www.s2is.org
International Journal on Smart Sensing and Intelligent Systems
ISSN 1178-5608

PARETO OPTIMAL ROBUST FEEDBACK LINEARIZATION CONTROL OF A NONLINEAR SYSTEM WITH PARAMETRIC UNCERTAINTIES

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Submitted: Dec. 5, 2014 ***Accepted: Feb. 17, 2014*** ***Published: Mar. 10, 2014***

Abstract- The problem of multi-objective robust feedback linearization controller design of nonlinear system with parametric uncertainties is solved in this paper. The main objective of this paper is to propose an optimal technique to design a robust feedback linearization controller with multi-objective genetic algorithm. A nonlinear system is considered as a benchmark and feedback linearization controller is designed for deterministic and probabilistic model of the benchmark. Three and four conflicting objective functions are used in Pareto design of feedback linearization controller for deterministic and probabilistic design, respectively. The simulation results reveal the effectiveness of the proposed method.

Index terms: Pareto, Feedback Linearization, Parametric Uncertainty, Robust Design.

I. INTRODUCTION

Feedback linearization is a nonlinear control method which is used a coordinate transformation to change a nonlinear system into a linear one [1]. The obtained linear system can be controlled by a linear control method, such as state feedback control. Therefore, the feedback linearization method can be classified into two parts: one part cancels out the nonlinearities of the plant, and other part controls the linearized system [2]. There are many methods to design such controllers which can be regarded as optimization problems of certain performance measures of the controlled systems [3]. A very effective means of solving such optimum design of controllers is genetic algorithms (GAs) [4, 5]. The simplicity and global characteristics of such evolutionary methods have been the main reasons for their extensive applications in off-line optimum control system design. However, a major restriction of feedback linearization method is that the system should be precisely known, in order to cancel the nonlinearities exactly [1]. In real control engineering problems, however, exact models of the systems are not often known [6]. Additionally, there are many sources of uncertainties such as parametric uncertainties that prevent the effectiveness of such feedback linearization [7]. Therefore, it is strongly needed to design a robust feedback linearization method which can stand the uncertainties and provides satisfactory both robust stability and robust performance in the presence of uncertainties and modeling error.

In this regard, many researches have been accomplished in robust feedback linearization method [8, 9]. Most of them are based on the worst case robust analysis and synthesis which use the most pessimistic value of the performance for a particular member of the set of uncertain models [10]. The conservatism involved in such approaches is not very desirable in robust control design and the most likely plants with respect to uncertainties should be considered [11]. There have been many efforts for designing robust controllers reducing the conservatism or accounting more for the most likely plants with respect to uncertainties. This idea results in propagating the probabilistic uncertainties through the uncertain model which builds the set of plants as the actual dynamic systems each with a separate probability density function (PDF) [12]. Therefore, such information regarding the likelihood of each plant allows a reliability-based design in which probability is incorporated in the robust design [13]. In this method, robustness and performance are stochastic variables [12]. Stochastic behavior of the system can be simulated by Monte-Carlo Simulation (MCS) [6].

Robustness and performance can be considered as objective functions with respect to the controller parameters in an optimization problem. Wang et al. [9] used genetic algorithms to design robust nonlinear controls, in which both stability and performances metrics was augmented for a single objective optimization problem. Since conflictions exist between robustness and performance metrics, choosing appropriate weighting factor in the cost function consisting of weighted quadratic sum of those non-commensurable objectives is inherently difficult and could be regarded as a subjective design concept. Moreover, the trade-offs existed between those objectives cannot be explored and it would be, therefore, impossible to choose an appropriate optimum design reflecting the compromise of the designer’s choice concerning the absolute values of objective functions. Therefore, this problem should be formulated as a multi objective optimization problem (MOP) so that the trade-offs between objectives can be found consequently [14, 15].

The aim of this paper is to use a multi-objective optimization approach to robustly design a feedback linearized controller for an uncertain nonlinear system. A nonlinear two masses and spring system is considered and the nonlinear controller will be designed for both deterministic and probabilistic situations. In this regard, three non-commensurable objective functions are considered for deterministic design and four non-commensurable objective functions are considered for probabilistic one. The comparison will be accomplished to show the robustness and superiority of the probabilistic design over the deterministic design in the presence of parametric uncertainties of the nonlinear model.

The organization of the paper is as follows. After a general introduction of the effect of fault on the power system, the usefulness and requirement of a fault current limiter is presented to the students which has been discussed in section II. The traditional ways of fixing fault currents in power system has been discussed in section III. In section IV, operating principle, design details, and experimental results of magnetic current limiter has been presented. The analysis and simulation results of high temperature superconducting fault current limiter has been discussed in section V. The lecture has been concluded in section VI.

II. ROBUST FEEDBACK LINEARIZATION DESIGN

Consider a nonlinear system as follow

$$\begin{aligned} x^{(n)} &= f(\mathbf{x}) + g(\mathbf{x})u(t) + q(\mathbf{x})w(t) \\ y &= h(\mathbf{x}) \end{aligned}, \qquad (1)$$

where $f(.)$, $g(.)$, and $q(.)$ are nonlinear functions, $\mathbf{x} = [x, \dot{x}, ..., x^{(n-1)}]^T = [x_1, x_2, ..., x_n]^T \in R^n$ is the state vector of the system, w is the bounded disturbance, and u and y are the input and output of the system, respectively. In the feedback linearization method, if the nominal system is feedback linearizable, there exists an algebraic transformation which is used to linearize nonlinear system [1]. The closed-loop system under the feedback linearization method is represented in the Figure 1. This control structure consists of two loops, with the inner loop achieving the linearization of the input-state (or input-output) relation, and the outer loop achieving the stabilization of the closed-loop dynamics [1].

