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Evaluations of some classes of the trigonometric moment integrals

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ABSTRACT

Four classes of the trigonometric moment integrals are evaluated in closed form in a simple and unified manner by making use of the contour integration in conjunction with the Cauchy integral theorem. In all cases, the closed contour of the same shape is used and it is shown that the integrals are expressible only in terms of the Hurwitz zeta function and elementary functions. A number of interesting (known or new) special cases and consequences of the main results are also considered.

1. Introduction and preliminaries

It is quite remarkable that, with exception of the sine and cosine function, very few integrals of other trigonometric functions can be found in the literature. For the sake of illustration, we note that in Section 2.5.26 of the standard reference text by Prudnikov et al. [11, pp. 436–438], among not more than half a dozen definite cotangent integrals, each of the following moments:

$$I_n=\int_0^{\pi/2} x^n \cot x\,dx=\left(\frac{\pi}{2}\right)^n\left(\frac{1}{n}-2\sum_{k=1}^{\infty}\frac{\zeta(2k)}{2^{2k}(n+2k)}\right)\quad (n\in\mathbb{N}) \tag{1.1}$$

and

$$J_n=\int_0^{\pi/4} x^n \cot x\,dx=\frac{1}{2}\left(\frac{\pi}{4}\right)^n\left(\frac{2}{n}-\sum_{k=1}^{\infty}\frac{\zeta(2k)}{4^{2k-1}(n+2k)}\right)\quad (n\in\mathbb{N}), \tag{1.2}$$

$\mathbb{N}:=\{1,2,3,\ldots\}$ being the set of natural numbers, is listed as the most general (see also [9, p. 456, Section 3.748]). Moreover, it has turned out that all these tabulated integrals are classical results and they were already recorded as long ago as 1867 [2, pp. 306–310, Tables 204–206] and 1891 [10, Tables 204–206]. Similarly, there are not many available tangent, cosecant and secant integrals, and almost all of the known ones are the nineteenth century results.

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In recent years, several general families of definite integrals involving the tangent or secant function have been evaluated in closed form by making use of some elementary arguments (see, for details, [7]). In this sequel, four classes of the trigonometric moment integrals (i.e., the cotangent, tangent, cosecant and secant moment integrals) are evaluated in closed form by applying the contour integration method in conjunction with the Cauchy integral theorem. In all cases, closed contour (introduced by Cho et al. [3]) of the same shape is used and it is shown that the integrals are expressible only in terms of the Hurwitz zeta function and elementary functions. In addition, several interesting (known or new) special cases and consequences of our main results are also considered.

The Hurwitz (or generalized) zeta function $\zeta(s,a)$ is an analytic function of s in the whole complex s-plane (except for a simple pole at $s=1$) and is defined by the following series [12, p. 88, Eq. (1)]:

$$\zeta(s,a)=\sum_{k=0}^{\infty}\frac{1}{(k+a)^s}\quad \big(\Re(s)>1;\ a\neq 0,-1,-2,\ldots\big), \tag{1.3}$$

whenever it converges and (by analytic continuation) elsewhere.

The Riemann zeta function $\zeta(s)$ is a special case of the Hurwitz zeta function [12, p. 96, Eq. (1)]

$$\zeta(s)=\zeta(s,1) \tag{1.4}$$

and it is also an analytic function in the whole complex s-plane, except for the simple pole at $s=1$.

Define the *alternating* counterpart of $\zeta(s,a)$ by

$$\eta(s,a)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+a)^s}\quad \big(\Re(s)>0\big) \tag{1.5}$$

and observe that there exists the following readily derivable relationship:

$$\eta(s,a)=\frac{1}{2^s}\left[\zeta\left(s,\frac{a}{2}\right)-\zeta\left(s,\frac{a+1}{2}\right)\right] \tag{1.6}$$

between $\zeta(s,a)$ and $\eta(s,a)$. The relationship (1.6) can be used to continue $\eta(s,a)$ analytically to the whole complex s-plane.

We shall also use the Dirichlet beta function $\beta(s)$ defined by [1, p. 807, Eq. (23.2.21)]

$$\beta(s)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^s}=\frac{1}{4^s}\left[\zeta\left(s,\frac{1}{4}\right)-\zeta\left(s,\frac{3}{4}\right)\right] \tag{1.7}$$

and the Dirichlet eta function $\eta(s)$ defined by [1, p. 807, Eq. (23.2.19)]

$$\eta(s):=\eta(s,1)=\big(1-2^{1-s}\big)\zeta(s). \tag{1.8}$$

In our present investigation, some or all of the following known identities involving $\zeta(s,a)$ and $\zeta(s)$ will be used *implicitly* [12, p. 88 *et seq.*]:

$$\zeta(s,1)=\zeta(s)=\frac{1}{2^s-1}\zeta\left(s,\frac{1}{2}\right),$$

$$\zeta(s)=\frac{1}{q^s}\sum_{j=1}^{q}\zeta\left(s,\frac{j}{q}\right)\quad (q\in\mathbb{N}),$$

and

$$\zeta(s,a)=\zeta(s,a+n)+\sum_{k=0}^{n-1}\frac{1}{(k+a)^s}\quad \big(\Re(s)>1;\ a\neq 0,-1,-2,\ldots\big),$$

the first two of which yield

$$\zeta\left(s,\frac{1}{3}\right)+\zeta\left(s,\frac{2}{3}\right)=\big(3^s-1\big)\zeta(s),$$

$$\zeta\left(s,\frac{1}{4}\right)+\zeta\left(s,\frac{3}{4}\right)=2^s\big(2^s-1\big)\zeta(s),$$

and

$$\zeta\left(s,\frac{1}{6}\right)+\zeta\left(s,\frac{5}{6}\right)=\big(2^s-1\big)\big(3^s-1\big)\zeta(s).$$

Finally, throughout this paper, G denotes the Catalan constant defined by

$$G := \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{1}{4^2}\left[\zeta\left(2, \frac{1}{4}\right) - \zeta\left(2, \frac{3}{4}\right)\right] = \frac{1}{8}\left[\zeta\left(2, \frac{1}{4}\right) - \pi^2\right], \tag{1.9}$$

so that

$$G \cong 0.915965594177219015 \cdots.$$

2. Statements of the main results

In what follows, in order to simplify our presentation, we shall use the following definition:

$$Z_q(s, x_q) := \begin{cases} \eta(s, x_q) & (q \text{ is odd}), \\ \zeta(s, x_q) & (q \text{ is even}), \end{cases} \tag{2.1}$$

$x_q := x(q)$ being any real sequence, and $\zeta(s, a)$ and $\eta(s, a)$ are the above-defined functions.

