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ROBUST p-MEDIAN PROBLEM IN CHANGING NETWORKS

Summary. The robust p-median problem in changing networks is a version of known discrete p-median problem in network with uncertain edge lengths where uncertainty is characterised by given interval. The uncertainty in edge lengths may appear in travel time along the edges in any network location problem. Several possible future scenarios with respect to the lengths of edges are presented. The planner will want a strategy of positioning p medians that will be working “as well as possible" over the future scenarios.

We present MILP formulation of the problem and the solution method based on exchange MILP heuristic. The cluster of each median is presented by rooted tree with the median as root. The performance of the proposed heuristic is compared to the optimal solution found via Gurobi solver for MILP models through some illustrative instances of Slovak road network in Žilina.

DAS PROBLEM DES p-MEDIANS IN DEN SICH WECHSELNDEN NETZEN

Resumé. Das Problem des P-Medians in den sich wechselnden Netzen ist eines der Versionen des bekannten diskreten Problems über P-Median im Netz mit nicht gewissen Abschnittlängen, wo die Unbestimmheit durch das gegebene Intervall angesetzt wird.Nicht gewisse Länge der Abschitte kann sich als Fahrtlänge in dem Gebiet des jeweiligen Lokationsproblem bestimmen. Wir führen einige Szenare mit Rücksicht auf Kantenlänge ein. Der Planer sucht die Strategie „möglichst guter“ Plazierung von P-Medianen mit Rücksicht auf zukünftige Szenare.

Wir stellen MILP-Formulierung des Problems und Lösungsverfahren vor, die auf der Tausch-Heuristik gegründet werden. Die zu jedem Median gehörende Ansammlung wird als der Baum mit Würzeln als Median präsentiert. Die Qualität der vorgeschlagenen Heuristik vergleichen wir mit der optimalen Lösung der erworbenen Gurobi-Solver für MILP-Modelle auf einigen Illustrationsinstanzen der Strassennetze in der Slowakischen Republik im Region Žilina.

1. INTRODUCTION

The p-median problem is one of the most widely studied problems in location theory. A detailed introduction to this problem and solution methods appears in Reese [8]. Janáček’s school in University of Žilina developed several original methods for practical applications. From the latest works we mention medical service planning (Janošíková [2]) and network design (Janáček [3], Janáček and Kvet [4]) which were an inspiration for this paper. An address data uncertainty for discrete optimization and network flow problems that allows controlling the degree of conservatism of the solution both practically and theoretically can be found in Bertsimas & Sim [6].

We formulate a p-median problem with uncertain edge lengths where uncertainty is characterized by given intervals. The uncertainty in edge lengths may influence travel time along the edges between stops of bus network in urban area. Several possible future scenarios with respect to the lengths of edges are presented.

We present MILP formulation of the problem and the solution method based on exchange MILP heuristic. The performance of the proposed heuristic is compared with optimum solution found via Gurobi solver for MILP models on some illustrative instances of Slovak road network in Žilina region.

We begin with modified uncertain formulation of the *p*-median problem in network proposed by Nikoofal & Sadjadi [5]. As in our capacitated model [7] the cluster of each median is represented by rooted tree in network with the median as roots.

2. ROBUST p-MEDIAN PROBLEM

We will suppose that a connected network with set of vertices V and set of edges E is represented by a weighted graph $G = (V, E, d, w)$. For each vertex $v_i \epsilon V$ value w_i (non-negative integer) is the weight associated with vertex v_i. Total demands in network are then equal to the sum of weights $w_0 = \sum_{v_i \epsilon V} w_i$.

Let d_{ij}be the length of edge $\{v_i, v_j\} \epsilon E$ which may take any value from known interval $[d_{ij}^L, d_{ij}^U]$. Note that in this notation we allow $0 \leq d_{ij}^L \leq d_{ij}^U$.

For model we define associated digraph $\vec{G} = (V_0, \vec{E}, \vec{d}, w)$ where:

- $V_0 = V \cup \{v_0\}$ with fictive vertex v_0,
- $\vec{E} = \{(v_i, v_0): v_i \in V\} \cup \{(v_i, v_j): i < j, \{v_i, v_j\} \in E\} \cup \{(v_i, v_j): j < i, \{v_i, v_j\} \in E\}$,
- $\vec{d}(i, j) = \vec{d}(j, i) =[d_{ij}^L, d_{ij}^U]$ for $\{v_i, v_j\} \in E, \vec{d}(v_i, v_0) = [0,0]$ for $v_i \in V$.

