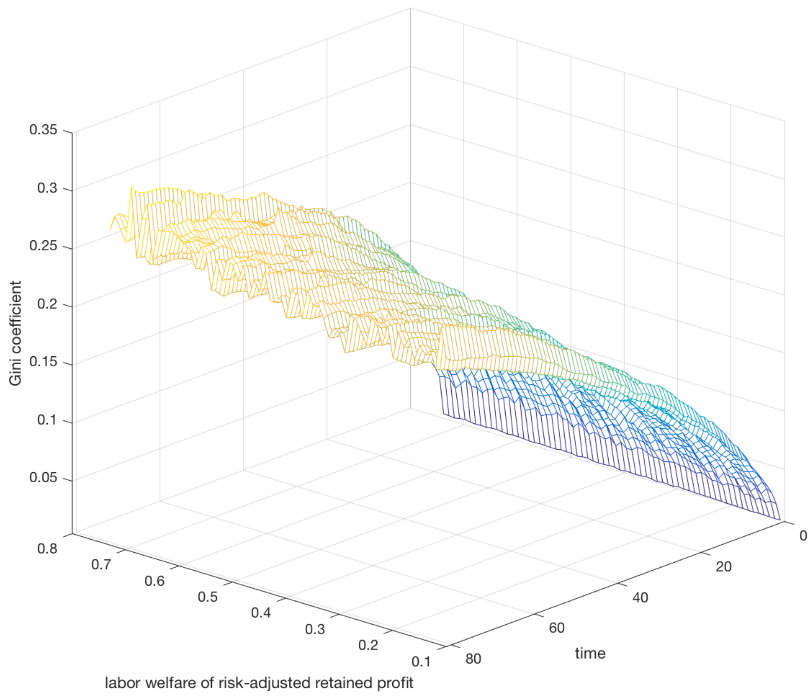


Figure 4.4: labor welfare share vs. Gini coefficient along time

Figure 4.4 shows the labor welfare level in the business sectors could be negative associated with wealth inequality. Intuitively, if the labor welfare level occupied an larger proportion of the risk-adjusted retained profit. If their retained earnings have a great difference among business sectors, then this leads the households who working for the high-profitable sector obtaining comparatively higher welfare than those who work for the low-profitability sector, amplifying their inequality of wealth. It can be suggested that increasing the labor welfare cannot relieve the wealth inequality but widen that. It may be not a reasonable approach to improve wealth inequality by rising labor welfare. Alternatively, improving the secondary distribution through fiscal aid or taxation system perhaps can be considered.

Table 4.6: Potential determinants of wealth inequality

Parameter	Symbol	Value
debt-financing share of output	ω	0.2
equity-financing share of output	d	0.2
change of human capital	δ	[0.05:0.30]
change of sector output	μ	0.02
worker's welfare in residual profit	k	0.2
household's risk aversion	γ	0.25

