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TORIC GEOMETRY OF G_2-MANIFOLDS

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Abstract. We consider G_2-manifolds with an effective torus action that is multi-Hamiltonian for one or more of the defining forms. The case of T^3-actions is found to be distinguished. For such actions multi-Hamiltonian with respect to both the three- and four-form, we derive a Gibbons-Hawking type ansatz giving the geometry on an open dense set in terms a symmetric 3×3-matrix of functions. This leads to particularly simple examples of explicit metrics with holonomy equal to G_2. We prove that the multi-moment maps exhibit the full orbit space topologically as a smooth four-manifold containing a trivalent graph as the image of the set of special orbits and describe these graphs in some complete examples.

Contents

1. Introduction

The Gibbons-Hawking ansatz [27] furnishes a way of constructing hyperKähler four-manifolds with circle symmetry. More generally, the classifications of complete hypertoric manifolds (see, e.g., [8, 17]) show that moment map techniques, similar to the Delzant construction of symplectic geometry, can be useful when exploring Ricci-flat metrics.

Metrics of holonomy G_2 are known to be Ricci-flat. What is perhaps less familiar is that also in this setting, one has a notion of (multi-)symplectic geometry [36, 37]. It is therefore natural to ask what should be the analogue of toric or hypertoric geometry in this context.

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The first question to consider is which tori can act in a multi-Hamiltonian way on a torsion-free G_2-manifold. We find in §2 that the torus must have rank between 2 and 4. A dimension count then reveals that the case that best mimics hypertoric geometry is when a three-torus is multi-Hamiltonian for both the defining three-form and its Hodge dual four-form: this is the only case where the dimension of the orbit space matches the dimension of the target space for the multi-moment map. This 'toric' case with an effective T^3-action enjoys several immediate properties in common with the standard toric and hypertoric situation. In particular, we see that all stabilisers are again connected subtori, in this case of dimension at most 2, and that the multi-moment maps provide local coordinates on the manifold of principal orbits, so an open dense set of M becomes a T^3-bundle over a four-manifold.

In §3, we derive the analogue of the Gibbons-Hawking ansatz for toric G_2-manifolds M. The crucial local datum is now a smooth positive definite section $V \in \Gamma(U, S^2(\mathbb{R}^3))$ on an open set in $U \subset \mathbb{R}^4$. This determines the curvature of the T^3-bundle and must satisfy a pair of PDEs: one is a divergence-free condition on V and the other system is a quasi-linear elliptic second order PDE. These differential operators are natural for the action of $\mathrm{GL}(3, \mathbb{R})$ resulting from change of basis for the Lie algebra $\mathfrak{t}^3$ of T^3, and are nearly uniquely specified by this property. The divergence-free equation is essentially one used in continuum mechanics.

The above description, in terms of V, applies at points that have trivial T^3-stabiliser. In §4, we obtain a good understanding of the differential topology near singular orbits. As in the hypertoric case, one finds that M/T^3 is homeomorphic to a smooth manifold. This is unlike the situation for toric symplectic manifolds where the orbit space is a manifold with corners [32]. Our main result is that such a homeomorphism is realised via the multi-moment maps. Furthermore, the image of the singular orbits in the four-manifold M/T^3 is a trivalent graph, whose edges are straight lines in multi-moment map coordinates. These results are obtained by first studying flat models, including $S^1 \times \mathbb{C}^3$, where the graph has a single vertex where three edges meet, and $T^2 \times \mathbb{R} \times \mathbb{C}^2$, where the graph has one edge and no vertex.

Our distinguished case of G_2-manifolds that are multi-Hamiltonian for T^3 has the good feature that there are non-trivial complete examples with full holonomy G_2. Indeed, the Bryant-Salamon G_2-structure on the spin bundle of S^3 [13] is such an example, as are the generalisations in [11, 5, 10]. We study the Bryant-Salamon example in some detail, showing how it fits into the general framework. In particular, the associated trivalent graph is connected with two vertices and the multi-moment map provides a global homeomorphism $M/T^3 \to \mathbb{R}^4$.

If one is willing to compromise on completeness, our approach produces particularly simple Riemannian metrics with (restricted) holonomy equal to G_2, see Examples 5.2 and 5.5.

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2. G_2-MANIFOLDS WITH MULTI-HAMILTONIAN TORUS ACTIONS

Let M be a connected 7-manifold. A G_2-structure on M is determined by a 3-form φ that is pointwise linearly equivalent to the form

$$\varphi_0 = e^{123} - e^1(e^{45} + e^{67}) - e^2(e^{46} + e^{75}) - e^3(e^{47} + e^{56}),$$

where $E_1, \dots, E_7$ is a basis of $V \cong \mathbb{R}^7$, $e^1, \dots, e^7$ is its dual basis of V^*, wedge signs are suppressed and $e^{123} = e^1 \wedge e^2 \wedge e^3$, etc. We shall sometimes refer to $E_1, \dots E_7$ (and its dual) as an *adapted basis*.

The $\mathrm{GL}(V)$-stabiliser of φ_0 is the compact 14-dimensional Lie group $G_2 \subset \mathrm{SO}(V)$. In fact, φ_0 uniquely determines both the inner product $g_0 = \sum_{j=1}^7 (e^j)^2$ and volume element $\mathrm{vol}_0 = e^{1234567}$ via the relation

$$6 g_0(X, Y)\,\mathrm{vol}_0 = (X \lrcorner \varphi_0) \wedge (Y \lrcorner \varphi_0) \wedge \varphi_0,$$

for all $X, Y \in V$ (see [12]). Correspondingly, φ determines a metric g and a volume form vol on M. From this, it also follows that we have an additional dual 4-form, $*\varphi$, pointwise equivalent to

$$*\varphi_0 = e^{4567} - e^{23}(e^{45} + e^{67}) - e^{31}(e^{46} + e^{75}) - e^{12}(e^{47} + e^{56}).$$

We also get a cross-product operation via $g(X \times Y, Z) = \varphi(X, Y, Z)$. Three-dimensional subspaces of T_pM closed under the cross-product are *associative*, their orthogonal complements are *co-associative*.

Following standard terminology, we say that (M, φ) is a G_2-*manifold* if the G_2-structure is torsion-free, hence the (restricted) holonomy group $\mathrm{Hol}_0(g)$ is contained in $G_2 \subset \mathrm{SO}(7)$. This implies g is Ricci-flat. It is well-known [23] that being torsion-free, in this context, is equivalent to the condition that φ is closed and co-closed.

We are interested in G_2-manifolds that come with an effective action of a torus T^k on M that preserves φ, hence also $*\varphi$ and the metric g. Such an action gives us a map

$$\xi\colon \mathbb{R}^k \cong \mathfrak{t}^k \to \mathfrak{X}(M), \tag{2.1}$$

which is a Lie algebra anti-homomorphism. Subsequently, we shall often write ξ_p for the image of ξ at $p \in M$. This is a subspace of T_pM of dimension at most k.

Definition 2.1 ([37, Def. 3.5])**.** Let N be a manifold equipped with a closed $(k+1)$-form α, and G an Abelian Lie group acting on N preserving α. A *multi-moment map* for this action is an invariant map $\nu\colon N \to \Lambda^k \mathfrak{g}^*$ such that

$$d\langle \nu, W\rangle = \xi(W) \lrcorner \alpha,$$

for all $W \in \Lambda^k \mathfrak{g}$; here $\xi(W) \in \Gamma(\Lambda^k TM)$ is the unique multi-vector determined by W via ξ.

We say that such a torus symmetry on a G_2-manifold is *multi-Hamiltonian* if there are multi-moment maps associated with (φ, T^k) and/or $(*\varphi, T^k)$. This requires that $k \geqslant 2$ for non-triviality. A discussion of circle invariant G_2-metrics can be found in [2], and such metrics were also at the heart of the constructions in [25].

Given an effective torus action by T^k on (M, φ), it is obvious that $k \leqslant 7$ as we have the following well-known observation:

Lemma 2.2. *Let N be an n-manifold with an effective action of a torus T^k. Then $k \leqslant n$ and the principal stabiliser is trivial.*

Proof. It suffices to prove the final statement. As T^k is Abelian, conjugation is trivial. Therefore different isotropy subgroups H_p belong to different isotropy types. It follows that the principal stabiliser can be obtained as the intersection of all stabilisers, $\bigcap_{p\in N} H_p$, and so is the trivial group by effectiveness of the action. □

If N is a compact Ricci-flat manifold, then each Killing vector field is parallel [9]. It follows by [7, Cor. 6.67] that (N, h) has a finite cover in the form of a Riemannian product $T^\ell \times N_1^{n-\ell}$, some $k \leqslant \ell \leqslant n$, of a flat torus and compact simply-connected Ricci-flat manifold N_1. In particular, for a compact G_2-manifold with an effective T^k-action, $\mathrm{Hol}_0(g)$ is a proper subgroup of G_2. From Berger's classification [6], it follows that the restricted holonomy is trivial, $SU(2)$ or $SU(3)$. Correspondingly, we must have $\ell = 7$, $\ell = 3$ or $\ell = 1$, respectively.

As our main interest is the case of full holonomy, we will often concentrate on the case when M is non-compact.

Focusing on multi-Hamiltonian actions, we have already established that our torus must have rank between 2 and 7. It turns out there are further restrictions.

Proposition 2.3. *If T^k acts effectively on a G_2-manifold and is multi-Hamiltonian, then $2 \leqslant k \leqslant 4$.*

The proof of Proposition 2.3 is an immediate consequence of Lemmas 2.4 and 2.5 below.

Lemma 2.4. *Suppose W is a 5-dimensional subspace of (V, φ_0). Then W contains both associative and co-associative subspaces.*

Proof. Choose an orthonormal basis E_1, E_2 for $W^\perp$. Then $E_3 = E_1 \times E_2$ lies in W. Thus W contains the co-associative subspace $\langle E_1, E_2, E_3\rangle^\perp$. Furthermore, E_1, E_2, E_3 can be extended to a G_2 adapted basis for V. For this basis $E_4 \times E_7 = E_3$, so $\langle E_3, E_4, E_7\rangle$ is an associative subspace of W. □

The following observation states that a necessary condition for an action to be multi-Hamiltonian is that the orbits are 'isotropic'.

Lemma 2.5. *If a torus action of T^k on N is multi-Hamiltonian for a closed differential form α of degree $r \leqslant k$, then $\alpha|_{\Lambda^r \xi} \equiv 0$.*

If $b_1(N) = 0$, this condition is also sufficient for the T^k-action to be multi-Hamiltonian.

Proof. Consider the fundamental vector fields $\xi(V_1), \dots, \xi(V_{r-1})$ associated with vectors $V_1, \dots V_{r-1} \in \mathfrak{t}^k \cong \mathbb{R}^k$, and let ν' be a component of the multi-moment map $\nu\colon N \to \Lambda^{r-1}\mathfrak{t}^k$ that satisfies

$$d\nu' = \alpha(\xi(V_1), \dots, \xi(V_{r-1}), \cdot\,).$$

By invariance of the multi-moment map, we have for any $V_r \in \mathfrak{t}^k$ that

$$0 = \mathcal{L}_{\xi(V_r)}\nu' = \xi(V_r) \lrcorner\, d\nu' = \alpha(\xi(V_1), \dots, \xi(V_r)).$$

It follows that α vanishes on $\Lambda^r\xi$, as required.

As T^k preserves α, the 1-form $\alpha(\xi(V_1), \dots, \xi(V_{r-1}), \cdot\,)$ is closed and therefore exact, say equal to $d\nu'$, when $b_1(N) = 0$. The condition $\alpha|_{\Lambda^r\xi} \equiv 0$ implies invariance of ν', since T^k is connected. □

The upshot of Proposition 2.3 is that there are potentially 7 possible cases that can occur: T^2 multi-Hamiltonian for φ, T^3 multi-Hamiltonian for either φ or $*\varphi$, T^3 multi-Hamiltonian for both φ and $*\varphi$, T^4 acts multi-Hamiltonian for φ or $*\varphi$, and T^4 acts multi-Hamiltonian for both φ and $*\varphi$. In reality, the last situation cannot occur as we shall explain below.

Let $M_0 \subset M$ denote subset of points p such that the map ξ of (2.1) is injective. It follows by Lemma 2.2 that M_0 is open and dense, since it contains the set of principal orbits M_0'. Note that M_0' is the total space of a principle T^k-bundle.

2.1. **Two-torus actions.** This case was studied in [36], so we shall only give a brief summary.

Given a multi-Hamiltonian action for φ, the multi-moment map ν is an invariant scalar function $M \to \Lambda^2(\mathfrak{t}^2)^* \cong \mathbb{R}$. For $t \in \nu(M)$, if the action of T^2 is free on the level set $\nu^{-1}(t)$, then the reduction $N = \nu^{-1}(t)/T^2$ is a 4-manifold carrying three symplectic forms of the same orientation, induced by

$$U_1 \lrcorner\, \varphi, \quad U_2 \lrcorner\, \varphi \quad \text{and} \quad U_1 \wedge U_2 \lrcorner *\varphi,$$

where U_i generate the T^2-action. In interesting cases this triple is not hyperKähler, but does fit in to the framework of [24].

Conversely, the G_2-manifold (M, φ) can be recovered from the 4-manifold N by building a two-torus bundle over it. One then equips the total space of this bundle with a suitable $SU(3)$-structure and reconstructs the original G_2-holonomy manifold via an adapted 'Hitchin flow'.

Known complete G_2-manifolds with a multi-Hamiltonian T^2-action include the Bryant-Salamon metrics on the space of anti-self-dual 2-forms over a complete self-dual positive Einstein manifold [13].

2.2. **Three-torus actions.** The main interest here will be for actions that are multi-Hamiltonian for both φ and $*\varphi$, so that we have multi-moment maps $(\nu, \mu)\colon M \to \mathbb{R}^3 \times \mathbb{R}$. This is the only case in which the dimension of M/T^k matches that of the target space for the multi-moment maps. Being multi-Hamiltonian for φ, it follows by Lemma 2.5 that $\varphi|_{\Lambda^3\xi} \equiv 0$. This condition was studied in [29, §IV], where it where it is shown that G_2 acts transitively on the set of such three-planes. Indeed, for $p \in M_0$, for any orthonormal $X_2, X_3 \in \xi_p$, there is an adapted basis where these correspond to E_6 and E_7. The G_2-stabiliser of $\{E_6, E_7\}$ is an $SU(2)$ acting on $\langle E_2, E_3, E_4, E_5\rangle \cong \mathbb{C}^2$. Using this action, we see that we can extend to a basis X_1, X_2, X_3 of ξ_p and have X_1 identified with E_5. Now $\hat\theta_i = e^{i+4}$, $i = 1, 2, 3$, are dual to X_1, X_2, X_3: $\hat\theta_i(X_j) = \delta_{ij}$ and $\hat\theta_i(X) = 0$ for $X \perp \langle X_1, X_2, X_3\rangle$. Putting

$$\alpha_i = X_j \wedge X_k \lrcorner \varphi = -e^i, \quad \beta = X_1 \wedge X_2 \wedge X_3 \lrcorner *\varphi = -e^4,$$

where $(ijk) = (123)$, corresponding to the differentials of the multi-moment maps at p, the G_2-structure at $p \in M_0$ takes the form:

$$\begin{aligned} \varphi &= -\alpha_{123} - \alpha_1(\beta\hat\theta_1 - \hat\theta_{23}) - \alpha_2(\beta\hat\theta_2 - \hat\theta_{31}) - \alpha_3(\beta\hat\theta_3 - \hat\theta_{12}), \\ *\varphi &= \hat\theta_{123}\beta + \alpha_{23}(\beta\hat\theta_1 - \hat\theta_{23}) + \alpha_{31}(\beta\hat\theta_2 - \hat\theta_{31}) + \alpha_{12}(\beta\hat\theta_3 - \hat\theta_{12}). \end{aligned} \tag{2.2}$$

We shall return to this expression later on, in §3, refining it to give a G_2-analogue of the Gibbons-Hawking ansatz.

