1-1-2019

# Applications of Geometric and Spectral Methods in Graph Theory

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# Applications of geometric and spectral methods in graph theory

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A Dissertation

Presented to

the Faculty of Natural Sciences and Mathematics

University of Denver

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In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

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by

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June 2019

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Title: Applications of geometric and spectral methods in graph theory
Advisor: Paul Horn
Degree Date: June 2019

ABSTRACT

Networks, or graphs, are useful for studying many things in today's world. Graphs can be used to represent connections on social media, transportation networks, or even the internet. Because of this, it's helpful to study graphs and learn what we can say about the structure of a given graph or what properties it might have. This dissertation focuses on the use of the probabilistic method and spectral graph theory to understand the geometric structure of graphs and find structures in graphs. We will also discuss graph curvature and how curvature lower bounds can be used to give us information about properties of graphs.

A rainbow spanning tree in an edge-colored graph is a spanning tree in which each edge is a different color. Carraher, Hartke, and Horn showed that for $n$ and $C$ large enough, if $G$ is an edge-colored copy of $K_n$ in which each color class has size at most $n/2$, then $G$ has at least $\lfloor n/(C\log n)\rfloor$ edge-disjoint rainbow spanning trees. Here we show that spectral graph theory can be used to prove that if $G$ is *any* edge-colored graph with $n$ vertices in which each color appears on at most $\delta\lambda_1/2$ edges, where $\delta \geq C\log n$ for $n$ and $C$ sufficiently large and $\lambda_1$ is the second-smallest eigenvalue of the normalized Laplacian matrix of $G$, then $G$ contains at least $\left\lfloor \frac{\delta\lambda_1}{C\log n}\right\rfloor$ edge-disjoint rainbow spanning trees.

We show how curvature lower bounds can be used in the context of understanding (personalized) PageRank, which was developed by Brin and Page. PageRank ranks the importance of webpages near a seed webpage, and we are interested in how this importance

diffuses. We do this by using a notion of graph curvature introduced by Bauer, Horn, Lin, Lippner, Mangoubi, and Yau.

ACKNOWLEDGEMENTS

First of all, I would like to thank Paul Horn for supporting me and encouraging me. I would not be here without him, and I am so very thankful for his help and guidance.

I would also like to thank the members of my committee: Natasha Dobrinen, Mei Yin, and Jennifer Hoffman. I am so grateful for Professor Dobrinen. I could not have asked for a better mentor, and I will be forever grateful for her encouragement and advice. Mei made my time at DU so much more enjoyable, and I always looked forward to our conversations.

Thank you, also, to my parents. I would not have been able to do any of this without you.

Most of all, I want to thank Luke – I cannot imagine doing this without him. He always believed in me, and I cannot put into words how much his support has meant to me.

TABLE OF CONTENTS

# CHAPTER 1: INTRODUCTION

## 1.1 Overview

Graphs, or networks, are interesting but complex objects that we would like to better understand. Understanding geometric properties of graphs allows us to prove results about their structure and about substructures they contain. Two ways of doing this are by using spectral graph theory and curvature lower bounds. For example, we can use spectral graph theory, along with the probabilistic method to show that graphs contain certain substructures.

The probabilistic method, pioneered by Paul Erdős, allows one to establish the existence of combinatorial objects by demonstrating that (in a suitable probability space) the probability that a random object has the desired property is positive. It enables us to show the existence of objects without explicitly constructing them. The probabilistic method has become a central tool in modern combinatorics. It has been used to make advances in Ramsey theory, graph theory, and number theory. Probabilistic methods can also be used to obtain asymptotic results, which offer insight into unsolved problems or conjectures. The study of randomly constructed objects, such as random graphs, has proven to be an important topic on its own. It has also played a big role in the study of complex networks.

Spectral graph theory relies on using information about eigenvalues of matrices associated with graphs to understand their properties. For example, we might be interested in understanding geometric properties of a graph, such as whether or not the graph has a bottleneck (that is, a sparse cut). The Cheeger, or isoperimetric, constant is a graph parameter that measures bottlenecking. Determining this value is NP-hard, partly because there are exponentially many cuts of a graph. However, a result known as Cheeger's inequality for graphs relates eigenvalues of certain matrices (such as the normalized Laplacian matrix) to the Cheeger constant so that if one is small, then the other is small. Similarly, eigenvalues also give information about other geometric properties of graphs such as diameter and distances between subsets of vertices. Spectral graph theory also has ties to randomness. The spectrum of the normalized Laplacian matrix is related to the the mixing of random walks, and spectral information certifies 'pseudo-randomness' of edge distributions of graphs.

Cheeger's inequality for graphs is related to the study of manifolds. In [15], Cheeger showed that an eigenvalue of the Laplace-Beltrami operator of a manifold is related to the isoperimetric constant of the manifold. This shows that, in some sense, graphs "act like" discrete manifolds. This motivates us to use the study of manifolds to better understand graphs. Notions of curvature of manifolds are well-studied, and it is natural to try to extend notions of curvature to graphs in meaningful ways. This dissertation will discuss recently developed notions of graph curvature. Spectral graph theory gives us a way to capture global geometric properties of a graph from just a few eigenvalues, which are global quantities that depend on the structure of the entire graph. Graph curvature, on the other hand, is a *local* property which, on graphs, depends only on vertices and their second neighborhoods. However, curvature lower bounds can also be used to give us information about global properties of our graphs. (This is similar to how notions of curvature can be used in Riemannian geometry.)

Both the spectrum of a matrix and graph curvature, then, provide enough information for applying probabilistic methods to understand general classes of graphs. Here we will introduce both spectral graph theory and graph curvature and show how they can be used to understand the geometry and structure of graphs. Understanding the structure of a graph allows us to answer a lot of interesting questions. For example, we show how spectral information can be used to say something about the number of rainbow spanning trees in general graphs. We also use a notion of graph curvature to understand the diffusion of PageRank on a graph.

In Chapter 2 we will discuss rainbow spanning trees in graphs and show how one can use spectral graph theory to have enough information about our graph in order to give a meaningful bound on rainbow spanning trees in general graphs. Chapter 3 will focus on graph curvature and how we can use curvature lower bounds in order to better understand PageRank.

## 1.2 Preliminaries

### 1.2.1 Definitions, Notation, and Terminology

Graphs, or networks, are collections of vertices and edges where each edge joins two vertices. If two vertices $u$ and $v$ have an edge between them, then we say that $u$ and $v$ are *adjacent* and write $u \sim v$. The *degree* of a vertex, $v$, is the number of vertices that are adjacent to $v$, and is denoted $\deg(v)$. For a graph $G$, the volume of a subset $X \subset V(G)$, denoted by $\mathrm{Vol}(X)$, is defined as follows:

$$\mathrm{Vol}(X) = \sum_{v \in X} \deg(v).$$

For a subset, $S$, of $V(G)$, $\overline{S}$ denotes the complement of $S$, and we denote by $e(S, \overline{S}) = |E(S, \overline{S})|$ the number of edges with one end in $S$ and the other in $\overline{S}$.

## 1.3 Spectral Graph Theory

We are interested in learning information about the structure of graphs. In order to do this, we can associate some matrix with a graph. Then our hope is that the eigenvalues of that matrix give us information about properties of our graph. There are several natural matrices to associate with graphs. In this section, we will discuss the adjacency matrix, the degree matrix, the Laplacian matrix, and the normalized Laplacian matrix. Each of these matrices gives us information about our graph. However, much of this dissertation focuses on the normalized Laplacian matrix for reasons that we will discuss later.

### 1.3.1 Adjacency Matrix

The adjacency matrix of a graph with $n$ vertices is the $n \times n$ matrix with rows and columns indexed by vertices where the entry corresponding to vertices $u$ and $v$ has a one if $u$ and $v$ have an edge between them, and a zero if they do not. For example, if we consider the graph, $G$, in Figure 1.1, we see that the adjacency matrix is

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The adjacency matrix is a real, symmetric matrix and has $n$ eigenvalues, $\lambda_1 \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$.

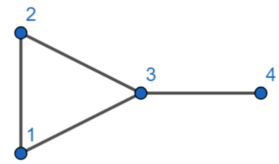



Figure 1.1: $G$

It follows from linear algebra that

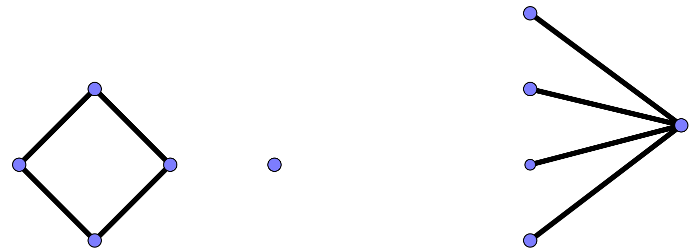
$$\lambda_1(A) = \max_{x \in \mathbb{R}^n} \frac{x^T A x}{x^T x}.$$

This is a useful identity, and can be used to show relationships between $\lambda_1(A)$ and other quantities related to the graph.

Eigenvalues of $A$ capture many graph properties, but not all. For instance, they tell us how many edges are in the graph, but they do not tell us if the graph is connected in general. Recall that if an $n \times n$ matrix, $B$, is symmetric, then the sum of its eigenvalues is equal to its trace, $\text{Tr}(B)$. This implies that

$$\sum_{i=1}^{n} \lambda_i^2(A) = \text{Tr}(A^2) = \sum_{i=1}^{n} \deg(v_i) = 2e(G),$$

so we can determine the number of edges in the graph if we know the eigenvalues of the adjacency matrix. In order to see that eigenvalues of $A$ do not give us information about whether or not a graph is connected, notice that in Figure 1.2, there are two graphs – one of the graphs is connected, and the other is not. However, the adjacency matrices of these graphs have the same eigenvalues. It would be nice to be able to recover information about the connectedness of a graph from the spectrum of a matrix. The adjacency matrix can't



(a) Disconnected graph

(b) Connected graph

Figure 1.2: Two graphs with adjacency matrix eigenvalues $2$, $-2$, and $0^3$

give us this information, but some of the matrices we introduce later can tell us whether or not our graph is connected.

### 1.3.2 Degree Matrix

The degree matrix of a graph is the matrix with the degrees of vertices along the diagonal and zeros everywhere else. For example, the degree matrix of the graph in Figure 1.1 is

$$D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The degree matrix is not very interesting on its own, but it is useful for defining the Laplacian matrix.

### 1.3.3 Laplacian Matrix

The Laplacian matrix (which is sometimes called the *combinatorial Laplacian matrix*) is defined to be $L = D - A$. $L$ has $n$ eigenvalues, $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. It turns out that we can use the eigenvalues of the Laplacian matrix to say something interesting about certain substructures of our graph. A *tree* is a connected graph without cycles. A *spanning tree* in a graph is a tree that includes every vertex of the graph. We will later be interested in looking for certain kinds of spanning trees in graphs, and the following theorem shows an important connection between spanning trees and the Laplacian matrix.

**Theorem 1.3.1** (Kirchhoff's Matrix Tree Theorem)**.** *The number of spanning trees of a graph is the determinant of any principal submatrix of* $L$*.*

(A principal submatrix of $L$ is a square submatrix obtained by deleting a row and column associated with a vertex, $v$.)

The following corollary of the Matrix Tree Theorem tells us that the number of spanning trees of a graph is related to the eigenvalues of the Laplacian matrix.

**Corollary 1.3.2.** *If the eigenvalues of* $L$ *are* $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$*, then the number of spanning trees of* $G$ *is*

$$\frac{1}{n}\left(\lambda_2 \cdot \lambda_3 \cdots \lambda_n\right).$$

This result is very interesting because it shows that eigenvalues can be helpful in counting spanning trees in graphs. In Chapter 2 we will be looking for certain kinds of spanning trees in edge-colored graphs, and Corollary 1.3.2 might prompt the reader to think that spectral methods could be helpful when looking for trees in graphs.

An alternate definition of the Laplacian matrix comes from a matrix called the *oriented edge-incidence matrix*. If $n = |V(G)|$ and $m = |E(G)|$, then the oriented edge-incidence matrix, $B$, is an $m \times n$ matrix. Choose an arbitrary ordering of the edges of the graph and assign an orientation to the edges. (The matrix $L$ is independent of the orientation.) If $v_i v_j$

is the $k$th edge in the ordering, then the $k$th row of $B$ has a $1$ in the $i$th column and entry $-1$ in the $j$th column and zeros in all other entries. (If the oriented edge originates at $v_j$ and terminates at $v_i$, then there will be a $-1$ in the $i$th column and $1$ in the $j$th column.) Then $L = B^T B$. Using this definition, we can learn more information about the eigenvalues of $L$. Note that for a vector, $\varphi$, we have

$$\begin{aligned}\varphi^T L\varphi &= \varphi^T B^T B\varphi \\ &= (B\varphi)^T (B\varphi) \\ &= \sum_{v_i \sim v_j} \left(\varphi(v_i) - \varphi(v_j)\right)^2. \qquad (1.1)\end{aligned}$$

From this, we see that $L$ is a positive semi-definite matrix. In particular, if $\varphi$ is an eigenvector, then $\sum_{v_i \sim v_j} \left(\varphi(v_i) - \varphi(v_j)\right)^2 = \lambda \cdot ||\varphi||_2^2$ for an eigenvalue, $\lambda$. This implies that all eigenvalues of $L$ are non-negative.

Note that $0$ is always an eigenvalue of $L$ since $\varphi \equiv 1$ implies that $\varphi^T L\varphi = 0$. So $\mathbb{1}$ is an eigenvector corresponding to the eigenvalue $0$. Also, by looking at (1.1), we see that if $\varphi$ is constant on components of the graph, then $\varphi^T L\varphi = 0$. Conversely, (1.1) is zero if and only of $\varphi(v_i) = \varphi(v_j)$ for all edges $v_i v_j$. Hence, eigenvectors corresponding to $0$ are constant on components.

### 1.3.4 Normalized Laplacian Matrix

The normalized Laplacian matrix, $\mathcal{L}$, which was popularized by Chung (see [23]), is the Laplacian matrix normalized by the degrees. That is,

$$\mathcal{L} = D^{-1/2}(D - A)D^{-1/2}.$$

Since $D$ is a diagonal matrix, the matrix $D^{1/2}$ is the matrix obtained from $D$ by raising each entry to the $1/2$ power. When talking about the normalized Laplacian matrix, we always assume that there are no isolated vertices.

The matrix $\mathcal{L}$ is a real, $n \times n$ symmetric matrix, so it has $n$ real eigenvalues, $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ with orthonormal eigenvectors $\varphi_0, \cdots, \varphi_{n-1}$. The eigenvalues of the normalized Laplacian matrix tell us a lot about the structure of our graph. It is always that case that $\lambda_0 = 0$. In fact, just as for $L$, the number of eigenvalues that are zero is the number of connected components of the graph. So $\lambda_1 > 0$ if and only if the graph is connected. Also, we know that $\lambda_1 \leq \frac{n}{n-1}$ and $\lambda_{n-1} \geq \frac{n}{n-1}$. Additionally, $\lambda_1 \leq 1$ except if $G$ is complete. The largest eigenvalue of the normalized Laplacian also gives us information about the graph. We always have that $\lambda_{n-1} \leq 2$, and $\lambda_{n-1} = 2$ if and only if the graph has a bipartite component.

For the remainder of this dissertation, $\lambda_i = \lambda_i(\mathcal{L})$.

Let $\varphi$ be an arbitrary column vector from $V(G) \to \mathbb{R}$. The quotient $\dfrac{\varphi^T \mathcal{L} \varphi}{\varphi^T \varphi}$ is called the *Rayleigh quotient*. We can use the Rayleigh quotient to say something about the eigenvalues of our graph. Notice that

$$\begin{aligned}\frac{\varphi^T \mathcal{L} \varphi}{\varphi^T \varphi} &= \frac{\varphi^T D^{-1/2} B^T B D^{-1/2} \varphi}{\varphi^T \varphi} \\ &= \frac{\sum_{x \sim y} \left( \varphi(x) \deg(x)^{-1/2} - \varphi(y) \deg(y)^{-1/2} \right)^2}{\sum_{x \in V(G)} \varphi(x)^2}.\end{aligned}$$

Define $f(x) = \varphi(x)\deg(x)^{-1/2}$. Then $\varphi(x) = f(x)\deg(x)^{1/2}$. Then

$$\frac{\varphi^T \mathcal{L} \varphi}{\varphi^T \varphi} = \frac{\sum_{x \sim y}(f(x) - f(y))^2}{\sum_x f(x)^2 \deg(x)}. \tag{1.2}$$

The Courant-Fischer Theorem says that the second-smallest eigenvalue of $\mathcal{L}$ can be found by looking at a minimum over this Rayleigh quotient:

$$\begin{aligned}
\lambda_1 &= \min_{\varphi \perp \varphi_0} \frac{\varphi^T \mathcal{L} \varphi}{\varphi^T \varphi} \\
&= \min_{\varphi : \sum_v \varphi(v)\sqrt{\deg(v)} = 0} \frac{\varphi^T \mathcal{L} \varphi}{\varphi^T \varphi} \\
&= \min_{f : \sum_v f(v)\deg(v) = 0} \frac{\sum_{x \sim y}(f(x) - f(y))^2}{\sum_x f(x)^2 \deg(x)} \text{ by (1.2).}
\end{aligned}$$

So

$$\lambda_1 = \min_{f \perp D\mathbb{1}} \frac{\sum_{x \sim y}(f(x) - f(y))^2}{\sum_x f(x)^2 \deg(x)}. \tag{1.3}$$

(Such an $f$ that achieves this minimum is called a *harmonic eigenfunction*.)

The normalized Laplacian matrix is related to random walks on graphs. The transition probability matrix for a simple random walk, $D^{-1}A$, is similar to the matrix $I - \mathcal{L}$ since $D^{1/2}(D^{-1}A)D^{-1/2} = D^{-1/2}AD^{-1/2} = I - \mathcal{L}$. The matrices $D^{-1/2}AD^{-1/2}$ and $D^{-1}A$ hence share the same eigenvalues, and we can use what we know about the eigenvalues of $\mathcal{L}$ to answer questions about the convergence of random walks on graphs. This is related to the study of heat dispersion on graphs, which we will touch on in Chapter 3.

### 1.3.5 Normalized Laplace Operator

In Chapter 3, the principal matrix that we will consider is the normalized Laplace operator

$$\Delta = I - D^{-1}A,$$

where $D$ is the diagonal matrix of vertex degrees and $D^{-1}A$ is the transition probability matrix for simple random walk.

As a quick observation, note that $\Delta$ is non-positive semidefinite. This is contrary to usual sign conventions in graph theory, but is the proper sign convention for the Laplace-Beltrami operator in Riemannian manifolds, the analogy to which we emphasize here. Also note that this matrix is (up to sign) the unsymmetrized version of the normalized Laplacian, $\mathcal{L} = (D^{-1/2}AD^{-1/2}) - I$.

### 1.3.6 Expander Mixing Lemma

Since we are interested in understanding the structure of a graph, it is often helpful to have a bound on the number of edges between two subsets of vertices. It turns out that we can do this using the *spectral gap* of the normalized Laplacian matrix, $\sigma(G) = \max\{|\lambda_1 - 1|, |\lambda_{n-1} - 1|\}$.

**Theorem 1.3.3** (Expander Mixing Lemma)**.** *Suppose $G$ is a graph and $\sigma(G) = \max\{|\lambda_1 - 1|, |\lambda_{n-1} - 1|\}$. For any subsets $X, Y \subseteq V(G)$, let $e(X,Y)$ be the number of edges with one end in $X$ and one in $Y$. Then*

$$\left| e(X,Y) - \frac{\mathrm{Vol}(X) \cdot \mathrm{Vol}(Y)}{\mathrm{Vol}(G)} \right| \leq \sigma \cdot \sqrt{\mathrm{Vol}(X) \cdot \mathrm{Vol}(Y)}.$$

Note that $\dfrac{\mathrm{Vol}(X)\cdot\mathrm{Vol}(Y)}{\mathrm{Vol}(G)}$ is the expected number of edges between $X$ and $Y$ in a random graph with the same degree sequence as $G$; this makes precise the comment from the introduction that spectral information certifies 'pseudo-randomness' of edge-distributions of graphs.

As the proof of Theorem 1.3.3 reflects a common way of using spectral graph theory to understand graph structure, we record the proof here.

