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CONSENSUS OF CONTINUOUS-TIME MULTI-AGENT SYSTEMS UNDER NOISY MEASUREMENT

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Abstract- In this paper, we consider the consensus problem of first-order continuous-time multi-agent systems with fixed topology and time-varying topology in the presence of measurement noises. It is assumed that each agent can only obtain the information from its neighbors, and the information is corrupted by white noises. For the case of fixed topology, it is shown that consensus can be reached asymptotically in mean square provided the interaction topology has a spanning tree. For the case of time-varying topology, with the assumption that each interaction topology is balanced and strongly connected, consensus can be reached asymptotically in mean square as well. The convergence analysis is given by studying the reduced-order system with the help of stochastic Lyapunov analysis. Simulation results are presented to illustrate the theoretical results.

Index terms: Multi-agent systems, consensus, measurement noise, stochastic systems.

I. INTRODUCTION

In recent years, distributed coordination control of multi-agent systems has received considerable attention from various research communities. This is mainly due to its wide applications in many practical areas such as rendezvous, formation control, flocking, attitude alignment, and sensor networks[1-4]. As a critical issue in the coordination control of multi-agent systems, consensus means that all agents eventually reach an agreement via local interaction. Problems of consensus have been studied extensively.

In most existing works, the problem of consensus have been investigated in a noisy-free environment, namely, it has assumed that each agent can obtain accurate information from its neighbors. However, in many practical situations, such information is often corrupted by various noises. Hence it is more practical to consider consensus problems under noisy environment, and it has attracted the attention of some researchers[5-11]. In [5], average-consensus problems of first-order continuous-time multi-agent systems under measurement noises have been investigated. They have proved that mean square average-consensus can be reached if the interaction topology is balanced and has a spanning tree. In [6], the authors have extended the results of [5] to fixed and directed topology cases. In [9], they have investigated the consensus problem of multi-agent systems (MASs) with imperfect communication both in channels and in actuators, and used a Markov chain approach to describe the occurrence of the two types missing data in a unified framework. A sufficient consensus condition has been first obtained in terms of linear matrix inequalities. Then, based on this condition, a novel controller design method has been developed such that the MASs with imperfect communication reaches mean-square consensus. In [10], the authors have investigated the leader-following tracking consensus problem for high-order nonlinear dynamical multi-agent systems with switching topology and communication delay under noisy environments. In reality, the interaction topology may be directed and time-varying due to unreliable or limited communication/sensing range. So it is more practical to study the case when the interaction topology is time-varying[12-14]. To the best of our knowledge, there is few works concerning the consensus problem of continuous-time multi-agent systems with time-varying directed topologies and measurement noises.

In this paper, we focus on the consensus problem of first-order continuous-time multi-agent systems with fixed topology and time-varying topology under noisy measurements. Comparing with [5], this paper makes the following contributions. We prove that, as long as the interaction topology has a spanning tree which is a much weaker condition, the consensus can be reached asymptotically in mean square in the case of fixed topology. In addition, the case of time-varying topology is also taken into account in this paper. It is shown that the average-consensus can be achieved asymptotically in mean square with the assumption that each interaction topology is balanced and strongly connected. The convergence analysis is given by studying the reduced-order system with the help of stochastic Lyapunov function and matrix theory.

The following notations are used throughout this paper. Let I be an identity matrix with appropriate dimension, and **1** be a column vector of all ones with appropriate dimension. For a given matrix $A \in R^{n\times n}$, A^T denotes its transpose, $\|A\|$ denotes its Frobenius norm, and $\mathrm{tr}(A)$ denotes its trace. A matrix $A \in R^{n\times n}$ is said to be positively stable if all of its eigenvalues have a positive real part. For a given set S, $\chi(S)$ denotes its indicator function. For a given random variable ξ, $E(\xi)$ denotes its mathematical expectation, $\mathrm{Var}(\xi)$ denotes its variance. For given real numbers a and b, $a \wedge b$ denotes $\min\{a,b\}$.

II. PROBLEM FORMULATION

Consider a multi-agent system consisting of n agents labeled 1 through n. The interaction topology of n agents can be conveniently described by a digraph $G=(V,E)$, where $V=\{1,\cdots,n\}$ is the set of nodes and $E \subseteq V\times V$ is the set of edges of the graph. An edge of G is denoted by (i,j), representing that agent j can directly receive information from agent i. In this case, i is called the parent node of j and j the child node of i. Node j is a neighbor of node i if $(j,i)\in E$, where $j \neq i$. Denote the neighbors of node i by $N_i \subset V$. A path in G is a sequence $i_0,\cdots,i_m$ of distinct nodes such that $(i_{j-1},i_j)\in E$ for all $1\leq j \leq m$. A digraph G is said to be strongly connected if there is a path between any pair of distinct nodes. A directed tree is a digraph, where every node, except the root, has exactly one parent. A spanning tree of a digraph is a directed tree formed by graph edges that connect all the nodes of the graph[15].