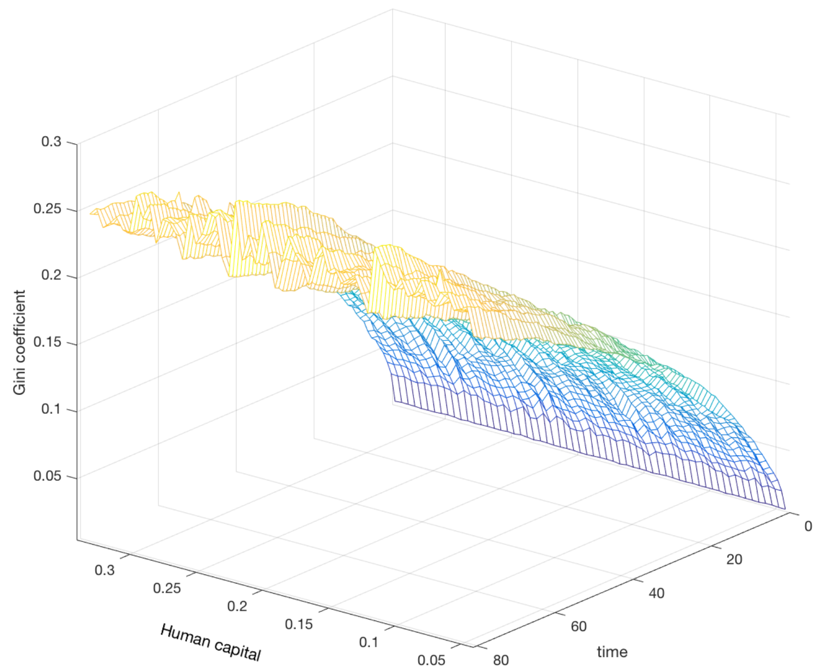


Figure 4.5: human capital vs. Gini coefficient along time

It shows that human capital accumulation insignificantly affects on inequality of wealth.

Table 4.7: Potential determinants of wealth inequality

Parameter	Symbol	Value
debt-financing share of output	ω	0.2
equity-financing share of output	d	0.2
change of human capital	δ	0.1
change of sector output	μ	0.02
worker's welfare in residual profit	k	0.2
household's risk aversion	γ	[0.05:0.35]

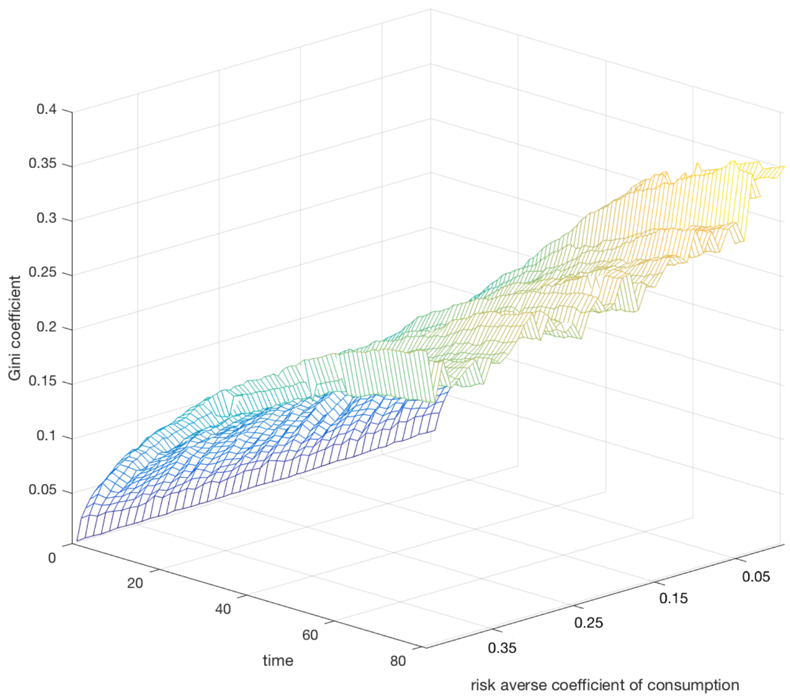


Figure 4.6: risk-averse coefficient of consumption vs. Gini coefficient along time

Figure 4.6 demonstrates that under the optimal consumption level, Eq. (4.57), if the homogenous risk aversion of consumptions is higher (less risk-averse), the wealth inequality in the economy tends to be larger. It may be explained that under less risk aversion, the richer prefers to consume more luxury good, however if they are high-risk averse, then they will spend on luxury good and accumulate wealth faster that other households. This illustration is opposite to the result shown in the previous chapter (Figure 3.7) that risk averse level in household's consumption has no significant impacts on wealth inequality. The intrinsic reason why the simulated results between two chapters are opposite, could caused by the household's income source has been switched from the exogenously given to the endogenous progressing from the technological progress, productive factors and output growth. Intuitively, the higher risk aversion coefficient (γ) in the quadratic form utility function, corresponds to that households obtain higher utility at the same consumptions, corresponding less risk-averse. Assuming all households are centralised to one fixed utility targeting level in consumption, then the less γ benefits richer spend relatively less share of wealth to maintain the averaging utility target, and so have more wealth left for saving. Therefore, it leads to the wealth inequality deteriorating more fiercely when households are less risk averse in consumption.

