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SUJATHA DISTRIBUTION AND ITS APPLICATIONS

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ABSTRACT

In this paper a new one-parameter lifetime distribution named "Sujatha Distribution" with an increasing hazard rate for modelling lifetime data has been suggested. Its first four moments about origin and moments about mean have been obtained and expressions for coefficient of variation, skewness, kurtosis and index of dispersion have been given. Various mathematical and statistical properties of the proposed distribution including its hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, and stress-strength reliability have been discussed. Estimation of its parameter has been discussed using the method of maximum likelihood and the method of moments. The applications and goodness of fit of the distribution have been discussed with three real lifetime data sets and the fit has been compared with one-parameter lifetime distributions including Akash, Shanker, Lindley and exponential distributions.

Key words: lifetime distributions, Akash distribution, Shanker distribution, Lindley distribution, mathematical and statistical properties, estimation of parameter, goodness of fit.

1. Introduction

The analyses and modelling of lifetime data are crucial in all applied sciences including engineering, medical science, insurance and finance, among other things. Although a number of lifetime distributions have evolved in statistical literature, including exponential, Lindley, Akash, Shanker, gamma, lognormal and Weibull distributions, amongst other things, each has its own advantages and disadvantages in modelling lifetime data. The exponential, Lindley, Akash, Shanker and Weibull distributions are more popular than the gamma and the lognormal distributions because the survival functions of the gamma and the lognormal distributions cannot be expressed in closed forms and both require numerical integration. Although Akash, Shanker, Lindley and exponential distributions are of one parameter, Akash, Shanker and Lindley distributions have

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an advantage over the exponential distribution that the exponential distribution has a constant hazard rate, whereas Akash, Shanker, and Lindley distributions have a monotonically increasing hazard rate. Further, Akash, Shanker, and Lindley distributions have many interesting mathematical and statistical properties in terms of shape, moments, skewness, kurtosis, hazard rate function: mean residual life function, stochastic ordering, mean deviations, order statistics, Bonferroni and Lorenz curves, entropy measure, stress-strength reliability and index of dispersion.

The probability density function (p.d.f.) and the cumulative distribution function (c.d.f.) of Lindley (1958) distribution are given by

$$f_1(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x} \quad ;x>0,\ \theta>0 \tag{1.1}$$

$$F_1(x;\theta)=1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x} \ ;x>0,\theta>0 \tag{1.2}$$

The density (1.1) is a two-component mixture of an exponential distribution with scale parameter θ and a gamma distribution with shape parameter 2 and scale parameter θ with their mixing proportions $\frac{\theta}{\theta+1}$ and $\frac{1}{\theta+1}$ respectively. A detailed study on its various mathematical properties, estimation of parameter and application showing the superiority of Lindley distribution over exponential distribution for the waiting times before service of the bank customers has been done by Ghitany *et al.* (2008). The Lindley distribution has been generalized, extended and modified along with its applications in modelling lifetime data from different fields of knowledge by different researchers including Zakerzadeh and Dolati (2009), Nadarajah *et al.* (2011), Deniz and Ojeda (2011), Bakouch *et al.* (2012), Shanker and Mishra (2013 a, 2013 b), Shanker and Amanuel (2013), Shanker *et al*. (2013), Elbatal *et al.* (2013), Ghitany *et al.* (2013), Merovci (2013), Liyanage and Pararai (2014), Ashour and Eltehiwy (2014), Oluyede and Yang (2014), Singh *et al.* (2014), Shanker *et al* (2015, 2016 a, 2016 b), Sharma *et al.* (2015 a, 2015 b), Alkarni (2015), Pararai *et al.* (2015), Abouammoh *et al.* (2015), among others.

Shanker (2015 a) has introduced one-parameter Akash distribution for modelling lifetime data defined by its p.d.f. and c.d.f.

$$f_2(x;\theta)=\frac{\theta^3}{\theta^2+2}\left(1+x^2\right)e^{-\theta x} \quad ;x>0,\ \theta>0 \tag{1.3}$$

$$F_2(x;\theta)=1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x} \ ;x>0,\theta>0 \tag{1.4}$$

Shanker (2015 a) has shown that density (1.3) is a two-component mixture of an exponential distribution with scale parameter θ and a gamma distribution with shape parameter 3 and a scale parameter θ with their mixing proportions $\frac{\theta^2}{\theta^2+2}$ and $\frac{2}{\theta^2+2}$ respectively. Shanker (2015 a) has discussed its various mathematical and statistical properties including its shape, moment generating function, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic orderings, mean deviations, distribution of order statistics, Bonferroni and Lorenz curves, Renyi entropy measure, stress-strength reliability, among other things. Shanker *et al* (2015 c) has a detailed study on modelling of various lifetime data from different fields using Akash, Lindley and exponential distributions and concluded that Akash distribution has some advantage over Lindley and exponential distributions. Further, Shanker (2015 c) has obtained Poisson mixture of Akash distribution named Poisson-Akash distribution (PAD) and discussed its various mathematical and statistical properties, estimation of its parameter and applications for various count data sets.