As in the hypertoric case, there are no points with discrete stabiliser. In particular, M_0 is the total space of a principal T^3-bundle over the corresponding orbit space.

Lemma 2.6. *Suppose T^3 acts effectively on a manifold M with G_2-structure φ so that the orbits are isotropic, $\varphi|_{\Lambda^3\xi_p} = 0$. Then each isotropy group is connected and of dimension at most two; hence trivial, a circle or T^2.*

Proof. Let $p \in M$ have isotropy group $H \leqslant T^3$. Then H is an Abelian group acting on $V = T^\perp$, where $T = T_p(T^3 \cdot p)$ is the tangent space to the orbit. As $T^3 \cdot p$ has an neighbourhood that can be identified with the normal bundle $T^3 \times_H V$ and this neighbourhood necessarily intersects principal orbits, the action on V is faithful. Adding the trivial H-module T to V, we have that the H-action on $T_pM = T \oplus V$ preserves the G_2-structure. As G_2 has rank 2, we get $\dim H \leqslant 2$.

If $\dim H = 0$, then at p, then any generators U_1, U_2, U_3 of the T^3 have the property that their cross products span T_pM. As the T^3-action preserves the G_2-structure, this implies that H fixes every element of T_pM. Thus H is trivial.

For $\dim H = 1$, the space T is spanned by two linearly independent vectors U_1 and U_2. It follows that H preserves the non-zero vector $U_1 \times U_2$ in V and must act as a subgroup of $SU(2)$ on the orthogonal complement. Thus H is a one-dimensional Abelian subgroup of $SU(2)$. This forces the identity component H_0 to be a maximal torus of $SU(2)$, so conjugate to $T^1 = \{\mathrm{diag}(\exp(i\theta), \exp(-i\theta)) \mid \theta \in \mathbb{R}\}$. But any matrix in $SU(2)$ commuting with T^1 is diagonal, so belongs to T^1. Thus $H \cong T^1$, which is connected.

When $\dim H = 2$, then H is a subgroup of $SU(3)$, so its identity component is a maximal torus. Again this conjugate to a group of diagonal matrices $\mathrm{diag}(\exp(i\theta), \exp(i\varphi), \exp -i(\theta+\varphi))$ and any other matrix commuting with this group is of this form. Thus $H \cong T^2$ and is connected. □

The classical example of a complete G_2-holonomy manifold with a multi-Hamiltonian T^3-action is the spin bundle of S^3 equipped with its Bryant-Salamon structure [13], see §5.1.2. Additional complete examples can be found in [11, 5, 10].

2.3. **Four-torus actions.** If a torus T^4 is multi-Hamiltonian for φ, then the multi-moment map has 6 components as its image is in $\Lambda^2(\mathfrak{t}^4)^* \cong \mathbb{R}^6$.

Lemma 2.7. *Suppose (M,φ) admits an effective T^4-action that is multi-Hamiltonian for φ. If $p \in M_0$, then $\xi_p \subset T_pM$ is co-associative.*

Proof. Take a pair E_1, E_2 of orthonormal vectors in ξ_p. As $\varphi|_{\Lambda^3\xi} \equiv 0$, we have that $E_3 = E_1 \times E_2$ lies in $\xi_p^\perp$. We may extend E_1, E_2, E_3 to an adapted basis $E_1, \dots, E_7$. Using the stabiliser $SU(2)$ of E_1, E_2 in G_2, we may ensure that $E_4 \in \xi_p$. Then the relations $E_1 \times E_4 = E_5$ and $E_2 \times E_4 = E_6$ give $\xi_p^\perp = \langle E_3, E_5, E_6 \rangle$, and so $\xi_p = \langle E_1, E_2, E_4, E_7 \rangle$. In particular, $\xi_p^\perp$ is associative and ξ_p is co-associative. □

A local description of G_2-manifolds with T^4-symmetry whose orbits are co-associative is given in [3], and also discussed in [20]. Essentially these correspond to positive minimal immersions into $\mathbb{R}^{3,3} \cong H^2(T^4)$, and this in turn is the image of the multi-moment map.

If T^4 is multi-Hamiltonian for $*\varphi$, we get a multi-moment map with 4 components as it has values in $\Lambda^3 \mathfrak{t}^4 \cong \mathbb{R}^4$.

Lemma 2.8. *Suppose T^4 acts effectively on (M,φ) and is multi-Hamiltonian for $*\varphi$. If $p \in M_0$, then the 4-dimensional subspace $\xi_p \leqslant T_pM$ contains an associative subspace. In particular, the action can not be multi-Hamiltonian for φ.*

Proof. Choose a pair of orthonormal vectors $E_1, E_2 \in \xi_p$ and extend these to an adapted basis for T_pM. As before, we may now use the stabiliser $SU(2) \leqslant G_2$ of E_1, E_2 to ensure that $E_4 \in \xi_p$. Now $*\varphi|_{\Lambda^4\xi} \equiv 0$ implies that $E_7 = E_1 \times E_2 \times E_4$ lies in $\xi_p^\perp$. Therefore, $\xi_p = \langle E_1, E_2, E_4, v \rangle$ with v a unit vector in $\langle E_3, E_5, E_6 \rangle$.

As $\langle E_3, E_4, E_7 \rangle$ is associative, there is a circle subgroup of G_2 that acts via multiplication by e^{it} on $\mathbb{C}^2 \cong \langle E_1 + iE_2, E_5 + iE_6 \rangle$. Using this, we may ensure that $v \in \langle E_3, E_5 \rangle$. Writing $v = xE_3 + yE_5$, we find that $E_1 \times v = -xE_2 + yE_4$ so that ξ_p contains the associative subspace $\langle E_1, xE_2 - yE_4, xE_3 + yE_5 \rangle$. □

All currently known examples of complete G_2-manifolds with a multi-Hamiltonian action of T^4 have reduced holonomy.

3. Toric G_2-manifolds: local characterisation

Motivated by the discussion in §2, we introduce the following terminology:

Definition 3.1. A *toric G_2-manifold* is a torsion-free G_2-manifold (M,φ) with an effective action of T^3 multi-Hamiltonian for both φ and $*\varphi$.

The purpose of this section is to derive an analogue of the Gibbons-Hawking ansatz [27, 28] for toric G_2-manifolds, more specifically obtaining a local form for a toric G_2-structure and describing the torsion-free condition in these terms. An independent derivation of such equations with an extension to $SU(2)$-actions was obtained by [14] after our announcement [43].

So assume (M, φ) is a toric G_2-manifold, with T^3 acting effectively. Let U_1, U_2, U_3 be infinitesimal generators for the T^3-action, then these give a basis for $\xi_p \leqslant T_pM$ for each $p \in M_0$. Denote by $\theta = (\theta_1, \theta_2, \theta_3)^t$ the dual basis of $\xi_p^* \leqslant T_p^*M$:

$$\theta_i(U_j) = \delta_{ij} \quad \text{and} \quad \theta(X) = 0 \quad \text{for all } X \perp U_1, U_2, U_3.$$

For brevity we write θ_{ab} for $\theta_a \wedge \theta_b$, etc.

Let $\nu = (\nu_1, \nu_2, \nu_3)^t$ and μ be the associated multi-moment maps; these satisfy

$$\begin{aligned} d\nu_i &= U_j \wedge U_k \lrcorner \varphi = (U_j \times U_k)^\flat, \qquad (ijk) = (123), \\ d\mu &= U_1 \wedge U_2 \wedge U_3 \lrcorner *\varphi. \end{aligned}$$

It follows from §2.2 that $(d\nu, d\mu)$ has full rank on M_0 and induces a local diffeomorphism $M_0/T^3 \to \mathbb{R}^4$. We define a 3×3-matrix B of inner products given by

$$B_{ij} = g(U_i, U_j),$$

and on M_0 we put $V = B^{-1} = \det(B)^{-1} \operatorname{adj}(B)$.

In these terms, we have the following local expression for the G_2-structure:

Proposition 3.2. *On M_0, the 3-form φ and 4-form $*\varphi$ are*

$$\varphi = -\det(V) d\nu_{123} + d\mu \wedge d\nu^t \operatorname{adj}(V)\theta + \underset{ijk}{\mathfrak{S}} \theta_{ij} \wedge d\nu_k,$$

$$*\varphi = \theta_{123} d\mu + \tfrac{1}{2\det(V)} \big(d\nu^t \operatorname{adj}(V)\theta\big)^2 + \det(V) d\mu \wedge \underset{ijk}{\mathfrak{S}} \theta_i \wedge d\nu_{jk}.$$

The associated G_2-metric is given by

$$g = \tfrac{1}{\det V} \theta^t \operatorname{adj}(V)\theta + d\nu^t \operatorname{adj}(V) d\nu + \det(V) d\mu^2. \tag{3.1}$$

We note that M_0 comes with a co-associative foliation with T^3-symmetry whose leaves are specified by setting ν equal to a constant. The corresponding distribution is given by the kernel of $d\nu_{123}$. In particular, the restriction of $*\varphi$ to the each leaf is $\theta_{123} d\mu$.

Proof. We start by choosing an auxiliary symmetric matrix $A > 0$ such that $A^2 = B^{-1}$ which is possible as B is positive definite. Then we set $X_i = \sum_{j=1}^3 A_{ij} U_j$ and observe that

$$g(X_i, X_j) = (ABA)_{ij} = (A^2B)_{ij} = \delta_{ij},$$

showing that the triplet (X_1, X_2, X_3) is orthonormal. It follows that we can apply the formulae (2.2) for φ and $*\varphi$.

We make the identification $\mathbb{R}^3 \cong \Lambda^2\mathbb{R}^3$ via contraction with the standard volume form. Then if we let $\Lambda^2 A$ denote the induced action of A on $\Lambda^2\mathbb{R}^3$, we can get

$$\Lambda^2 A = \det(A) A^{-1}.$$

In these terms, we have that

$$\alpha = (\Lambda^2 A) d\nu, \quad \beta = \det(A) d\mu \quad \text{and} \quad \hat{\theta} = A^{-1}\theta = \tfrac{1}{\det(A)} (\Lambda^2 A)\theta.$$

Turning to the expressions for the G_2 three-form, we start by noting that

$$\alpha_{123} = \det(\Lambda^2 A) d\nu_{123}$$

and that $\alpha_q(\beta\hat{\theta}_q - \hat{\theta}_{rs})$ equals

$$\sum_{i=1}^{3} (\Lambda^2 A)_{qi} d\nu_i \Big(\sum_{j=1}^{3} (\Lambda^2 A)_{qj} d\mu \theta_j - \det(B) \sum_{k,\ell=1}^{3} (\Lambda^2 A)_{rk} (\Lambda^2 A)_{s\ell} \theta_{k\ell} \Big),$$

where $(qrs) = (123)$. Summing these terms gives

$$\begin{aligned}\varphi = &- \det(\Lambda^2 A) d\nu_{123} + d\mu \sum_{i,j=1}^{3} d\nu_i (\Lambda^2 A)^2_{ij} \theta_j \\ &+ \det(B) \sum_{i,k,\ell=1}^{3} (\Lambda^2 A)_{1i} (\Lambda^2 A)_{2k} (\Lambda^2 A)_{3\ell} (d\nu_i \theta_{k\ell} + d\nu_k \theta_{\ell i} + d\nu_\ell \theta_{ik}),\end{aligned}$$

which is simplified by observing that the expression in the second line above reduces to give $d\nu_1\theta_{23} + d\nu_2\theta_{31} + d\nu_2\theta_{12}$, as required by the multi-moment map relations. The asserted expression for φ therefore follows by noting that $(\Lambda^2 A)^2 = B/\det(B) = \operatorname{adj}(V)$.

To rephrase the 4-form expression, we observe that

$$\hat{\theta}_{123}\beta = \theta_{123} d\mu,$$

consistent with the multi-moment map condition, and that $\alpha_{rs}(\beta\hat{\theta}_q - \hat{\theta}_{rs})$ equals

$$\sum_{i,j=1}^{3} (\Lambda^2 A)_{ri} (\Lambda^2 A)_{sj} d\nu_{ij} \Big(\sum_{k=1}^{3} (\Lambda^2 A)_{qk} d\mu \theta_k - \tfrac{1}{\det(A)^2} \sum_{k,\ell=1}^{3} (\Lambda^2 A)_{rk} (\Lambda^2 A)_{s\ell} \theta_{k\ell} \Big)$$

for $(qrs) = (123)$. Upon summation, this quickly gives the stated expression for $*\varphi$.

Finally, for the metric we have

$$\begin{aligned} g &= \hat{\theta}^t\hat{\theta} + \alpha^t\alpha + \beta^2 = (A^{-1}\theta)^t A^{-1}\theta + (\Lambda^2 A d\nu)^t \Lambda^2 A d\nu + \det(A)^2 d\mu^2 \\ &= \theta^t \big(\tfrac{1}{\det(V)} \operatorname{adj}(V) \big) \theta + d\nu^t \operatorname{adj}(V) d\nu + \det(V) d\mu^2, \end{aligned}$$

as claimed. □

Remark 3.3. The expression for $*\varphi$ may also be written as

$$*\varphi = \theta_{123} d\mu - \mathop{\mathfrak{S}}_{ijk} \mathop{\mathfrak{S}}_{pqr} V_{ip} d\nu_{jk} \theta_{qr} + \det(V) d\mu \wedge \mathop{\mathfrak{S}}_{ijk} \theta_i \wedge d\nu_{jk}. \qquad (3.2)$$

Remark 3.4. In the above, we have a natural action of $GL(3, \mathbb{R})$, corresponding to changing the basis of $\mathfrak{t}^3$. This action can sometimes be used to simplify arguments as it allows us to assume V is diagonal or the identity matrix at a given point provided only the $\mathbb{R}^3 = \widetilde{T^3}$ action is of relevance.

