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# A BAYESIAN INFERENCE OF MULTIPLE STRUCTURAL BREAKS IN MEAN AND ERROR VARIANCE IN PANEL AR (1) MODEL

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## ABSTRACT

This paper explores the effect of multiple structural breaks to estimate the parameters and test the unit root hypothesis in panel data time series model under Bayesian perspective. These breaks are present in both mean and error variance at the same time point. We obtain Bayes estimates for different loss function using conditional posterior distribution, which is not coming in a closed form, and this is approximately explained by Gibbs sampling. For hypothesis testing, posterior odds ratio is calculated and solved via Monte Carlo Integration. The proposed methodology is illustrated with numerical examples.

**Key words**: panel data model, autoregressive model, structural break, MCMC, posterior odds ratio.

## 1. Introduction

Statistical inference of panel data time series model received great attention in the last several decades in both econometrics and statistics literature. The main idea behind the use of panel data time series model is to overcome the difficulty of unobserved variation in cross-section data sets over individual units as well as variation, which may change the structure also. It was assumed that this change was taken by some observations at a fixed and common time point in each series referred to as break point. Thus, structural break concept in panel data set-up is important to handle the permanent effects in the series and impacts other simultaneous variables. For this, an extensive literature concentrates on testing, estimation and detection of the existence of single or multiple structural breaks from univariate to multivariate time series. Bai and Perron (1998, 2003) considered the problem of estimation and testing for break point in linear model and determine the number of breaks using double maximum tests. They have also further addressed various issues such as estimation and testing number of

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breaks, forming confidence interval related to multiple linear regression with multiple structural break. Altissimo and Corradi (2003) suggested an approach for detecting and estimation of the number of shifts in mean of a nonlinear process, which is having dependent and heterogeneous observations. They proposed a new estimator for long run variance, which was consistent in the presence of breaks and verified via a simulation exercise. Li (2004) applied quasi-Bayesian approach to detect the number and position of structural breaks in China's GDP and labour productivity data using predictive likelihood information criterion.

Apart from the above literature, which mainly dealt with the classical approach, a generalized form of estimation and testing the structural break by using Bayesian inference is less explored. Geweke and Jiang (2011) developed Bayesian approach to modelling in-sample structural breaks and forecasting out-of-sample breaks. Eo (2012) used Bayesian approach to estimate the number of breaks in autoregressive regressions with structural breaks in intercept, persistence, and residual variance. A model selection criterion was also considered to select the best model from U.S. GDP deflator data. Aue and Horvath (2013) discussed several approaches for estimating the parameter and locating multiple break points. They considered CUSUM procedure as well as likelihood statistic to adjust the serial dependence in presence of structural break. Recently, Melighotsiduo *et al.* (2017) suggested a Bayesian approach for autoregressive model allowing multiple structural changes in both mean and error variance of economic series occurring at unknown times, and Bayesian unit root testing is also proposed.

In current scenario, a growing literature on estimation and testing of multiple structural breaks in generalized univariate model such as panel data as well as multivariate time series model. A partial list of contributions in multiple structural breaks include Sugita (2006), Liu *et al.* (2011), Jin *et al.* (2013), Preuss *et al.* (2015) and Eo and Morley (2015) to analysis the procedure for detection and estimation of change point in vector error correction model, panel data model. In recent time, detection and estimation of multiple change points in panel data with interactive fixed effect and dynamic structure is introduced. Li *et al.* (2016) through penalized principal component (PPC) estimation procedure with an adaptive group fused LASSO.

An overview of the above description, this paper provides a general methodology to estimate and inference for panel data model under the presence of multiple change points in mean and error variance parameters. Our approach provides a flexible way to the interpretation of the result in real situation because in most economic and time series data are varying by trend and variance component. If one considers a break in mean also, then the impact of the series changes due to both type of break versus no break point. Thus, a Bayesian approach is introduced to capture the impact of break points in the panel data model. For Bayes estimation, we apply both symmetric and asymmetric loss function to posterior density in order to get better estimators and compare them with ordinary least square estimator. In addition, we also examine the model selection criterion to find the appropriate model, which may or may not contain multiple break points in a real data set.

## 2. Model Specification

Let {$y_{it}$ , t=1,2,…,T; i=1,2…,n} be a panel data time series model having *B* multiple break points in mean and error variance where breaks occur in both parameters in the same location. In that case our panel data model can be expressed as

$$y_{it} = \begin{cases} \rho y_{i,t-1} + (1-\rho)\mu_{i1} + \sigma_1 \varepsilon_{it}; & T_0 < t \le T_1 \\ \vdots & \vdots \\ \rho y_{i,t-1} + (1-\rho)\mu_{ij} + \sigma_j \varepsilon_{it}; & T_{j-1} < t \le T_j \\ \vdots & \vdots \\ \rho y_{i,t-1} + (1-\rho)\mu_{i,B+1} + \sigma_{B+1} \varepsilon_{it}; & T_B < t \le T_{B+1} = T \end{cases} \quad (1)$$

for j = 1,2,…, B and where n denotes number of cross sectional units, $\rho$ is the autoregressive coefficient, $\mu_j$ is a ( n × 1 ) vector of mean coefficients at j[th] division and $\varepsilon_{it}$ are assumed to be independent and normally distributed with zero mean and division specific variance $\sigma_j^2$. This is a partial structural change model since the parameter ρ is not subject to shifts and is estimated using the entire sample space. The model in (1) can also be casted in the form of matrix notation with **.*** Kronecker delta product indicating element by element array multiplication, Z as the nT × (B+1) matrix whose j[th] column is equal to one if $T_{j-1} < t \le T_j$ and zero otherwise, and consider mean and residual variance parameters as a vector form.

$$y = \rho y_{-1} + (1-\rho)L\mu + \varepsilon \;\; ; \quad \varepsilon \sim N(0, I_{nT} \cdot{*}S)$$

where

$$\underset{nT\times 1}{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}; \; \underset{T\times 1}{y_i} = \begin{pmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{iT} \end{pmatrix}; \; \underset{n(B+1)\times 1}{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{B+1} \end{pmatrix}; \; \underset{n\times 1}{\mu_j} = \begin{pmatrix} \mu_{1j} \\ \mu_{2j} \\ \vdots \\ \mu_{nj} \end{pmatrix}; \; \underset{(B+1)\times 1}{\sigma} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_{B+1} \end{pmatrix}$$

$$\underset{nT\times n(B+1)}{L} = \begin{pmatrix} I_n \otimes l_{T_1} & 0 & \cdots & 0 \\ 0 & I_n \otimes l_{T_2-T_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_n \otimes l_{T-T_B} \end{pmatrix}; \; Z = \begin{pmatrix} l_{T_1} & 0 & \cdots & 0 \\ 0 & l_{T_2-T_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & l_{T-T_B} \end{pmatrix}; \; S = Z\sigma$$

Our study attempts to estimate the parameters in structural break model under Bayesian framework and test the unit root hypothesis by using posterior odds ratio. Under unit root case, model (1) reduces to a pure structural change model where all the model's coefficients are subject to change

$$\Delta y_{it} = \begin{cases} \sigma_1 \varepsilon_{it}; & T_0 < t \le T_1 \\ \vdots & \vdots \\ \sigma_j \varepsilon_{it}; & T_{j-1} < t \le T_j \\ \vdots & \vdots \\ \sigma_{B+1} \varepsilon_{it}; & T_B < t \le T_{B+1} \end{cases} \tag{2}$$

As mentioned, if we follow the usual approach defined in the literature to test for stationarity model reduces by (2) under the null hypothesis $H_0$: ρ = 1 is difference stationary with multiple breaks in error variance against the alternative hypothesis $H_1$: ρ ϵ S, series is stationary with multiple breaks in mean as well as error variance.

