Applied Mathematics Letters 22 (2009) 217–220

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Values of the derivatives of the cotangent at rational multiples of π ☆

Djurdje Cvijović

Atomic Physics Laboratory, Vinča Institute of Nuclear Sciences, P.O. Box 522, 11001 Belgrade, Serbia

ARTICLE INFO

Article history:
Received 13 December 2006
Received in revised form 2 January 2008
Accepted 24 March 2008

Keywords:
Cotangent function
Simpson's series multisection formula
Bernoulli polynomials
Bernoulli numbers
Hurwitz zeta function
Polylogarithm

ABSTRACT

By elementary arguments, we deduce closed-form expressions for the values of all derivatives of the cotangent function at rational multiples of π. These formulae are considerably simpler than similar ones which were found in a different manner by Kölbig. Also, we show that the values of $\cot^{(n)}(\pi x)$, $n \in N$, at $x = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}$ and $\frac{5}{6}$ are expressible in terms of the values of the Bernoulli polynomials alone.

1. Introduction

Recently, Adamchik [2, p. 4, Eq. 26] completely solved a long-standing problem of finding a closed-form expression for the higher derivatives of the cotangent function [2,3,6]. In this sequel to the work of Adamchik, by elementary arguments, we deduce that the values of the derivatives of the cotangent at rational multiples of π can be expressed as finite sums involving the Bernoulli polynomials, the sine or cosine functions and known constants. These formulae are considerably simpler than similar ones which were found in a different manner by Kölbig [6, p. 8, Theorem 4]. Also, as an immediate corollary of this result, we find that the values of $\cot^{(n)}(\pi x)$, $n \in N$, at $x = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}$ and $\frac{5}{6}$ are expressible in terms of the values of the Bernoulli polynomials alone.

2. Statement of the results

In what follows, we shall use the function given by

$$l_\nu(x) = \sum_{k=1}^{\infty} \frac{\left(e^{2\pi i x}\right)^k}{k^\nu} = Li_\nu(e^{2\pi i x}) \quad (x \in \mathbb{R}) \tag{1}$$

where

$$Li_\nu(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^\nu} \tag{2}$$

$$(\nu \in \mathbb{C}, |z| < 1; \Re(\nu) > 0, |z| \le 1, z \ne 1; \Re(\nu) > 1, |z| \le 1)$$

☆ The present investigation was supported by the Ministry of Science and Environmental Protection of the Republic of Serbia under research project number 142025.

E-mail address: djurdje@vin.bg.ac.yu.

0893-9659/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2008.03.013

is the *polylogarithm* $Li_\nu(z)$ [5, p. 30, Eq. 1. 11 (14)]. Observe that

$$\Re[l_\nu(x)] = \sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{k^\nu} \quad \text{and} \quad \Im[l_\nu(x)] = \sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{k^\nu} \quad (x \in \mathbb{R}). \tag{3}$$

The *Bernoulli polynomial* of degree n in x, $B_n(x)$, is, as usual, given by the generating function [5, p. 36, Eq. 1.13 (2)]

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi). \tag{4}$$

and $B_n = B_n(0)$ is the n-th *Bernoulli number* [5, p. 35, Eq. 1.13 (1)].

The *Hurwitz (or generalized)* zeta function $\zeta(s, a)$ is defined as an analytic continuation of the series [5, p. 24, Eq. 1.10 (1)]

$$\zeta(s, a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \quad (\Re(s) > 1;\, a \neq 0, -1, -2, \dots,). \tag{5}$$

Our results are as follows.

Theorem. *If n, p and $q \in \mathbb{N}$ and p and q are such that $1 \le p < q$, then*

$$\text{(a)}\ \frac{d^{2n-1}}{dx^{2n-1}} \cot(\pi x)|_{x=p/q} = (-1)^n \frac{(2\pi q)^{2n-1}}{n} \sum_{s=1}^{q} B_{2n}(s/q) \cos(s2\pi p/q),$$

$$\text{(b)}\ \frac{d^{2n}}{dx^{2n}} \cot(\pi x)|_{x=p/q} = (-1)^{n-1} \frac{2(2\pi q)^{2n}}{2n+1} \sum_{s=1}^{q} B_{2n+1}(s/q) \sin(s2\pi p/q),$$

where $B_n(x)$ is the Bernoulli polynomial of degree n.