4.4 Conclusion and future studies

In this chapter, a complex economic system has been built, where satisfies the optimal consumption framework to an economic system with a completed financial market where investor following general investment strategies, among the industrial productions have distances to or on the global technological frontier, with industrial allocation under competitive equilibrium. Household?s revenues is constituted by both labour income and financial capital by allocating their saving to the range of business sectors, adaptively during each development stages. Moreover, based on the neoclassical economic growth developed by Mankiw et al. (1992), both financial capital gains and labor income are endogenously driven by the technological progress (TFP) of each business sectors, the growth rate of output, productive factors and human capital. At each development stage, the attributions of labor income and financial capital gain follows the contingent claim analysis. The simulated analysis in the Chapter 4 illustrates that the evolution of wealth inequality is endogenously driven by the financial structure (equity financing vs. debt financing), labor force's welfare level and household's risk aversion in consumption, conversely the output and human capital accumulation illustrate no significant effects on wealth inequality. Moreover, the whole sets of simulations, indicating wealth in- equality issues should be more and more fierce, accompanying economic development throughout the whole dynasty. If we believe the technological progress promotes the economic growth, then it can be suggested technological progress could be one of intrinsic cause raising inequality.

In respond to the robustness of the results suggested in this chapter, prospectively for each simulation from under Table 4.3 to Table 4.6, beside the incremental interval of varying economic variable, such as the change rate of sector output in Table 4.4, the other economic parameters have also segmentally been adjusted, to see whether the observed result could be sensitive to the freezing parameters or not. Our finding is that the mainstream has not been influenced. However, it should be highlighted, the historical parameters require to be calibrated inside the economic complexity in the future work.

Appendix A

Estimation of decomposed volatility

The observable price of stock $S(t)$, index $IN(t)$ and market $M(t)$ respectively satisfy

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma_s dZ_s(t) \tag{A.1}$$

$$\frac{dIN(t)}{IN(t)} = \mu_{IN} dt + \sigma_{IN} dZ_{IN}(t) \tag{A.2}$$

$$\frac{dM(t)}{M(t)} = \mu_M dt + \sigma_M dZ_3(t) \tag{A.3}$$

where μ, μ_{IN} and μ_M are the drift rate of each process. σ_s, σ_{IN} and σ_M are the volatility factor corresponding to the dynamics of $S(t)$, $IN(t)$ and $M(t)$. $Z_s(t)$, $Z_{IN}(t)$ and $Z_3(t)$ are the Brownian motions.

In section 2.1, the stock retrun volatility is decomposed into three components respectively attributed by the firm, index and market level information. Therefore, the diffusion $\sigma_s dZ_s(t)$ of stock price dynamic can be split to be three independent diffusions as

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma_f dZ_1(t) + \sigma_{in} dZ_2(t) + \sigma_M dZ_3(t) \tag{A.4}$$

where $\sigma_f dZ_1(t)$ is the endogenous diffusion driven by the firm level information, $\sigma_{in} dZ_2(t)$ and $\sigma_M dZ_3(t)$ are the exogenous diffusions driven by the index and market level information. Following the similar procedure, splitting index volatility into two parts driven by index and market level information. Thus the diffusion $\sigma_{IN} dZ_{IN}(t)$

can be split to be two as

$$\frac{dIN(t)}{IN(t)} = \mu_{IN}dt + \sigma_{in}dZ_2(t) + \sigma_M dZ_3(t) \tag{A.5}$$

where $\sigma_{in}dZ_2(t)$ and $\sigma_M dZ_3(t)$ motivated by the endogenous information from index and exogenous information from market level respectively.