*Proof of Expander Mixing Lemma.* Let $0=\lambda_0\le\lambda_1\le\cdots\le\lambda_{n-1}$ be eigenvalues of $\mathcal{L}$, and let $\frac{\mathbf{d}^{1/2}}{\sqrt{\mathrm{Vol}(G)}}=\varphi_0,\varphi_1,\ldots,\varphi_{n-1}$ be the orthonormal eigenvalues of $\mathcal{L}$. Then

$$\begin{aligned}
e(X,Y) &= \mathbb{1}_X A \mathbb{1}_Y\\
&= \mathbb{1}_X^T D^{1/2} D^{-1/2} A D^{-1/2} D^{1/2}\mathbb{1}_Y\\
&= \mathbb{1}_X^T D^{1/2}(I-\mathcal{L})D^{1/2}\mathbb{1}_Y\\
&= \mathbb{1}_X^T D^{1/2}\left(\sum_{i=0}^{n-1}(1-\lambda_i)\varphi_i\varphi_i^T\right)D^{1/2}\mathbb{1}_Y\\
&= \mathbb{1}_X^T D^{1/2}\left(1\cdot\frac{\mathbf{d}^{1/2}}{\sqrt{\mathrm{Vol}(G)}}\cdot\frac{(\mathbf{d}^{1/2})^T}{\sqrt{\mathrm{Vol}(G)}}+\sum_{i=1}^{n-1}(1-\lambda_i)\varphi_i\varphi_i^T\right)D^{1/2}\mathbb{1}_Y\\
&= \frac{\mathrm{Vol}(X)\,\mathrm{Vol}(Y)}{\mathrm{Vol}(G)}+\mathbb{1}_X^T D^{1/2}\left(\sum_{i=1}^{n-1}(1-\lambda_i)\varphi_i\varphi_i^T\right)D^{1/2}\mathbb{1}_Y.
\end{aligned}$$

Let $\langle u,v\rangle$ denote the dot product of the vectors $u$ and $v$. Then

$$\begin{aligned}
\left|e(X,Y)-\frac{\mathrm{Vol}(X)\,\mathrm{Vol}(Y)}{\mathrm{Vol}(G)}\right| &= \left|\mathbb{1}_X^T D^{1/2}\left(\sum_{i=1}^{n-1}(1-\lambda_i)\varphi_i\varphi_i^T\right)D^{1/2}\mathbb{1}_Y\right|\\
&= \left|\sum_{i=1}^{n-1}(1-\lambda_i)\left\langle D^{1/2}\mathbb{1}_X,\varphi_i\right\rangle\left\langle D^{1/2}\mathbb{1}_Y,\varphi_i\right\rangle\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sigma \sum_{i=1}^{n-1} \left|\left\langle D^{1/2}\mathbb{1}_X, \varphi_i \right\rangle\right| \cdot \left|\left\langle D^{1/2}\mathbb{1}_Y, \varphi_i \right\rangle\right| \\
&\leq \sigma \sqrt{\left(\mathrm{Vol}(X) - \frac{(\mathrm{Vol}(X))^2}{\mathrm{Vol}(G)}\right)\left(\mathrm{Vol}(Y) - \frac{(\mathrm{Vol}(Y))^2}{\mathrm{Vol}(G)}\right)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by Cauchy-Schwartz} \\
&\leq \sigma \sqrt{\frac{\mathrm{Vol}(X)\,\mathrm{Vol}(\overline{X})\,\mathrm{Vol}(Y)\,\mathrm{Vol}(\overline{Y})}{(\mathrm{Vol}(G))^2}} \\
&\leq \sigma \sqrt{\mathrm{Vol}(X)\,\mathrm{Vol}(Y)}.
\end{aligned}
$$

□

### 1.3.7 Cheeger's Inequality

One of the most important bits of geometric information certified by the spectrum is expansion: the number of edges leaving subsets. For instance, when looking for disjoint spanning structures in graphs, a sparse cut limits the number we can hope to find. It turns out that sparse cuts in our graph are related to an eigenvalue of the normalized Laplacian matrix through Cheeger's inequality. Before discussing Cheeger's inequality, we introduce some necessary notation.

For a subset $S$, of $V(G)$, we define $h_G(S) = \dfrac{|E(S,\overline{S})|}{\min\{\mathrm{Vol}(S), \mathrm{Vol}(\overline{S})\}}$. The *Cheeger constant* (or *isoperimetric constant*), $h_G$ is then defined by

$$h_G = \min_S h_G(S).$$

Determining $h_G$ is computationally difficult, but is related to the smallest eigenvalue through the following result known as *Cheeger's inequality*.

**Theorem 1.3.4** (Cheeger's inequality [23]). *If $G$ is a connected graph and $\lambda_1$ is the second-smallest eigenvalue of the normalized Laplacian of $G$, then*

$$\frac{h_G^2}{2} < \lambda_1 \leq 2h_G.$$

What Theorem 1.3.4 tells us is that we have a sparse cut in $G$ if and only if $\lambda_1$ is small. Hence $\lambda_1$ is a measure of sparse cuts in the graph. This is very helpful information about the structure that we can use to our advantage.

The lower bound of this inequality is analogous to Cheeger's inequality in Riemannian geometry [15]. The first graph theoretical result was by Dodziuk and Karp [30], for infinite graphs. Later versions were proved for regular graphs [3]. It is interesting to note that not just are the statements of the Cheeger inequality similar in Riemannian geometry and graph theory, but even the proofs are similar.

Notice that as a consequence of Cheeger's inequality, we have the following inequality for a subset $S \subseteq V(G)$ with $\mathrm{Vol}(S) \leq \frac{1}{2}\mathrm{Vol}(G)$:

$$e(S, \overline{S}) \geq \frac{\lambda_1}{2}\mathrm{Vol}(S). \tag{1.4}$$

The upper bound on $\lambda_1$ from Cheeger's inequality is not too difficult to prove. (This upper bound is analagous to Buser's inequality in Riemannian geometry.) For the proof, we use (1.3).

*Proof that* $\lambda_1 \le 2h_G$. Let $S \subseteq V(G)$ such that $h_G = \frac{e(S,\overline{S})}{\mathrm{Vol}(S)}$. Let

$$f(x) = \begin{cases} \frac{1}{\mathrm{Vol}(S)} & \text{if } x \in S \\ -\frac{1}{\mathrm{Vol}(\overline{S})} & \text{if } x \in \overline{S}. \end{cases}$$

Since

$$\begin{aligned} \sum_{v \in V(G)} f(v)\deg(v) &= \sum_{v \in S} f(v)\deg(v) + \sum_{v \in \overline{S}} f(v)\deg(v) \\ &= \frac{1}{\mathrm{Vol}(S)} \sum_{v \in S} \deg(v) - \frac{1}{\mathrm{Vol}(\overline{S})} \sum_{v \in \overline{S}} \deg(v) \\ &= 1 - 1 \\ &= 0, \end{aligned}$$

we know that $f \perp D\mathbb{1}$. So by (1.3), we have

$$\lambda_1 \le \frac{\sum_{u \sim v}(f(u) - f(v))^2}{\sum_v (f(v))^2 \cdot \deg(v)}.$$

If $u$ and $v$ are both in $S$ or if $u$ and $v$ are both in $\overline{S}$, then $(f(u) - f(v))^2 = 0$.

If $u \in S$ and $v \in \overline{S}$ (or vice-versa), then

$$(f(u) - f(v))^2 = \left( \frac{1}{\mathrm{Vol}(S)} + \frac{1}{\mathrm{Vol}(\overline{S})} \right)^2.$$

Also,

$$
\begin{aligned}
\sum_{v}(f(v))^2\cdot\deg(v) &= \sum_{v\in S}(f(v))^2\cdot\deg(v)+\sum_{v\in\overline{S}}(f(v))^2\cdot\deg(v)\\
&= \frac{1}{(\mathrm{Vol}(S))^2}\cdot\sum_{v\in S}\deg(v)+\frac{1}{(\mathrm{Vol}(\overline{S}))^2}\cdot\sum_{v\in\overline{S})^2}\deg(v)\\
&= \frac{1}{(\mathrm{Vol}(S))^2}\cdot\mathrm{Vol}(S)+\frac{1}{(\mathrm{Vol}(\overline{S}))^2}\cdot\mathrm{Vol}(\overline{S})\\
&= \frac{1}{\mathrm{Vol}(S)}+\frac{1}{\mathrm{Vol}(\overline{S})}.
\end{aligned}
$$

So

$$
\begin{aligned}
\lambda_1 &\le \frac{\sum_{u\sim v}(f(u)-f(v))^2}{\sum_{v}(f(v))^2\cdot\deg(v)}\\
&= \frac{e(S,\overline{S})\left(\frac{1}{\mathrm{Vol}(S)}+\frac{1}{\mathrm{Vol}(\overline{S})}\right)^2}{\frac{1}{\mathrm{Vol}(S)}+\frac{1}{\mathrm{Vol}(\overline{S})}}\\
&= e(S,\overline{S})\left(\frac{1}{\mathrm{Vol}(S)}+\frac{1}{\mathrm{Vol}(\overline{S})}\right)\\
&\le \frac{2e(S,\overline{S})}{\min\{\mathrm{Vol}(S),\mathrm{Vol}(\overline{S})\}}\\
&= 2h_G.
\end{aligned}
$$

□

CHAPTER 2: RAINBOW SPANNING TREES

## 2.1 Introduction

Spectral graph theory and graph curvature are very helpful tools that can help us generalize results about complete graphs. Often we would like to be able to answer questions about general graphs that are not necessarily complete. When we are working with the complete graph, we have much more information with which to work. However, geometric information can be used to better understand the structure of graphs and extend results to non-complete graphs. In this chapter we are interested in using spectral methods in order to generalize results about the number of rainbow spanning trees that can be found in edge-colored complete graphs. For an edge-colored graph $G$, a *rainbow spanning tree* of $G$ is a spanning tree in which each edge is a different color.

Our motivation is the following conjecture of Brualdi and Hollingsworth.

**Conjecture 2.1** ([11])**.** *If $K_n$ (for $n \geq 6$ and $n$ even) is edge-colored such that each color class is a perfect matching, then there is a decomposition of the edges into $n/2$ edge-disjoint rainbow spanning trees*

Progress was slow: Brualdi and Hollingsworth proved that in any such edge-colored $K_n$, there are at least two edge-disjoint rainbow spanning trees. Krussel, Marshall, and Verrall [50] showed that there are at least three edge-disjoint rainbow spanning trees. Horn [41] showed that under these hypotheses, a postitve fraction of the graph can be covered by edge-disjoint rainbow spanning trees. Fu, Lo, Perry, and Rodger [35] gave a constructive proof that in a properly edge-colored $K_n$ (where $n$ is even) there is a decomposition of the edges into at least $\lfloor\sqrt{3n+9}/3\rfloor$ edge-disjoint rainbow spanning trees. While asymptotically weaker, this result holds for all values of $n$. Pokrovskiy and Sudakov [57] later showed that in a properly edge-colored complete graph there are at least $\frac{n}{9}-6$ edge-disjoint rainbow spanning trees, and very recently Glock, Kühn, Montgomery, and Osthus [37] settled the conjecture of Brualdi and Hollingsworth for sufficiently large $n$. In fact, they showed that under the hypotheses, there is a decomposition of the edges into *isomorphic* edge-disjoint rainbow spanning trees, settling a conjecture of Constantine ([24], [25]).

Kaneko, Kano, and Suzuki strengthened the conjecture of Brualdi and Hollingsworth with the following.

**Conjecture 2.2** ([47]). *If $G$ is a properly edge-colored $K_n$ where $n \geq 6$ and $n$ is even, then $G$ contains $\lfloor n/2\rfloor$ edge-disjoint rainbow spanning trees.*

Also related, Akbari and Alipour showed in [1] that if $G$ is an edge-colored $K_n$ ($n \geq 5$) in which each color appears at most $n/2$ times, then $G$ contains at least two edge-disjoint rainbow spanning trees. Carraher, Hartke, and Horn showed in [14] that for $n$ and $C$ sufficiently large, if $G$ is an edge-colored copy of $K_n$ in which each color appears less than $n/2$ times, then $G$ contains at least $\lfloor n/(C\log n)\rfloor$ edge-disjoint rainbow spanning trees. Independently, Pokrovskiy and Sudakov [57] and Balogh, Liu, and Montgomery [5] showed that under the conditions of the conjecture of Kaneko, Kano, and Suzuki, a positive fraction of the graph can be covered by edge-disjoint rainbow spanning trees. In fact, under these conditions, Pokrovskiy and Sudakov showed that this can be done with isomorphic rainbow-spanning trees. Recently, Montgomery, Pokrovskiy, and Sudakov [54] showed that a properly edge-colored $K_n$ contains $(1-o(1))\frac{n}{2}$ edge-disjoint rainbow spanning trees.

There are a number of results about rainbow structures other than spanning trees in edge-colored graphs. Kano and Li did a survey of many results and conjectures about such structures in [48]. Brualdi and Hollingsworth [12] looked at edge-colored complete bipartite graphs and proved results about when such graphs contain rainbow forests or trees. Constantine [25] showed that for $p$ prime ($p > 2$), there is some proper edge-coloring of the complete graph $K_p$ such that there is a partition of the edges of $K_p$ into rainbow hamiltonian cycles. He also showed that for certain values of $n$, there is a proper edge-coloring of $K_n$ such that there is a partition of the edges of $K_n$ into isomorphic rainbow spanning trees. Rainbow cycles in graphs have also been studied. Albert, Frieze, and Reed [2] showed that for $n$ sufficiently large, if $K_n$ is edge-colored such that each color appears less that $n/64$ times, then there is a rainbow hamiltonian cycle. (Rue gave a correction of this constant – see [34].) Frieze and Krivelevich [34] proved that there is a constant $c > 0$ such that an edge-coloring of $K_n$ in which each color appears at most $\max\{cn, 1\}$ times contains a rainbow cycle of length $k$ for each $3 \leq k \leq n$.

Here we show that for *any* edge-colored graph $G$ on sufficiently many vertices and with large enough minimum degree, we can give a lower bound on the number of edge-disjoint rainbow spanning trees in $G$. This lower bound will depend on the second-smallest eigenvalue of the normalized Laplacian matrix.

**Theorem 2.1.1.** *If $G$ is an edge-colored graph with minimum degree $\delta \geq C \log n$ (for $C$ and $n$ sufficiently large) in which each color class has size at most $\delta\lambda_1/2$, then $G$ contains at least $\left\lfloor \frac{\delta\lambda_1}{C \log n} \right\rfloor$ edge-disjoint rainbow spanning trees.*

**Remark:** We have not attempted to optimize the constant, $C$, but the result holds for $C \geq 1500$ and $n \geq 14398$. (The limiting factor comes from an inequality in Lemma 2.3.1.)

We note that $\frac{\delta}{2}$ is essentially the best possible bound on the size of the color classes that we could hope for, and there are graphs for which $\lambda_1 = 1$, so $\frac{\delta\lambda_1}{2} = \frac{\delta}{2}$. Let $C_1, ..., C_s$ be the color classes and let $c_i = |C_i|$. If $c_i \leq \frac{\delta}{2}$ for each $i$, then notice that on the one hand,

$$\begin{aligned} |E(G)| &= \sum_{i\in[s]} c_i \\ &\leq s \cdot \frac{\delta}{2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} |E(G)| &= \frac{1}{2} \sum_{v\in V(G)} \deg(v) \\ &\geq \frac{1}{2} \sum_{v\in V(G)} \delta \\ &= \frac{1}{2} n\delta. \end{aligned}$$

So $\frac{1}{2}n\delta \leq s\frac{\delta}{2}$, which implies that $n \leq s$. If $s < n-1$, then no rainbow spanning tree would be possible. So this bound on the size of the color classes is a natural condition to consider. If we let colors show up more than $\frac{\delta}{2}$ times, then we might run into trouble and not be able to find even one rainbow spanning tree. For example, if the host graph is a cycle, then if colors are allowed to have multiplicity greater that $1$ we may not be able to find a rainbow spanning tree.

We emphasize that our theorem works for both regular and irregular graphs; use of spectral methods frequently restrict results to apply only for irregular graphs. We also emphasize that the colorings considered in our theorem need not be proper – there is only a restriction on the multiplicity of a color. Another advantage of our approach is that it uses only the smallest eigenvalue. Results in extremal combinatorics using spectral graph theory frequently assume strong control on both the smallest (non-trivial) and largest eigenvalue of $\mathcal{L}$ as such gives stronger pseudo-random properties of the edge set of a graph via the expander mixing lemma. We also note that our result does not actually require $\lambda_1$ to be close to $1$ (another common requirement), although our result is certainly strongest if $\lambda_1$ is close to $1$. Some additional comments regarding the hypothesis of our results are given in Section 2.5.

The chapter is organized as follows: Sections 2.2 and 2.3 introduce definitions and preliminary results. The proof of Theorem 2.1.1 is in Section 2.4, and we conclude in Section 2.5 with some discussion, along with some applications of Theorem 2.1.1 to particular classes of graphs where it yields particularly strong results.

## 2.2 Definitions and Background

The general outline of our proof is as follows: We partition our original graph $G$ into graphs $G_1, G_2, \ldots, G_q$ for an appropriately chosen integer $q$, then show that each $G_i$ contains a rainbow spanning tree. In order to avoid bias in our partition, it turns out to useful to construct the partition *randomly*. That is, for each edge we uniformly select a $G_i$ to place it into; hence we place an edge into $G_i$ uniformly with probability $1/q$. Each edge is placed independently.

In order to analyze this, we require a criterion to verify that the $G_i$ contain rainbow spanning trees (with high probability) along with a method to get structural information out of our (general) graph $G$. The remainder of this section discusses these basic tools.

In order to show that a graph has a rainbow spanning tree, we use the following proposition, originally due to Schrijver [58].

**Proposition 2.2.1.** *A graph $G$ has a rainbow spanning tree if and only if for every partition $\mathcal{P}$ of $V(G)$ into $t$ parts, there are at least $t-1$ different colors represented between the parts of $\mathcal{P}$.*

Broersma and Li [10] showed that the Matroid Intersection Theorem [31] can be used to determine the largest rainbow spanning forest in a graph. (See [58].) Schrijver [58] showed that the conditions of the Matroid Intersection Theorem are equivalent to the necessary and sufficient conditions from Proposition 2.2.1 for the existence of a rainbow spanning tree. Suzuki [59] and Carraher and Hartke [13] provided additional graph theoretical proofs of this result.

Our strategy is to take our random partition of the edges of $G$ and prove some structural results that hold with high probability (Lemma 2.3.1 below). Then we show deterministically that each graph satisfies Proposition 2.2.1. The strategy is similar to that of [14], with additional technical difficulties given from the fact that our underlying graph is not complete and, instead, we only have spectral information to understand the geometry of the host graph $G$. In some sense our primary new difficulty is to extract sufficient geometric information from the spectrum to push the analysis through.

We will frequently use the fact that if $A_1, ..., A_\ell$ are events, then

$$\mathbb{P}\left[\bigcup_{i=1}^{\ell} A_\ell\right] \leq \sum_{i=1}^{\ell} \mathbb{P}[A_i],$$

where $\mathbb{P}(X)$ for an event $X$ denotes the probability that $X$ occurs. We also use the following Chernoff bounds.

**Lemma 2.2.2** ([16]). *If $\lambda > 0$ and $X_i$ are independent random variables with*

$$\mathbb{P}(X_i = 1) = p_i,\ \mathbb{P}(X_i = 0) = 1 - p_i$$

*and $X = \sum_i X_i$, ($\mathbb{E}[X] = \sum_i p_i$) then*

$$\mathbb{P}[X \leq \mathbb{E}[X] - \lambda] \leq \exp\left(-\frac{\lambda^2}{2\mathbb{E}[X]}\right)$$

*and*

$$\mathbb{P}[X \geq \mathbb{E}[X] + \lambda] \leq \exp\left(-\frac{\lambda^2}{2(\mathbb{E}[X] + \lambda/3)}\right).$$

## 2.3 Preliminary Results

### Standard Assumptions

Throughout the remainder of the chapter, we use several conventions to simplify the discussion and make the statements of lemmas more readable. The following *standard assumptions* for parameters underlie the lemmas and theorems for the remainder of the chapter:

- There are $q = \lfloor \frac{\delta\lambda_1}{C \log n} \rfloor$ graphs, $G_1, \ldots, G_q$. Edges are placed into exactly one

  $G_j$, each with probability $p$.

- There are color classes $C_1, \ldots, C_s$ and $|C_i| = c_i$,

- $e_j(S, \bar{S})$ is the number of edges in $G_j$ with one edge in $S$ and the other in $\bar{S}$.

- Parameters have the following values:

  $$\delta \geq C \log n$$
  $$p = \frac{C \log n}{\delta\lambda_1}$$
  $$\epsilon = 0.1$$
  $$\text{for each } i \in [s] : \ 1 \leq c_i \leq \frac{\delta\lambda_1}{2}$$

  $C$ is a sufficiently large constant. (This is an absolute constant C. For instance, C=1500 works.)