Remark 1. Although, throughout the text of this paper, we use $\zeta(s)$, $\beta(s)$, $\eta(s, a)$, $\eta(s)$ as well as $Z_q(s, x_q)$, it is clear that all of the deduced expressions could be expressed only in terms of the Hurwitz zeta function $\zeta(s, a)$, since these aforementioned functions are all related to $\zeta(s, a)$ [see Eqs. (1.4), (1.7), (1.6) and (1.8)].

It is assumed that n, p and q are arbitrary positive integers, $\lfloor x \rfloor$ denotes the greatest integer not exceeding the real number x and an empty sum is understood (as usual) to be nil.

Now we are ready to state our main results as Theorems 1 and 2 below.

Theorem 1. *Let $\zeta(s)$, $\zeta(s, a)$ and $\eta(s, a)$ be, respectively, the functions defined by* (1.4), (1.3) *and* (1.5), *respectively.*

(i) *If $0 < \frac{p}{q} < 1$, then*

$$\begin{aligned} \int_0^{p\pi/q} \phi^n \cot\phi \, d\phi = {} & \left(\frac{p\pi}{q}\right)^n \ln\left[2\sin\left(\frac{p\pi}{q}\right)\right] \\ & + (-1)^{\lfloor n/2 \rfloor}\left\{1 + (-1)^n\right\} 2^{-(n+1)} n! \zeta(n+1) + 2n!\left(\frac{p\pi}{q}\right)^{n+1} \\ & \cdot \left[\sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \sum_{l=1}^{q} \frac{(-1)^{k-1}}{(n-2k+1)!} \frac{1}{(2p\pi)^{2k}} \sin\left(\frac{2lp\pi}{q}\right) \zeta\left(2k, \frac{l}{q}\right) \right. \\ & \left. + \sum_{k=1}^{\lfloor n/2 \rfloor} \sum_{l=1}^{q} \frac{(-1)^{k-1}}{(n-2k)!} \frac{1}{(2p\pi)^{2k+1}} \cos\left(\frac{2lp\pi}{q}\right) \zeta\left(2k+1, \frac{l}{q}\right)\right]; \end{aligned} \tag{2.2}$$

(ii) *Let Z_q be given by* (2.1). *If $0 < \frac{p}{q} < \frac{1}{2}$, then*

$$\begin{aligned} \int_0^{p\pi/q} \phi^n \tan\phi \, d\phi = {} & -\left(\frac{p\pi}{q}\right)^n \ln\left[2\cos\left(\frac{p\pi}{q}\right)\right] \\ & + (-1)^{\lfloor n/2 \rfloor}\left\{1 + (-1)^n\right\} 2^{-(n+1)} \left(1 - 2^{-n}\right) n! \zeta(n+1) + 2n!\left(\frac{p\pi}{q}\right)^{n+1} \\ & \cdot \left[\sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \sum_{l=1}^{q} \frac{(-1)^{k-1}}{(n-2k+1)!} \frac{(-1)^{l-1}}{(2p\pi)^{2k}} \sin\left(\frac{2lp\pi}{q}\right) Z_q\left(2k, \frac{l}{q}\right) \right. \\ & \left. + \sum_{k=1}^{\lfloor n/2 \rfloor} \sum_{l=1}^{q} \frac{(-1)^{k-1}}{(n-2k)!} \frac{(-1)^{l-1}}{(2p\pi)^{2k+1}} \cos\left(\frac{2lp\pi}{q}\right) Z_q\left(2k+1, \frac{l}{q}\right)\right]. \end{aligned} \tag{2.3}$$

Theorem 2. *Let $\zeta(s)$, $\zeta(s,a)$ and $\eta(s,a)$ denote the functions defined as in* (1.4), (1.3) *and* (1.5), *respectively.*

(i) *If* $0 < \frac{p}{q} < 1$, *then*

$$\begin{aligned}\int_0^{p\pi/q} \phi^n \csc\phi \, d\phi &= \left(\frac{p\pi}{q}\right)^n \ln\left[\tan\left(\frac{p\pi}{2q}\right)\right] \\ &\quad + (-1)^{\lfloor n/2 \rfloor}\left\{1+(-1)^n\right\} n!\left(1-2^{-(n+1)}\right)\zeta(n+1) + 2\cdot n!\left(\frac{p\pi}{q}\right)^{n+1} \\ &\quad \cdot \left[\sum_{k=1}^{\lfloor (n+1)/2 \rfloor}\sum_{l=1}^{q} \frac{(-1)^{k-1}}{(n-2k+1)!}\frac{1}{(2p\pi)^{2k}}\sin\left(\frac{(2l-1)p\pi}{q}\right)\zeta\left(2k,\frac{2l-1}{2q}\right)\right. \\ &\quad \left. + \sum_{k=1}^{\lfloor n/2 \rfloor}\sum_{l=1}^{q} \frac{(-1)^{k-1}}{(n-2k)!}\frac{1}{(2p\pi)^{2k+1}}\cos\left(\frac{(2l-1)p\pi}{q}\right)\zeta\left(2k+1,\frac{2l-1}{2q}\right)\right];\end{aligned} \tag{2.4}$$

(ii) *Let Z_q be given by* (2.1) *and let $\beta(s)$ be the function defined as in* (1.7). *If* $0 < \frac{p}{q} < \frac{1}{2}$, *then*

$$\begin{aligned}\int_0^{p\pi/q} \phi^n \sec\phi \, d\phi &= \left(\frac{p\pi}{q}\right)^n \ln\left[\tan\left(\frac{p\pi}{2q}+\frac{\pi}{4}\right)\right] \\ &\quad + (-1)^{\lfloor (n+1)/2 \rfloor}\left\{1-(-1)^n\right\} n!\beta(n+1) + 2n!\left(\frac{p\pi}{q}\right)^{n+1} \\ &\quad \cdot \left[\sum_{k=1}^{\lfloor (n+1)/2 \rfloor}\sum_{l=1}^{q} \frac{(-1)^{k-1}}{(n-2k+1)!}\frac{(-1)^{l-1}}{(2p\pi)^{2k}}\cos\left(\frac{(2l-1)p\pi}{q}\right)Z_q\left(2k,\frac{2l-1}{2q}\right)\right. \\ &\quad \left. - \sum_{k=1}^{\lfloor n/2 \rfloor}\sum_{l=1}^{q} \frac{(-1)^{k-1}}{(n-2k)!}\frac{(-1)^{l-1}}{(2p\pi)^{2k+1}}\sin\left(\frac{(2l-1)p\pi}{q}\right)Z_q\left(2k+1,\frac{2l-1}{2q}\right)\right].\end{aligned} \tag{2.5}$$

Several immediate consequences of the application of our integral formulas and some illustrative special cases will be given in Section 4. Here we present a few sets of illustrative examples.

