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ON A FAIR MANIFOLD FARE RATING ON A LONG TRAFFIC LINE

Summary. The paper studies the possibilities to design a fair manifold tariff on a long traffic line. If a single tariff is used on a long bus or railway line, passengers travelling long distances are favoured at the expense of those travelling short distances. The fairest approach to tariff is setting an individual tariff for every origin–destination relation of line stops that expresses real travel costs. However, sometimes the individual tariff is too complicated and is therefore replaced by double-, triple- or manifold tariff. This paper shows how to design a manifold tariff in order to minimize unfairness to passengers.

1. INTRODUCTION

Suppose we are given a long bus line with a single tariff. Some passengers travel long distances while others take short trips. Such a single tariff is advantageous for the first-mentioned passengers. However, a single tariff is inconvenient for short-trip passengers. Hamacher and Schöbel tried to solve this problem by dividing serviced area into zones in [2, 3]. Another way to improve fairness is to introduce a double tariff – fare x for passengers travelling at most k laps and fare y for passengers travelling more than k laps. A generalization of double tariff is the approach where passengers are sorted into several groups by the number of laps traveled and determine fare for every group. This paper shows how to design such a fair manifold tariff.

2. A MATHEMATICAL MODELAND ITS SOLUTION

Suppose we are given a bus line L with n bus stops. Lap of a line L is a segment of a bus line between two successive bus stops. Suppose, we are given a line L with n bus stops and the following input data:

R_i – the number of passengers travelling exactly along i laps of the line L ($i = 1,2, \dots, n-1$)

t_i – ideal, but for some reasons inapplicable, distance tariff for passengers travelling along exactly i laps of the line L

K – the number of tariff regions

the numbers $0 = k_0 < k_1 < k_2 < \cdots < k_K = n-1$ – boundary numbers of tariff regions – all passengers travelling along i laps where $k_{i-1} < j \leq k_i$ pay the same fare.

Let $x_1, x_2, \dots, x_K$ be unknown variables with the following meaning:

x_1 – manifold tariff fare for passengers traveling at most k_1 laps

x_2 – manifold tariff fare for passengers traveling from $(k_1 + 1)$ to k_2 laps

x_i – manifold tariff fare for passengers traveling from $(k_{i-1} + 1)$ to k_i laps

x_K – manifold tariff fare for passengers traveling from $(k_{K-1} + 1)$ to k_K laps; $k_K = n-1$

Total fare on line L in the case of ideal fair tariff is

$$F = \sum_{i=1}^{n-1} t_i R_i \qquad (1)$$

Total fare on line L in the case of multifold tariff is

$$F_m = \sum_{r=1}^{K} \sum_{i=k_{r-1}+1}^{k_r} x_r R_i \qquad (2)$$

A bus provider wants to keep the income from double tariff the same as the one from ideal fair tariff, i.e.:

$$F = F_m \qquad (3)$$

$$\sum_{i=1}^{n-1} t_i R_i = \sum_{r=1}^{K} \sum_{i=k_{r-1}+1}^{k_r} x_r R_i \qquad (4)$$

There are several ways to express unfairness u_i to a passenger travelling exactly i laps. A comprehensive survey of attitudes to fairness is presented in [1]. We decided to use the second power of the difference between the ideal fare and the multifold fare of a single passenger travelling exactly i laps:

$$u_i = \begin{cases} (x_1 - t_i)^2 & \text{if} & 0 < i \le k_1 \\ (x_2 - t_i)^2 & \text{if} & k_1+1 < i \le k_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ (x_K - t_i)^2 & \text{if} & k_{K-1}+1 < i \le k_K \end{cases} \qquad (5)$$

There are several reasons for this selection. The perception of unfairness is not linear. People are willing to ignore small injustices, but are sensitive to large discrepancies. The simplest model of nonlinear increasing perception of unfairness is a quadratic function.

Another motive to use the second power of differences is a similarity with the linear regression procedure where parameters of the regression line are obtained by minimization of the sum of squared differences. Last but not least, the resulting mathematical model allows an exact mathematical solution based on well-known mathematical methods.

Total unfairness to all passengers travelling exactly i laps is

$$R_i u_i = R_i (x_r - t_i)^2 \qquad (6)$$

where r is such that $i \in (k_{r-1}, k_r\rangle$.