3.1. **The torsion-free condition.** Whilst it is true that any toric G_2-manifold can be expressed as in Proposition 3.2, the G_2-structure captured by these formulae is not automatically torsion-free.

Computing $d\varphi$ and $d*\varphi$ involves the exterior derivatives of θ. By our observations in §2.2, we may think of θ as a connection 1-form and its exterior derivative

$$d\theta = \omega = (\omega_1, \omega_2, \omega_3)^t$$

is therefore a curvature 2-form (and as such represents an integral cohomology class). In terms of our parameterisation for the base space, via multi-moment maps, we can write the curvature components of ω in the form

$$\omega_\ell = \mathop{\mathfrak{S}}_{ijk}(z^i_\ell d\nu_i d\mu + w^i_\ell d\nu_{jk}).$$

For convenience, we collect these curvature coefficients in two 3×3 matrices $Z = (z^i_j)$ and $W = (w^i_j)$.

Closedness of φ now becomes:

$$\begin{aligned} 0 = &-d\det(V) \wedge d\nu_{123} + d\mu(d\nu)^t \operatorname{adj}(V)\omega + d\mu(d\nu)^t d(\operatorname{adj}(V)) \wedge \theta \\ &+ \mathop{\mathfrak{S}}_{ijk}(\omega_i d\nu_j - \omega_j d\nu_i)\theta_k. \end{aligned} \tag{3.3}$$

More explicitly, by wedging with $d\nu_i$, these equations completely determine the 9 curvature functions z^j_i:

$$z^\ell_i = \frac{\partial \operatorname{adj}(V)_{k\ell}}{\partial \nu_j} - \frac{\partial \operatorname{adj}(V)_{j\ell}}{\partial \nu_k} \tag{3.4}$$

where $(ijk) = (123)$. Note, in particular, that the above expressions imply that Z is traceless, $\operatorname{tr}(Z) = 0$.

In addition, upon wedging with $d\mu$, we see that equation (3.3) forces W to be symmetric, $w^i_j = w^j_i$. Finally, it follows by wedging (3.3) with θ_{123} that

$$\left\langle \operatorname{adj}(V), \frac{\partial V}{\partial \mu} - W \right\rangle = 0, \tag{3.5}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $M_3(\mathbb{R}) \cong \mathbb{R}^9$.

Addressing co-closedness of φ, we use (3.2) to get

$$\begin{aligned} 0 &= d*\varphi \\ &= \mathop{\mathfrak{S}}_{ijk} \omega_i \theta_{jk} d\mu - \mathop{\mathfrak{S}}_{ijk} \mathop{\mathfrak{S}}_{pqr} dV_{ip} \wedge d\nu_{jk}\theta_{qr} - \mathop{\mathfrak{S}}_{ijk} \mathop{\mathfrak{S}}_{pqr} V_{ip} d\nu_{jk}(\omega_q\theta_r - \theta_q\omega_r) \\ &\quad + d(\det(V)) \wedge d\mu \mathop{\mathfrak{S}}_{ijk} \theta_i d\nu_{jk}. \end{aligned} \tag{3.6}$$

The curvature functions w^i_j are computed from the wedge product of (3.6) with $d\nu_i\theta_j$ to be

$$w^j_i = \frac{\partial V_{ij}}{\partial \mu} \tag{3.7}$$

and it follows that equation (3.5) automatically holds. If instead we wedge (3.6) with $d\mu\theta_i$ we find that

$$\sum_{i=1}^{3} \frac{\partial V_{ij}}{\partial \nu_i} = 0 \qquad j = 1,2,3. \tag{3.8}$$

We shall occasionally refer to this first order underdetermined elliptic PDE system as the 'divergence-free' condition. Coincidentally, (3.8) appears in the study of (linear) elasticity in continuum mechanics, expressing that the stress tensor is divergence-free (see, e.g., [21, 22]). This equation together with the expression for $\operatorname{adj} V$ allows us to rewrite the coefficients z^i_j as

$$z^i_\ell = \sum_{a=1}^{3} \frac{\partial V_{j\ell}}{\partial \nu_a} V_{ka} - \frac{\partial V_{k\ell}}{\partial \nu_a} V_{ja} \qquad (ijk) = (123). \tag{3.9}$$

One may now check that there are no further relations from (3.3) or (3.6).

There are only 6 additional equations, arising from the condition $d\omega = 0$. Using (3.7), (3.8) and (3.9), these equations can be expressed in the form of a second order non-linear elliptic PDE without zeroth order terms:

$$L(V) + Q(dV) = 0. \tag{3.10}$$

Here the operator L is given by

$$L = \frac{\partial^2}{\partial \mu^2} + \sum_{i,j} V_{ij} \frac{\partial^2}{\partial \nu_i \partial \nu_j},$$

and so has the same principal symbol as the Laplacian for the metric $d\mu^2 + d\nu^t B d\nu$, which, up to a conformal factor of $\det(V)$, is the same as the restriction of the G$_2$-metric (3.1) to the horizontal space. The operator Q is the quadratic form in dV given explicitly by

$$Q(dV)_{ij} = -\sum_{a,b=1}^{3} \frac{\partial V_{ia}}{\partial \nu_b} \frac{\partial V_{jb}}{\partial \nu_a}.$$

In summary, we have that the torsion-free condition determines Z and W together with three first order equations and six second order equations. We therefore have the following local description of toric G$_2$-manifolds.

Theorem 3.5. *Any toric* G$_2$*-manifold can be expressed in the form of Proposition 3.2 on the open dense subset of principal orbits for the* T^3*-action.*

Conversely, given a principal T^3*-bundle over an open subset* $\mathcal{U} \subset \mathbb{R}^4$*, parameterised by* (ν, μ)*, together with* $V \in \Gamma(\mathcal{U}, S^2(\mathbb{R}^3))$ *that is positive definite at each point. Then the total space comes equipped with a* G$_2$*-structure of the form given in Proposition 3.2. This structure is torsion-free, hence toric, if and only if the curvature matrices* Z *and* W *are determined by* V *via* (3.4) *and* (3.7)*, respectively, and* V *satisfies the divergence-free condition* (3.8) *together with the non-linear second order elliptic system* (3.10)*.* □

Using this characterisation, it is not difficult to construct many explicit incomplete examples of toric G$_2$-manifolds (see §5.2).

As one would expect, solutions with V constant are trivial in the following sense:

Corollary 3.6. *A toric* G_2*-manifold with* V *constant is flat and hence locally isometric to* $\mathbb{R}^7$*.*

Proof. If V is constant, we may assume $V \equiv 1$. Now $\det(V) = 1$ everywhere and therefore $M_0 = M$. Consequently, by Proposition 3.2, we have a global orthonormal co-frame $e^1, \dots, e^7$ satisfying $de^i = 0$ for all $1 \leqslant i \leqslant 7$. □

Let us conclude this section by remarking that (3.8) can be integrated to obtain what in a sense may be seen as an analogue of the local potential for hypertoric manifolds (cf. [8]). The following observation is also known from continuum mechanics.

Proposition 3.7. *Assume that* $V \in \Gamma(\mathcal{U}, S^2(\mathbb{R}^3))$ *satisfies* (3.8)*, with* $\mathcal{U} \subset \mathbb{R}^3$ *simply-connected. Then there exists* $A \in \Gamma(\mathcal{U}, S^2(\mathbb{R}^3))$ *such that*

$$V_{ii} = \frac{\partial^2 A_{jj}}{\partial v_k^2} + \frac{\partial^2 A_{kk}}{\partial v_j^2} - 2\frac{\partial^2 A_{jk}}{\partial v_j \partial v_k}, \quad V_{ij} = \frac{\partial^2 A_{ik}}{\partial v_j \partial v_k} + \frac{\partial^2 A_{jk}}{\partial v_k \partial v_i} - \frac{\partial^2 A_{ij}}{\partial v_k^2} - \frac{\partial^2 A_{kk}}{\partial v_i \partial v_j}, \tag{3.11}$$

where $(ijk) = (123)$*.*

Proof. We begin by noting that equation (3.8) can be written more concisely as $d *_3(Vdv) = 0$, where $v = (v_1, v_2, v_3)^t$ and $*_3$ is the flat Hodge star operator with respect to v. It follows that $*_3 Vdv$ is exact, i.e., $Vdv = *_3 d(Wdv)$ for some $W \in \Gamma(\mathcal{U}, M_3(\mathbb{R}))$. The symmetry of V is then

$$\frac{\partial W_{iq}}{\partial v_p} - \frac{\partial W_{ip}}{\partial v_q} = \frac{\partial W_{js}}{\partial v_r} - \frac{\partial W_{jr}}{\partial v_s} \qquad (jpq) = (123) = (irs).$$

For $i = j$ this relation is trivial. For $i \neq j$, order i and j and take k such that $(ijk) = (123)$. Then $p = k$, $q = i$, $r = j$, $s = k$, so the symmetry is

$$-\frac{\partial(W_{ii} + W_{jj})}{\partial v_k} + \frac{\partial W_{ik}}{\partial v_i} + \frac{\partial W_{jk}}{\partial v_j} = 0.$$

This is the same as

$$d *_3(\widetilde{W} dv) = 0,$$

where $\widetilde{W} = W^T - (\operatorname{tr} W)1_3$, which is a divergence-free condition. Thus $\widetilde{W} dv = *_3 d(Adv)$, for some $A \in \Gamma(\mathcal{U}, M_3(\mathbb{R}))$. It follows that A determines the symmetric matrix V. In detail, we have $\widetilde{W}_{ij} = \partial A_{iq}/\partial v_p - \partial A_{ip}/\partial v_q$, $(jpq) = (123)$, so using $W = \widetilde{W}^T - \frac{1}{2}(\operatorname{tr} \widetilde{W})1_3$, we get

$$\begin{aligned} V_{ij} = \frac{\partial W_{iq}}{\partial v_p} - \frac{\partial W_{ip}}{\partial v_q} = \frac{\partial}{\partial v_p}\Big(\frac{\partial A_{qs}}{\partial v_r} - \frac{\partial A_{qr}}{\partial v_s} - \tfrac{1}{2}\delta_{iq} \sum_{t=1}^{3} \big(\frac{\partial A_{tv}}{\partial v_u} - \frac{\partial A_{tu}}{\partial v_v}\big)\Big) \\ - \frac{\partial}{\partial v_q}\Big(\frac{\partial A_{ps}}{\partial v_r} - \frac{\partial A_{pr}}{\partial v_s} - \tfrac{1}{2}\delta_{ip} \sum_{t=1}^{3} \big(\frac{\partial A_{tv}}{\partial v_u} - \frac{\partial A_{tu}}{\partial v_v}\big)\Big), \end{aligned}$$

$(jpq) = (123) = (irs) = (tuv)$. To simplify this, consider separately the cases where $i = j$ and where $i \neq j$. First for $i = j$, we get $p = r$, $q = s$ distinct from i, so

$$V_{ii} = \frac{\partial^2 A_{rr}}{\partial v_s^2} + \frac{\partial^2 A_{ss}}{\partial v_r^2} - \frac{\partial^2 (A_{sr} + A_{rs})}{\partial v_r \partial v_s}.$$

For $i \neq j$, again rearrange and introduce k so that $(ijk) = (123)$. Then $p = k$, $q = i$, $r = j$, $s = k$, and

$$\begin{aligned} V_{ij} = {} & \frac{\partial}{\partial \nu_k}\Big(\frac{\partial A_{ik}}{\partial \nu_j} - \frac{\partial A_{ij}}{\partial \nu_k} - \tfrac{1}{2}\delta_{ii}\sum_{t=1}^{3}\big(\frac{\partial A_{tv}}{\partial \nu_u} - \frac{\partial A_{tu}}{\partial \nu_v}\big)\Big) \\ & - \frac{\partial}{\partial \nu_i}\Big(\frac{\partial A_{kk}}{\partial \nu_j} - \frac{\partial A_{kj}}{\partial \nu_k} - \tfrac{1}{2}\delta_{ik}\sum_{t=1}^{3}\big(\frac{\partial A_{tv}}{\partial \nu_u} - \frac{\partial A_{tu}}{\partial \nu_v}\big)\Big), \end{aligned}$$

which reduces to an expression that only depends on the symmetric part of A, so we may take A to be symmetric. □

Note that the right-hand side of (3.11) is not elliptic, so a rewriting of Theorem 3.5 looses ellipticity of that system. The papers [22, 21] contain a description of the kernel of $A \mapsto V(A)$.

3.2. **Digression: natural PDEs for toric G_2-manifolds.** As we have already seen, toric G_2-manifolds come with an associated action of $\mathrm{GL}(3,\mathbb{R})$. Thus a way of approaching equation (3.10), is to understand how L and Q transform with respect to this action.

The general linear group $\mathrm{GL}(3,\mathbb{R})$ acts by changing the basis of $\mathfrak{t}^3$ and so of $\xi_p \cong \mathbb{R}^3$, $p \in M_0$. It is useful to write $\mathrm{GL}(3,\mathbb{R}) \cong \mathbb{R}^\times \times \mathrm{SL}(3,\mathbb{R})$ and accordingly express irreducible representations in the form $\ell^p\Gamma_{a,b}$, where $\Gamma_{a,b}$ is an irreducible representation of $\mathrm{SL}(3,\mathbb{R})$ (see, e.g., [4]) and ℓ is the standard one-dimensional representation of $\mathbb{R}^\times \to \mathbb{R}\setminus\{0\}$ given by $t \mapsto t$. As an example, this means that we have for $p \in M_0$ that $\xi_p = \ell^1\Gamma_{0,1}$.

So let $U = (\mathbb{R}^3)^* = \ell^{-1}\Gamma_{1,0}$, viewed as a representation of $\mathrm{GL}(3,\mathbb{R})$. Then $V \in S^2(U) = \ell^{-2}\Gamma_{2,0}$. The collection of first order partial derivatives $V^{(1)} = (V_{ij,k}) = (\partial V_{ij}/\partial \nu_k)$ is then an element of $S^2(U) \otimes \ell^{-3}U^* = \ell^{-4}\Gamma_{2,0}\otimes\Gamma_{0,1}$. As a $\mathrm{GL}(3,\mathbb{R})$ representation this decomposes as

$$S^2(U)\otimes \ell^{-3}U^* = \ell^{-4}\Gamma_{1,0} \oplus \ell^{-4}\Gamma_{2,1},$$

with the projection to $\Gamma_{1,0}$ being just the contraction $S^2(\Gamma_{1,0})\otimes\Gamma_{0,1} \to \Gamma_{1,0}$, and $\Gamma_{2,1}$ denoting the kernel of this map. The divergence-free equation (3.8) just says this contraction is zero, so $V^{(1)} \in \ell^{-4}\Gamma_{2,1}$.