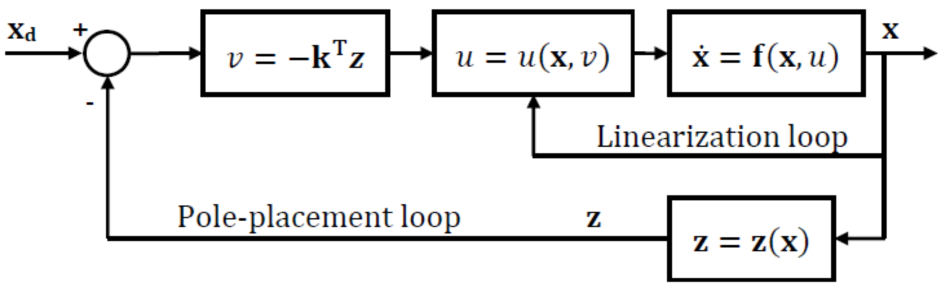

Figure 1. Feedback Linearization [1]

The linearized system equation with respect to disturbance w(t) can be written as

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{b}v + \mathbf{C}w + \mathbf{b}\psi(\mathbf{z})w, \qquad (2)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{n \times n}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n \times 1}. \qquad (3)$$

The new linearizing state z can be defined as $\mathbf{z} = [h(\mathbf{x}), L_f h(\mathbf{x}), \ldots, L_f^{n-1} h(\mathbf{x})]^T$, and $L_f^k h(\mathbf{x})$ is the *Lie* derivative of $L_f^{k-1} h(\mathbf{x})$ along the vector field *f*, and C is the $n \times 1$ constant matrix and $\psi(.)$ is a nonlinear function [1]. The linear control law can be determined by a linear control design such as pole placement method. The nonlinear feedback control law based on the feedback linearization method can be then obtained as

$$u = \alpha(\mathbf{x}) + \beta(\mathbf{x})v, \tag{4}$$

where $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ can be derived as

$$\alpha(\mathbf{x}) = -\frac{L_f^n h(\mathbf{x})}{L_g L_f^{n-1} h(\mathbf{x})}, \tag{5}$$

$$\beta(\mathbf{x}) = \frac{1}{L_g L_f^{n-1} h(\mathbf{x})}. \tag{6}$$

A closed-loop system in the feedback linearization method is stable with respect to the disturbance w(t) if a positive-definite matrix **P** can be found from the *Riccati* equation with the design parameters **Q** and r [7]

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}A - r^{-1}\mathbf{Pbb}^T\mathbf{P} + \mathbf{Q} = 0, \tag{7}$$

where, **Q** is a positive-definite matrix and r is the positive real number. Therefore, the linear control law can be given as

$$v = -r^{-1}\mathbf{b}^T\mathbf{Pz} - \delta|\psi|^2\mathbf{b}^T\mathbf{Pz}. \tag{8}$$

In order to optimally design a feedback linearization controller the design vector $d = \{Q, r, \delta\}$ should be found appropriately.

If there is any uncertainty in the model parameters in feedback linearization method, it will cause error in the new state and the control input, and the output of the system may deteriorate accordingly [1]. Therefore, it is beneficial for control engineers to design controllers robustly.

In the stochastic robust control design, an uncertain nonlinear system with the set of uncertain parameters, **p**, is considered so that the deterministic nonlinear system given in (1) can be re-written in a new form with parametric uncertainties as follows

$$\begin{aligned} x^{(n)} &= f(\mathbf{x},\mathbf{p}) + g(\mathbf{x},\mathbf{p})u(t) + q(\mathbf{x},\mathbf{p})w(t) \\ y &= h(\mathbf{x},\mathbf{p}). \end{aligned} \tag{9}$$

In the stochastic robust design, the uncertain parameters, **p**, considered as probabilistic uncertainties which have PDFs, for example, normal distribution. Therefore, the propagation of the uncertain parameters through the system provides some probabilistic metrics such as random variables (e.g., settling time, maximum overshoot …), and random processes (e.g., step response, Bode or Nyquist diagram …) in a control system design [12], [16, 17]. This situation leads to variations in the performance of main system. Therefore, it is very desirable to find robust design whose performance variation in the presence of uncertainties is low. Generally, there exist two approaches addressing the stochastic robustness issue, namely, robust design optimization (RDO) and reliability-based design optimization (RBDO) [18]. Both approaches represent non-deterministic optimization formulations in which the probabilistic uncertainty is incorporated into the stochastic optimal design process. In this work, the reliability-based design optimization method is used to design a robust nonlinear controller. In this method, random samples are generated assuming some pre-defined probabilistic distributions for uncertain parameters. The system is then simulated with each of these randomly generated samples and the percentage of cases produced in failure region defined by a limit state function approximately reflects the probability of failure.

Let X be a random variable, then the prevailing model for uncertainties in stochastic randomness is the probability density function (PDF), $f_X(x)$ or equivalently by the cumulative distribution function (CDF), $F_X(x)$, where the subscript X refers to the random variable. This can be given by

$$F_X(x) = \Pr(X \le x) = \int_{-\infty}^{x} f_X(x)dx \tag{10}$$

where Pr(.) is the probability that an event ($X \le x$) will occur.

In the reliability-based design, it is required to define reliability-based metrics via some inequality constraints. Therefore, in the presence of uncertain parameters of plant ($\mathbf{p}$) whose PDF or CDF can be given by $f_{\mathbf{p}}(\mathbf{p})$ or $F_{\mathbf{p}}(\mathbf{p})$, respectively, the reliability requirements can be given as

$$P_f^i(\mathbf{p}) = \Pr(g_i(\mathbf{p}) \le 0) = \varepsilon, \tag{11}$$

where, P_f^i denotes the probability of failure of the i^{th} reliability measure, $g(\mathbf{p})$ is the limit state function which separates the failure region $g(\mathbf{p}) \le 0$ from the safe region $g(\mathbf{p}) > 0$. In the reliability-based design ε should be minimized. Therefore, taking into consideration the stochastic distribution of uncertain parameters ($\mathbf{p}$) as $f_{\mathbf{p}}(\mathbf{p})$, equation (13) can now be evaluated for each probability function as

$$P_f^i(\mathbf{p}) = \Pr(g_i(\mathbf{p}) \le 0) = \int_{g_i(\mathbf{p}) \le 0} f_{\mathbf{p}}(\mathbf{p}) d\mathbf{p}, \tag{12}$$

This integral is, in fact, very complicated particularly for systems with complex $g(\mathbf{p})$ [19] and MCS is alternatively used to approximate equation (14). Based on this method the probability using sampling technique can be estimated using

$$P_f(\mathbf{p}) = \frac{1}{N} \sum_{i=1}^{N} I_{g(\mathbf{p})} \mathrm{F}_i(\mathbf{x}, u, \mathbf{p}), \tag{13}$$

where, $\mathrm{F}_i(.)$ is the i^{th} nonlinear system that is simulated by MCS. Also, $I_{g(\mathbf{p})}$ is an indicator which causes the probability of failure to be obtained numerically, and defined as

$$I_{g(\mathbf{p})} = \begin{cases} 0 & g(\mathbf{p}) > 0 \\ 1 & g(\mathbf{p}) \le 0 \end{cases}. \tag{14}$$

Therefore, the probability of failure which means the failure occurred is the number of samples in the failure region divided by the total number of samples. Evidently, such estimation of P_f approaches to the actual value in the limit as $N \to \infty$ [9]. However, there have been many research activities on sampling techniques to reduce the number of samples keeping a high level of accuracy. Alternatively, the quasi-MCS has now been increasingly accepted as a better sampling technique which is also known as Hammersley Sequence Sampling (HSS) [19]. In this work, HSS has been used to generate samples for probability estimation of failures. In a RBDO problem, the probability of failure of some metrics should be minimized using an optimization method. In a multi-objective optimization of a RBDO problem presented in this paper, however, there are different conflicting robust metrics that should be minimized simultaneously. In this work, multi-objective Pareto genetic algorithm of MATLAB is used for RBDO of robust feedback linearization control design.