## 3. Bayesian Inference

In this section, we discuss issues related to the estimation and inference about the parameters and testing of unit root hypothesis. In order to perform Bayesian inference we need the likelihood function and specify prior distribution for the model parameters. Posterior probability is obtained by using sample information contained in the likelihood function combined with the joint prior distribution. The likelihood function for this model is

$$L(\rho,\mu,\sigma \mid y) = (2\pi)^{-\frac{nT}{2}} \prod_{j=1}^{B+1}\left(\sigma_j^{-n(T_j - T_{j-1})}\right) \exp\left[-\frac{1}{2}\sum_{j=1}^{B+1}\left\{\frac{1}{\sigma_j^2}\sum_{i=1}^{n}\sum_{t=T_{j-1}}^{T_j}\left(y_{it} - \rho y_{i,t-1} - (1-\rho)\mu_{ij}\right)^2\right\}\right] \tag{3}$$

For panel data model generally normal prior distribution is considered for $\mu_{ij} \sim N\left(\gamma_{ij}, \sigma_j^2\right)$, for error variance ($\sigma_j^2$) assume conjugate inverted gamma prior *IG($c_j$ , $d_j$)* and uniform prior is taken for autoregressive coefficient (ρ), see [Schotman and Van Dijk (1991) and Phillips (1991)]. The joint prior distribution is given as

$$\pi\left(\rho.\mu,\sigma^2\right) = \frac{(2\pi)^{-\frac{n(B+1)}{2}}}{1-l} \prod_{j=1}^{B+1}\left(\frac{d_j^{c_j}}{\Gamma c_j}\left(\sigma_j^2\right)^{-c_j-1-\frac{n}{2}}\right) \exp\left[-\sum_{j=1}^{B+1}\frac{1}{\sigma_j^2}\left\{d_j + \frac{1}{2}\sum_{i=1}^{n}\left(\mu_{ij} - \gamma_{ij}\right)^2\right\}\right] \tag{4}$$

### 3.1. Bayesian Estimation via Gibbs Sampling

Given the likelihood function and prior density defined by $eq^n$ (3) and $eq^n$ (4), the posterior distribution is given by

$$\pi(\rho,\mu,\sigma\mid y)\propto\prod_{j=1}^{B+1}\left(\frac{d_j^{c_j}}{\Gamma c_j}\left(\sigma_j^2\right)^{-\left[\frac{n\left(T_j-T_{j-1}+1\right)}{2}+c_j+1\right]}\right)\exp\left[-\frac{1}{2}\sum_{j=1}^{B+1}\left\{\frac{1}{\sigma_j^2}\left(\sum_{i=1}^{n}\sum_{t=T_{j-1}}^{T_j}\left(y_{it}-\rho y_{i,t-1}-(1-\rho)\mu_{ij}\right)^2+\sum_{i=1}^{n}\left(\mu_{ij}-\gamma_{ij}\right)^2+2d_j\right)\right\}\right] \quad 5)$$

The posterior distribution in (5) is very complicated and hence no closed form inference appears to be possible. For Bayesian estimation, we proceed via Gibbs sampler, a MCMC method, proposed by Geman and Geman (1984). The Gibbs sampler procedure, which we used, is described by Wang and Zivot (2000) in a time series regression model with multiple structural breaks. By means of this procedure, it gives a chain of estimated parameters values, which is frequently obtained by conditional probability distribution. Here, our aim is to generate a sequence of random variables from the conditional probability distribution using the current value of the parameters. For this we have derived the form of conditional posterior distributions given below:

$$\pi\left(\mu_{ij}\mid\rho,\sigma_j^2,\underline{y}\right)\sim N\left(\frac{(1-\rho)\sum_{t=T_{j-1}}^{T_j}\left(y_{it}-\rho y_{i,t-1}\right)+\gamma_{ij}}{(1-\rho)^2\left(T_j-T_{j-1}\right)+1},\frac{\sigma_j^2}{(1-\rho)^2\left(T_j-T_{j-1}\right)+1}\right) \quad (6)$$

$$\pi\left(\sigma_j^2\mid\rho,\mu_{ij},\underline{y}\right)\sim IG\left(\frac{n\left(T_j-T_{j-1}\right)+n}{2}+c_j,d_j+\frac{1}{2}\sum_{i=1}^{n}\left(\sum_{t=T_{j-1}}^{T_j}\left(y_{it}-\rho y_{i,t-1}-(1-\rho)\mu_{ij}\right)^2+\left(\mu_{ij}-\gamma_{ij}\right)^2\right)\right) \quad (7)$$

$$\pi\left(\rho\mid\mu_{ij},\sigma_j^2,\underline{y}\right)\sim TN\left(\sum_{j=1}^{B+1}\frac{\sum_{i=1}^{n}\sum_{t=T_{j-1}}^{T_j}\left(y_{it}-\mu_{ij}\right)\left(y_{i,t-1}-\mu_{ij}\right)}{\sum_{i=1}^{n}\sum_{t=T_{j-1}}^{T_j}\left(y_{it-1}-\mu_{ij}\right)^2},\sum_{j=1}^{B+1}\frac{\sigma_j^2}{\sum_{i=1}^{n}\sum_{t=T_{j-1}}^{T_j}\left(y_{it-1}-\mu_{ij}\right)^2},l,1\right) \quad (8)$$

Using the generated samples from the above posteriors, Bayes estimates of the parameter are evaluated by different loss functions under Gibbs sampling algorithm. A loss function is a decision rule to select the best estimator and represent each of the possible estimates. Here, we consider squared error (symmetric) loss function as well as entropy (asymmetric) loss function for getting better understanding of the Bayesian estimation. Under squared error and entropy loss function, Bayes estimator are $E(\theta\mid\underline{x})$ and $\left[E\left(\theta^{-1}\mid\underline{x}\right)\right]^{-1}$.

