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FULL ORDER UNKNOWN INPUTS OBSERVER FOR MULTIPLE TIME-DELAY SYSTEMS

[1] Seifeddine Ben Warrad and [2] Olfa Boubaker

National Institute of Applied Sciences and Technology

INSAT, Centre Urbain Nord BP. 676 – 1080 Tunis Cedex, Tunisia

Emails : [1] ben.warrad_seifeddine@hotmail.com [2] olfa.boubaker@insat.rnu.tn

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Abstract- In this paper, a design of a full-order observer for linear time-invariant (LTI) multivariable systems with multiple time-delays and unknown inputs (UI) is proposed. The main idea is to reduce the problem of the unknown input observer (UIO) for systems with multiple time-delays to that of a standard one. To that purpose, the orthogonal collocation method is used to transform the infinite dimensional model of the delayed system described by a set of linear partial differential equations (PDEs) to a finite dimensional one described by a set of linear ordinary differential equations (ODEs). Even using an approximation method, the asymptotic stability of the UIO is well proven. The efficiency of the proposed algorithm is shown using the quadruple-tank benchmark. The two cases of minimum and non-minimum phase models are considered.

Index terms*:* Asymptotic stability, Delay systems, Observer, Unknown Inputs.

I. INTRODUCTION

In the past few years, the emergence of delays in dynamical systems was diagnosed as one of the major causes of loss of stability and degradation of performances and robustness [1, 2, 3, 4]. The systems represented by a set of delayed ordinary differential equations (ODEs) are called time-delay systems and, in many cases, considered as distributed parameter systems (DPSs) [5]. They are generally described by a set of partial differential equations (PDEs) and belong to the family of infinite dimensional systems. Their analysis and synthesis methods are not belonging, in that case, to those of standard ones. Massive research activities have been developed to solve stability and stabilization problems of delay systems (see for example [6, 7, 8, 9, 10, 11, 12, 13, 14] and references therein). In that context, many interests have been also given to systems with multiple time-delays [15, 16, 17, 18, 19] for which the well-known limitations are the presence of delays either in the states of the plant, in the inputs as well in the outputs [20, 21]. The problem of estimating state variables is of much significance in many applications [22, 23]. For linear time-invariant (LTI) multivariable systems with multiple time-delays, only several elaborated results can be found in this framework [21, 24, 25, 26]. One of the rare research papers dealing with the design problem of unknown inputs observers (UIO) is found in [27].

In this paper, we propose a different approach to address UIO design for LTI systems with multiple time- delays. The basic idea is to transform the problem of UIO for LTI system with multiple time-delays to that of a standard one. Even using an approximation method for modeling the multiple time-delay system, the asymptotic stability of the UIO is well proven.

The paper is organized as follows: In section II, the design problem is stated. Section III, presents the mathematical model of the LTI system with multiple time-delays. Section IV, gives necessary and sufficient conditions for asymptotic stability of the finite dimensional UIO. In section V, the efficiency of the proposed approach is shown through simulation results for the quadruple tank benchmark.

II. PROBLEM STATEMENT

Consider the following class of the infinite dimensional LTI multivariable systems with multiple time-delays and unknown inputs described by [28]:

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + A_1x(t-\tau_1) + B_0u(t-\tau_2) + B_1u(t-\tau_3) + Dd(t) \\ y(t) &= Cx(t) \end{aligned}$$

where $x(t) \in \Re^{n}$, $u(t) \in \Re^{m}$, $y(t) \in \Re^{p}$ and $d(t) \in \Re^{q}$ are the state vector, the control vector, the output vector and the unknown input vector, respectively. $A_0 \in \Re^{n\times n}$, $A_1 \in \Re^{n\times n}$, $B_0 \in \Re^{n\times m}$, $B_1 \in \Re^{n\times m}$, $C \in \Re^{p\times n}$ and $D \in \Re^{n\times q}$ are constant matrices with appropriate dimensions and, τ_1, τ_2 and τ_3 are known and constant delays of the system (1). Consider now from (1), an approximated finite dimensional LTI system with free-time delays described by:

$$\begin{aligned} \dot{\chi}(t) &= \tilde{A}\chi(t) + \tilde{B}u(t) + \tilde{D}d(t) \\ \tilde{y}(t) &= \tilde{C}\chi(t) \end{aligned} \tag{2}$$

where $\chi(t) \in \Re^{n'}$, $u(t) \in \Re^{m}$, $\tilde{y}(t) \in \Re^{p}$ and $d(t) \in \Re^{q}$ are the state vector, the control vector, the output vector and the unknown input vector, respectively. $\tilde{A} \in \Re^{n'\times n'}$, $\tilde{B} \in \Re^{n'\times m}$, $\tilde{C} \in \Re^{p\times n'}$ and $\tilde{D} \in \Re^{n'\times q}$ are known constant matrices with appropriate dimensions.

The objective is to design a finite dimensional UIO for the linear system (1) described by:

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + L\tilde{y}(t) + Gu(t) \\ \hat{\chi}(t) &= \zeta(t) - E\tilde{y}(t) - Qr(t) \end{aligned} \tag{3}$$

where $\zeta(t) \in \Re^{n'}$; $\hat{\chi}(t) \in \Re^{n'}$ is the estimated state vector. The approximation error, $r(t)$, defined between the DPS system (1) and the system (2), is given by:

$$r(t) = \chi(t) - Tx(t) \tag{4}$$

where $T \in \Re^{n'\times n}$ is a constant matrix. Q is the observer constant gain. N, L, G and E are constant matrices of appropriate dimensions to be designed. The last design problem is solved under the following assumptions:

(1) $p \geq q$

(2) $\operatorname{rank}(\tilde{D}) = q$

(3) $\operatorname{rank}(\tilde{C}) = p$

(4) The pair $\left(\tilde{A}, \tilde{C}\right)$ is observable at least detectable.

Notation

In the following, we consider the following error variables and vectors:

$r(t)$: Approximation error vector between the DPS (1) and the lumped system (2).

$e_N(z,t)$: Interpolation error.

$e(t)$: Observer error vector.