The cluster of each median is presented by rooted tree with median as root. So the solution can be represented by a forest with p trees. After adding the oriented edges from medians to fictive vertex we can find the solution in the form of spanning tree in digraph $\vec{G}$. To make the interpretation easier to grasp we use the following notation

- $V_i^+ = \{v_j \in V_0: (v_i, v_j) \in \vec{E}\}$ – set of successors of vertex $v_i \in V$
- $V_i^- = \{v_j \in V_0: (v_j, v_i) \in \vec{E}\}$ – set of predecessors of vertex $v_i \in V$

Inspired by the paper of Nikoofal & Sadjadi [5] we define the scalar deviation of distance

$d_{ij} \epsilon [d_{ij}^L, d_{ij}^U], d_{ij}^L < d_{ij}^U$as parameter

$$s_{ij} = \frac{d_{ij} - d_{ij}^L}{d_{ij}^U - d_{ij}^L} \tag{1}$$

that has values from interval [0,1]. In case $d_{ij}^L = d_{ij}^U$ we define $s_{ij} = 0$. Then we can define a budget of uncertainty in the following sense: The total scaled variance of uncertainty parameter of the edge length cannot exceed same threshold Γthat is not necessarily an integer, i.e.

$$\sum_{\{v_i, v_j\} \in E} s_{ij} \leq \Gamma. \tag{2}$$

The value of threshold Γcan change from zero, when all edges take their lower value, to $|E|$ which is associated with the worst case scenario (feasible lengths of edges).

Let x_{ij} be the number of units elements flowing through arc $(v_i, v_j) \epsilon \overrightarrow{E}$ and let t_{ij} be corresponding binary variable with value 1 if this arc lie in the forest. The value 1 of binary variable y_i locates median to vertex $v_i \epsilon V$. Now we can formulate our robust version of p-median problem in network $\vec{G}$ as following mixed integer linear problem (RMILP):

$$\sum_{(v_i,v_j)\in\vec{E}} (d_{ij}^L x_{ij} + z_{ij}) + \mathrm{q}\Gamma \rightarrow minimum \tag{3}$$

$$\sum_{v_i\in V} y_i = p \tag{4}$$

$$\sum_{i\in V} x_{i0} = w_0 \tag{5}$$

$$\sum_{j\in V_i^+} x_{ij} = \sum_{j\in V_i^-} x_{ji} + w_i \qquad \forall v_i \in \mathrm{V} \tag{6}$$

$$\sum_{j\in V_i^+} t_{ij} = 1 \qquad \forall v_i \in \mathrm{V} \tag{7}$$

$$x_{ij} \geq w_i t_{ij} \qquad \forall (v_i, v_j) \in \vec{E} \tag{8}$$

$$x_{i0} \leq w_0 y_i \qquad \forall v_i \in \mathrm{V} \tag{9}$$

$$q + z_{ij} \geq (d_{ij}^U - d_{ij}^L) x_{ij} \qquad \forall (v_i, v_j) \in \vec{E} \tag{10}$$

$$q, z_{ij} \geq 0 \qquad \forall (v_i, v_j) \in \vec{E} \tag{11}$$

$$x_{ij} \geq 0, integer \qquad \forall (v_i, v_j) \in \vec{E} \tag{12}$$

$$y_i \in \{0,1\} \qquad \forall v_i \in \mathrm{V} \tag{13}$$

$$t_{ij} \in \{0,1\} \qquad \forall (v_i, v_j) \in \vec{E} \tag{14}$$

The objective function (3) minimizes the sum of expected allocated lengths in the form of weighted flow from vertices to assigned medians and multiple q and given threshold Γ. The constraints (4) and (11) limit the number of medians to p. The constraint (5) forces assignment of all demands in vertices. The value of flows to fictive vertex is definite via constraint (6). Constraints (7), (8) and (14) ensure that demands in vertices cannot be assigned to different clusters. The constraint (9) guarantees that every median has at most one father to fictive vertex and so the solution is an oriented spanning tree with root v_0. The last constraint (10) provides for proportional part of threshold budget for every used edge of network. Last constraints (11)-(14) are obligatory.

3. MILP EXCHANGE HEURISTIC

Computational experiments with larger instances showed that we need some heuristic. Our heuristic approach is based on modified version of the RMILP formulation, where feasible solutions are strongly reduced. The main idea of the MILP exchange heuristic is the following: Let us have a set of p candidates of median $M = \{v_{[1]}, v_{[2]}, \dots, v_{[p]}\}$. Then we can request that at most 2 vertices of the set M will be replaced by some of the vertices from set *V-M*, if we add the constraint

$$\sum_{v_i\in M} y_i \geq p - 2. \tag{15}$$

Now we can find via MILP solver the heuristic set of p-median defined by the set

$$P = \{v_i \in V:\ y_i = 1, st. (3), (4), \dots, (15)\}. \tag{16}$$

The heuristic ends when $M = P$otherwise we continue with $M := P$ solving (16).

4. COMPUTATIONAL EXPERIMENTS

Our experiments were conducted on HP XW660 Workstation (8-core Xeon 3GHz, RAM 16GB) with OS Linux (Debian/Jessie). We use Python-based tools, namely the programming language Python 2.7 [9] with modules NumPy and SciPy[10], Matplotlib [11], IPython [12].

Let us say that all parameters of tested instances were integer-valued. In the real-word applications this can be achieved by quantifying the lengths of edges and demands in nodes of network into several integer levels (there are seldom more that few hundreds levels).