Example Set 1. Let G be the Catalan constant (1.9). Then

(1) $$\int_0^{\pi/2} \phi^2 \cot\phi \, d\phi = \frac{\pi^2}{4}\ln 2 - \frac{7}{8}\zeta(3);$$

(2) $$\int_0^{\pi/3} \phi^2 \cot\phi \, d\phi = \frac{\pi^2}{18}\ln 3 - \frac{2\pi}{27\sqrt{3}} - \frac{13}{18}\zeta(3) + \frac{\pi}{9\sqrt{3}}\zeta\left(2,\frac{1}{3}\right);$$

(3) $$\int_0^{\pi/4} \phi^2 \cot\phi \, d\phi = \frac{\pi}{4}G + \frac{\pi^2}{32}\ln 2 - \frac{35}{64}\zeta(3);$$

(4) $$\int_0^{\pi/6} \phi^2 \cot\phi \, d\phi = -\frac{\pi^3}{27\sqrt{3}} - \frac{1}{3}\zeta(3) + \frac{\pi}{72\sqrt{3}}\left[\zeta\left(2,\frac{1}{3}\right) + \zeta\left(2,\frac{1}{6}\right)\right].$$

Example Set 2. Each of the following integral formulas holds true:

(1) $$\int_0^{\pi/2} \phi^2 \csc\phi \, d\phi = 2G\pi - \frac{7}{2}\zeta(3);$$

(2) $$\int_0^{\pi/3} \phi^2 \csc\phi\, d\phi = -\frac{2\pi^3}{9\sqrt{3}} - \frac{\pi^2}{18}\ln 3 - \frac{35}{18}\zeta(3) + \frac{\pi}{9\sqrt{3}}\zeta\left(2, \frac{1}{6}\right);$$

(3) $$\int_0^{\pi/4} \phi^2 \csc\phi\, d\phi = -\frac{\pi^3}{8\sqrt{2}}G + \frac{\pi^2}{16}\ln(\sqrt{2}-1) - \left(\frac{7}{2} + \frac{7\sqrt{2}}{4}\right)\cdot\zeta(3) + \frac{\pi}{32\sqrt{2}}\left[\zeta\left(2, \frac{1}{8}\right) + \zeta\left(2, \frac{3}{8}\right)\right] + \frac{1}{64\sqrt{2}}\left[\zeta\left(3, \frac{1}{8}\right) + \zeta\left(3, \frac{7}{8}\right)\right];$$

(4) $$\int_0^{\pi/6} \phi^2 \csc\phi\, d\phi = \frac{2}{27}G\pi + \frac{\pi^2}{36}\ln(2-\sqrt{3}) - \frac{7}{2}\zeta(3) + \frac{\pi}{432}\left[\zeta\left(2, \frac{1}{12}\right) + \zeta\left(2, \frac{5}{12}\right) - \zeta\left(2, \frac{7}{12}\right) - \zeta\left(2, \frac{11}{12}\right)\right] + \frac{1}{288\sqrt{3}}\left[\zeta\left(3, \frac{1}{12}\right) - \zeta\left(3, \frac{5}{12}\right) + \zeta\left(3, \frac{7}{12}\right) - \zeta\left(3, \frac{11}{12}\right)\right].$$

Remark 2. The existence of the cotangent and cosecant integrals in (2.2) and (2.4) is assured, since the integrands involved have no other singularities on $[0, \phi]$ $(0 < \phi < 1)$ except for the *removable* singularity at $\phi = 0$.

Remark 3. It is noteworthy that Theorems 1 and 2 enable the closed-form evaluation of the given integrals in terms of a finite combination of values of the Hurwitz zeta function and elementary functions (see Remark 1). Moreover, it is clear that Theorems 1 and 2 can also be rewritten in terms of the polygamma function $\psi^{(n)}(z)$, since there exists the following relationship between the Hurwitz zeta function $\zeta(s, a)$ and the polygamma function $\psi^{(n)}(z)$ [1, p. 260, Eq. (6.4.101)]:

$$\psi^{(n)}(z) = (-1)^{n+1} n!\,\zeta(n+1, z) \quad (z \neq 0, -1, -2, \ldots;\ n \in \mathbb{N}). \tag{2.6}$$

Remark 4. It is interesting to note that, as is shown in Example Sets 1* and 2* below, even modern-day algorithms for symbolic integration work rather unsatisfactorily on the integrals described by Theorems 1 and 2. PolyGamma[n, z], i.e., the polygamma function $\psi^{(n)}(z)$ given by (2.6), is not problematic here, since it is, in essence, the Hurwitz zeta function $\zeta(n, z)$, but it is hard to see how it is possible, in general, to reduce the polylogarithm function PolyLog[n, z] to $\zeta(n, z)$.

Example Set 1***.** Consider the integral formulas (1) to (4) in Example Set 1. The results of integration by means of *Mathematica* 6.0 (Wolfram Research) are given below:

$$Out[1] = \frac{1}{8}\pi^2\,\mathrm{Log}[4] - \frac{7\,\mathrm{Zeta}[3]}{8},$$

$$Out[2] = \frac{i\pi^3}{54} + \frac{1}{54}\pi^2\,\mathrm{Log}[27] + \frac{1}{3}i\pi\,\mathrm{PolyLog}\big[2, -(-1)^{1/3}\big] - \frac{13\,\mathrm{Zeta}[3]}{18},$$

$$Out[3] = \frac{\mathrm{Catalan}\,\pi}{4} + \frac{1}{64}\pi^2\,\mathrm{Log}[4] - \frac{35\,\mathrm{Zeta}[3]}{64},$$

$$Out[4] = -\frac{i\pi^3}{216} + \frac{1}{6}i\pi\,\mathrm{PolyLog}\big[2, -(-1)^{2/3}\big] - \frac{\mathrm{Zeta}[3]}{3}.$$

Example Set 2***.** Evaluation of the integral (3) in Example Set 2 yields in the following result:

$$Out[3] = \frac{\mathrm{Catalan}\,\pi}{8} + \frac{i\pi^3}{384} + \frac{1}{16}\pi^2\,\mathrm{Log}\big[1-(-1)^{1/4}\big] - \frac{1}{16}\pi^2\,\mathrm{Log}\big[1+(-1)^{1/4}\big] - \frac{1}{128}(-1)^{3/4}\pi\,\mathrm{PolyGamma}\left[1, \frac{1}{8}\right] + \frac{1}{128}(-1)^{1/4}\pi\,\mathrm{PolyGamma}\left[1, \frac{3}{8}\right] + \frac{1}{128}(-1)^{3/4}\pi\,\mathrm{PolyGamma}\left[1, \frac{5}{8}\right] - \frac{1}{128}(-1)^{1/4}\pi\,\mathrm{PolyGamma}\left[1, \frac{7}{8}\right] + \frac{1}{2}i\pi\,\mathrm{PolyLog}\left[2, -\frac{1+i}{\sqrt{2}}\right] + 2\,\mathrm{PolyLog}\big[3, (-1)^{1/4}\big] - 2\,\mathrm{PolyLog}\left[3, -\frac{1+i}{\sqrt{2}}\right] - \frac{7\,\mathrm{Zeta}[3]}{2}.$$

Remark 5. It is not assumed in Theorems 1 and 2 above that the fraction p/q is a reduced fraction; however, it is clear that, when p/q is in a reduced form, then the formulas give the simplest expressions.