III. THE FINITE DIMENSIONAL MODEL

In this section, the finite dimensional model (2) will be elaborated. Many approximated methods are used in the literature to transform time-delay systems on standard models without delays. The most known is the padé approximation [29, 30, 31] which gives approximated systems with very high dimension. In this paper, we present a different approach, the orthogonal collocation method having the advantage to give lower dimensional lumped systems [32]. First, the LTI multivariable system with multiple time-delays and unknown inputs will be transformed into a DPS described by a set of linear PDEs. Then, using the orthogonal collocation method, the DPS is reduced in a lumped one described by a set of ODEs.

a. Modeling each delayed variable by a PDE

Each delayed variable can be modeled as a DPS described by a PDE as follows [5]:

$$\frac{\partial \omega(z,t)}{\partial t} = -\frac{1}{\tau}\frac{\partial \omega(z,t)}{\partial z} \qquad (5)$$

with the boundary condition:

$$v(t) = \omega(0, t) \tag{6}$$

and the output equations:

$$v(t - \tau) = \omega(1, t) \tag{7}$$

where t and z are time and pseudo-space variables, respectively. $v(t)$, $\omega(z, t)$ and $v(t-\tau)$ are the input, the state variable and the output of the delay block, respectively. τ is a constant time delay.

b. Transformation of PDEs on ODEs

The PDE described by (5) augmented by the boundary conditions (6) and the output equations (7) can be transformed into ODEs using functional approximation methods [30, 31]. Within the framework of weighted residuals methods, the orthogonal collocation method can be useful. The orthogonal collocation method [32, 33, 34] is particular suited for digital computation when compared with other approximation methods. The orthogonal collocation offers the advantages of a fairly easy implementation: the residual function is minimized without integral or averaging computation. Another benefit lies in that the nature and dimension of state variables remain unchanged after reduction. The principle of the orthogonal collocation method is to search an approximation in the form ODEs using the collocation formula given by:

$$\omega(z, t) = \sum_{i=0}^{N+1} L_i(z)\omega_i(t) \tag{8}$$

where $\omega(t)$ denote the approximation and L_i is the Lagrange interpolation polynomials evaluated at the (N+2) collocation points collocation points chosen on the pseudo-spatial interval [0, 1]. The Lagrange polynomials $L_i(z)$ have the following property:

$$\begin{cases} L_i(z_j) = 1 & \text{if} \quad i = j \\ L_i(z_j) = 0 & \text{if} \quad i \neq j \end{cases} \tag{9}$$

The collocation points $\left(z_i,\ i=0,1,\cdots,N\right)$ are chosen as follows

$$z_0 = 0 < z_1 < \cdots < z_{N-1} < z_N = 1 \tag{10}$$

The N internal collocation points are obtained by calculating the zeros of orthogonal Jacobi polynomials $P_N^{(\alpha,\beta)}$ having the following property [35]:

$$P_N^{(\alpha,\beta)} = \left(z - g_N\left(\alpha,\beta\right)\right) P_{N-1}^{(\alpha,\beta)} h_N\left(\alpha,\beta\right) P_{N-2}^{(\alpha,\beta)} \tag{11}$$

with $P_0^{(\alpha,\beta)} = 1$ and where coefficients $h_N\left(\alpha,\beta\right)$

$$h_N\left(\alpha,\beta\right) := \begin{cases} \text{if } N > 2 \\ \dfrac{\left(N-1\right)\left(N+\alpha-1\right)\left(N+\alpha+\beta-1\right)}{\left(2N+\alpha+\beta-1\right)\left(2N+\alpha+\beta-2\right)^2\left(2N+\alpha+\beta-3\right)} \\ \text{if } N = 2 \\ \dfrac{\left(\alpha+1\right)\left(\beta+1\right)}{\left(\alpha+\beta+2\right)^2\left(\alpha+\beta+3\right)} \end{cases} \tag{12}$$

and

$$g_N\left(\alpha,\beta\right) := \begin{cases} 0 \quad \text{otherwise} \\ \dfrac{1}{2}\left(1 - \dfrac{\alpha^2-\beta^2}{\left(2N+\alpha+\beta-1\right)^2 - 1}\right) & \text{if } N > 1 \\ \dfrac{\beta+1}{\alpha+\beta+2} & \text{if } N = 1 \end{cases} \tag{13}$$

where α, β are two constant parameters affecting the position of the collocation points.

c. Approximation error analysis

Referring to equations (5) where L_i is the Lagrange interpolation polynomials evaluated at the

N+1 collocation points collocation points chosen on the pseudo-spatial interval [0,1] such that $z_0 < z_1 < ... < z_N = 1$ and using the Cauchy formula involves an upper error bounded for such approximations. Indeed, the interpolation error is given by [35]:

$$e_N(z,t) = w(z,t) - \sum_{i=0}^{N} w_i(t) P_i(z) \tag{14}$$

We assume the unknown solution $w(z,t)$ is sufficiently continuously differentiable, we obtain:

$$e_N(z,t) := v(z) \frac{w_z^{N+1}(\eta(z),t)}{(N+1)!} \tag{15}$$

where

$$v(z) := \prod_{i=0}^{N} (z - z_i) \quad \text{and} \quad \eta(z) \in [-1,+1] \tag{16}$$

According to [33], Chebyshev polynomials obtained for $\beta = \alpha = -\frac{1}{2}$, has the minimal norm of $v(z)$ as:

$$\|v(z)\|_\infty = \left\| \prod_{k=1}^{N-1} (z - z_k^*) \right\|_\infty = 2^{-(N-2)} \tag{17}$$

Thus, the minimal norm of the interpolation error is given by:

$$\|e_N\|_\infty \le \frac{\left\| w_z^{*(N+1)}(.,t) \right\|_\infty}{(N+1)!2^{N-2}} \tag{18}$$

where

$$z_k^* := \cos\left(\pi \frac{2k+1}{2(N-1)}\right) \quad \text{for} \;\; k = 0 \ldots N-2 \tag{19}$$

We can then conclude that interpolation error is always bounded when $\beta = \alpha = -0.5$.

d. The finite dimensional Model

In this section, we apply the collocation approximation method for each delayed variable of the vectors $x(t-\tau_1)$, $u(t-\tau_2)$ and $u(t-\tau_3)$. The following 3(N+1) finite dimensional equations can be then obtained [31]:

$$\dot{\omega}_1(t) = -\frac{1}{\tau_1}\bar{A}_1\omega_1 + \frac{1}{\tau_1}\bar{B}_1 x(t) \quad (20)$$

$$\dot{\omega}_2(t) = -\frac{1}{\tau_2}\bar{A}_2\omega_2 + \frac{1}{\tau_2}\bar{B}_2 u(t) \quad (21)$$

$$\dot{\omega}_3(t) = -\frac{1}{\tau_3}\bar{A}_3\omega_3 + \frac{1}{\tau_3}\bar{B}_3 u(t) \quad (22)$$

augmented by the following outputs :