**Remark:** Note that $qp \leq 1$, and strict inequality is possible as a result of the floor. This technically means that there is a (small) chance that edges won't be in any $G_j$, but it is more convenient than keeping the floor throughout and does not materially change the result. Also, we have chosen $\epsilon = 0.1$, but any sufficiently small value of $\epsilon$ will suffice.

In order to show that each $G_j$ has a rainbow spanning tree, we use Proposition 2.2.1.

We proceed by proving some preliminary results. We begin by establishing some properties of each of the graphs $G_j$ constructed above.

**Lemma 2.3.1.** *Under the standard assumptions, for every $j \in [q]$, the edge sets, $E_j$ of $G_j$ satisfy*

*(i) For every $i \in [s]$, $|E_j \cap C_i| \leq (1+\epsilon)\frac{C \log n}{2}$.*

*(ii) For every set $S \subseteq V(G)$,*

$$e_j(S, \bar{S}) \geq (1-\epsilon)\mathbb{E}[e_j(S, \bar{S})].$$

*(iii) For every vertex $v \in V(G)$,*

$$\deg_{G_j}(v) \geq \frac{(1-\epsilon)C}{\lambda_1} \log n.$$

*simultaneously with probability at least $1 - n^{-2}$, assuming $n$ is sufficiently large.*

*Proof.* Fix a color $i \in [s]$.

To prove (i), note that $\mathbb{E}[|E_j \cap C_i|] = pc_i \leq \frac{C}{2} \log n$. Using Lemma 2.2.2 with $\lambda = \epsilon \frac{C \log n}{2}$ implies that

$$\begin{aligned} \mathbb{P}\left(|E_j \cap C_i| \geq (1+\epsilon)\frac{C \log n}{2}\right) &\leq \exp\left(-\frac{\epsilon^2 C \log n}{2(1+\epsilon/6)}\right) \\ &\leq \exp\left(-\frac{\epsilon^2 C \log n}{3}\right) \\ &\leq n^{-5} \text{ for } C \geq \frac{15}{\epsilon^2}. \end{aligned}$$

Part $(iii)$ is merely a (useful) special case of $(ii)$, so it suffices to prove $(ii)$. For part $(ii)$, notice that for any set $S$, either $\text{Vol}(S) \leq \frac{1}{2}\text{Vol}(G)$ or $\text{Vol}(\overline{S}) \leq \frac{1}{2}\text{Vol}(G)$, and hence it suffices to prove it for sets with $\mathrm{Vol}(S) \leq \frac{1}{2}\mathrm{Vol}(G)$. Fix a set $S$ of size $k$ and with volume at most $\frac{1}{2}\mathrm{Vol}(G)$. Then by (1.4), $e(S,\bar{S}) \geq \frac{\lambda_1}{2}\mathrm{Vol}(S) \geq \frac{\lambda_1 \delta k}{2}$. Hence $\mathbb{E}[e_j(S,\bar{S})] \geq \frac{Ck\log n}{2}$. Applying the Chernoff bounds with $\lambda = \epsilon\mathbb{E}[e_j(S,\bar{S})]$ yields

$$
\begin{aligned}
\mathbb{P}\Big(e_j(S,\bar{S}) \leq (1-\epsilon)\mathbb{E}[e_j(S,\bar{S})]\Big) &\leq \exp\left(-\frac{\epsilon^2}{2}\mathbb{E}[e_j(S,\bar{S})]\right) \\
&\leq \exp\left(-\frac{C\epsilon^2}{4}k\log n\right). \qquad (2.1)
\end{aligned}
$$

Let $\mathcal{B}$ denote the event that there exists a set $S$ which doesn't satisfy the conclusion of part (ii). A union bound over $k$ and $S$ of size $k$ yields

$$
\begin{aligned}
\mathbb{P}(\mathcal{B}) &\leq \sum_{k=1}^{n} \sum_{\substack{S\subseteq V(G) \\ S:|S|=k \\ \mathrm{Vol}(S)\leq \mathrm{Vol}(G)/2}} \mathbb{P}\Big(e_j(S,\bar{S}) \leq (1-\epsilon)\mathbb{E}[e_j(S,\bar{S})]\Big) \\
&\leq \sum_{k=1}^{n} \binom{n}{k} \exp\left(-\frac{C\epsilon^2}{4}k\log n\right) \\
&\leq \sum_{k=1}^{n} \exp\left(-\left(\frac{C\epsilon^2}{4}-1\right)k\log n\right) \\
&\leq n^{-4}.
\end{aligned}
$$

Here the second inequality follows from (2.1) and the fact that there are at most $\binom{n}{k}$ sets of size $k$ satisfying $\mathrm{Vol}(S) \leq \mathrm{Vol}(G)/2$, the third from the simple bound that $\binom{n}{k} \leq n^k$, and the last inequality holds assuming that $C$ is sufficiently large.

A union bound over all $j \in [q]$ and all color classes $i \in [s]$ yields the result. □

Lemma 2.3.1 provides lower bounds on the number of edges leaving a set, and upper bounds on the number of edges in a particular color in each of our graphs $G_j$. In order to apply Proposition 2.2.1 to then prove that the graphs contain rainbow spanning trees, we thus must study the number of edges between parts. This requires some care.

Suppose $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$ is partition of $V(G)$ into $t$ parts. We use the notation

$$e_j(\mathcal{P}) = \frac{1}{2} \sum_{i \in [t]} e_j(P_i, \bar{P}_i)$$

to denote the total number of edges between parts in the graph $G_j$. (We denote the number of edges between parts in $G$ by $e(\mathcal{P})$.)

The following Lemma is then immediate.

**Lemma 2.3.2.** *Let $\mathcal{P}$ be a partition of $V(G)$ into $t$ parts. If each color appears at most $k$ times in $G_j$ and $e_j(\mathcal{P}) \geq (t-2)k + 1$, then there are at least $t-1$ colors between parts of $\mathcal{P}$ in $G_j$.*

We often use this in the following way.

**Corollary 2.3.3.** *Let $\mathcal{P}$ be a partition of $V(G)$ into $t$ parts. If*

$$e_j(\mathcal{P}) \geq (t-2)(1+\epsilon)\frac{C \log n}{2} + 1,$$

*then there are at least $t-1$ colors between parts of $\mathcal{P}$ in $G_j$.*

Using the Cheeger inequality to lower bound the number of edges leaving a set proves insufficient for our goals, at least for small sets. For a set $S \subseteq V(G)$, let

$$f(S) = \max\left\{\frac{\lambda_1}{2}\operatorname{Vol}(S), \operatorname{Vol}(S) - 2\binom{|S|}{2}\right\}. \tag{2.2}$$

Both quantities serve as a lower bound for $e(S, \bar{S})$ and hence,

$$\mathbb{E}[e_j(S, \overline{S})] \geq pf(S).$$

In order to be able to apply Lemma 2.3.2 to verify the hypothesis of Proposition 2.2.1, we need to be somewhat careful when minimizing the number of edges crossing a partition. We accomplish this as follows:

**Lemma 2.3.4.** *Assume the standard assumptions hold. Suppose $\mathcal{P} = \{P_1, ..., P_t, P_\star\}$ is a partition of $[n]$, satisfying $|P_1| \leq |P_2| \leq \cdots \leq |P_t|$ and satisfying either that*

*(a)* $\mathrm{Vol}(P_\star) > \frac{1}{2}\mathrm{Vol}(G)$*, or*

*(b)* $P_\star = \emptyset$ *and* $\mathrm{Vol}(P_i) \leq \frac{1}{2}\mathrm{Vol}(G)$ *for* $1 \leq i \leq t$*.*

*Let $M = 1 + \delta - \frac{\lambda_1 \delta}{2}$, and $t' \leq t$ denote the largest index such that $|P_{t'}| \leq M$. Finally, set $N' = \sum_{i=1}^{t'} |P_i|$. Then there exist unique integers $x := x(|P_1|, \cdots, |P_t|)$ and $1 < x^\star \leq M$ satisfying $N' = x + M(t' - x - 1) + x^\star$. For this integer $x$:*

$$2e(\mathcal{P}) \geq \left(\frac{\lambda_1}{2}\mathrm{Vol}\left(\bigcup_{i=1}^{t} P_i\right) + \delta x\left(1 - \frac{\lambda_1}{2}\right)\right) + e(P_\star, \bar{P}_\star) \tag{2.3}$$

$$\geq \frac{\lambda_1 \delta(n - |P_\star| - x)}{2} + \delta x + e(P_\star, \bar{P}_\star) \tag{2.4}$$

*Furthermore, $x \geq t - \lfloor \frac{n-t}{M-1} \rfloor - 1$.*

*Proof.* Let $\mathcal{P}_1 = \{P_i \in \mathcal{P} : |P_i| \leq M\}$; then $t' = |\mathcal{P}_1|$. Let $\mathcal{P}_1'$ be a partition of $\bigcup_{P_i \in \mathcal{P}_1} P_i$ into $|\mathcal{P}_1|$ parts, each of size $1$ or $M$, with possibly one set with size $x^\star$, where $1 < x^\star \leq M$.

Then $\sum_{P_i \in \mathcal{P}_1} \binom{|P_i|}{2} \leq \sum_{P_i' \in \mathcal{P}_1'} \binom{|P_i'|}{2}$. Let $x = |\{P_i' \in \mathcal{P}_1' : |P_i'| = 1\}|$. Note that $x$ is the number of parts of size one, so $x \cdot 1 + x^\star + (|\mathcal{P}_1| - (x+1))M = N'$.

Also, $x + x^\star + M(t - x - 1) \leq n$. From this, we know that $x(M-1) \geq Mt - n - M + x^\star = (M-1)t - (n-t) - (M - x^\star)$, so

$$\begin{aligned} x &\geq t - \frac{n-t}{M-1} - \frac{M-x^\star}{M-1} \\ &\geq t - \frac{n-t}{M-1} - 1. \end{aligned}$$

This proves the "furthermore" statement in the lemma.

Observe,

$$\begin{aligned} \sum_{P_i \in \mathcal{P}_1} e(P_i, \overline{P}_i) &\geq \sum_{P_i \in \mathcal{P}_1} \left[ \operatorname{Vol}(P_i) - 2\binom{|P_i|}{2} \right] \\ &\geq \sum_{P_i' \in \mathcal{P}_1'} \left[ \operatorname{Vol}(P_i') - 2\binom{|P_i'|}{2} \right]. \end{aligned}$$

Note that $\operatorname{Vol}(P_i) \geq \delta|P_i|$ for each $i \in [t]$, so if $\delta|P_i| - 2\binom{|P_i|}{2} \geq \frac{\lambda_1}{2}\delta|P_i|$, then $\operatorname{Vol}(P_i) - 2\binom{|P_i|}{2} \geq \frac{\lambda_1}{2}\operatorname{Vol}(P_i)$, since $\frac{\lambda_1}{2} < 1$. By our choice of $M$, all parts of size at least $M$ satisfy this inequality as $M$ was chosen so that $\delta M - 2\binom{M}{2} = \frac{\lambda_1 \delta M}{2}$. Thus,

$$\begin{aligned} \sum_{P_i \in \mathcal{P}_1} e(P_i, \overline{P}_i) &\geq \sum_{P_i' \in \mathcal{P}_i : |P_i'| = 1} \left( \operatorname{Vol}(P_i') - 2\binom{1}{2} \right) + \sum_{P_i' \in \mathcal{P}_1' : |P_i'| > 1} \frac{\lambda_1}{2} \operatorname{Vol}(P_i') \\ &= \sum_{P_i' \in \mathcal{P}_1' : |P_i'| = 1} \left( \frac{\lambda_1}{2} \operatorname{Vol}(P_i') + \left(1 - \frac{\lambda_1}{2}\right) \operatorname{Vol}(P_i') \right) \end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{P_i' \in \mathcal{P}_1' : |P_i'| > 1} \frac{\lambda_1}{2} \operatorname{Vol}(P_i') \\
&\geq \sum_{P_i' \in \mathcal{P}_1'} \frac{\lambda_1}{2} \operatorname{Vol}(P_i') + x\delta \left(1 - \frac{\lambda_1}{2}\right) \\
&= \sum_{P_i \in \mathcal{P}_1} \frac{\lambda_1}{2} \operatorname{Vol}(P_i) + x\delta \left(1 - \frac{\lambda_1}{2}\right).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
2e(\mathcal{P}) &\geq \sum_{P_i \in \mathcal{P}_1} \frac{\lambda_1}{2} \operatorname{Vol}(P_i) + x\delta \left(1 - \frac{\lambda_1}{2}\right) + \sum_{P_i \in \mathcal{P} \setminus \mathcal{P}_1} \frac{\lambda_1}{2} \operatorname{Vol}(P_i) + e(P_\star, \overline{P}_\star) \\
&= \sum_{i \in [t]} \frac{\lambda_1}{2} \operatorname{Vol}(P_i) + x\delta \left(1 - \frac{\lambda_1}{2}\right) + e(P_\star, \overline{P}_\star) \\
&\geq \frac{\lambda_1}{2} \text{Vol}(\cup P_i) + x\delta \left(1 - \frac{\lambda_1}{2}\right) + e(P_\star, \overline{P}_\star) \\
&\geq \frac{\lambda_1}{2} \delta(n - |P_\star| - x) + \delta x + e(P_\star, \overline{P}_\star).
\end{aligned}
$$

□

## 2.4 Proof of Theorem 2.1.1

Our strategy now is, in principle, simple: We use Lemma 2.3.4 along with Lemma 2.3.1 $(iii)$ to prove that there are sufficiently many edges leaving any partition that Lemma 2.3.2 will allow us to apply Proposition 2.2.1 in each of our graphs $G_j$. Unfortunately, while this straightforward approach works (with some effort) for partitions into not too many parts, it breaks down as the number of parts gets very close to $n$. We handle these at the end in a slightly different way.

### 2.4.1 Partitions with a large part

**Lemma 2.4.1.** *Under the standard assumptions, for all partitions* $\mathcal{P} = \{P_1, ..., P_t\}$ *such that there exists a part,* $P_\star \in \{P_1, ..., P_t\}$ *with* $\mathrm{Vol}(P_\star) \geq \frac{1}{2}\mathrm{Vol}(G)$*, we have that*

$$(1-\epsilon)\mathbb{E}[e_j(\mathcal{P})] > \frac{(1+\epsilon)(t-2)C\log n}{2}.$$

*Proof.* Fix a partition $\mathcal{P} = \{P_1, ..., P_t\}$ such that there exists a $P_\star \in \{P_1, ..., P_t\}$ with $\mathrm{Vol}(P_\star) \geq \frac{1}{2}\mathrm{Vol}(G)$. Without loss of generality, let us assume that $P_\star = P_t$. Note that $\mathbb{E}[e_j(\mathcal{P})] = pe(\mathcal{P})$. Observe,

$$\begin{aligned}\frac{2}{p}\mathbb{E}\left[e_j(\mathcal{P})\right] &= \sum_{i\in[t-1]} e(P_i, \overline{P}_i) + e(P_\star, \overline{P}_\star)\\ &\geq \delta x + \frac{\lambda_1\delta(N-x)}{2} + e(P_\star, \overline{P}_\star) \text{ by Lemma 2.3.4, where } N = n - |P_\star|.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{2}{p}\mathbb{E}\left[e_j(\mathcal{P})\right] &\geq \delta x + \frac{\lambda_1\delta(N-x)}{2} + \frac{\lambda_1}{2}\mathrm{Vol}(\overline{P}_\star) \text{ by (1.4) and the fact that}\\ &\qquad\qquad \mathrm{Vol}(P_\star) > \frac{\mathrm{Vol}(G)}{2}\\ &\geq \delta x + \frac{\lambda_1\delta(N-x)}{2} + \frac{\lambda_1\delta}{2}N \text{ since } |\overline{P}_\star| = N\\ &\geq \delta x + \frac{\lambda_1\delta(2N-x)}{2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{2}{p}\mathbb{E}\left[e_j(\mathcal{P})\right] &\geq \left(\delta - \frac{\lambda_1\delta}{2}\right)x + \lambda_1\delta N \\ &\geq \lambda_1\delta\left[\frac{1}{2}x + N\right].\end{aligned}$$

Notice that $N \geq x + 2(t-1-x)$ since $N$ is equal to $|\overline{P}_\star|$, so it must be at least the number of parts of size one plus 2 times the number of parts of size bigger than one. From this, one obtains $x \geq 2(t-1) - N$. Thus,

$$\begin{aligned}\frac{1}{2}x + N &\geq \frac{1}{2}(2(t-1) - N) + N \\ &= t - 1 + \frac{N}{2}.\end{aligned}$$

Let $N = \alpha t$. Then $t - 1 + \frac{N}{2} = \left(1 + \frac{\alpha}{2}\right)t - 1$. Since $N \geq t - 1$, we have that $\alpha \geq 1/2$. This implies that

$$\mathbb{E}[e_j(\mathcal{P})] \geq \frac{C\log n}{2}\left(\frac{5}{4}t - 1\right).$$

Therefore,

$$(1-\epsilon)\mathbb{E}[e_j(\mathcal{P})] \geq (1-\epsilon)\frac{C\log n}{2}\left(\frac{5}{4}t - 1\right),$$

and for $0 < \epsilon < \frac{1}{9} < \frac{t+4}{9t-4}$ we have that

$$(1-\epsilon)\frac{C\log n}{2}\left(\frac{5}{4}t-1\right) > \frac{(1+\epsilon)(t-2)C\log n}{2}.$$

$\square$

By Lemma 2.3.1, we know that $e_j(\mathcal{P}) \geq (1-\epsilon)\mathbb{E}[e_j(\mathcal{P})]$ with probability at least $1-n^{-2}$. So we have shown that $e_j(\mathcal{P}) > \frac{(1+\epsilon)(t-2)C\log n}{2}$ with probability at least $1-n^{-2}$. By Corollary 2.3.3, each graph $G_j$ satisfies Schrijver's condition with probability at least $1-n^{-2}$.

### 2.4.2 Partitions where $2 \leq t \leq \frac{(1-\epsilon)}{(1+4\epsilon)}n$.

**Lemma 2.4.2.** *Under the standard assumptions, for all partitions $\mathcal{P} = \{P_1, ..., P_t\}$ where $\mathrm{Vol}(P_i) \leq \frac{1}{2}\mathrm{Vol}(G)$ for each $i \in [t]$ and $2 \leq t \leq \frac{(1-\epsilon)}{(1+4\epsilon)}n$, we have that*

$$(1-\epsilon)\mathbb{E}[e_j(\mathcal{P})] > \frac{(1+\epsilon)(t-2)C\log n}{2}.$$

*Proof.* Fix a partition $\mathcal{P} = \{P_1, \ldots, P_t\}$. Note that $\mathbb{E}[e_j(\mathcal{P})] = pe(\mathcal{P})$.

**Case 1:** $2 \leq t < \frac{(1-\epsilon)}{(1+\epsilon)}\frac{n}{2}$.

In this case, we may apply (1.4) directly to each part of the partition. Observe,

$$2\mathbb{E}[e_j(\mathcal{P})] = p\sum_{i\in[t]} e(P_i, \overline{P_i})$$

$$
\begin{aligned}
&\geq p \sum_{i \in [t]} \frac{\lambda_1 \delta}{2} |P_i| \\
&= p \frac{\lambda_1 \delta}{2} n \\
&= \frac{Cn \log n}{2}.
\end{aligned}
$$

Thus,

$$
\mathbb{E}[e_j(\mathcal{P})] \geq \frac{Cn \log n}{4}.
$$

Since $t \leq \frac{(1-\epsilon)}{(1+\epsilon)} \frac{n}{2}$, we have that

$$
\frac{(1-\epsilon)n}{4} > \frac{(1+\epsilon)(t-2)}{2}.
$$

This implies that

$$
\begin{aligned}
(1-\epsilon)\mathbb{E}[e_j(\mathcal{P})] &\geq \frac{(1-\epsilon)nC \log n}{4} \\
&> \frac{(1+\epsilon)(t-2)C \log n}{2}.
\end{aligned}
$$

**Case 2:** $\dfrac{(1-\epsilon)}{(1+\epsilon)} \dfrac{n}{2} \leq t \leq \dfrac{(1-\epsilon)}{(1+4\epsilon)} n.$

Let $M = 1 + \delta - \dfrac{\lambda_1 \delta}{2}$, as in Lemma 2.3.4. Observe,

$$
\begin{aligned}
\frac{2}{p}\mathbb{E}[e_j(\mathcal{P})] &\geq \delta x + \frac{\lambda_1 \delta}{2}(n - x) \text{ by Lemma 2.3.4} \\
&\geq \left(\delta - \frac{\lambda_1 \delta}{2}\right) x + \frac{\lambda_1 \delta}{2} n
\end{aligned}
$$

$$\geq \frac{\delta}{2}\left(t - \left\lfloor \frac{n-t}{M-1} \right\rfloor - 1\right) + \frac{\lambda_1 \delta}{2} n \quad \text{as } x \geq t - \left\lfloor \frac{n-t}{M-1} \right\rfloor - 1 \text{ by}$$

Lemma 2.3.4 and since $\lambda_1 \leq 1$

$$\geq \lambda_1 \delta \left( \frac{t+n}{2} - o(n) \right)$$

since $\lambda_1 \leq 1$, $M \geq \frac{\delta}{2}$, and $\delta \gg C \log n$.

