Robust Fusion Algorithms for Linear Dynamic System with Uncertainty

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Abstract- In this paper, two robust fusion algorithms for a linear system with observation uncertainty are proposed. The first algorithm is based on the classical median function and the second one uses relative distances between local estimates and their median value. In the view of estimation accuracy, the proposed fusion algorithms can be robust against uncertainty measurements since median can avoid extremely big or small values. This fact is verified from comparative analysis using numerical examples.

I. INTRODUCTION

Recently, the interest of multisensory data fusion has been increased to improve the accuracy of estimation and system states. Related with a multisensory fusion, two basic fusion architectures are well known: centralized and decentralized (or distributed) fusion. The distributed fusion is considered more challenging, and thus has studied. Several distributed fusion architectures, and their corresponding techniques have been previously discussed and presented in [1-3]. Consequently, the optimal mean-square linear fusion formulas representing the weighted sums of local estimates with matrix and scalar weights and corresponding explicit and implicit formulas for the weights have been reported in [4-7].

However, the above mean-square fusion formulas yield inaccurate fusion estimates when local estimates contain uncertainty, because the uncertainty affects statistical information such as local estimation error covariance, measurement error variance, and so on. For this reason, the classical fusion formulas are not applicable to a real application on uncertainty measurements.

Therefore, to overcome these problems, we propose two robust fusion algorithms; median fusion and weighted fusion using relative distances between the median and local estimates. Focusing on the robust fusion property, we suggest comparison examples with classical algorithm, equally weighted fusion. Since the proposed algorithms depend only on the values of local estimates, the fusion estimates using them can be little influenced by uncertainty.

This paper is organized as follows. In Section II, the fusion estimation problem is setting and the main goal is presented. In Section III, two robust fusion algorithms are proposed with explicit formulas. Comparative analysis is given using numerical examples demonstrating the concrete accuracies of the proposed fusion algorithms in Section IV. Finally, a brief conclusion is given in Section V.

II. PROBLEM SETTING

Let us consider a discrete-time linear dynamic system with N sensors having uncertainties, which is described by

$$\begin{aligned} x_{k+1} &= F_k x_k + G_k w_k, \quad k = 0,1,\ldots, \\ y_k^{(i)} &= H_k^{(i)} x_k + v_k^{(i)}, \quad i = 1,\ldots,N, \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ and $y_k^{(i)} \in \mathbb{R}^{m_i}$ are unknown system state vector and observation (sensor) vector, $x_0 \sim \mathbb{N}(\bar{x}_0, P_0)$, $w_k \in \mathbb{R}^q \sim \mathbb{N}(0, Q_k)$ and $v_k^{(i)} \in \mathbb{R}^{m_i} \sim \mathbb{N}\left(0, R_k^{(i)}\right)$ are the zero-mean white Gaussian noises, and $F_k \in \mathbb{R}^{n\times n}$, $G_k \in \mathbb{R}^{n\times q}$, and $H_k^{(i)} \in \mathbb{R}^{m_i\times n}$ are transition matrix, noise gain and observation matrix, respectively. In the observation noise $v_k^{(i)}$, the corresponding error-variances $R_k^{(i)}$ contains known value $\bar{R}_k^{(i)}$ and uncertainty $\Delta R_k^{(i)}$, i.e., $R_k^{(i)} = \bar{R}_k^{(i)} + \Delta R_k^{(i)}$.

For individual (local) sensor $y_k^{(i)}$, the system (1) can be divided into N subsystems with the common state x_k. One subsystem is described as

$$x_{k+1} = F_k x_k + G_k w_k, \quad y_k^{(i)} = H_k^{(i)} x_k + \bar{v}_k^{(i)}, \tag{2}$$

where the index "i" is fixed, and $\bar{v}_k^{(i)} \sim \mathbb{N}(0, \bar{R}_k^{(i)})$..