The probability density function and the cumulative distribution function of Shanker distribution introduced by Shanker (2015 b) are given by

$$f_3(x;\theta)=\frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x} \quad ;x>0,\ \theta>0 \tag{1.5}$$

$$F_3(x,\theta)=1-\frac{(\theta^2+1)+\theta x}{\theta^2+1}e^{-\theta x} \quad ;x>0,\theta>0 \tag{1.6}$$

Shanker (2015 b) has shown that density (1.5) is a two-component mixture of an exponential distribution with scale parameter θ and a gamma distribution with shape parameter 2 and a scale parameter θ with their mixing proportions $\frac{\theta^2}{\theta^2+1}$ and $\frac{1}{\theta^2+1}$ respectively. Shanker (2015 b) has discussed its various mathematical and statistical properties including its shape, moment generating function, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic orderings, mean deviations, distribution of order statistics, Bonferroni and Lorenz curves, Renyi entropy measure, stress-strength reliability, among other things. Further, Shanker (2015 d) has obtained Poisson mixture of Shanker distribution named Poisson-Shanker distribution (PSD) and discussed its various mathematical and statistical properties, estimation of its parameter and applications for various count data sets.

In this paper we have proposed a new continuous distribution, which is better than Akash, Shanker, Lindley and exponential distributions for modelling lifetime data by considering a three-component mixture of an exponential distribution with scale parameter θ, a gamma distribution with shape parameter 2 and scale parameter θ, and a gamma distribution with shape parameter 3 and scale parameter θ with their mixing proportions $\frac{\theta^2}{\theta^2+\theta+2}$, $\frac{\theta}{\theta^2+\theta+2}$ and $\frac{2}{\theta^2+\theta+2}$, respectively. The probability density function (p.d.f.) of a new one-parameter lifetime distribution can be introduced as

$$f_4(x;\theta)=\frac{\theta^3}{\theta^2+\theta+2}\left(1+x+x^2\right)e^{-\theta x}\quad ;x>0,\ \theta>0 \tag{1.7}$$

We would call this new one-parameter continuous lifetime distribution "Sujatha distribution (S.D)". The corresponding cumulative distribution function (c.d.f.) of Sujatha distribution (1.7) is obtained as

$$F_4(x,\theta)=1-\left[1+\frac{\theta x\left(\theta x+\theta+2\right)}{\theta^2+\theta+2}\right]e^{-\theta x};x>0,\theta>0 \tag{1.8}$$

The graphs of the p.d.f. and the c.d.f. of Sujatha distribution (1.7) for different values of θ are shown in Figures 1(a) and 1(b).

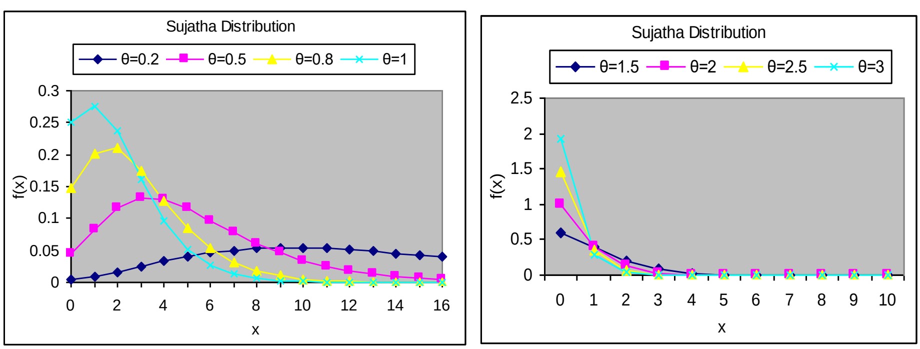

Figure 1(a). Graphs of p.d.f. of Sujatha distribution for selected values of parameter