$$x(t-\tau_1) = \omega_1(1,t) = \bar{C}_1\omega_1(t) \quad (23)$$

$$u(t-\tau_2) = \omega_2(1,t) = \bar{C}_2\omega_2(t) \quad (24)$$

$$u(t-\tau_3) = \omega_3(1,t) = \bar{C}_3\omega_3(t) \quad (25)$$

where for k=1,2,3

$$\omega_{k,i} = \begin{bmatrix} \omega_{k,i_1} & \omega_{k,i_2} & \cdots & \omega_{k,i_{N+1}} \end{bmatrix}^T$$

$$\omega_{k,j} = \begin{bmatrix} \omega_{k,j_1} & \omega_{k,j_2} & \cdots & \omega_{k,j_{N+1}} \end{bmatrix}^T$$

$$\bar{A}_k = \begin{bmatrix} a_{k,ij} \end{bmatrix} = \left.\frac{dL_j(z)}{dz}\right|_{z=z_i} \in \Re^{(N+1)\times(N+1)},\ i,j = 1,\cdots,N+1,$$

$$\bar{B}_1 = \begin{bmatrix} b_{k,i1} & \cdots & b_{k,i_n} \end{bmatrix} = \begin{bmatrix} \left.\frac{dL_0(z)}{dz}\right|_{z=z_i} & \cdots & \left.\frac{dL_0(z)}{dz}\right|_{z=z_i} \end{bmatrix} \in \Re^{(N+1)\times n},\ i = 1,\cdots,N+1$$

$$\bar{B}_2 = \bar{B}_3 = \begin{bmatrix} \left.\frac{dL_0(z)}{dz}\right|_{z=z_i} & \cdots & \left.\frac{dL_0(z)}{dz}\right|_{z=z_i} \end{bmatrix} \in \Re^{(N+1)\times m}$$

$$\bar{C}_1 = \begin{bmatrix} c_{1,ij} \end{bmatrix}^T \in \Re^{n\times(N+1)} \quad c_{1,ij} = \begin{cases} 0 & \text{if} \quad j = 1,\cdots,N \\ 1 & \text{if} \quad j = N+1 \end{cases},$$

$$\bar{C}_2 = \bar{C}_3 = \begin{bmatrix} c_{2,ij} \end{bmatrix}^T = \begin{bmatrix} c_{3,ij} \end{bmatrix}^T \in \Re^{m\times(N+1)}$$

$$c_{2,ij} = c_{3,ij} = \begin{cases} 0 & \text{if} \quad j = 1,\cdots,N \\ 1 & \text{if} \quad j = N+1 \end{cases}$$

Thus, using the equations (23), (24) and (25), the system (1) can be written as:

$$\dot{x}(t) = A_0 x(t) + A_1\bar{C}_1\omega_1 + B_0\bar{C}_2\omega_2 + B_1\bar{C}_3\omega_3 + Dd(t) \quad (26)$$

Let $\chi^T = \left[x^T(t) \quad \omega_1^T(t) \quad \omega_2^T(t) \quad \omega_3^T(t) \right]$ the state vector of the new augmented system and $\tilde{y} = \tilde{C}\chi = [C \quad \tilde{C}_1 \quad \tilde{C}_2 \quad \tilde{C}_3]^T \chi$ its output vector. Imposing the constraint n = N+1, we can deduce that the dynamics of the lumped system can be written as:

$$\begin{aligned} \dot{\chi}(t) &= \tilde{A}\chi(t) + \tilde{B}u(t) + \tilde{D}d(t) \\ \tilde{y}(t) &= \tilde{C}\chi(t) \end{aligned} \tag{27}$$

where

$$\tilde{A} = \begin{bmatrix} A_0 & A_1\bar{C}_1 & B_0\bar{C}_2 & B_1\bar{C}_3 \\ \frac{1}{\tau_1}\bar{B}_1 & -\frac{1}{\tau_1}\bar{A}_1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\tau_2}\bar{A}_2 & 0 \\ 0 & 0 & 0 & -\frac{1}{\tau_3}\bar{A}_3 \end{bmatrix} \in \Re^{4n\times 4n}, \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\tau_2}\bar{B}_2 \\ \frac{1}{\tau_3}\bar{B}_3 \end{bmatrix} \in \Re^{4n\times m},$$

$$\tilde{C} = \begin{bmatrix} C & \bar{C}_1 & \bar{C}_2 & \bar{C}_3 \end{bmatrix} \in \Re^{p\times 4n}, \ \tilde{D} = \begin{bmatrix} D \\ 0_{(3n\times q)} \end{bmatrix}$$

IV. THE ROBUST OBSERVER

Theorem 1

The finite dimensional observer (3) converges asymptotically to the system (1) for all $T \in \Re^{4n\times 4n}$ if the following conditions hold:

- The matrix N is Hurwitz
- The approximation error dynamics $r(t)$, given by (4), are described by the asymptotically stable system:

$$\dot{r} = Nr \tag{28}$$

- There exists a matrix $P = \left(I + E\tilde{C}\right) \in \Re^{4n\times 4n}$ such that the following equations are satisfied:

$$NP + L\tilde{C} - P\tilde{A} = 0 \tag{29}$$

$$G - P\tilde{B} = 0 \tag{30}$$

$$P\tilde{D} = 0 \tag{31}$$

Proof

Define the observer reconstruction error by:

$$e = \hat{\chi} - \chi \tag{32}$$

We can write that

$$\zeta = e + \chi + E\tilde{y} + Qr \tag{33}$$

The dynamics of the observer error are then given by

$$\dot{e} = \dot{\zeta} - E\dot{\tilde{y}} - Q\dot{r} - \dot{\chi} \tag{34}$$

Using (3) and (2), it can be shown that (34) is equivalent to:

$$\dot{e} = N\zeta + L\tilde{C}\chi + Gu - E\tilde{C}\left[\tilde{A}\chi + \tilde{B}u + \tilde{D}d\right] - Q\dot{r} - \left[\tilde{A}\chi + \tilde{B}u + \tilde{D}d\right] \tag{35}$$

Substituting (32) into (35) gives

$$\dot{e} = N\left[e + P\chi + Qr\right] + L\tilde{C}\chi + Gu - \left[E\tilde{C}\tilde{A} + \tilde{A}\right]\chi - \left[E\tilde{C}\tilde{B} + \tilde{B}\right]u - \left[E\tilde{C}\tilde{D} + \tilde{D}\right]d - Q\dot{r} \tag{36}$$

when $P = \left(I + E\tilde{C}\right)$, equation (36) can be written as

$$\dot{e} = Ne + \left[NP + L\tilde{C} - P\tilde{A}\right]\chi + \left[G - P\tilde{B}\right]u - P\tilde{D}d - Q\left(\dot{r} - Nr\right) \tag{37}$$