Then using the subsystem (2), the local estimate $\hat{x}_k^{(i)}$ and corresponding error-covariance $P_k^{(ii)}$ can be described by the Kalman filter equations [8]:

$$
\begin{aligned}
&\hat{x}_k^{(i)} = F_k \hat{x}_{k-1}^{(i)} + K_k^{(i)} \left(y_k^{(i)} - H_k^{(i)} F_k \hat{x}_{k-1}^{(i)} \right), \quad \hat{x}_0^{(i)} = \bar{x}_0, \\
&M_k^{(ii)} = F_k P_{k-1}^{(ii)} F_k^T + G_k Q_k G_k^T, \quad P_0^{(ii)} = P_0, \\
&K_k^{(i)} = M_k^{(ii)} H_k^{(i)^T} \left[H_k^{(i)} M_k^{(ii)} H_k^{(i)^T} + \bar{R}_k^{(i)} \right]^{-1}, \\
&P_k^{(ii)} = \left(I_n - K_k^{(i)} H_k^{(i)} \right) M_k^{(ii)},
\end{aligned} \tag{3}
$$

where $K_k^{(i)}$ is a local Kalman gain, the superscript T represents the matrix transpose, and I_n is an $n \times n$ identity matrix.

After using (3) for $i = 1, ..., N$, we have N local estimates $\hat{x}_k^{(1)}, ..., \hat{x}_k^{(N)}$ and corresponding local error-covariances $P_k^{(11)}, \dots, P_k^{(NN)}$. The error-covariances $P_k^{(11)}, \dots, P_k^{(NN)}$ are vital factors for the classical linear fusion algorithm [4-7]. In (3), local error-covariances $P_k^{(11)}, \dots, P_k^{(NN)}$ are calculated using $R_k^{(i)}$. However, practically, we calculate $P_k^{(11)}, \dots, P_k^{(NN)}$ using $\bar{R}_k^{(i)}$ since $\Delta R_k^{(i)}$ is unknown, and thus $P_k^{(11)}, \dots, P_k^{(NN)}$ are not accurate when $\Delta R_k^{(i)} \neq 0$. For such reason, the classical fusion algorithm is not applicable to the uncertainty measurements.

Therefore, in a multisensory environment with uncertainty measurements, we propose two robust fusion algorithms which do not use the local error-covariances $P_k^{(11)}, \dots, P_k^{(NN)}$. The details are given in the next section.

III. ROBUST FUSION ALGORITHMS

A. Median Fusion

Median fusion (MDF) algorithm is based on a *median* function. The median function $\mathbf{med}(X)$ is defined as

$$
\mathbf{med}(X) = \begin{cases} X_k, & \text{if } \ m = 2k+1, \\ (X_k + X_{k+1})/2, & \text{if } \ m = 2k, \end{cases} \tag{4}
$$

where $X = \{X_1, ..., X_m\}$, $m \ge 2$. Then, the general function (4) is applied to MDF algorithm.

Suppose that we have N local estimates for an unknown vector x_k,

$$\hat{x}_k^{(i)} = \left[\hat{x}_{1,k}^{(i)} \dots \hat{x}_{n,k}^{(i)}\right]^T \in \mathbb{R}^n, \quad i = 1, ..., N. \tag{5}$$

Next, we create sets of estimates $S_{j,k}$,

$$S_{j,k} = \left\{\hat{x}_{j,k}^{(1)}, ..., \hat{x}_{j,k}^{(N)}\right\}, \quad j = 1, ..., n. \tag{6}$$

Then, the fusion estimate $\hat{x}_k^{MDF}$ can be defined by using (4). We have

$$\hat{x}_k^{MDF} = \begin{bmatrix} \mathbf{med}\left(S_{1,k}\right) \\ \vdots \\ \mathbf{med}\left(S_{n,k}\right) \end{bmatrix}. \tag{7}$$

Since $\hat{x}_k^{MDF}$ depends only on median values of local estimates $\hat{x}_k^{(1)}, \dots, \hat{x}_k^{(N)}$, it can avoid extremely big or small values. This is the reason why MDF is robust against uncertainty measurements.

B. Weighted Fusion using Distances

Let us consider distances $d_k^{(1)}, ..., d_k^{(N)}$ between the median (7) and all local estimates $\hat{x}_k^{(1)}, \dots, \hat{x}_k^{(N)}$, i.e.,

$$d_k^{(i)} = \left[d_{1,k}^{(i)} \dots d_{n,k}^{(i)}\right]^T, \quad d_{m,k}^{(i)} = \left|\hat{x}_{m,k}^{(i)} - \hat{x}_{m,k}^{MDF}\right|, \quad m = 1, ..., n, \quad i = 1, ..., N. \tag{8}$$