III. ROBUST CONTROL DESIGN FOR A NONLINEAR SPRING-MASS SYSTEM

In this section, a multi-objective robust controller design is described. The problem is a two mass and spring benchmark which includes two masses (m_1, m_2), a linear spring (k_1), and a nonlinear cubic spring (k_2).The control signal applies to m_1 and unit impulse disturbance exerts on m_2 (Figure 2). The state space representation of the system is [7]

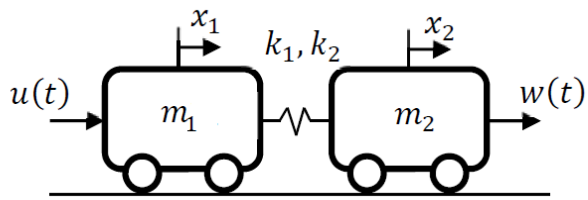

Figure 2. A nonlinear spring-mass system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ -\frac{k_1}{m_1}(x_1 - x_2) - \frac{k_2}{m_1}(x_1 - x_2)^3 \\ \frac{k_1}{m_2}(x_1 - x_2) + \frac{k_2}{m_2}(x_1 - x_2)^3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix} w, \tag{15}$$

$$y = x_2. \tag{16}$$

In this system the uncertain parameters include $\mathbf{p} = [m_1, m_2, k_1, k_2]$. The nominal value of them are $m_1^0 = m_2^0 = 1$, $k_1^0 = 1.25$, and $k_2^0 = -0.15$. The corresponding constraints are considered as follows

$$\begin{aligned} &0.5 < k_1 < 2, \quad -0.5 < k_2 < 0.2, \\ &0.5 < m_1 < 1.5, \quad 0.5 < m_2 < 1.5. \end{aligned} \tag{17}$$

The input-output linearization of this single input/output system can be accomplished by using $z_1 = x_2$. The equations of this procedure which are adopted from [7] are given as

$$z_1 = h(\mathbf{x}) = x_2,$$

$$z_2 = L_f h = x_4,$$

$$z_3 = L_f^2 h = \frac{k_1^0}{m_2^0}\left(x_1 - x_2\right) + \frac{k_2^0}{m_2^0}\left(x_1 - x_2\right)^3,$$

$$z_4 = L_f^3 h = \left(x_3 - x_4\right)\left[\frac{k_1^0}{m_2^0} + \frac{3k_2^0}{m_2^0}\left(x_1 - x_2\right)^2\right]. \tag{18}$$

As a result of the above transformation, the linear equation can be written as

$$\begin{aligned}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= z_3 + \frac{1}{m_2^0} w, \\
\dot{z}_3 &= z_4, \\
\dot{z}_4 &= L_f^4 h + L_g L_f^3 h u + L_q L_f^3 h w,
\end{aligned} \tag{19}$$

where,

$$\begin{aligned}
L_f^4 h = {} & \frac{6k_2^0}{m_2^0}\left(x_1 - x_2\right)\left(x_3 - x_4\right)^2 - \left(\frac{1}{m_1^0} + \frac{1}{m_2^0}\right)\left[\frac{k_1^0}{m_2^0} + \frac{3k_2^0}{m_2^0}\left(x_1 - x_2\right)^2\right] \\
& \times \left[k_1^0\left(x_1 - x_2\right) + k_2^0\left(x_1 - x_2\right)^3\right],
\end{aligned} \tag{20}$$

$$L_g L_f^3 h = \frac{1}{m_1^0 m_2^0}\left[k_1^0 + 3k_2^0\left(x_1 - x_2\right)^2\right], \tag{21}$$

$$L_q L_f^3 h = -\frac{1}{\left(m_2^0\right)^2}\left[k_1^0 + 3k_2^0\left(x_1 - x_2\right)^2\right]. \tag{22}$$

Then transformed system can be formulated as

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{b}v + \mathbf{C}w - \frac{3k_2^0}{\left(m_2^0\right)^2}\left(x_1 - x_2\right)^2 \mathbf{b}w, \tag{23}$$

where

$$\mathbf{C} = \begin{bmatrix} 0 \\ 1/m_2^0 \\ 0 \\ -k_1^0/\left(m_2^0\right)^2 \end{bmatrix}. \tag{24}$$

Then the control law of the system can be written as

$$u = \frac{1}{L_g L_f^3 h}\left(-L_f^4 h - r^{-1}\mathbf{b}^T\mathbf{Pz} - \delta(x_1 - x_2)^2\mathbf{b}^T\mathbf{Pz}\right). \tag{25}$$

In this study, the appropriate values of the parameters of the control law are designed by multi-objective optimization. Four objective functions are considered to design robust optimal nonlinear controller for this benchmark. In the first place, the most important goal of the robust controller design is the robust stability which implies that all the closed-loop systems remain stable in the presence of any uncertainty.

Definition. For the input control signal $u(t)$ that leads the system to stability, there should exist a class KL function β and a class K_∞ function γ such that [20]

$$|y(t)| \le \beta(|x^0|, t) + \gamma(\|u\|_\infty), \tag{26}$$

holds for all solutions. This system is *input-to-output stable* [20]. $|\cdot|$ denotes Euclidean norm and $\|\cdot\|_\infty$ denotes infinity norm.