If subtract (A.1) from (A.4) and (A.2) from (A.5) and write the result in a discrete form, it gives

$$\sigma_s Z_s(t)\sqrt{\Delta t} = \sigma_f Z_1(t)\sqrt{\Delta t} + \sigma_{in}Z_2(t)\sqrt{\Delta t} + \sigma_M Z_3(t)\sqrt{\Delta t} \tag{A.6}$$

$$\sigma_{IN}Z_{IN}(t)\sqrt{\Delta t} = \sigma_{in}Z_2(t)\sqrt{\Delta t} + \sigma_M Z_3(t)\sqrt{\Delta t} \tag{A.7}$$

Taking variance on (A.6) and (A.7) give

$$\begin{aligned}\mathrm{Var}(\sigma_s Z_s(t)\sqrt{\Delta t}) &= \mathrm{Var}(\sigma_f Z_1(t)\sqrt{\Delta t}) + \mathrm{Var}(\sigma_{in}Z_2(t)\sqrt{\Delta t}) \\ &+\mathrm{Var}(\sigma_M Z_3(t)\sqrt{\Delta t}) + 2\sigma_f\sigma_{in}\mathrm{Cov}(Z_1(t)Z_2(t)\sqrt{\Delta t}) \\ &+2\sigma_{in}\sigma_M\mathrm{Cov}(Z_2(t)Z_3(t)\sqrt{\Delta t}) + 2\sigma_f\sigma_M\mathrm{Cov}(Z_1(t)Z_3(t)\sqrt{\Delta t})\end{aligned} \tag{A.8}$$

$$\begin{aligned}\mathrm{Var}(\sigma_{IN}Z_{IN}(t)\sqrt{\Delta t}) &= \mathrm{Var}(\sigma_{in}Z_2(t)\sqrt{\Delta t}) + \mathrm{Var}(\sigma_M Z_3(t)\sqrt{\Delta t}) \\ &+2\sigma_{in}\sigma_M\mathrm{Cov}(Z_2(t)Z_3(t)\sqrt{\Delta t})\end{aligned} \tag{A.9}$$

The independent Brownian motions contributes their covariances are zero but $\mathrm{Var}(Z_{IN}(t)) = \mathrm{Var}(Z_2(t)) = \mathrm{Var}(Z_3(t))$, then (A.8) and (A.9) can be rewritten to be

$$\sigma_s^2 = \sigma_f^2 + \sigma_{in}^2 + \sigma_M^2 \tag{A.10}$$

$$\sigma_{IN}^2 = \sigma_{in}^2 + \sigma_M^2 \tag{A.11}$$

By taking variance on the discrete form of (A.1), (A.2) and (A.3), after reordering, offers

$$\sigma_s = \sqrt{N \cdot \mathrm{Var}(R_S(t))} \tag{A.12}$$

$$\sigma_{IN} = \sqrt{N \cdot \mathrm{Var}(R_{IN}(t))} \tag{A.13}$$

$$\sigma_M = \sqrt{N \cdot \mathrm{Var}(R_M(t))} \tag{A.14}$$

Subjected to (A.10), (A.11), (A.12) and (A.13), it yields

$$\sigma_f = \sqrt{\mid \sigma_s^2 - \sigma_{IN}^2 \mid} = \sqrt{N \cdot \mid \mathrm{Var}(R_S(t)) - \mathrm{Var}(R_{IN}(t)) \mid} \tag{A.15}$$

Subjected to (A.11), (A.13) and (A.14), it yields

$$\sigma_{in} = \sqrt{\mid \sigma_{IN}^2 - \sigma_M^2 \mid} = \sqrt{N \cdot \mid \mathrm{Var}(R_{IN}(t)) - \mathrm{Var}(R_M(t)) \mid} \tag{A.16}$$

Appendix B

Other historical parameter estimation

$$
\begin{aligned}
\rho_i &= \frac{\mathrm{Cov(\Delta Z_i(t)\Delta W(t))}}{\sqrt{\mathrm{Var}(\Delta Z_i(t))\mathrm{Var}(\Delta W(t))}}, \quad i = 1,2,3 \\
\Delta Z_1(t) &= \frac{R_S(t) - \overline{R}_S - R_{IN}(t) + \overline{R}_{IN}}{\sqrt{N\cdot \mid \mathrm{Var}(R_S(t)) - \mathrm{Var}(R_{IN}(t)) \mid}} \\
\Delta Z_2(t) &= \frac{R_{IN}(t) - \overline{R}_{IN} - R_M(t) + \overline{R}_M}{\sqrt{N\cdot \mid \mathrm{Var}(R_{IN}(t)) - \mathrm{Var}(R_M(t)) \mid}} \\
\Delta Z_3(t) &= \frac{R_M(t) - \overline{R}_M}{\sqrt{N \cdot \mathrm{Var}(R_M(t))}} \\
\Delta W(t) &= \frac{x(t+1) - x(t) - N^{-1}k(\widehat{\theta}(t) - x(t))}{\sqrt{N \cdot \mathrm{Var}(\Delta x(t))}}
\end{aligned} \tag{B.1}
$$

