Elliptic $\mathfrak{p}$-units and the equivariant Tamagawa Number Conjecture

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Abstract

A main objective of algebraic number theory is finding relations between special values of L-functions and arithmetic invariants. In this research area the equivariant Tamagawa Number Conjecture (short: eTNC) is a very broad generalization of, for example, the analytic class number formula and the refined Birch and Swinnerton-Dyer conjecture. It is only proven for very few cases like the case of Tate motives with underlying Galois groups of absolutely abelian extensions by work of D. Burns, C. Greither and M. Flach.

Mimicking their proof, W. Bley was able to show the p-part of the eTNC for Tate motives at $s = 0$, where the underlying Galois group is that of an abelian extension of an imaginary quadratic field k if p is an odd split prime in k not dividing the class number h_k. One vital ingredient in this proof is an analogue of a result of D. Solomon concerning the construction of p-units in absolutely abelian fields and the computation of their valuation in the split imaginary quadratic situation.

In this dissertation we construct $\mathfrak{p}$-units, where $\mathfrak{p}$ is a prime ideal above p, and compute their valuation if $p > 3$ is a non-split prime in the imaginary quadratic field k and $p \nmid h_k$ as well as some partial results for $p = 2, 3$. Additionally, we are able to recover the results of D. Solomon (resp. W. Bley) for the cyclotomic (resp. the split imaginary quadratic) situation with this new method. Then we show the usefulness of this result for proving the eTNC in two ways:

First, we prove the Iwasawa-theoretic Mazur-Rubin-Sano conjecture for some abelian extensions where p is non-split in the imaginary quadratic number field, which is in turn a condition for proving the eTNC for Tate motives at $s = 0$ with a theorem of D. Burns, M. Kurihara and T. Sano.

Secondly, building on the proof of W. Bley for the split case, we show how to reduce the validity of the p-part of the eTNC in the inert case for $p > 2$ and $p \nmid h_k$ to three conditions incorporating work on the equivariant main conjecture for imaginary quadratic number fields.

Zusammenfassung

Ein Hauptziel der Algebraischen Zahlentheorie ist das Finden von Beziehungen zwischen speziellen Werten von L-Funktionen und arithmetischen Invarianten. In diesem Teilgebiet ist die äquivariante Tamagawazahlvermutung (kurz: eTNC) eine sehr umfassende Verallgemeinerung unter anderem von der analytischen Klassenzahlformel und der verfeinerten Vermutung von Birch und Swinnerton-Dyer. Sie ist nur in wenigen Fällen bewiesen, wie beispielsweise von D. Burns, C. Greither und M. Flach im Fall von Tatemotiven, denen die Galoisgruppe einer absolut-abelschen Erweiterung zugrunde liegt.

Basierend auf Ideen in deren Beweis konnte W. Bley den p-Anteil der eTNC für Tatemotive bei $s = 0$ zeigen, welchen die Galoisgruppe einer abelschen Erweiterung eines imaginär-quadratischen Zahlkörpers k zugrunde liegt (im Fall, dass p eine ungerade Primzahl ist, die in k zerlegt, und p nicht die Klassenzahl h_k teilt). Ein essentieller Bestandteil der Beweisführung ist ein Analogon eines Resultats von D. Solomon über die Konstruktion von p-Einheiten in absolut-abelschen Zahlkörpern sowie die Berechnung ihrer Bewertungen in der zerlegten imaginär-quadratischen Situation.

In dieser Dissertation konstruieren wir $\mathfrak{p}$-Einheiten, wobei $\mathfrak{p}$ ein Primideal über p ist und $p > 3$ eine im imaginär-quadratischen Zahlkörper k nicht-zerlegte Primzahl mit $p \nmid h_k$ ist (für $p = 2, 3$ zeigen wir einige Teilresultate). Zusätzlich sind wir in der Lage, das Resultat von D. Solomon (bzw. W. Bley) im zyklotomischen (bzw. zerlegten imaginär-quadratischen) Fall mit unseren neuen Methoden erneut zu beweisen. Anschließend zeigen wir den Nutzen unseres neuen Resultats im Zusammenhang mit der eTNC auf zwei Arten:

Erstens zeigen wir die Iwasawa-theoretische Mazur-Rubin-Sano-Vermutung für einige Erweiterungen, bei denen p im imaginär-quadratischen Zahlkörper nicht-zerlegt ist. Diese Vermutung ist wiederum eine Voraussetzung in einem Theorem von D. Burns, M. Kurihara und T. Sano, welches zeigt, wie man unter der Annahme einiger Hypothesen die eTNC für Tatemotive bei $s = 0$ folgern kann.