Total unfairness to all passengers travelling more than k_{r-1} and at most k_r laps (i.e. that of all passengers with fare x_r) is

$$\sum_{i=k_{r-1}+1}^{k_r} R_i u_i = \sum_{i=k_{r-1}+1}^{k_r} R_i (x_r - t_i)^2 \qquad (7)$$

Total unfairness to all passengers on the considered line can be calculated as follows:

$$U(x_1, x_2, \dots, x_K) = \sum_{r=1}^{K} \sum_{i=k_{r-1}+1}^{k_r} R_i (x_r - t_i)^2 \qquad (8)$$

Our next goal is to determine the given number K and fixed numbers

$$0 = k_0 < k_1 < k_2 < \dots < k_K = n - 1 \qquad (9)$$

representing tariff regions $(k_{r-1}, k_r\rangle, r = 1,2,\dots,K$ fares $x_1, x_2, \dots, x_K$ which minimize the total unfairness to all passengers. A necessary constraint is to retain unchanged the total income on the considered line. This leads to the following mathematical problem:
Minimize

$$U(x_1, x_2, \dots, x_K) = \sum_{r=1}^{K} \sum_{i=k_{r-1}+1}^{k_r} R_i (x_r - t_i)^2 \qquad (10)$$

subject to

$$\sum_{i=1}^{k} x_i R_i = \sum_{r=1}^{K} \sum_{i=k_{r-1}+1}^{k_r} x_r R_i \qquad (11)$$

This formulation is a constrained extreme problem solvable by the Lagrange Multiplier Method.

The next procedure of minimizing (1) subject to (2) is slightly technical and a reader who is not interested in the mathematical solution can skip to formulas (7) and (8) expressing optimum fare and minimal total unfairness.

Denote

$$F(x_1, x_2, \ldots, x_K, \lambda) = U(x_1, x_2, \ldots, x_K) - \lambda\left[\sum_{r=1}^{K} \sum_{i=k_{r-1}+1}^{k_r} x_r R_i - \sum_{i=1}^{n-1} t_i R_i\right] \quad (12)$$

$$F(x_1, x_2, \ldots, x_K, \lambda) = \sum_{r=1}^{K} \sum_{i=k_{r-1}+1}^{k_r} R_i(x_r - t_i)^2 - \lambda\left[\sum_{r=1}^{K} \sum_{i=k_{r-1}+1}^{k_r} x_r R_i - \sum_{i=1}^{n-1} t_i R_i\right] (13)$$

Formula (1) for $U(x_1, x_2, \ldots, x_K)$ defines a differentiable function on R^K (where R is the set of all real numbers). The Lagrange Multipier Theorem asserts that if $U(x_1, x_2, \ldots, x_K)$ achieves a minimum on R2 subject to (2), then the minimum is necessarily achieved at a point where all partial derivatives

$$\frac{\partial F(x_1, x_2, \ldots, x_K, \lambda)}{\partial \lambda} = 0 \quad \text{and} \quad \frac{\partial F(x_1, x_2, \ldots, x_K, \lambda)}{\partial x_r} = 0 \quad \text{for all } r = 1, 2, \ldots, K.$$

Let us see where the Lagrange Multiplier method tells us to look for an optimal solution. It holds that:

$$\frac{\partial F(x_1, x_2, \ldots, x_K, \lambda)}{\partial \lambda} = \left[\sum_{r=1}^{K} \sum_{i=k_{r-1}+1}^{k_r} x_r R_i - \sum_{i=1}^{n-1} t_i R_i\right] = \sum_{r=1}^{K} x_r \sum_{i=k_{r-1}+1}^{k_r} R_i - \sum_{i=1}^{n-1} t_i R_i = 0 \quad (14)$$

$$\frac{\partial F(x_1, x_2, \ldots, x_K, \lambda)}{\partial x_r} = 2\sum_{i=k_{r-1}+1}^{k_r} R_i (x_r - t_i) - \lambda \sum_{i=k_{r-1}+1}^{k_r} R_i = 0 \quad (15)$$

It follows from equation (15):

$$\frac{\partial F(x_1, x_2, \ldots, x_K, \lambda)}{\partial x_r} = 2x_r \sum_{i=k_{r-1}+1}^{k_r} R_i - 2\sum_{i=k_{r-1}+1}^{k_r} R_i t_i - \lambda \sum_{i=k_{r-1}+1}^{k_r} R_i =$$
$$= (2x_r - \lambda)\sum_{i=k_{r-1}+1}^{k_r} R_i - 2\sum_{i=k_{r-1}+1}^{k_r} R_i t_i = 0 \quad (16)$$

the solution of (5) is as follows:

$$x_r = \frac{\sum_{i=k_{r-1}+1}^{k_r} R_i t_i}{\sum_{i=k_{r-1}+1}^{k_r} R_i} + \frac{\lambda}{2} \quad (17)$$