Following maximized log-likelihood estimation, the converging speed k and cointegration equilibrium θ can be estimated by

$$
\begin{aligned}
k &= -T \ln \left\{ \frac{T\sum_{t=0}^{T-1} x(t)x(t+1) - (x(T) - x(0)) \cdot \sum_{t=0}^{T-1} x(t) - \left(\sum_{t=0}^{T-1} x(t)\right)^2}{T\sum_{t=0}^{T-1} x^2(t) - \left(\sum_{t=0}^{T-1} x(t)\right)^2} \right\} \\
\theta &= \frac{n}{1-m}
\end{aligned} \tag{B.2}
$$

$$
\begin{aligned}
m &= \frac{T\sum_{t=0}^{T-1}[x(t)x(t+1)] - [x(T) - x(0)]\sum_{t=0}^{T-1} x(t) - [\sum_{t=0}^{T-1} x(t)]^2}{T\sum_{t=0}^{T-1} x^2(t) - \left[\sum_{t=0}^{T-1} x(t)\right]^2} \\
n &= \frac{x(T) - x(0) + \sum_{t=0}^{T-1} x(t) - m\sum_{t=0}^{N-1} x(t)}{T}
\end{aligned} \tag{B.3}
$$

$$
\begin{aligned}
\eta &= \sqrt{\frac{2k\delta^2}{1-m^2}} \\
\delta &= \frac{1}{T}\left[x^2(T) - x^2(0) + (1+m^2)\sum_{t=0}^{T-1} x^2(t) - 2m\sum_{t=0}^{T-1} x(t)x(t+1) - Tn\right]
\end{aligned}
\tag{B.4}
$$

Appendix C

Trading volume

Table C.1: Trading volume of stocks in Eurozone-US portfolio (In-Sample)

Stock Name	Average daily volume(lead)	Average daily volume(lag)
AEGON	12,220,541	951,200
Alcatel-lucent	31,637,984	19,144,920
Anheuser-Busch InBev SA	2,203,982	1,105,417
ArcelorMittal SA	9,315,133	6,074,824
Eni SpA	17,749,943	691,509
Fresenius Medical Care	789,666	174,543
Orange	9,328,530	1,061,169
RELX NV	4,089,247	213,449
Sanofi	3,202,636	2,571,929
SAP SE	3,869,732	1,664,125
STMicroelectronics NV	6,031,119	1,911,053
Telecom Italia S.p.A	82,464,466	358,539
Tenaris S.A	2,900,515	1,711,129
TOTAL S.A	6,045,863	3,013,808
Unilever	5,303,551	2,334,112

Table C.2: Trading volume of stocks in UK-US portfolio (In-Sample)

Stock Name	Average daily volume(lead)	Average daily volume(lag)
ARM holdings plc	5,323,268	2,524,225
AstraZeneca plc	2,773,519	2,766,091
Barclays plc	50,227,723	3,452,195
BHP Billiton plc	8,589,135	1,321,290
BP plc	32,533,117	9,171,087
British American Tobacco plc	2,843,520	238,162
BT Group plc	18,745,776	265,420
Carnival plc	818,275	184,814
Diageo plc	4,281,529	581,543
GlaxoSmithKline plc	8,247,496	2,674,625
HSBC Holding plc	26,440,983	2,130,255
InterContinental Hotels Group	973,734	239,187
Lloyds Banking Group plc	159,523,022	3,160,685
National Grid plc	7,407,353	485,207
Pearson plc	2,376,736	338,290
Prudential plc	5,648,959	254,555
Randgold Resources Ltd	454,319	852,553
Rio Tinto plc	5,242,438	3,130,434
Royal Bank of Scotland Group	11,716,216	659,719
Royal Dutch Shell plc-A	3,976,278	2,623,036
Royal Dutch Shell plc-B	4,416,276	1,085,237
Shire plc	1,754,137	704,677
Smith&Newphew plc	2,465,972	300,293
Unilever plc	2,598,871	1,269,066
Vodafone Group plc	82,326,525	7,496,278