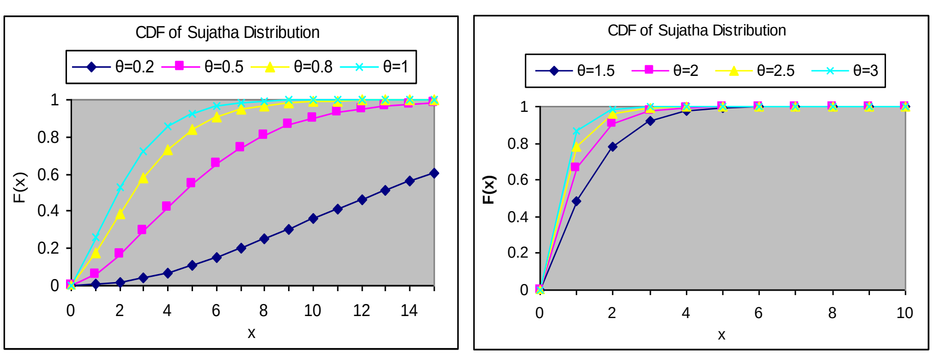

Figure 2(a). Graphs of c.d.f. of Sujatha distribution for selected values of parameter

2. Moment generating function, moments and associated measures

The moment generating function of Sujatha distribution (1.7) can be obtained as

$$M_X(t) = \frac{\theta^3}{\theta^2+\theta+2}\int_0^\infty e^{-(\theta-t)x}\left(1+x+x^2\right)dx$$

$$= \frac{\theta^3}{\theta^2+\theta+2}\left[\frac{1}{\theta-t}+\frac{1}{(\theta-t)^2}+\frac{2}{(\theta-t)^3}\right]$$

$$= \frac{\theta^3}{\theta^2+\theta+2}\left[\frac{1}{\theta}\sum_{k=0}^{\infty}\left(\frac{t}{\theta}\right)^k+\frac{1}{\theta^2}\sum_{k=0}^{\infty}\binom{k+1}{k}\left(\frac{t}{\theta}\right)^k+\frac{2}{\theta^3}\sum_{k=0}^{\infty}\binom{k+2}{k}\left(\frac{t}{\theta}\right)^k\right]$$

$$= \sum_{k=0}^{\infty}\frac{\theta^2+(k+1)\theta+(k+1)(k+2)}{\left(\theta^2+\theta+2\right)}\left(\frac{t}{\theta}\right)^k$$

The r moment about origin $\mu_r{}'$ obtained as the coefficient of $\frac{t^r}{r!}$ in $M_X(t)$, of Sujatha distribution (1.7) has been obtained as

$$\mu_r{}' = \frac{r!\left[\theta^2+(r+1)\theta+(r+1)(r+2)\right]}{\theta^r\left(\theta^2+\theta+2\right)} \quad ; r=1,2,3,4,...$$

The first four moments about origin of Sujatha distribution (1.7) are thus obtained as

$$\mu_1{}' = \frac{\theta^2+2\theta+6}{\theta\left(\theta^2+\theta+2\right)}, \qquad \mu_2{}' = \frac{2\left(\theta^2+3\theta+12\right)}{\theta^2\left(\theta^2+\theta+2\right)},$$

$$\mu_3{}' = \frac{6\left(\theta^2+4\theta+20\right)}{\theta^3\left(\theta^2+\theta+2\right)}, \qquad \mu_4{}' = \frac{24\left(\theta^2+5\theta+30\right)}{\theta^4\left(\theta^2+\theta+2\right)}$$

Using the relationship between moments about mean and the moments about origin, the moments about mean of Sujatha distribution (1.7) are obtained as

$$\mu_2 = \frac{\theta^4 + 4\theta^3 + 18\theta^2 + 12\theta + 12}{\theta^2\left(\theta^2 + \theta + 2\right)^2}$$

$$\mu_3 = \frac{2\left(\theta^6 + 6\theta^5 + 36\theta^4 + 44\theta^3 + 54\theta^2 + 36\theta + 24\right)}{\theta^3\left(\theta^2 + \theta + 2\right)^3}$$

$$\mu_4 = \frac{3\left(3\theta^8 + 24\theta^7 + 172\theta^6 + 376\theta^5 + 736\theta^4 + 864\theta^3 + 912\theta^2 + 480\theta + 240\right)}{\theta^4\left(\theta^2 + \theta + 2\right)^4}$$

The coefficient of variation $\left(C.V\right)$, coefficient of skewness $\left(\sqrt{\beta_1}\right)$, coefficient of kurtosis $\left(\beta_2\right)$, index of dispersion $\left(\gamma\right)$ of Sujatha distribution (1.7) are thus obtained as

$$C.V = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\theta^4 + 4\theta^3 + 18\theta^2 + 12\theta + 12}}{\theta^2 + 2\theta + 6}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2\left(\theta^6 + 6\theta^5 + 36\theta^4 + 44\theta^3 + 54\theta^2 + 36\theta + 24\right)}{\left(\theta^4 + 4\theta^3 + 18\theta^2 + 12\theta + 12\right)^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\left(3\theta^8 + 24\theta^7 + 172\theta^6 + 376\theta^5 + 736\theta^4 + 864\theta^3 + 912\theta^2 + 480\theta + 240\right)}{\left(\theta^4 + 4\theta^3 + 18\theta^2 + 12\theta + 12\right)^2}$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{\theta^4 + 4\theta^3 + 18\theta^2 + 12\theta + 12}{\theta\left(\theta^2 + \theta + 2\right)\left(\theta^2 + 2\theta + 6\right)}$$

The over-dispersion, equi-dispersion, under-dispersion of Sujatha, Akash, Shanker, Lindley and exponential distributions for varying values of their parameter θ are presented in Table 1.