The weighted adjacency matrix of digraph G is denoted by $A=[a_{ij}]\in R^{n\times n}$, where $a_{ij}>0$ if $(j,i)\in E$ and $a_{ij}=0$ otherwise. The in-degree of node i is defined as $\deg_{in}(i)=\sum_{j=1}^{n} a_{ij}$ and the out-degree is defined as $\deg_{out}(i)=\sum_{j=1}^{n} a_{ji}$. Its degree matrix $D=\mathrm{diag}\{d_1,\ldots,d_n\}$ is a diagonal matrix, whose diagonal elements $d_i=\deg_{in}(i)$. G is called a balanced digraph, if $\deg_{in}(i)=\deg_{out}(i), i=1,2,\ldots,n$. The Laplacian associated with the digraph G is defined as

$$L=D-A.$$

Below is an important property of Laplacian L associated with G.

Lemma 1[15] Zero is an eigenvalue of L, and **1** is the associated right eigenvector. In addition, zero is a simple eigenvalue of L and all the other eigenvalues have positive real parts if and only if the digraph G has a spanning tree.

Owing to the existence of disturbances and subjecting to communication range limitations, the interaction topology may change dynamically. To describe the variable topology, we define a piecewise constant switching signal $\sigma(t):[0,+\infty)\to\Lambda=\{1,2,\ldots,N\}$, where N denotes the total number of all possible graphs describing the interconnection topology. We assume Λ is a finite set. For convenience of exposition, suppose that the time-interval $[0,\infty)$ is constituted by an infinite sequence of nonempty, non-overlapping, and contiguous subintervals $[t_k,t_{k+1})$ for $k=0,1,\ldots,$with $t_0=0$, and during each of such subintervals, the interaction topology described by $G_{\sigma(t_k)}$ does not change.

In this paper, we consider the following first-order continuous-time system of n agents:

$$\dot{x}_i(t)=u_i(t), i=1,\ldots,n, \quad (1)$$

where $x_i(t)\in R$ and $u_i(t)\in R$ are the state and control input of agent i ,respectively.

For each agent, we assume that the obtained state of its neighbors is corrupted by white noises. In this case, the state of agent j obtained by agent i can be described as:

$$y_{ji}(t)=x_j(t)+\varrho_{ji}n_{ji}(t), j\in N_i(t), \quad (2)$$

where $N_i(t)$ is the neighbor set of agent i at time t, $w_{ji}(t)$ is an independent normal white noise, $\varrho_{ji}\geq 0$ is the noise intensity. Meanwhile, we assume that each agent knows its own state exactly.

Our control goal is to let the states of all the agents converge to a common value in the sense of mean square. For this end, we use the following distributed protocol:

$$u_i(t) = a(t)\sum_{j=1}^{n} a_{ij}(t)(y_{ji}(t) - x_i(t)), i = 1,\ldots,n, \tag{3}$$

where $a(t):[0,\infty) \to (0,\infty)$ is piecewise continuous, called consensus-gain function. In order to reduce the detrimental effect of the noise, we assume that the consensus-gain function satisfies the following assumptions.

(A1) $\int_0^\infty a(s)ds = \infty$

(A2) $\int_0^\infty a^2(s)ds < \infty$

Remark 1.Similar to the condition on the step size in stochastic approximation [16], Assumption (A2) implies that $a(t) \to 0$ as $t \to \infty$, which further implies that the impact of the noise will be attenuated as time goes on; Assumption (A1) means that $a(t)$ cannot decrease too fast, otherwise the agents may prematurely converge to different individual limits.

III. MAIN RESULTS

A. Fixed topology

In this subsection, we consider the case of fixed topology. It was shown that having a spanning tree is the minimum requirement to guarantee consensus in the case of fixed topology without measurement noises. As a matter of fact, having a spanning tree implies that there is at least an agent whose information can be shared by all the other agents. Naturally, if the interaction topology has a spanning tree, even in the presence of measurement noises consensus could still be reached in certain sense by means of appropriate protocol.