Thus,

$$(1-\epsilon)\mathbb{E}[e_j(\mathcal{P})] \geq (1-\epsilon)\frac{C \log n}{2}\left(\frac{t+n}{2} - o(n)\right).$$

Notice that

$$\begin{aligned}
(1-\epsilon)\left(\frac{t+n}{2}\right) - (1-\epsilon)t &= \frac{(1-\epsilon)n - (1+3\epsilon)t}{2} \\
&\leq \frac{(1-\epsilon)n - (1+3\epsilon)\left(\frac{1-\epsilon}{1+4\epsilon}\right)n}{2} \text{ since } t \leq \frac{1-\epsilon}{1+4\epsilon}n \\
&= cn \text{ for some positive constant } c.
\end{aligned}$$

So, $(1-\epsilon)\left(\frac{t+n}{2}\right) = (1+\epsilon)t + \Omega(n)$, and

$$\begin{aligned}
(1-\epsilon)\frac{C\log n}{2}\left(\frac{t+n}{2} - o(n)\right) &\geq \frac{(1+\epsilon)tC\log n}{2} \\
&\geq \frac{(1+\epsilon)(t-2)C\log n}{2}
\end{aligned}$$

for $n$ and $C$ sufficiently large. Note that for the sake of clarity, we identified the $\frac{n-1}{M-1}$ and constant terms as a '$o(n)$' term. A tedious (but relatively simple) computation shows that the inequality is satisfied for a relatively mild $n$, depending on $C$. □

By Corollary 2.3.3, we have that for partitions $\mathcal{P} = \{P_1, ..., P_t\}$ of $V(G)$ into $t$ parts with $2 \leq t \leq \frac{(1-\epsilon)}{(1+4\epsilon)}n$ and $\operatorname{Vol}(P_i) \leq \frac{1}{2}\operatorname{Vol}(G)$ for each $i \in [t]$, Schrijver's condition is satisfied with probability at least $1 - n^{-2}$.

### 2.4.3 Partitions where $\frac{(1-\epsilon)}{(1+4\epsilon)}n \leq t < n - 6$.

**Lemma 2.4.3.** *Assume that the standard assumptions hold and that for each $i \in [t]$, $\operatorname{Vol}(P_i) < \frac{1}{2}\operatorname{Vol}(G)$ and $\frac{(1-\epsilon)}{(1+4\epsilon)}n < t < n - 6$. Then there are at least $t$ colors between parts with probability at least $1 - n^{-2}$.*

Notice that this is a corollary of the following lemma.

**Lemma 2.4.4.** *Assume that the standard assumptions hold, and let $\Pi$ be the set of partitions of $[n]$ into $t$ parts, where $\frac{(1-\epsilon)}{(1+4\epsilon)}n < t < n - 6$. Let $C_1, ..., C_s$ be the color classes of $G$, and let $c_i = |C_i|$. Let $\mathcal{C}$ be the set of collections of $s - t$ color classes. For a partition, $\mathcal{P} \in \Pi$ and $\mathfrak{C} \in \mathcal{C}$, let $\mathcal{B}_{\mathcal{P},\mathfrak{C}}$ be the event that none of the $s - t$ color classes in $\mathfrak{C}$ show up between the parts of $\mathcal{P}$. Then*

$$\mathbb{P}\left[\bigcup_{\mathcal{P}\in\Pi}\bigcup_{\mathfrak{C}\in\mathcal{C}}\mathcal{B}_{\mathcal{P},\mathfrak{C}}\right] \leq n^{-2}$$

*for $n$ and $C$ sufficiently large.*

**Remark:** This implies that there are at least $t$ colors between the parts in $\Pi$ of size $t$. Technically, we only need $t-1$ colors between parts, but proving this result as stated, where $\mathcal{C}$ consists of collections of $s-t$ color classes instead of $s-(t-1)$ color classes, makes an already messy computation slightly cleaner.

*Proof.* For $k, \ell \in \mathbb{N}$, $\binom{[k]}{\ell}$ denotes the collection of all subsets of $[k]$ of size $\ell$. Fix a partition, $\mathcal{P} \in \Pi$, and for each $i \in [s]$, let $\mathcal{C}_i^{\mathcal{P}}$ be the number of edges of color $i$ between parts in $\mathcal{P}$.

Then

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{\mathfrak{C} \in \mathcal{C}} \mathcal{B}_{\mathcal{P},\mathfrak{C}}\right) &\leq \sum_{\mathfrak{C} \in \mathcal{C}} \mathbb{P}\left(\mathcal{B}_{\mathcal{P},\mathfrak{C}}\right) \\
&= \sum_{\mathfrak{C} \in \mathcal{C}} \prod_{C_i \in \mathfrak{C}} (1-p)^{|C_i| - |C_i \cap \mathcal{P}|} \\
&= \sum_{\mathfrak{C} \in \mathcal{C}} \prod_{C_i \in \mathfrak{C}} (1-p)^{\mathcal{C}_i^{\mathcal{P}}} \\
&= \sum_{I \in \binom{[s]}{s-t}} (1-p)^{\sum_{i \in I} \mathcal{C}_i^{\mathcal{P}}} \\
&\leq \sum_{I \in \binom{[s]}{s-t}} \exp\left(-p \sum_{i \in I} \mathcal{C}_i^{\mathcal{P}}\right).
\end{aligned}$$

So we want to consider $\displaystyle\sum_{I\in\binom{[s]}{s-t}} \exp\left(-p\sum_{i\in I}\mathcal{C}_i^{\mathcal{P}}\right)$. Let

$$f\left(x_1,...,x_s;t\right)=\sum_{I\in\binom{[s]}{s-t}} \exp\left(-p\sum_{i\in I}x_i\right).$$

We begin with an observation:

**Claim.** Let $k$ be the smallest integer such that $e(\mathcal{P})\geq(s-k)\lambda_1\delta/2$. Then

$$f\left(\mathcal{C}_1^{\mathcal{P}},\cdots\mathcal{C}_s^{\mathcal{P}};t\right)\leq f\left(\underbrace{0,\cdots,0}_{k\text{ times}},\underbrace{\frac{\lambda_1\delta}{2},\cdots,\frac{\lambda_1\delta}{2}}_{s-k\text{ times}};t\right).$$

To see this, observe that by convexity,

$$f\left(\mathcal{C}_1^{\mathcal{P}},\cdots\mathcal{C}_s^{\mathcal{P}};t\right)\leq f\left(\underbrace{0,\cdots0}_{k-1\text{ times}},x^*,\underbrace{\frac{\lambda_1\delta}{2},\cdots,\frac{\lambda_1\delta}{2}}_{s-k\text{ times}};t\right),$$

where $0\leq x^*<\frac{\lambda_1\delta}{2}$. (The details are as in [14].) Note that the integer $k$ was chosen so that $(s-k)\frac{\lambda_1\delta}{2}+x^*=e(\mathcal{P})$. The claim then follows from monotonicity of $f$ in its variables.

Let $x$ be as in Lemma 2.3.4. Then

$$f\left(\mathcal{C}_1^{\mathcal{P}},...,\mathcal{C}_s^{\mathcal{P}};t\right)=\sum_{I\in\binom{[s]}{s-t}} \exp\left(-p\sum_{i\in I}\mathcal{C}_i^{\mathcal{P}}\right)\tag{2.5}$$

$$\leq \sum_{r=\max\{0,t-(s-k)\}}^{\min\{t,k\}} \binom{k}{r}\binom{s-k}{s-t-k+r} \exp\left(-p(s-k-t+r)\frac{\lambda_1\delta}{2}\right). \tag{2.6}$$

This inequality follows from the claim, noting that in (2.5), the sum is bounded above by a sum where each of the $\mathcal{C}_i^{\mathcal{P}}$ are $0$ or $\frac{\lambda_1\delta}{2}$ and that choosing $s-t$ colors consists of choosing $k-r$ of the classes of size zero, and $s-t-(k-r)$ classes of size $\frac{\lambda_1\delta}{2}$. The number of ways of doing this (by symmetry of the binomial coefficients) is $\binom{k}{r}\binom{s-k}{s-t-k+r}$. Note that writing them in this form is most convenient for bounding the binomial coefficients using the inequality $\binom{n}{k} \leq n^k$.

$$\begin{aligned}
(2.6) &\leq \sum_{r=\max\{0,t-(s-k)\}}^{\min\{t,k\}} \exp\left(r\log k + (s-k-t+r)\log(s-k) - \tfrac{C\log n}{2}(s-k-t+r)\right) \\
&\leq n\exp\left((s-k-t)\log(s-k) - \frac{C\log n(s-k-t)}{2}\right) \\
&\leq \exp\left(\log n + (s-k-t)\left(\log(n^2) - \frac{C\log n}{2}\right)\right) \\
&= \exp\left(\log n\left(1-(s-k-t)\left(\frac{C-4}{2}\right)\right)\right). \qquad (2.7)
\end{aligned}$$

Here, in the first inequality we use that $r\log k + r\log(s-k) - \frac{Cr\log n}{2} \leq 0$ for $C$ sufficiently large. The second follows as $s \leq n^2$ and bringing the $n$ to the exponent and the final equality by algebra.

To continue we need to use a lower bound on $s-k$ to show that the exponent in (2.7) is negative. To that end, note

$$\begin{aligned}
s-k &\geq \frac{e(\mathcal{P})}{\lambda_1\delta/2}-1 \\
&\geq \frac{|E(G)|}{\delta}+\frac{x}{\lambda_1}\left(1-\frac{\lambda_1}{2}\right)-1 \text{ by Lemma 2.3.4} \\
&\geq \frac{|E(G)|}{\delta}+\left(t-\frac{n-t}{M-1}-1\right)\left(1-\frac{\lambda_1}{2}\right)\frac{1}{\lambda_1}-1 \text{ by Lemma 2.3.4} \\
&\geq \frac{|E(G)|}{\delta}+\frac{1}{2}\left(t-\frac{n-t}{M-1}-3\right) \text{ as } \left(1-\frac{\lambda_1}{2}\right)\frac{1}{\lambda_1}\geq\frac{1}{2} \\
&\geq \frac{n}{2}+\frac{1}{2}\left(t-\frac{n-t}{M-1}-3\right). \qquad (2.8)
\end{aligned}$$

Using (2.8) in (2.7) we continue:

$$\begin{aligned}
(2.7) &\leq \exp\left(\log n\left(1-\left(\frac{n}{2}+\frac{1}{2}\left(t-\frac{n-t}{M-1}-3\right)-t\right)\left(\frac{C-4}{2}\right)\right)\right) \\
&= \exp\left(-\log n\left(\left((n-t)\left(\frac{1}{2}-\frac{1}{2(M-1)}\right)-\frac{3}{2}\right)\left(\frac{C-4}{2}\right)-1\right)\right) \\
&\leq \exp\left(-\log n\left(\left(\frac{n-t}{4}-\frac{3}{2}\right)\left(\frac{C-4}{2}\right)-1\right)\right).
\end{aligned}$$

Here, the second line followed by algebra (and in particular, combining the $\frac{n}{2}$, $\frac{t}{2}$ and $-t$ terms) and factoring out $n-t$, while the final inequality follows as $\frac{1}{2(M-1)}\leq\frac{1}{4}$.

Now, observe that the number of partitions of the vertices into $t$ parts is at most

$$\begin{aligned}
\binom{n}{t}t^{n-t} &= \binom{n}{n-t}t^{n-t} \\
&\leq n^{n-t}t^{n-t} \\
&\leq \exp(2(n-t)\log(n)).
\end{aligned}$$

Thus, a union bound shows that

$$\mathbb{P}\left[\bigcup_{\mathcal{P}\in\Pi}\bigcup_{\mathfrak{C}\in\mathcal{C}}\mathcal{B}_{\mathcal{P},\mathfrak{C}}\right] \leq \exp\left(2(n-t)\log(n) - \log n\left(\left(\frac{n-t}{4}-\frac{3}{2}\right)\left(\frac{C-4}{2}\right)-1\right)\right)$$
$$= \exp\left(-\log n\left(\left(\frac{(n-t)-6}{4}\right)\left(\frac{C-4}{2}\right)-1-2(n-t)\right)\right).$$

So long as $n-t\geq 7$ and $C$ is sufficiently large this exponent is less than $-2\log n$. Hence for $n-t\geq 7$,

$$\mathbb{P}\left[\bigcup_{\mathcal{P}\in\Pi}\bigcup_{\mathfrak{C}\in\mathcal{C}}\mathcal{B}_{\mathcal{P},\mathfrak{C}}\right] \leq n^{-2}.$$

□

### 2.4.4 Partitions where $n-6\leq t\leq n$.

**Lemma 2.4.5.** *Assume that the standard assumptions hold, and let* $t\in\{n-6,n-5,n-4,n-3,n-2,n-1,n\}$*. Then there are at least* $t-1$ *colors between parts with probability at least* $1-n^{-2}$.

*Proof.* Fix a partition with $t$ parts where $t\in\{n-6,n-5,n-4,n-3,n-2,n-1,n\}$. We want to show that there are at least $t-1$ colors between the parts of our partition. Unfortunately it is impossible to prove a lower bound on the number of edges in a *particular* color class $C_i$ in some $G_j$ since $C_i$ may be too small. To circumvent this, instead of considering each individual color class, we combine color classes to create pseudocolor classes. As shown in [14], we can construct $n-1$ pseudocolor classes $D_1,...,D_{n-1}$ such that for each

$k \in [n-1]$,

$$D_k = \left(\bigcup_{j=1}^{\ell} D_j\right) \setminus \left(\bigcup_{i=1}^{k-1} D_i\right),$$

where $\ell$ is the least integer such that $\left|\left(\bigcup_{j=1}^{\ell} C_j\right) \setminus \left(\bigcup_{i=1}^{k-1} D_i\right)\right| \geq n/4$. Fix $i \in [n-1]$ and $j \in [t]$. Let $Z_i^{(j)} = |E(G_i) \cap D_j|$. Then

$$\mathbb{E}[Z_i^{(j)}] \geq p\frac{n}{4} = \frac{nC\log n}{4\lambda_1\delta} \geq \frac{nC\log n}{4\delta} \geq \frac{C}{4}\log n.$$

Observe,

$$\begin{aligned}
\mathbb{P}\left(Z_i^{(j)} \leq \frac{C}{8}\log n\right) &\leq \mathbb{P}\left(Z_i^{(j)} \leq \frac{1}{2}\mathbb{E}[Z_i^{(j)}]\right) \\
&\leq \exp\left(-\frac{1}{8}\mathbb{E}[Z_i^{(j)}]\right) \text{ by Lemma 2.2.2} \\
&\leq \exp\left(-\frac{1}{8}\frac{C}{4}\log n\right) \\
&= \exp\left(-\frac{C\log n}{32}\right) \\
&\leq n^{-4}
\end{aligned}$$

for $C$ and $n$ sufficiently large. Thus,

$$\begin{aligned}
\bigcup_{j\in[t]}\bigcup_{i\in[n-1]} \mathbb{P}\left(Z_i^{(j)} \leq \frac{C}{8}\log n\right) &\leq t(n-1)n^{-4} \\
&< n^2 n^{-4} \\
&= n^{-2}.
\end{aligned}$$

This shows that in each $G_j$ there are at least $\frac{C}{8}\log n$ edges within each pseudocolor class with probability at least $1 - n^{-2}$. It remains to show that, in the range that $t \geq n - 6$, that the edges of these pseudocolor classes cannot be contained entirely within parts. This holds, however, as with so many parts there are only at most $n - t$ parts of size greater than one, and none of these can have size larger than $n - t + 1$ – indeed, the most edges within a part in this regime is $\binom{7}{2} = 21$, which occurs when $t = n - 6$ and one part has size $7$ and the others size $1$. Since $\frac{C}{8}\log n > 21$ (for $n$ and $C$ large enough) all $n - 1$ pseudocolor classes are represented, and there are at least $t - 1$ colors between parts with probability at least $1 - n^2$. □

*Proof of Theorem 2.1.1.* Theorem 2.1.1 follows almost immediately from the previous lemmas and Schrijver's condition (2.2.1). Indeed, we have shown that for any $1 \leq t \leq n$ every partition into $t$ parts satisfies Schrijver's condition with probability at least $1 - n^{-2}$. This follows by combining Lemmas 2.3.1, 2.3.2, 2.4.1, and 2.4.2 (for $2 \leq t \leq \frac{(1-\epsilon)}{(1+4\epsilon)}n$), by Lemma 2.4.3 (for $\frac{(1-\epsilon)}{(1+4\epsilon)}n \leq t \leq n - 6$) and by Lemma 2.4.5 (for $n - 6 \leq t \leq n$). A union bound over values of $t$ completes the proof. □

## 2.5 Applications and Discussion

While Theorem 2.1.1 applies to all sufficiently large graphs (as a function of $\lambda_1$) it is strongest when $\lambda_1$ is close to one. This is when the requirements on the color classes are weakest and the conclusion is strongest. Fortunately there are some graph classes satisfying this. The only graphs with $\lambda_1 = 1$ are complete bipartite graphs. The corollary below follows immediately from Theorem 2.1.1 since $\lambda_1(K_{n,m}) = 1$.

**Corollary 2.5.1.** *Let $G$ be an edge-colored copy of $K_{n,m}$ where $m \geq n$ and $n \geq C\log(n+m)$ for $n, m,$ and $C$ sufficiently large in which each color appears on at most $n/2$ edges. Then $G$ contains at least $\lfloor \frac{n}{C\log(n+m)} \rfloor$ edge-disjoint rainbow spanning trees.*

While non-complete graphs have $\lambda_1 < 1$, there are several natural classes of graphs which have $\lambda_1$ close to one. First, consider random $d$-regular graphs. Friedman, Kahn, and Szemerédi gave a bound on the eigenvalues of such graphs with $d$ fixed in [33]. This was a combination of two papers – one by Friedman, and the other by Kahn and Szemerédi. Their techniques were different, and in [9], Broder, Frieze, Suen, and Upfal showed that Kahn and Szemerédi's technique could be applied to more dense random $d$-regular graphs. More recently, Cook, Goldstein, and Johnson improved the range at which the eigenvalue bound was known.

**Theorem 2.5.2** ([26])**.** *Let $A$ be the adjacency matrix of a uniform random $d$-regular graph on $n$ vertices. Let $\lambda_0(A) \geq \cdots \geq \lambda_{n-1}(A)$ be the eigenvalues of $A$, and let $\lambda(A) = \max\{\lambda_1(A), -\lambda_{n-1}(A)\}$. For any $C_0, K > 0$, there exists $\alpha > 0$ such that if $1 \leq d \leq C_0(n^{2/3})$, then $\mathbb{P}(\lambda(A) \leq \alpha\sqrt{d}) \geq 1 - n^{-K}$ for $n$ sufficiently large.*

In a $d$-regular graph, we have that $\lambda_1(\mathcal{L}) = 1 - \frac{1}{d}\lambda_1(A)$. Therefore, this result gives us a lower bound on $\lambda_1(\mathcal{L})$, which we can use to apply Theorem 2.1.1.

**Corollary 2.5.3.** *Let $G$ be an edge-colored uniform random $d$-regular graph in which $C\log n \leq d \leq Cn^{2/3}$ (for $C$ and $n$ sufficiently large). Then there exists $\alpha > 0$ such that if each color class has size at most $d\left(1 - \frac{\alpha}{\sqrt{d}}\right)/2$, then $G$ contains at least $\left\lfloor \frac{d-\alpha\sqrt{d}}{C\log n} \right\rfloor$ edge-disjoint rainbow spanning trees with high probability.*

Our result applies to some graphs with very skewed degree distributions. The graph $G_{n,p}$ is the graph on $n$ vertices in which each edge appears with probability $p$. This can be generalized in the following way. For a sequence $\mathbf{w} = (w_1, \cdots, w_n)$, let $\rho = \frac{1}{\sum_{i=1}^{n} w_i}$. Then $G(\mathbf{w})$ is a random graph in which we label the vertices $v_1, ..., v_n$, and the edge $v_i v_j$ appears with probability $w_i w_j \rho$ [21]. (Here, we allow for loops.) In the graph $G(\mathbf{w})$, it is easy to see that $\mathbb{E}[\deg(v_i)] = \omega_i$. Notice that if we take $\mathbf{w} = (np, \cdots np)$, we get $G(\mathbf{w}) = G_{n,p}$.