Table 1. Over-dispersion, equi-dispersion and under-dispersion of Sujatha, Akash, Shanker, Lindley and exponential distributions for varying values of their parameter θ

Distribution	Over-dispersion $\left(\mu<\sigma^2\right)$	Equi-dispersion $\left(\mu=\sigma^2\right)$	Under-dispersion $\left(\mu>\sigma^2\right)$
Sujatha	$\theta<1.364271174$	$\theta=1.364271174$	$\theta>1.364271174$
Akash	$\theta<1.515400063$	$\theta=1.515400063$	$\theta>1.515400063$
Shanker	$\theta<1.171535555$	$\theta=1.171535555$	$\theta>1.171535555$
Lindley	$\theta<1.170086487$	$\theta=1.170086487$	$\theta>1.170086487$
Exponential	$\theta<1$	$\theta=1$	$\theta>1$

3. Hazard rate function and mean residual life function

Let X be a continuous random variable with p.d.f. $f\left(x\right)$ and c.d.f. $F\left(x\right)$. The hazard rate function (also known as the failure rate function) and the mean residual life function of X are respectively defined as

$$h\left(x\right)=\lim_{\Delta x\to 0}\frac{P\left(X<x+\Delta x\middle|X>x\right)}{\Delta x}=\frac{f\left(x\right)}{1-F\left(x\right)} \tag{3.1}$$

and $$m\left(x\right)=E\left[X-x\middle|X>x\right]=\frac{1}{1-F\left(x\right)}\int_x^\infty\left[1-F\left(t\right)\right]\,dt \tag{3.2}$$

The corresponding hazard rate function, $h\left(x\right)$ and the mean residual life function, $m\left(x\right)$ of Sujatha distribution are thus obtained as

$$h\left(x\right)=\frac{\theta^3\left(1+x+x^2\right)}{\theta^2\left(1+x+x^2\right)+2\theta x+\theta+2} \tag{3.3}$$

and $$m\left(x\right)=\frac{\theta^2+\theta+2}{\left[\left(\theta^2+\theta+2\right)+\theta x\left(\theta x+\theta+2\right)\right]e^{-\theta x}}\int_x^\infty\left[1+\frac{\theta t\left(\theta t+\theta+2\right)}{\theta^2+\theta+2}\right]e^{-\theta t}dt$$

$$=\frac{\theta^2\left(x^2+x+1\right)+2\theta\left(2x+1\right)+6}{\theta\left[\left(\theta^2+\theta+2\right)+\theta x\left(\theta x+\theta+2\right)\right]} \tag{3.4}$$

It can be easily verified that $h(0)=\dfrac{\theta^3}{\theta^2+\theta+2}=f(0)$ and $m(0)=\dfrac{\theta^2+2\theta+6}{\theta\left(\theta^2+\theta+2\right)}=\mu_1'$. The graphs of $h(x)$ and $m(x)$ of Sujatha distribution (1.7) for different values of its parameter are shown in Figures 4(a) and 4(b).

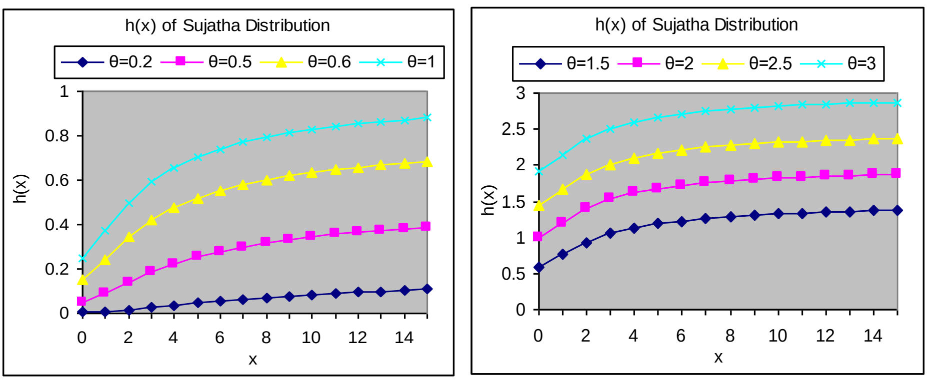

Figure 4(a). Graphs of $h(x)$ of Sujatha distribution for selected values of parameter