Thus, in the case of stochastic robust design, the limit state function to define the probability of instability will be represented by

$$g_{ins}(\mathbf{p}) = \beta(|x^0|, t) + \gamma(\|u\|_\infty) - |y(t)|. \tag{27}$$

The probability of instability can now be computed as

$$\mathrm{Pr}_{ins} = \sum_{i=1}^{N} I_{g_{ins}} \, \mathrm{F}_i(\mathbf{x}, u, \mathbf{p}). \tag{28}$$

The robust performance is another criterion for designing nonlinear controllers. In this study robust performance includes settling time and overshoot. Their limit state functions are defined as

$$g_{ts}(\mathbf{p}) = 7 - Ts, \tag{29}$$

$$g_{os}(\mathbf{p}) = 1 - os, \tag{30}$$

where, Ts and OS are settling time and overshoot of the output (x_2), respectively. Therefore, the probability of the failure for each of them can be defined as

$$\mathrm{Pr}_{ts} = \sum_{i=1}^{N} I_{g_{ts}} \, \mathrm{F}_i(\mathbf{x}, u, \mathbf{p}), \tag{31}$$

$$\Pr_{os} = \sum_{i=1}^{N} I_{g_{os}} \mathrm{F}_i(\mathbf{x}, u, \mathbf{p}). \quad (32)$$

The control effort plays a pivotal role in the control design, and in many practical cases, there is a limitation on control effort which must be considered by control engineers. The limit state function of this criterion is

$$g_u(\mathbf{p}) = 1 - \|u\|_\infty , \quad (33)$$

and the failure probability of the control effort can be formulated as

$$\Pr_u = \sum_{i=1}^{N} I_{g_u} \mathrm{F}_i(\mathbf{x}, u, \mathbf{p}). \quad (34)$$

These robust metrics should be minimized simultaneously using multi-objective optimization. In the next section the results of such multi-objective optimization procedure will be given.

IV. RESULT

In order to show the supremacy of the robust design, the multi-objective optimization will be accomplished for both determinist and probabilistic feedback linearization design.

In the first place, it is obvious that displaying more than two objective functions to demonstrate the trade-off is not feasible. In this way, several multidimensional visualization methods are proposed [21, 22]. One of the methods which leads to comprehensive analysis of the Pareto front is called *Level Diagrams* method [22] which is used in this paper to visualize the Pareto fronts of the multi-objective optimization.

In this method, each point of Pareto front must be normalized between 0 and 1 based on its minimum and maximum values [22]

$$J_i^M = \max J_i \; , \; J_i^m = \min J_i \; , \; i = 1, \ldots, n, \quad (35)$$

$$\overline{J}_i = \frac{J_i - J_i^m}{J_i^M - J_i^m}, \quad (36)$$

Provided that the origin of the n-dimensional space is considered as the ideal point, the distance of the each Pareto front point can be used for comparison. In this paper, Euclidean norm of all objective functions ($\|\overline{J}\|_2 = \sqrt{\sum_{i=1}^{n} \overline{J}_i^2}$) is used for this purpose. To represent the Pareto front, Y axis is specified for Euclidean norm of all objective functions and X axis is specified for each objective function; therefore, each objective function has its own graphical representation whilst Y axis of each graph would be the same.

a. Deterministic Design

The three objective functions are now considered simultaneously in a Pareto optimization process to obtain some important trade-offs among the conflicting objectives. In a deterministic design approach, the vector of objective functions to be optimized in a Pareto sense is given as follow

$$\vec{\mathbf{J}} = \left[Ts, OS, \|u\|_{\infty}\right]. \tag{37}$$

In the deterministic design, in order to prevent instability, if a system is unstable, each objective function is coerced to ∞.

The evolutionary process of the multi-objective optimization is accomplished with a population size of 100 which has been chosen with crossover probability P_c and mutation probability P_m as 0.8 and 0.01, respectively. A total number of 120 non-dominated optimum design points have been obtained for optimal feedback linearized controller. The results of the 3-objective optimization process are shown in Figure 3.

It is evident from figure 3, that there is confliction between settling time and control effort, also, between overshot and control effort. The *V* shape of the Pareto fronts confirms this situation. Take for illustration, the system whose settling time is low has the worse value of control effort. This situation is very evident in the *Level Diagrams* of both overshoot and infinity norm of control effort, because the system with lowest value of overshoot has the maximum value of the control effort and vice versa. Therefore the promising results show the confliction between each pair of objective functions which can be found from this figure. Also, the circled point which is the vertices of the *V* layout of each Pareto front (the nearest point to ideal) is very close to the minimum value of each objective function, hence, it can be chosen as a pivotal trade-off point. The objective functions and design variables for minimum value of each objective function and for minimum value of the 2-norm function are given in Table 1.