It is well known [36] that $\lambda_1(G_{n,p}) \geq 1 - \frac{2}{\sqrt{np}}$ with high probability. So for all $\epsilon > 0$, $\lambda_1(G_{n,p}) \geq 1 - \epsilon$ for $n$ sufficiently large. Also, $\delta(G_{n,p}) \geq (1-\epsilon)np$ if $np \gg \log^2 n$. These results also apply to irregular graphs. For instance, consider $G(\mathbf{w})$. Fix $\epsilon > 0$. If $\mathbf{w}_{\min} \gg \log^2 n$, then $\delta \geq (1-\epsilon)\mathbf{w}_{\min}$ with high probability for $n$ large enough. Also, if $\mathbf{w}_{\min} \gg \log^2 n$, then $\lambda_1(G(\mathbf{w})) \geq 1 - \epsilon$ with high probability. This is implied by the following result of Chung, Lu, and Vu.

**Theorem 2.5.4** ([22]). *For a random graph with given expected degrees, if the minimal expected degree $\mathbf{w}_{\min}$ satisfies $\mathbf{w}_{\min} \gg \log^2 n$, then almost surely the eigenvalues of the Laplacian satisfy*

$$\max_{i \neq 0} |1 - \lambda_i| \leq (1 + o(1)) \frac{4}{\sqrt{\overline{\mathbf{w}}}} + \frac{g(n) \log^2 n}{w_{\min}},$$

*where $\overline{\mathbf{w}} = \frac{\sum_{i=1}^{n} w_i}{n}$ is the average expected degree and $g(n)$ is a function tending to infinity (with $n$) arbitrarily slowly.*

This bound on $\lambda_1(G(\mathbf{w}))$ gives us the following corollary of Theorem 2.1.1.

**Corollary 2.5.5.** *Fix $\epsilon > 0$. Assume that $\mathbf{w}_{\min} \gg \log^2 n$ and $G(\mathbf{w})$ is edge-colored so that each color class has size at most $\frac{\mathbf{w}_{\min}(1-\epsilon)}{2}$. Then for $n$ and $C$ sufficiently large, a graph $G \in G(\mathbf{w})$ contains at least $\left\lfloor \frac{\mathbf{w}_{\min}(1-\epsilon)}{C \log n} \right\rfloor$ edge-disjoint rainbow spanning trees with probability $1 - o(1)$.*

One can also wonder about the sharpness of our result and the dependence on $\lambda_1$. Let $S \subseteq V(G)$ such that $h_G = \frac{e(S,\overline{S})}{\mathrm{Vol}(S)}$. By Cheeger's inequality, $\lambda_1 > \frac{1}{2}\left(\frac{e(S,\overline{S})}{\mathrm{Vol}(S)}\right)^2$. So if $\lambda_1$ is very small, then $e(S,\overline{S})$ is much smaller than $\mathrm{Vol}(S)$. That is, there is a sparse cut in the graph. A sparse cut limits the number of disjoint spanning trees the graph can contain, let alone the number of disjoint rainbow spanning trees. Also, if $S \subseteq V(G)$ with $\mathrm{Vol}(S) \leq \frac{1}{2}\mathrm{Vol}(G)$, Cheeger's inequality implies that $e(S,\overline{S}) \geq \frac{\lambda_1}{2}\mathrm{Vol}(S)$. From this we can see that if $e(S,\overline{S})$ is small, then $\lambda_1$ is also small. However, it is not clear whether or not this dependence on $\lambda_1$ can be improved.

**Question 2.5.1.** *Is it possible to improve this dependence on $\lambda_1$? Is it possible to replace $\lambda_1$ with $\sqrt{\lambda_1}$, say, in Theorem 2.1.1?*

On the other hand, certainly no more than $\delta$ edge disjoint spanning trees are possible in any graph with minimum degree $\delta$. It seems plausible that the logarithmic factor could be removed in our lower bound on the number of rainbow spanning trees. In the more specialized situation of proper edge colorings of $K_n$, this is what [41] does.

In our proof of the main result, $\lambda_1$ was mostly used to lower bound the isoperimetric constant by Cheeger's inequality. It seems likely that $\lambda_1$ could be replaced by the isoperimetric constant. However, the isoperimetric constant is practically impossible to compute so stating the hypothesis in terms of $\lambda_1$ seems most natural.

It's also possible that the bound on the size of the color classes could be improved. It is clear that if color classes are allowed to be larger than $\frac{\delta}{2}$ in a $\delta$-regular graph, then rainbow spanning trees can be avoided entirely. In particular, for complete bipartite graphs (where $\lambda_1 = 1$) our size bound on color classes is correct. It is less clear that the factor of $\lambda_1$ appearing in our bound is actually required. We suspect that this dependence can be somewhat weakened, though it is unclear to us exactly what the dependence (if any!) on $\lambda_1$ the size of the color classes should have. Our proof rather naturally leads to bounding color classes by $\frac{\lambda_1 \delta}{2}$ as we have done.

Finding edge-disjoint rainbow spanning trees also has potential applications in communication networks. Consider a communication network with $n$ nodes. It would be helpful to find a way to connect all of the nodes in the cheapest way possible. A tree is a way of connecting nodes "cheaply", and that tree should be spanning if we want every node in the network to be connected. A rainbow spanning tree in this setting could represent a way of connecting all the nodes where the colors of the edges represent different wavelengths or frequencies. Showing that such a communications network has many edge-disjoint rainbow spanning trees shows that it is robust, in a certain sense.

It is also worth mentioning that the method of proving Theorem 2.1.1 is more interesting than the result itself. In Theorem 2.1.1 we used spectral graph theory to extend a result about the complete graph to general graphs. These methods can be used to help answer extremal questions on general classes of graphs.

## 2.6 Related Questions

Erdős and Gallai [32] showed that if a graph, $G$, has average degree bigger than $k-2$, then $G$ contains a path on $k$ vertices. This is related to the Erdős-Sós Conjecture:

**Conjecture 2.3.** *Let $G$ be a graph, and let $\overline{d}$ denote the average degree of $G$. If $\overline{d} > k-2$, then $G$ contains every tree on $k$ vertices as a subgraph.*

This motivates the question of what rainbow structures edge-colored graphs contained. If $G$ is a properly edge-colored complete graph, then we have that $\overline{d} > n-2$. However, $G$

does not necessarily contain a rainbow Hamiltonian path [53]. However, we can still ask what one can say along these lines.

**Question 2.6.1.** *Suppose you have a properly edge-colored graph, $G$, with average degree $\overline{d}$. What is the largest $k = k(\overline{d})$ such that $G$ contains every rainbow tree on up to $k$ vertices?*

Related to this, Montgomery, Pokrovskiy, and Sudakov [55] proved the following result.

**Theorem 2.6.1** ([55]). *Assume that $\epsilon, \frac{1}{k} \gg \frac{1}{n} > 0$ and $G$ is the complete graph on $n$ vertices and is edge-colored so that each vertex is incident to at most $k$ edges of any one color. Then $G$ contains a rainbow copy of every tree with at most $(1-\epsilon)n/k$ vertices.*

Instead of looking into average degree, we could, instead, ask what trees a graph is guaranteed to contain if we know something about the minimum degree of the graph. In fact, if we have information about the average degree of a graph, then we can make the following observation.

**Observation 2.6.2.** *If a graph $G$ has average degree at least $d$, then $G$ contains a subgraph with minimum degree at least $\frac{d}{2}$.*

This can be proved by induction on $|V(G)|$. Let $G$ be a graph with $n$ vertices and average degree at least $d$. If the minimum degree of $G$ is at least $\frac{d}{2}$, then we are done. So assume $\delta(G) < \frac{d}{2}$. Let $v \in V(G)$ with $\deg(v) < \frac{d}{2}$. Let $G'$ be the graph obtained by deleting $v$. Let $\overline{d}(G')$ denote the average degree of $G'$, and let $\deg_H(u)$ denote the degree of a vertex $u$ in a graph $H$.

Notice that

$$\begin{aligned}
\overline{d}(G') &= \frac{2|E(G)| - 2\deg_G(v)}{n-1} \\
&> \frac{2|E(G)| - d}{n-1} \quad \text{since } \deg_G(v) < \frac{d}{2} \\
&\geq \frac{2|E(G)| - \frac{2|E(G)|}{n}}{n-1} \\
&= 2|E(G)| \left( \frac{1 - \frac{1}{n}}{n-1} \right) \\
&= 2|E(G)| \left( \frac{\frac{n-1}{n}}{n-1} \right) \\
&= 2|E(G)| \left( \frac{1}{n} \right) \\
&= \frac{2|E(G)|}{n} \\
&\geq d.
\end{aligned}$$

So by induction, $G'$ contains a subgraph with minimum degree at least $\frac{d}{2}$, so $G$ contains a subgraph with minimum degree at least $\frac{d}{2}$.

**Question 2.6.2.** *Suppose $G$ is a properly edge-colored graph with minimum degree $d$. What is the largest $k = k(d)$ such that $G$ contains every rainbow tree on up to $k$ vertices?*

It is not hard to find a $k$ such that $G$ contains every rainbow tree on up to $k$ vertices, as evidenced by the following proposition. However, it seems unlikely that this is the largest.

**Proposition 2.6.3.** *If $G$ is a properly edge-colored graph with minimum degree at least $d$, then $G$ contains every rainbow tree on up to $\left\lfloor \frac{d}{2} \right\rfloor + 1$ vertices.*

*Proof.* We proceed by induction on the size of trees. Assume $G$ contains every rainbow tree on up to $t$ vertices where $1 \leq t \leq \lfloor d/2 \rfloor$, and let $T$ be a tree on $\lfloor d/2 \rfloor + 1$ vertices. Let

$x$ be a leaf of $T$, and assume $x \sim y$ for $y$ in $T$. Delete $x$. Then the remaining tree, $T'$, has $\lfloor d/2 \rfloor$ vertices. By induction, we can embed $T'$ in $G$. We handle the cases that $d$ is even and odd separately.

- If $d$ is even, then $T'$ has $\frac{d}{2}$ vertices. Since the minimum degree of $G$ is at least $d$, we know that the vertex $v$ has at least $d - \left(\frac{d}{2} - 1\right) = \frac{d}{2} + 1$ neighbors that are not in $T'$. Also, since $T'$ has $\frac{d}{2}$ vertices, we know that there are $\frac{d}{2} - 1$ colors represented in $T'$. Since $\left(\frac{d}{2} + 1\right) - \left(\frac{d}{2} - 1\right) = 2$, there are two neighbors of $v$ that can be added to the tree without repeating any colors.
- If $d$ is odd, then $T'$ has $\frac{d-1}{2}$ vertices. So the vertex $v$ has at least $d - \left(\frac{d-1}{2} - 1\right) = \frac{d+3}{2}$ neighbors that are not in $T'$. Also, there are $\frac{d-1}{2} - 1 = \frac{d-3}{2}$ colors represented in $T'$. Notice that $\frac{d+3}{2} - \frac{d-3}{2} = 3$, so there are 3 neighbors of $v$ that can be added to the tree without repeating a color.

□

**Rainbow Matchings**

It is also interesting to consider substructures other than rainbow trees in graphs, such as rainbow matchings. A *rainbow matching* in an edge-colored graph is a matching in which each edge is a different color. Rainbow matchings are well-studied.

Fu [62] showed that a properly-colored complete graph on an even number of vertices contains a rainbow matching. Wang and Li [61] showed that if $G$ is an edge-colored graph in which each vertex has color degree at least $k$, then $G$ has a rainbow matching of cardinality $\left\lceil \frac{5k-3}{12} \right\rceil$. They conjectured the following:

**Conjecture 2.4** ([61])**.** *Let $G$ be an edge-colored graph, and let $\hat{d}(v)$ denote the color degree of a vertex $v \in V(G)$. Suppose that $\hat{d}(v) \geq k \geq 4$ for each $v \in V(G)$. Then $G$ has a rainbow matching with $\left\lceil \frac{k}{2} \right\rceil$ edges.*

In [51], LeSaulnier, et al., showed that if $G$ is an edge-colored graph, then $G$ has a rainbow matching of size at least $\left\lfloor \frac{\hat{\delta}(G)}{2} \right\rfloor$, where $\hat{\delta}(G) = \min_{v \in V(G)} \hat{d}(v)$. They also proved the following theorem:

**Theorem 2.6.4** ([51])**.** *Each condition below guarantees that an edge-colored graph $G$ has a rainbow matching of size at least $\left\lceil \frac{\hat{\delta}(G)}{2} \right\rceil$.*

*(a) $G$ contains more than $\frac{3(\hat{\delta}(G)-1)}{2}$ vertices.*

*(b) $G$ is triangle-free.*

*(c) $G$ is properly edge-colored, $G \neq K_4$, and $|V(G)| \neq \hat{\delta}(G) + 2$.*

Wang asked the following question:

**Question 2.6.3** ([60])**.** *Is there a function $f(n)$ such that for each properly colored graph, $G$, with $|V(G)| \geq f(\delta(G))$, $G$ must contain a rainbow matching of size $\delta(G)$?*

Wang proved that if a graph $G$ is properly edge-colored and has at least $\frac{8\delta}{5}$ vertices, then $G$ has a rainbow matching of size at least $\left\lfloor \frac{3\delta}{5} \right\rfloor$. Wang also showed that if $G$ is a properly edge-colored triangle-free graph, then $G$ has a rainbow matching of size at least $\left\lfloor \frac{2\delta}{3} \right\rfloor$.

Diemunsch, et al. [28] answered this question of Wang and showed that if $G$ is a properly edge-colored graph with $|V(G)| \geq \frac{98\delta(G)}{23}$, then $G$ contains a rainbow matching of size $\delta(G)$. Gyárfás and Sárközy [38] improved this result by showing that if $G$ is a properly edge-colored graph with $|V(G) \geq 4\delta(G) - 3$, then $G$ contains a rainbow matching of size $\delta(G)$.

## 2.7 Conclusion

In this chapter, we successfully generalized a result related to the Brualdi-Hollingsworth conjecture to general graphs. In particular, we showed that edge colored graphs with bounded color class size have many edge-disjoint rainbow spanning trees.

The most important part of this work is that it illustrates that spectral graph theory gives us enough information about the structure of a general graph to apply probabilistic methods and extend a result about the complete graph to say something about substructures in general graphs. Recent results have improved the bounds on edge-disjoint rainbow spanning trees in edge colored complete graphs, and it would be interesting to consider different techniques that would allow Theorem 2.1.1 to be improved. In particular, it would be good to remove the log factor.

(Part of this chapter is based on the paper "Many edge-disjoint rainbow spanning trees in general graphs" [43], which is joint work with Paul Horn. This paper has been submitted.)

# CHAPTER 3: PAGERANK

## 3.1 Introduction/Background

Personalized PageRank, developed by Brin and Page [8] ranks the importance of webpages 'near' a seed. PageRank can be thought of in a variety of ways, but one of the most important points of view of PageRank is that it is a the distribution of a random walk allowed to diffuse for a geometrically distributed number of steps. A key parameter in PageRank, then, is the 'jumping' or 'teleportation' constant which controls the expected length of the involved random walks. As it controls the length, it controls locality – that is, how far from the seed the random walk is (likely) willing to stray.

When the jumping constant is small, the involved walks are (on average) short, and the mass of the distribution will remain concentrated near the seed. As the jumping constant increases, then the involved walk will (likely) be much longer. This will allow the random walk to mix, and the involved distribution will tend towards the stationary distribution of the random walk. As the PageRank of individual vertices (for a fixed jumping costant) can be thought of as a measure of importance to the seed, then as the jumping constant increases this importance diffuses.

Here, we are interested in how this importance diffuses as the jumping constant increases. This diffusion is related to the geometry of the graph; in particular the importance can get 'caught' by small cuts. This partially accounts for PageRank's importance

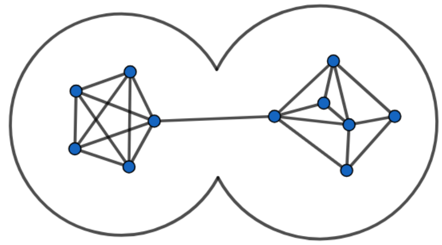


Figure 3.1: Sparse cut

in web search but has other uses as well – for instance Andersen, Chung and Lang use PageRank to implement local graph partitioning algorithms in [4].

This chapter seeks to understand the diffusion of influence (as the jumping constant changes) in analogy to the diffusion of heat. The study of solutions to the heat equation $\Delta u = \frac{\partial}{\partial t} u$ on both graphs and manifolds has a long history, motivated by its close ties to geometric properties of graphs. The geometry of a graph is very important when considering the diffusion of heat. If a graph has a sparse cut, as in Figure 3.1, then it will take longer for heat to diffuse. This is similar to the diffusion of influence when considering PageRank. A particularly useful way of understanding positive solutions to the heat equation is through curvature lower bounds. Knowing that a graph satisfies certain curvature lower bounds can tell us that our graph does not contain a sparse cut.

On graphs, the relationship between heat flow and PageRank has been noticed and used several times: Chung [17] introduced the notion of heat kernel PageRank and used it to improve the algorithm of Anderson, Chung, Lang for graph partitioning and it is well known that solutions to the heat equation reflect the diffusion of a continuous time random walk. In [17], Chung used solutions of the heat equation, and what we do here is essentially replacing PageRank with solutions to the heat equations.

Curvature lower bounds can be used to prove 'gradient estimates', which bound how heat diffuses locally in space and time and which can be integrated to obtain Harnack inequalities. Most classical of these is the Li-Yau inequality [52], which (in its simplest form) states that if $u$ is a positive solution to the heat equation, $\frac{\partial}{\partial t}u = \Delta u$, on a non-negatively curved $n$-dimensional compact manifolds, then $u$ satisfies

$$\frac{|\nabla u|}{u^2} - \frac{u_t}{u} \leq \frac{n}{2t}. \tag{3.1}$$

It was difficult to find a discrete version of this inequality. One reason for this is that the Laplace does not satisfy the chain rule. However, in the graph setting, Bauer, et al. proved a gradient estimate for the heat kernel on graphs in [6]. In this chapter we aim to prove a similar inequality for PageRank. Our gradient estimate, which is formally stated as Theorem 3.3.3 below, is proved using the exponential curvature dimension inequality $CDE$, introduced by Bauer et. al. This is a new notion of curvature, and it is beneficial to us because it effectively 'bakes in' the chain rule.

Although much has been done with curvature of manifolds, there isn't necessarily a clear way of defining curvature on graphs. Ollivier [56] introduced a notion of Ricci curvature of Markov chains on metric spaces, and Lin, Lu, and Yau modified Ollivier's definition of curvature in order to study other classes of graphs.

In some ways, our inequality is more closely related to another inequality of Hamilton [39] which bounds merely $\frac{|\nabla u|}{u^2}$, and was established for graphs by Horn in [40]. Other related works establish gradient estimates for eigenfunctions for the Laplace matrix; these include [19].

This chapter is organized as follows: In the next section we introduce definitions for both PageRank and the graph curvature notions used. We further establish a useful 'time parameterization' for PageRank, which allows us to think of increasing the jumping constant as increasing a time parameter, and makes our statements and proofs cleaner. In Section 3 we prove a gradient estimate for PageRank. In Section 4 we use this gradient estimate to prove a Harnack-type inequality that allows us to compare PageRank at two vertices in a graph.

## 3.2 Preliminaries

### 3.2.1 PageRank

(Personalized) PageRank was introduced as a ranking mechanism [45], to rank the importance of webpages with respect to a seed. To define personalized PageRank, we introduce the following operator which we call the PageRank operator. This operator, $P(\alpha)$, is defined as follows:

$$P(\alpha) = (1-\alpha)\sum_{k=0}^{\infty} \alpha^k W^k,$$

where $W = D^{-1}A$ is the transition probability matrix for a simple random walk. Here the parameter $\alpha$ is known as the jumping or teleportation constant. For a finite $n$-vertex graph, $P(\alpha)$ is a square matrix; the *personalized PageRank vector* of a vector $u : V \to \mathbb{R}$ is

$$u^T P(\alpha) = (1-\alpha)\sum_{k=0}^{\infty} \alpha^k u^T W^k$$

PageRank can then be viewed as the distribution of a geometric sum of the distribution of simple random walks: that is, the expected distribution of a simple random walk of length geometrically distributed with parameter $1-\alpha$ starting at initial distribution $u$. As $\alpha \to 1$, the expected length of this geometric random walk tends to infinity, and the resulting distribution tends to the limiting distribution of a simple random walk (which, for a finite graph is proportional to the degree.)