Table C.3: Trading volume of stocks in China A-H share portfolio

Stock Name	Average daily volume(lead)	Average daily volume(lag)
Air China Ltd	38,352,809	14,600,094
Aluminum of China Co Ltd	64,612,605	25,735,846
Anhui Conch Cement Co Ltd	31,470,627	12,275,738
Bank of China Ltd	214,324,348	328,338,758
Bank of Communication Co Ltd	127,000,452	36,917,281
China CITIC Bank Co Ltd	60,126,805	45,196,555
China Eastern Airlines Co Ltd	46,710,415	13,334,904
China Life Insurance Co Ltd	26,603,850	39,392,931
China Merchants Bank Co Ltd	93,097,864	23,777,840
China Shenhua Energy Co Ltd	28,982,030	17,548,524
China Southern Airlines Co Ltd	66,072,075	18,907,438
Industrial&Commercial Bank of China	154,743,210	282,553,486
Tsingtao Brewery Co Ltd	3,541,230	1,508,603
Zijin Ming Group Co Ltd	151,937,626	40,618,173

Appendix D

Proof of existence of the constraint 1

Defining the individual's accumulative consumptions adjusted by state-price density from 0 to t as

$$J(t) \triangleq \int_0^t H_0(\tau)c_k(\tau)d\tau. \tag{D.1}$$

Also denoting the conditional expectation of the risk adjusted total consumptions adaptive to the information $\mathcal{F}(t)$, as $M(t)$

$$M(t) \triangleq \mathbb{E}\left[\int_0^T H_0(\tau)c_k(\tau)d\tau \mid \mathcal{F}(t)\right], t \in [0, T]. \tag{D.2}$$

According to the martingale representation theorem, $M(t)$ almost surely satisfies

$$M(t) = c + \int_0^t \varphi'(\tau)dW(\tau). \tag{D.3}$$

Under $\mathcal{F}(t)$-adaptive, the individual's current prospect on the future accumulative consumptions adjusted by state-price density $C_f(t)$ satisfies:

$$\begin{aligned} C_f(t) &\triangleq \frac{1}{Z_0(t)}\mathbb{E}_t\left[\int_t^T H_0(\tau)c_k(\tau)d\tau \mid \mathcal{F}(t)\right] \\ &= \frac{1}{Z_0(t)}\left[M(t) - J(t)\right]. \end{aligned} \tag{D.4}$$

Additionally, applying Ito's formula on $Z(t)^{-1}$ subjected to Eq. (3.1), it has

$$dZ(t)^{-1} = \|\theta(t)\|^2 Z(t)^{-1}dt + Z(t)^{-1}\theta(t)'dW(t). \tag{D.5}$$

Based on Eq. (3.1), (D.1), (D.3) and (D.5), applying Ito's formula on C_f (script t is omitted) then yields

$$
\begin{aligned}
dC_f &= Z^{-1}\left[dM - dJ\right] + (M-J)\, d(Z^{-1}) + \left[dM - dJ\right] d(Z^{-1}) \\
&= Z^{-1}\left\{(\varphi' dW - H_0 c_k dt) + (M - J + \varphi' dW - H_0 c_k dt)(\|\theta\|^2 dt + \theta' dW)\right\} \\
&= Z^{-1} H_0 \left\{-c_k dt + H_0\left[\varphi'(\theta dt + dW) + (M-J)\theta'(\theta dt + dW)\right]\right\},
\end{aligned}
\tag{D.6}
$$

with defining

$$
\begin{aligned}
H_0[\varphi'\theta + (M-J)\theta'\theta] &\triangleq \mathcal{E}_k(t,\cdot), \\
H_0[\varphi' + (M-J)\theta'] &\triangleq \sigma(t, X_k, \cdot),
\end{aligned}
\tag{D.7}
$$

then subjected to Eq. (3.2), the (D.6) can be rewritten as

$$
dC_f(t) = -e^{-rt} c_k(t) dt + e^{-rt}\mathcal{E}_k(t,\cdot) dt + e^{-rt}\sigma(t, X_k, \cdot) dW(t). \tag{D.8}
$$