Substitution for x_r from (17) into equation (14) gives

$$\sum_{r=1}^{K} x_r \sum_{i=k_{r-1}+1}^{k_r} R_i - \sum_{i=1}^{n-1} t_i R_i = \sum_{r=1}^{K} \sum_{i=k_{r-1}+1}^{k_r} R_i t_i + \frac{\lambda}{2}\sum_{r=1}^{K} \sum_{i=k_{r-1}+1}^{k_r} R_i -$$
$$\sum_{i=1}^{n-1} t_i R_i == \sum_{i=1}^{n-1} t_i R_i + \frac{\lambda}{2}\sum_{r=1}^{K} \sum_{i=k_{r-1}+1}^{k_r} R_i - \sum_{i=1}^{n-1} t_i R_i = \frac{\lambda}{2}\sum_{i=1}^{n-1} R_i = 0 \quad (18)$$

It follows from the last equation that $\lambda = 0$, and therefore it holds for optimum fares x_r^* minimizing total unfairness

$$x_r^* = \frac{\sum_{i=k_{r-1}+1}^{k_r} R_i t_i}{\sum_{i=k_{r-1}+1}^{k_r} R_i} \text{ for } r = 1, 2, \ldots, K \quad (19)$$

To guarantee that the function $U(x_1, x_{2, \ldots,} x_K)$ achieves minimum at point $(x_1^*, x_2^*, \ldots, x_K^*)$ (19), it is necessary to show that all following second partial derivatives are greater than zero.

Indeed, it holds that:

$$\frac{\partial F(x_1, x_2, \ldots, x_K, \lambda)}{\partial x_r^2} = 2\sum_{i=k_{r-1}+1}^{k_r} R_i > 0, \qquad r = 1, 2, \ldots, K \quad (20)$$

However, fares given by formula (19) are optimal for fixed tariff range borders

$$0 = k_0 < k_1 < k_2 < \ ... \ < k_K = n - 1 \qquad (21)$$

and corresponding total unfairness is

$$U^* = U^*(k_1, k_2, \ldots, k_K) = \sum_{j=1}^{n-1} R_j u_j = \sum_{r=1}^{K} \sum_{j=k_{r-1}+1}^{k_r} R_j \left(\frac{\sum_{i=k_{r-1}+1}^{k_r} R_i t_i}{\sum_{i=k_{r-1}+1}^{k_r} R_i} - t_j \right)^2 \qquad (22)$$

The second step of our proposed procedure is to find numbers $0 = k_0 < k_1 < k_2 < \ ... \ < k_K = n - 1$ for which is $U^* = U^*(k_1, k_2, \ldots, k_K)$ minimal.

Notice that the optimum x_r depends only on the boundaries of the r-th tariff region and numbers of passengers belonging to this region.

Let us define an acyclic digraph $G = (V, A, C)$ with vertex set defined as

$$V = \{(0,0), (K,n)\} \cup \{(r,i) \mid r = 1,2, \ldots, K - 1, \quad i = 1,2, \ldots, n - 1\} \qquad (23)$$

and arc set

$$A = \{((r,i)(r+1,j)) \mid (r,i) \in V, (r+1,j) \in V, i < j\} \qquad (24)$$

The arc cost is defined as follows:

$$c((0,0)(1,j)) = 0 \qquad (25)$$

$$c((r,i)(r+1,j)) = \sum_{j=k_{r-1}+1}^{k_r} R_j \left(\frac{\sum_{i=k_{r-1}+1}^{k_r} R_i t_i}{\sum_{i=k_{r-1}+1}^{k_r} R_i} - t_j \right)^2 \qquad (26)$$