Corollary 1. *Assume that $n \in \mathbb{N}$ and let B_n be the n-th Bernoulli number. We have*

$$\text{(a)}\ \frac{d^{2n-1}}{dx^{2n-1}} \cot(\pi x)|_{x=1/2} = (-1)^n \frac{(4\pi)^{2n-1}}{n} [B_{2n} - B_{2n}(1/2)] = (-1)^n \frac{(2\pi)^{2n-1}(2^{2n}-1)}{n} B_{2n};$$

$$\text{(b)}\ \frac{d^{2n-1}}{dx^{2n-1}} \cot(\pi x)|_{x=1/3} = \frac{d^{2n-1}}{dx^{2n-1}} \cot(\pi x)|_{x=2/3} = (-1)^n \frac{(6\pi)^{2n-1}}{n} [B_{2n} - B_{2n}(1/3)] = (-1)^n \frac{(2\pi)^{2n-1}(3^{2n}-1)}{2n} B_{2n};$$

$$\text{(c)}\ \frac{d^{2n-1}}{dx^{2n-1}} \cot(\pi x)|_{x=1/4} = \frac{d^{2n-1}}{dx^{2n-1}} \cot(\pi x)|_{x=3/4} = (-1)^n \frac{(8\pi)^{2n-1}}{n} [B_{2n} - B_{2n}(1/2)] = (-1)^n \frac{(4\pi)^{2n-1}(2^{2n}-1)}{n} B_{2n};$$

$$\begin{aligned}\text{(d)}\ \frac{d^{2n-1}}{dx^{2n-1}} \cot(\pi x)|_{x=1/6} &= \frac{d^{2n-1}}{dx^{2n-1}} \cot(\pi x)|_{x=5/6} = (-1)^n \frac{(12\pi)^{2n-1}}{n} [B_{2n} - B_{2n}(1/2) - B_{2n}(1/3) + B_{2n}(1/6)] \\ &= (-1)^n \frac{(2\pi)^{2n-1}(2^{2n}-1)(3^{2n}-1)}{2n} B_{2n}.\end{aligned}$$

Corollary 2. *Assume that $n \in \mathbb{N}$ and let $B_n(x)$ be the Bernoulli polynomial of degree n. We have*

$$\text{(a)}\ \frac{d^{2n}}{dx^{2n}} \cot(\pi x)|_{x=1/2} = 0;$$

$$\text{(b)}\ \frac{d^{2n}}{dx^{2n}} \cot(\pi x)|_{x=1/3} = -\frac{d^{2n}}{dx^{2n}} \cot(\pi x)|_{x=2/3} = (-1)^{n-1} \frac{2(6\pi)^{2n}\sqrt{3}}{2n+1} B_{2n+1}(1/3);$$

$$\text{(c)}\ \frac{d^{2n}}{dx^{2n}} \cot(\pi x)|_{x=1/4} = -\frac{d^{2n}}{dx^{2n}} \cot(\pi x)|_{x=3/4} = (-1)^{n-1} \frac{4(8\pi)^{2n}}{2n+1} B_{2n+1}(1/4);$$

$$\text{(d)}\ \frac{d^{2n}}{dx^{2n}} \cot(\pi x)|_{x=1/6} = -\frac{d^{2n}}{dx^{2n}} \cot(\pi x)|_{x=5/6} = (-1)^{n-1} \frac{2(12\pi)^{2n}\sqrt{3}}{2n+1} [B_{2n+1}(1/6) + B_{2n+1}(1/3)].$$

3. Proof of the results

In our proof of the theorem we shall use some known results:

(1) (Simpson's series multisection formula) [7, p. 131]. Let $f(x) = \sum_{k=1}^{\infty} a_k x^k$ and let $\omega = e^{2\pi i/q}$ ($i := \sqrt{-1}$; $q \in \mathbb{N}$). Then for any integer p, $1 \le p \le q$, we have $q \sum_{k=0}^{\infty} a_{p+qk} x^{p+qk} = \sum_{s=1}^{q} \omega^{-sp} f(\omega^s x)$.