Since the corresponding fusion weights are selected by the fact that they are inversely proportional to the distance (8), the specific weighted formula is given by

$$W_k^{(i)} = \mathrm{diag}\left(w_{1,k}^{(i)}, ..., w_{n,k}^{(i)}\right), \quad w_{m,k}^{(i)} = d_{m,k}^{(i)-2} \sum_{s=1}^{N} \left(d_{m,k}^{(s)}\right)^{-2}, \quad d_{m,k}^{(i)} \ne 0, \quad m = 1, ..., n, \quad i = 1, ..., N. \tag{9}$$

Then, the fusion formula is defined as

$$\hat{x}_k^{WFD} = \sum_{i=1}^{N} W_k^{(i)} \hat{x}_k^{(i)}. \tag{10}$$

Note that if $d_{m,k}^{(i)} = 0$ in (8), then directly $w_{m,k}^{(i)} = 1$ in (9). Thus, when N is odd and $x_k \in \mathbb{R}$ ($n = 1$) is scalar state, the fusion estimate $\hat{x}_k^{WFD}$ is fully identical to $\hat{x}_k^{MDF}$. This fact is also verified in Section IV.

IV. COMPARISON EXAMPLES

Let us consider the following scalar signal model with N sensors, i.e.,

$$\begin{aligned} x_{k+1} &= 0.9x_k + w_k, \quad k = 0,1,\dots,T_k, \\ y_k^{(i)} &= x_k + v_k^{(i)}, \quad i = 1,\dots,N, \end{aligned} \tag{11}$$

where $T_k = 20$, $x_0 \sim \mathbb{N}(0,1)$, $w_k \sim \mathbb{N}(0,1)$, $v_k^{(i)} \sim \mathbb{N}\left(0, r^{(i)}\right)$, and $r^{(i)} = \left|\cos(i)\right| + \alpha^{(i)}$, $\alpha^{(i)}$ is a constant uncertainty in an i-th sensors error.

To compare two robust fusion algorithms, MDF and WFD, we consider the classical non-robust algorithm known as *average* (AVR), i.e.,

$$\hat{x}_k^{AVR} = \frac{1}{N}\left(\hat{x}_k^{(1)} + \cdots + \hat{x}_k^{(N)}\right). \tag{12}$$

Next, numerical simulations with 2000 Monte-Carlo runs are performed in five cases. All cases have restrict conditions respectively, such as the number of sensors N and the values of uncertainties $\alpha^{(i)}$. According to the conditions of each case, we compare the concrete mean square errors (MSEs) of fusion filters based on MDF, WFD and AVR, which are given by

$$P_k^{MDF} = E\left(x_k - \hat{x}_k^{MDF}\right)^2, \; P_k^{WFD} = E\left(x_k - \hat{x}_k^{WFD}\right)^2, \quad P_k^{AVR} = E\left(x_k - \hat{x}_k^{AVR}\right)^2. \tag{13}$$

Case A: 3 Sensors without uncertainty; $N = 3$, $\alpha^{(i)} = 0$, $i = 1,2,3$

In this case, 3 sensors without uncertainties are considered. Figure 1 shows concrete MSEs of two robust fusion estimates $\hat{x}_k^{MDF}, \hat{x}_k^{WFD}$ and non-robust estimation $\hat{x}_k^{AVR}$.