Let us note that the cost of the arc $((r,i)(r+1,j))$ is the total unfairness to all passengers travelling more than k_{r-1} and at most k_r laps.
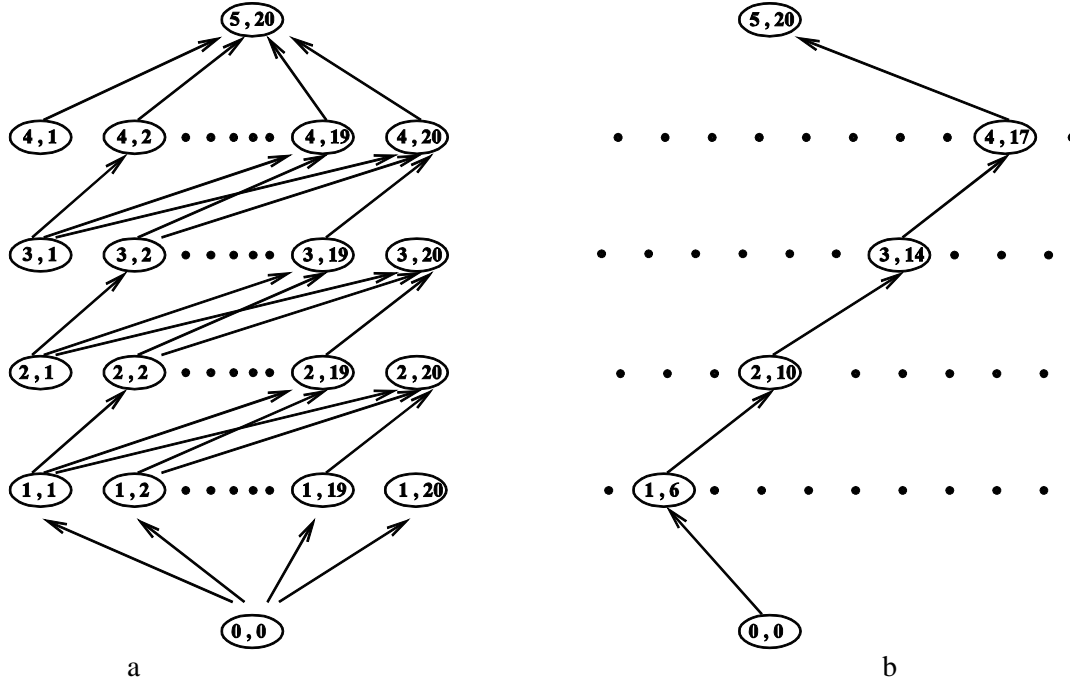
Fig. 1. a) Digraph $= (V, A, C)$, b) A path in G corresponds to 5 tariff region boundaries $k_1 = 6, k_2 = 10, k_3 = 14, k_4 = 17, k_5 = 20$

Every path form vertex $(0,0)$ to vertex (K, n) in digraph $G = (V, A, C)$ is in the form

$$(0,0), (1, k_1), (2, k_2), \ldots, (K - 1, k_{K-1}), (K, n) \qquad (27)$$

with the following length

$$0 + \sum_{j=1}^{k_1} R_j \left(\frac{\sum_{i=1}^{k_1} R_i t_i}{\sum_{i=1}^{k_1} R_i} - t_j \right)^2 + \sum_{j=k_1+1}^{k_2} R_j \left(\frac{\sum_{i=k_1+1}^{k_2} R_i t_i}{\sum_{i=k_1+1}^{k_2} R_i} - t_j \right)^2 +$$

$$+ \sum_{j=k_2+1}^{k_3} R_j \left(\frac{\sum_{i=k_2+1}^{k_3} R_i t_i}{\sum_{i=k_2+1}^{k_3} R_i} - t_j \right)^2 \ldots + \sum_{j=k_{K-1}+1}^{k_K} R_j \left(\frac{\sum_{i=k_{K-1}+1}^{k_K} R_i t_i}{\sum_{i=k_{K-1}+1}^{k_K} R_i} - t_j \right)^2 =$$

$$\sum_{r=1}^{K} \sum_{j=k_{r-1}+1}^{k_r} R_j \left(\frac{\sum_{i=k_{r-1}+1}^{k_r} R_i t_i}{\sum_{i=k_{r-1}+1}^{k_r} R_i} - t_j \right)^2 = U^*(k_1, k_2, \ldots, k_K) \tag{28}$$

The shortest path from vertex $(0,0)$ to vertex (K, n) determines a series

$$0 = k_0 < k_1 < k_2 < \ldots < k_K = n - 1 \tag{29}$$

which minimizes total unfairness $U^*(k_1, k_2, \ldots, k_K)$. We have just reduced the problem of minimizing $U^*(k_1, k_2, \ldots, k_K)$ to a shortest path problem in diagraph $G = (V, A, C)$.