Applying protocol (3) to system (1) yields:

$$\dot{x}_i(t) = a(t)\sum_{j=1}^{n} a_{ij}(y_{ji}(t) - x_i(t)), i = 1,\ldots,n. \tag{4}$$

Let α_i be the i th row of the adjacency matrix A of digraph G. Denote $x(t) = (x_1(t),\ldots,x_n(t))^T \in R^n$, $\Sigma_i = \text{diag}(\sigma_{1i},\ldots,\sigma_{ni}) \in R^{n\times n}$, $D = \text{diag}(\alpha_1^T\Sigma_1,\ldots,\alpha_n^T\Sigma_n)$ is an $n\times n^2$

dimensional block diagonal matrix, $\omega_i(t)=(\omega_{1i},\dots,\omega_{ni})^T\in R^n$, $\omega(t)=(\omega_1^T(t),\dots,\omega_n^T(t))^T\in R^{n^2}$. Then system (4) can be written in a compact form:

$$\dot{x}(t)=-a(t)Lx(t)+a(t)D\omega(t). \tag{5}$$

Since that L is not positive stable, most existing results in stochastic approximation cannot be directly applied to the convergence analysis of system (5). To deal with this problem, we need the following lemma.

Lemma2.[7] Suppose that G has a spanning tree. Denote $C(L)=\{B\in R^{n\times(n-1)}\,|\,\text{span}\{B\}=\text{span}\{L\}\}$, where L is the Laplician of G, $\text{span}(B)$ and $\text{span}(L)$ denote the span of the column vectors of B and L, respectively. For any given $Q_1\in C(L)$, the matrix $Q=(1,Q_1)$ is nonsingular and

$$Q^{-1}LQ=\begin{pmatrix}0 & \\ & H\end{pmatrix}$$

where H is positive stable. In addition, let $Q^{-1}=\begin{pmatrix}q^T\\ Q_2\end{pmatrix}$, where $q\in R^n$, then $q^TL=0$ and $q^T1=1$.

Note that the vector q in Lemma 2 is unique. Denote $y(t)=Q^{-1}x(t)$ and write $y(t)=(y_1(t),\hat{y}^T(t))^T$, where $\hat{y}(t)\in R^{n-1}$. From (5) we have

$$\dot{y}_1(t)=a(t)q^TD\omega(t), \tag{6}$$

$$\dot{\hat{y}}(t)=-a(t)H\hat{y}(t)+a(t)Q_2D\omega(t), \tag{7}$$

and we have the following relationship:

$$x(t)=y_1(t)1+Q_1\hat{y}(t); \tag{8}$$

$$y_1(t)=q^Tx(t), \tag{9}$$

which will be used in the sequel.

Let us agree to say that system (7) is the reduced-order system of (5). Noticing that H is positive stable, stochastic Lyapunov analysis can be employed to the convergence analysis of the reduced-order system. Next, we shall study the reduced-order system by using stochastic Lyapunov analysis.

Theorem 1**.** Consider system (7). Assume G has a spanning tree and Assumptions (A1) and (A2) hold, then

$$\lim_{t\to\infty} E\|\hat{y}(t)\|^2 = 0.$$

Proof. For convenience, we write (7) in the form of stochastic differential equation:

$$d\hat{y}(t) = -a(t)H\hat{y}(t)dt + a(t)Q_2DdW(t), \tag{10}$$

where $W(t)$ is an n^2-dimensional standard Brownian motion.

Since H is positive stable, by Lyapunov theorem, there exists a positive definite matrix P such that

$$PH + H^TP = I.$$

Choose stochastic Lyapunov function

$$V(t) = \hat{y}^T(t)P\hat{y}(t).$$

By use of Itô formula, from (10) we have

$$dV(t) = -a(t)\hat{y}^T(t)\hat{y}(t)dt + C_0a^2(t)dt + 2a(t)\hat{y}^T(t)PQ_2DdW(t), \tag{11}$$

where $C_0 = \mathrm{tr}(PQ_2DD^TQ_2^T)$. It is clear that

$$\frac{1}{\lambda_{\max}(P)}V(t) \le \hat{y}^T(t)\hat{y}(t) \le \frac{1}{\lambda_{\min}(P)}V(t), \tag{12}$$

where $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and minimum eigenvalue of P, which leads to

$$dV(t) \le -\frac{a(t)}{\lambda_{\max}(P)}V(t)dt + C_0a^2(t)dt + 2a(t)\hat{y}^T(t)PQ_2DdW(t). \tag{13}$$

We claim that

$$E\left[\int_{t_0}^{t} a(s)\hat{y}^T(s)PQ_2DdW(s)\right] = 0, \forall t \ge t_0. \tag{14}$$

To establish this fact, let $\tau_\delta^{t_0} = \inf\{t \ge t_0 : V(t) \ge \delta\}$ for any $t_0 \ge 0, T \ge t_0$, where δ is a given positive number if $V(t) \ge \delta$ for some $t \in [t_0, T]$; otherwise $\tau_\delta^{t_0} = T$. From (13), we have

$$\begin{aligned}
&E[V(t\wedge\tau_\delta^{t_0})\chi_{t\le\tau_\delta^{t_0}}] - E[V(t_0)]\\
&\le -\frac{1}{\lambda_{\max}(P)}\int_{t_0}^{t} a(s)V(s\wedge\tau_\delta^{t_0})\chi_{t\le\tau_\delta^{t_0}}ds + C_0\int_{t_0}^{t} a^2(s)ds\\
&\le C_0\int_{t_0}^{t} a^2(s)ds.
\end{aligned}$$