Zweitens, aufbauend auf dem Beweis von W. Bley im zerlegten Fall, zeigen wir, wie man die Gültigkeit des p-Anteils der eTNC im trägen Fall für ungerade Primzahlen p mit $p \nmid h_k$ auf drei Annahmen reduzieren kann, wobei wir auch bisherige Arbeiten zur äquivarianten Hauptvermutung für imaginär-quadratische Zahlkörper betrachten.

*«Après cela il y aura, j'espère,
des gens qui trouveront leur profit
à déchiffrer tout ce gâchis.»*

Évariste Galois (1811–1832)

Chapter 1

Introduction

Prologue The decision to write a dissertation in mathematics implies the hubris of thinking one could contribute something new to this noble discipline. In order to have any chance of doing that one has to commit substantial time and effort in trying to solve a very abstract question. So one regularly comes across the following question: 'Why do we study this particular problem?'

As there are not many practical applications for the topics contained in this dissertation (yet[1]), it is our opinion that one should look at the development of mathematics for motivation. In mathematics this is particularly rewarding since, in comparison to other sciences, the knowledge we inherit from our ancestors is surviving untarnished. So the following introduction has two parts: The first part[2] contains a short sketch of the history of that part of number theory we are concerned with later on. It consists of, at first, seemingly unrelated strands of narrative which will come together at the end. The second part is more of a conventional introduction giving an overview of what can be found in this dissertation.

1.1 A short, subjective history of number theory - from Fermat to the eTNC

From Fermat to Euler. It is said that P. Fermat (1607-1665) got hooked on number theory after picking up a newly published Latin translation of the ancient Greek work of Diophantus - the moment A. Weil ([Wei07, p.1]) calls the birth of modern number theory. Fermat studied questions which can be formulated very easily: Which forms can primes have? What are the integral solutions for x and y of Diophantine equations like $x^2 - Ny^2 = \pm 1$? Are there non-trivial integral solutions to the equation $x^n + y^n = z^n$ for $n \geq 3$? Regarding the latter, he claimed that he can prove that there are none. However, his contemporaries did not share his enthusiasm and it seems that nobody wanted to pick up the baton. So one had to wait until 1729 when C. Goldbach (1690-1764) wrote to his friend L. Euler (1707-1783) about Fermat's assertion that all integers of the form $2^{2^n} + 1$ are primes. This assertion, which Euler later showed to be wrong through proving that $2^{2^5} + 1$ is not a prime, lured Euler to thinking about number theoretical questions.

[1] Not to repeat Hardy's mistake cf. [Har40]

[2] This part may be a bit unconventional as it is more of an essay-style text than a rigorous mathematical text, but we hope it is nevertheless enjoyable to the reader.

The Basler Problem. One of these questions Euler considered is called the *Basler Problem*: When $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$, what is the value of $\zeta(2)$?[3] In 1735 he succeeded by proving that $\zeta(2) = \frac{\pi^2}{6}$ and in another proof of this assertion he showed that there is an (Euler) product expansion for $\zeta(s)$. This means $\zeta(s) = \prod_p (1-p^{-s})^{-1}$, where the product ranges over all rational primes. In 1739, he even showed that

$$\zeta(2k) = (-1)^{k-1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}, \tag{1.1}$$

where B_{2k} are the Bernoulli numbers and $k \geq 1$, and from that formula it is easy to obtain

$$\zeta(2k)\pi^{-2k} \in \mathbb{Q}. \tag{1.2}$$

We remark here that (1.2) is an archetypical example of a phenomenon we will concern ourselves with further below.