### 3.2. Testing Unit Root Hypothesis via Posterior Odds Ratio

In a hypothesis testing problem, one is generally interested in testing the stationary condition of a model. Here, null hypothesis is used as a unit root hypothesis against the alternative of a stationary model. In Bayesian framework, testing is often convenient to summarize the information in terms of posterior odds ratio. The posterior odds ratio is the ratio of posterior probability under null versus alternative hypothesis with the product of prior odds, notation given as:

$$\beta_{01} = \frac{p_0}{1-p_0}\frac{P(y\,|\,H_0)}{P(y\,|\,H_1)} \tag{9}$$

**Theorem:** To test the null hypothesis that $y_{it}$ is a non-stationary I(1) process, i.e. $\rho$=1 in equation (2), against the alternative hypothesis that $y_{it}$ is a stationary I(0) process, i.e. $\rho \in S$ in equation (1). The posterior odds ratio can be constructed according to equation (9)

$$\beta_{01} = \frac{p_0}{1-p_0}\frac{\prod_{j=1}^{B+1}\left[d_j+\frac{1}{2}\sum_{i=1}^{n}\sum_{t=T_{j-1}}^{T_j}\left(y_{it}-y_{i,t-1}\right)^2\right]^{-\frac{n\left(T_j-T_{j-1}\right)+2c_j}{2}}}{\frac{1}{1-l}\int_l^1\prod_{j=1}^{B+1}\left[A_j(\rho)\right]^{-\frac{n}{2}}\left[C_j(\rho)\right]^{-\frac{n\left(T_j-T_{j-1}\right)+2c_j}{2}}d\rho} \tag{10}$$

where

$$A_j(\rho) = (1-\rho)^2\left(T_j-T_{j-1}\right)+1$$

$$B_{ij}(\rho) = (1-\rho)\sum_{t=T_{j-1}}^{T_j}\left(y_{it}-y_{i,t-1}\right)+\gamma_{ij}$$

$$C_j(\rho) = d_j+\frac{1}{2}\sum_{i=1}^{n}\left(\sum_{t=T_{j-1}}^{T_j}\left(y_{it}-y_{i,t-1}\right)^2+\gamma_{ij}^2-\frac{\left[B_{ij}(\rho)\right]^2}{A_j(\rho)}\right)$$

**Proof:** The proof of the theorem is given in the appendix.

In the equation (10), closed form expression of posterior odds ratio is not obtained. Therefore, we use an alternative technique as Monte Carlo integration for approximately solving the integrals and get the value of posterior odds ratio.

## 4. Simulation Study

In this section, we conduct a set of simulated experiments to evaluate the performance of our model and compare different estimators based on Monte Carlo simulation. To estimate the model parameters, assume that the number of breaks and the location of break points are known so that the remaining objectives in equation (1) are estimated via an iterative procedure. In simulation experiment we have generated artificial time series from our model with varying numbers of structural breaks at the same time points in mean and error variance parameters. We are starting with the initial observation $y_{0i}$ = (10, 15, 20) to generate panel data time series from the suggested model having three panel (n=3) and each panel contains T observations. For better interpretation, we took different size of time series T = (50, 75, 100) and also varying autoregressive coefficients $\rho$ = (0.9, 0.95, 0.99). The number of possible structural break (B) has been 3. Thus, the disturbances $\varepsilon_{it}$ are generated as i.i.d. for all *i* and *j* with four different variance, namely $\left(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2\right) = (0.1, 0.2, 0.3, 0.4)$. For inverse gamma prior distribution with hyper parameters is to known. For numerical purpose we have taken as $c_j = d_j = 0.01$ for all break points. In the case of normal prior, hyper prior mean is equal to mean of the generated series at every break point interval ($T_{j-1}$, $T_j$) with parallel variance given in disturbances term. The true value of mean term for each panel having four partitions is written as ($\mu_{11}$, $\mu_{12}$, $\mu_{13}$) = (14, 16, 18); ($\mu_{21}$, $\mu_{22}$, $\mu_{23}$) = (20, 22, 24); ($\mu_{31}$, $\mu_{32}$, $\mu_{33}$) = (26, 28, 30); ($\mu_{41}$, $\mu_{42}$, $\mu_{43}$) = (32, 34, 36). All results are based on 5000 replications. From the generated sample, we obtained Bayes estimate of parameters and compared the performance with ordinary least square (OLS) estimate. We report the estimated value and its mean square error in Table-4.1 to 4.3.

**Table-4.1**. Estimators with varying time series at $\rho$ = 0.9

| | T=50 | | | T=75 | | | T=100 | | |
|---|---|---|---|---|---|---|---|---|---|
| | OLS | SELF | ELF | OLS | SELF | ELF | OLS | SELF | ELF |
| $\rho$ | 0.8607 | 0.9037 | 0.9036 | 0.8724 | 0.9049 | 0.9048 | 0.8763 | 0.9043 | 0.9043 |
| | 0.0025 | 0.0000 | 0.0000 | 0.0012 | 0.0001 | 0.0001 | 0.0009 | 0.0001 | 0.0001 |
| $\mu_{11}$ | 13.7381 | 13.9853 | 13.9785 | 13.8519 | 13.9959 | 13.9893 | 13.7567 | 13.9815 | 13.9749 |
| | 0.5840 | 0.0104 | 0.0107 | 0.5529 | 0.0098 | 0.0100 | 0.5706 | 0.0091 | 0.0094 |
| $\mu_{12}$ | 16.1184 | 15.9989 | 15.9929 | 16.0870 | 15.9938 | 15.9880 | 16.1338 | 16.0026 | 15.9969 |
| | 0.7121 | 0.0124 | 0.0125 | 0.5515 | 0.0106 | 0.0107 | 0.6236 | 0.0110 | 0.0111 |
| $\mu_{13}$ | 18.2558 | 17.9770 | 17.9717 | 18.2331 | 17.9833 | 17.9782 | 18.2917 | 17.9964 | 17.9912 |
| | 0.7394 | 0.0104 | 0.0106 | 0.7099 | 0.0108 | 0.0110 | 0.7042 | 0.0100 | 0.0101 |
| $\mu_{21}$ | 18.9938 | 20.0110 | 20.0014 | 19.4758 | 20.0186 | 20.0098 | 19.6677 | 20.0093 | 20.0015 |
| | 2.6156 | 0.0100 | 0.0099 | 1.2051 | 0.0235 | 0.0234 | 0.6942 | 0.0386 | 0.0385 |
| $\mu_{22}$ | 20.8791 | 21.9925 | 21.9838 | 21.4667 | 22.0115 | 22.0035 | 21.7912 | 22.0295 | 22.0224 |
| | 2.6572 | 0.0151 | 0.0154 | 1.5227 | 0.0293 | 0.0292 | 0.6849 | 0.0359 | 0.0356 |
| $\mu_{23}$ | 23.1734 | 24.0115 | 24.0035 | 23.5620 | 24.0168 | 24.0094 | 23.7996 | 24.0274 | 24.0209 |
| | 2.2536 | 0.0116 | 0.0114 | 1.2840 | 0.0278 | 0.0275 | 0.5568 | 0.0325 | 0.0322 |
| $\mu_{31}$ | 24.9505 | 26.0408 | 26.0295 | 25.5998 | 26.0509 | 26.0410 | 25.7780 | 26.0333 | 26.0244 |
| | 4.4370 | 0.0257 | 0.0248 | 1.7431 | 0.0381 | 0.0373 | 0.7004 | 0.0381 | 0.0376 |
| $\mu_{32}$ | 26.7468 | 28.0131 | 28.0026 | 27.6054 | 28.0537 | 28.0445 | 27.7914 | 28.0448 | 28.0364 |
| | 4.8589 | 0.0230 | 0.0228 | 1.7379 | 0.0385 | 0.0377 | 0.9369 | 0.0557 | 0.0550 |
| $\mu_{33}$ | 28.6660 | 30.0059 | 29.9961 | 29.5123 | 30.0316 | 30.0230 | 29.6804 | 30.0104 | 30.0027 |
| | 5.1925 | 0.0218 | 0.0218 | 1.7849 | 0.0359 | 0.0355 | 0.8159 | 0.0444 | 0.0443 |
| $\mu_{41}$ | 30.9830 | 32.0348 | 32.0234 | 31.1368 | 31.9948 | 31.9832 | 31.4541 | 32.0210 | 32.0093 |
| | 3.6570 | 0.0363 | 0.0358 | 3.9274 | 0.0448 | 0.0450 | 2.8567 | 0.0384 | 0.0382 |
| $\mu_{42}$ | 32.9950 | 34.0339 | 34.0232 | 33.5021 | 34.0500 | 34.0391 | 33.6176 | 34.0352 | 34.0243 |
| | 4.4146 | 0.0376 | 0.0369 | 2.5846 | 0.0370 | 0.0361 | 2.8885 | 0.0380 | 0.0374 |
| $\mu_{43}$ | 34.8637 | 36.0233 | 36.0132 | 35.1567 | 36.0032 | 35.9929 | 35.4250 | 36.0242 | 36.0139 |
| | 3.7348 | 0.0342 | 0.0340 | 3.4651 | 0.0386 | 0.0388 | 2.5830 | 0.0353 | 0.0349 |
| $\sigma_1^2$ | 0.1071 | 0.1085 | 0.0988 | 0.1042 | 0.1053 | 0.1014 | 0.1038 | 0.1046 | 0.1005 |
| | 0.0007 | 0.0007 | 0.0000 | 0.0004 | 0.0005 | 0.0000 | 0.0005 | 0.0005 | 0.0000 |
| $\sigma_2^2$ | 0.2104 | 0.2094 | 0.1961 | 0.2065 | 0.2069 | 0.2052 | 0.2031 | 0.2037 | 0.2030 |
| | 0.0040 | 0.0034 | 0.0000 | 0.0017 | 0.0015 | 0.0001 | 0.0006 | 0.0006 | 0.0000 |
| $\sigma_3^2$ | 0.3121 | 0.3189 | 0.3028 | 0.3002 | 0.3031 | 0.2920 | 0.3054 | 0.3060 | 0.2984 |
| | 0.0075 | 0.0071 | 0.0003 | 0.0034 | 0.0034 | 0.0001 | 0.0018 | 0.0018 | 0.0000 |
| $\sigma_4^2$ | 0.4058 | 0.4116 | 0.4027 | 0.4140 | 0.4193 | 0.3987 | 0.4252 | 0.4241 | 0.4221 |
| | 0.0055 | 0.0055 | 0.0004 | 0.0078 | 0.0078 | 0.0002 | 0.0100 | 0.0094 | 0.0011 |