Writing $C_f(t)$ in an integral form with the limit from 0 to T

$$
\begin{aligned}
C_f(T) = C_f(0) - \int_0^T e^{-rt} c_k(t) dt + \int_0^T e^{-rt}\mathcal{E}_k(t,\cdot) dt \\
+ \int_0^T e^{-rt}\sigma(t, X_k, \cdot) dW(t),
\end{aligned}
\tag{D.9}
$$

substituting the $C_f(0)$ subjected to integral form of (D.4) into (D.9). Moreover at the life end, the individual's prospect for her left consumption will be zero, $C_f(T) = 0$, then (D.9) can be arranged as

$$
\begin{aligned}
\mathbb{E}\int_0^T H_0(t) c_k(t) dt = \int_0^T e^{-rt} c_k(t) dt - \int_0^T e^{-rt}\mathcal{E}_k(t,\cdot) dt \\
- \int_0^T e^{-rt}\sigma(t, X_k, \cdot) dW(t).
\end{aligned}
\tag{D.10}
$$

By considering money time value, based on (3.16) the integral form of $X_k(t)$ from 0 to T is

$$
\begin{aligned}
e^{-rT} X_k(T) = X_k(0) - \int_0^T e^{-rt} c_k(t) dt + \int_0^T e^{-rt}\mathcal{E}_k(t,\cdot) dt \\
+ \int_0^T e^{-rt}\sigma(t, X_k, \cdot) dW(t),
\end{aligned}
\tag{D.11}
$$

then (D.10) and (D.11) contribute

$$\mathbb{E}\int_0^T H_0(\tau)c_k(\tau)d\tau = X_k(0) - e^{-rT}X_k^c(T).,\tag{D.12}$$

In addition, subjected to Eq. (3.2) it almost sure that

$$e^{-rT}X_k(T) \geq \mathbb{E}\left[H_0(T)X_k(T)\right],\tag{D.13}$$

lastly, (D.12) and (D.13) yields the constraint 1, Eq. (3.15).

Appendix E

Proof for household's wealth framework

Recalling the risk deflator process, Eq. (3.1):

$$dZ(t) = -Z(t)\theta(t)'dW(t) \tag{E.1}$$

applying Ito's formula on $ln(Z(t))$ yields:

$$Z(t) = e^{\left(-\int_0^t \theta(\tau)'dW(\tau) - \frac{1}{2}\int_0^t \|\theta(\tau)\|^2 d\tau\right)} \tag{E.2}$$

then defining $H_0(t)$ as state-price density, contains both the discounted factor and risk deflator:

$$H_0(t) = e^{-rt}Z(t) = e^{-\left(rt + \int_0^t \theta(\tau)'dW(\tau) + \frac{1}{2}\int_0^t \|\theta(\tau)\|^2 d\tau\right)} \tag{E.3}$$

recalling the agent's wealth process (4.4) with risk-neutral brownian motion W_0 and rearranging it with considering time discount factor to be integral form:

$$e^{-rt}X_k(t) = X_k(0) + \int_{(0,t)}^{t} e^{-r\tau}\left[\epsilon_k(\tau) - c_k(\tau)\right]d\tau + \int_0^t e^{-r\tau}\pi'(\tau)\sigma(\tau)dW_0(\tau) \tag{E.4}$$

applying Ito's formula on $(Z(t)e^{-rt}X_k(t))$ with (E.1) and (E.4):

$$\begin{aligned} d\left(Z(t)e^{-rt}X_k(t)\right) = {} & Z(t)e^{-rt}\left\{\left[\epsilon_k(t) - c_k(t)\right]dt + \pi'(t)\sigma(t)dW_0(t)\right\} \\ & -e^{-rt}X_k(t)Z(t)\theta'(t)dW(t) - Z(t)\theta'(t)dW(t) \\ & \cdot\left\{e^{-rt}\left[\epsilon_k(t) - c_k(t)\right]dt + e^{-rt}\pi'(t)\sigma(t)dW_0(t)\right\} \end{aligned} \tag{E.5}$$