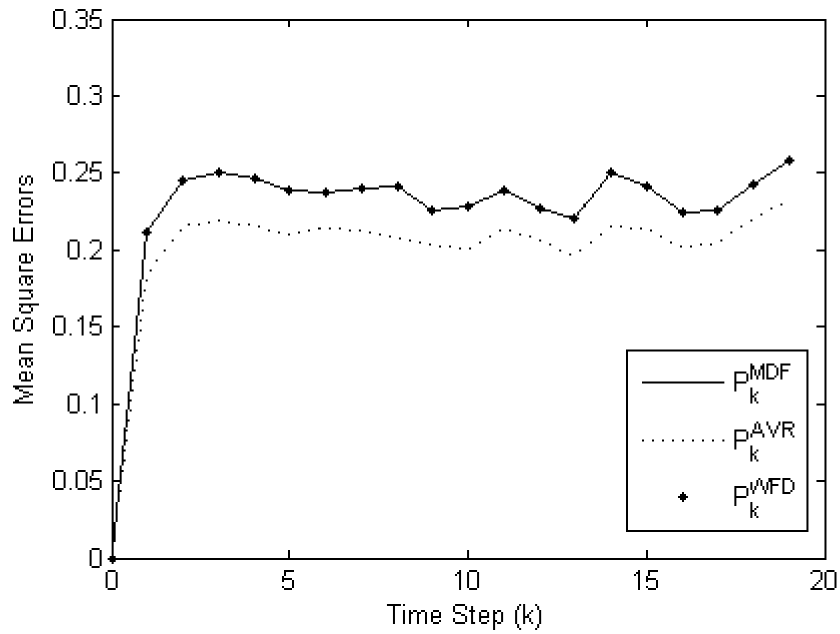

Fig. 1: MSEs in Case A

As discussed in Section III, P_k^{MDF} is identical to P_k^{WFD} when $N=3$ is odd. This fact is shown in Figure 1. Moreover, we observe that the accuracy of $\hat{x}_k^{AVR}$ is better, because all sensors measure the signal x_k without uncertainty.

Case B: 3 Sensors (Only 1 sensor with uncertainty); $N=3$, $\alpha^{(1)}=50$, $\alpha^{(i)}=0$, $i=2,3$

Differently from the Case A, one sensor transmits the measured data with uncertainty. Under this condition, Figure 2 shows the different result from that of Case A.

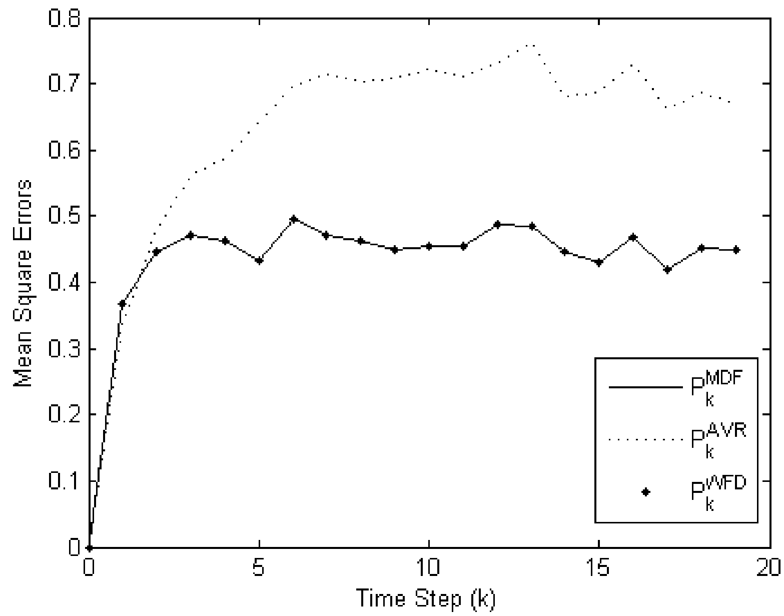

Fig. 2: MSEs in Case B

From Figure 2, we observe that $P_k^{MDF} = P_k^{WFD}$, and P_k^{AVR} is bigger than P_k^{MDF} and P_k^{WFD}. This phenomenon is caused by uncertainty $\alpha^{(1)}$.

Therefore, we confirm that $\hat{x}_k^{MDF}$ and $\hat{x}_k^{WFD}$ are more robust and accurate than $\hat{x}_k^{AVR}$ on uncertainty measurements.

Case C: 4 Sensors (Only 1 sensor with uncertainty); $N = 4$, $\alpha^{(1)} = 50$, $\alpha^{(i)} = 0$, $i = 2,3,4$

In this case, $N = 4$ is even. Differently from Case A, B, two MSEs P_k^{MDF} and P_k^{WFD} are not identical as discussed in Section III. Figure 3 illustrates the MSEs P_k^{MDF}, P_k^{AVR} and P_k^{WFD}.