Magnum opus of Gauss. Euler still felt the need to justify his efforts in number theory, which led him to announce that they will be to 'the whole benefit of analysis'. But then things changed quickly for number theory, so that several decades later C.F. Gauss (1777-1855) already proclaimed that 'mathematics is the queen of science and arithmetic is the queen of mathematics'. He himself contributed a lot to this new standing of number theory. In his famous *Disquisitiones Arithmeticae* - abbreviated by D.A. - he summarised the number theory known then and included several of his own results. One of these was a proof of the quadratic reciprocity law, a vital source of motivation for Gauss for studying number theory. Among the many topics contained in this monumental work we want to pick up the topic of binary quadratic forms, which is contained in Chapter 5 of D.A. There he developed a notion of when two such forms are equivalent and counted the number of equivalence classes corresponding to a fixed discriminant D, the so-called class number $h(D)$. For example, Gauss gave a list of negative discriminants with class number one and claimed that this list is complete.

A connection between two worlds. In 1837, G.L. Dirichlet (1805-1859) picked up a conjecture which Euler stated in 1783, namely that there are infinitely many primes in an arithmetic progression, i.e. for two coprime numbers a and m there are infinitely many prime numbers contained in the sequence $(a + n \cdot m)_{n \in \mathbb{N}}$. From a modern point of view the result is still interesting, but what really made head-waves were the tools Dirichlet developed because in order to prove the theorem he introduced *Dirichlet characters* and the *Dirichlet L-function.*

A question we have not asked yet is: What are the analytic properties of $\zeta(s)$ and where can it be defined? In his only number-theoretic paper [Rie60] B. Riemann (1826-1866) showed that $\zeta(s)$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ and that there is a functional equation.[4] In analogy of the ζ-function R. Dedekind (1831-1916) defined a similar function for a general number field K and showed that this function also has an Euler product expansion,

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\mathcal{N}(\mathfrak{a})^s} = \prod_{\mathfrak{p}} (1 - \mathcal{N}(\mathfrak{p})^{-s})^{-1}, \tag{1.3}$$

where $\mathcal{N}$ denotes the ideal norm, the sum ranges over all the non-zero integral ideals of the ring of integers $\mathcal{O}_K$ of K and the product over all the prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$. Dedekind also showed

[3] As one sees here for the first time exemplified, we will use modern notation and definitions throughout this overview which were, most of the time, not known to the mathematicians we are talking about.

[4] A side note in the paper was that the zeroes of this ζ-function should be at negative even integers and complex numbers that have real part 1/2.

that this function has a simple pole at $s = 1$ and that it converges absolutely for $Re(s) > 1$. The pinnacle of Dedekind's work in this direction was that he succeeded in proving the *analytic class number formula* in the 1870's:

$$\lim_{s\to 1}(s-1)^{-1}\zeta_K(s) = h_K R_K \frac{2^{r_1}(2\pi)^{r_2}}{w_K|d_K|^{1/2}}, \tag{1.4}$$

where R_K is the regulator, w_K the number of roots of unity, d_K the discriminant and r_1 and r_2 the number of real and complex places of K, respectively.

Prehistory of cyclotomic fields. A classic problem going back to at least the ancient Greeks is the possibility of the construction of a regular n-polygon solely with ruler and compass. In 1796, younger than 20 by then, Gauss showed this to be possible for $n = 17$. In D.A. he even showed a sufficient condition for a general n: the odd prime factors of n are distinct Fermat primes[5]. Although this is certainly an impressive result, the methods he used had even greater impact. Gauss considered in Chapter 7 of D.A. what we nowadays would denote by $\mathbb{Q}(\zeta_n)$, where ζ_n is a root of the equation $x^n - 1 = 0$, and call a cyclotomic field. He also showed that every quadratic number field lies in such a cyclotomic field and also remarked that the cyclotomic theory should have an analogue using the lemniscate and other transcendental functions.

Influx of analysis. Going beyond the results of Gauss, L. Kronecker (1823-1891) claimed in 1853 that every abelian extension of $\mathbb{Q}$ is contained in a cyclotomic field, an assertion nowadays called *Theorem of Kronecker-Weber.* Over the next years Kronecker obtained some results connecting complex multiplication of elliptic functions with abelian extensions of imaginary quadratic number fields. This culminated in a letter to Dedekind in 1880 where he admitted that it is his 'liebster Jugendtraum' to prove that all abelian equations with coefficients in imaginary quadratic number fields are exhausted by those which come from the theory of elliptic functions.