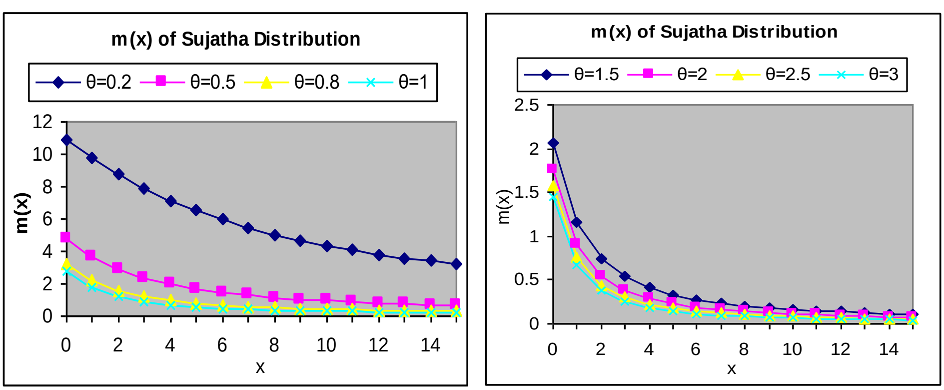

Figure 4(b). Graphs of $m(x)$ of Sujatha distribution for selected values of parameter

It is also obvious from the graphs of $h(x)$ and $m(x)$ that $h(x)$ is a monotonically increasing function of x and θ, whereas $m(x)$ is a monotonically decreasing function of x, and θ.

4. Stochastic orderings

Stochastic ordering of positive continuous random variables is an important tool for judging the comparative behaviour of continuous distributions. A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order $\left(X \le_{st} Y\right)$ if $F_X(x) \ge F_Y(x)$ for all x

(ii) hazard rate order $\left(X \le_{hr} Y\right)$ if $h_X(x) \ge h_Y(x)$ for all x

(iii) mean residual life order $\left(X \le_{mrl} Y\right)$ if $m_X(x) \le m_Y(x)$ for all x

(iv) likelihood ratio order $\left(X \le_{lr} Y\right)$ if $\dfrac{f_X(x)}{f_Y(x)}$ decreases in x.

The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions

$$\begin{array}{c} X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{mrl} Y \\ \Downarrow \\ X \le_{st} Y \end{array}$$

Sujatha distribution is ordered with respect to the strongest 'likelihood ratio' ordering as shown in the following theorem:

Theorem: Let X ~ Sujatha distribution$\left(\theta_1\right)$ and Y ~ Sujatha distribution$\left(\theta_2\right)$. If $\theta_1 > \theta_2$, then $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

Proof: We have

$$\frac{f_X(x)}{f_Y(x)} = \frac{\theta_1^3\left(\theta_2^2+\theta_2+2\right)}{\theta_2^3\left(\theta_1^2+\theta_1+2\right)} e^{-(\theta_1-\theta_2)x} \quad ; \; x>0$$

Now

$$\log\frac{f_X(x)}{f_Y(x)} = \log\left[\frac{\theta_1^3\left(\theta_2^2+\theta_2+2\right)}{\theta_2^3\left(\theta_1^2+\theta_1+2\right)}\right] - \left(\theta_1-\theta_2\right)x$$

This gives $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} = -(\theta_1 - \theta_2)$

Thus, for $\theta_1 > \theta_2$, $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} < 0$. This means that $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

5. Deviations from mean and median

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and the median. These are known as the mean deviation about the mean and the mean deviation about the median and are defined by

$$\delta_1(X) = \int_0^\infty |x-\mu|\, f(x)\,dx \text{ and } \delta_2(X) = \int_0^\infty |x-M|\, f(x)\,dx, \text{ respectively,}$$

where $\mu = E(X)$ and $M = \text{Median }(X)$.

The measures $\delta_1(X)$ and $\delta_2(X)$ can be calculated using the following relationships

$$\begin{aligned}\delta_1(X) &= \int_0^\mu (\mu - x) f(x)\,dx + \int_\mu^\infty (x-\mu) f(x)\,dx \\ &= \mu F(\mu) - \int_0^\mu x\, f(x)\,dx - \mu\left[1-F(\mu)\right] + \int_\mu^\infty x\, f(x)\,dx \\ &= 2\mu F(\mu) - 2\mu + 2\int_\mu^\infty x\, f(x)\,dx \\ &= 2\mu F(\mu) - 2\int_0^\mu x\, f(x)\,dx \end{aligned} \tag{5.1}$$