Our experiments were tried for two types of instances generated on Slovak road network in Žilina– small (|V|=100, |E|=124) and medium (|V|=500, |E|=620). For every instance ten thresholds were generated evenly from $\Gamma = 0$ to $\Gamma = |E|$. The corresponding uncertainty budget b_Γ is computed, reference to (1), as value determined via the solution *x*,*z* of the RMILP:

$$b_\Gamma(x, z) = \sum_{(v_i,v_j)\in\vec{E}:x_{ij}>0} \frac{z_{ij}}{x_{ij}} \left(d_{ij}^U - d_{ij}^L\right) \tag{17}$$

Expected allocated length versus values of uncertainty budget is presented on the fig. 1 and 2.

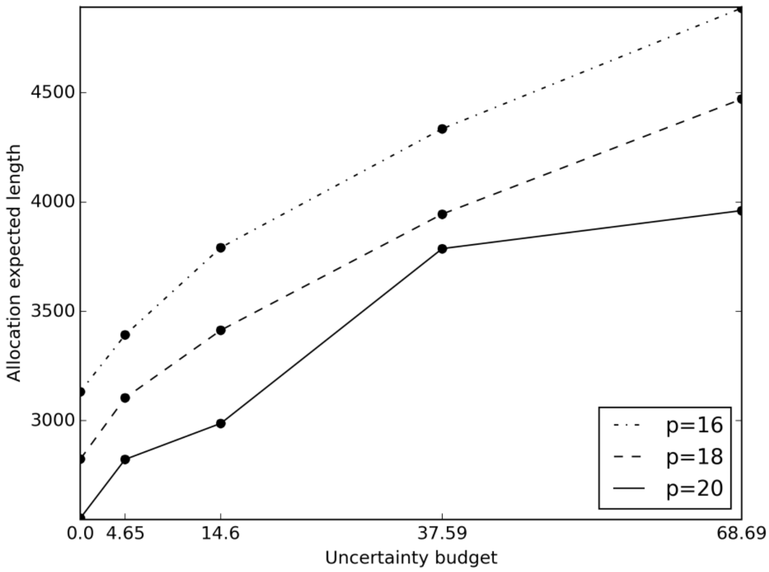

Fig. 1. Expected allocated length for small instance
Abb. 1. Erwartete durchschnittliche Länge für kleine Instanzen

Note that the expected allocated length is only the first sum of the objective function (3). With 20 runs of MILP solutions for five distinct parameters p we achieved very good solution times given below in table 1. Number in parentheses indicates count of cases where the heuristic found an optimal solution.

Table 1

Mean runtime for optimum and heuristic solutions

Instances	small	medium
Optimal solution (min:sec.)	1:42	68:37
Heuristic solution (min:sec.)	0:53 [7]	13:37 [5]

It should be noted that Gurobi solver is able to use all the processor cores, so the solution time on on-core processor will be much greater. We use the academic licence of Gurobi, which is the full version of solver without any constraints on size of problem instances or the number of processors (on multicore computers).

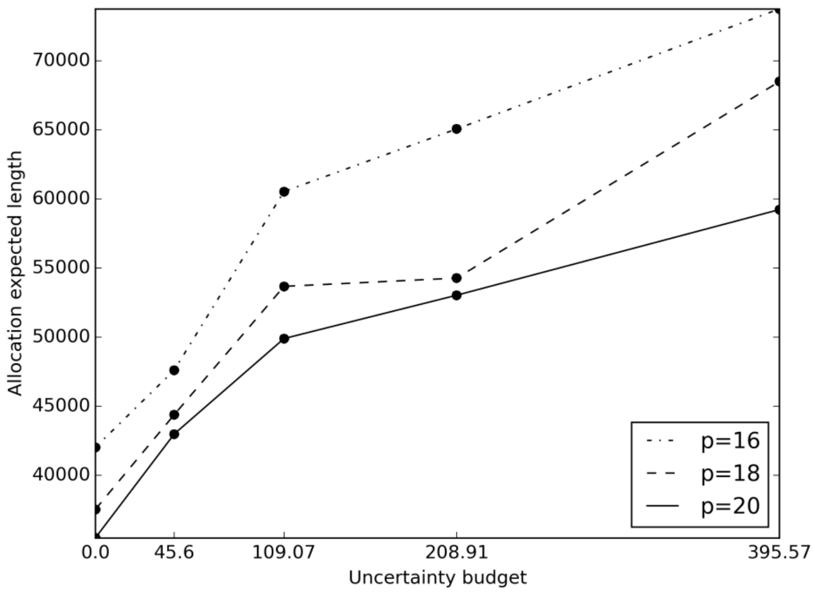

Fig. 2. Expected allocated length for medium instance
Abb. 2. Erwartete durchschnittliche Länge fúr mittlere Instanzen

5. CONCLUSION

We have considered the version of the robust p-median problem where vertices weights are deterministic and edge lengths and edges lengths are uncertain and belong to known intervals. We modified the Nikoofal & Sadjadi’s formulation of the *p*-median problem so that the cluster of each median is represented by rooted tree in network with the median as root. We present MILP formulation of the problem and the solution method based on the MILP heuristic where at most two vertices of the candidate p-median set can be exchanged. We have shown by computational experiments that proposed heuristic is comparable with optimum solution via Gurobi solver for MILP models for tree size types of instances on Slovak road network in Žilina region.

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