It follows that there exists a constant $\Delta_{t_0,T}$, such that

$$E[V(t\wedge\tau_\delta^{t_0})\lambda_{t\le\tau_\delta^{t_0}}] \le \Delta_{t_0,T}.$$

Then, by Fatou lemma, we have

$$\sup_{t_0 \le t \le T} E[V(t)] \le \Delta_{t_0,T}.$$

Therefore,

$$\begin{aligned} & E\left[\int_{t_0}^{t} a^2(s)V(s)ds\right] \\ & \le \sup_{t_0 \le s \le t} E[V(s)]\int_0^T a^2(s)ds \\ & < \infty. \end{aligned}$$

At the same time, note that

$$\begin{aligned} & E\left[\int_{t_0}^{t} a^2(s)\left\|\hat{y}^T(t)PQ_2D\right\|^2 ds\right] \\ & \le C_1 E\left[\int_{t_0}^{t} a^2(s)V(s)ds\right], \end{aligned}$$

where $C_1 = \frac{1}{\lambda_{\min}(P)}\left\|PQ_2D\right\|^2$. By the property of Ito integral , (14) is verified.

From (13), we have

$$\begin{aligned} & E[V(t)] - E[V(t_0)] \\ & \le -\frac{1}{\lambda_{\max}(P)}\int_{t_0}^{t} a(s)E[V(s)]ds + C_0\int_{t_0}^{t} a^2(s)ds. \end{aligned}$$

By the comparison theorem, we have

$$\begin{aligned} & E[V(t)] \\ & \le E[V(t_0)]\exp\left\{-\frac{1}{\lambda_{\max}(P)}\int_{t_0}^{t} a(s)ds\right\} \\ & \quad + C_0\int_{t_0}^{t} a^2(s)\exp\left\{-\frac{1}{\lambda_{\max}(P)}\int_{s}^{t} a(u)du\right\}ds. \end{aligned}$$

Noticing that

$$\begin{aligned} & \int_{t_0}^{t} a^2(s)\exp\left\{-\frac{1}{\lambda_{\max}(P)}\int_{s}^{t} a(u)du\right\}ds \\ & \le \exp\left\{-\frac{1}{\lambda_{\max}(P)}\int_{t_0}^{t} a(s)ds\right\}\int_{t_0}^{t} a^2(s)V(s)ds, \end{aligned}$$

recalling Assumptions (A1) and (A2), we have

$$\lim_{t\to\infty}\int_{t_0}^{t} a^2(s)\exp\left\{-\frac{1}{\lambda_{\max}(P)}\int_s^t a(u)du\right\}ds = 0$$

At the same time, recalling Assumption (A1) again, we have

$$\lim_{t\to\infty}\exp\left\{-\frac{1}{\lambda_{\max}(P)}\int_{t_0}^{t} a(s)ds\right\} = 0.$$

As a result,

$$\lim_{t\to\infty} E[V(t)] = 0.$$

From (12), the conclusion follows.

Theorem 2. Consider system (5). Assume G has a spanning tree and Assumptions (A1) and (A2) hold, then the n agents reach consensus asymptotically in mean square. That is, there is a random variable x^*, such that

$$\lim_{t\to\infty} E\left|x_i(t) - x^*\right|^2 = 0, i = 1,\dots,n. \tag{15}$$

In addition, $E[x^*] = q^T x(0), \mathrm{var}(x^*) = q^T DD^T q\int_0^\infty a^2(s)ds$.

Proof. Using the fact that $\hat{y}^T(t)Q_1^T Q_1\hat{y}(t) \le \|Q_1\|^2 \hat{y}^T(t)\hat{y}(t)$, invoking Theorem 1, we have

$$\lim_{t\to\infty} E\|Q_1\hat{y}(t)\|^2 = 0.$$

Then, it follows from (8) that

$$\lim_{t\to\infty} E\|x(t) - y_1(t)1\|^2 = 0. \tag{16}$$

It follows from (5) and Lemma 2 that

$$q^T\dot{x}(t) = a(t)q^T D\omega(t).$$

Recalling (9), we have

$$y_1(t) = q^T x(0) + q^T D\int_0^t a(s)dW(s).$$

Let

$$x^* = q^T x(0) + q^T D\int_0^\infty a(s)dW(s), \tag{17}$$

which is well defined since $\int_0^\infty a^2(s)ds < \infty$. Therefore, by the Ito isometry, we have

$$\begin{aligned}&\lim_{t\to\infty} E(y_1(t)-x^*)^2\\&=\lim_{t\to\infty} E\left(q^T D\int_t^\infty dW(s)\right)^2\\&=\lim_{t\to\infty} q^T DD^T q\int_t^\infty a^2(s)ds\\&=0.\end{aligned}$$

This together with Theorem 1 leads to (15).