Imposing to the error r to follow the dynamics of an asymptotic stable system (28) and if the conditions (29), (30) and (31) are satisfied, we have:

$$\lim_{t\to\infty} \dot{e}\left(t\right) = 0 \tag{38}$$

The asymptotic stability of the UIO (3) is then well proven. On the other hand, following the Darouach and Zasadzinsk works [36, 37, 38] we can state the following:

Theorem 2

For the LTI multivariable system (2), the full-order observer (3) exists if and only if:

(1) $\operatorname{rank}(\tilde{C}\tilde{D}) = q$

(2) $\operatorname{rank}(\tilde{D}) = q$

(3) $\operatorname{rank}\begin{bmatrix} sP - P\tilde{A} \\ \tilde{C} \end{bmatrix} = n' \quad \forall s \in C,\ \operatorname{Re}(s) \geq 0$

Theorem 3

If the condition (1) holds and $\operatorname{rank}(P) = n - q$, the following three conditions are equivalent:

(1) The pair $\left(P\tilde{A}, \tilde{C}\right)$ is observable, at least detectable

(2) $\operatorname{rank}\begin{bmatrix} sP - P\tilde{A} \\ \tilde{C} \end{bmatrix} = 4n \quad \forall s \in C,\ \operatorname{Re}(s) \geq 0$

(3) $\operatorname{rank}\begin{bmatrix} sP - \tilde{A} & \tilde{D} \\ \tilde{C} & 0 \end{bmatrix} = 4n + q \quad \forall s \in C,\ \operatorname{Re}(s) \geq 0$

We may conclude that, according to the above two theorems, only three existence conditions must be fulfilled. If the theorem 2 and the assumption (1) are satisfied, then it is possible to define a generalized inverse of the matrix $\left(\tilde{C}\tilde{D}\right)$ as a matrix $\left(\tilde{C}\tilde{D}\right)^{+}$ such that $\left(\tilde{C}\tilde{D}\right)\left(\tilde{C}\tilde{D}\right)^{+}\left(\tilde{C}\tilde{D}\right) = \left(\tilde{C}\tilde{D}\right)$. From the condition (31), we easily obtain the expression of matrix E:

$$E = -\tilde{D}\left(\tilde{C}\tilde{D}\right)^{+} \tag{39}$$

where $\left(\tilde{C}\tilde{D}\right)^{+} = \left[\left(\tilde{C}\tilde{D}\right)\left(\tilde{C}\tilde{D}\right)\right]^{-1}\left(\tilde{C}\tilde{D}\right)^{T}$. Let:

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}, P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \tilde{C}^{T} = \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{bmatrix}$$

The UIO (3) can be designed according the following theorem:

Theorem 4: Observer design

If the rank conditions in theorem 2 are satisfied, $p \geq q$ and the pair $(\tilde{A}, \tilde{C})$ is observable, the UIO (3) can be designed for the multivariable LTI with UI (2) if the following conditions hold:

$$N = \begin{bmatrix} N_{11} & \left(\tilde{F}_{12} - \left(\tilde{F}_{11} - \tilde{F}_{12}P_{22}^{+}P_{21}\right)\left(\tilde{C}_1 - \tilde{C}_2P_{22}^{+}P_{21}\right)^{+}\tilde{C}_2\right)P_{22}^{+} \\ N_{21} & \left(\tilde{F}_{22} - \left(\tilde{F}_{21} - \tilde{F}_{22}P_{22}^{+}P_{21}\right)\left(\tilde{C}_1 - \tilde{C}_2P_{22}^{+}P_{21}\right)^{+}\tilde{C}_2\right)P_{22}^{+} \end{bmatrix} \quad (40)$$

$$L = \begin{bmatrix} \left(\tilde{F}_{11} - \tilde{F}_{12}P_{22}^{+}P_{21}\right)\left(\tilde{C}_1 - \tilde{C}_2P_{22}^{+}P_{21}\right)^{+} \\ \left(\tilde{F}_{21} - \tilde{F}_{22}P_{22}^{+}P_{21}\right)\left(\tilde{C}_1 - \tilde{C}_2P_{22}^{+}P_{21}\right)^{+} \end{bmatrix} \quad (41)$$

$$G = P\tilde{B} = \left[I_{4n} - \tilde{D}\left(\tilde{C}D\right)^{+}\tilde{C}\right]\tilde{B} \quad (42)$$

$$E = -\tilde{D}\left(\tilde{C}\tilde{D}\right)^{+} \quad (43)$$

where

$$\begin{aligned} \tilde{F}_{11} &= P_{11}\tilde{A}_{11} + P_{12}\tilde{A}_{21} - N_{11}P_{11}, \\ \tilde{F}_{12} &= P_{11}\tilde{A}_{12} + P_{12}\tilde{A}_{22} - N_{11}P_{12}, \\ \tilde{F}_{21} &= P_{21}\tilde{A}_{11} + P_{22}\tilde{A}_{21} - N_{21}P_{11}, \\ \tilde{F}_{22} &= P_{21}\tilde{A}_{12} + P_{22}\tilde{A}_{22} - N_{21}P_{21} \end{aligned} \quad (44)$$

and N_{11} and N_{21} are random matrices.

Proof:

Therefore, inequality (29) can be written in the following form:

$$\begin{aligned} N_{11}P_{11} + N_{12}P_{21} + L_1\tilde{C}_1 - P_{11}\tilde{A}_{11} - P_{12}\tilde{A}_{21} &= 0_{(4n-p)\times(4n-p)} \\ N_{11}P_{12} + N_{12}P_{22} + L_1\tilde{C}_2 - P_{11}\tilde{A}_{12} - P_{12}\tilde{A}_{22} &= 0_{(4n-p)\times p} \\ N_{21}P_{11} + N_{22}P_{21} + L_2\tilde{C}_1 - P_{21}\tilde{A}_{11} - P_{22}\tilde{A}_{21} &= 0_{p\times(4n-p)} \\ N_{21}P_{12} + N_{22}P_{22} + L_2\tilde{C}_2 - P_{21}\tilde{A}_{12} - P_{22}\tilde{A}_{22} &= 0_{p\times p} \end{aligned} \quad (45)$$

Using the equalities (44), the system (45) can be written as the two following systems:

$$\begin{aligned} N_{12}P_{21} + L_1\tilde{C}_1 &= \tilde{F}_{11} \\ N_{12}P_{22} + L_1\tilde{C}_2 &= \tilde{F}_{12} \end{aligned} \tag{46}$$

$$\begin{aligned} N_{22}P_{21} + L_2\tilde{C}_1 &= \tilde{F}_{21} \\ N_{22}P_{22} + L_2\tilde{C}_2 &= \tilde{F}_{22} \end{aligned} \tag{47}$$