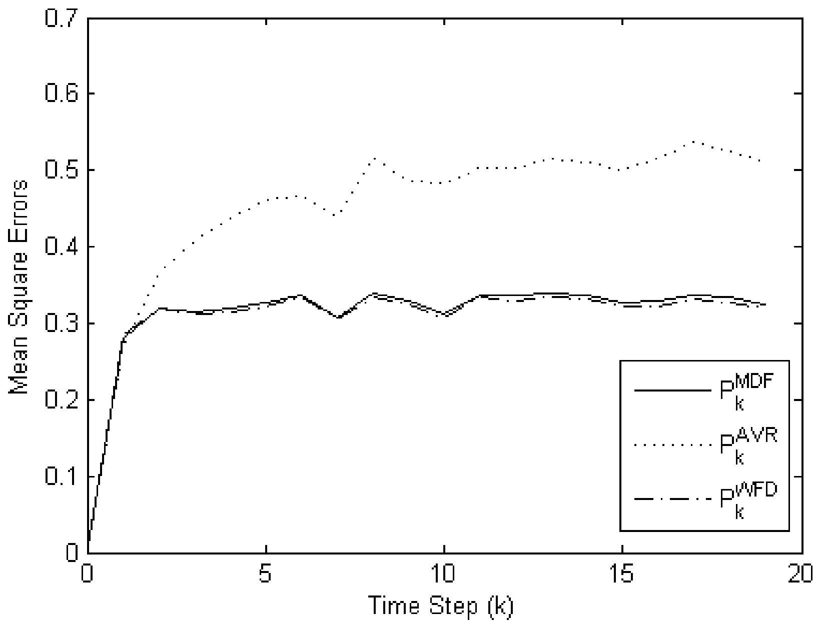

Fig. 3: MSEs in Case C

As shown in Figure 3, both P_k^{AVR} and P_k^{MDF} seem to represent same result, but P_k^{WFD} slightly more accurate than P_k^{MDF}. This means, $\hat{x}_k^{WFD}$ is the most accurate on uncertainty measurements.

Case D: 6 Sensors (2 sensors with uncertainty); $N = 6$, $\alpha^{(2)} = \alpha^{(5)} = 50$, $\alpha^{(i)} = 0$, $i = 1, 3, 4, 6$

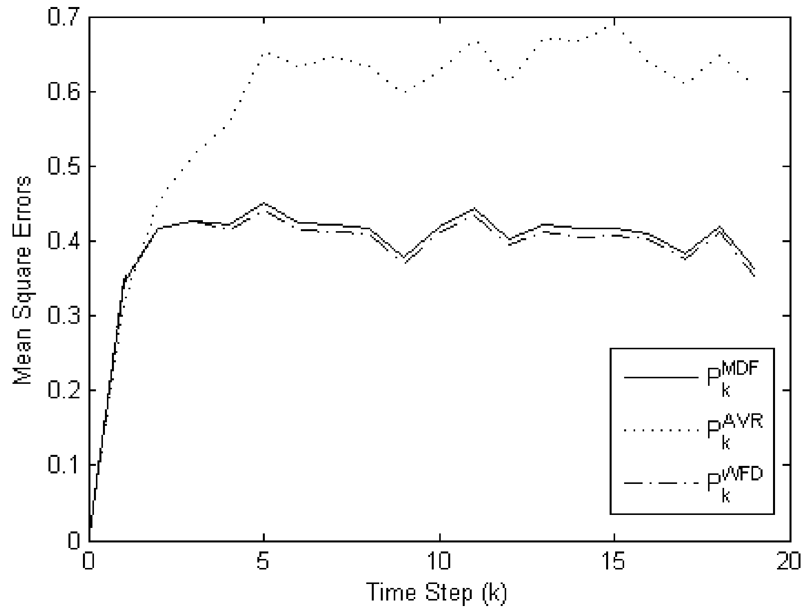

Figure 4: MSEs in Case D

In this case, The simulation condition is extended from that of the Case C; $N = 6$, two uncertainties $\alpha^{(2)}$, $\alpha^{(5)}$. However, even if the condition is extended, we observe the same result that $P_k^{AVR} > P_k^{MDF} > P_k^{WFD}$ as shown in Figure 4.

Therefore, we conclude that WFD is the best algorithm regardless of the number of sensors N for uncertainty measurements.

V. CONCLUSION

This paper focuses on two robust fusion algorithms WFD and MDF for a linear system with observation uncertainty. WFD and MDF Since these fusion algorithms do not consider system (signal) information affected by uncertainties, they can be robust than the classical fusion algorithm using average estimation.

Also, among proposed algorithms, WFD turn out the robust fusion algorithm under measurements system with uncertainty. These facts are supported by numerical examples demonstrating the concrete accuracies. Therefore, WFD and MDF are useful and applicable when uncertainty measurements are considered in real application.

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