In addition, from (17) we have

$$E[x^*]=q^T x(0).$$

$$\begin{aligned}Var(x^*) &= E\left(q^T D\int_0^\infty a(s)dW(s)\right)^2\\&= q^T DD^T q\int_0^\infty a^2(s)ds.\end{aligned}$$

Remark 2. Comparing with [5], here we only require that G has a spanning tree, which is much weaker than being balanced. In addition, if G is balanced and has a spanning tree, then $q=\frac{1}{n}\mathbf{1}$, which implies that the n agents reach average consensus asymptotically in mean square. That is, Theorem 3.3 in [5] is a special case of this theorem.

B. Time-varying topology

In this subsection, we consider the case of time-varying topology. Let $\alpha_i(t)$ be the ith row of the adjacency matrix $A_{\sigma(t)}$ of digraph $G_{\sigma(t)}$, and $\Sigma_i=\text{diag}(\varrho_{1i},\ldots,\varrho_{ni})\in R^{n\times n}$, where $\varrho_{ji}=0,\ j\notin N_i(t)$, $D_{\sigma(t)}=\text{diag}(\alpha_1^T\Sigma_1,\ldots,\alpha_n^T\Sigma_n)$ is an $n\times n^2$ dimensional block diagonal matrix. $\eta_i(t)=(n_{1i}(t),\ldots,n_{ni}(t))^T,\omega(t)=(\eta_1^T(t),\ldots,\eta_n^T(t))^T$. Applying protocol (3) to system (1), gives

$$\frac{dx(t)}{dt}=-a(t)L_{\sigma(t)}x(t)+a(t)D_{\sigma(t)}\omega(t) \tag{18}$$

It is a system driven by an N^2-dimensional standard white noise, which can be written in the form of the Itô stochastic differential equation

$$dx(t)=(-a(t)L_{\sigma(t)}x(t))dt+a(t)D_{\sigma(t)}dW(t)$$

Where $W(t)=(W_{11}(t),\ldots,W_{n1}(t),\ldots,W_{nn}(t))^T$ is an N^2-dimensional standard Brownian motion.To proceed, we need the following assumption:

(A3) $G_{\sigma(t)}$ is balanced and strongly connected for all $t\geq 0$.

Under (A3), there exists an orthogonal matrix $U=(\frac{1}{\sqrt{n}}\mathbf{1},U_1)$ such that

$$U^T L_{\sigma(t)} U = \begin{pmatrix} 0 & 0 \\ 0 & H_{\sigma(t)} \end{pmatrix}$$

Where $H_{\sigma(t)}$ is positively stable. Denote $y(t)=U^T x(t)$ and write $y(t)=(y_1(t),\hat{y}^T(t))^T$, where $\hat{y}(t)\in R^{n-1}$. From (18), we have

$$\frac{dy_1(t)}{dt}=a(t)\frac{1}{\sqrt{n}}1^T D_{\sigma(t)}\omega(t) \tag{19}$$

$$\frac{d\hat{y}(t)}{dt}=-a(t)H_{\sigma(t)}\hat{y}(t)+a(t)U_1^T D_{\sigma(t)}\omega(t) \tag{20}$$

and the following relationship:

$$x(t)=\frac{1}{\sqrt{n}}y_1(t)\mathbf{1}+U_1\hat{y}(t) \tag{21}$$

$$y_1(t)=\frac{1}{\sqrt{n}}\mathbf{1}^T x(t) \tag{22}$$

which will be used in the sequel.

Theorem 3.Consider system (20). Assume that Assumptions (A1)-(A3) hold, then

$$\lim_{t\to\infty} E\left\|\hat{y}(t)\right\|^2=0$$

Proof. Choose stochastic Lyapunov function

$$V(t)=\hat{y}^T(t)\hat{y}(t)$$

It follows from (20) and Itô formula that

$$\begin{aligned} dV(t)&=-a(t)\hat{y}^T(t)(H_{\sigma(t)}+H_{\sigma(t)}^T)\hat{y}(t)dt \\ &\quad+a^2(t)C_0dt+2a(t)\hat{y}^T(t)U_1^T D_{\sigma(t)}dW(t) \end{aligned}$$

where $C_0=\mathrm{tr}(U_1^T D_{\sigma(t)}D_{\sigma(t)}^T U_1)$. Since $G_{\sigma(t)}$ is balanced and strongly connected, $L_{\sigma(t)}+L_{\sigma(t)}^T$ is the laplacian of the mirror associated with $G_{\sigma(t)}$ [16], and hence zero is a simple eigenvalue of $L_{\sigma(t)}+L_{\sigma(t)}^T$. Thus, $H_{\sigma(t)}+H_{\sigma(t)}^T$ is positively definite. Therefore,

$$\begin{aligned} dV(t)&\le -a(t)\lambda^* V(t)dt+C_0^* a^2(t)dt \\ &\quad+2a(t)\hat{y}^T(t)U_1^T D_{\sigma(t)}dW(t) \end{aligned} \tag{23}$$