Remark 6. Several general families of definite integrals involving the tangent or secant function have been more recently evaluated in closed form by making use of the elementary arguments [7]. The integrals in (2.3) and (2.5) were then deduced as immediate corollaries of these results [7, p. 573, Corollary 2.4]. We note also that a recent result proven by Cho et al. [3, p. 465, Eqs. (20) and (22)] is similar to the formula (2.4), but it involves, in addition to elementary functions and $\zeta(s, a)$, the double sum [3, p. 565, Eq. (21)] which, in general, does not appear to be easily tractable. Cho et al. [3] computed several particular cases of this sum which enabled them to deduce a number of interesting special cases of the integrals in (2.5) [3, pp. 466–467, Eqs. (23) to (29)].

3. Proofs of the main results

At the outset, we note that the proofs of Theorems 1 and 2 require the following two known results [11, p. 491, Entry 2.6.5.9]:

$$\int_0^1 \frac{t^{\alpha-1}}{1-t^{\mu}}(\ln t)^n\,dt = (-1)^n \frac{n!}{\mu^{n+1}}\zeta\left(n+1, \frac{\alpha}{\mu}\right) \quad \big(\Re(\alpha) > 0;\ \Re(\mu) > 0\big) \tag{3.1}$$

and [11, p. 488, Entry 2.6.4.4]

$$\int_0^1 \frac{t^{\alpha-1}}{1+t^{\mu}}(\ln t)^n\,dt = (-1)^n \frac{n!}{\mu^{n+1}}\eta\left(n+1, \frac{\alpha}{\mu}\right) \quad \big(\Re(\alpha) > 0;\ \Re(\mu) > 0\big). \tag{3.2}$$

In order to derive the integral formulas (2.2) to (2.5), we use contour integration and the Cauchy integral theorem. We begin by recalling the Cauchy integral theorem (sometimes called the Cauchy–Goursat theorem) which states that the contour integral of a complex-valued function $\Phi(z)$ of the complex variable z vanishes whenever the contour is a piecewise smooth simple closed curve and $\Phi(z)$ is analytic on and inside the contour. The interested reader is referred to any standard text on *Complex Analysis* for further details.

In our proofs of Theorems 1 and 2, we shall first consider the following the complex-valued functions:

$$\left.\begin{matrix} f_{\csc}(z) \\ f_{\sec}(z) \end{matrix}\right\} = \frac{(\ln z)^n}{z^2 \mp 1} \quad \text{and} \quad \left.\begin{matrix} f_{\cot}(z) \\ f_{\tan}(z) \end{matrix}\right\} = \frac{z(\ln z)^n}{z^2 \mp 1} \quad (z \in \mathbb{C}), \tag{3.3}$$

which are so chosen because their integrals on the unit circle involve, for our purposes, the needed trigonometric functions. For instance, we have

$$2 \oint_{|z|=1} f_{\csc}(z)\,dz = i^n \int_0^{2\pi} \phi^n \csc\phi\,d\phi \quad \big(i := \sqrt{-1}\big).$$

Furthermore, all functions in (3.3) have no singularities that lie on and inside the contour Γ described below (see Fig. 1), and thus the Cauchy integral theorem can be applied.

Next, we integrate these functions in the positive (counter-clockwise) direction along the following contour in the complex z-plane (which will hereafter be denoted by Γ):

$$\begin{aligned}
&0 < \delta \leqq x \leqq 1-\eta && (0 < \eta < \delta); \\
&z = 1 + \eta e^{i\phi} && \left(\phi \text{ varies from } \pi \text{ to } \frac{\pi}{2}\right); \\
&z = e^{i\phi} && \left(\arctan\eta \leqq \phi \leqq \frac{p\pi}{q}\right); \\
&z = u e^{ip\pi/q} && (u \text{ varies from } 1 \text{ to } \delta); \\
&z = \delta e^{i\phi} && \left(\phi \text{ varies from } \frac{p\pi}{q} \text{ to } 0\right).
\end{aligned}$$

Moreover, in the case of the functions $f_{\cot}$ and $f_{\csc}$, we let

$$0 < \frac{p}{q} < 1,$$

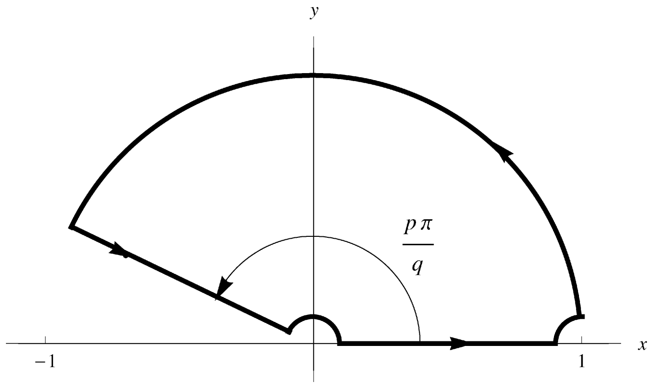

Fig. 1. Integration contour Γ.

and, in the case of the functions $f_{\tan}$ and $f_{\sec}$, we suppose that

$$0<\frac{p}{q}<\frac{1}{2}.$$

It is tacitly assumed that we work here with the *principal branch* of the logarithm function. This means that the branch of the logarithm function whose imaginary part lies in the semi-closed interval $(-\pi,\pi]$ is taken. We note that the contour Γ is analogous to the contours used by De Doelder [8] and introduced (more recently) by Cho et al. [3].