The operator Q is a symmetric quadratic operator on $V^{(1)}$ with values in $S^2(U)$. Thus we may think of $Q(dV)$ as an element of the space $\ell^6S^2(\Gamma_{2,1})^*\otimes S^2(\Gamma_{1,0})$. This space contains exactly one submodule isomorphic to ℓ^6 as $S^2(\Gamma_{1,0})^*$ is a submodule of $S^2(\Gamma_{2,1})^*$. Direct computations show that $Q(dV)$ belongs to ℓ^6.

Similarly, we may discuss the second order terms in (3.10). We have $V^{(2)} = (V_{ij,k\ell}) \in R = (S^2(U)\otimes S^2(\ell^{-3}U^*)) \cap (\ell^{-6}\Gamma_{2,1}\otimes\Gamma_{0,1})$. Now, ignoring the $\partial^2V/\partial\mu^2$ term, $L(V)$ is built from a product of V with $V^{(2)}$ and takes values in $S^2(U)$. So $L(V) \in S^2(U)^*\otimes R^*\otimes S^2(U)$. In this case, there are two submodules isomorphic to ℓ^6, but only one appears in $L(V)$, corresponding to the contractions

$$S^2(U^*)\otimes\Big(S^2(U^*)\otimes S^2(\ell^3U)\Big)\otimes S^2(U) \to \ell^6.$$

Contracting in this way is arguably the most natural choice.

Finally, addressing the terms of L involving $\partial^2 V/\partial\mu^2$, we have that $\partial/\partial\mu$ is an element of ℓ^{-3}, and therefore $\partial^2 V/\partial\mu^2$ belongs to $\ell^6 S^2(U)^* \otimes S^2(U)$. In fact, it is easy to see that $\partial^2 V/\partial\mu^2$ belongs to the one-dimensional summand isomorphic to ℓ^6 as we are tracing.

In conclusion, we have that L and Q are preserved up to scale by $\mathrm{GL}(3,\mathbb{R})$ change of basis, and this specifies Q uniquely.

Proposition 3.8. *Under the action of* $\mathrm{GL}(3,\mathbb{R})$*,* $L(V)$ *and* $Q(dV)$ *transform as elements of* ℓ^6*. Moreover, up to scaling,* Q *is the unique* $S^2(U)$*-valued quadratic form in* dV *with this property.* □

4. Behaviour near singular orbits

In our description of toric G_2-manifolds, we have so far been focusing on the regular part $M_0 \subset M$. We now turn to address what happens near a singular orbit for the T^3-action.

4.1. Flat models.

For a complete hyperKähler manifold with a tri-Hamiltonian action of T^n it is known that the hyperKähler moment map induces a homeomorphism $M/T^n \to \mathbb{R}^n$ (see [17, 42]). In this section, we establish the analogous result for toric G_2-manifolds for flat models with a singular orbit; later we will prove this in general. There are two cases to consider as the singular orbit can be either S^1 or T^2, corresponding to a stabiliser of dimension 2 or 1.

4.1.1. *Two-dimensional stabiliser.* Consider the flat model $M = S^1 \times \mathbb{C}^3$ equipped with the 3-form

$$\varphi = \tfrac{i}{2} dx \wedge (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3) + \mathrm{Re}(dz_1 \wedge dz_2 \wedge dz_3),$$

with dual 4-form

$$*\varphi = \mathrm{Im}(dz_1 \wedge dz_2 \wedge dz_3) \wedge dx - \tfrac{1}{8}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3)^2,$$

where $z_j = x_j + iy_j$, $j = 1,2,3$, are standard complex coordinates on $\mathbb{C}^3$.

There is a natural effective T^3-action on M: writing $T^3 = S^1 \times T^2$, the T^2 acts as a maximal torus of $\mathrm{SU}(3)$ on $\mathbb{C}^3$ and the remaining circle acts naturally on the S^1 factor. Correspondingly, we have generating vector fields given by

$$U_1 = \frac{\partial}{\partial x},\ U_2 = 2\,\mathrm{Re}\Big(i\Big(z_1\frac{\partial}{\partial z_1} - z_3\frac{\partial}{\partial z_3}\Big)\Big),\ U_3 = 2\,\mathrm{Re}\Big(i\Big(z_2\frac{\partial}{\partial z_2} - z_3\frac{\partial}{\partial z_3}\Big)\Big).$$

It follows that the matrix B is

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & |z_1|^2 + |z_3|^2 & |z_3|^2 \\ 0 & |z_3|^2 & |z_2|^2 + |z_3|^2 \end{pmatrix}$$

and so V takes the form

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (|z_2|^2 + |z_3|^2)/A & -|z_3|^2/A \\ 0 & -|z_3|^2/A & (|z_1|^2 + |z_3|^2)/A \end{pmatrix},$$

where $A = |z_1 z_2|^2 + |z_3 z_1|^2 + |z_2 z_3|^2$. We have that M_0 is the complement of the following sets: $M^{T^2} = S^1 \times \{0\}$ where the singular stabiliser is

$T^2 = \{1\} \times T^2 \leqslant S^1 \times T^2 = T^3$; $M^{S^1_i} = S^1 \times \{z_j = z_k = 0,\ z_i \neq 0\}$, $(ijk) = (123)$, which all have singular stabiliser circles $S^1_i \leqslant T^2 \leqslant T^3$.

For the multi-moment maps, we first compute

$$d\mu = U_1 \wedge U_2 \wedge U_3 \lrcorner *\varphi = d\,\mathrm{Im}(z_1z_2z_3),$$

giving that, up to addition of a constant, $\mu = \mathrm{Im}(z_1z_2z_3)$. Similarly, we find $\nu_1 = -\mathrm{Re}(z_1z_2z_3)$, from $U_2 \wedge U_3 \lrcorner \varphi$, and

$$d\nu_2 = U_3 \wedge U_1 \lrcorner \varphi = \tfrac{1}{2}d(|z_2|^2 - |z_3|^2).$$

So, again up to addition of a constant, $\nu_2 = \frac{1}{2}(|z_2|^2 - |z_3|^2)$. Finally, we have that $\nu_3 = -\frac{1}{2}(|z_1|^2 - |z_3|^2)$. Summarising, the multi-moment maps are

$$\nu_1 + i\mu = -\overline{z_1z_2z_3}, \quad \nu_2 = \tfrac{1}{2}(|z_2|^2 - |z_3|^2), \quad \nu_3 = -\tfrac{1}{2}(|z_1|^2 - |z_3|^2).$$

Proposition 4.1. *The multi-moment map $(\nu,\mu)\colon S^1 \times \mathbb{C}^3 \to \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$ induces a homeomorphism $(S^1 \times \mathbb{C}^3)/T^3 = \mathbb{C}^3/T^2 \to \mathbb{R}^4$.*

As the referee points out, this map $\mathbb{C}^3/T^2 \to \mathbb{R}^4$ has also been considered in [1].

Proof. Let us introduce some new variables. Putting $t = |z_3|^2$, we have $|z_1|^2 = t - a$, $|z_2|^2 = t - b$, where $a = 2\nu_3$ and $b = -2\nu_2$. For $c = |\mu|^2 + |\nu_1|^2 = |z_1|^2|z_2|^2|z_3|^2$, we have the relation

$$f(t) := t(t-a)(t-b) = c.$$

Note that f has zeros at 0, a and b. The constraints $|z_i|^2 \geqslant 0$, imply $t \geqslant x := \max\{0, a, b\}$. Now $f(t) \to \infty$ as $t \to \infty$, so $f([x,\infty)) = [0,\infty)$ and f is strictly monotone increasing on $[x,\infty)$. Thus $f(t) = c$ has a unique solution $t = t(a,b,c) \geqslant x$ for each $a, b \in \mathbb{R}$ and each $c \geqslant 0$.

Write $\rho\colon \mathbb{C}^3/T^2 \to \mathbb{R}^4$ for the map induced by (ν,μ). Given $(p,q) \in \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$, let $t = t(2p_3, -2p_2, q^2 + p_1^2)$, where $t(a,b,c)$ is defined above. Now $\rho(z_1,z_2,z_3) = (p,q)$ if and only if $(|z_1|^2, |z_2|^2, |z_3|^2) = (t - 2p_3, t + 2p_2, t)$ and $z_1z_2z_3 = (iq - p_1)$. One sees that these equations are consistent, ρ is surjective, and solutions are unique up to the action of $T^2 \leqslant \mathrm{SU}(3)$. Thus ρ is a continuous bijection $\mathbb{C}^3/T^2 \to \mathbb{R}^4$.

But $\mathbb{C}^3/T^2$ is homeomorphic to $\mathbb{R}^4$. Indeed, it follows from the results of [30] that S^5/T^2 is homeomorphic to S^3, so the claimed result follows by considering the cones on these spaces.

To be explicit, we note that $S^5 = \{(z_1,z_2,z_3) \mid |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\} = \{(t_1^{1/2}e^{iu}, t_2^{1/2}e^{iv}, t_3^{1/2}e^{iw}) \mid t_i \geqslant 0,\ t_1 + t_2 + t_3 = 1\}$ with T^2-action induced by $(e^{i\theta}, e^{i\phi}) \cdot (e^{iu}, e^{iv}, e^{iw}) = (e^{i(\theta+u)}, e^{i(\phi+v)}, e^{i(w-\theta-\phi)})$. Each T^2-orbit contains a representative with $u = v = w$. Furthermore, this representative is unique modulo $2\pi/3$ unless some t_i is zero, since $\theta + u = \phi + v = w - \theta - \phi \pmod{2\pi}$ implies the common value a satisfies $3a = u + v + w \pmod{2\pi}$ and each such a gives a unique solution for θ and ϕ mod 2π.

Topologically the two-simplex $\{(t_1,t_2,t_3) \mid t_i \geqslant 0,\ t_1 + t_2 + t_3 = 1\}$ is a unit disc $\{w \in \mathbb{C} \mid |w|^2 \leqslant 1\}$. The quotient S^5/T^2 has circle fibres over the interior of the disc that collapse to points on the boundary. Thus S^5/T^2 is topologically $\{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} = S^3$.

Now ρ is a continuous bijection $\mathbb{R}^4 = \mathbb{C}^3/T^2 \to \mathbb{R}^4$. By Brouwer's invariance of domain (see [35, Thm. 7.12]), it follows that ρ is a homeomorphism. □

4.1.2. *One-dimensional stabiliser.* The previous model contains points with stabiliser S^1, but we can also provide a simple standard model in this case. Let $M = (T^2 \times \mathbb{R}) \times \mathbb{C}^2$ with the 3-torus split as $T^3 = T^2 \times S^1$, the first T^2-factor acting on the corresponding torus in the first factor of M, and the S^1-factor acting as the maximal torus of $\mathrm{SU}(2)$ on $\mathbb{C}^2$. Introduce standard (local) coordinates x, y, u for $T^2 \times \mathbb{R}$ and (z, w) for $\mathbb{C}^2$.

The G_2 3-form may be written as

$$\begin{aligned}\varphi = du \wedge dx \wedge dy - du \wedge \tfrac{i}{2}(dz \wedge d\bar{z} + dw \wedge d\overline{w}) \\ - \operatorname{Re}((dx - idy) \wedge dz \wedge dw),\end{aligned}$$

with dual 4-form

$$\begin{aligned}*\varphi = \tfrac{1}{8}(dz \wedge d\bar{z} + dw \wedge d\overline{w})^2 + dx \wedge dy \wedge \tfrac{i}{2}(dz \wedge d\bar{z} + dw \wedge d\overline{w}) \\ + du \wedge \operatorname{Im}((dx - idy) \wedge dz \wedge dw).\end{aligned}$$

The generating vector fields are then

$$U_1 = \frac{\partial}{\partial x}, \quad U_2 = \frac{\partial}{\partial y}, \quad U_3 = -2\operatorname{Re}\Big(i\Big(z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w}\Big)\Big).$$

The matrix V is now

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/(|z|^2 + |w|^2) \end{pmatrix}.$$

We compute the multi-moment maps:

$$\begin{aligned}d\mu &= U_1 \wedge U_2 \wedge U_3 \lrcorner *\varphi = d(\tfrac{1}{2}(|z|^2 - |w|^2)), \\ d\nu_1 &= U_2 \wedge U_3 \lrcorner \varphi = d\operatorname{Re}(zw), \\ d\nu_2 &= U_3 \wedge U_1 \lrcorner \varphi = d\operatorname{Im}(zw), \\ d\nu_3 &= U_1 \wedge U_2 \lrcorner \varphi = du.\end{aligned}$$

Thus, we may take

$$\mu = \tfrac{1}{2}(|z|^2 - |w|^2), \quad \nu_1 + i\nu_2 = zw, \quad \nu_3 = u.$$

Note that, as expected, (μ, ν_1, ν_2) are just the standard hyperKähler moment maps for the action of S^1 on $\mathbb{H} = \mathbb{C}^2$. We know that this is essentially the Hopf fibration $S^3 \to S^2$ on distance spheres in $\mathbb{H} = \mathbb{R}^4$ and $\mathbb{R}^3$. Indeed

$$\mu^2 + \nu_1^2 + \nu_2^2 = \tfrac{1}{4}(|z|^4 - 2|z|^2|w|^2 + |w|^4) + |z|^2|w|^2 = \tfrac{1}{4}(|z|^2 + |w|^2)^2$$

so 3-spheres of radius r are mapped to 2-spheres of radius $r^2/2$. Again we get:

Proposition 4.2. *The multi-moment map $(\nu, \mu)\colon (T^2 \times \mathbb{R}) \times \mathbb{C}^2 \to \mathbb{R}^4$ induces a homeomorphism $((T^2 \times \mathbb{R}) \times \mathbb{C}^2)/T^3 = \mathbb{R} \times \mathbb{H}/S^1 \to \mathbb{R}^4$.* □

4.2. **Comparing with the flat models.** We now turn to general toric G_2-manifolds (M, φ). One way of obtaining a first feel for the behaviour of the multi-moment maps near singular stabilisers is by comparing with the flat models. In order to do so, it turns out useful to recall some basic facts about Killing fields.

4.2.1. *Killing vector fields.* If a vector field X on (M, g) is Killing, then this implies that ∇X is skew-adjoint, normalises the holonomy algebra and

$$\nabla^2_{A,B} X = -R_{X,A} B.$$

For the last result, cf. [33] (see also [7]), we use that X preserves the Levi-Civita connection,

$$[X, \nabla_A B] = \nabla_{[X,A]} B + \nabla_A [X, B] = \nabla_{[X,A]} B + \nabla_A \nabla_X B - \nabla_A \nabla_B X \quad (4.1)$$

to get

$$\begin{aligned} R_{X,A} B &= \nabla_X \nabla_A B - \nabla_A \nabla_X B - \nabla_{[X,A]} B = \nabla_X \nabla_A B - [X, \nabla_A B] - \nabla_A \nabla_B X \\ &= \nabla_{\nabla_A B} X - \nabla_A \nabla_B X = -\nabla^2_{A,B} X. \end{aligned}$$

It follows that at a zero p of X, we have $(\nabla^2 X)_p = 0$ and

$$\begin{aligned} (\nabla^3_{A,B,C} X)_p &= (-(\nabla_A (R_X))_B C)_p = (-(\nabla_A R)_{X,B} C - R_{\nabla_A X, B} C)_p \\ &= -(R_{\nabla_A X, B} C)_p. \end{aligned}$$

Note also that at such a p, the endomorphism $(\nabla X)_p$ on $T_p M$ gives the infinitesimal action of the one-parameter group generated by X.