Where $\lambda^* = \min_{t\geq 0}\{\lambda^+_{\min}(L_{\sigma(t)} + L^T_{\sigma(t)}) \mid \lambda^+_{\min}(L_{\sigma(t)} + L^T_{\sigma(t)})$ denotes the minimum nonzero eigenvalue of $L_{\sigma(t)} + L^T_{\sigma(t)}\}$, $C_0^* = \min_{t\geq 0}\left\{\mathrm{tr}(U_1^T D_{\sigma(t)} D^T_{\sigma(t)} U_1)\right\}$, which are well defined since Λ is a finite set.

We claim that

$$E\left[\int_{t_0}^t a(s)\hat{y}^T(s)U_1^T D_{\sigma(s)} dW(s)\right] = 0, \forall t \geq t_0 \tag{24}$$

To establish this fact, let $\tau_\delta^{t_0} = \inf\{t \geq t_0 : V(t) \geq \delta\}$ for any $t_0 \geq 0, T \geq t_0$, where δ is a given positive number if $V(t) \geq \delta$ for some $t \in [t_0, T]$; otherwise $\tau_\delta^{t_0} = T$. From (23), we have

$$\begin{aligned}
&E[V(t \wedge \tau_\delta^{t_0})\chi_{t\leq \tau_\delta^{t_0}}] - E[V(t_0)] \\
&\leq -\lambda^* \int_{t_0}^t a(s)E\left[V(s \wedge \tau_\delta^{t_0})\chi_{t\leq \tau_\delta^{t_0}}\right]ds + C_0^* \int_{t_0}^t a^2(s)ds \\
&\leq C_0^* \int_{t_0}^t a^2(s)ds.
\end{aligned}$$

It follows that there exists a constant $\Delta_{t_0,T}$, such that $E[V(t \wedge \tau_\delta^{t_0})\lambda_{t\leq \tau_\delta^{t_0}}] \leq \Delta_{t_0,T}$. Then, by Fatou lemma [18], we have $\sup_{t_0\leq t\leq T} E[V(t)] \leq \Delta_{t_0,T}$. Therefore,

$$\begin{aligned}
&E\left[\int_{t_0}^t a^2(s)V(s)ds\right] \\
&\leq \sup_{t_0\leq t\leq T} E[V(s)]\int_0^T a^2(s)ds \\
&< \infty.
\end{aligned}$$

At the same time, note that

$$\begin{aligned}
&E\left[\int_{t_0}^t a^2(s)\left\|\hat{y}^T(t)U_1^T D_{\sigma(s)}\right\|^2 ds\right] \\
&\leq C_1^* E\left[\int_{t_0}^t a^2(s)V(s)ds\right],
\end{aligned}$$

where $C_1^* = \frac{C_1}{\lambda_{\min}(P)}$ with $C_1 = \max_{t\geq 0}\{\left\|U_1^T D_{\sigma(t)}\right\|\}$. By the property of Ito integral [19], (24) is verified.

From (23), we have

$$\begin{aligned}
&E[V(t)] - E[V(t_0)] \\
&\leq -\lambda^* \int_{t_0}^t a(s)E[V(s)]ds + C_0^* \int_{t_0}^t a^2(s)ds
\end{aligned}.$$

By the comparison theorem [20], we have

$$E[V(t)] \leq E[V(t_0)] \exp\left\{-\lambda^* \int_{t_0}^t a(s)ds\right\} + C_0 \int_{t_0}^t a^2(s) \exp\left\{\lambda^* \int_s^t a(u)du\right\} ds.$$

Noticing that

$$\int_{t_0}^t a^2(s) \exp\left\{\lambda^* \int_s^t a(u)du\right\} ds \leq \exp\left\{-\lambda^* \int_{t_0}^t a(s)ds\right\} \int_{t_0}^t a^2(s) V(s) ds,$$

recalling Assumptions (A1) and (A2), we have

$$\lim_{t\to\infty} \int_{t_0}^t a^2(s) \exp\left\{\lambda^* \int_s^t a(u)du\right\} ds = 0$$

At the same time, recalling Assumption (A1) again, we have

$$\lim_{t\to\infty} \exp\left\{\lambda^* \int_{t_0}^t a(s)ds\right\} = 0.$$

As a result, $\lim_{t\to\infty} E[V(t)] = 0$, which implies the conclusion.