Proof of Theorem 1. Consider the contour integration of $f_{\cot}$ and $f_{\tan}$ in (3.3) along the contour Γ. Since both of our functions are analytic on Γ and within the region enclosed by Γ, in light of the Cauchy integral theorem, for the fixed p and q, we have

$$\lim_{\substack{\delta\downarrow 0\\ \eta\downarrow 0}}\Bigg[\int_{\delta}^{1-\eta}\frac{x(\ln x)^n}{x^2\mp 1}dx+i\int_{\pi}^{\pi/2}\frac{(1+\eta e^{i\phi})[\frac{1}{2}\ln(1+2\eta\cos\phi+\eta^2)+i\theta]^n}{(1+\eta e^{i\phi})^2\mp 1}\eta e^{i\phi}\,d\phi$$
$$+\frac{i^n}{2}\int_{\arctan\eta}^{p\pi/q}\left(\phi^n\cdot\begin{Bmatrix}\cot\phi\\ -\tan\phi\end{Bmatrix}+i\right)d\phi+e^{ip\pi/q}\int_{1}^{\delta}\frac{ue^{ip\pi/q}(\ln u+i\frac{p\pi}{q})^n}{u^2e^{2ip\pi/q}\mp 1}du$$
$$+i\int_{p\pi/q}^{0}\frac{\delta e^{i\phi}(\ln\delta+i\phi)^n}{\delta^2e^{2i\phi}\mp 1}\delta e^{i\phi}\,d\phi\Bigg]=0\quad\left(\begin{Bmatrix}0<\frac{p}{q}<1\\ 0<\frac{p}{q}<\frac{1}{2}\end{Bmatrix}\right),\tag{3.4}$$

where use is made of the following relationship:

$$1+\eta e^{i\phi}=\left(1+2\eta\cos\phi+\eta^2\right)^{\frac{1}{2}}e^{i\theta}\quad(0\leqq\theta\leqq\arctan\eta).$$

Upon taking the indicated limits, it is easy to show that

$$\int_0^1\frac{x(\ln x)^n}{x^2\mp 1}dx\pm\int_0^1\left(\ln u+i\frac{p\pi}{q}\right)^n\frac{ue^{i2p\pi/q}du}{1\mp u^2e^{2ip\pi/q}}+\frac{i^{n+1}}{2}\int_0^{p\pi/q}\phi^n\,d\phi$$
$$+\frac{i^n}{2}\int_0^{p\pi/q}\phi^n\begin{Bmatrix}\cot\phi\\ -\tan\phi\end{Bmatrix}d\phi=0\quad\left(\begin{Bmatrix}0<\frac{p}{q}<1\\ 0<\frac{p}{q}<\frac{1}{2}\end{Bmatrix}\right).\tag{3.5}$$

Next, since

$$\left(\ln u+i\frac{p\pi}{q}\right)^n=i^n\left(\frac{p\pi}{q}\right)^n\left[1+\sum_{k=1}^{n}(-1)^k\binom{n}{k}i^k\left(\frac{q}{p\pi}\right)^k(\ln u)^k\right],\tag{3.6}$$

we may rewrite the equations in (3.5) in the following forms:

$$\frac{1}{2}\int_0^{p\pi/q}\phi^n\cot\phi\,d\phi=\Re\left(I_n^1-I_n^2(p,q)-I_n^3(p,q)\right)\quad\left(0<\frac{p}{q}<1\right)\tag{3.7}$$

and

$$\frac{1}{2}\int_0^{p\pi/q}\phi^n\tan\phi\,d\phi=\Re\bigl(J_n^1-J_n^2(p,q)-J_n^3(p,q)\bigr)\quad\left(0<\frac{p}{q}<\frac{1}{2}\right), \tag{3.8}$$

where the integrals I_n^1, $I_n^2(p,q)$ and $I_n^3(p,q)$, and the integrals J_n^1, $J_n^2(p,q)$ and $J_n^3(p,q)$, are as given below.

Now, by observing that

$$0<\frac{p}{q}<1,$$

we evaluate the cotangent integral (3.7).

First, clearly, the integral I_n^1 given by

$$I_n^1=i^n\int_0^1\frac{x(\ln x)^n}{1-x^2}\,dx=i^n(-1)^n n!2^{-(n+1)}\zeta(n+1),$$

follows at once as a special case of (3.1) *with*

$$\alpha=2\quad\text{and}\quad\mu=2.$$

The integral $I_n^2(p,q)$ is elementary and it is not difficult to show that

$$\left(\frac{q}{p\pi}\right)^n I_n^2(p,q)=\int_0^1\frac{ue^{2ip\pi/q}}{1-u^2e^{2ip\pi/q}}\,du=-\frac{1}{2}\ln\left[2\sin\left(\frac{p\pi}{q}\right)\right]+i\left(\frac{\pi}{4}-\frac{p\pi}{2q}\right).$$

In the case of $I_n^3(p,q)$, we have

$$I_n^3(p,q)=e^{i2p\pi/q}\left(\frac{p\pi}{q}\right)^n\int_0^1\sum_{k=1}^n(-1)^k\binom{n}{k}i^k\left(\frac{q}{p\pi}\right)^k\frac{u(\ln u)^k}{1-u^2e^{2ip\pi/q}}.$$

We thus need only to verify the following evaluation:

$$\begin{aligned}
I_n^3(p,q)&=\left(\frac{p\pi}{q}\right)^n\sum_{k=1}^n(-1)^k\binom{n}{k}i^k\left(\frac{q}{p\pi}\right)^k\sum_{l=0}^{q-1}e^{(2l+2)ip\pi/q}\int_0^1\frac{u^{2l+1}(\ln u)^k}{1-u^{2q}}\,du\\
&=n!\left(\frac{p\pi}{q}\right)^{n+1}\sum_{k=1}^n\frac{i^k}{(n-k)!}\frac{1}{(2p\pi)^{k+1}}\sum_{l=1}^q e^{2lip\pi/q}\zeta\left(k+1,\frac{l}{q}\right)\\
&=n!\left(\frac{p\pi}{q}\right)^{n+1}\left[\sum_{k=1}^{\lfloor n/2\rfloor}\sum_{l=1}^q\frac{(-1)^k}{(n-2k)!}\frac{e^{2lip\pi/q}}{(2p\pi)^{2k+1}}\zeta\left(2k+1,\frac{l}{q}\right)\right.\\
&\quad\left.+\sum_{k=1}^{\lfloor (n+1)/2\rfloor}\sum_{l=1}^q\frac{i(-1)^{k-1}}{(n-2k+1)!}\frac{e^{2lip\pi/q}}{(2p\pi)^{2k}}\zeta\left(2k,\frac{l}{q}\right)\right],
\end{aligned}$$

which is obtained by making use of the elementary identity:

$$\frac{1}{1-u^2e^{2ip\pi/q}}=\frac{1}{1-u^{2q}}\sum_{l=0}^{q-1}u^{2l}\,e^{2ip\pi l/q}$$

and the integral in (3.1). Lastly, by making use of (3.7), and upon taking the real parts of I_n^1, $I_n^2(p,q)$ and $I_n^3(p,q)$, we arrive at the required formula (2.2).

Now, by considering the hypothesis that

$$0<\frac{p}{q}<\frac{1}{2},$$

we evaluate the tangent integral (3.8).