It has been noticed ([17],[18]) that PageRank has many similarities to the heat kernel $e^{t\Delta}$. Chung defined the notion of 'Heat Kernel PageRank' to exploit these similarities. In this work, we take inspiration in the opposite direction: we are interested in understanding the action of the PageRank operator in analogy to solutions of the heat equation.

The Laplace operator, $\Delta$, on a graph $G$ is defined at a vertex $x$ by

$$\Delta f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} (f(y) - f(x)).$$

In order to emphasize our point of view, we note that graph theorists view the heat kernel operator in two different ways: For a vector $u : V \to \mathbb{R}$ studying the evolution of

$$u^T e^{t\Delta}$$

as $t \to \infty$ is really studying the evolution of the *continuous time random walk* while studying the evolution of

$$e^{t\Delta} u$$

as $t \to \infty$ is studying the solutions to the heat equation

$$\Delta u = u_t.$$

The differing behavior of these two evolutions comes from the fact that (for irregular graphs) the left and right eigenvectors of $\Delta = I - W$ are different: the left Perron-Frobenius eigenvector of $\Delta$ is proportional to the degrees of a graph (as it captures the stationary distribution of the random walk) while the right Perron-Frobenius eigenvector is the constant vector. In particular as $t \to \infty$ the vector $e^{t\Delta}u$ tends to a constant. Physically, this represents the 'heat' on a graph evening out, and this regularization (and the rate of regularization) is related to a number of geometric features of a graph.

A similar feature holds for PageRank. As $\alpha \to 1$, $u^T P(\alpha)$ tends to a vector proportional to degrees, but $P(\alpha)u$ regularizes. In this chapter we study this regularization.

To see this regularization, notice that

$$\begin{aligned} W &= D^{-1}A \\ &= D^{-1/2}\left(D^{-1/2}AD^{-1/2}\right)D^{1/2} \\ &= D^{-1/2}\left(\sum_{i=0}^{n-1}\lambda_i\varphi_i\varphi_i^T\right)D^{1/2}, \end{aligned}$$

where $1 = \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$ are the eigenvalues of $D^{-1/2}AD^{-1/2}$ and $\varphi_0, \cdots, \varphi_{n-1}$ are the corresponding eigenvectors. Notice that $\varphi_0 = \frac{\mathbf{d}^{1/2}}{\sqrt{\mathrm{Vol}(G)}}$ is the eigenvector corresponding to $\lambda_0 = 1$, since

$$D^{-1/2}AD^{-1/2}\mathbf{d}^{1/2} = D^{-1/2}A\mathbb{1}$$

$$
\begin{aligned}
&= D^{-1/2}\mathbf{d} \\
&= \mathbf{d}^{1/2}.
\end{aligned}
$$

So

$$W^k = D^{-1/2}\left(\sum_{i=0}^{n-1} \lambda_i^k \varphi_i \varphi_i^T\right) D^{1/2}.$$

Thus,

$$
\begin{aligned}
P(\alpha) &= (1-\alpha)\sum_{k=0} \alpha^k W^k \\
&= (1-\alpha)\sum_{k=0}^{\infty} D^{-1/2}\left(\sum_{i=0}^{n-1}(\alpha\lambda_i)^k \varphi_i\varphi_i^T\right) D^{1/2} \\
&= (1-\alpha)\sum_{i=0}^{n-1} D^{-1/2}\left(\sum_{k=0}^{\infty}(\alpha\lambda_i)^k \varphi_i\varphi_i^T\right) D^{1/2} \text{ by Fubini's theorem} \\
&= (1-\alpha)\sum_{i=0}^{n-1} D^{-1/2}\left(\frac{1}{1-\alpha\lambda_i}\varphi_i\varphi_i^T\right) D^{1/2} \\
&= (1-\alpha)\left(\frac{1}{1-\alpha}D^{-1/2}\frac{\mathbf{d}^{1/2}\left(\mathbf{d}^{1/2}\right)^T}{\mathrm{Vol}(G)} + \sum_{i=1}^{n-1} D^{-1/2}\left(\frac{1}{1-\alpha\lambda_i}\varphi_i\varphi_i^T\right) D^{1/2}\right) \\
&= \frac{\mathbb{1}\cdot\mathbf{d}}{\mathrm{Vol}(G)} + \sum_{i=1}^{n-1}\frac{1-\alpha}{1-\alpha\lambda_i}D^{-1/2}\varphi_i\varphi_i^T D^{1/2}.
\end{aligned}
$$

(Recall the proof of Theorem 1.3.3, and note that this computation is very similar.)

Consider

$$\frac{1-\alpha}{1-\alpha\lambda_i}. \tag{3.2}$$

Notice that if $G$ is connected and not bipartite, then $\lambda_i < 1$ for $1 \leq i \leq n-1$. So $\dfrac{1-\alpha}{1-\alpha\lambda_i} \to 0$ as $\alpha \to 1$. As $\alpha \to 1$, the dominant term in $P(\alpha)$ becomes $\dfrac{\mathbb{1}\cdot\mathbf{d}}{\mathrm{Vol}(G)}$. Note that the smaller $\lambda_i$ are, the faster this tends to zero. This is Cheeger's inequality in action – if $\lambda_1 \approx 1$, then there is a sparse cut (since $D^{-1/2}AD^{-1/2} = I - \mathcal{L}$), so diffusion will take longer. If the $\lambda_i$ are far from $1$, then there won't be a sparse cut, so the diffusion will happen more quickly.

If $\varphi$ is a probability distribution, then

$$\varphi^T \frac{\mathbb{1} \cdot \mathbf{d}}{\mathrm{Vol}(G)} = \frac{\mathbf{d}}{\mathrm{Vol}(G)},$$

which is the stationary distribution for a random walk.

If, instead, we consider $P(\alpha)u$ for a vector $u$, then $\dfrac{\mathbb{1} \cdot \mathbf{d}}{\mathrm{Vol}(G)} u = c\mathbb{1}$, for a constant $c$. So $P(\alpha)$ regularizes, or "smooths out" as $\alpha \to 1$.

We note that the left and right action of the PageRank operator are closely related, and we study the left action versus the right action. For an undirected graph

$$u^T P(\alpha) = (P(\alpha)^T u)^T = (DP(\alpha)D^{-1}u)^T,$$

so that the regularization of $D^{-1}u$ can be translated into information on the 'mixing' of the personalized PageRank vector seeded at $u$.

To complete the analogy between $P(\alpha)u$ and $e^{t\Delta}u$, it is helpful to come up with a time parameterization $t = t(\alpha)$ so we can view the regularization as a function of 'time', in analogy to the heat equation. To do this in the best way, it is useful to think of $\alpha = \alpha(t)$ and compute $\dfrac{\partial}{\partial t} P_\alpha$.

**Proposition 3.2.1.**

$$\frac{\partial}{\partial t}P_\alpha = \frac{\alpha'}{(1-\alpha)^2}\Delta P_\alpha^2,$$

*where* $\Delta P_\alpha^2 = \Delta P_\alpha(P_\alpha)$.

*Proof.* Notice that

$$\begin{aligned}
\frac{\partial}{\partial t}P_\alpha &= \frac{\partial}{\partial t}(1-\alpha)(I-\alpha W)^{-1} \\
&= -\alpha'(I-\alpha W)^{-1} + (1-\alpha)\alpha' W(I-\alpha W)^{-2} \\
&= \alpha'((\alpha W - I)(I-\alpha W)^{-2} + (1-\alpha)W(I-\alpha W)^{-2} \\
&= \alpha'((W-I)(I-\alpha W)^{-2}) \\
&= \frac{\alpha'}{(1-\alpha)^2}\Delta P_\alpha^2.
\end{aligned}$$

□

This is remarkably close to the heat equation if $\alpha'(t) = (1-\alpha)^2$; solving this separable differential equation yields that $\alpha = \alpha(t) = 1 - \frac{1}{t+C}$. Since we desire a parameterization so that $\alpha(0) = 0$ and $\alpha \to 1$ as $t \to \infty$, this gives us that $C = 1$ from whence we obtain:

$$\boxed{\begin{aligned} \alpha(t) &= 1 - \frac{1}{t+1} \qquad (3.3) \\ t &= \frac{\alpha}{1-\alpha} \qquad (3.4) \end{aligned}}$$

We showed that $(3.2) \to 0$ as $\alpha \to 1$. If we used this time parameterization, we would get that

$$\begin{aligned}(3.2) &= \frac{1-\left(1-\frac{1}{t+1}\right)}{1-\left(1-\frac{1}{t+1}\right)\lambda_i} \\ &= \frac{\frac{1}{t+1}}{1-\lambda_i+\frac{1}{t+1}\lambda_i}.\end{aligned}$$

Since $\lambda_i$ is bounded away from 1 for $1 \leq i \leq n-1$, we have that

$$\frac{\frac{1}{t+1}}{1-\lambda_i+\frac{1}{t+1}\lambda_i} \to 0 \text{ as } t \to \infty.$$

Given the time parameterization in Equation 3.3, we get the following Corollary to Proposition 3.2.1.

**Corollary 3.2.2.**

$$\frac{\partial}{\partial t}P_\alpha = \Delta P_\alpha^2,$$

*where* $\Delta P_\alpha^2 = \Delta P_\alpha(P_\alpha)$.

*Proof.* From Proposition 3.2.1, we see that

$$\begin{aligned}\frac{\partial}{\partial t}P_\alpha &= \frac{\alpha'}{(1-\alpha)^2}\Delta P_\alpha^2 \\ &= \frac{\frac{1}{(t+1)^2}}{\left(\frac{1}{t+1}\right)^2}\Delta P_\alpha^2 \text{ by Equation 3.3.} \\ &= \Delta P_\alpha^2.\end{aligned}$$

□

Fix a vector $u : V \to \mathbb{R}$. From now on, we let

$$\boxed{f = P_\alpha u.} \tag{3.5}$$

**Lemma 3.2.3.** *For $f = P_\alpha u$ and $t = \dfrac{\alpha}{1-\alpha}$, we have that $\Delta f = \dfrac{f-u}{t}$.*

*Proof.* We know that $W = D^{-1}A$ and $\Delta = W - I$, so

$$\begin{aligned}
\Delta &= (W - I)(1-\alpha)(I - \alpha W)^{-1} \\
&= (W - \frac{1}{\alpha}I + \left(\frac{1-\alpha}{\alpha}\right) I)(1-\alpha)(I-\alpha W)^{-1} \\
&= -\frac{1}{\alpha}(I - \alpha W)(1-\alpha)(I-\alpha W)^{-1} + \frac{1-\alpha}{\alpha} \cdot (1-\alpha)(I-\alpha W)^{-1} \\
&= -\frac{1-\alpha}{\alpha} I + \frac{1-\alpha}{\alpha} P_\alpha \\
&= \frac{1-\alpha}{\alpha}(P_\alpha - I).
\end{aligned}$$

So

$$\begin{aligned}
\Delta f &= \Delta P_\alpha u \\
&= \frac{(1-\alpha)}{\alpha}(P_\alpha - I)u \\
&= \frac{f-u}{t}.
\end{aligned}$$

□

### 3.2.2 Graph curvature

In this chapter we study the regularization of $P(\alpha)u$ for an initial seed $u$ as $\alpha \to 1$. On one hand, the information about this regularization is contained in the spectral decomposition of the random walk matrix $W$. The eigenvalues of $P(\alpha)$ are determined by the eigenvalues of $W$: indeed, if $\lambda$ is an eigenvalue of $W$, then $\frac{1-\alpha}{1-\alpha\lambda_i}$ is an eigenvalue of $P_\alpha$. One may observe that, then, as $\alpha \to 1$ all eigenvalues of $P_\alpha$ tend to zero except for the eigenvalue, $1$ , of $W$ and this is what causes the regularization. Thus the difference between $P_\alpha u$ and the constant vector can be bounded in terms of (say) the infinity norms of eigenvectors of $P_\alpha$ and $\alpha$ itself.

On the other hand, curvature lower bounds (in graphs and manifolds) have proven to be important ways to understand the local evolution of solutions to the heat equation. As we have already noted important similarities between heat solutions and PageRank, we seek similar understanding in the present case. Curvature, for graphs and manifolds, gives a way of understanding the local geometry of the object. A manifold (or graph) satisfying a curvature lower bound at every point has a locally constrained geometry which allows a *local* understanding of heat flow through where a ‘gradient estimate’ can be proved. These gradient estimates can then be ‘integrated’ over space-time to yield Harnack inequalities which compare the ‘heat’ of different points at different times. This, in turn, can be used to establish geometric properties of the graph, which we will discuss in more detail later.

While a direct analogue of the Ricci curvature is not defined in a graph setting, a number of graph theoretical analogues have been developed recently in an attempt to apply geometrical ideas in the graph setting. In the context of proving gradient estimates of heat solutions, a new notion of curvature known as the *exponential curvature dimension inequality* was introduced in [6]. In order to discuss the exponential curvature dimension inequality, we first need to introduce some notation. Since it will show up frequently in computation, we define the following averaged sum.

**Definition 3.2.1.** *For $x \in V(G)$,*

$$\widetilde{\sum}_{y\sim x} h(x,y) = \frac{1}{\deg(x)} \sum_{y\sim x} h(x,y).$$

**Definition 3.2.2.** *The gradient form $\Gamma$ is defined by*

$$\begin{aligned} 2\Gamma(f,g)(x) &= (\Delta(f\cdot g) - f\cdot\Delta(g) - \Delta(f)\cdot g)(x) \\ &= \widetilde{\sum}_{y\sim x} (f(y)-f(x))(g(y)-g(x)). \end{aligned}$$

We write $\Gamma(f) = \Gamma(f,f)$.

In general, there is no "chain rule" that holds for the Laplacian on graphs. However, the following formula does hold for the Laplacian and will be useful to us:

$$\Delta f = 2\sqrt{f}\Delta\sqrt{f} + 2\Gamma(\sqrt{f}). \tag{3.6}$$

At the heart of the exponential curvature dimension inequality is an idea that had been used previously based on the Bochner formula. The Bochner formula reveals a connection between solutions to the heat equation and the curvature of a manifold. Bochner's formula tells us that if $M$ is a Riemannian manifold and $f$ is in $C^\infty(M)$, then

$$\frac{1}{2}\Delta|\nabla f|^2 = \langle \nabla f, \nabla\Delta f\rangle + ||\text{Hess} f||_2^2 + \text{Ric}(\nabla f, \nabla f).$$

The Bochner formula implies that for an $n$-dimensional manifold with Ricci curvature at least $K$, we have

$$\frac{1}{2}\Delta|\nabla f|^2 \geq \langle \nabla f, \nabla \Delta f\rangle + \frac{1}{n}(\Delta f)^2 + K|\nabla f|^2. \tag{3.7}$$

An important insight of Bakry and Emery was that an object satisfying an inequality like (3.7) could be used as a *definition* of a curvature lower bound even when curvature could not be directly defined. Such an inequality became known as a *curvature dimension inequality*, or the CD inequality.

Klartag, Kozma, Ralli, and Tetali [49] used this notion of curvature and showed that Cayley graphs of abelian groups, the complete graph, the symmetric group, $S_n$, with all transpositions, and slices of the hypercube satisfy certain curvature lower bounds. In [20], Chung, Lin, and Yau proved a Harnack inequality for finite connected graphs that satisfied this curvature lower bound. The CD inequality has applications for graphs, but it is badly-suited for studying diffusion.

Bauer, et al. [6] introduced a modification of the CD inequality that defines a new notion of curvature on graphs that we will use here – the exponential curvature inequality.

**Definition 3.2.3.** *A graph is said to satisfy the* **exponential curvature dimension inequality** $CDE(n, K)$ *if, for all positive* $f : V \to \mathbb{R}$ *and at all vertices* $x \in V(G)$ *satisfying* $(\Delta f)(x) < 0$

$$\Delta\Gamma(f) - 2\Gamma\left(f, \frac{\Delta f^2}{2f}\right) \geq \frac{2}{n}(\Delta f)^2 + 2K\Gamma(f), \tag{3.8}$$

*where the inequality in* (3.8) *is taken pointwise.*

While the inequality (3.8) may seem somewhat unwieldy it, as shown in [6], arises from 'baking in' the chain rule and is actually equivalent to the standard curvature dimension inequality (3.7) in the setting of diffusive semigroups (where the Laplace operator satisfies the chain rule.) Additionally, in [6], it is shown that some graphs including the Ricci flat graphs of Chung and Yau satisfy $CDE(n, 0)$ (and hence are non-negatively curved for this curvature notion) and some general curvature lower bounds are given.

An important observation is that this notion of curvature only requires looking at the second neighborhood of a graph, and hence this kind of curvature is truly a local property (and hence a curvature lower bound can be certified by only inspecting second neighborhoods of vertices.)

### 3.3 Gradient Estimate for PageRank

Our main result will make use of the following lemma from [6], and we include its simple proof for completeness.

**Lemma 3.3.1** ([6]). *Let $G(V, E)$ be a (finite or infinite) graph, and let $f, H : V \times \{t^\star\} \to \mathbb{R}$ be functions. If $f \geq 0$ and $H$ has a local maximum at $(x^\star, t^\star) \in V \times \{t^\star\}$, then*

$$\Delta(fH)(x^\star, t^\star) \leq (\Delta f)H(x^\star, t^\star).$$

*Proof.* Observe that

$$\begin{aligned}
\Delta(fH)(x^\star, t^\star) &= \widetilde{\sum}_{y\sim x^\star}(f(y, t^\star)H(y, t^\star) - f(x^\star, t^\star)H(x^\star, t^\star)) \\
&\leq \widetilde{\sum}_{y\sim x^\star}(f(y, t^\star)H(x^\star, t^\star) - f(x^\star, t^\star)H(x^\star, t^\star)) \\
&= (\Delta f)H(x^\star, t^\star).
\end{aligned}$$

$\square$

Our goal is to show that $\frac{\Gamma(\sqrt{f})}{\sqrt{f \cdot M}} \leq \frac{C(t)}{t}$ for some function $C(t)$. However, $\frac{C(t)}{t}$ is badly behaved as $t \to 0$. The way that we handle this is by showing that $H := t \cdot \frac{\Gamma(\sqrt{f})}{\sqrt{f \cdot M}} \leq C(t)$. If $H$ is a function from $V \times [0, \infty) \to \mathbb{R}$, then instead consider $H$ as a function from $V \times [0, T] \to \mathbb{R}$ for some $T > 0$. Then, by compactness, there is a point $(x^\star, t^\star)$ in $V \times [0, T]$ at which $H(x, t)$ is maximized. At this maximum, we know that $\Delta H \leq 0$ and $\frac{\partial}{\partial t} H \geq 0$. Since $\mathcal{L} = \Delta - \frac{\partial}{\partial t}$, this implies that at the maximum point, $\mathcal{L} H \leq 0$. Using the CDE inequality, along with some other lemmas and an identity, we are able to relate $H^2$ with itself. This allows us to find an upper bound for $H$, and thus for $\frac{\Gamma(\sqrt{f})}{\sqrt{f \cdot M}}$. Our situation is a little easier, because we consider a fixed $t$.

**Lemma 3.3.2.** *Let $G$ be a graph, and suppose $0 \leq f(x) \leq M$ for all $x \in V(G)$ and $t \in [0, \infty)$, and let $H = \frac{t\Gamma(\sqrt{f})}{\sqrt{f \cdot M}}$. Then*

$$\Delta \sqrt{f} = \frac{f - u}{2t\sqrt{f}} - \frac{\sqrt{M} H}{t}.$$

*Proof.* Using (3.6), we get that $\Delta \sqrt{f} = \dfrac{\Delta f - 2\Gamma(\sqrt{f})}{2\sqrt{f}}$. Thus,

$$\begin{aligned} \Delta \sqrt{f} &= \frac{\Delta f}{2\sqrt{f}} - \frac{\Gamma(\sqrt{f})}{\sqrt{f}} \\ &= \frac{f - u}{2t\sqrt{f}} - \frac{\sqrt{M} H}{t} \text{ by Lemma 3.2.3.} \end{aligned}$$

$\square$

At the heart of the proof of the Li-Yau inequality on manifolds is the identity

$$\Delta \log u = \frac{\Delta u}{u} - |\nabla \log u|^2 = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2}.$$

The Li-Yau inequality on graphs [6] uses the identity

$$\frac{\Delta\sqrt{u}}{\sqrt{u}} = \frac{\Delta u}{u} - \frac{|\nabla\sqrt{u}|^2}{u}.$$

Lemma 3.3.2 is similar to these other identities and the CDE inequality allows us to exploit this relationship.