The solutions of system (46) are given by:

$$\begin{aligned} N_{12} &= \left(\tilde{F}_{12} - \left(\tilde{F}_{11} - \tilde{F}_{12}P_{22}^{+}P_{21}\right)\left(\tilde{C}_1 - \tilde{C}_2P_{22}^{+}P_{21}\right)^{+}\tilde{C}_2\right)P_{22}^{+} \\ L_1 &= \left(\tilde{F}_{11} - \tilde{F}_{12}P_{22}^{+}P_{21}\right)\left(\tilde{C}_1 - \tilde{C}_2P_{22}^{+}P_{21}\right) \end{aligned} \tag{46}$$

whereas the solutions of system (47) are given by:

$$\begin{aligned} N_{22} &= \left(\tilde{F}_{22} - \left(\tilde{F}_{21} - \tilde{F}_{22}P_{22}^{+}P_{21}\right)\left(\tilde{C}_1 - \tilde{C}_2P_{22}^{+}P_{21}\right)^{+}\tilde{C}_2\right)P_{22}^{+} \\ L_2 &= \left(\tilde{F}_{21} - \tilde{F}_{22}P_{22}^{+}P_{21}\right)\left(\tilde{C}_1 - \tilde{C}_2P_{22}^{+}P_{21}\right) \end{aligned} \tag{47}$$

Finally, for random matrices N_{11} and N_{21}, matrices N, L, G and E can be found solving (46) and (47) and using (40)-(43).

In this section, the UIO is reduced in to a standard one where the unknown input vector will not interfere in the observer equations. The designed observer is of higher dimension than the delayed system (1) since it will be based on the near model (2).

The last approach is in part inspired from [39] for the simplicity and direct design for high dimension models compared to the O'Reilly's observer [40] or Hui and Zak observer [41].

V. APPLICATION

a. The quadruple- tank process

This section briefly described the quadruple-tank benchmark extensively used in the literature (see for example [21, 42, 43, 44, 45, 46, 47]) and shown by Figure 1. The process comports four interconnected water tanks and two pumps.

The target is to control the level in the two lower tanks using the two pumps, where S_i is the cross-section of tank i and a_i is a parameter area of the pipe flowing out from tank i. γ_1 is ratio of water diverting to tank 1 and tank 4 and γ_2 is ratio of water diverting to tank 2 and tank 3. h_i is the level of water in tank i, ϑ_1 and ϑ_2 are manipulated inputs.

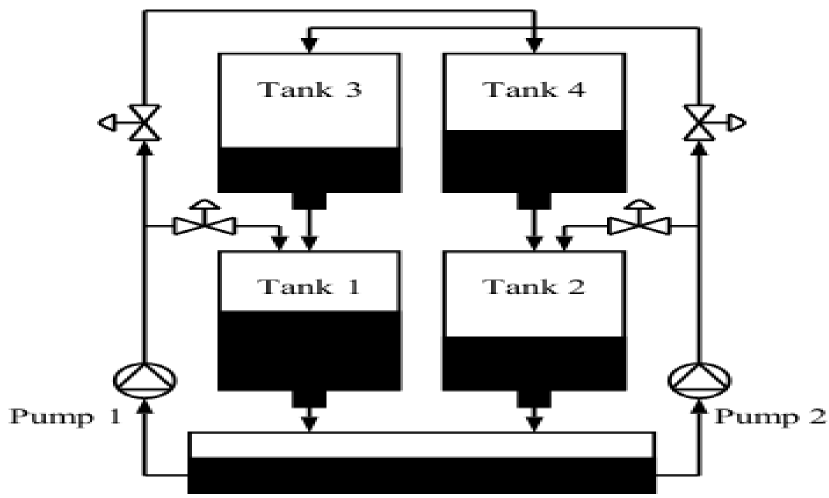


Figure 1. The quadruple tank process

b. The quadruple-tank process model

The nonlinear model of the process is described by [44]:

$$\begin{aligned}
\frac{dh_1}{dt} &= -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_1 k_1 v_1}{A_1} \\
\frac{dh_2}{dt} &= -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_2 k_2 v_2}{A_2} \\
\frac{dh_3}{dt} &= -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_2) k_2 v_2}{A_3} \\
\frac{dh_4}{dt} &= -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_1) k_1 v_1}{A_4}
\end{aligned} \tag{48}$$

Let $x=[h_1 \quad h_2 \quad h_3 \quad h_4]^T$ the state vector, $u=\left[\vartheta_1 \quad \vartheta_2\right]^T$ the control vector and $y=[y_1 \quad y_2]^T$ is the output vector. Furthermore, to have a more realistic description of the process, let take into account transport delays between valves and tanks. The differential equations of the quadruple-tank process with time-delays will be described by:

$$\begin{aligned}
\frac{dh_1}{dt} &= -\frac{a_1}{A_1}\sqrt{2gh_1}+\frac{a_3}{A_1}\sqrt{2gh_3(t-t_1)}+\frac{\gamma_1 k_1 v_1(t-t_5)}{A_1}\\
\frac{dh_2}{dt} &= -\frac{a_2}{A_2}\sqrt{2gh_2}+\frac{a_4}{A_2}\sqrt{2gh_4(t-t_2)}+\frac{\gamma_2 k_2 v_2(t-t_6)}{A_2}\\
\frac{dh_3}{dt} &= -\frac{a_3}{A_3}\sqrt{2gh_3}+\frac{(1-\gamma_2)k_2 v_2(t-t_3)}{A_3}\\
\frac{dh_4}{dt} &= -\frac{a_4}{A_4}\sqrt{2gh_4}+\frac{(1-\gamma_1)k_1 v_1(t-t_4)}{A_4}
\end{aligned} \tag{49}$$

Assume that the transport delays t_i , i=1,…,6 are perfectly symmetric, we can write that $t_1=t_2$, $t_3=t_4$ and $t_5=t_6$. Linearizing the non linear system (49) around the equilibrium point, one can obtain the MIMO LTI model with multiple delays (1) with:

$$A_0=\begin{bmatrix} -\frac{a_1}{A_1}\sqrt{\frac{g}{2h_{10}}} & 0 & 0 & 0\\ 0 & -\frac{a_2}{A_2}\sqrt{\frac{g}{2h_{20}}} & 0 & 0\\ 0 & 0 & -\frac{a_3}{A_3}\sqrt{\frac{g}{2h_{30}}} & 0\\ 0 & 0 & 0 & -\frac{a_4}{A_4}\sqrt{\frac{g}{2h_{40}}} \end{bmatrix}$$

$$A_1=\begin{bmatrix} 0 & 0 & \frac{a_3}{A_1}\sqrt{\frac{g}{2h_{30}}} & 0\\ 0 & 0 & 0 & \frac{a_4}{A_2}\sqrt{\frac{g}{2h_{40}}}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}, B_0=\begin{bmatrix} \frac{\gamma_1 k_1}{A_1} & 0\\ 0 & \frac{\gamma_2 k_2}{A_2}\\ 0 & 0\\ 0 & 0 \end{bmatrix},\ B_1=\begin{bmatrix} 0 & 0\\ 0 & 0\\ 0 & \frac{(1-\gamma_2)k_2}{A_3}\\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{bmatrix} \tag{50}$$

with $\tau_1 = t_1$, $\tau_2 = t_5$ and $\tau_3 = t_3$.

c. Minimum phase and non minimum phase numerical models

Using the numerical values of the process parameters found in Table 1 and Table 2 [43], one can obtain the minimum phase model and the minimum phase model of the quadruple-tank process with multiple delays.