**Table-4.2.** Estimators with varying time series at $\rho$ = 0.95

| | T=50 | | | T=75 | | | T=100 | | |
|---|---|---|---|---|---|---|---|---|---|
| | OLS | SELF | ELF | OLS | SELF | ELF | OLS | SELF | ELF |
| $\rho$ | 0.9277 | 0.9511 | 0.9511 | 0.9239 | 0.9511 | 0.9510 | 0.9231 | 0.9515 | 0.9515 |
| | 0.0010 | 0.0000 | 0.0000 | 0.0012 | 0.0000 | 0.0000 | 0.0011 | 0.0000 | 0.0000 |
| $\mu_{11}$ | 14.2813 | 14.0055 | 13.9979 | 14.3639 | 14.0031 | 13.9955 | 14.2784 | 14.0008 | 13.9934 |
| | 2.8644 | 0.0026 | 0.0026 | 2.8140 | 0.0026 | 0.0027 | 2.4136 | 0.0021 | 0.0021 |
| $\mu_{12}$ | 17.0431 | 16.0055 | 15.9989 | 16.8423 | 15.9981 | 15.9916 | 17.0321 | 16.0007 | 15.9942 |
| | 4.9016 | 0.0027 | 0.0027 | 4.3718 | 0.0029 | 0.0029 | 3.7536 | 0.0022 | 0.0022 |
| $\mu_{13}$ | 19.2725 | 17.9970 | 17.9910 | 19.2403 | 17.9869 | 17.9811 | 19.5766 | 17.9947 | 17.9889 |
| | 5.8300 | 0.0026 | 0.0027 | 5.7341 | 0.0021 | 0.0022 | 6.0293 | 0.0021 | 0.0022 |
| $\mu_{21}$ | 19.1892 | 20.0027 | 19.9902 | 19.1421 | 20.0079 | 19.9981 | 19.0485 | 20.0086 | 19.9988 |
| | 18.5998 | 0.0029 | 0.0030 | 6.4266 | 0.0088 | 0.0087 | 3.9955 | 0.0082 | 0.0082 |
| $\mu_{22}$ | 21.1867 | 21.9990 | 21.9876 | 21.3103 | 22.0017 | 21.9928 | 21.2755 | 21.9997 | 21.9907 |
| | 10.5883 | 0.0025 | 0.0027 | 4.6515 | 0.0079 | 0.0080 | 3.2040 | 0.0079 | 0.0080 |
| $\mu_{23}$ | 23.8031 | 24.0070 | 23.9966 | 23.6473 | 24.0044 | 23.9962 | 23.7322 | 24.0062 | 23.9980 |
| | 11.7370 | 0.0029 | 0.0028 | 3.8351 | 0.0074 | 0.0074 | 3.0148 | 0.0089 | 0.0088 |
| $\mu_{31}$ | 23.5217 | 25.9969 | 25.9830 | 23.9895 | 25.9922 | 25.9803 | 24.5652 | 26.0025 | 25.9912 |
| | 30.2210 | 0.0041 | 0.0043 | 9.4940 | 0.0073 | 0.0076 | 6.4169 | 0.0145 | 0.0146 |
| $\mu_{32}$ | 26.4880 | 28.0077 | 27.9947 | 26.7673 | 28.0165 | 28.0056 | 26.7426 | 28.0081 | 27.9977 |
| | 22.0991 | 0.0037 | 0.0037 | 9.3507 | 0.0119 | 0.0117 | 5.5741 | 0.0151 | 0.0151 |
| $\mu_{33}$ | 28.4393 | 29.9977 | 29.9856 | 28.7290 | 30.0130 | 30.0027 | 28.7848 | 30.0002 | 29.9904 |
| | 25.9163 | 0.0038 | 0.0039 | 7.4535 | 0.0120 | 0.0119 | 5.2416 | 0.0146 | 0.0148 |
| $\mu_{41}$ | 29.9873 | 31.9863 | 31.9745 | 30.8228 | 32.0145 | 32.0025 | 31.0121 | 32.0101 | 31.9987 |
| | 10.5213 | 0.0195 | 0.0201 | 8.4309 | 0.0336 | 0.0335 | 4.5547 | 0.0305 | 0.0305 |
| $\mu_{42}$ | 32.9922 | 34.0512 | 34.0402 | 32.9787 | 34.0263 | 34.0151 | 33.0487 | 34.0116 | 34.0008 |
| | 8.6453 | 0.0235 | 0.0224 | 5.3532 | 0.0246 | 0.0242 | 4.1663 | 0.0297 | 0.0296 |
| $\mu_{43}$ | 34.7339 | 36.0139 | 36.0035 | 35.2679 | 36.0439 | 36.0333 | 35.0919 | 36.0133 | 36.0032 |
| | 8.8365 | 0.0193 | 0.0192 | 5.1718 | 0.0286 | 0.0278 | 4.1124 | 0.0292 | 0.0291 |
| $\sigma_1^2$ | 0.1087 | 0.1058 | 0.1019 | 0.1077 | 0.1048 | 0.1017 | 0.1065 | 0.1047 | 0.0999 |
| | 0.0011 | 0.0010 | 0.0000 | 0.0007 | 0.0007 | 0.0000 | 0.0008 | 0.0008 | 0.0000 |
| $\sigma_2^2$ | 0.2566 | 0.2515 | 0.2244 | 0.1889 | 0.2040 | 0.1978 | 0.2057 | 0.2030 | 0.2000 |
| | 0.0126 | 0.0100 | 0.0019 | 0.0016 | 0.0017 | 0.0002 | 0.0015 | 0.0014 | 0.0000 |
| $\sigma_3^2$ | 0.3739 | 0.3679 | 0.3372 | 0.3182 | 0.3137 | 0.2980 | 0.3094 | 0.3062 | 0.3036 |
| | 0.0265 | 0.0211 | 0.0016 | 0.0049 | 0.0048 | 0.0003 | 0.0026 | 0.0026 | 0.0000 |
| $\sigma_4^2$ | 0.3961 | 0.4001 | 0.3946 | 0.4132 | 0.4091 | 0.4032 | 0.3940 | 0.4016 | 0.3984 |
| | 0.0043 | 0.0043 | 0.0002 | 0.0035 | 0.0034 | 0.0001 | 0.0024 | 0.0024 | 0.0000 |