**Theorem 3.3.3.** *Let $G$ be a graph satisfying $CDE(n, 0)$. Suppose $0 \le f \le M$ for all $x \in V(G)$ and $t \in (0, \infty)$. Then*

$$\frac{\Gamma(\sqrt{f})}{\sqrt{f \cdot M}} \le \frac{n+4}{n+2} \cdot \frac{1}{t} + 2\sqrt{\frac{n}{n+2}} \cdot \frac{1}{\sqrt{t}}.$$

Notice that a true Li-Yau-type inequality would have a time derivative. However, proving this in space is just as strong as it would be with the time derivative.

*Proof.* Let $H = \dfrac{t\Gamma(\sqrt{f})}{\sqrt{f \cdot M}}$. Fix $t > 0$. Let $(x^\star, t)$ be a point in $V \times \{t\}$ such that $H(x, t)$ is maximized. All of the following computations are made at the point $(x^\star, t)$. In order to apply the exponential curvature dimension inequality to $\sqrt{f}$, we must have that $\Delta\sqrt{f} < 0$.

If $\Delta\sqrt{f} \geq 0$, then by Lemma 3.3.2 we get that $\dfrac{f-u}{2t\sqrt{f}} - \dfrac{\sqrt{M}H}{t} \geq 0$. Thus,

$$\frac{\sqrt{M}H}{t} \leq \frac{f-u}{2t\sqrt{f}} \leq \frac{\sqrt{f}}{2t},$$

which implies that

$$H \leq \frac{\sqrt{f}}{2\sqrt{M}} \leq \frac{1}{2}.$$

So we can assume $\Delta\sqrt{f} < 0$, which allows us to use the inequality in (3.8).

Then we have that

$$\begin{aligned}
(\Delta\sqrt{f})H &\geq \Delta(\sqrt{f}H) \text{ by Lemma 3.3.1} \\
&\geq \frac{t}{\sqrt{M}}\left(\frac{2}{n}\left(\Delta\sqrt{f}\right)^2 + 2\Gamma\left(\sqrt{f}, \frac{\Delta f}{\sqrt{f}}\right)\right) \text{ by (3.8).}
\end{aligned}$$

Thus,

$$\left(\Delta\sqrt{f}\right)H \geq \frac{t}{\sqrt{M}}\left(\frac{2}{n}\left(\frac{f-u}{2t\sqrt{f}} - \frac{\sqrt{M}H}{t}\right)^2 + \frac{2}{t}\Gamma(\sqrt{f}) - \frac{2}{t}\Gamma\left(\sqrt{f}, \frac{u}{\sqrt{f}}\right)\right),$$

using Lemma 3.2.3 and the fact that $\Gamma$ is bilinear. Notice that

$$\begin{aligned}
&\frac{2}{t}\Gamma(\sqrt{f}) - \frac{2}{t}\Gamma\left(\sqrt{f}, \frac{u}{\sqrt{f}}\right) \\
&= \frac{2}{t}\left(\widetilde{\sum}_{y\sim x}(\sqrt{f(y)} - \sqrt{f(x)})^2 - \widetilde{\sum}_{y\sim x}(\sqrt{f(y)} - \sqrt{f(x)})\left(\frac{u(y)}{\sqrt{f(y)}} - \frac{u(x)}{\sqrt{f(x)}}\right)\right) \\
&\geq -\frac{2}{t}\widetilde{\sum}_{y\sim x}(\sqrt{f(y)} - \sqrt{f(x)})\left(\frac{u(y)}{\sqrt{f(y)}} - \frac{u(x)}{\sqrt{f(x)}}\right).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\left(\Delta\sqrt{f}\right)H \geq \frac{2}{\sqrt{M}nt}&\left(\frac{(f-u)^2}{2\sqrt{f}}-\frac{(f-u)\sqrt{M}H}{\sqrt{f}}+MH^2\right)\\
&+\frac{2\Gamma(\sqrt{f})}{\sqrt{M}}-\frac{2}{\sqrt{M}}\Gamma\left(\sqrt{f},\frac{u}{\sqrt{f}}\right)\\
&\geq \frac{2\big(-\sqrt{f}\sqrt{M}H+MH^2\big)}{\sqrt{M}nt}\\
&\quad -\frac{1}{\sqrt{M}}\widetilde{\sum_{y\sim x}}(\sqrt{f(y)}-\sqrt{f(x)})\left(\frac{u(y)}{\sqrt{f(y)}}-\frac{u(x)}{\sqrt{f(x)}}\right)\\
&\geq \frac{2\left(MH^2-\sqrt{f}\sqrt{M}H\right)}{\sqrt{M}nt}\\
&\quad -\frac{1}{\sqrt{M}}\widetilde{\sum_{y\sim x}}\left(u(x)\left(1-\sqrt{\frac{f(y)}{f(x)}}\right)+u(y)\left(1-\sqrt{\frac{f(x)}{f(y)}}\right)\right)\\
&\geq \frac{2\left(MH^2-\sqrt{f}\sqrt{M}H\right)}{\sqrt{M}nt}-\sqrt{M}.
\end{aligned}
$$

By Lemma 3.3.2, we have that

$$
\begin{aligned}
\Delta\sqrt{f} &= \frac{f-u}{2t\sqrt{f}}-\frac{\sqrt{M}H}{t}\\
&\leq \frac{\sqrt{f}}{2t}-\frac{\sqrt{M}H}{t}.
\end{aligned}
$$

So we have that $(\Delta\sqrt{f})H \leq \dfrac{\sqrt{f}H}{2t}-\dfrac{\sqrt{M}H^2}{t}$.

This implies that

$$\frac{\sqrt{f}H}{2t} - \frac{\sqrt{M}H^2}{t} \geq \frac{2\sqrt{M}H^2}{nt} - \frac{2\sqrt{f}H}{nt} - 2\sqrt{M}.$$

Combining terms, we get

$$\left(\frac{\sqrt{f}}{2t} + \frac{2\sqrt{f}}{nt}\right) H + 2\sqrt{M} \geq \left(\frac{2\sqrt{M}}{nt} + \frac{\sqrt{M}}{t}\right) H^2.$$

Multiplying by $t/\sqrt{M}$ yields the inequality

$$\left(\frac{\sqrt{f}}{2\sqrt{M}} + \frac{2\sqrt{f}}{n\sqrt{M}}\right) H + 2t \geq \left(\frac{2}{n} + 1\right) H^2,$$

which implies

$$\left(\frac{1}{2} + \frac{2}{n}\right) H + 2t \geq \left(\frac{2}{n} + 1\right) H^2$$

since $\dfrac{\sqrt{f}}{\sqrt{M}} \leq 1$. Thus,

$$\begin{aligned} H^2 &\leq \tfrac{\left(\frac{1}{2}+\frac{2}{n}\right)H}{\left(1+\frac{2}{n}\right)} + \tfrac{2t}{\left(1+\frac{2}{n}\right)} \\ &= C_1 \cdot H + C_2 \cdot t \end{aligned} \tag{3.9}$$

for constants $C_1 = C_1(n)$ and $C_2 = C_2(n)$.

If $C_1 H \geq C_2 \cdot t$, then $H^2 \leq 2C_1 H$, which implies that $H \leq 2C_1$. Thus,

$$\frac{\Gamma(\sqrt{f})}{\sqrt{f \cdot M}} \leq \frac{2C_1}{t}.$$

If $C_2 \cdot t > C_1 H$, then $H^2 \leq 2C_2 t$, so $H \leq \sqrt{2C_2 t}$. Therefore,

$$\frac{\Gamma(\sqrt{f})}{\sqrt{f \cdot M}} \leq \frac{\sqrt{2C_2 t}}{t} = \frac{\sqrt{2C_2}}{\sqrt{t}}.$$

Since

$$C_1 = \frac{\frac{1}{2} + \frac{2}{n}}{1 + \frac{2}{n}}$$

and

$$C_2 = \frac{2}{1 + \frac{2}{n}},$$

we have that

$$\begin{aligned}
\frac{\Gamma(\sqrt{f})}{\sqrt{f \cdot M}} &\leq \frac{2C_1}{t} + \frac{\sqrt{2C_2}}{\sqrt{t}} \\
&= 2 \cdot \left( \frac{\frac{1}{2} + \frac{2}{n}}{1 + \frac{2}{n}} \right) \cdot \frac{1}{t} + \sqrt{\frac{4}{1 + \frac{2}{n}}} \cdot \frac{1}{\sqrt{t}} \\
&= \frac{1 + \frac{4}{n}}{1 + \frac{2}{n}} \cdot \frac{1}{t} + \sqrt{\frac{4}{1 + \frac{2}{n}}} \cdot \frac{1}{\sqrt{t}} \\
&= \frac{n+4}{n+2} \cdot \frac{1}{t} + 2\sqrt{\frac{n}{n+2}} \cdot \frac{1}{\sqrt{t}}.
\end{aligned}$$

$\square$

**Remark:** In a typical applicataion of the maximum principal, we maximize over $[0, T]$ and then use information from the time derivative. Here, we don't do this. This is important because of the form of the inequality 3.9. Because of the dependence of this inequality on the time where the maximum occurs, if the $t^\star$ maximizing the function over all $[0,T]$ is considered, then the result will depend on $t^\star$, giving a bound like $H \leq \dfrac{\sqrt{2C_2 t^\star}}{t}$. However, since we are able to do the computation at $t$, this problem does not arise.

**Corollary 3.3.4.**

- *If* $0 < t \leq \dfrac{1}{4} + \dfrac{3n+8}{2n^2+4n}$, *then* $\dfrac{\Gamma(\sqrt{f})}{\sqrt{f \cdot M}} \leq \dfrac{2(n+4)}{(n+2)} \cdot \dfrac{1}{t}$.
- *If* $t \geq \dfrac{1}{4} + \dfrac{3n+8}{2n^2+4n}$, *then* $\dfrac{\Gamma(\sqrt{f})}{\sqrt{f \cdot M}} \leq 4\sqrt{\dfrac{n}{n+2}} \cdot \dfrac{1}{\sqrt{t}}$.

*Proof.* Let $A = 2C_1$ and $B = \sqrt{2C_2}$. We are interested in knowing when $\frac{A}{t} = \frac{B}{\sqrt{t}}$. This is equivalent to $\sqrt{t} = \frac{A}{B}$, so

$$\begin{aligned}
t &= \frac{A^2}{B^2} \\
&= \frac{(2C_1)^2}{(\sqrt{2C_2})^2} \\
&= \frac{4C_1^2}{2C_2} \\
&= \frac{2\left(\frac{\frac{1}{2}+\frac{2}{n}}{1+\frac{2}{n}}\right)^2}{\left(\frac{2}{1+\frac{2}{n}}\right)} \\
&= \frac{\left(\frac{1}{2}+\frac{2}{n}\right)^2}{1+\frac{2}{n}} \\
&= \frac{\frac{1}{4}+\frac{2}{n}+\frac{4}{n^2}}{1+\frac{2}{n}} \\
&= \frac{\frac{n^2}{4}+2n+4}{n^2+2n} \\
&= \frac{\frac{1}{4}(n^2+2n)+\frac{3n}{2}+4}{n^2+2n} \\
&= \frac{1}{4} + \frac{\frac{3n}{2}+4}{n^2+2n} \\
&= \frac{1}{4} + \frac{3n+8}{2n^2+4n}.
\end{aligned}$$

If $t \leq \frac{1}{4} + \frac{3n+8}{2n^2+4n}$, then $\frac{A}{t} \geq \frac{B}{\sqrt{t}}$, so

$$\frac{\Gamma(\sqrt{f})}{\sqrt{f \cdot M}} \leq 2 \cdot \frac{A}{t} = \frac{2(n+4)}{(n+2)} \cdot \frac{1}{t}.$$

If $t \geq \frac{1}{4} + \frac{3n+8}{2n^2+4n}$, then $\frac{A}{t} \leq \frac{B}{\sqrt{t}}$, so

$$\frac{\Gamma(\sqrt{f})}{\sqrt{f \cdot M}} \leq 2 \cdot \frac{B}{\sqrt{t}} = 4 \cdot \sqrt{\frac{n}{n+2}} \cdot \frac{1}{\sqrt{t}}.$$

□

### 3.4 Harnack-Type Inequality

We can use Theorem 3.3.1 to prove a result comparing PageRank at two vertices in a graph depending on the distance between them. This result is similar to a *Harnack inequality*. The classical form of a Harnack inequality is the following.

**Proposition 3.4.1** ([6]). *Suppose $G$ is a graph satisfying $CDE(n,0)$. Let $T_1 < T_2$ be real numbers, and let $\mathrm{d}(x,y)$ denote the distance between $x, y \in V(G)$. If $u$ is a positive solution to the heat equation on G, then*

$$u(x,T_1) \leq u(y,T_2)\left(\frac{T_2}{T_1}\right)^n \exp\left(\frac{4Dd(x,y)^2}{T_2 - T_1}\right),$$

*where $D = \max_{v \in V(G)} \deg(v)$.*

A result like this allows one to compare heat at different points and different times. This can make it possible to deduce geometric information about the graph, such as bottlenecking. Delmotte [27] showed that Harnack inequalities do not only allow us to compare heat at different points in space and time – they also have geometric consequences, such as volume doubling and satisfying the Poincaré inequality. Horn, Lin, Liu, and Yau [42] completed the work of Delmotte by proving that even more geometric information can be obtained from Harnack inequalities.

Using Theorem 3.3.3, we are able to relate PageRank at different vertices, but our result is not quite of the right form to be a Harnack inequality. In Theorem 3.3.3, it would be better if we had an $f$ instead of $\sqrt{f \cdot M}$ in the denominator. Since we do not, this makes proving a "Harnack-type" inequality more difficult. (What we do is similar to what Horn does in [40].)

To prove our Harnack-type inequality, we will use a lemma comparing PageRank at adjacent vertices. From now on, we will consider $t$ fixed and write $f(x)$ instead of $f(x,t)$. If a vertex, $w$, is adjacent to a vertex, $z$, then we want to lower bound $\sqrt{f(z)}$ by a function only involving $f(w)$. The trick to this is to rewrite $\sqrt{\frac{f(w)}{f(z)}}$ so that we can use Theorem 3.3.3 in order to get rid of the '$\sqrt{f(z)}$' in the denominator.

**Lemma 3.4.2.** *If $w \sim z$, then*

$$\sqrt{\frac{f(w)}{f(z)}} \leq \frac{2CD\sqrt{M}}{\sqrt{t}} \cdot \frac{1}{\sqrt{f(w)}} + 2,$$

*where $D = \max_{v \in V(G)} \deg(v)$.*

*Proof.* If $\sqrt{f(z)} \geq \frac{1}{2}\sqrt{f(w)}$, then $\sqrt{\frac{f(w)}{f(z)}} \leq 2 \leq \frac{2CD\sqrt{M}}{\sqrt{t}} \cdot \frac{1}{\sqrt{f(w)}} + 2$.

If $\sqrt{f(z)} < \frac{1}{2}\sqrt{f(w)}$, then

$$\begin{aligned}
\sqrt{\frac{f(w)}{f(z)}} &= \frac{\sqrt{f(w)} - \sqrt{f(z)} + \sqrt{f(z)}}{\sqrt{f(z)}} \\
&= \frac{\sqrt{f(w)} - \sqrt{f(z)}}{\sqrt{f(z)}} + 1 \\
&= \frac{D(\sqrt{f(w)} - \sqrt{f(z)})^2}{D\sqrt{f(z)}(\sqrt{f(w)} - \sqrt{f(z)})} + 1 \\
&\leq \frac{CD\sqrt{M}}{\sqrt{t}} \cdot \frac{1}{\sqrt{f(w)} - \sqrt{f(z)}} + 1 \\
&\leq \frac{CD\sqrt{M}}{\sqrt{t}} \cdot \frac{2}{\sqrt{f(w)}} + 1 \text{ since } \sqrt{f(z)} < \frac{1}{2}\sqrt{f(w)} \\
&\leq \frac{2CD\sqrt{M}}{\sqrt{t}} \cdot \frac{1}{\sqrt{f(w)}} + 2.
\end{aligned}$$

□

Using this, we can prove Theorem 3.4.3.

**Theorem 3.4.3.** *Let $G$ be a graph satisfying $CDE(n, 0)$. If dist$(x, y) = d$, where $d \geq 2$, then*

$$\frac{1}{\sqrt{f(y)}} \leq 4^{2^d-2} \cdot \max_{k=0,d} \left\{ \frac{A^{2^k-1}}{(\sqrt{f(x)})^{2^k}} \right\},$$

*where $A = \frac{4CD\sqrt{M}}{\sqrt{t}}$.*

*Proof.* We proceed by induction on $d$.

If $d = 2$, then let $z \in V(G)$ such that $x \sim z$ and $z \sim y$. Then by Lemma 3.4.2,

$$
\begin{aligned}
\frac{1}{\sqrt{f(y)}} &\leq \frac{2C\sqrt{M}}{\sqrt{t}f(z)} + \frac{2}{\sqrt{f(z)}} \\
&\leq \max\left\{\frac{A}{f(z)}, \frac{4}{\sqrt{f(z)}}\right\} \\
&\leq \max\left\{A \cdot \max\left\{\frac{A^2}{(f(x))^2}, \frac{4^2}{f(x)}\right\}, 4 \cdot \max\left\{\frac{A}{f(x)}, \frac{4}{\sqrt{f(x)}}\right\}\right\} \\
&= \max\left\{\frac{A^3}{(f(x))^2}, \frac{4^2 A}{f(x)}, \frac{4^2}{\sqrt{f(x)}}\right\} \\
&\leq 4^2 \cdot \max\left\{\frac{A^3}{(f(x))^2}, \frac{A}{f(x)}, \frac{1}{\sqrt{f(x)}}\right\}.
\end{aligned}
$$

Assume the result holds for $d \geq 2$. Let $x, y \in V(G)$ with $\text{dist}(x, y) = d + 1$. Let $z \in V(G)$ such that $z \sim y$ and $\text{dist}(x, z) = d$. Then by Lemma 3.4.2, we have that

$$
\begin{aligned}
\frac{1}{\sqrt{f(y)}} &\leq \frac{2C\sqrt{M}}{\sqrt{t}f(z)} + \frac{2}{\sqrt{f(z)}} \\
&\leq \max\left\{\frac{A}{f(z)}, \frac{4}{\sqrt{f(z)}}\right\} \\
&\leq \max\left\{A \cdot 4^{2^{d+1}-2^2} \cdot \max_{k=0,d}\left\{\frac{A^{2^{k+1}-2}}{(\sqrt{f(x)})^{2^{k+1}}}\right\}, 4 \cdot 4^{4^{2^d-2}} \cdot \max_{k=0,d}\left\{\frac{A^{2^k-1}}{(\sqrt{f(x)})^{2^k}}\right\}\right\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{by the induction hypothesis} \\
&= \max\left\{4^{2^{d+1}-2^2} \cdot \max_{k=0,d}\left\{\frac{A^{2^{k+1}-1}}{(\sqrt{f(x)})^{2^{k+1}}}\right\}, 4^{2^d-1} \cdot \max_{k=0,d}\left\{\frac{A^{2^k-1}}{(\sqrt{f(x)})^{2^k}}\right\}\right\} \\
&\leq 4^{2^{d+1}-2} \cdot \max\left\{\max_{k=0,d+1}\left\{\frac{A^{2^k-1}}{(\sqrt{f(x)})^{2^k}}\right\}, \max_{k=0,d}\left\{\frac{A^{2^k-1}}{(\sqrt{f(x)})^{2^k}}\right\}\right\}
\end{aligned}
$$

$$= 4^{2^{d+1}-2} \cdot \max_{k=0,d+1} \left\{ \frac{A^{2^k-1}}{(\sqrt{f(x)})^{2^k}} \right\},$$

where the last step follows since

$$\max_{k=0,d} \left\{ \frac{A^{2^k-1}}{(\sqrt{f(x)})^{2^k}} \right\} \leq \max_{k=0,d+1} \left\{ \frac{A^{2^k-1}}{(\sqrt{f(x)})^{2^k}} \right\}.$$

To see this, notice that $\frac{A^{2^k-1}}{(\sqrt{f(x)})^{2^k}} = \left( \frac{A}{\sqrt{f(x)}} \right)^{2k} \cdot \frac{1}{A}$. This function of $k$ is either increasing or decreasing, so the maximum over the interval $0 \leq k \leq t+1$ is achieved at either $k = 0$ or $k = d + 1$. $\square$

### 3.5 Discussion

In order to prove a Harnack-type inequality, we first needed to look at adjacent vertices. If $x \sim y$, then we wanted to lower bound $f(y)$ by a function of $x$. The classical way of doing this is to lower bound the log term $\log\left(\sqrt{\frac{f(y)}{f(x)}}\right)$. We can do this by using the power series expansion for $\ln(1 + x)$: For $|x| < 1$:

$$\log(1 + x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}. \tag{3.10}$$

Notice that for $-1 < x < 0$,

$$\begin{aligned} \ln(1 + x) &= -\left( |x| + \frac{|x|^2}{2} + \frac{|x|^3}{3} + \cdots \right) \\ &\geq -(|x| + |x|^2 + |x|^3 + \cdots) \end{aligned}$$

$$= -\frac{|x|}{1-|x|}.$$

**Lemma 3.5.1.** *Let $G$ be a graph satisfying $CDE(n,0)$. For adjacent vertices $x$ and $y$, we have that*

$$f(y) \geq f(x) \exp\left(-\frac{6CD\sqrt{M}}{\sqrt{t}\sqrt{f(x)}} - 1\right),$$

*where $D = \max_{v \in V(G)} \deg(v)$.*

*Proof.* If $f(y) \geq \frac{f(x)}{e}$, then we are done since

$$\begin{aligned} f(y) &\geq f(x)\exp(-1) \\ &\geq f(x) \exp\left(-\frac{6C\sqrt{M}}{\sqrt{t}\sqrt{f(x)}} - 1\right). \end{aligned}$$

So assume $f(y) < \frac{f(x)}{e}$.