Theorem 4. Consider system (18). Assume that Assumptions (A1)-(A3) hold, then the n agents reach mean square average-consensus, namely, there exists a random variable x^* such that

$$\lim_{t\to\infty} E\,|\,x_i(t) - x^*\,|^2 = 0, i = 1, \ldots, n,$$

and

$$E[x^*] = \frac{1}{n}\sum_{i=1}^n x_i(0)$$

Proof Using the fact that $\hat{y}^T(t) U_1^T U_1 \hat{y}(t) \leq \left\|U_1^T U_1\right\| \hat{y}^T(t)\hat{y}(t)$, invoking Theorem 3, we have $\lim_{t\to\infty} E\left\|U_1 \hat{y}(t)\right\|^2 = 0.$ Then, it follows from (21) that

$$\lim_{t\to\infty} E\left\|x(t) - \frac{1}{\sqrt{n}} y_1(t)\mathbf{1}\right\|^2 = 0 \tag{25}$$

It follows from (18) that $\mathbf{1}^T \frac{dx}{dt} = a(t)\mathbf{1}^T D_{\sigma(t)}\omega(t).$ Then, we have

$$\mathbf{1}^T x(t) = \mathbf{1}^T x(0) + \sum_{i=0}^{m-1} \mathbf{1}^T D_{\sigma(t_i)} \int_{t_i}^{t_{i+1}} a(s)dW(s) + \mathbf{1}^T D_{\sigma(t_m)} \int_{t_m}^t a(s)dW(s),\ t_m < t \leq t_{m+1}. \tag{26}$$

Let $x^* = \lim_{t\to\infty} \frac{1}{\sqrt{n}} y_1(t) = \lim_{t\to\infty} \frac{1}{n}\mathbf{1}^T x(t)$, which is well defined by recalling $\int_0^\infty a^2(s)ds < \infty$, and we have

$$\begin{aligned} & E\left|\frac{1}{n}\mathbf{1}^T x(t) - x^*\right|^2 \\ &= \frac{1}{n} E\left|\mathbf{1}^T D_{\sigma(t_m)} \int_t^{t_{m+1}} a(s)dW(s) + \sum_{i=m+1}^{\infty} \mathbf{1}^T D_{\sigma(t_i)} \int_{t_i}^{t_{i+1}} a(s)dW(s)\right|^2 \\ &\le \frac{1}{n} \max_{t\ge 0} \left\{\mathbf{1}^T D_{\sigma(t)} D_{\sigma(t)}^T \mathbf{1} \int_t^\infty a^2(s)ds\right\}. \end{aligned}$$

Thus, again by Assumption (A2) we have $\lim_{t\to\infty} E\left|\frac{1}{n} 1^T x(t) - x^*\right|^2 = 0$. Noticing that $E[x^*] = \frac{1}{n}\mathbf{1}^T x(0)$, from (25), the conclusion follows.

IV. NUMERICAL EXAMPLE

In this section, two examples are provided to illustrate our theoretical results. The two examples consider the case of fixed topology and time-varying topology, respectively. Choose $a(t) = \frac{1}{1+t}$. It is clear that Assumptions (A1) and (A2) are satisfied. Suppose that $a_{ij} = 1$ when $(j,i) \in E$, $a_{ij} = 0$ otherwise, and the noise intensity $\varrho_{ji} = 1$ when $a_{ij} = 1$.

Example 1. Consider a multi-agent system consisting of four agents with the interaction topology described as Figure 1. It is clear that G has a spanning tree. Let the initial positions of agents be $x_1(0) = 1, x_2(0) = 2, x_3(0) = 3, x_4(0) = -1$. It can be seen from Figure 2 that all agents reach consensus asymptotically under noisy measurements.