**Table-4.3.** Estimators with varying time series at $\rho$ = 0.99

| | T=50 | | | T=75 | | | T=100 | | |
|---|---|---|---|---|---|---|---|---|---|
| | OLS | SELF | ELF | OLS | SELF | ELF | OLS | SELF | ELF |
| $\rho$ | 0.9887 | 0.9900 | 0.9900 | 0.9896 | 0.9900 | 0.9900 | 0.9896 | 0.9900 | 0.9900 |
| | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $\mu_{11}$ | 14.0541 | 13.9998 | 13.9995 | 13.8541 | 13.9996 | 13.9993 | 13.9496 | 13.9991 | 13.9988 |
| | 2.5280 | 0.0000 | 0.0000 | 2.2032 | 0.0000 | 0.0000 | 1.4473 | 0.0000 | 0.0000 |
| $\mu_{12}$ | 15.7119 | 15.9994 | 15.9992 | 16.1693 | 16.0004 | 16.0002 | 16.1484 | 16.0006 | 16.0004 |
| | 2.7599 | 0.0000 | 0.0000 | 1.7276 | 0.0000 | 0.0000 | 1.9185 | 0.0000 | 0.0000 |
| $\mu_{13}$ | 18.0323 | 18.0016 | 18.0014 | 17.9392 | 18.0001 | 17.9998 | 18.0924 | 18.0009 | 18.0006 |
| | 2.8263 | 0.0000 | 0.0000 | 2.4351 | 0.0001 | 0.0001 | 2.0293 | 0.0000 | 0.0000 |
| $\mu_{21}$ | 20.0894 | 20.0014 | 20.0011 | 20.1550 | 20.0008 | 20.0005 | 19.7357 | 19.9988 | 19.9985 |
| | 15.3991 | 0.0001 | 0.0001 | 5.6088 | 0.0001 | 0.0001 | 4.1294 | 0.0001 | 0.0001 |

**Table-4.3.** Estimators with varying time series at ρ = 0.99 (cont.)

| | T=50 | | | T=75 | | | T=100 | | |
|---|---|---|---|---|---|---|---|---|---|
| | OLS | SELF | ELF | OLS | SELF | ELF | OLS | SELF | ELF |
| $\mu_{22}$ | 22.1373 | 21.9999 | 21.9996 | 21.8658 | 22.0008 | 22.0005 | 22.0746 | 22.0001 | 21.9998 |
| | 17.3947 | 0.0001 | 0.0001 | 6.3083 | 0.0001 | 0.0001 | 3.4212 | 0.0001 | 0.0001 |
| $\mu_{23}$ | 24.1136 | 24.0001 | 23.9998 | 23.8898 | 23.9995 | 23.9992 | 24.1808 | 24.0002 | 23.9999 |
| | 20.6617 | 0.0001 | 0.0001 | 3.6968 | 0.0001 | 0.0001 | 4.7187 | 0.0001 | 0.0001 |
| $\mu_{31}$ | 25.4894 | 26.0006 | 26.0002 | 25.9236 | 26.0004 | 26.0001 | 25.8274 | 25.9991 | 25.9988 |
| | 50.4575 | 0.0001 | 0.0001 | 9.9667 | 0.0001 | 0.0001 | 9.8900 | 0.0001 | 0.0001 |
| $\mu_{32}$ | 27.7662 | 27.9993 | 27.9991 | 27.8165 | 28.0000 | 27.9997 | 27.5075 | 28.0013 | 28.0010 |
| | 37.2489 | 0.0001 | 0.0001 | 15.0037 | 0.0001 | 0.0001 | 11.5979 | 0.0001 | 0.0001 |
| $\mu_{33}$ | 29.7095 | 30.0006 | 30.0003 | 29.7752 | 30.0009 | 30.0006 | 30.0822 | 29.9988 | 29.9985 |
| | 36.9404 | 0.0001 | 0.0001 | 12.4540 | 0.0001 | 0.0001 | 7.6263 | 0.0001 | 0.0001 |
| $\mu_{41}$ | 31.7953 | 32.0001 | 31.9998 | 32.1870 | 32.0006 | 32.0004 | 31.7316 | 31.9987 | 31.9985 |
| | 72.6862 | 0.0001 | 0.0001 | 7.7369 | 0.0001 | 0.0001 | 2.9958 | 0.0001 | 0.0001 |
| $\mu_{42}$ | 33.7731 | 34.0002 | 33.9999 | 33.9305 | 34.0000 | 33.9998 | 33.8511 | 34.0027 | 34.0024 |
| | 72.3302 | 0.0001 | 0.0001 | 6.8616 | 0.0001 | 0.0001 | 2.8177 | 0.0001 | 0.0001 |
| $\mu_{43}$ | 35.9221 | 35.9988 | 35.9985 | 35.9559 | 35.9998 | 35.9995 | 35.8189 | 36.0004 | 36.0002 |
| | 75.6086 | 0.0001 | 0.0001 | 6.9744 | 0.0001 | 0.0001 | 2.9922 | 0.0001 | 0.0001 |
| $\sigma_1^2$ | 0.1191 | 0.1088 | 0.1006 | 0.1221 | 0.1121 | 0.0993 | 0.1153 | 0.1122 | 0.1017 |
| | 0.0024 | 0.0020 | 0.0000 | 0.0034 | 0.0017 | 0.0003 | 0.0024 | 0.0016 | 0.0001 |
| $\sigma_2^2$ | 0.2396 | 0.2082 | 0.1962 | 0.2306 | 0.2288 | 0.1845 | 0.2405 | 0.2236 | 0.2025 |
| | 0.0088 | 0.0077 | 0.0002 | 0.0097 | 0.0093 | 0.0003 | 0.0122 | 0.0071 | 0.0001 |
| $\sigma_3^2$ | 0.3452 | 0.3208 | 0.2999 | 0.3418 | 0.3215 | 0.2642 | 0.3502 | 0.3313 | 0.2817 |
| | 0.0172 | 0.0123 | 0.0002 | 0.0227 | 0.0139 | 0.0018 | 0.0253 | 0.0207 | 0.0009 |
| $\sigma_4^2$ | 0.3953 | 0.4094 | 0.3891 | 0.3920 | 0.4080 | 0.4013 | 0.4067 | 0.4005 | 0.4011 |
| | 0.0036 | 0.0032 | 0.0004 | 0.0017 | 0.0018 | 0.0000 | 0.0012 | 0.0012 | 0.0001 |