(2) (Abel's theorem) [4, p. 148]. Let $f(x) = \sum_{k=1}^{\infty} a_k x^k$, $|x| < 1$. If $\sum_{k=1}^{\infty} a_k$ converges then $\lim_{x \to 1-} f(x) = \sum_{k=1}^{\infty} a_k$.

(3) (The Fourier series for $B_n(x)$). Let $B_n(x)$ be the Bernoulli polynomial as defined in (4). We have [5, p. 38, Eq. 1.13 (15)]

$$B_{2n-1}(x) = (-1)^n \frac{2(2n-1)!}{(2\pi)^{2n-1}} \sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{k^{2n-1}} \quad (0 < x < 1, n = 1; 0 \le x \le 1, n = 2, 3, \ldots), \tag{6}$$

and [5, p. 37, Eq. 1.13 (14)]

$$B_{2n}(x) = (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{k^{2n}} \quad (0 \le x \le 1, n \in \mathbb{N}). \tag{7}$$

(4) (Reflection formulae for $\zeta(s, a)$) (see [5, p. 44, Eq. 1.16 (4)] in conjunction with [5, p. 45, Eq. 1.17 (9)]) If $n \in \mathbb{N}$ and $0 < x < 1$, then

$$\zeta(2n+1, x) - \zeta(2n+1, 1-x) = \frac{\pi}{(2n)!} \cot(\pi x)^{(2n)}, \tag{8}$$

$$\zeta(2n, x) + \zeta(2n, 1-x) = -\frac{\pi}{(2n-1)!} \cot(\pi x)^{(2n-1)}. \tag{9}$$

In the proof of Corollary 1 we shall need the following relations [1, pp. 805–806, Eqs. 23.1.19–23.1.24]:

$$B_{2n}(0) = B_{2n}(1) = B_{2n}; \tag{10}$$

$$B_{2n}(1/2) = (2^{1-2n} - 1)B_{2n}; \tag{11}$$

$$B_{2n}(1/3) = B_{2n}(2/3) = (1/2)(3^{1-2n} - 1)B_{2n}; \tag{12}$$

$$B_{2n}(1/4) = B_{2n}(3/4) = 2^{-2n}(2^{1-2n} - 1)B_{2n}; \tag{13}$$

$$B_{2n}(1/6) = B_{2n}(5/6) = (1/2)(2^{1-2n} - 1)(3^{1-2n} - 1)B_{2n}. \tag{14}$$

Proof of Theorem. First, we shall show that, for $p, q \in \mathbb{N}$, $1 \le p \le q$, and $\Re(\nu) > 1$, the following holds:

$$\zeta(\nu, p/q) = q^{\nu-1} \sum_{s=1}^{q} \omega^{-sp} Li_\nu(\omega^s), \quad \omega = e^{2\pi i/q}. \tag{15}$$

Clearly, by Abel's theorem and the definition of $\zeta(s, a)$ in (5), the relation in (15) follows at once from

$$\sum_{k=0}^{\infty} \frac{q x^{p+qk}}{(p+qk)^\nu} = \sum_{s=1}^{q} \omega^{-sp} Li_\nu(\omega^s x)$$

which is obtained by making use of Simpson's multisection formula on $Li_\nu(z)$, $\Re(\nu) > 1$.

Second, let $l_\nu(x)$ be given by (1) and let $p, q \in \mathbb{N}$, $1 \le p \le q-1$. Then

$$\zeta(\nu, p/q) - \zeta(\nu, 1-p/q) = 2q^{\nu-1} \sum_{s=1}^{q} \Im[l_\nu(s/q)] \sin(s2\pi p/q), \tag{16}$$

$$\zeta(\nu, p/q) + \zeta(\nu, 1-p/q) = 2q^{\nu-1} \sum_{s=1}^{q} \Re[l_\nu(s/q)] \cos(s2\pi p/q). \tag{17}$$