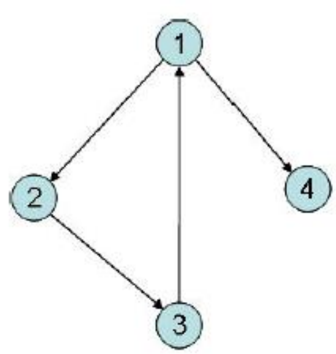

Figure 1. Interaction topology G

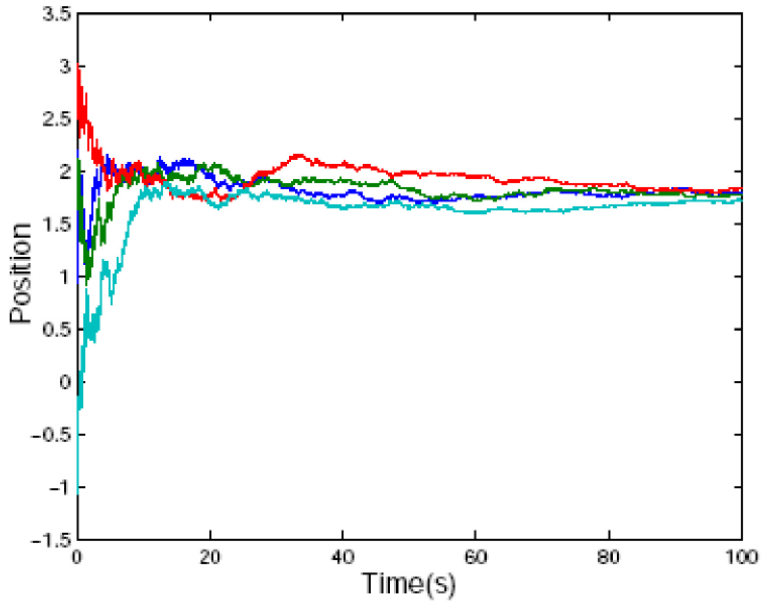

Figure 2. Curves of states of Example 1.

Example 2. Consider a multi-agent system consisting of four agents. Suppose that the interaction topology is time-varying of switching period 1 between two graphs $G_i(i=1,2)$ described as Figure 3. It is clear that Assumption (A3) holds. Let the initial position of agents be $x_1(0)=2, x_2(0)=1, x_3(0)=-3, x_4(0)=4$. Figure 4 shows that the four agents reach consensus asymptotically.

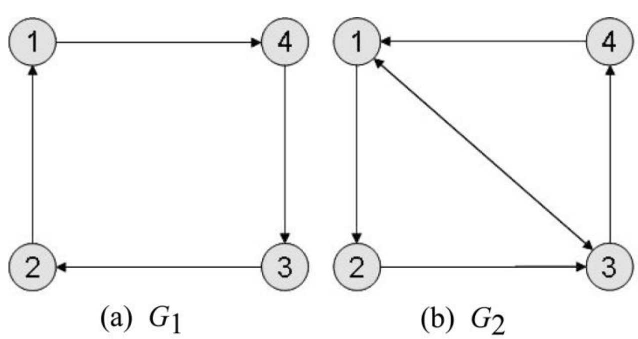

Figure 3. Switching topologies: G_1 and G_2

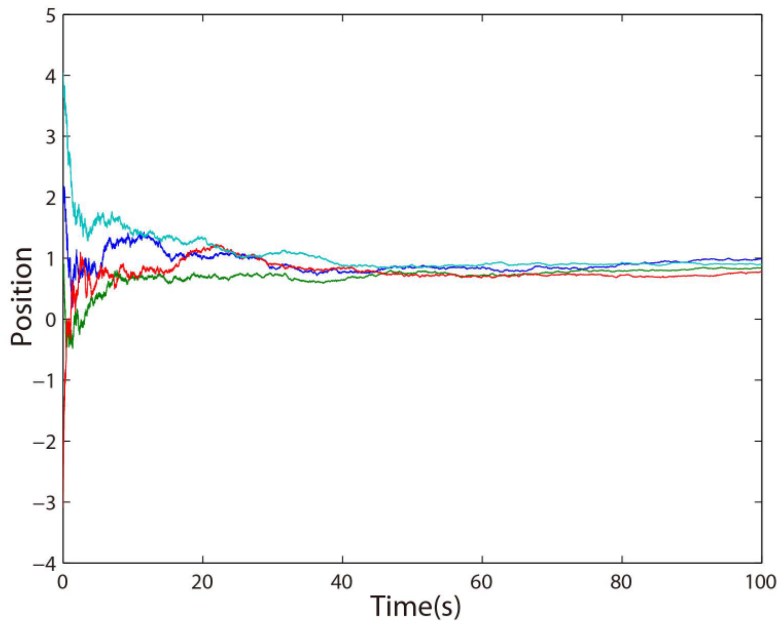

Figure 4. Curves states of Example 2.

V. CONCLUSIONS

In this paper, consensus problems of first-order continuous-time multi-agent systems with fixed and time-varying topology in the presence of measurement noises are investigated. Stochastic Lyapunov analysis and matrix theory are employed in the convergence analysis of the multi-agent system. Through studying the reduced-order system, we proved that consensus can be reached asymptotically in mean square for both fixed and time-varying topology cases. In the case of fixed topology, we only require that the interaction topology has a spanning tree, which is a much weaker condition. In the case of time-varying topology, with the assumption that each interaction topology is balanced and strongly connected, average consensus can be reach asymptotically in mean square.

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