If X and Y are two commuting Killing vector fields with $X_p = 0$, then we claim that the endomorphisms ∇X and ∇Y commute at p. To see this, let A be an arbitrary vector field. Then at p, we have $\nabla_X \cdot = 0$, so using (4.1) gives

$$\begin{aligned} [\nabla X, \nabla Y]_p (A) &= (\nabla_{\nabla_A Y} X - \nabla_{\nabla_A X} Y)_p = ([\nabla_A Y, X] - \nabla_{\nabla_A X} Y)_p \\ &= (\nabla_{[A,X]} Y + \nabla_A [Y, X] - \nabla_{\nabla_A X} Y)_p = (\nabla_{\nabla_X A} Y)_p = 0, \end{aligned}$$

as claimed.

Finally, for a vector field X preserving φ, we get that X is Killing and

$$\begin{aligned} 0 = L_X \varphi = d(X \lrcorner \varphi) &= \mathbf{a}\varphi(\nabla X, \cdot\,, \cdot\,) \\ &= \varphi(\nabla X, \cdot\,, \cdot\,) + \varphi(\,\cdot\,, \nabla X, \cdot\,) + \varphi(\,\cdot\,, \cdot\,, \nabla X), \end{aligned}$$

which shows that $\nabla X \in \mathfrak{g}_2$.

4.2.2. *Near points with two-dimensional stabiliser.* Let $p \in M$ be a point with $\mathrm{Stab}_{T^3}(p) \cong T^2$. We may identify $T_p M$ linearly with $\mathbb{R} \times \mathbb{C}^3 = T_{(1,0)}(S^1 \times \mathbb{C}^3)$ in the standard model of §4.1.1, so that the G_2-forms agree at this point. We have an equivariant diffeomorphism between a neighbourhood of $0 \in T_p M$ and a neighbourhood of $p \in M$ via the local tubular model $T^3 \times_{\mathrm{Stab}(p)} \mathbb{C}^3 \cong T^3/T^2 \times \mathbb{C}^3$, the map on the $\mathbb{C}^3$ part being given by the Riemannian exponential map. The elements of $\mathrm{Stab}(p)$ act on $\mathbb{R} \times \mathbb{C}^3$ linearly as a maximal torus in $SU(3)$. We may choose our linear identification so this is the standard diagonal subgroup and may choose our generators U_2, U_3 for $\operatorname{Stab}(p)$ so that

$$(\nabla U_2)_p = \operatorname{diag}(i, 0, -i), \quad (\nabla U_3)_p = \operatorname{diag}(0, i, -i)$$

in this model.

Let us now specify a choice of U_1. We note that the T^3-orbit of p is $T^3/\operatorname{Stab}(p) \times \{0\}$ in the local model. This orbit is the fixed point set of $\operatorname{Stab}(p)$, so is totally geodesic. For any U generating $T^3/\operatorname{Stab}(p)$, we thus have $(\nabla_U U)_p \in \mathbb{R}U$. But $(\nabla U)_p$ is an element of $\mathfrak{g}_2 \subset \mathfrak{so}(7)$, so $(\nabla_U U)_p = 0$. As the splitting $\mathbb{R} \times \mathbb{C}^3$ is orthogonal, it follows that $(\nabla U)_p \in \mathfrak{su}(3)$. Now each U_i vanishes at p, so the endomorphisms $(\nabla U_i)_p$ commute with $(\nabla U)_p$, by §4.2.1. As $(\nabla U_2)_p$, $(\nabla U_3)_p$ generate a maximal torus of $\mathfrak{su}(3)$, it follows that $(\nabla U)_p = a(\nabla U_2)_p + b(\nabla U_3)_p$, for some $a, b \in \mathbb{R}$. Putting $U_1 = U - aU_2 - bU_3$, we still have that U_1 generates $T^3/\operatorname{Stab}(p)$ and get $(\nabla U_1)_p = 0$. If we wish, we may assume that $(U_1)_p$ is of length 1.

Now consider the multi-moment maps. For ν_2, we have

$$(\nabla \nu_2)_p = (d\nu_2)_p = (U_3 \wedge U_1 \lrcorner \varphi)_p = 0,$$

since $(U_3)_p = 0$. Similarly $\nabla \nu_3 = 0 = \nabla \nu_1 = \nabla \mu$ at p. Furthermore,

$$\begin{aligned}(\nabla^2 \nu_2)_p &= \big((\nabla\varphi)(U_3, U_1, \cdot\,) + \varphi(\nabla U_3, U_1, \cdot\,) + \varphi(U_3, \nabla U_1, \cdot\,)\big)_p \\ &= \varphi(\nabla U_3, U_1, \cdot\,)_p\end{aligned}$$

agrees with the flat model at p. Similarly for $(\nabla^2 \nu_3)_p$. For ν_1, we have

$$(\nabla^2 \nu_1)_p = \big(\varphi(\nabla U_2, U_3, \cdot\,) + \varphi(U_2, \nabla U_3, \cdot\,)\big)_p = 0,$$

as both U_2 and U_3 vanish at p. Similarly, $(\nabla^2 \mu)_p = 0$.

For third order derivatives, we have

$$(\nabla^3 \nu_2)_p = \big(\varphi(\nabla^2 U_3, U_1, \cdot\,) + 2\varphi(\nabla U_3, \nabla U_1, \cdot\,) + \varphi(U_3, \nabla^2 U_1, \cdot\,)\big)_p = 0,$$

since $(\nabla^2 U_3)_p = 0$ by §4.2.1, and $(\nabla U_1)_p = 0$ by our choice of U_1. Similarly, $(\nabla^3 \nu_3)_p = 0$. On the other hand,

$$\begin{aligned}(\nabla^3 \nu_1)_p &= \big(\varphi(\nabla^2 U_2, U_3, \cdot\,) + 2\varphi(\nabla U_2, \nabla U_3, \cdot\,) + \varphi(U_2, \nabla^2 U_3, \cdot\,)\big)_p \\ &= 2\varphi(\nabla U_2, \nabla U_3, \cdot\,)_p,\end{aligned}$$

which agrees with the flat model, as does $(\nabla^3 \mu)_p$.

Let us now compute fourth order derivatives. Firstly,

$$\begin{aligned}(\nabla^4 \nu_2)_p &= \big(\varphi(\nabla^3 U_3, U_1, \cdot\,) + 3\varphi(\nabla^2 U_3, \nabla U_1, \cdot\,) \\ &\qquad + 3\varphi(\nabla U_3, \nabla^2 U_1, \cdot\,) + \varphi(U_3, \nabla^3 U_1, \cdot\,)\big)_p \\ &= \varphi(\nabla^3 U_3, U_1, \cdot\,)_p + 3\varphi(\nabla U_3, \nabla^2 U_1, \cdot\,)_p \\ &= -\varphi(R_{\nabla U_3, \cdot}\,\cdot\,, U_1, \cdot\,)_p - 3\varphi(\nabla U_3, R_{U_1, \cdot}\,\cdot\,, \cdot\,)_p,\end{aligned}$$

with a similar expression for $(\nabla^4 \nu_3)_p$. For ν_1 and μ, the same type of computation gives $(\nabla^4 \nu_1)_p = 0 = (\nabla^4 \mu)_p$. In conclusion, we have shown:

Lemma 4.3. *Let $p \in M$ be a point with stabiliser T^2 whose infinitesimal generators are U_2, U_3. Then the multi-moment maps ν_2, ν_3 agree with the flat model to order 3 and ν_1, μ agree with the flat model to order 4.* □

4.2.3. *Near points with one-dimensional stabiliser.* In this case, we need less detailed information. Let $p \in M$ have $\mathrm{Stab}_{T^3}(p) \cong S^1$. We take the infinitesimal generator for this stabiliser to be U_3. Let U_1 and U_2 be two vector fields of the T^3 action that generate the quotient $T^3/\operatorname{Stab}(p) \cong T^2$. We take them to be of unit length and orthogonal at p. Then U_1 and U_2 are invariant under U_3 as is their G_2-cross-product $U_1 \times U_2 = \varphi(U_1, U_2, \cdot\,)^\sharp$. We have $T_pM = \mathbb{R}^3 \times \mathbb{C}^2$ linearly, with $\mathbb{R}^3 = \langle U_1, U_2, U_1 \times U_2 \rangle_p$ and $\mathbb{C}^2$ the orthogonal complement. This identification may be chosen so that $(\nabla U_3)_p$ acts as the element $\mathrm{diag}(i, -i)$ in $\mathfrak{su}(2)$ on $\mathbb{C}^2$. The local model is $T^3 \times_{\mathrm{Stab}(p)} (\mathbb{R} \times \mathbb{C}^2) \cong (T^2 \times \mathbb{R}) \times \mathbb{C}^2$, with $T^2 \times \mathbb{R} \times \{0\}$ the fixed point set of U_3, so totally geodesic. Now $d\nu_3 = (U_1 \times U_2)^\flat$ is non-zero and therefore provides a transverse coordinate to a six-dimensional level set, and $d\nu_1 = 0 = d\nu_2 = d\mu$ are zero at p. The three second derivatives $\nabla^2\nu_1$, $\nabla^2\nu_2$ and $\nabla^2\mu$ are specified by U_i, $i = 1, 2$, and ∇U_3 at p and so all agree with the standard flat model at p.

4.2.4. *Images of singular orbits.* First consider a point p with stabiliser S^1. The previous section provides an integral basis U_1, U_2, U_3 of $\mathfrak{t}^3$ with $(U_3)_p = 0$. Furthermore, this is true for all points of $T^2 \times \mathbb{R}$ in the local model. It follows that ν_1, ν_2 and μ are constant on this set, and so the image under (ν, μ) of this family of singular orbits is a straight line parameterised by the values of ν_3.

Now for points p with T^2-stabiliser, these lie on a circle T^3p. The normal bundle is modelled on $\mathbb{C}^3$ and there are three families of points with stabiliser S^1. These families meet at p and correspond to the complex coordinate axes in $\mathbb{C}^3$. There is thus an integral basis U_1, U_2, U_3 of $\mathfrak{t}^3$ with $U_2 = 0 = U_3$ at p and such that U_2, U_3 and $-U_2 - U_3$ generate the S^1 stabilisers of the three families. The images of the families under (ν, μ) all have the same constant μ- and ν_1-coordinates, and provide the three half-lines meeting at the image of p lying in ν_3, ν_2 or $(\nu_2 - \nu_3)$ constant.

Summarising, we have:

Lemma 4.4. *For $p \in M \setminus M_0$, we have $\operatorname{rank} B_p \leqslant 2$. The image in M/T^3 of the union $M \setminus M_0$ of singular orbits consists of trivalent graphs lying in sets $\mu =$ constant with edges that are straight lines of rational slope in the ν-coordinates. At each vertex the three primitive integral slope vectors sum to zero, in particular these edges lie a plane.* □

4.3. **Deforming to the flat model.** Let φ be a torsion-free G_2-structure on the ball $B_2(0) \subset \mathbb{R}^7$ with centre 0 and radius 2. Choose linear coordinates $(x_1, \dots, x_7)$ on $\mathbb{R}^7$ so that $\varphi|_0 = \varphi_0|_0$, where φ_0 is the standard constant coefficient G_2-form. Our aim is to construct a family of torsion-free G_2-structures φ_t, $t \in (0, 1]$, with $\varphi_1 = \varphi$, and with φ_t converging to φ_0 on $\overline{B_1(0)}$ in each C^k-norm.

For $t \in (0,1]$, define a linear diffeomorphism $\lambda_t\colon \mathbb{R}^7 \to \mathbb{R}^7$ by $\lambda_t(x) = tx$. Note that $\lambda_t^*\varphi_0 = t^3\varphi_0$, so let us take φ_t to be

$$\varphi_t = t^{-3}\lambda_t^*\varphi, \qquad \text{for } t \in (0,1].$$

We have $\varphi = \varphi_0 + \psi$ where $\psi \in \Omega^3(B_2(0))$ is smooth and has $\psi|_0 = 0$. It follows that

$$\psi = \sum_{|I|=3} f_I dx_I,$$

where $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge dx_{i_3}$, for $I = (i_1, i_2, i_3) \in \{1, \dots, 7\}^3$, and f_I is smooth with $f_I(0) = 0$. We may therefore write $f_I(x) = \sum_{k=1}^7 x_k h_{I,k}(x)$ with $h_{I,k}$ smooth. We have $\lambda_t^*\psi = \sum_I (\lambda_t^* f_I) t^3 dx_I$ and $(\lambda_t^* f_I)(x) = \sum_k t x_k h_{I,k}(tx)$, so $\|\lambda_t^* f_I\|_{C^0} \leqslant t\|f_I\|_{C^0}$. Thus putting $\psi_t = t^{-3}\lambda_t^*\psi$, so $\varphi_t = \varphi_0 + \psi_t$, we get $\|\psi_t\|_{C^0} \leqslant t\|\psi\|_{C^0}$. Thus $\varphi_t \to \varphi_0$ in $C^0(\overline{B_1(0)})$ as $t \searrow 0$.

The Riemannian metric g_t defined by φ_t satisfies

$$g_t = t^{-2}\lambda_t^* g,$$

where $g = g_1$. The same types of computations as above show that $g_t \to g_0 = \sum_{i=1}^7 dx_i^2$ in C^0 as $t \searrow 0$. Let ∇^t be the Levi-Civita connection of g_t and write its Christoffel symbols as $(\Gamma_t)_{ij}^k$. We claim that $\nabla^t \to \nabla^0$, meaning that $(\Gamma_t)_{ij}^k \to 0$, as $t \searrow 0$.

We have

$$(g_t)_{ij}(x) = \delta_{ij} + t\sum_{k=1}^7 x_k h_{ijk}(tx)$$

for some smooth functions h_{ijk}. Thus

$$\frac{\partial}{\partial x_\ell}(g_t)_{ij}(x) = t h_{ij\ell}(tx) + t^2 \sum_{k=1}^7 x_k \frac{\partial h_{ijk}}{\partial x_\ell}(tx)$$

and

$$(g_t^{-1})_{ij}(x) = \delta_{ij} + t\sum_{k=1}^7 x_k \tilde{h}_{ijk}(tx),$$

for some smooth functions $\tilde{h}_{ijk}$. This gives

$$2(\Gamma_t)_{ij}^k(x) = t(h_{ij\ell} + h_{ji\ell} - h_{\ell ij})(tx) + O(t^2)$$

and hence $(\Gamma_t)_{ij}^k \to 0$, as claimed.