and

$$\begin{aligned}\delta_2(X) &= \int_0^M (M - x) f(x)\,dx + \int_M^\infty (x-M) f(x)\,dx \\ &= M\,F(M) - \int_0^M x\, f(x)\,dx - M\left[1-F(M)\right] + \int_M^\infty x\, f(x)\,dx \\ &= -\mu + 2\int_M^\infty x\, f(x)\,dx \end{aligned}$$

$$= \mu - 2\int_0^M x\, f(x)\, dx \tag{5.2}$$

Using p.d.f. (1.7) and expression for the mean of Sujatha distribution (1.7), we get

$$\int_0^\mu x\, f_4(x)\, dx = \mu - \frac{\left\{\theta^3\left(\mu^3+\mu^2+\mu\right)+\theta^2\left(3\mu^2+2\mu+1\right)+2\theta\left(3\mu+1\right)+6\right\}e^{-\theta\mu}}{\theta\left(\theta^2+\theta+2\right)} \tag{5.3}$$

$$\int_0^M x\, f_4(x)\, dx = \mu - \frac{\left\{\theta^3\left(M^3+M^2+M\right)+\theta^2\left(3M^2+2M+1\right)+2\theta\left(3M+1\right)+6\right\}e^{-\theta M}}{\theta\left(\theta^2+\theta+2\right)} \tag{5.4}$$

Using expressions from (5.1), (5.2), (5.3) and (5.4), and after some mathematical simplifications, the mean deviation about the mean, $\delta_1(X)$ and the mean deviation about the median, $\delta_2(X)$ of Sujatha distribution are obtained as

$$\delta_1(X) = \frac{2\left[\theta^2\left(\mu^2+\mu+1\right)+2\theta\left(2\mu+1\right)+6\right]e^{-\theta\mu}}{\theta\left(\theta^2+\theta+2\right)} \tag{5.5}$$

and

$$\delta_2(X) = \frac{2\left[\theta^3\left(M^3+M^2+M\right)+\theta^2\left(3M^2+2M+1\right)+2\theta\left(3M+1\right)+6\right]e^{-\theta M}}{\theta\left(\theta^2+\theta+2\right)} - \mu \tag{5.6}$$

6. Bonferroni and Lorenz curves and indices

The Bonferroni and Lorenz curves (Bonferroni, 1930) and Bonferroni and Gini indices have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curves are defined as

$$B(p)=\frac{1}{p\mu}\int_0^q x\,f(x)\,dx=\frac{1}{p\mu}\left[\int_0^\infty x\,f(x)\,dx-\int_q^\infty x\,f(x)\,dx\right]=\frac{1}{p\mu}\left[\mu-\int_q^\infty x\,f(x)\,dx\right] \tag{6.1}$$

and

$$L(p)=\frac{1}{\mu}\int_0^q x\,f(x)\,dx=\frac{1}{\mu}\left[\int_0^\infty x\,f(x)\,dx-\int_q^\infty x\,f(x)\,dx\right]=\frac{1}{\mu}\left[\mu-\int_q^\infty x\,f(x)\,dx\right] \tag{6.2}$$

respectively, or equivalently as

$$B(p)=\frac{1}{p\mu}\int_0^p F^{-1}(x)\,dx \tag{6.3}$$

and

$$L(p)=\frac{1}{\mu}\int_0^p F^{-1}(x)\,dx \tag{6.4}$$

respectively, where $\mu=E(X)$ and $q=F^{-1}(p)$.

The Bonferroni and Gini indices are thus defined as

$$B=1-\int_0^1 B(p)\,dp \tag{6.5}$$

and

$$G=1-2\int_0^1 L(p)\,dp \tag{6.6}$$

respectively.

Using p.d.f. of Sujatha distribution (1.7), we get

$$\int_q^\infty x\,f_4(x)\,dx=\frac{\left\{\theta^3\left(q^3+q^2+q\right)+\theta^2\left(3q^2+2q+1\right)+2\theta\left(3q+1\right)+6\right\}e^{-\theta q}}{\theta\left(\theta^2+\theta+2\right)} \tag{6.7}$$

Now using equation (6.7) in (6.1) and (6.2), we get

$$B(p)=\frac{1}{p}\left[1-\frac{\left\{\theta^3\left(q^3+q^2+q\right)+\theta^2\left(3q^2+2q+1\right)+2\theta\left(3q+1\right)+6\right\}e^{-\theta q}}{\theta^2+2\theta+6}\right] \tag{6.8}$$

and $$L(p)=1-\frac{\left\{\theta^3\left(q^3+q^2+q\right)+\theta^2\left(3q^2+2q+1\right)+2\theta\left(3q+1\right)+6\right\}e^{-\theta q}}{\theta^2+2\theta+6} \tag{6.9}$$