We want to lower bound $f(y)$ by a function purely of $x$. Observe,

$$\begin{aligned} \log\left(\sqrt{\frac{f(y)}{f(x)}}\right) &= \log\left(1 - \frac{\sqrt{f(x)} - \sqrt{f(y)}}{\sqrt{f(x)}}\right) \\ &\geq \frac{-\left(\frac{\sqrt{f(x)}-\sqrt{f(y)}}{\sqrt{f(x)}}\right)}{1-\left(\frac{\sqrt{f(x)}-\sqrt{f(y)}}{\sqrt{f(x)}}\right)} \\ &= \frac{-(\sqrt{f(x)} - \sqrt{f(y)})}{\sqrt{f(y)}} \\ &= -\frac{D\left(\sqrt{f(x)} - \sqrt{f(y)}\right)^2}{D\sqrt{f(y)}} \cdot \frac{1}{\sqrt{f(x)} - \sqrt{f(y)}} \\ &\geq -\frac{CD\sqrt{M}}{\sqrt{t}} \cdot \frac{1}{\sqrt{f(x)} - \sqrt{f(y)}}, \end{aligned}$$

where the last inequality follows from Theorem 3.3.3.

Since $f(y) < \frac{f(x)}{e}$, we have that $\sqrt{f(y)} < \dfrac{\sqrt{f(x)}}{\sqrt{e}}$. So

$$\begin{aligned}
-\frac{CD\sqrt{M}}{\sqrt{t}} \cdot \frac{1}{\sqrt{f(x)} - \sqrt{f(y)}} &\geq -\frac{CD\sqrt{M}}{\sqrt{t}} \cdot \frac{1}{\sqrt{f(x)}\left(1 - \frac{1}{\sqrt{e}}\right)} \\
&\geq -\frac{CD\sqrt{M}}{\sqrt{t}} \cdot \frac{1}{\sqrt{f(x)}\left(1 - \frac{2}{3}\right)} \\
&= -\frac{3CD\sqrt{M}}{\sqrt{t}\sqrt{f(x)}}.
\end{aligned}$$

From this, we see that

$$\sqrt{\frac{f(y)}{f(x)}} \geq \exp\left(-\frac{3CD\sqrt{M}}{\sqrt{t}\sqrt{f(x)}}\right),$$

so

$$f(y) \geq f(x) \exp\left(-\frac{6CD\sqrt{M}}{\sqrt{t}\sqrt{f(x)}}\right).$$

□

This is the type of result we were hoping to get, and this allows us to get the following Harnack-type result.

**Theorem 3.5.2.** *Let* $w_1(x) = \exp\left(\dfrac{-6C\sqrt{M}}{\sqrt{t}\sqrt{f(x)}} - 1\right)$*, and for* $k > 1$*, let*

$$w_k(x) = \exp\left(\frac{-6C\sqrt{M}}{\sqrt{t}\sqrt{f(x)}\prod_{i=1}^{k-1}\sqrt{w_i(x)}} - 1\right).$$

*For vertices* $x, y \in V(G)$ *with dist*$(x, y) = n$*,*

$$f(y) \geq f(x) \cdot \prod_{i=1}^{n} w_i(x).$$

*Proof.* We proceed by induction on $n$.

If $n = 1$, then the result holds by Lemma 3.5.1.

Assume the result holds for $n \geq 1$. Let $x, y \in V(G)$ with dist$(x, y) = n+1$. Let $z \in V(G)$ such that $z$ is adjacent to $x$, and dist$(y, z) = n$.

By Lemma 3.5.1, we have that

$$f(z) \geq f(x) \cdot w_1(x).$$

We can also see that for each $i \in \{1, ..., n\}$, $w_i(z) \geq w_{i+1}(x)$.

By the induction hypothesis,

$$\begin{aligned} f(y) &\geq f(x) \cdot \prod_{i=1}^{n} w_i(z) \\ &\geq f(x) \cdot w_1(x) \cdot \prod_{n=2}^{n+1} w_i(x) \\ &= f(x) \cdot \prod_{i=1}^{n+1} w_i(x). \end{aligned}$$

□

## 3.6 Conclusion

We have shown that graph curvature can be used to give us enough information about the geometry of a graph to allow us to say something about how PageRank diffuses. Using curvature lower bounds, which give us local information about a graph, we were able to prove a gradient estimate for PageRank and use that to compare PageRank at any two vertices. An interesting future direction would be to fix the scaling in Theorem 3.3.3 and get an $f$ instead of $\sqrt{M \cdot f}$ in the denominator.

(This chapter is based on the paper "A gradient estimate for PageRank" [44], which is joint work with Paul Horn.)

BIBLIOGRAPHY

[1] S. Akbari and A. Alipour. Multicolored trees in complete graphs. *J. Graph Theory*, 54(3):221–232, 2007.

[2] M. Albert, A. Frieze, and B. Reed. Multicoloured Hamilton cycles. *Electron. J. Combin.*, 2:Research Paper 10, approx. 13 pp. 1995.

[3] N. Alon. Eigenvalues and expanders. *Combinatorica*, 6(2):83–96, 1986. Theory of computing (Singer Island, Fla., 1984).

[4] R. Andersen, F. Chung, and K. Lang. Local partitioning for directed graphs using PageRank. *Internet Math.*, 5(1-2):3–22, 2008.

[5] J. Balogh, H. Liu, and R. Montgomery. Rainbow spanning trees in properly coloured complete graphs. *Discrete Appl. Math.*, 247:97–101, 2018.

[6] F. Bauer, P. Horn, Y. Lin, G. Lippner, D. Mangoubi, and S.-T. Yau. Li-Yau inequality on graphs. *J. Differential Geom.*, 99(3):359–405, 2015.

[7] Kirk Boyer, Lauren M. Nelsen, Luke L. Nelsen, Florian Pfender, Elizabeth Reiland, and Ryan Solava. Erdős–Szekeres on-line. https://arxiv.org/pdf/1804.05952.pdf, 2018.

[8] S. Brin and L. Page. The anatomy of a large-scale hypertextual web search engine. *Computer Networks and ISDN Systems*, 30(1):107 – 117, 1998. Proceedings of the Seventh International World Wide Web Conference.

[9] A. A. Broder, A. M. Frieze, S. Suen, and E. Upfal. Optimal construction of edge-disjoint paths in random graphs. In *Proceedings of the Fifth Annual ACM-SIAM Symposium on Discrete Algorithms (Arlington, VA, 1994)*, pages 603–612. ACM, New York, 1994.

[10] H. Broersma and X. Li. Spanning trees with many or few colors in edge-colored graphs. *Discuss. Math. Graph Theory*, 17(2):259–269, 1997.

[11] R. A. Brualdi and S. Hollingsworth. Multicolored trees in complete graphs. *J. Combin. Theory Ser. B*, 68(2):310–313, 1996.

[12] R. A. Brualdi and S. Hollingsworth. Multicolored forests in complete bipartite graphs. *Discrete Math.*, 240(1-3):239–245, 2001.

[13] J. M. Carraher and S. G. Hartke. Eulerian circuits with no monochromatic transitions in edge-colored digraphs. *SIAM J. Discrete Math.*, 27(4):1924–1939, 2013.

[14] J. M. Carraher, S. G. Hartke, and P. Horn. Edge-disjoint rainbow spanning trees in complete graphs. *European J. Combin.*, 57:71–84, 2016.

[15] J. Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. pages 195–199, 1970.

[16] H. Chernoff. A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations. *Ann. Math. Statistics*, 23:493–507, 1952.

[17] F. Chung. The heat kernel as the pagerank of a graph. *Proceedings of the National Academy of Sciences*, 104(50):19735–19740, 2007.

[18] F. Chung. PageRank as a discrete Green's function. In *Geometry and analysis. No. 1*, volume 17 of *Adv. Lect. Math. (ALM)*, pages 285–302. Int. Press, Somerville, MA, 2011.

[19] F. Chung, Y. Lin, and S.-T. Yau. Harnack inequalities for graphs with non-negative Ricci curvature. *J. Math. Anal. Appl.*, 415(1):25–32, 2014.

[20] F. Chung, Y. Lin, and S.-T. Yau. Harnack inequalities for graphs with non-negative Ricci curvature. *J. Math. Anal. Appl.*, 415(1):25–32, 2014.

[21] F. Chung and L. Lu. *Complex graphs and networks*, volume 107 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006.

[22] F. Chung, L. Lu, and V. Vu. The spectra of random graphs with given expected degrees. *Internet Math.*, 1(3):257–275, 2004.

[23] F. R. K. Chung. *Spectral graph theory*, volume 92 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997.

[24] Gregory M. Constantine. Multicolored parallelisms of isomorphic spanning trees. *Discrete Math. Theor. Comput. Sci.*, 5(1):121–125, 2002.

[25] Gregory M. Constantine. Edge-disjoint isomorphic multicolored trees and cycles in complete graphs. *SIAM J. Discrete Math.*, 18(3):577–580, 2004/05.

[26] N. Cook, L. Goldstein, and T. Johnson. Size biased couplings and the spectral gap for random regular graphs. *arXiv:1510.06013 [math.PR]*, 2015.

[27] T. Delmotte. Parabolic Harnack inequality and estimates of Markov chains on graphs. *Rev. Mat. Iberoamericana*, 15(1):181–232, 1999.

[28] J. Diemunsch, M. Ferrara, A. Lo, C. Moffatt, F. Pfender, and P. S. Wenger. Rainbow matchings of size $\delta(G)$ in properly edge-colored graphs. *Electron. J. Combin.*, 19(2):Paper 52, 11, 2012.

[29] Jennifer Diemunsch, Nathan Graber, Lucas Kramer, Victor Larsen, Lauren M. Nelsen, Luke L. Nelsen, Devon Sigler, Derrick Stolee, and Charlie Suer. Color-blind index in graphs of very low degree. *Discrete Appl. Math.*, 225:122–129, 2017.

[30] J. Dodziuk and L. Karp. Spectral and function theory for combinatorial Laplacians. In *Geometry of random motion (Ithaca, N.Y., 1987)*, volume 73 of *Contemp. Math.*, pages 25–40. Amer. Math. Soc., Providence, RI, 1988.

[31] J. Edmonds. Submodular functions, matroids, and certain polyhedra. In *Combinatorial Structures and their Applications (Proc. Calgary Internat. Conf., Calgary, Alta., 1969)*, pages 69–87. Gordon and Breach, New York, 1970.

[32] P. Erdős and T. Gallai. On maximal paths and circuits of graphs. *Acta Math. Acad. Sci. Hungar*, 10:337–356 (unbound insert), 1959.

[33] J. Friedman, J. Kahn, and E. Szemerédi. On the second eigenvalue of random regular graphs. In *Proceedings of the 21st Annual ACM Symposium on Theory of Computing*, number 3, pages 587–598, 1989.

[34] A. Frieze and M. Krivelevich. On rainbow trees and cycles. *Electron. J. Combin.*, 15(1):Research paper 59, 9, 2008.

[35] H.-L. Fu, Y.-H. Lo, K. E. Perry, and C. A. Rodger. On the number of rainbow spanning trees in edge-colored complete graphs. *Preprint*, 2016.

[36] Z. Füredi and J. Komlós. The eigenvalues of random symmetric matrices. *Combinatorica*, 1(3):233–241, 1981.

[37] S. Glock, D. Kühn, R. Montgomery, and D. Osthus. Decompositions into isomorphic rainbow spanning trees. *arXiv:1903.04262 [math.CO]*, 2019.

[38] A. Gyárfás and G. N. Sárközy. Rainbow matchings and cycle-free partial transversals of Latin squares. *Discrete Math.*, 327:96–102, 2014.

[39] R. S. Hamilton. A matrix Harnack estimate for the heat equation. *Comm. Anal. Geom.*, 1(1):113–126, 1993.

[40] P. Horn. A spacial gradient estimate for solutions to the heat equation on graphs. *Siam J. Discrete Math. (to appear)*.

[41] P. Horn. Rainbow spanning trees in complete graphs colored by one factorizations. *Preprint*, 2013.

[42] P. Horn, Y. Lin, S. Liu, and S.-T. Yau. Volume doubling, poincaré inequality and gaussian heat kernel estimate for non-negatively curved graphs.

[43] P. Horn and L. M. Nelsen. Many edge-disjoint rainbow spanning trees in general graphs. *Preprint*, 2017.

[44] P. Horn and L. M. Nelsen. A gradient estimate for pagerank. *Preprint*, 2019.

[45] G. Jeh and J. Widom. Scaling personalized web search. pages 271–279, 2003.

[46] Rafał Kalinowski, Monika Pilśniak, Jakub Przybyło, and Mariusz Woźniak. Can colour-blind distinguish colour palettes? *Electron. J. Combin.*, 20(3):Paper 23, 12, 2013.

[47] A. Kaneko, M. Kano, and K. Suzuki. Three edge disjoint multicolored spanning trees in complete graphs. *Preprint*, 2003.

[48] M. Kano and X. Li. Monochromatic and heterochromatic subgraphs in edge-colored graphs—a survey. *Graphs Combin.*, 24(4):237–263, 2008.

[49] B. Klartag, G. Kozma, P. Ralli, and P. Tetali. Discrete curvature and abelian groups. *Canad. J. Math.*, 68(3):655–674, 2016.

[50] J. Krussel, S. Marshall, and H. Verrall. Spanning trees orthogonal to one-factorizations of $K_{2n}$. *Ars Combin.*, 57:77–82, 2000.

[51] T. D. LeSaulnier, C. Stocker, P. S. Wenger, and D. B. West. Rainbow matching in edge-colored graphs. *Electron. J. Combin.*, 17(1):Note 26, 5,, 2010.

[52] P. Li and S.-T. Yau. On the parabolic kernel of the Schrödinger operator. *Acta Math.*, 156(3-4):153–201, 1986.

[53] M. Maamoun and H. Meyniel. On a problem of G. Hahn about coloured Hamiltonian paths in $K_{2n}$. *Discrete Math.*, 51(2):213–214, 1984.

[54] R. Montgomery, A. Pokrovskiy, and B. Sudakov. Decompositions into spanning rainbow structures.

[55] R. Montgomery, A. Pokrovskiy, and B. Sudakov. Embedding rainbow trees with applications to graph labelling and decomposition. *arXiv:1803.03316 [math.CO]*, 2018.

[56] Y. Ollivier. Ricci curvature of Markov chains on metric spaces. *J. Funct. Anal.*, 256(3):810–864, 2009.

[57] A. Pokrovskiy and B. Sudakov. Linearly many rainbow trees in properly edge-coloured complete graphs. *J. Combin. Theory Ser. B*, 132:134–156, 2018.

[58] A. Schrijver. *Combinatorial optimization. Polyhedra and efficiency. Vol. B*, volume 24 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2003. Matroids, trees, stable sets, Chapters 39–69.

[59] K. Suzuki. A necessary and sufficient condition for the existence of a heterochromatic spanning tree in a graph. *Graphs Combin.*, 22(2):261–269, 2006.

[60] G. Wang. Rainbow matchings in properly edge colored graphs. *Electron. J. Combin.*, 18(1):Paper 162, 7, 2011.

[61] G. Wang and H. Li. Heterochromatic matchings in edge-colored graphs. *Electron. J. Combin.*, 15(1):Research Paper 138, 10, 2008.

[62] David E. Woolbright and Hung-Lin Fu. On the existence of rainbows in 1-factorizations of $K_{2n}$. *J. Combin. Des.*, 6(1):1–20, 1998.

# CHAPTER A: APPENDIX

This thesis contains results about the structure and geometry of graphs. The main results show how the spectral graph theory and graph curvature can be used to prove interesting results about graphs. Other work has been done by the author in other areas of combinatorics and graph theory, including graph coloring and combinatorial games. [1]

## A.1 Color-blind index

This is work done with J. Diemunsch, N. Graber, L. Kramer, V. Larsen, L.L. Nelsen, D. Sigler, D. Stolee, and C. Suer. This paper has been published in *Discrete Applied Mathematics* [29].

Let $c : E(G) \to \{1, ..., k\}$ be an edge-coloring (not necessarily proper). For a vertex, $v$, let $\overline{c}(v) = (a_1, ..., a_k)$, where $a_i = |\{u : uv \in E(G),\ c(uv) = i\}|$. Reorder the sequence $\overline{c}(v)$ in non-decreasing order to obtain $c^\star(v) = (d_1, ..., d_k)$.

When $c^\star$ induces a proper vertex coloring we say that $c$ is *color-blind distinguishing*. The minimum $k$ for which there exists a color-blind distinguishing edge coloring $c : E(G) \to \{1, ..., k\}$ is the *color-blind index of* $G$, which we denote by $\text{dal}(G)$.

There are a number of results that show that $\text{dal}(G)$ is small for certain graphs $G$. In [29], we show the following:

[1]These collaborations began as part of the 2014 and 2016 Rocky Mountain–Great Plains Graduate Research Workshops in Combinatorics, supported in part by NSF-DMS Grants #1427526, #1604458, #1604773, #1604697 and #1603823.

**Theorem A.1.1** (Diemunsch, et al., [29]). *Determining if dal$(G) = 2$ is NP-complete, even under the promise that dal$(G) \in \{2, 3\}$.*

A result of Kalinowski, Pilśniak, Pryzbyło, and Woźniak gives a class of graphs with such a promise.

**Theorem A.1.2** ([46]). *If $G$ is a $k$-regular bipartite graph with $k \geq 2$, then dal$(G) \leq 3$.*

In this case, determining if dal$(G) = 2$ is equivalent to asking if $k$-uniform, $k$-regular hypergraphs are 2-colorable.

One can also think about what happens when we have graphs that are far from being bipartite. We look at 3-regular graphs in which every vertex is in at least one 3-cycle, and investigate the color-blind index of such graphs. An example of such a graph is an odd cycle of diamonds. A diamond is made up of two triangles that share an edge. An odd cycle of diamonds is a 3-regular graph with an odd number of diamonds that are connected in a cycle.

**Theorem A.1.3** (Diemunsch, et al., [29]). *Let $G$ be a connected 3-regular graph where every vertex is in at least one 3-cycle of $G$. Then $G$ has a color-blind coloring iff $G$ is not an odd cycle of diamonds. When $G$ is not an odd cycle of diamonds, then dal$(G) \leq 3$.*

### A.2 Erdős-Szekeres Online

This is work done with K. Boyer, L.L. Nelsen, F. Pfender, E. Reiland, and R. Solava. This paper has been submitted [7].

In 1935, Erdős and Szekeres proved that the minimum number of points in the plane which definitely contain an increasing subset of $m$ points or a decreasing subset of $k$ points (as ordered by their $x$-coordinates) is $(m-1)(k-1)+1$. We consider their result from an on-line game perspective: Let points be determined one-by-one with player A first determining the $x$-coordinate and then player B determining the $y$-coordinate. We would like to know the minimum number of points such that player A can force an increasing subset of $m$ points or a decreasing subset of $k$ points. We introduce this as the *Erdős-Szekeres on-line number*, and denote it by ESO$(m, k)$.

In [7], we show that ESO$(m, k) < (m-1)(k-1)+1$ for $m, k \geq 3$, and ESO$(m, k) \geq \lfloor \frac{k}{2} \rfloor (m-k+5) - 3$ for $m \geq k \geq 4$. We also determine ESO$(m, 3)$ up to an additive constant.

**Theorem A.2.1** (Boyer, et al., [7])**.** *ESO*$(m, 3) = m + (6m)^{1/3} + O(1)$. *Specifically,*

$$m + (6m)^{1/3} - 2 < \textit{ESO}(m, 3) < m + (6m)^{1/3} + 3.$$