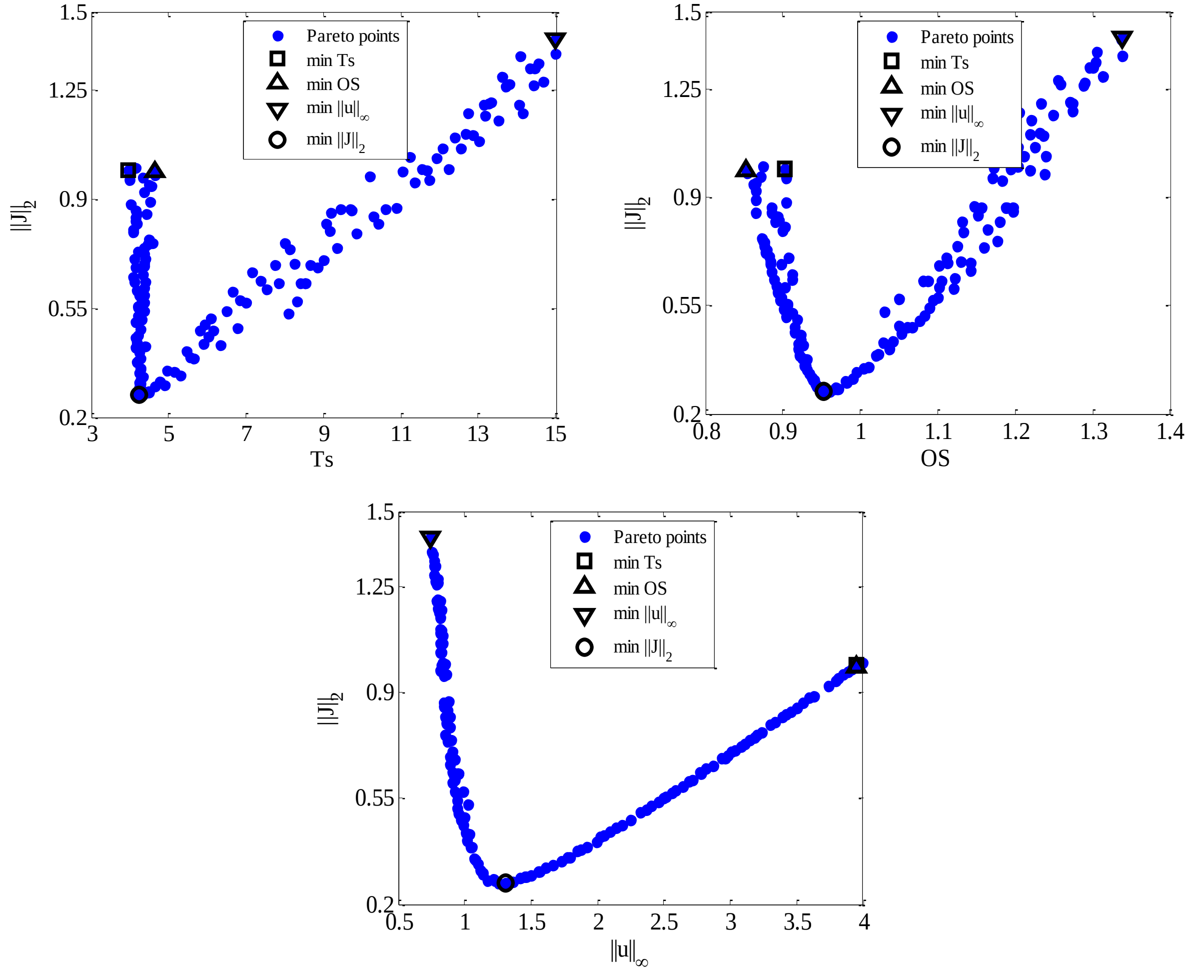

Figure 3: 2-Norm *Level Diagrams* of Pareto fronts for determinist design

Table 1. The values of objective functions and the corresponding design variables of the selected optimum points.

Optimum Points	Design Variables						Objective Functions			
	q_1	q_2	q_3	q_4	r	δ	Ts	OS	$\|u\|_\infty$	$\|J\|_2$
min Ts	1.967	0.692	0.324	0.183	0.223	0.362	3.960	0.903	3.951	0.992
min OS	2.028	1.863	0.440	0.975	0.052	0.667	4.640	0.852	3.949	0.987
min $\|u\|_\infty$	0.030	0.921	1.122	0.561	0.375	0.793	15	1.338	0.745	1.414
min $\|J\|_2$	2.020	0.718	0.371	0.387	0.383	0.407	4.240	0.953	1.304	0.270

The response of each considered optimal controller against unit impulse disturbance is shown in Figure 4. In addition, the corresponding control efforts are shown in Figure 5.

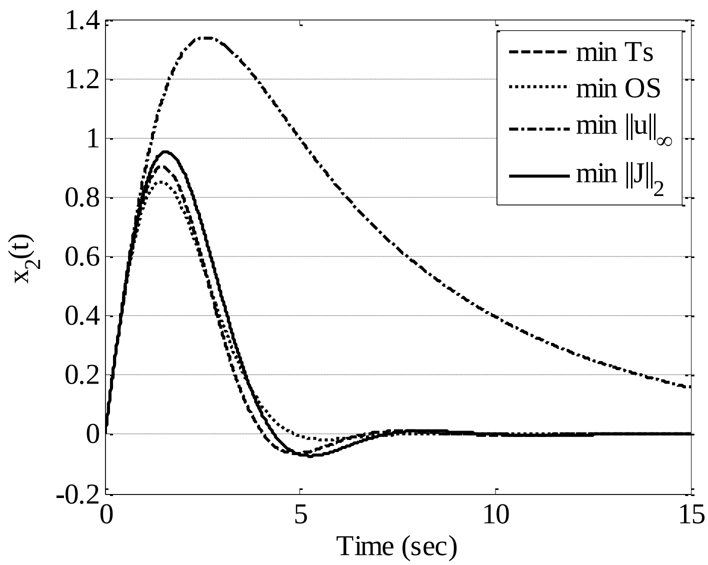

Figure 4: The responses of the optimum selected points with respect to the unit impulse

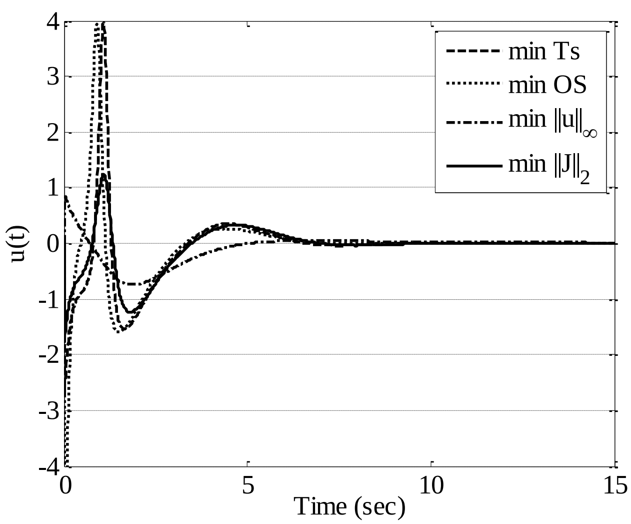

Figure 5: The control efforts of the optimum selected points with respect to the unit impulse

According to figures 4 and 5, designing high-performance and cost effective controllers is a very complex subject which could be practically impossible, because the systems which have a good time response need high cost control effort, and, on the other hand, the system whose control effort is low has the adverse time response.

Each state of the controlled system with the circled point ($\min \|J\|_2$ or the trade-off design) is shown in Figure 6. It is obvious that the system is input-state stable system because all of the states are asymptotically stable, and they have acceptable settling times.