The integral J_n^1 is the special case of (3.2):

$$J_n^1=i^n\int_0^1\frac{x(\ln x)^n}{1+x^2}\,dx=(-1)^n i^n n!2^{-(n+1)}\bigl(1-2^{-n}\bigr)\zeta(n+1),$$

$J_n^2(p,q)$ is readily deducible as follows:

$$\left(\frac{q}{p\pi}\right)^n J_n^2(p,q)=\int_0^1 \frac{ue^{2ip\pi/q}}{1+u^2e^{2ip\pi/q}}\,du=\frac{1}{2}\ln\left[2\cos\left(\frac{p\pi}{q}\right)\right]+i\frac{\pi p}{2q},$$

while the integral $J_n^3(p,q)$ is given by

$$J_n^3(p,q)=\left(\frac{p\pi}{q}\right)^n\int_0^1\sum_{k=1}^n(-1)^k\binom{n}{k}i^k\left(\frac{q}{p\pi}\right)^k\frac{ue^{2ip\pi/q}(\ln u)^k}{1+u^2e^{2ip\pi/q}}.$$

In view of the following elementary identity:

$$\frac{1}{1+u^2e^{2ip\pi/q}}=\frac{1}{1-(-1)^q u^{2q}}\sum_{l=0}^{q-1}(-1)^l u^{2l}e^{2ip\pi l/q},$$

and the integrals in (3.1) and (3.2), it is not difficult to show that

$$\begin{aligned}J_n^3(p,q)&=\left(\frac{p\pi}{q}\right)^n\sum_{k=1}^n(-1)^k\binom{n}{k}i^k\left(\frac{q}{p\pi}\right)^k\\&\quad\cdot\sum_{l=0}^{q-1}(-1)^l e^{(2l+2)ip\pi/q}\int_0^1\frac{u^{2l+1}(\ln u)^k}{1-(-1)^q u^{2q}}\,du\\&=n!\left(\frac{p\pi}{q}\right)^{n+1}\left[\sum_{k=1}^{\lfloor n/2\rfloor}\sum_{l=1}^{q}\frac{(-1)^{k-1}}{(n-2k)!}\frac{(-1)^l}{(2p\pi)^{2k+1}}e^{2lip\pi/q}Z_q\left(2k+1,\frac{l}{q}\right)\right.\\&\quad\left.+\sum_{k=1}^{\lfloor (n+1)/2\rfloor}\sum_{l=1}^{q}\frac{(-1)^{k-1}}{(n-2k+1)!}\frac{i(-1)^{l-1}}{(2p\pi)^{2k}}e^{2lip\pi/q}Z_q\left(2k,\frac{l}{q}\right)\right],\end{aligned}$$

where Z_q [see Eq. (2.1)] denotes either $\zeta(s,a)$ or $\eta(s,a)$ depending upon the parity of q. Finally, the proposed integral formula (2.3) follows by making use of (3.8), and upon taking the real parts of J_n^1, $J_n^2(p,q)$ and $J_n^3(p,q)$.

This completes the proof of Theorem 1. □

Proof of Theorem 2. Our proof of Theorem 2 runs along the same lines as the proof of Theorem 1.

Here, we consider the integration of the complex functions $f_{\csc}$ and $f_{\sec}$ (3.3) around the above-described closed contour Γ. In this way, by the Cauchy integral theorem, we arrive at the expressions analogous to those in (3.4) and, upon taking the limits, we find for the fixed p and q that

$$\int_0^1\frac{(\ln x)^n}{x^2\mp 1}\,dx\pm e^{ip\pi/q}\int_0^1\left(\ln u+i\frac{p\pi}{q}\right)^n\frac{du}{1\mp u^2e^{2ip\pi/q}}+\frac{i^n}{2}\int_0^{p\pi/q}\phi^n\left\{\begin{matrix}\csc\phi\\ i\sec\phi y\end{matrix}\right\}d\phi=0\quad\left(\left\{\begin{matrix}0<\frac{p}{q}<1\\ 0<\frac{p}{q}<\frac{1}{2}\end{matrix}\right\}\right),\tag{3.9}$$

which, in turn, and in view of (3.6), can be rewritten as follows:

$$\frac{1}{2}\int_0^{p\pi/q}\phi^n\csc\phi\,d\phi=\Re\big(\mathcal{I}_n^1-\mathcal{I}_n^2(p,q)-\mathcal{I}_n^3(p,q)\big)\quad\left(0<\frac{p}{q}<1\right)\tag{3.10}$$

and

$$\frac{1}{2}\int_0^{p\pi/q}\phi^n\sec\phi\,d\phi=\Re\big(\mathcal{J}_n^1+\mathcal{J}_n^2(p,q)+\mathcal{J}_n^3(p,q)\big)\quad\left(0<\frac{p}{q}<\frac{1}{2}\right),\tag{3.11}$$

$\mathcal{I}_n^1$, $\mathcal{I}_n^2(p,q)$ and $\mathcal{I}_n^3(p,q)$, and $\mathcal{J}_n^1$, $\mathcal{J}_n^2(p,q)$ and $\mathcal{J}_n^3(p,q)$, being the integrals given below.

Now, in order to evaluate the cosecant integral (2.4), we use the following results:

$$\mathcal{I}_n^1=(-1)^n i^n\int_0^1\frac{(\ln x)^n}{1-x^2}\,dx=i^n n!\big(1-2^{-(n+1)}\big)\zeta(n+1),$$

$$\mathcal{I}_n^2(p,q)\left(\frac{q}{p\pi}\right)^n=\int_0^1\frac{e^{ip\pi/q}}{1-u^2e^{2ip\pi/q}}\,du=\frac{i\pi}{4}-\frac{1}{2}\ln\left[\tan\left(\frac{p\pi}{2q}\right)\right],$$

and

$$\mathcal{I}_n^3(p,q)=\left(\frac{p\pi}{q}\right)^n\int_0^1\sum_{k=1}^n(-1)^k\binom{n}{k}i^k\left(\frac{q}{p\pi}\right)^k\frac{e^{ip\pi/q}(\ln u)^k}{1-u^2e^{2ip\pi/q}}\,du$$
$$=n!\left(\frac{p\pi}{q}\right)^{n+1}\left[\sum_{k=1}^{\lfloor n/2\rfloor}\sum_{l=1}^{q}\frac{(-1)^k}{(n-2k)!}\frac{e^{(2l-1)ip\pi/q}}{(2p\pi)^{2k+1}}\zeta\left(2k+1,\frac{2l-1}{2q}\right)\right.$$
$$\left.+\sum_{k=1}^{\lfloor (n+1)/2\rfloor}\sum_{l=1}^{q}\frac{i(-1)^{k-1}}{(n-2k+1)!}\frac{e^{(2l-1)ip\pi/q}}{(2p\pi)^{2k}}\zeta\left(2k,\frac{2l-1}{2q}\right)\right],$$

and we readily obtain the required formula by making use of (3.10) in conjunction with taking the real parts of $\mathcal{I}_n^1$, $\mathcal{I}_n^2(p,q)$ and $\mathcal{I}_n^3(p,q)$.