Table-4.1-4.3 shows the behaviour of ordinary least square (OLS) and Bayes estimators of parameter with varying values of time series at different autoregressive coefficient. It can be easily seen that MSEs of all estimators decrease as the sample size of series increases. The difference between the estimated value and the true value in OLS is large, which explains that average bias is maximum as compared to Bayes estimator. The Bayes estimator under both loss functions performs better because of additional information given about the parameter. A better estimated value for autoregressive coefficient, mean term and error variance is obtained by ELF as compared to SELF due to less MSE for low value of ρ. For a high value of ρ, Bayes estimator obtained under different loss function is equally applicable to estimate the parameters since both the estimators show more or less same magnitudes for their MSE. An increase in the number of breaks points, MSE also increases because of the length of the segment is small and take less observation to estimate the parameters.

After estimation of structural break parameters, testing of the unit root is considered. We can calculate posterior odds ratio values with different values of ρ and varying size of the series, which are reported in Table-4.4. The table shows that as the value of ρ increases, POR reduces to zero and this rejects the null hypothesis, i.e. unit root hypothesis. Thus, the model which contains multiple breaks in mean and variance have a stationary model for this simulated series.

**Table-4.4.** Posterior odds ratio with varying ρ and T

| T | ρ=0.90 | ρ=0.92 | ρ=0.94 | ρ=0.96 | ρ=0.98 |
|---|---|---|---|---|---|
| 50 | 0.0750 | 0.0416 | 0.0245 | 0.0098 | 0.0037 |
| 60 | 0.0645 | 0.0412 | 0.0291 | 0.0118 | 0.0044 |
| 70 | 0.0723 | 0.0447 | 0.0316 | 0.0149 | 0.0060 |
| 80 | 0.0647 | 0.0441 | 0.0338 | 0.0159 | 0.0067 |

## 5. Real Data Analysis

To provide a practical application of our model and verify the result obtained by simulation study, we apply our proposed work to a real data set. We use agricultural production and productivity of various crops of food grains data set consisting of 60 years' time series of Rice (R), Wheat (W) and Coarse Cereals (CC) variables for the annually book of "*Handbook of Statistics on Indian Economy*" from 1954-55 to 2014-15. The source of food grains data set is taken through Ministry of Agriculture & Farmers Welfare, Government of India by Reserve Bank of India and the book was published by Data Management and Dissemination Division (DMDD), Department of Statistics and Information Management (DSIM), Reserve Bank of India (RBI). This book provides statistical data on a wide range of economic and financial indicators related to national income variable, output, prices, money, banking, financial markets, etc. To determine the number of break points and their positions in food grains data set, we can use "*strucchange*" package developed by Zeileis *et al.* (2002) in R-language. The command "*breakpoints*" is considered to suggest the number of break points and identify their location in respective individual series. The results are reported in Table-5.1, which is given below:

**Table-5.1: Structural break point present in different cereals**

| Break Point | Rice | Wheat | Coarse Cereals |
|---|---|---|---|
| $T_1$ | 12 | 12 | 12 |
| $T_2$ | 22 | 22 | 33 |
| $T_3$ | 33 | 30 | 51 |
| $T_4$ | 42 | 39 | |
| $T_5$ | 51 | 51 | |

From the above table one can observe that each series could not contain equal number of break points and their positions also differ from one series to another because of finding the break points individually. After applying the procedure we observe that each series includes two similar break points 12 and 51, which is near and far from the series. The remaining break points (22, 30, 33, 39, 42) mainly occur in between these points. For analysis, we make different combinations of break points containing various numbers of breaks. There are seven break points established; 127 combinations included single as well as multiple breaks. To identify a suitable model for this data set by using log-likelihood function, Akaike information criterion (AIC) and Bayes information criterion (BIC) define how many break points and their location is present in the model. For examination, consider only starting eight combinations, which have minimum AIC and BIC values, as shown in the Table-5.2.

**Table-5.2.** Break point detection

| Break point | POR | ρ | AIC | BIC | Log L |
|---|---|---|---|---|---|
| (22,42,51) | 0.00878 | 0.9803 | 1433.0263 | 1474.3162 | -703.5131 |
| (22,30,51) | 0.01617 | 0.9778 | 1454.7726 | 1496.0625 | -714.3863 |
| (22,30,42,51) | 0.00362 | 0.9732 | 1460.8775 | 1511.6959 | -714.4387 |
| (22,42) | 0.06997 | 0.9808 | 1471.8446 | 1513.6061 | -725.9223 |
| (12,30,51) | 0.01872 | 0.9749 | 1477.9163 | 1519.2062 | -725.9581 |
| (12,30,42,51) | 0.00745 | 0.9693 | 1487.6784 | 1538.4968 | -727.8392 |
| 22 | 0.51915 | 0.9881 | 1524.1465 | 1546.3796 | -755.0733 |
| (22,30,42) | 0.01195 | 0.9738 | 1524.5874 | 1565.8773 | -749.2937 |

By using information criterion, Table-5.2 gives appropriate conclusion about the number of break points and their positions to obtain a suitable model for this data set. The table shows that data follow a PAR(1) model having three break points (22, 42, 51) because of minimum AIC and BIC. Considering higher number of break points, i.e. 5, 6 or 7 in the series, no combination occurs in the last eight observations because the increase in the break point is inconvenient to partition the series in small segments. The maximum number of break points is 4 in this table, which can be considered after 3 break points. When the break point position occurs at only single point, i.e. 22, then AIC and BIC have large value as compared to two break points (22, 42). If we think about these two break points, AIC and BIC values may or may not be larger than three or four break points at different locations. If positions occur mostly at 22 and 51, statistic values is minimum in our combination so that mean ${}^{7}C_{i}$, I = 1 to 7. Minimum difference between break points may increase the statistic values, which directly reject these points of the model. An increase in the number of break points, posterior odds ratio value tends to zero, which concludes our model, which contains multiple breaks in mean and variance, is a better model compared to no-break model, i.e. PAR(1) model. The table also shows that data series is a stationary series for any combination of break point considered in the model. As a break point increases, the value of POR tends to zero, which concludes the model is stationary and no unit root is present in the model to make this as a difference stationary.

Once we acquire the number of break points and their positions in the proposed model, use this to estimate the parameters of the model for the data set using Gibbs procedure. The results are summarized in the Table-5.3. The table gives an appropriate value of the estimate parameter for the food grains data set in different types of estimation technique.