Now using equations (6.8) and (6.9) in (6.5) and (6.6), the Bonferroni and Gini indices of Sujatha distribution are obtained as

$$B=1-\frac{\left\{\theta^3\left(q^3+q^2+q\right)+\theta^2\left(3q^2+2q+1\right)+2\theta\left(3q+1\right)+6\right\}e^{-\theta q}}{\theta^2+2\theta+6} \tag{6.10}$$

$$G=-1+\frac{2\left\{\theta^3\left(q^3+q^2+q\right)+\theta^2\left(3q^2+2q+1\right)+2\theta\left(3q+1\right)+6\right\}e^{-\theta q}}{\theta^2+2\theta+6} \qquad 6.11)$$

7. Stress-strength reliability

The stress-strength reliability of a component illustrates the life of the component which has random strength X that is subjected to a random stress Y. When the stress of the component Y applied to it exceeds the strength of the component X, the component fails instantly, and the component will function satisfactorily until $X > Y$. Therefore, $R = P(Y < X)$ is a measure of the component reliability and is known as stress-strength reliability in statistical literature. It has extensive applications in almost all areas of knowledge especially in engineering such as structures, deterioration of rocket motors, static fatigue of ceramic components, aging of concrete pressure vessels, etc.

Let X and Y be independent strength and stress random variables with Sujatha distribution (1.7) with parameter θ_1 and θ_2 respectively. Then the stress-strength reliability R of Sujatha distribution can be obtained as

$$R = P\left(Y < X\right) = \int_0^\infty P\left(Y < X \mid X = x\right) f_X\left(x\right) dx$$

$$= \int_0^\infty f_4\left(x;\theta_1\right) F_4\left(x;\theta_2\right) dx$$

$$= 1 - \frac{\theta_1^3 \begin{bmatrix} \theta_2^6 + \left(4\theta_1 + 3\right)\theta_2^5 + \left(6\theta_1^2 + 10\theta_1 + 13\right)\theta_2^4 + \left(4\theta_1^3 + 12\theta_1^2 + 33\theta_1 + 18\right)\theta_2^3 \\ + \left(\theta_1^4 + 6\theta_1^3 + 29\theta_1^2 + 26\theta_1 + 40\right)\theta_2^2 + \left(\theta_1^3 + 11\theta_1^2 + 10\theta_1 + 20\right)\theta_1\theta_2 \\ + 2\left(\theta_1^2 + \theta_1 + 2\right)\theta_1^2 \end{bmatrix}}{\left(\theta_1^2 + \theta_1 + 2\right)\left(\theta_2^2 + \theta_2 + 2\right)\left(\theta_1 + \theta_2\right)^5}$$

8. Estimation of the parameter

8.1. Maximum Likelihood Estimation of the Parameter

Let $\left(x_1, x_2, x_3, \ldots, x_n\right)$ be random sample from Sujatha distribution (1.7). The likelihood function L is given by

$$L = \left(\frac{\theta^3}{\theta^2 + \theta + 2}\right)^n \prod_{i=1}^{n}\left(1 + x_i + x_i^2\right) e^{-n\theta\bar{x}}$$

The natural log likelihood function is thus obtained as

$$\ln L = n \ln\left(\frac{\theta^3}{\theta^2 + \theta + 2}\right) + \sum_{i=1}^{n} \ln\left(1 + x_i + x_i^2\right) - n\theta\bar{x}$$

Now $\dfrac{d \ln L}{d\theta} = \dfrac{3n}{\theta} - \dfrac{n\left(2\theta + 1\right)}{\theta^2 + \theta + 2} - n\bar{x}$, where $\bar{x}$ is the sample mean.

The maximum likelihood estimate, $\hat{\theta}$ of θ of Sujatha distribution (1.7) is the solution of the equation $\dfrac{d \ln L}{d\theta} = 0$ and is given by solution of the following cubic equation

$$\bar{x}\theta^3 + \left(\bar{x} - 1\right)\theta^2 + 2\left(\bar{x} - 1\right)\theta - 6 = 0 \tag{8.1.1}$$

8.2. Method of Moment Estimation (MOME) of the Parameter

Equating the population mean of Sujatha distribution to the corresponding sample mean, the method of moment (MOM) estimate, $\tilde{\theta}$, of θ is the same as given by equation (8.1.1).

9. Applications and goodness of fit

Since Sujatha, Akash, Shanker, and Lindley distributions have an increasing hazard rate and exponential distribution has a constant hazard rate, Sujatha distribution has been fitted to some data sets to test its goodness of fit over Akash, Shanker, Lindley and exponential distributions. In this section, we present the fitting of Sujatha distribution using maximum likelihood estimate to three real lifetime data sets and compare its goodness of fit with Akash, Shanker, Lindley and exponential distributions. The following three real lifetime data sets have been considered for goodness of fit of distributions.