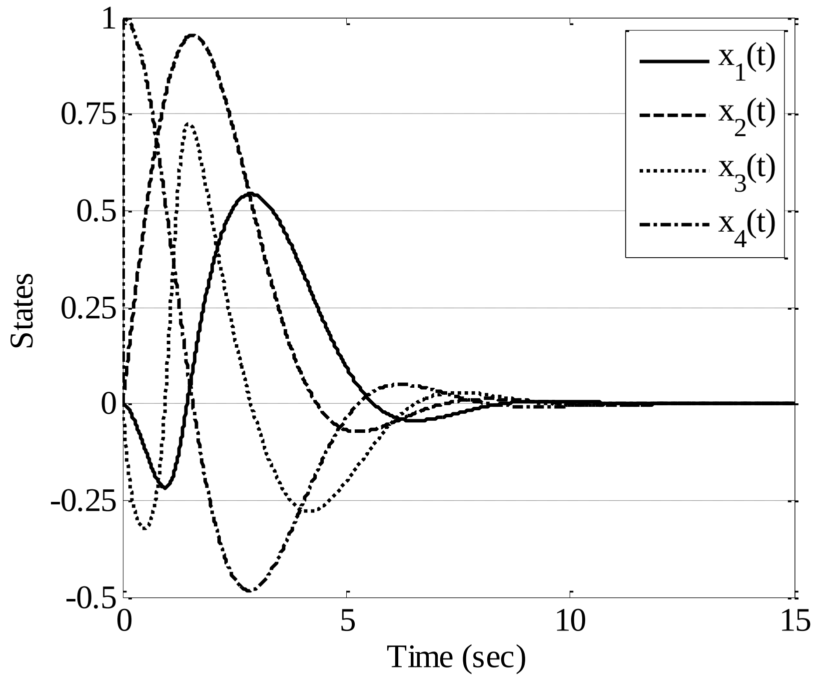

Figure 6: Time response of each state of the point with minimum 2-norm

b. Probabilistic Design

In this section the stochastic design will be accomplished to obtain robust feedback linearization controller. In this way, four objective functions are now considered simultaneously in a Pareto optimization process to obtain some important trade-offs among the conflicting objectives. In a robust design approach, the vector of objective functions to be optimized in a Pareto sense is given as follow

$$\vec{\mathbf{J}} = [\Pr_{ins}, \Pr_{ts}, \Pr_{os}, \Pr_{u}], \qquad (38)$$

whose elements are computed by equations (28), (31), (32), and (34), respectively, in the quasi-Monte Carlo simulation process. In this study, two kinds of uncertainty are considered, namely uniform distribution and Gaussian distribution. In uniform distribution the range of system parameters have been listed in equation (17), and for Gaussian distribution, mean and variance of each parameter are chosen which have the same range as those of the uniform distribution. The parameters that describe the Gaussian distribution are listed in Table 2.

The evolutionary process of the multi-objective optimization for both uniform and Gaussian distributions are accomplished with a population size of 100 which has been chosen with crossover probability P_c and mutation probability P_m as 0.8 and 0.01, respectively. The optimization process of the robust controller is accomplished for 200 Monte Carlo evaluations using HSS distribution for each candidate control law during the evolutionary process.

Table 2. Uncertainty model

	Mean	Variance
k_1	1.25	0.065
k_2	-0.15	0.017
m_1	1	0.027
m_2	1	0.027

b.i Multi-objective optimization results for Gaussian distribution

A total number of 72 non-dominated optimum design points have been obtained for robust feedback linearized controller with the considered Gaussian distribution for uncertain parameters. Since, the value of probability of instability ($\Pr_{ins}$) of all non-dominated optimum points has been found equal to zero, the results of the 4-objective optimization process, consequently, correspond to a 3-objective optimization process which is shown in Figure 7.

Also, the circled point in Pareto front which has the lowest value of the 2-norm, can be selected as a significant trade-off point. This point which is located on the vertices of the *V* layout form of the *Level Diagram* has the low value of $\Pr_{ts}$ and the moderate value of two other objective functions. For comparison, the objective functions and design variables for minimum value of each objective function and for minimum value of 2-norm are given in Table 3.

It is obvious from the numerical results in Table 3 that the systems whose $\Pr_{ts}$ and $\Pr_{os}$ are low, have the 100% failure in the control signal requirement.

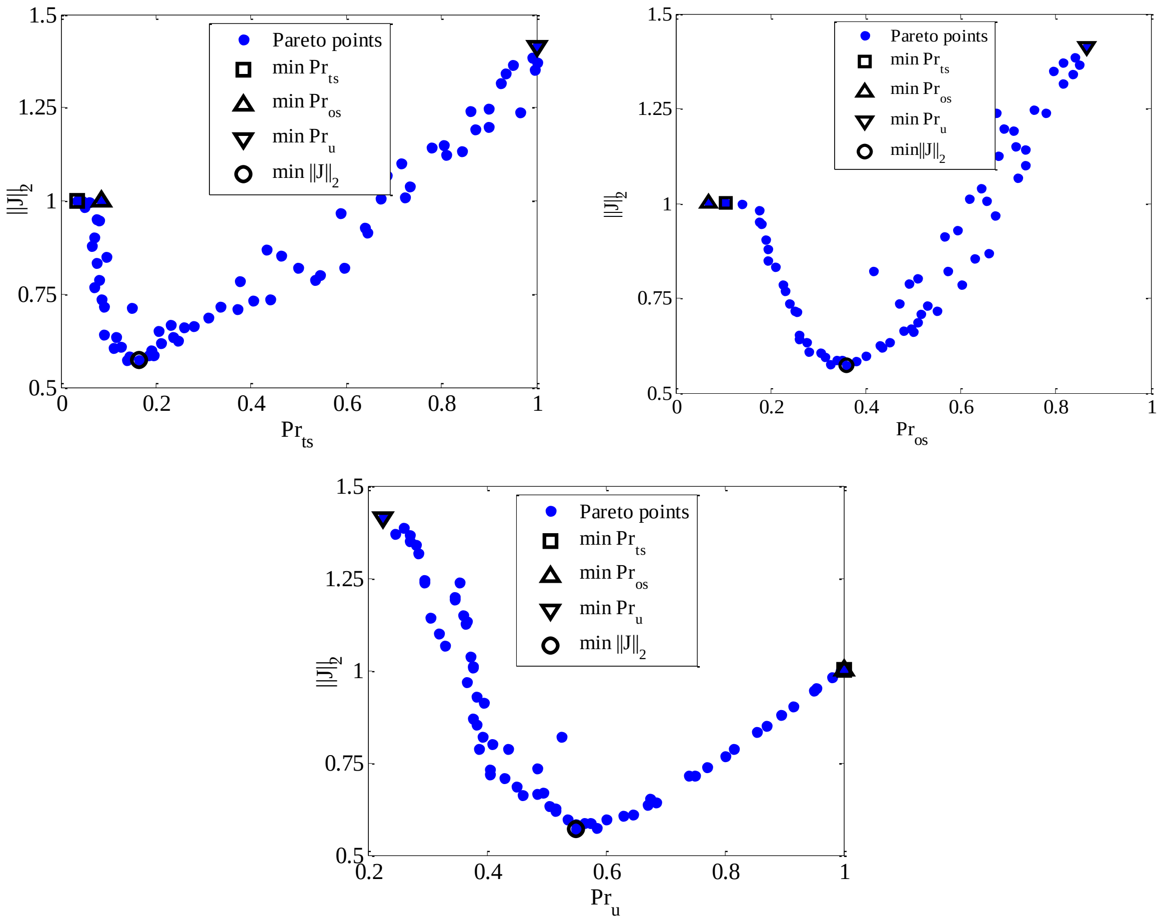

Figure 7: 2-Norm *Level Diagrams* of Pareto fronts for probabilistic design with Gaussian distribution

Table 3. The values of objective functions and the corresponding design variables of the selected optimum points.