Similarly, in order to derive the integral formula (2.5), the following integrals are needed:

$$\mathcal{J}_n^1=(-1)^n i^{n+1}\int_0^1\frac{(\ln x)^n}{1+x^2}\,dx=i^{n+1}n!\beta(n+1),$$

$$i\mathcal{J}_n^2(p,q)\left(\frac{q}{p\pi}\right)^n=\int_0^1\frac{e^{ip\pi/q}}{1+u^2e^{2ip\pi/q}}du=\frac{\pi}{4}+\frac{i}{2}\log\left[\tan\left(\frac{p\pi}{2q}+\frac{\pi}{4}\right)\right],$$

and

$$\mathcal{J}_n^3(p,q)=-i\left(\frac{p\pi}{q}\right)^n\int_0^1\sum_{k=1}^n(-1)^k\binom{n}{k}i^k\left(\frac{q}{p\pi}\right)^k\frac{e^{ip\pi/q}(\ln u)^k}{1+u^2e^{2ip\pi/q}}\,du$$
$$=n!\left(\frac{p\pi}{q}\right)^{n+1}\left[\sum_{k=1}^{\lfloor n/2\rfloor}\sum_{l=1}^{q}\frac{(-1)^{k-1}}{(n-2k)!}\frac{i(-1)^{l-1}}{(2p\pi)^{2k+1}}e^{(2l-1)ip\pi/q}Z_q\left(2k+1,\frac{2l-1}{2q}\right)\right.$$
$$\left.+\sum_{k=1}^{\lfloor (n+1)/2\rfloor}\sum_{l=1}^{q}\frac{(-1)^{k-1}}{(n-2k+1)!}\frac{(-1)^{l-1}}{(2p\pi)^{2k}}e^{(2l-1)ip\pi/q}Z_q\left(2k,\frac{2l-1}{2q}\right)\right].$$

We then employ (3.11) and take the real parts of $\mathcal{J}_n^1$, $\mathcal{J}_n^2(p,q)$ and $\mathcal{J}_n^3(p,q)$. This completes the proof of Theorem 2. □

4. Special cases and consequences

In this section, we give several immediate consequences and some illustrative examples of the application of our integral formulas derived in Section 2.

The cotangent integrals. Let

$$Q_n^{\cot}=(-1)^{\lfloor n/2\rfloor}\left\{1+(-1)^n\right\}n!2^{-(n+1)}\zeta(n+1).$$

Some special cases of the cotangent integrals in (2.2) are given by

$$\int_0^{\pi/2}\phi^n\cot\phi\,d\phi=Q_n^{\cot}+\left(\frac{\pi}{2}\right)^n\ln 2+2n!\left(\frac{\pi}{2}\right)^{n+1}\sum_{k=1}^{\lfloor n/2\rfloor}\frac{(-1)^{k-1}}{(n-2k)!}\frac{V_{2k+1}}{(2\pi)^{2k+1}},\tag{4.1}$$

where

$$V_{2k+1}=\left(2-2^{2k+1}\right)\zeta(2k+1);\tag{4.2}$$

$$\int_0^{\pi/3}\phi^n\cot\phi\,d\phi=Q_n^{\cot}+\frac{1}{2}\left(\frac{\pi}{3}\right)^n\ln 3+n!\left(\frac{\pi}{3}\right)^{n+1}$$
$$\cdot\left(\sqrt{3}\sum_{k=1}^{\lfloor (n+1)/2\rfloor}\frac{(-1)^{k-1}}{(n-2k+1)!}\frac{A_{2k}^{\cot}}{(2\pi)^{2k}}+\sum_{k=1}^{\lfloor n/2\rfloor}\frac{(-1)^{k-1}}{(n-2k)!}\frac{B_{2k+1}^{\cot}}{(2\pi)^{2k+1}}\right),\tag{4.3}$$

where

$$A_{2k}^{\cot} = 2\zeta\left(2k, \frac{1}{3}\right) + \left(1 - 3^{2k}\right)\zeta(2k) \quad \text{and} \quad B_{2k+1}^{\cot} = \left(3 - 3^{2k+1}\right)\zeta(2k+1);$$

and

$$\int_0^{\pi/4} \phi^n \cot\phi \, d\phi = Q_n^{\cot} + \frac{1}{2}\left(\frac{\pi}{4}\right)^n \ln 2 + 2n!\left(\frac{\pi}{4}\right)^{n+1} \cdot \left(\sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{(-1)^{k-1}}{(n-2k+1)!} \frac{U_{2k}}{(2\pi)^{2k}} + \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(-1)^{k-1}}{(n-2k)!} \frac{V_{2k+1}}{(2\pi)^{2k+1}}\right), \tag{4.4}$$

where V_{2k+1} is given by (4.2) and U_{2k} is given by

$$U_{2k} = 2\zeta\left(2k, \frac{1}{4}\right) + 2^{2k}\left(1 - 2^{2k}\right)\zeta(2k). \tag{4.5}$$

Regarding the integrals (4.1) and (4.4), it is a well-known classical result (cf. [2, pp. 306–310, Tables 204–206] and [10, Tables 204–206]) that they can be evaluated by means of infinite series as in (1.1) and (1.2). A much newer result is the finite series evaluation of these integrals: Crandall and Buhler [6, p. 280] obtained (4.1), while (4.1) and (4.4) can be deduced as special cases of the integrals which were given by Srivastava et al. [13, p. 834, Eqs. (2.14) and (2.15)]. Formula (4.3) is presumably a new result (see also a recent work on the evaluation of the Euler and related sums by Choi and Srivastava [5]).

Now, by making use of (4.1), it is easy to deduce the following results:

$$\int_0^{\pi/2} \phi \cot\phi \, d\phi = \frac{\pi}{2} \ln 2;$$

$$\int_0^{\pi/2} \phi^2 \cot\phi \, d\phi = \frac{\pi^2}{4} \ln 2 - \frac{7}{8}\zeta(3);$$

$$\int_0^{\pi/2} \phi^3 \cot\phi \, d\phi = \frac{\pi^3}{8} \ln 2 - \frac{9}{16}\pi\zeta(3);$$

$$\int_0^{\pi/2} \phi^4 \cot\phi \, d\phi = \frac{\pi^4}{16} \ln 2 - \frac{9}{16}\pi^2\zeta(3) + \frac{93}{32}\zeta(5);$$

$$\int_0^{\pi/2} \phi^5 \cot\phi \, d\phi = \frac{\pi^5}{32} \ln 2 - \frac{15}{32}\pi^3\zeta(3) + \frac{225}{64}\pi\zeta(5).$$