Now note that $0 = \nabla^t\varphi_t = \nabla^t\varphi_0 + \nabla^t\psi_t$, so $\nabla^t\psi_t = -\nabla^t\varphi_0 \to 0$ in C^0 as $t \searrow 0$. It follows that $\varphi_t \to \varphi_0$ in C^1. Iterating, noting that each derivative adds an extra factor of t, we get the claimed convergence in C^k.

If U is a linear symmetry of $\mathbb{R}^7$ that preserves φ, then it is also a symmetry of φ_t, since U commutes with dilations. Furthermore, if $X = \sum_{i=1}^7 v_i \partial/\partial x_i$ is a constant coefficient vector field preserving φ then it is also a symmetry of φ_t. Indeed, the one-parameter group generated by X is $T_s(x) = x + sv$. Now, for any $f \in C^\infty(V)$ we have $(\lambda_t)_* X = tX$. This gives

$$\begin{aligned} L_X\varphi_t &= t^{-3} L_X \lambda_t^*\varphi = t^{-3}\bigl(X \lrcorner d\lambda_t^*\varphi + d(X \lrcorner \lambda_t^*\varphi)\bigr) \\ &= t^{-2}\lambda_t^* L_X\varphi = 0, \end{aligned}$$

which is the claimed symmetry.

Note that we now also get that the multi-moment maps converge to those of flat space as $t \searrow 0$.

4.4. **Identifications of the quotients.** Consider a compact group G acting linearly on a finite-dimensional vector space V. A main result of [40], cf. [38], is that any smooth G-invariant function is necessarily a smooth function of any set of generators for the ring of G-invariant polynomials on V. Suppose $\sigma_1, \dots, \sigma_k$ is a minimal set of such polynomial generators, meaning that no subset generates. Then the statement gives that σ induces a diffeomorphism of $V/\,\mathrm{G}$ with $\sigma(V) \subset \mathbb{R}^k$ with respect to the 'smooth structures': a function on $V/\,\mathrm{G}$ is smooth if its pull-back to V is smooth; a function on $\sigma(V)$ is smooth if it has local extensions to smooth functions in open $\mathbb{R}^k$-neighbourhoods of each point.

In our cases we are interested in two models:

(i) $\mathrm{G} = S^1$ acting on $V = \mathbb{R}^4 = \mathbb{C}^2$ as a maximal torus in $\mathrm{SU}(2)$, and
(ii) $\mathrm{G} = T^2$ action on $V = \mathbb{R}^6 = \mathbb{C}^3$ as a maximal torus in $\mathrm{SU}(3)$.

Let us consider each of these in turn. For (i), let (z, w) be standard complex coordinates. Then S^1 acts as $e^{i\theta}(z, w) = (e^{i\theta} z, e^{-i\theta} w)$. The invariant polynomials are generated by $(\sigma_1, \dots, \sigma_4)$:

$$\sigma_1 + i\sigma_2 = zw, \quad \sigma_3 = \tfrac{1}{2}(|z|^2 - |w|^2), \quad \sigma_4 = \tfrac{1}{2}(|z|^2 + |w|^2).$$

Note that these satisfy the relations

$$\sigma_4 \geqslant 0, \qquad \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = \sigma_4^2. \tag{4.2}$$

For (ii), write (z_1, z_2, z_3) for the standard coordinates in the flat model, as above. This time the ring of polynomial invariants is generated by five elements

$$\sigma_1 + i\sigma_2 = -\overline{z_1 z_2 z_3}, \quad \sigma_3 = \tfrac{1}{2}(|z_2|^2 - |z_3|^2), \quad \sigma_4 = \tfrac{1}{2}(|z_3|^2 - |z_1|^2),$$
$$\sigma_5 = |z_3|^2,$$

satisfying the relations

$$\sigma_5 \geqslant \max\{0, -2\sigma_3, 2\sigma_4\}, \qquad \sigma_1^2 + \sigma_2^2 = \sigma_5(\sigma_5 + 2\sigma_3)(\sigma_5 - 2\sigma_4). \tag{4.3}$$

We have chosen our generators in such a way that $\sigma_1, \dots, \sigma_{k-1}$ correspond to the relevant multi-moment maps in the flat models. Our work in §4.1 on the flat models shows that in both cases the map $\sigma(V) \to \mathbb{R}^{k-1}$ given by $(\sigma_1, \dots, \sigma_{k-1}, \sigma_k) \mapsto (\sigma_1, \dots, \sigma_{k-1})$ is a homeomorphism. For the non-flat cases, we have the multi-moment maps giving us invariant functions that agree with $\sigma_1, \dots, \sigma_{k-1}$ to certain orders. As Schwarz gives that $V/\,\mathrm{G}$ is diffeomorphic to $\sigma(V)$, the aim is now to show that these still give homeomorphisms to $\sigma(V) \to \mathbb{R}^{k-1}$. For the case of one-dimensional stabilisers this is what [8] does, albeit in a hyperKähler context, but the local model is the same. We discuss this briefly as preparation for the six-dimensional case.

For the four-dimensional model we may proceed as follows. Let V denote the slice with its S^1 action. Write $\pi\colon V \to V/S^1$ for the projection. Use $W = \mathbb{R}^4 = U \times \mathbb{R}$ with $U = \mathbb{R}^3$. Let F_0 be the linear projection $W \to U$. Let $S = \sigma(V) \subset W$ be the semi-algebraic set given by (4.2).

On the four-dimensional slice V, we have (restrictions of) the multi-moment map functions ν_1, ν_2 and μ. Collect these into a single function $m = (\nu_1, \nu_2, \mu)\colon V \to \mathbb{R}^3$. This is a smooth invariant function, so by Schwarz it is induced by a smooth function on S. Write

$$m = f \circ \sigma, \qquad f\colon S \to \mathbb{R}^3.$$

Note that f smooth means it extends to a smooth function in a neighbourhood of any given point; we use the same name for a choice of such smooth extension in a neighbourhood of $0 \in W$.

By §4.2.3, we know that the first two covariant derivatives at the origin of ν_1, ν_2 and μ agree with those of σ_1, σ_2 and σ_3, respectively. So m agrees with $m_0 = (\sigma_1, \sigma_2, \sigma_3)$ to order 2 near the origin and $f = F_0 + \tilde{f}$ with $\tilde{f}$ smooth. In the slice coordinates at the origin, $\tilde{f} \circ \sigma$ vanishes to order 2 and all the σ_i have degree 2, so $\tilde{f}$ vanishes to order 1 in σ. In other words

$$\tilde{f}(\sigma) = \sum_{i,j=1}^{4} \sigma_i \sigma_j f_{ij}(\sigma),$$

where each f_{ij} is smooth. In particular, the derivative of $\tilde{f}$ has norm bounded above by $c\|\sigma\|$ on this neighbourhood and the mean value theorem gives

$$\|\tilde{f}(x) - \tilde{f}(y)\| \leqslant c(\|x\| + \|y\|)\|x - y\|. \tag{4.4}$$

Consider points q_1 and q_2 in the slice near near the fixed point $p = 0$. Write $x = \sigma(q_1)$, $y = \sigma(q_2)$. Then

$$\begin{aligned} \|m(q_1) - m(q_2)\| &= \|f(x) - f(y)\| = \|F_0(x) - F_0(y) + \tilde{f}(x) - \tilde{f}(y)\| \\ &\geqslant \|F_0(x) - F_0(y)\| - c(\|x\| + \|y\|)\|x - y\|. \end{aligned} \tag{4.5}$$

But $F_0^{-1}(a) = (a, \|a\|) \in S$ and

$$\begin{aligned} \|x - y\| &= \big\|(F_0(x), \|F_0(x)\|) - (F_0(y), \|F_0(y)\|)\big\| \\ &\leqslant 2\|F_0(x) - F_0(y)\| \end{aligned}$$

gives

$$\begin{aligned} \|m(q_1) - m(q_2)\| &\geqslant \tfrac{1}{2}\|x - y\| - c(\|x\| + \|y\|)\|x - y\| \\ &\geqslant \big(\tfrac{1}{2} - c(\|x\| + \|y\|)\big)\|x - y\|. \end{aligned}$$

So for $\|x\|, \|y\| \leqslant 1/(8c)$, we have $\|m(q_1) - m(q_2)\| \geqslant \|x - y\|/4$, proving that m is injective on orbits in a neighbourhood of the origin. Invoking Brouwer's invariance of domain, gives that m induces a homeomorphism of the quotient space in a neighbourhood of the origin.

Let us turn to the six-dimensional models. Let V be the slice with its T^2 action and write $\pi\colon V \to V/T^2$ for the projection map. Let $W = \mathbb{R}^5 = U \times \mathbb{R}$ with $U = \mathbb{R}^4$ and write $F_0\colon W \to U$ for the linear projection. The vector space W contains the semi-algebraic set $S = \sigma(V)$ given by (4.3). Write $m = (\nu_1, \mu, \nu_2, \nu_3)\colon V \to \mathbb{R}^4$ for the collection of multi-moment maps. By Schwarz, $m = f \circ \sigma$ for a smooth $f\colon S \to \mathbb{R}^4$. On V, the first four derivatives of ν_1 and μ, and the first three derivatives of ν_2 and ν_3, agree with those of σ_1, σ_2, σ_3 and σ_4, respectively. Noting that any homogeneous polynomial in σ_i of degree 2 is at least of degree 4 in the $z_i, \overline{z_i}$, we thus have $f = F_0 + \tilde{f}$ with

$$\tilde{f}(\sigma) = \sum_{i,j=1}^{5} \sigma_i \sigma_j f_{ij}(\sigma)$$

for some smooth functions f_{ij}. This gives the estimates (4.4) and (4.5) on some neighbourhood S_0 of $0 \in S$.

Now consider points $x = \sigma(q)$ satisfying (4.3). To estimate x_5, note that

$$x_5(x_5 + 2x_3)(x_5 - 2x_4) \geqslant (x_5 - \max\{0, -2x_3, 2x_4\})^3$$

so, as $x_5 \geqslant 0$, we have

$$\begin{aligned} |x_5| &\leqslant (x_1^2 + x_2^2)^{1/3} + \max\{0, -2x_3, 2x_4\} \\ &\leqslant (x_1^2 + x_2^2)^{1/3} + 2(x_3^2 + x_4^2)^{1/2}. \end{aligned}$$

For $\|(x_1, x_2, x_3, x_4)\| < 1$, we have

$$\begin{aligned} \|x\| = \|(F_0(x), x_5)\| &\leqslant \|F_0(x)\| + |x_5| \\ &\leqslant \|F_0(x)\| + \|F_0(x)\|^{2/3} + 2\|F_0(x)\| \\ &\leqslant \|F_0(x)\|^{2/3}(3\|F_0(x)\|^{1/3} + 1) \leqslant 4\|F_0(x)\|^{2/3}. \end{aligned}$$

So on $S_0 \cap B_1(0)$ this gives

$$\begin{aligned} \|m(q)\| = \|f(x)\| &\geqslant \|F_0(x)\| - c\|x\|^2 \\ &\geqslant \left(\tfrac{1}{4}\|x\|\right)^{3/2} - c\|x\|^2 = \|x\|^{3/2}\left(\tfrac{1}{8} - c\|x\|^{1/2}\right). \end{aligned}$$

Thus for $\|x\| \leqslant 1/(256c^2)$ we have that $\|m(q)\| > \|x\|^{3/2}/16$. This implies that 0 is the only point in this neighbourhood $W_0 = \{x \in S_0 \cap B_1(0) \mid \|x\| < 1/(256c^2)\}$ that maps to 0 under m.

Now consider a family φ_t of T^3-invariant torsion-free G_2-structures on $S^1 \times \sigma^{-1}(W_0)$ with $\varphi_1 = \varphi$, the structure we are interested in, and φ_0 the flat G_2-structure that coincides with φ at 0. Such a family was constructed in §4.3 and the discussion there shows that $f_t \to f_0 = F_0$ as $t \searrow 0$. Moreover the bound c_t above for f_t also has $c_t \searrow 0$ and in particular $c = c_1 \geqslant c_t$ for all $t < 1$.

Let us consider the Brouwer degrees of these maps, cf. [39, 18]: let $W_1 \subset\subset W_0$ be an open ball containing 0; for $f\colon W_0 \to \mathbb{R}^4$ of class C^2 the *Brouwer degree* is

$$d_B[f, W_1] = \int_{W_1} \chi(\|f(x)\|) J_f(x)\, dx,$$

where $J_f = \det Df$ is the Jacobian of f and $\chi\colon [0,\infty) \to [0,\infty)$ is continuous, has the closure of its support contained in $(0, \inf_{x \in \partial W_1} \|f(x)\|)$ and satisfies $\int_{\mathbb{R}^4} \chi(\|x\|)\, dx = 1$. This definition extends to continuous functions f by approximating them uniformly via smooth functions, and the degree is homotopy invariant; it agrees with the topological degree of the map $f/\|f\|\colon \partial W_1 \to S^3$. For $z \notin f(\partial W_1)$, the Brouwer degree of f at z is $d_B[f, W_1, z] = d_B[f(\,\cdot\,) - z, W_1]$. At regular values z, the number $d_B[f, W_1, z]$ counts the points x in $f^{-1}(z) \cap W_1$ with the signs of $J_f(x)$. Any homeomorphism has $d_B[f, W_1, z] = \pm 1$.

Now $F_0 = f_0$ is a homeomorphism $S \to \mathbb{R}^4$ and has degree $+1$ at all points. Furthermore, S is the set set of $(\sigma_1, \dots, \sigma_5) \in \mathbb{R}^5$ satisfying (4.3). Differentiating this equation we have

$$p_5 d\sigma_5 = \sum_{i=1}^{4} p_i d\sigma_i$$

with

$$\begin{aligned} p_1 = 2\sigma_1, \quad p_2 = 2\sigma_2, \quad p_3 = -2\sigma_5(\sigma_5 - 2\sigma_4), \quad p_4 = 2\sigma_5(\sigma_5 + 2\sigma_3), \\ p_5 = (\sigma_5 + 2\sigma_3)(\sigma_5 - 2\sigma_4) + \sigma_5(\sigma_5 - 2\sigma_4) + \sigma_5(\sigma_5 + 2\sigma_3) \\ = |z_1 z_2|^2 + |z_3 z_1|^2 + |z_2 z_3|^2, \end{aligned}$$

where (z_1, z_2, z_3) are the coordinates on $V = \mathbb{C}^3$. In particular, σ_5 is a smooth function of $(\sigma_1, \dots, \sigma_4)$ off the locus $p_5 = 0$ which is the image of the set on which two of the z_i are zero, i.e., the image of the complex coordinate axes of V. But this is just the locus of points with T^3-stabiliser of dimension at least 1 and so is specified purely by the group action. Off this locus dm_t has rank 4 and so the same is true of df_t. In particular, off this locus df_t is a local diffeomorphism. Furthermore, on the locus but away from 0, we have S^1-stabilisers and from the four-dimensional models we know that f_t is a local homeomorphism.