3. EXPERIMENTS

We have used real data from public transport in the Slovakian town of Martin to compute optimum manifold fares. The public transport system in Martin serves circa 40,000 passengers in one working day. The longest trip has 26 bus stops and the histogram of traveling distance (number of traveling laps) is shown in Fig. 2.
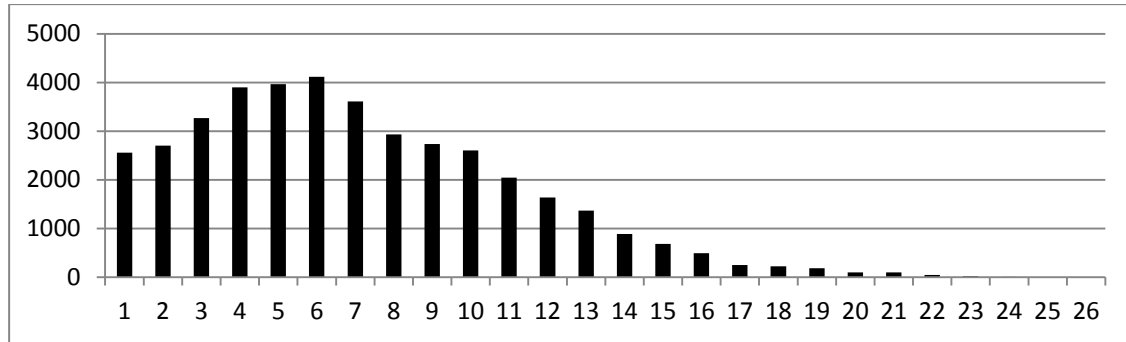

Fig. 2. Histogram of traveling distances of passengers in Martin

3.1. Ideal fare

We have exactly 40,409 sold tickets in our dataset and the price of a basic ticket in Martin for a single tariff public transport system is $0.60 €$ today. Total receipts without discount amounted to $24,245.40 €$ Suppose fixed and variable costs are divided in the ratio of 50:50; the ideal fare for a passenger i traveling d_i loops can be computed using the formula $t_i = 0.30 + 0.0426 d_i$.

3.2. Optimal manifold fares

The proposed algorithm was implemented in the C# programming language using Microsoft Visual Studio. The LabelSet algorithm was used to search for an optimal solution as the shortest path in the directed graph of zone strategies. Optimal results for the public transport of Martin with 2, 3, 4 and 5 zones are shown in Table 1.

Table 1

Optimal manifold fares

Number of tariff regions	Tariff region	Laps	Fare	Unfairness
2	1	1 – 8	0.50 €	407.9
	2	9 – 26	0.80 €	
3	1	1 – 5	0.44 €	201.2
	2	6 – 10	0.63 €	
	3	11 – 26	0.87 €	
4	1	1 – 4	0.42 €	119.9
	2	5 – 8	0.57 €	
	3	9 – 13	0.75 €	
	4	14 – 26	0.98 €	
5	1	1 – 3	0.39 €	77.1
	2	4 – 6	0.51 €	
	3	7 – 9	0.64 €	
	4	10 – 13	0.78 €	
	5	14 – 26	0.98 €	

Acknowledgements

References

1. Bertsimas, D. & Farias, V.F. & Trichakis, N. The price of fairness. *Operations Research.* 2011. Vol. 59(1). P. 17-31.
2. Černá, A. & Černý, J. *Teorie řízení a rozhodování v dopravních systémech* [In Czech: *Theory of Management and Decision Making in Transport Systems*.] Jan Perner Institute. 2004. ISBN 80-86530-15-9.
3. Černá, A. Optimalizace tarifů – časovéj ízdenky. [In Czech: Tariff Optimization – Season Tickets]. *Sci. Pap. U. of Pardubice.* 1999. Vol. B 4. P. 267-270.
4. Hamacher, H.W. & Schöbel, A. Design of Zone Tariff Systems in Public Transportation. *Operations Research.* November–December 2004. Vol. 52. No. 6. P. 897-908. ISSN 0030-364X.
5. Koháni, M. Zone partitioning problem with given prices and number of zones in counting zones tariff system. *SOR´13: proceedings of the 12th International Symposium on Operational Research.* Dolenjske Toplice, Slovenia (Zadnik Stirn, L. & Žerovnik, J. & Povh, J. & Drobne, S. & Lisec, A. & eds.). Slovenian Society Informatika, Section for Operational Research. Ljubljana. 2013. P. 75-80.
6. Koháni, M. Impact of changes in fares and demand to the tariff zones design. *Proceedings of 31st international conference - Mathematical methods in economics.* Jihlava, Czech Republic. College of Polytechnics. 2013. P. 417-422.
7. Majer, T. & Palúch, S. Several Attitudes to a Fair Double Rate on a Bus Line. *Proceeding of 31st international conference Mathematical methods in economics.* Olomouc, Czech Republic. 2014.
8. Palúch, S. On a fair fare rating on a bus line. *Communications: Scientific Letters of the University of Žilina.* 2013. Vol. 15. No. 1. P. 25-28. ISSN 1335-4205.

9. Schöbel, A. Zone planning in public transportation. In: Bianco, L. & Toth, P. eds. *Advanced Methods in Transportation Analysis.* 1996. Berlin, Germany. Springer Verlag. P. 117-134.