Table 1. Parameter values of the quadruple-tank process

Parameter	**Description**	**Value**	**unit**
A_1, A_3	Area of the tanks	28	$[cm^2]$
A_2, A_4	Area of the tanks	32	$[cm^2]$
a_1, a_3	Area of the outlet pipes	0.071	$[cm^2]$
a_2, a_4	Area of the outlet pipes	0.057	$[cm^2]$
k_c	Constant	0.5	[V/cm]
g	Acceleration due to gravity	9.81	$[cm^2/s]$

Table 2. Operating parameters of minimum and non-minimum phase models

Parameter	Description	Minimum phase values	Non-minimum phase values	Unit
h_1^0, h_2^0	Steady-state value for the water level i	12.4, 12.7	12.6 , 13	[cm]
h_3^0, h_4^0	Steady-state value for the water level i	1.8 , 1.4	4.8 , 4.9	[cm]
ϑ_1^0, ϑ_2^0	Voltage applied to pump i	3.00, 3.00	3.15 , 3.15	[V]
k_1, k_2	Gain from pump i	3.33, 3.35	3.14 , 3.29	[cm^3/V.s]
γ_1, γ_2	Fraction of flow going to tank i from pump i	0.7 , 0.6	0.43 , 0.34	-

The minimum phase model is described by (1) where:

$$A_0 = \begin{bmatrix} -0.1978 & 0 & 0 & 0 \\ 0 & -0.1406 & 0 & 0 \\ 0 & 0 & -0.0753 & 0 \\ 0 & 0 & 0 & -0.0467 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & 0.0753 & 0 \\ 0 & 0 & 0 & 0.0467 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0.0833 & 0 \\ 0 & 0.0628 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0.0479 \\ 0.0312 & 0 \end{bmatrix}$$

The non-minimum phase model is described by (1) where:

$$A_0 = \begin{bmatrix} -0.1993 & 0 & 0 & 0 \\ 0 & -0.1422 & 0 & 0 \\ 0 & 0 & -0.1230 & 0 \\ 0 & 0 & 0 & -0.0873 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & 0.1230 & 0 \\ 0 & 0 & 0 & 0.0873 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0.0482 & 0 \\ 0 & 0.0350 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0.0775 \\ 0.0559 & 0 \end{bmatrix}$$

d. The orthogonal collocation method

To ensure a compromise between the model complexity and the accuracy of the approximation, we opted for the following numerical values N=3, $\beta = \alpha = -0.5$. Note also that N=3 is imposed by the observer design problem. In this case, the collocated points are given by:

$z_0 = 0, z_1 = 0.1727$, $z_2 = 0.5$, $z_3 = 0.8273$ and $z_4 = 1$.

The finite dimensional system (27) is defined by the following matrices:

$$\bar{A}_1 = \begin{bmatrix} 0 & 3.4915 & -1.5275 & 0.5180 \\ -2.6732 & -0.0000 & 2.6732 & -0.7500 \\ 1.5275 & -3.4915 & -0.0000 & 2.4820 \\ -2.8203 & 5.3333 & -13.5130 & 10 \end{bmatrix}$$

$$\bar{B}_1 = \begin{bmatrix} -2.4820 & -2.4820 & -2.4820 & -2.4820 \\ 0.7500 & 0.7500 & 0.7500 & 0.7500 \\ -0.5180 & -0.5180 & -0.5180 & -0.5180 \\ 1 & 1 & 1 & 1 \end{bmatrix} \bar{C}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\bar{A}_2 = \begin{bmatrix} 0 & 3.4915 & -1.5275 & 0.5180 \\ -2.6732 & -0.0000 & 2.6732 & -0.7500 \\ 1.5275 & -3.4915 & -0.0000 & 2.4820 \\ -2.8203 & 5.3333 & -13.5130 & 10 \end{bmatrix}, \bar{B}_2 = \begin{bmatrix} -2.4820 & -2.4820 \\ 0.75 & 0.75 \\ -0.5180 & -0.5180 \\ 1 & 1 \end{bmatrix},$$

$$\bar{C}_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\bar{A}_3 = \begin{bmatrix} 0 & 3.4915 & -1.5275 & 0.5180 \\ -2.6732 & -0.0000 & 2.6732 & -0.7500 \\ 1.5275 & -3.4915 & -0.0000 & 2.4820 \\ -2.8203 & 5.3333 & -13.5130 & 10 \end{bmatrix}, \qquad \bar{B}_3 = \begin{bmatrix} -2.4820 & -2.4820 \\ 0.75 & 0.75 \\ -0.5180 & -0.5180 \\ 1 & 1 \end{bmatrix} \qquad ,$$

$$\bar{C}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

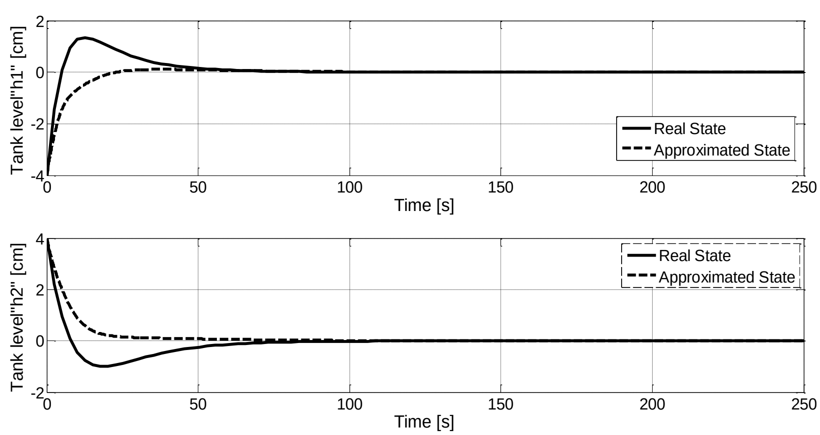
e. Simulation results

Simulation results for the two case studies (minimum and non-minimum phase models) are conducted for the initial condition $\chi(0)=\begin{bmatrix}-4 & 4 & 6 & -5 & 0_{3n\times q}\end{bmatrix}^T$ and the constant time-delays $\tau_1=5s$, $\tau_2=2s$ and $\tau_3=4s$. The input signal is given by $u(t)=\sin(2\pi ft)$ while the unknown input is given by $d(t)=0.3\sin(2\pi ft)$ and its related constant matrix is given by $\tilde{D}=\begin{bmatrix}D & 0_{(3n\times q)}\end{bmatrix}^T$ providing that $D=\begin{bmatrix}40 & 50 & 0 & 60\end{bmatrix}^T$.