And there was hope for such a project because on the analytic side of the problem were also major developments happening. Going back to 1847, G. Eisenstein (1823-1852) proved several properties of the Δ-, ϕ- and j-functions and started what we today would call the theory of Eisenstein series. In 1862, K. Weierstrass (1815-1897) defined his $\wp$-, ζ- and σ-functions and expressed the $\wp$-function in terms of Eisenstein series. In 1877, Dedekind introduced his η-function and proved a transformation formula for it. Also in an article about elliptic functions, Kronecker discovered a limit formula: For $\tau = x + y\cdot i$ with $y > 0$ and $s \in \mathbb{C}$ we have[6]

$$\sum_{(m,n)\neq(0,0)\in\mathbb{Z}^2} \frac{y^s}{|m\tau+n|^{2s}} = \frac{\pi}{s-1} + 2\pi(\gamma - \log(2) - \log(\sqrt{y}|\eta(\tau)|^2)) + O(s-1), \tag{1.5}$$

where γ is the Euler constant and $\eta(\tau)$ is the value of the Dedekind eta function. This formula is called *Kronecker's first limit formula.*

Hilbert's summary and predictions. In 1896, D. Hilbert (1862-1943) carefully studied the known instances of algebraic number theory and wrote an exposition of almost all known results of number theory in his own formulation. This exposition is known as 'Zahlbericht'. In this work, led by analogies to Riemann surfaces, he conjectured that for any number field K there is a unique extension L over K such that the Galois group of L/K is isomorphic to the class group of K, L/K is unramified at all places, every abelian extension of K with this property is

[5] i.e. primes of the form $2^{2^m}+1$

[6] at first for $Re(s) > 1$ and then analytically continued to $\mathbb{C}$

a subfield of L, for any prime $\mathfrak{p}$ of K, the residue field degree at $\mathfrak{p}$ is the order of $[\mathfrak{p}]$ in the class group of K and every ideal of K is principal in L. A number field satisfying these properties is now called a *Hilbert class field*.

Another consolidating effort of Hilbert was his list of 23 problems presented at the occasion of the International Congress of Mathematicians in 1900. His 12th problem is concerned with the explicit construction of an abelian extension of a number field, citing the model cases of $\mathbb{Q}$ and imaginary quadratic number fields, but only one of these cases was proven at the time he stated this problem.

Then the last two developments got entangled because after studying Hilbert's 'Zahlbericht' T. Takagi (1875-1960) decided that he wanted to do algebraic number theory. He started working on 'Kronecker's Jugendtraum' and accomplished some partial results in 1903 essentially solving the case $\mathbb{Q}(i)$. In this direction R. Fueter (1880-1950) in [Fue14] proved 'Kronecker's Jugendtraum' for abelian extensions of imaginary quadratic number fields of odd degree. Closely connected to these results is a result of Fueter from 1910 in [Fue10], where he uses methods of Dedekind and limit formulas like Kronecker's in (1.5) to obtain class number formulas for abelian extensions of imaginary quadratic number fields.

Takagi revolutionizes class field theory. Takagi also started thinking about generalizing the properties of Hilbert class fields and even dared to contemplate that maybe every abelian extension is a class field, which was originally only considered for imaginary quadratic base fields. In giving a new definition of a class field using norms of ideals instead of splitting laws and also incorporating infinite places into the modulus he was able to show this vast generalization of the known ideas by then. Indeed, the main results of his work published in [Tak20] were his *Existence Theorem* (which asserts that for an ideal group H there is a class field over K), the *Isomorphism Theorem* (which says that if H is an ideal group with modulus $\mathfrak{m}$ and class field L, and $I_K(\mathfrak{m})$ the group of all ideals coprime to $\mathfrak{m}$ then there is an isomorphism $\mathrm{Gal}(L/K) \cong I_K(\mathfrak{m})/H$) and the *Completeness Theorem* (which says that any finite abelian extension of K is a class field). As if this had not been enough, Takagi fulfilled 'Kronecker's Jugendtraum' in this momentous work as well. This was obviously a big breakthrough but not yet utterly satisfying. Takagi had proved the Isomorphism Theorem by reducing the problem to the cyclic case and using the fact that two cyclic groups of equal order are isomorphic, so there was no explicit isomorphism given.