**Table-5.3.** Estimates Value Using Real Data Set

| Parameter | OLS | SELF | ELF |
|---|---|---|---|
| $\rho$ | 0.7786 | 0.7788 | 0.7788 |
| $\mu_{11}$ | 42.6226 | 41.6685 | 41.6810 |
| $\mu_{12}$ | 21.3355 | 20.8544 | 20.8631 |
| $\mu_{13}$ | 26.8882 | 26.2820 | 26.2926 |
| $\mu_{21}$ | 72.9170 | 71.2050 | 71.2258 |
| $\mu_{22}$ | 57.7628 | 56.4043 | 56.4220 |
| $\mu_{23}$ | 31.0778 | 30.3408 | 30.3538 |
| $\mu_{31}$ | 94.2339 | 91.1963 | 91.2235 |
| $\mu_{32}$ | 77.5137 | 75.0114 | 75.0358 |
| $\mu_{33}$ | 34.8718 | 33.7356 | 33.7521 |
| $\mu_{41}$ | 109.4301 | 105.7798 | 105.8104 |
| $\mu_{42}$ | 95.1015 | 91.9243 | 91.9550 |
| $\mu_{43}$ | 46.2301 | 44.6730 | 44.6951 |
| $\sigma_1^2$ | 12.9941 | 12.9820 | 12.9262 |
| $\sigma_2^2$ | 25.5067 | 25.4830 | 25.3736 |
| $\sigma_3^2$ | 51.1556 | 51.1081 | 50.8886 |
| $\sigma_4^2$ | 25.9380 | 25.9139 | 25.8027 |

## 6. Conclusions

This paper deals with multiple structural breaks, which are present in mean and error variance in panel AR (1) model. Bayesian framework is used for estimating and testing the unit root hypothesis. Bayesian estimator gives better estimated value of the parameter as compared to OLS estimator in simulation as well as in real data. Testing of the hypothesis gives appropriate conclusion about the simulated series, which is stationary, and this is also verified by the real data set at each combination of break points. Break point identification is also done in real data set by using information criterion. This model may be extended to panel AR (p) model with similar types of breaks as well as to VAR model.

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## APPENDIX

In this appendix, we have derived the posterior probability with the help of likelihood function and prior distribution, which are given below:

(A.1) For alternative hypothesis $H_1 : \rho \in S, \mu_{i1} \neq \mu_{i2}, \sigma_1^2 \neq \sigma_2^2$ , expression of posterior probability can be derived with the help of likelihood function (3) and prior distribution given (4), which is given as

$$P(y \mid H_1)$$

$$= \int_l^1 \int_{R_{B+1}^+} \int_{R_{n(B+1)}} L(\rho, \mu, \sigma \mid y) p(\mu) p(\sigma) p(\rho) d\mu d\sigma d\rho$$

$$= \int_l^1 \int_{R_{B+1}^+} \int_{R_{n(B+1)}} (2\pi)^{-\frac{nT}{2}} \prod_{j=1}^{B+1} \left( \sigma_j^{-n(T_j - T_{j-1})} \right) \exp\left[ -\frac{1}{2} \sum_{j=1}^{B+1} \left\{ \frac{1}{\sigma_j^2} \sum_{i=1}^{n} \sum_{t=T_{j-1}}^{T_j} \left( y_{it} - \rho y_{i,t-1} - (1-\rho)\mu_{ij} \right)^2 \right\} \right]$$

$$\frac{d_1^{c_1}}{\Gamma c_1} \frac{d_2^{c_2}}{\Gamma c_2} \cdots \frac{d_{B+1}^{c_{B+1}}}{\Gamma c_{B+1}} \left(\sigma_1^2\right)^{-c_1-1} \left(\sigma_2^2\right)^{-c_{21}-1} \cdots \left(\sigma_{B+1}^2\right)^{-c_{B+1}-1} \exp\left[ -\frac{d_1}{\sigma_1^2} \cdots - \frac{d_{B+1}}{\sigma_{B+1}^2} \right] (2\pi)^{-\frac{n(B+1)}{2}}$$

$$\sigma_1^{-n} \cdots \sigma_{B+1}^{-n} \exp\left[ -\frac{1}{2\sigma_1^2} \sum_{i=1}^{n} (\mu_{i1} - \gamma_{i1})^2 \cdots - \frac{1}{2\sigma_{B+1}^2} \sum_{i=1}^{n} \left(\mu_{i,B+1} - \gamma_{i,B+1}\right)^2 \right] d\mu d\sigma d\rho$$

$$= \int_l^1 \int_{R_{B+1}^+} \int_{R_{n(B+1)}} \frac{(2\pi)^{-\frac{n(T+B+1)}{2}}}{1-l} \prod_{j=1}^{B+1} \left( \frac{d_j^{c_j}}{\Gamma c_j} \left(\sigma_j^2\right)^{-\left[\frac{n(T_j - T_{j-1}+1)}{2} + c_j + 1\right]} \right) \exp\left[ -\frac{1}{2} \sum_{j=1}^{B+1} \left\{ \frac{1}{\sigma_j^2} \left( \sum_{i=1}^{n} \sum_{t=T_{j-1}}^{T_j} \left( y_{it} - \rho y_{i,t-1} \right)^2 \right. \right. \right.$$

$$- 2(1-\rho) \sum_{i=1}^{n} \sum_{t=T_{j-1}}^{T_j} \left( y_{it} - \rho y_{i,t-1} \right) \mu_{ij} + (1-\rho)^2 \left(T_j - T_{j-1}\right) \sum_{i=1}^{n} \mu_{ij}^2 + \sum_{i=1}^{n} \mu_{ij}^2 - 2 \sum_{i=1}^{n} \gamma_{ij} \mu_{ij}$$

$$\left. \left. \left. + \sum_{i=1}^{n} \gamma_{ij}^2 + 2d_j \right) \right\} \right] d\mu d\sigma d\rho$$

$$= \int_l^1 \int_{R_{B+1}^+} \int_{R_{n(B+1)}} \frac{(2\pi)^{-\frac{n(T+B+1)}{2}}}{1-l} \prod_{j=1}^{B+1} \left( \frac{d_j^{c_j}}{\Gamma c_j} \left(\sigma_j^2\right)^{-\left[\frac{n(T_j - T_{j-1}+1)}{2} + c_j + 1\right]} \right) \exp\left[ -\frac{1}{2} \sum_{j=1}^{B+1} \left\{ \frac{1}{\sigma_j^2} \left( \sum_{i=1}^{n} \sum_{t=T_{j-1}}^{T_j} \left( y_{it} - \rho y_{i,t-1} \right)^2 \right. \right. \right.$$

$$+ 2d_j + \sum_{i=1}^{n} \gamma_{ij}^2 + \left( (1-\rho)^2 \left(T_j - T_{j-1}\right) + 1 \right) \sum_{i=1}^{n} \mu_{ij}^2$$

$$\left. \left. \left. - 2 \sum_{i=1}^{n} \left( (1-\rho) \sum_{t=T_{j-1}}^{T_j} \left( y_{it} - \rho y_{i,t-1} \right) + \gamma_{ij} \right) \mu_{ij} \right) \right\} \right] d\mu d\sigma d\rho$$

Let us consider

$$A_j = (1-\rho)^2 \left(T_j - T_{j-1}\right) + 1$$

$$B_{ij} = (1-\rho) \sum_{t=T_{j-1}}^{T_j} \left(y_{it} - \rho y_{i,t-1}\right) + \gamma_{ij}$$