The state variables (thank levels) and their approximation variables via collocation method for the minimum and non-minimum phase models are shown by Figure 2 and Figure 3, respectively.

Figure 4 shows the observer estimation errors for the minimum phase model case study for different values Q=0, Q=0.3, Q=0.6 and Q=0.9 whereas observer estimation errors for the second case study are shown by Figures 5.

It is noted that for the two case studies, the UIO converges asymptotically but more rapidly for the first case study.



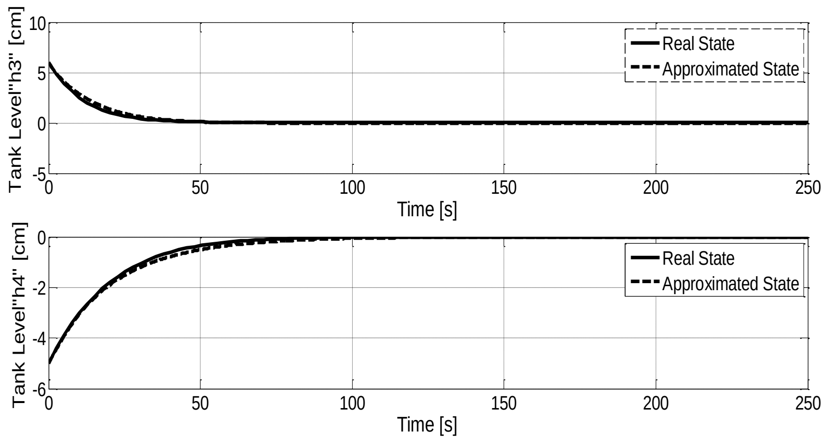


Figure 2. Tank levels and their approximation state variables via orthogonal collocation method: Minimum phase model case study

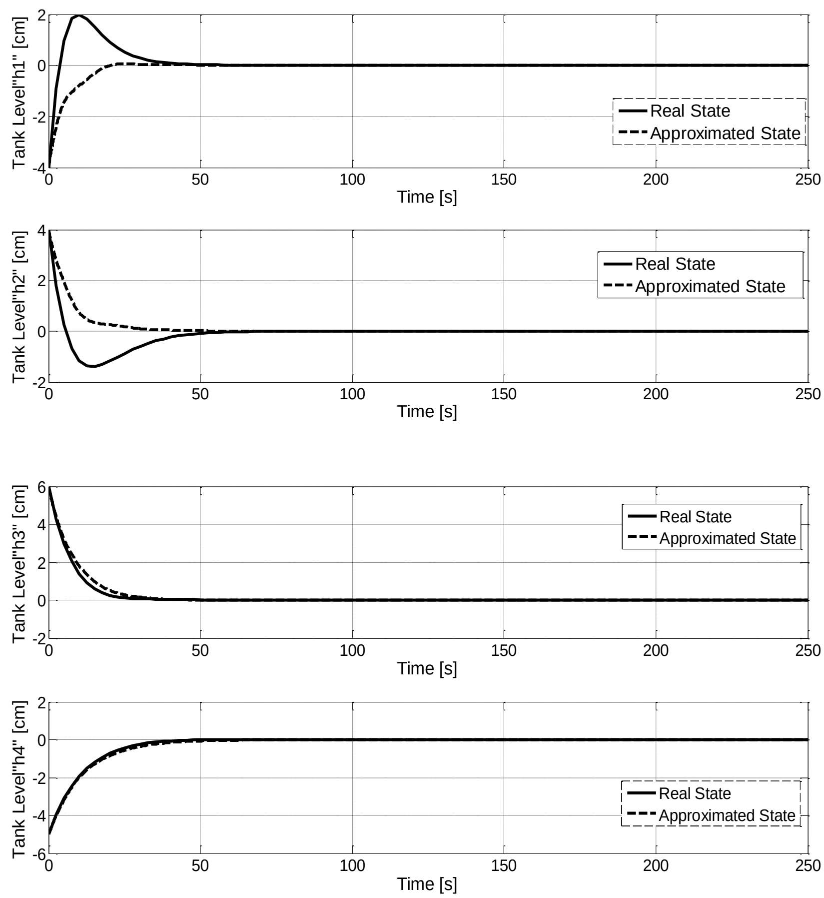


Figure 3. Tank levels and their approximation state variables via orthogonal collocation method: Non-minimum phase model case study