Indeed, starting from (15) we obtain

$$\zeta(\nu, p/q) = q^{\nu-1} \sum_{s=1}^{q} (\Re[l_\nu(s/q)] + i\Im[l_\nu(s/q)]) e^{-s2\pi i p/q} = q^{\nu-1} \sum_{s=1}^{q} [\Phi_1(s) + i\Phi_2(s)] = q^{\nu-1} \sum_{s=1}^{q} \Phi_1(s) \quad (1 \le p \le q), \tag{18}$$

where

$$\Phi_1(s) = \Re[l_\nu(s/q)] \cos(s2\pi p/q) + \Im[l_\nu(s/q)] \sin(s2\pi p/q),$$
$$\Phi_2(s) = \Im[l_\nu(s/q)] \cos(s2\pi p/q) - \Re[l_\nu(s/q)] \sin(s2\pi p/q),$$

i.e. we obtain that the imaginary part vanishes. To prove this, note that for $\Phi_2(s)$ we have that $\Phi_2(q) = 0$ and $\Phi_2(q-s) = -\Phi_2(s)$ $(1 \le s \le q-1)$ since (see Eq. (3))

$$\Re[l_\nu(1-x)] = \Re[l_\nu(x)] \quad \text{and} \quad \Im[l_\nu(1-x)] = -\Im[l_\nu(x)],$$

and therefore $\sum_{s=1}^{q-1} \Phi_2(s) = 0$, so Eq. (18) is valid. Now, by a simple trigonometric consideration, from (18) we have

$$\zeta(\nu, 1-p/q) = q^{\nu-1} \sum_{s=1}^{q} (\Re[l_\nu(s/q)] \cos(s2\pi p/q) - \Im[l_\nu(s/q)] \sin(s2\pi p/q)) \quad (1 \le p \le q-1). \tag{19}$$

Here, the case $p = q$ should be excluded, considering that for $\zeta(s, a)$ we must have $a \neq 0$ (see the definition of $\zeta(s, a)$ in (5)). Finally, by combining (18) and (19) we arrive at the proposed formulae in (16) and (17).

Third, by applying the Fourier expansions in (6) and (7) in conjunction with (3), from (12) and (13) we have

$$\zeta(2n+1, p/q) - \zeta(2n+1, 1-p/q) = (-1)^{n-1} q^{2n} \frac{(2\pi)^{2n+1}}{(2n+1)!} \sum_{s=1}^{q} B_{2n+1}(s/q) \sin(s2\pi p/q) \tag{20}$$

$$\zeta(2n, p/q) + \zeta(2n, 1-p/q) == (-1)^{n-1} q^{2n-1} \frac{(2\pi)^{2n}}{(2n)!} \sum_{s=1}^{q} B_{2n}(s/q) \cos(s2\pi p/q). \tag{21}$$

Lastly, the assertions of the theorem follow upon comparing the reflection formulae for $\zeta(s, a)$ in (8) and (9) and our expressions in (20) and (21). □

Proof of Corollaries. It is straightforward to verify that, in view of [1, pp. 804, Eq. 23.1.8],

$$B_m(1-x) = (-1)^m B_m(x) \quad (m = 0, 1, 2, \ldots), \tag{22}$$

the formulae given by Corollary 2 follow directly from the theorem. In order to deduce the formulae given by Corollary 1 we apply the theorem and make use of the identity (22) and the relations (10) through (14). □

References

[1] M. Abramowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables, in: NBS Applied Mathematics Series, vol. 55, National Bureau of Standards, Washington, DC, 1964.
[2] V.S. Adamchik, On the Hurwitz function for rational arguments, Appl. Math. Comput. 187 (2007) 3–12.
[3] T.M. Apostol, Dirichlet L-functions and character power sums, J. Number Theory 2 (1970) 223–234.
[4] T.J.I'A. Bromwich, An Introduction to the Theory of Infinite Series, AMS Chelsea Publishing, Providence, 2005.
[5] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher Transcendental Functions, vol. I, McGraw-Hill, New York, 1953.
[6] K.S. Kölbig, The polygamma function and the derivatives of the cotangent function for rational arguments, CERN-IT-Reports, CERN-CN-96-005, 1996.
[7] J. Riordan, Combinatorial Identities, Wiley, New York, 1968.