Then we may write

$$P\left(y \mid H_1\right)$$

$$= \int_l^1 \int_{R_{B+1}^+} \int_{R_{n(B+1)}} \frac{(2\pi)^{-\frac{n(T+B+1)}{2}}}{1-l} \prod_{j=1}^{B+1} \left( \frac{d_j^{c_j}}{\Gamma c_j} \left(\sigma_j^2\right)^{-\left[\frac{n\left(T_j - T_{j-1}+1\right)}{2} + c_j + 1\right]} \right) \exp\left[ -\frac{1}{2} \sum_{j=1}^{B+1} \left\{ \frac{1}{\sigma_j^2} \left( \sum_{i=1}^{n} \sum_{t=T_{j-1}}^{T_j} \left(y_{it} - \rho y_{i,t-1}\right)^2 + \sum_{i=1}^{n} \gamma_{ij}^2 + 2d_j + A_j \sum_{i=1}^{n} \left( \mu_{ij} - \frac{B_{ij}}{A_j} \right)^2 - \sum_{i=1}^{n} \frac{B_{ij}^2}{A_j} \right) \right\} \right] d\mu \, d\sigma \, d\rho$$

$$= \int_l^1 \int_{R_{B+1}^+} \frac{(2\pi)^{-\frac{nT}{2}}}{1-l} \prod_{j=1}^{B+1} \left( \frac{d_j^{c_j} \left(\sigma_j^2\right)^{-\left[\frac{n\left(T_j - T_{j-1}\right)}{2} + c_j + 1\right]}}{\left(A_j\right)^{\frac{n}{2}} \Gamma c_j} \right) \exp\left[ -\frac{1}{2} \sum_{j=1}^{B+1} \left\{ \frac{1}{\sigma_j^2} \left( \sum_{i=1}^{n} \sum_{t=T_{j-1}}^{T_j} \left(y_{it} - \rho y_{i,t-1}\right)^2 + \sum_{i=1}^{n} \gamma_{ij}^2 + 2d_j - \sum_{i=1}^{n} \frac{B_{ij}^2}{A_j} \right) \right\} \right] d\sigma \, d\rho$$

$$= \int_l^1 \int_{R_{B+1}^+} \frac{(2\pi)^{-\frac{nT}{2}}}{1-l} \prod_{j=1}^{B+1} \left( \frac{d_j^{c_j} \left(\sigma_j^2\right)^{-\left[\frac{n\left(T_j - T_{j-1}\right)}{2} + c_j + 1\right]}}{\left(A_j\right)^{\frac{n}{2}} \Gamma c_j} \right) \exp\left[ -\sum_{j=1}^{B+1} \frac{C_{ij}}{\sigma_j^2} \right] d\sigma \, d\rho$$

$$= \int_l^1 \frac{(2\pi)^{-\frac{nT}{2}}}{1-l} \prod_{j=1}^{B+1} \frac{d_j^{c_j} \Gamma\left( \frac{n\left(T_j - T_{j-1}\right)}{2} + c_j \right)}{\Gamma c_j \left(A_j\right)^{\frac{n}{2}} C_j^{\frac{n\left(T_j - T_{j-1}\right)}{2} + c_j}} d\rho$$

where

$$C_{ij} = d_j + \frac{1}{2} \left( \sum_{i=1}^{n} \sum_{t=T_{j-1}}^{T_j} \left(y_{it} - \rho y_{i,t-1}\right)^2 + \sum_{i=1}^{n} \gamma_{ij}^2 - \sum_{i=1}^{n} \frac{B_{ij}^2}{A_j} \right)$$

(A.2) Under unit root hypothesis $H_0 : \rho = 1, \mu_{i1} \neq \mu_{i2}, \sigma_1^2 \neq \sigma_2^2$, the joint likelihood function is given as

$$L(\sigma \mid y) = \prod_{j=1}^{B+1} L(\sigma_j \mid y) = \prod_{j=1}^{B+1} \left( (2\pi)^{-\frac{n(T_j - T_{j-1})}{2}} \sigma_j^{-n(T_j - T_{j-1})} \exp\left[ -\frac{1}{2\sigma_j^2} \sum_{i=1}^{n} \sum_{t=T_{j-1}}^{T_j} \varepsilon_{it}^2 \right] \right)$$

$$= (2\pi)^{-\frac{nT}{2}} \prod_{j=1}^{B+1} \left( \sigma_j^{-n(T_j - T_{j-1})} \right) \exp\left[ -\frac{1}{2} \sum_{j=1}^{B+1} \left\{ \frac{1}{\sigma_j^2} \sum_{i=1}^{n} \sum_{t=T_{j-1}}^{T_j} \Delta y_{it}^2 \right\} \right]$$

By using similar mathematical manipulations as above, we get the posterior probability

$$P(y \mid H_0)$$

$$= \int_{R_{B+1}^+} L(\sigma \mid y) p(\sigma) d\sigma$$

$$= \int_{R_{B+1}^+} (2\pi)^{-\frac{nT}{2}} \prod_{j=1}^{B+1} \left( \sigma_j^{-n(T_j - T_{j-1})} \right) \exp\left[ -\frac{1}{2} \sum_{j=1}^{B+1} \left\{ \frac{1}{\sigma_j^2} \sum_{i=1}^{n} \sum_{t=T_{j-1}}^{T_j} \Delta y_{it}^2 \right\} \right] \frac{d_1^{c_1}}{\Gamma c_1} \frac{d_2^{c_2}}{\Gamma c_2} \cdots \frac{d_{B+1}^{c_{B+1}}}{\Gamma c_{B+1}}$$
$$\left(\sigma_1^2\right)^{-c_1 - 1} \left(\sigma_2^2\right)^{-c_{21} - 1} \cdots \left(\sigma_{B+1}^2\right)^{-c_{B+1} - 1} \exp\left[ -\frac{d_1}{\sigma_1^2} - \frac{d_2}{\sigma_2^2} \cdots - \frac{d_{B+1}}{\sigma_{B+1}^2} \right] d\sigma$$

$$= \int_{R_{B+1}^+} (2\pi)^{-\frac{nT}{2}} \prod_{j=1}^{B+1} \left( \frac{d_j^{c_j}}{\Gamma c_j} \left(\sigma_j^2\right)^{-\left[\frac{n(T_j - T_{j-1})}{2} + c_j + 1\right]} \right) \exp\left[ -\sum_{j=1}^{B+1} \left\{ \frac{1}{\sigma_j^2} \left( \frac{1}{2} \sum_{i=1}^{n} \sum_{t=T_{j-1}}^{T_j} \Delta y_{it}^2 + d_j \right) \right\} \right] d\sigma$$

$$= (2\pi)^{-\frac{nT}{2}} \prod_{j=1}^{B+1} \frac{d_j^{c_j} \Gamma\left( \frac{n(T_j - T_{j-1})}{2} + c_j \right)}{\Gamma c_j \left[ \frac{1}{2} \sum_{i=1}^{n} \sum_{t=T_{j-1}}^{T_j} \Delta y_{it}^2 + d_j \right]^{\frac{n(T_j - T_{j-1})}{2} + c_j}}$$