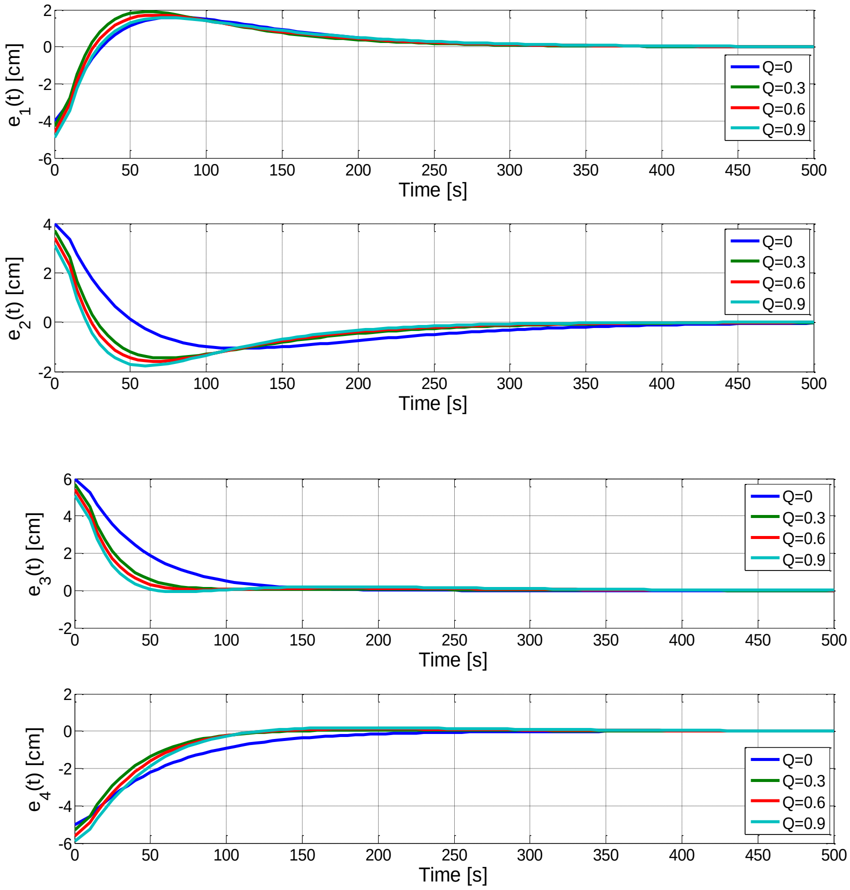
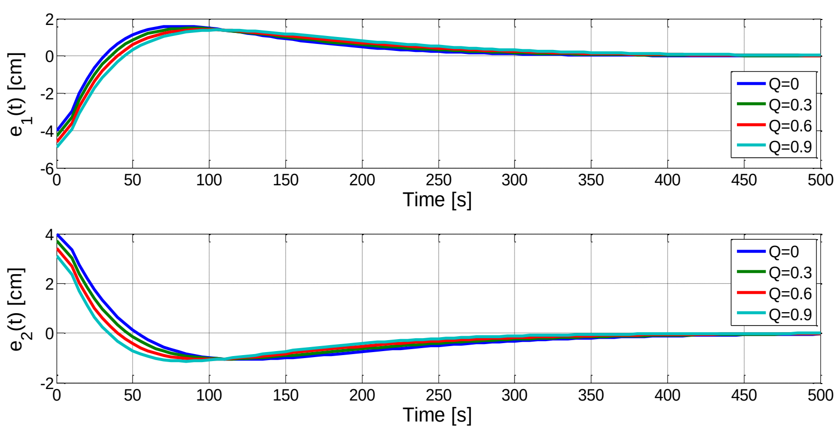


Figure 4. Observer error dynamics: Minimum phase model case study



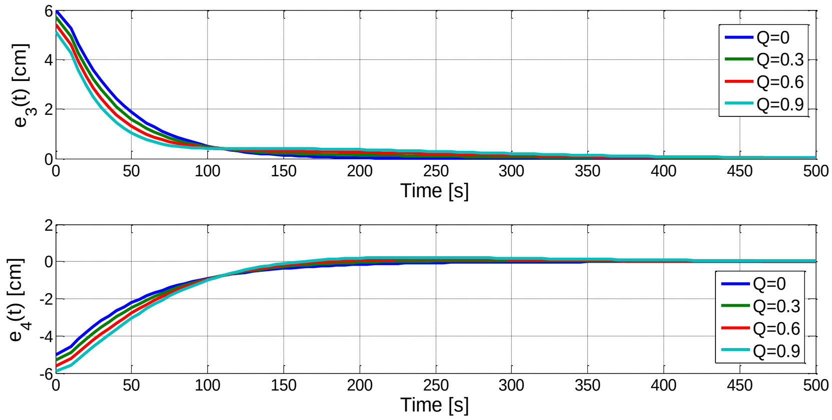


Figure 5. Observer error dynamics: Non-Minimum phase model case study

Figures 6 and 7 represent the error dynamics (28) related to the approximation error vector $r(t)$, defined between the time-delay system (1) and the lumped system (2) for the two case studies, respectively. For $i=1,2,3,4$, r_i (t) represent the estimation errors between the level of water in the tank i and its approximated level using the orthogonal collocation method.

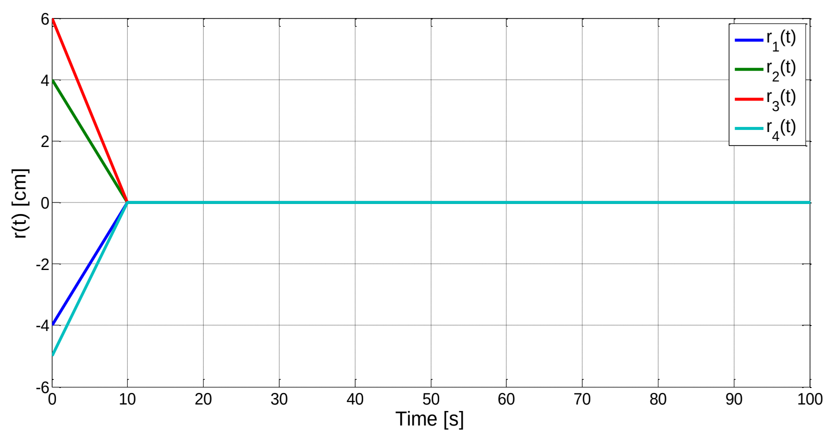


Figure 6. Error dynamics $\dot{r}(t)=Nr(t)$: Minimum phase model case study

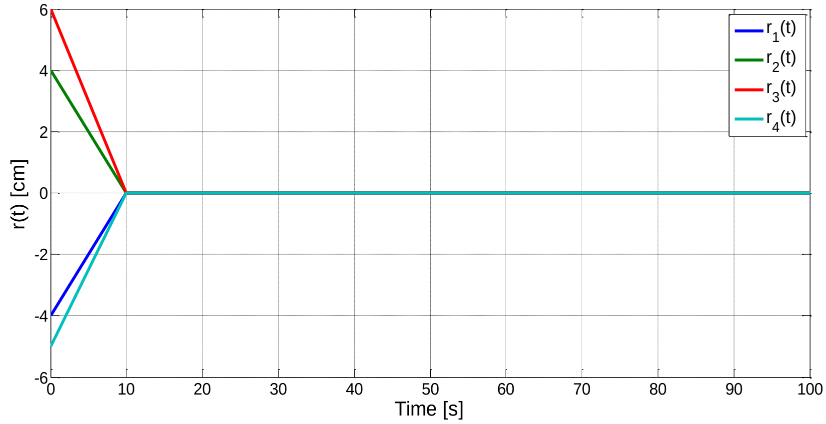


Figure 7. Error dynamics $\dot{r}(t) = Nr(t)$: Non-minimum phase model case study

VI. CONCLUSION

In this paper, we have presented a robust observer design for MIMO LTI systems with multiple time-delays and unknown inputs. A numerical example of a quadruple-tank benchmark has been used to illustrate the efficiency of the proposed method. The two case studies of minimum and non minimum phase models are used to show the asymptotic stability and the robustness of the designed observer.

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