Optimum Points	**Design Variables**						**Objective Functions**			
	q_1	q_2	q_3	q_4	r	δ	Pr_{ts}	Pr_{os}	Pr_u	$\left\|J\right\|_2$
min Pr_{ts}	2.773	2.245	1.428	0.285	0.105	1.662	0.035	0.105	1	1.001
min Pr_{os}	3.083	2.270	0.992	0.218	0.112	1.479	0.085	0.070	1	1.001
min Pr_u	0.189	1.092	1.278	0.841	0.463	0.354	1	0.865	0.225	1.414
min $\left\|J\right\|_2$	1.206	0.987	0.959	0.325	0.285	0.416	0.165	0.360	0.550	0.571

The stochastic response of each mentioned optimum point is shown in Figure 8. In this figure, upper interval and lower interval illustrate that all simulated response for 200 samples are laid between them. Also, the stochastic response of the proposed controller from [7] is shown in Figure 8.

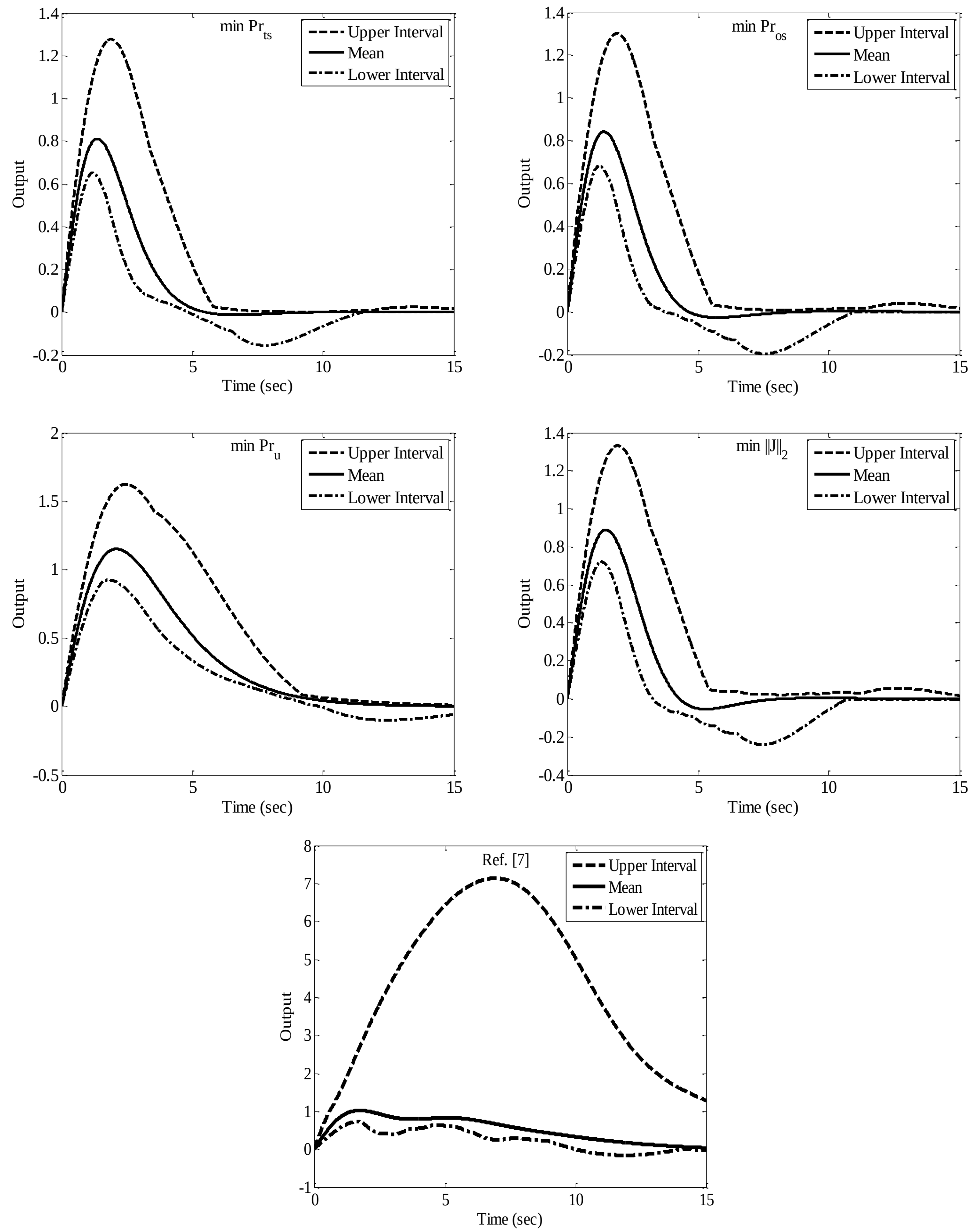

Figure 8: Probabilistic responses

According to Figure 8, both points with the lowest value of $\Pr_{ts}$ and $\Pr_{os}$ have the same stochastic time responses with respect to the unit impulse. In contrast, the system whose $\Pr_{u}$ is the lowest has the worse stochastic time responses in comparison with other points. Also, the point with the lowest value of 2-norm which can be considered as an outstanding trade-off point has the acceptable stochastic time responses. In addition, it can be seen that the stochastic time response of the proposed controller by [7] is worse than that of proposed controller by this paper.
The PDFs of each stochastic variable for each selected optimum point are shown in Figure 9.
It can be concluded from Figure 9 that the system with the lowest value of $\Pr_{u}$ has the settling time bigger than 7 seconds. As it was shown in Figure 8, both systems with the lowest value of $\Pr_{ts}$ and $\Pr_{os}$ have the same stochastic time response and same $\Pr_{u}$ and the maximum control effort are same for both of them.