Now homotopy invariance combined with the fact that $f_t^{-1}(0) \cap W_1 = \{0\}$ implies that each f_t has degree $+1$ and at smooth points the local degrees are also $+1$. It follows that on the smooth locus inside W_1 the maps f_t are one-to-one for all $t \in [0, 1]$. However, the image $(p_5 = 0) \setminus \{0\}$ consists of three half-lines each determined the group action, in particular by which copy of $S^1 \subset T^2$ is the corresponding stabiliser. On this set m_t is still a local homeomorphism and so is monotone on each half-line. As f_t is local homeomorphism it follows that the local degrees at these points are also $+1$. Thus f_t is injective on W_1. Using Brouwer's invariance of domain, we conclude that f is a homeomorphism from W_1 to a neighbourhood of $0 \in \mathbb{R}^4$.

Summarising the above analysis, we have shown:

Theorem 4.5. *Let (M, φ) be a toric G_2-manifold. Then M/T^3 is homeomorphic to a smooth four-manifold. Moreover, the multi-moment map (ν, μ) induces a local homeomorphism $M/T^3 \to \mathbb{R}^4$.* □

5. Explicit examples of toric G_2-manifolds

We now turn to write down some explicit examples of toric G_2-manifolds.

5.1. **Some complete examples.**

In this section, we describe some known non-flat complete examples of toric G_2-manifolds.

5.1.1. *Holonomy* $\mathrm{SU}(3)$*:* $M = S^1 \times T^*S^3$. Before turning to a concrete example, it seems worthwhile explaining how it arises as a particular case of a more general construction of toric G_2-manifolds with holonomy in $\mathrm{SU}(3)$. So assume we have a 6-manifold N with vanishing first Betti number and equipped with a Calabi-Yau structure (σ, Ψ). If there is an effective

T^2-action on N preserving σ and $\Psi = \psi + i\hat{\psi}$, then we have invariant scalar functions $(\nu, \mu)\colon N \to \mathbb{R}^4$ that satisfy the relations

$$d\nu_1 = \psi(U_2, U_3, \,\cdot\,), \quad d\nu_2 = -\sigma(U_3, \,\cdot\,), \quad d\nu_3 = \sigma(U_2, \,\cdot\,),$$
$$d\mu = -\hat{\psi}(U_2, U_3, \,\cdot\,),$$

where U_2, U_3 are generators for the torus action. We can now consider the torsion-free product G_2-structure on $M = S^1 \times N$ given by

$$\varphi = dx \wedge \sigma + \psi, \quad *\varphi = \hat{\psi} \wedge dx + \tfrac{1}{2}\sigma^2.$$

Clearly, (M, φ) is toric with $T^3 = S^1 \times T^2$ acting in the obvious way and associated multi-moment maps (ν, μ). Theorem 4.5 now implies that N/T^2 is locally homeomorphic to $\mathbb{R}^4$ and Lemma 4.4 implies that the trivalent graphs lie in the surfaces (ν_1, μ) constant.

For (N, σ, Ψ) as above there is a special Lagrangian foliation (of an open dense subset) with T^2-symmetry. The leaves are given by fixing (ν_2, ν_3, μ) to be constant. The corresponding distribution is given by the kernel of $d\mu \wedge d\nu_{23}$, and the restriction of ψ to each leaf is $\theta_{23} \wedge d\nu_1$.

As a concrete example of the above, one can take $N = T^*S^3$ with its Stenzel Calabi-Yau structure [41]. For our purposes, it is more convenient to identify N with the complex sphere

$$Q = \Big\{ z \in \mathbb{C}^4 \,\Big|\, \sum_{j=0}^{3} z_j^2 = 1 \Big\},$$

following [31]. Specifically, one has the SO(4)-equivariant diffeomorphism

$$T^*S^3 \ni (p, v) \mapsto \cosh(\|v\|)p + i\sinh(\|v\|)\frac{v}{\|v\|} \in Q,$$

(see [45]). In terms of Q, the Kähler 2-form is given by $\sigma = d\alpha$, where

$$\alpha(X)_z = \tfrac{1}{2} f'(|z|^2)\,\mathrm{Im}(X^t \bar{z}), \qquad X \in T_z Q,\ z \in Q,$$

with f satisfying the following differential equation:

$$((f_u)^3)_u = 3k(\sinh u)^2,$$

for some constant $k > 0$. The holomorphic volume form can be computed as

$$\Psi(X_1, X_2, X_3)_z = dz_{0123}(z, X_1, X_2, X_3),$$

for $X_1, X_2, X_3 \in T_z Q$ and $z \in Q$.

For the T^2-action, we consider $T^2 \subset \mathrm{SO}(4)$ generated by the vector fields

$$U_2(z) = (-z_1, z_0, 0, 0), \quad U_3(z) = (0, 0, -z_3, z_2).$$

In accordance with [31, Thm. 5.2] one finds that the multi-moment maps are

$$\nu_1 + i\mu = \tfrac{1}{2}(\bar{z}_0^2 + \bar{z}_1^2), \quad \nu_2 = -f'(|z|^2)\,\mathrm{Im}(z_2\bar{z}_3), \quad \nu_3 = f'(|z|^2)\,\mathrm{Im}(z_0\bar{z}_1).$$

Many other examples are to be found in [1, 34] and related works.

5.1.2. *The cone over $S^3 \times S^3$ and its deformation.* As mentioned in §2.2, one example of a complete toric G_2-manifold with holonomy equal to G_2 is the spin bundle over S^3 equipped with its Bryant-Salamon structure. It may be viewed as a deformation of the cone over $S^3 \times S^3$ with its nearly Kähler structure. In both cases, one can describe the G_2-structure in terms of one-parameter families of left-invariant half-flat $SU(3)$-structures on $S^3 \times S^3 \cong Sp(1) \times Sp(1) \subset \mathbb{H} \times \mathbb{H}$.

To make this concrete, let us take $\{(i,0),(j,0),(-k,0),(0,i),(0,j),(0,-k)\}$ as our basis of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \cong T_1(S^3 \times S^3)$. Correspondingly, the tangent space at $(p,q) \in S^3 \times S^3$ has basis

$$\begin{aligned} E_1(p,q) &= (pi,0), \quad E_2(p,q) = (pj,0), \quad E_3(p,q) = (-pk,0), \\ F_1(p,q) &= (0,qi), \quad F_2(p,q) = (0,qj), \quad F_3(p,q) = (0,-qk). \end{aligned} \tag{5.1}$$

If we let $e^1, \dots, f^3$ denote the dual co-frame, then $de^i = 2e^{jk}$ and $df^i = 2f^{jk}$, $(ijk) = (123)$.

We have an almost effective action of $Sp(1)^3$ on $S^3 \times S^3$ given by

$$((h,k,\ell),(p,q)) \mapsto (hp\ell^{-1}, kq\ell^{-1})$$

that preserves the half-flat $SU(3)$-structures of interest (cf. [15]). By choosing a maximal torus S^1 in each $Sp(1)$, we obtain an almost effective action of T^3. Considering the quotient of T^3 by $\mathbb{Z}_2 = \{\pm(1,1,1)\}$, we get an effective action of a torus T^3. For concreteness, let us choose each maximal torus $T^1 \subset Sp(1)$ to be of the form $\{e^{i\theta} \mid \theta \in \mathbb{R}\}$. In this case, we have generating vector fields given by

$$U_1(p,q) = (ip,0), \quad U_2(p,q) = (0,iq), \quad U_3(p,q) = (-pi,-qi).$$

Following [19], we can express these vector fields in terms of (5.1) via

$$\begin{aligned} U_1(p,q) &= \langle \bar{p}ip, i\rangle E_1(p,q) + \langle \bar{p}ip, j\rangle E_2(p,q) - \langle \bar{p}ip, k\rangle E_3(p,q), \\ U_2(p,q) &= \langle \bar{q}iq, i\rangle F_1(p,q) + \langle \bar{q}iq, j\rangle F_2(p,q) - \langle \bar{q}iq, k\rangle F_3(p,q), \\ U_3(p,q) &= -E_1(p,q) - F_1(p,q), \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\operatorname{Im}\mathbb{H} \cong \mathbb{R}^3$. Note that each of the maps $p \mapsto \bar{p}ip$, $q \mapsto \bar{q}iq$ is a standard Hopf fibration $\pi_H \colon S^3 \to S^2 \subset \operatorname{Im}\mathbb{H}$. We see that the span of the U_1, U_2, U_3 is 3-dimensional, unless $p,q \in \pi_H^{-1}(\{\pm i\}) = \{e^{i\theta}, je^{i\theta} \mid \theta \in \mathbb{R}\}$.

The nearly Kähler structure on $S^3 \times S^3$ can be expressed as

$$\begin{aligned} \sigma &= \tfrac{2}{3\sqrt{3}}(e^1f^1 + e^2f^2 + e^3f^3), \\ \psi &= \tfrac{4}{9\sqrt{3}}(e^{23}f^1 + e^{31}f^2 + e^{12}f^3 - e^1f^{23} - e^2f^{31} - e^3f^{12}), \\ \hat\psi &= \tfrac{4}{27}(-2e^{123} - 2f^{123} + e^1f^{23} + e^2f^{31} + e^3f^{12} + e^{23}f^1 + e^{31}f^2 + e^{12}f^3). \end{aligned}$$

Specifically this means that (σ,ψ) defines an $SU(3)$-structure satisfying $d\sigma = 3\psi$ and $d\hat\psi = -2\sigma^2$. As mentioned above, T^3 acts effectively, preserving the nearly Kähler structure, and we have associated multi-moment maps $(\tilde\nu,\tilde\mu) \colon S^3 \times S^3 \to \mathbb{R}^4$ for the pair of closed forms (ψ,σ^2). As $d\sigma = 3\psi$ and $d\hat\psi = -2\sigma^2$, it is particularly easy to compute the maps $(\tilde\nu,\tilde\mu)$: by [37, Prop. 3.1] we have that $\tilde\nu_i = \frac{1}{3}\sigma(U_j,U_k)$ and $\tilde\mu = \frac{1}{2}\hat\psi(U_1,U_2,U_3)$.

The conical G$_2$-structure on $\mathbb{R}_+ \times S^3 \times S^3$ is given by

$$\varphi_C = dr \wedge r^2\sigma + r^3\psi = d(\tfrac{1}{3}r^3\sigma), \quad *\varphi_C = r^3\hat{\psi} \wedge dr + \tfrac{1}{2}r^4\sigma^2 = d(-\tfrac{1}{4}r^4\hat{\psi}).$$

It follows that

$$U_i \wedge U_j \lrcorner \varphi = 3r^2\tilde{\nu}_k dr + r^3 d\tilde{\nu}_k = d(r^3\tilde{\nu}_k)$$

and

$$U_1 \wedge U_2 \wedge U_3 \lrcorner *\varphi = 2r^3\tilde{\mu}dr + \tfrac{1}{2}r^4 d\tilde{\mu} = d(\tfrac{1}{2}r^4\tilde{\mu}).$$

So in terms of nearly Kähler data, the multi-moment maps $(\nu^C, \mu^C)\colon \mathbb{R}_+ \times S^3 \times S^3 \to \mathbb{R}^4$ are given by $(\nu^C, \mu^C) = (r^3\tilde{\nu}, \frac{r^4}{2}\tilde{\mu})$. Explicitly,

$$\nu_1^C(r,(p,q)) = \tfrac{2r^3}{9\sqrt{3}}\langle \bar{q}iq, i\rangle, \quad \nu_2^C(r,(p,q)) = \tfrac{2r^3}{9\sqrt{3}}\langle \bar{p}ip, i\rangle,$$

$$\nu_3^C(r,(p,q)) = \tfrac{2r^3}{9\sqrt{3}}\langle \bar{p}ip, \bar{q}iq\rangle,$$

$$\mu^C(r,(p,q)) = \tfrac{2r^4}{27}\big(\langle \bar{p}ip, j\rangle\langle \bar{q}iq, k\rangle - \langle \bar{p}ip, k\rangle\langle \bar{q}iq, j\rangle\big).$$

From the remarks about Hopf-fibrations, it is clear that (ν^C, μ^C) induces a map $\mathbb{R}_+ \times S^2 \times S^2 \to \mathbb{R}^4$ given by

$$(r,(v,w)) \mapsto \tfrac{2r^3}{9\sqrt{3}}\big(\langle v, i\rangle, \langle w, i\rangle, \langle v, w\rangle, \tfrac{2r}{\sqrt{3}}(\langle v, j\rangle\langle w, k\rangle - \langle v, k\rangle\langle w, j\rangle)\big).$$

Turning now to the Bryant-Salamon solution on the spin bundle of S^3, we begin by observing that this can be written in the form

$$\varphi_{BS} = -\tfrac{4}{3\sqrt{3}}\epsilon(e^{123} - f^{123}) + d(\tfrac{1}{3}(r^3 - \epsilon)\sigma),$$

$$*\varphi_{BS} = \tfrac{4}{9}\epsilon dr \wedge (e^{123} + f^{123}) + (r^3 - \epsilon)\hat{\psi} \wedge dr + \tfrac{1}{2}r(r^3 - 4\epsilon)\sigma^2,$$

for some $\epsilon > 0$ (see, e.g., [11]). Then, building on the computations from the nearly Kähler case, we find that the multi-moment maps for the toric Bryant-Salamon manifold are

$$\nu_1^{BS}(r,(p,q)) = \tfrac{2}{9\sqrt{3}}(r^3 - 4\epsilon)\langle \bar{q}iq, i\rangle,$$

$$\nu_2^{BS}(r,(p,q)) = \tfrac{2}{9\sqrt{3}}(r^3 - 4\epsilon)\langle \bar{p}ip, i\rangle,$$

$$\nu_3^{BS}(r,(p,q)) = \tfrac{2}{9\sqrt{3}}(r^3 - \epsilon)\langle \bar{p}ip, \bar{q}iq\rangle,$$

$$\mu^{BS}(r,(p,q)) = \tfrac{2}{27}r(r^3 - 4\epsilon)\big(\langle \bar{p}ip, j\rangle\langle \bar{q}iq, k\rangle - \langle \bar{p}ip, k\rangle\langle \bar{q}iq, j\rangle\big).$$

In this case, the matrix V has inverse given by

$$V^{-1} = \begin{pmatrix} \frac{4(r^3-\epsilon)}{9r} & -\frac{\sqrt{3}}{r}\frac{2\epsilon+r^3}{r^3-\epsilon}\nu_3^{BS} & -\frac{\sqrt{3}}{r}\nu_2^{BS} \\ -\frac{\sqrt{3}}{r}\frac{2\epsilon+r^3}{r^3-\epsilon}\nu_3^{BS} & \frac{4(r^3-\epsilon)}{9r} & -\frac{\sqrt{3}}{r}\nu_1^{BS} \\ -\frac{\sqrt{3}}{r}\nu_2^{BS} & -\frac{\sqrt{3}}{r}\nu_1^{BS} & \frac{4(r^3-4\epsilon)}{9r} \end{pmatrix}.$$

We obtain the values of the multi-moment map on the zero section of the spin bundle by continuity. Away from this zero section, the points with one-dimensional stabilisers map to the straight lines $(\varepsilon_1 t, \varepsilon_2 t, \varepsilon_1\varepsilon_2(t+k), 0)$, where $\varepsilon_i \in \{\pm 1\}$, $k = 2\epsilon/(3\sqrt{3})$ and $t > 0$. The limit $t \searrow 0$ gives points with stabiliser T^2 and the preimages of the interior of the line segment from $(0,0,-k,0)$ to $(0,0,k,0)$ is also a family of points with one-dimensional stabiliser. The image of the singular orbits is thus of the form $\succ\!\!-\!\!\prec$.