The cosecant integrals. Let

$$Q_n^{\csc} = (-1)^{\lfloor n/2 \rfloor}\left\{1 + (-1)^n\right\} n!\left(1 - 2^{-(n+1)}\right)\zeta(n+1).$$

Some special cases of the cosecant integrals in (2.4) are given by

$$\int_0^{\pi/2} \phi^n \csc\phi \, d\phi = Q_n^{\csc} + 2n!\left(\frac{\pi}{2}\right)^{n+1} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{(-1)^{k-1}}{(n-2k+1)!} \frac{U_{2k}}{(2\pi)^{2k}}, \tag{4.6}$$

where U_{2k} is given by (4.5);

$$\int_0^{\pi/3} \phi^n \csc\phi \, d\phi = Q_n^{\csc} - \frac{1}{2}\left(\frac{\pi}{3}\right)^n \ln 3 + n!\left(\frac{\pi}{3}\right)^{n+1} \cdot \left(\sqrt{3} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{(-1)^{k-1}}{(n-2k+1)!} \frac{A_{2k}^{\csc}}{(2\pi)^{2k}} + \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(-1)^{k-1}}{(n-2k)!} \frac{B_{2k+1}^{\csc}}{(2\pi)^{2k+1}}\right), \tag{4.7}$$

where

$$A_{2k}^{\csc} = 2\zeta\left(2k, \frac{1}{6}\right) + \left(1 - 2^{2k}\right)\left(3^{2k} - 1\right)\zeta(2k)$$

and

$$B_{2k+1}^{\csc} = \left(1 - 2^{2k+1}\right)\left(3 - 3^{2k+1}\right)\zeta(2k+1);$$

and

$$\int_0^{\pi/4} \phi^n \csc\phi \, d\phi = Q_n^{\csc} + \left(\frac{\pi}{4}\right)^n \ln\left(\sqrt{2} - 1\right) + n!\left(\frac{\pi}{4}\right)^{n+1}\sqrt{2}$$

$$\cdot \left(\sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{(-1)^{k-1}}{(n-2k+1)!} \frac{C_{2k}^{\csc}}{(2\pi)^{2k}} + \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(-1)^{k-1}}{(n-2k)!} \frac{D_{2k+1}^{\csc}}{(2\pi)^{2k+1}} \right), \tag{4.8}$$

where

$$C_{2k}^{\csc} = 2\zeta\left(2k, \frac{1}{8}\right) + 2\zeta\left(2k, \frac{3}{8}\right) + \left(4^{2k} - 8^{2k}\right)\zeta(2k) \tag{4.9}$$

and

$$D_{2k+1}^{\csc} = 2\zeta\left(2k+1, \frac{1}{8}\right) + 2\zeta\left(2k+1, \frac{7}{8}\right) + \left(4^{2k+1} - 8^{2k+1}\right)\zeta(2k+1). \tag{4.10}$$

We remark that the above cosecant integral formulas resemble those derived earlier by Cho et al. [3, pp. 465–467, Eqs. (23) and (24), and Eqs. (26) to (29)], but their results also contain the above-mentioned double sums as an *additional* term.

Now, by making use of (4.6), it is easy to deduce the following results:

$$\int_0^{\pi/2} \phi \csc\phi \, d\phi = 2G; \tag{4.11}$$

$$\int_0^{\pi/2} \phi^2 \csc\phi \, d\phi = 2\pi G - \frac{7}{2}\zeta(3); \tag{4.12}$$

$$\int_0^{\pi/2} \phi^3 \csc\phi \, d\phi = \frac{3}{2}\pi^2 G + \frac{\pi^4}{8} - \frac{3}{32}\zeta\left(4, \frac{1}{4}\right); \tag{4.13}$$

$$\int_0^{\pi/2} \phi^4 \csc\phi \, d\phi = \pi^3 G + \frac{\pi^5}{4} + \frac{93}{2}\zeta(5) - \frac{3}{16}\pi\zeta\left(4, \frac{1}{4}\right); \tag{4.14}$$

$$\int_0^{\pi/2} \phi^5 \csc\phi \, d\phi = \frac{5}{8}\pi^4 G + \frac{\pi^6}{16} - \frac{15}{64}\pi^2\zeta\left(4, \frac{1}{4}\right) + \frac{15}{128}\zeta\left(6, \frac{1}{4}\right), \tag{4.15}$$

in terms of the Catalan constant G given by (1.9).

The integral in (4.11) is recorded in the work of (for instance) Choi and Srivastava [4, p. 101, Eq. (2.41)] and (4.12) was first evaluated by De Doelder [8]. We have failed to find, in the literature, the integrals presented in (4.13) to (4.15).

The tangent and secant integrals. We now list the following special cases of the integrals in (2.4) and (2.5):

$$\int_0^{\pi/4} \phi^n \tan\phi \, d\phi = Q_n^{\tan} - \frac{1}{2}\left(\frac{\pi}{4}\right)^n \ln 2 + 2n!\left(\frac{\pi}{4}\right)^{n+1}$$

$$\cdot \left(\sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{(-1)^{k-1}}{(n-2k+1)!} \frac{U_{2k}}{(2\pi)^{2k}} - \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(-1)^{k-1}}{(n-2k)!} \frac{V_{2k+1}}{(2\pi)^{2k+1}} \right), \tag{4.16}$$

where

$$Q_n^{\tan} = (-1)^{\lfloor n/2 \rfloor}\left\{1 + (-1)^n\right\} n! 2^{-(n+1)}\left(1 - 2^{-n}\right)\zeta(n+1),$$

and the coefficients U_{2k} and V_{2k+1} are, respectively, given by (4.5) and (4.2);

$$\int_0^{\pi/4} \phi^n \sec\phi \, d\phi = Q_n^{\sec} + \left(\frac{\pi}{4}\right)^n \ln\left(\sqrt{2}+1\right) + n!\left(\frac{\pi}{4}\right)^{n+1}\sqrt{2} \cdot \left(\sum_{k=1}^{\lfloor (n+1)/2\rfloor} \frac{(-1)^{k-1}}{(n-2k+1)!}\frac{C_{2k}^{\csc}}{(2\pi)^{2k}} - \sum_{k=1}^{\lfloor n/2\rfloor} \frac{(-1)^{k-1}}{(n-2k)!}\frac{D_{2k+1}^{\csc}}{(2\pi)^{2k+1}}\right), \tag{4.17}$$

where

$$Q_n^{\sec} = (-1)^{\lfloor (n+1)/2\rfloor}\left\{1-(-1)^n\right\} n! 4^{-(n+1)}\left[\zeta\left(n+1, \frac{1}{4}\right) - \zeta\left(n+1, \frac{3}{4}\right)\right],$$

and the coefficients $C_{2k}^{\csc}$ and $D_{2k+1}^{\csc}$ are given by (4.9) and (4.10), respectively.

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