Data set 1: This data set represents the lifetime data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross and Clark (1975, P. 105).

1.1 1.4 1.3 1.7 1.9 1.8 1.6 2.2 1.7 2.7 4.1 1.8
1.5 1.2 1.4 3 1.7 2.3 1.6 2

Data Set 2: This data set is the strength data of glass of the aircraft window reported by Fuller *et al.* (1994):

18.83	20.80	21.657	23.03	23.23	24.05	24.321	25.5	25.52	25.80
26.69	26.77	26.78	27.05	27.67	29.90	31.11	33.2	33.73	33.76
33.89	34.76	35.75	35.91	36.98	37.08	37.09	39.58	44.045	
45.29	45.381								

Data Set 3: The following data represent the tensile strength, measured in GPa, of 69 carbon fibres tested under tension at gauge lengths of 20 mm (Bader and Priest, 1982):

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958
1.966	1.997	2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179
2.224	2.240	2.253	2.270	2.272	2.274	2.301	2.301	2.359	2.382
2.382	2.426	2.434	2.435	2.478	2.490	2.511	2.514	2.535	2.554
2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684	2.697	2.726
2.770	2.773	2.800	2.809	2.818	2.821	2.848	2.880	2.954	3.012
3.067	3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.585	

In order to compare the goodness of fit of Sujatha, Akash, Shanker, Lindley and exponential distributions, $-2\ln L$, AIC (Akaike Information Criterion), AICC (Akaike Information Criterion Corrected), BIC (Bayesian Information Criterion), and K-S Statistics (Kolmogorov-Smirnov Statistics) of distributions for three real lifetime data sets have been computed and presented in Table 2. The formulae for computing AIC, AICC, BIC, and K-S Statistics are as follows:

$AIC = -2\ln L + 2k$, $AICC = AIC + \frac{2k(k+1)}{(n-k-1)}$, $BIC = -2\ln L + k\ln n$ and $D = \sup_x |F_n(x) - F_0(x)|$, where k = the number of parameters, n = the sample size, and $F_n(x)$ = the empirical distribution function.

Table 2. MLE's, $-2\ln L$, AIC, AICC, BIC, and K-S Statistics of the fitted distributions of data sets 1, 2 and 3

	Model	Parameter estimate	$-2\ln L$	AIC	AICC	BIC	K-S statistic
Data 1	Sujatha	1.136745	57.50	59.50	59.72	60.49	0.309
	Akash	1.156923	59.52	61.52	61.74	62.51	0.320
	Shanker	0.803867	59.78	61.78	61.22	62.77	0.315
	Lindley	0.816118	60.50	62.50	62.72	63.49	0.341
	Exponential	0.526316	65.67	67.67	67.90	68.67	0.389
Data 2	Sujatha	0.09561	241.50	243.50	243.64	244.94	0.270
	Akash	0.097062	240.68	242.68	242.82	244.11	0.266
	Shanker	0.064712	252.35	254.35	254.49	255.78	0.326
	Lindley	0.062988	253.99	255.99	256.13	257.42	0.333
	Exponential	0.032455	274.53	276.53	276.67	277.96	0.426
Data 3	Sujatha	0.936119	221.61	223.61	223.67	225.84	0.319
	Akash	0.964726	224.28	226.28	226.34	228.51	0.348
	Shanker	0.658029	233.01	235.01	235.06	237.24	0.355
	Lindley	0.659000	238.38	240.38	240.44	242.61	0.390
	Exponential	0.407941	261.74	263.74	263.80	265.97	0.434

The best fit of the distribution is the distribution which corresponds to the lower values of $-2\ln L$, AIC, AICC, BIC, and K-S statistics. It is obvious from the fitting of distributions for three data sets in the Table 2 that Sujatha distribution provides better fit than Akash, Shanker, Lindley and exponential distributions for modelling lifetime data in data sets 1 and 3, whereas Akash distribution provides slightly better fit than Sujatha distribution in data set 2.

10. Concluding remarks

A new lifetime distribution named “Sujatha distribution” with an increasing hazard rate has been introduced to model lifetime data. Its moment generating function, moments about origin, moments about mean and expressions for skewness and kurtosis have been given. Various interesting mathematical and statistical properties of Sujatha distribution such as its hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves, and stress-strength reliability have been discussed. The method of maximum likelihood and the method of moments for estimating its parameter have been discussed. Three examples of real lifetime data sets have been presented to show the applications and goodness of fit of Sujatha distribution with Akash, Shanker, Lindley and exponential distributions.

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