For r fixed large, $(\nu, \mu/r)$ essentially induces the map $(x,z,y,w) \mapsto (x, y, xy + \|z\|\|w\| \cos\theta, \|z\|\|w\| \sin\theta)$, where $(x,z),(y,w) \in S^2 \subset \mathbb{R} \times \mathbb{C}$ and θ is the oriented angle from z to w. On the quotient space this map is thus a homeomorphisms of topological three spheres and of global degree 1. From the general theory, we know (ν, μ) has local degree $+1$, so we conclude that the multi-moment map is injective on the orbit space. However, varying the parameter r, we get a deformation retract to the ellipsoids to the line segment $\{(0,0,t,0) \mid t \in [-k,k]\}$, so the multi-moment map is onto. We conclude that the multi-moment map is a homeomorphism from the T^3 orbit space of the spin bundle onto $\mathbb{R}^4$.

Remark 5.1. After completing this paper, Foscolo, Haskins and Nordstöm [26] have constructed many new examples of G_2-manifolds, including several examples with T^3-symmetry. For some of these, we find that the corresponding trivalent graphs are planar (see [44]), even though the holonomy group is the whole of G_2.

5.2. **Ansätze simplifying the PDEs.** From a PDE viewpoint a particular challenge is the fact that the characterisation of toric G_2-manifolds involves the coupled system consisting of both first order PDEs (3.8) and a second order system (3.10). In the following, we shall study some special cases that circumvent this complicating issue. This allows us to construct many explicit (but generally incomplete) examples of toric G_2-manifolds. In particular, we find that simple polynomial solutions in the variables (ν, μ) can lead to metrics with holonomy equal to G_2.

5.2.1. *One variable dependence.* Let us assume that V depends only on the variable μ, so $\partial V/\partial \nu_i = 0$, $i = 1,2,3$. Then $Z \equiv 0$. The condition that $d\omega = 0$ now yields that $\partial^2 V_{ij}/\partial\mu^2 = 0$. So V is linear in μ and thus W is constant.

Example 5.2. Taking $V = \operatorname{diag}(\mu,\mu,\mu)$ gives a solution defined for all $\mu > 0$. In this case, the associated G_2-metric takes the form

$$g = \tfrac{1}{\mu}(\theta_1^2 + \theta_2^2 + \theta_3^2) + \mu^2(d\nu_1^2 + d\nu_2^2 + d\nu_3^2) + \mu^3 d\mu^2,$$

where $d\theta_i = d\nu_j \wedge d\nu_k$, $(ijk) = (123)$.

This metric has (restricted) holonomy equal to G_2 as can be seen, e.g., by computing the Riemannian curvature: regarded as a 2-form $\Omega = (\Omega_{ij})$ on $T^3 \times \mathcal{U}$ with values in an associated $\mathfrak{g}_2$-bundle, the span of Ω_{ij}, $1 \leqslant i \leqslant j \leqslant 7$, has dimension 14.

From the viewpoint of complete metrics, this situation turns out to be less interesting.

Proposition 5.3. *Suppose $V = V(\mu)$. If (M, φ) is complete, then it is flat and hence locally isometric to $\mathbb{R}^7$.*

Proof. By Corollary 3.6, it suffices to show that completeness forces V to be a constant matrix. So let us assume V is not constant.

After adding a constant to μ, if necessary, we may assume that $V(0) > 0$ and then it follows by Remark 3.4 that we can take $V(0) = 1_3$. In fact, using the action of $GL(3,\mathbb{R})$ on $S^2(\mathbb{R}^3)$, we can even assume V has the form $V(\mu) = \operatorname{diag}(\lambda_1\mu+1, \lambda_2\mu+1, \lambda_3\mu+1)$ where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$.

As V is not constant, there is $\lambda_i \neq 0$ such that the rank of V drops (the first time) when $\mu = -1/\lambda_i$. By Lemma 4.4, we cannot be approaching a point $p \in M \setminus M_0$, i.e, a singular orbit, as we have $\det(B) \to \infty$. To show that this implies incompleteness of the G_2-metric, we use the criterion of [16, Lem. 1]: we look for a finite length curve not contained in any compact set.

In the base space of our T^3-bundle, we have a curve γ, defined on $(-1/\lambda_i, 0]$, corresponding to a curve parameterised by the μ-coordinate. Let $p \in M_0$ be a point projecting to $\gamma(0)$ and $\tilde{\gamma}$ the horizontal lift of γ with $\tilde{\gamma}(0) = p$. Clearly, the curve $\tilde{\gamma}\colon (-1/\lambda_i, 0] \to M_0$ has finite length, but is not contained in any compact set. □

In the cases where V depends only on one of the variables ν_i, similar arguments and conclusions apply.

5.2.2. *Orthogonal Killing vectors.* Let us assume $V_{ij} = 0$ for all $i \neq j$, i.e., the generating vector fields for the torus action are orthogonal. The G_2-metric now takes the form

$$g = \tfrac{1}{V_{11}}\theta_1^2 + \tfrac{1}{V_{22}}\theta_2^2 + \tfrac{1}{V_{33}}\theta_3^2 + V_{11}V_{22}V_{33}\big(d\mu^2 + \tfrac{1}{V_{11}}d\nu_1^2 + \tfrac{1}{V_{22}}d\nu_2^2 + \tfrac{1}{V_{33}}d\nu_3^2\big).$$

In this case, W is diagonal with non-zero entries given by $w_j^j = \partial V_{jj}/\partial\mu$, and Z has zeros on the diagonal and off-diagonal entries given by

$$z_i^j = -V_{kk}\frac{\partial V_{ii}}{\partial \nu_k}, \quad z_j^i = V_{kk}\frac{\partial V_{jj}}{\partial \nu_k},$$

with $(ijk) = (123)$.

The divergence-free condition (3.8) tells us that $\partial V_{ii}/\partial\nu_i = 0$, for $i = 1,2,3$. Then the condition $d\omega = 0$ is given by the equations

$$\frac{\partial^2 V_{ii}}{\partial \mu^2} + V_{jj}\frac{\partial^2 V_{ii}}{\partial \nu_j^2} + V_{kk}\frac{\partial^2 V_{ii}}{\partial \nu_k^2} = 0 \qquad (ijk) = (123) \tag{5.2}$$

together with

$$\frac{\partial V_{ii}}{\partial \nu_j}\frac{\partial V_{jj}}{\partial \nu_i} = 0 \tag{5.3}$$

for $i \neq j$.

Assume now that one has $\partial V_{ii}/\partial\nu_j \neq 0$, for some $j \neq i$. Without loss of generality, we can take $\partial V_{11}/\partial\nu_2 \neq 0$, which forces $\partial V_{22}/\partial\nu_1 = 0$. So V_{22} is a function of ν_3 and μ alone. By differentiating the equation (5.2), for $i = 2$, we then find that

$$\frac{\partial V_{33}}{\partial \nu_1}\frac{\partial^2 V_{22}}{\partial \nu_3^2} = 0 = \frac{\partial V_{33}}{\partial \nu_2}\frac{\partial^2 V_{22}}{\partial \nu_3^2}.$$

So either $\partial^2 V_{22}/\partial\nu_3^2$ vanishes identically, or there is an open set where $\partial V_{33}/\partial\nu_i = 0$, for $i = 1,2,3$.

In the first case, V_{22}, as a function of ν_3, has non-vanishing derivative of order at most one and so is either constant or linear in that variable. Correspondingly, we have $\partial V_{22}/\partial\nu_3 = 0$ or $\partial V_{22}/\partial\nu_3 \neq 0$, respectively.

If $\partial V_{22}/\partial \nu_i = 0$, $i = 1,2,3$, the additional information captured by (5.3) is that either $\partial V_{11}/\partial \nu_3 = 0$ or $\partial V_{33}/\partial \nu_1 = 0$ in an open neighbourhood. If $\partial V_{22}/\partial \nu_3 \neq 0$, then (5.3) moreover tells us that $\partial V_{33}/\partial \nu_2 = 0$.

Considering the case where $\partial V_{22}/\partial \nu_3 \neq 0$ and

$$\frac{\partial V_{11}}{\partial \nu_3} = 0 = \frac{\partial V_{22}}{\partial \nu_1} = \frac{\partial V_{33}}{\partial \nu_2},$$

(5.2) reduces to the equations

$$\frac{\partial^2 V_{11}}{\partial \mu^2} + V_{22}\frac{\partial^2 V_{11}}{\partial \nu_2^2} = 0, \qquad \frac{\partial^2 V_{33}}{\partial \mu^2} + V_{11}\frac{\partial^2 V_{33}}{\partial \nu_1^2} = 0.$$

Differentiating the first of these expressions with respect to ν_3, we find that V_{11} is (at most) linear in ν_2. Similarly, from differentiating the second equation above with respect to ν_2, we find that either V_{33} is (at most) linear in ν_1 otherwise $\partial V_{11}/\partial \nu_2 = 0$ on an open set.

After possibly relabelling indices, the above considerations imply that there are two ways to satisfy (5.2) and (5.3). The first one is to have each V_{ii} (at most) a linear function in two variables as follows:

$$V_{11} = V_{11}(\mu, \nu_2), \quad V_{22} = V_{22}(\mu, \nu_3), \quad V_{33} = V_{33}(\mu, \nu_1). \tag{5.4}$$

From the viewpoint of complete metrics this is less interesting:

Proposition 5.4. *If (M, φ) is complete with V diagonal and its entries satisfy* (5.4)*, then (M, φ) is flat and hence locally isometric to $(\mathbb{R}^7, \varphi_0)$.*

Proof. This is essentially proved in the same way as Proposition 5.3. We may assume that $V(0) > 0$. Consequently, we can write V in the form:

$$\operatorname{diag}(\epsilon_1\nu_2\mu + \kappa_1\nu_2 + \lambda_1 + 1, \epsilon_2\nu_3\mu + \kappa_2\nu_3 + \lambda_2 + 1, \epsilon_3\nu_1\mu + \kappa_3\nu_1 + \lambda_3 + 1).$$

By considering suitable curves (corresponding to $(0, \mu)$, $(\nu_1, 0)$ etc.), we arrive at the asserted conclusion. $\square$

The second and more interesting possibility is to have $\partial V_{33}/\partial \nu_i = 0$, $i = 1,2,3$, together with

$$\frac{\partial V_{22}}{\partial \nu_1} = 0 = \frac{\partial V_{22}}{\partial \nu_2}, \quad \frac{\partial V_{11}}{\partial \nu_1} = 0.$$

In this case, (5.2) corresponds to the following elliptic hierarchy:

$$\frac{\partial^2 V_{11}}{\partial \mu^2} + V_{22}\frac{\partial^2 V_{11}}{\partial \nu_2^2} + V_{33}\frac{\partial^2 V_{11}}{\partial \nu_3^2} = 0, \quad \frac{\partial^2 V_{22}}{\partial \mu^2} + V_{33}\frac{\partial^2 V_{22}}{\partial \nu_3^2} = 0, \quad \frac{\partial^2 V_{33}}{\partial \mu^2} = 0. \tag{5.5}$$

So again V_{33} is at most a linear function of μ, and V is independent of ν_1. This means, in particular, that U_2 and U_3 have no zeros, i.e., there are no points with T^2-isotropy, and points with S^1-isotropy lie above disjoint lines parallel to the ν_1-axis.

When V_{33} is constant, which we can take to be 1, the G_2-metric is a product

$$g = \theta_3^2 + \tfrac{1}{V_{11}}\theta_1^2 + \tfrac{1}{V_{22}}\theta_2^2 + V_{11}V_{22}\big(d\mu^2 + \tfrac{1}{V_{11}}d\nu_1^2 + \tfrac{1}{V_{22}}d\nu_2^2 + d\nu_3^2\big),$$

so the holonomy reduces to a subgroup of $SU(3)$.

Reducing the holonomy further, one obvious solution to the elliptic system in this case is given by taking $V_{22} = 1 = V_{33}$ and $V = V_{11}(\mu, \nu_2, \nu_3)$ to be a harmonic function on an $\mathbb{R}^3$. Then the associated G_2-holonomy metric is given by

$$g = \theta_2^2 + \theta_3^2 + d\nu_1^2 + \tfrac{1}{V}\theta_1^2 + V\,(d\mu^2 + d\nu_2^2 + d\nu_3^2).$$

This has the form of a product of a flat metric on (an open set of) $T^2 \times \mathbb{R}$ and a hyperKähler metric on an S^1-bundle over (an open set of) $\mathbb{R}^3$.

Excluding these cases of reduced holonomy, we are thus left with analysing the equations:

$$\frac{\partial^2 V_{11}}{\partial \mu^2} + V_{22}\frac{\partial^2 V_{11}}{\partial \nu_2^2} + \mu\frac{\partial^2 V_{11}}{\partial \nu_3^2} = 0, \qquad \frac{\partial^2 V_{22}}{\partial \mu^2} + \mu\frac{\partial^2 V_{22}}{\partial \nu_3^2} = 0,$$

having set $V_{33}(\mu) = \mu$.

As the following example shows, it is easy to find local (incomplete) solutions to these equations that have full holonomy.

Example 5.5. By writing down (ν, μ) as a power series and solving (5.5), we get solutions on trivial bundles $T^3 \times \mathcal{U}$, where $\mathcal{U} \subset \mathbb{R}^4$ is an appropriate open subset. As an example of such a solution we can take

$$V_{11}(\nu_2, \nu_3, \mu) = 2\mu^5 - 15\mu^2\nu_3^2 - 5\nu_2^2, \quad V_{22}(\nu_3, \mu) = \mu^3 - 3\nu_3^2, \quad V_{33}(\mu) = \mu.$$

As in Example 5.2, one checks by explicit computations that the associated metric has (restricted) holonomy equal to G_2.

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