assuming $W(t)$ and $W_0(t)$ are independent and based on (E.3), rewriting (E.5) to be:

$$\begin{aligned} dH_0(t)X_k(t) = {} & H_0(t)\left[\epsilon_k(t) - c_k(t)\right]dt + H_0(t)\pi'_k\sigma(t)dW_0(t) \\ & -H_0(t)X_k(t)\theta'(t)dW(t) \end{aligned} \tag{E.6}$$

In addition, recalling the market risk vector, Eq. (4.5) that

$$\theta(t) = \sigma(t)'(\sigma(t)\sigma(t)')^{-1}(b(t) - r\mathbf{1}_n). \tag{E.7}$$

where $Z(t)$ is to reconcile the price to be risk-neutral by releasing the market risk premium. Additionally, the particular driving force of risk sources of each security $W(t)$ also can be reconciled to be risk-neutral $W_0(t)$ by the Girsanov's theorem:

$$W_0(t) = W(t) + \int_0^t \theta(\tau)d\tau \tag{E.8}$$

subject to the equation (E.7) and (E.8), reconverting the risk-neutral $W_0(t)$ in the (E.6) back to $W(t)$:

$$\begin{aligned} dH_0(t)X_k(t) = H_0(t)\left[\epsilon_k(t) - c_k(t) + \pi'(t)(b(t) - r\mathbf{1}_n)\right]dt \\ + H_0(t)[\sigma'(t)\pi(t) - X_k(t)\theta(t)]'dW(t) \end{aligned} \tag{E.9}$$

Finally we obtain the present value of risk-neutral wealth of every household's in an integral form as:

$$\begin{aligned} H_0(t)X_k(t) + \int_0^t H_0(\tau)\left[c_k(\tau) - \epsilon_k(\tau) - \pi'(t)(b(\tau) - r\mathbf{1}_n)\right]d\tau \\ = \int_0^t H_0(\tau)[\sigma'(\tau)\pi(\tau) - X_k(\tau)\theta(\tau)]'dW(\tau) \end{aligned} \tag{E.10}$$

and

$$\begin{aligned} H_0(T)X_k(T) + \int_0^T H_0(\tau)\left[c_k(\tau) - \epsilon_k(\tau) - \pi'(t)(b(\tau) - r\mathbf{1}_n)\right]d\tau \\ = \int_0^T H_0(\tau)[\sigma'(\tau)\pi(\tau) - X_k(\tau)\theta(\tau)]'dW(\tau) \end{aligned} \tag{E.11}$$

In addition, the aggregated wealth accumulated through the dynasty is

$$\mathcal{G}_k = \mathbb{E}(H_0(T)X_k(T)) - X_k(0)^+ \tag{E.12}$$

(E.11) and (E.12) together give

$$\begin{aligned} \mathcal{G}_k + X_k(0)^+ + \mathbb{E}_t\left[\int_t^T H_0(s)\left[c_k(s) - \epsilon_k(s) - \pi'(t)(b(s) - r\mathbf{1}_n)\right]ds\right] \\ = -\int_0^t H_0(\tau)\left[c_k(\tau) - \epsilon_k(\tau) - \pi'(t)(b(\tau) - r\mathbf{1}_n)\right]d\tau \end{aligned} \tag{E.13}$$

In the end, by taking conditional expectation and denoting the generalized income $\mathcal{E}_k$ contains both labor income ϵ_k and expected the capital gain vector $\pi'(t)(b(\tau) - r\mathbf{1}_n)$ (zero in the no financial investment scenario), then subject to (E.10) and (E.13), the generalized expression of each household's wealth is

$$X_k(t) = \frac{1}{H_0(t)} \left\{ \mathbb{E}_t \left[\int_t^T H_0(s) \left[c_k(s) - \mathcal{E}_k(s) \right) \right] ds \right] + X_k(0)^+ + \mathcal{G}_k \right\} \tag{E.14}$$

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