b.ii Multi-objective optimization results for uniform distribution

A total number of 65 non-dominated optimum design points have been obtained for robust feedback linearized controller with the considered uniform distribution for uncertain parameters. Unlike the Gaussian distribution, the value of probability of instability ($\Pr_{ins}$) of some non-dominated optimum points has not been equal to zero. The results of the 4-objective optimization process are shown in Figure 10.
The maximum probability of instability in this design is equal to 36%, it means that among 200 HSS exist 72 unstable plants. Like as a prior design, there is no confliction between $\Pr_{ts}$ and $\Pr_{os}$. It is evident from Figure 10 that there is confliction between robust stability and robust performance, due to the fact that the systems with low $\Pr_{ins}$ have worse value of $\Pr_{ts}$ and $\Pr_{os}$, on the other hand, the systems with the low value of $\Pr_{ts}$ and $\Pr_{os}$ have the worse value of $\Pr_{ins}$. In addition, both $\Pr_{ts}$ and $\Pr_{os}$ have confliction with $\Pr_{u}$. Therefore, the systems whose $\Pr_{ts}$ and $\Pr_{os}$ are low have the high value of $\Pr_{u}$, and the system with the lowest value of $\Pr_{u}$ has the worse value of $\Pr_{ts}$ and $\Pr_{os}$.
The circled point in Pareto front which has the lowest value of the 2-norm and is located on the vertices of the *V* layout form of the *Level Diagram* has the moderate value of each objective function; therefore, it can be considered as an outstanding optimum point. For comparison, the

objective functions and design variables for minimum value of each objective function and for minimum value of 2-norm are given in Table 4.

Table 4. The values of objective functions and the corresponding design variables of the selected optimum points.

Optimum Points	Design Variables						Objective Functions				
	q_1	q_2	q_3	q_4	r	δ	$\Pr_{ins}$	$\Pr_{ts}$	$\Pr_{os}$	$\Pr_u$	$\Vert J \Vert_2$
min $\Pr_{ins}$	0.135	1.608	0.565	0.400	0.946	0.398	0.005	0.969	0.768	0.404	1.390
min $\Pr_{ts}$	2.562	1.659	1.955	0.402	0.101	1.009	0.32	0.147	0.162	1	1.338
min $\Pr_{os}$	3.337	1.621	1.049	0.365	0.103	0.871	0.36	0.195	0.117	1	1.415
min $\Pr_u$	0.126	1.423	0.794	0.349	0.469	0.366	0.02	1	0.729	0.311	1.368
min $\Vert J \Vert_2$	2.106	1.579	0.950	0.507	0.275	0.670	0.145	0.362	0.380	0.538	0.698

The numerical results in Table 3 and Table 4 show that the time response performance of the systems with normal distribution are better that those of uniform distribution, because of the fact that the uniform distribution uncertainties are more stringent than normal ones.

The stochastic response of each mentioned optimum point is shown in Figure 11. Their stochastic behaviors are simulated for 200 samples, and upper bound and lower bound show the region of the responses. Also, the stochastic response of the proposed controller from [7] is shown in Figure 8.

From Figure 11, the systems with the lowest value of $\Pr_{ins}$ and $\Pr_u$ have the worse behaviors of the stochastic time responses with respect to the unit impulse. Also, the point with the lowest value of 2-norm which can be considered as an outstanding trade-off point has the acceptable stochastic time responses. It is obvious from Figure 11 that the proposed controller by [7] has the worse time response in comparison with proposed controller in this paper. In addition, according to the Figure 8 and Figure 11, the systems with the Gaussian distribution have the better time responses in comparison with the systems with the uniform distribution. It seems due to the fact that the uniform distribution is more stringent than Gaussian one.

The PDFs of each stochastic variable for each selected optimum point are shown in Figure 12.

It can be concluded from Figure 12 that the systems with the lowest value of $\Pr_{ins}$ and $\Pr_u$ have the unacceptable settling time and overshoot. In addition, the systems with the lowest value of $\Pr_{ts}$ and $\Pr_{os}$ need big control effort signal.

In order to compare the robustness of each design, the trade-off point of each robust design is considered. Controller A and B are considered as the trade-off points of the systems with Gaussian probability distribution and uniform probability distribution design, respectively. Table 5 shows the comparison of the robustness of the two controllers. Each evaluation data is based on 20,000 quasi Monte Carlo simulations, against either uniform or Gaussian evaluation distributions. Also, the results of the proposed controllers by Wang et al. [7] are given in Table 5. The controller C was designed for Gaussian probability distribution, and D was designed for uniform probability distribution.

Table 5. Comparison between different designs.

	$\Pr_{ins}$		$\Pr_{ts}$		$\Pr_{os}$		$\Pr_{u}$	
	Gaussian	Uniform	Gaussian	Uniform	Gaussian	Uniform	Gaussian	Uniform
Controller A	0.015	0.159	0.228	0.421	0.383	0.461	0.501	0.665
Controller B	0.023	0.122	0.171	0.373	0.275	0.376	0.526	0.458
Controller C	0.013	0.035	1	1	0.548	0.571	0.471	0.305
Controller D	0.014	0.032	1	1	0.540	0.561	0.396	0.371

As should be expected, if the controller is designed by the uniform probability distribution, it acts more robust than the controller with Gaussian one. Take for illustration, controller B which has been designed under uniform distribution, produces lower cost functions in comparison with controller A. But controller A whose probability distribution is Gaussian has better $\Pr_{ins}$ and $\Pr_{u}$ than controller A when they are evaluated by Gaussian probability distribution.

Although the controllers which were designed by Wang et al. al [7] have the low value $\Pr_{ins}$ and $\Pr_{u}$, the corresponding $\Pr_{ts}$ and $\Pr_{os}$ are worse than the proposed controllers. Because the parameters of the controller proposed by [7] were designed by minimizing a single objective function in which the weighting factor for $\Pr_{ins}$ has been chosen as 1 and others chosen as 0.01. Therefore they have low value of $\Pr_{ins}$ and it was shown in this paper that the system with low value of $\Pr_{ins}$ has also low value of $\Pr_{u}$.

V. CONCLUSIONS

This paper proposed an optimal robust design method for the feedback linearization method using a multi-objective optimization method. In this work, robust stability and robust performance metrics are used to define the objective functions. The parameters of the nonlinear system are as uncertain parameters with the known probability distribution. Two types of uncertainties were considered for the nonlinear systems. The multi-objective optimization of the nonlinear controller led to the discovering some important trade-offs among those objective functions. The design procedure in this paper can be applied very easily to any nonlinear systems with feedback linearization control. The multi-objective GAs of this work for the Pareto optimization of feedback linearization controllers using some non-commensurable objective functions is very promising and can be generally used in the optimum design of real-world complex control systems with uncertainties.

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