Artin L-function. At about the same time E. Hecke (1887-1947) in [Hec17] showed that the Dedekind ζ-function for a number field K has a meromorphic continuation to $\mathbb{C}$, satisfies a functional equation and has a simple pole at $s = 1$. So for an extension L/K the quotient ζ_L/ζ_K is meromorphic on $\mathbb{C}$. If the extension is abelian, one already knew that this quotient is even an entire function, because it was possible to express it in terms of Weber[7] L-functions of non-trivial characters. But E. Artin (1898-1962) wanted to know if the same thing was true for non-abelian extensions. On his path to discover L-functions of not necessarily abelian representations of Galois groups he made a definition which was also helpful in the abelian case. In [Art24] he defined an L-function for a finite abelian extension L/K with Galois group G: For $\chi \in \widehat{G}$ and $Re(s) > 1$, define $L(s,\chi) = \prod_{\mathfrak{p}}(1 - \chi(Fr_{\mathfrak{p}})\mathcal{N}(\mathfrak{p})^{-s})^{-1}$, where $Fr_{\mathfrak{p}}$ is the Frobenius element and the product ranges over all prime ideals $\mathfrak{p}$ of K which are unramified in L. But now one has two L-functions, from Weber and Artin, which are defined on characters of isomorphic groups, so it is natural to ask for an explicit isomorphism which identifies possibly these L-functions. The first thing that comes to one's mind is the map $\mathfrak{p} \mapsto Fr_{\mathfrak{p}}$. For this map, extended multiplicatively, Artin in [Art27] was able to show that it gives an explicit isomorphism in the Isomorphism

[7] H. Weber (1842-1913)

Theorem. This theorem is called *Artin reciprocity law* because it also subsumes all the classical reciprocity laws.

In [Art30], Artin gave a definition of a more general L-function: Let L/K be a Galois extension of number fields with Galois group G and let (ρ, V) be a representation of G. Then we set

$$L_{L/K}(\rho, s) = \prod_{\mathfrak{p}} (\det(1 - Fr_{\mathfrak{P}} \mathcal{N}(\mathfrak{p})^{-s}; V^{I_{\mathfrak{P}}}))^{-1}, \tag{1.6}$$

where the product ranges over all prime ideals of K, $\mathfrak{P}$ is a prime ideal of L above $\mathfrak{p}$, $Fr_{\mathfrak{P}}$ the corresponding Frobenius element and $I_{\mathfrak{P}}$ the inertia group.[8] Nowadays we call this an *Artin L-function.*

On the shoulders of Hensel and Kummer. In the mathematical world of Dedekind and Hilbert number theory had been the study of algebraic number fields and Hilbert's 'Zahlbericht' was a manifestation thereof. The ideas of E. Kummer (1810-1893) and Kronecker were somehow eclipsed by the glory and success of the Dedekind-Hilbert approach to number theory. Though there remained results and methods of Kummer that were not well-embedded in the existing theories, as for example his famous *Kummer's Congruence*: For p prime and $l, k \in 2\mathbb{Z}_+$ with $(p-1) \nmid l$ or $(p-1) \nmid k$ we have

$$B_l/l \equiv B_k/k \bmod p \quad \text{if } l \equiv k \bmod (p-1), \tag{1.7}$$

where B_i are again Bernoulli numbers. Or also the fact that a prime p is irregular, i.e. p does not divide $h(\mathbb{Q}(\zeta_p))$, if and only if p divides one of the numerators of $\zeta(-1), \dots, \zeta(4-p)$. Even more generally, one can define generalized Bernoulli numbers B_χ^m and show that they occur as values of Dirichlet L-functions $L(s, \chi)$ at odd negative integers:

$$L(1-m, \chi) = -\frac{B_\chi^m}{m}. \tag{1.8}$$

For any prime p the generalized Bernoulli numbers satisfy certain p-adic congruences which are called *generalized Kummer congruences.* T. Kubota and H.-W. Leopoldt (1927-2011) in [KL64] observed that those congruences can be interpreted in such a way that the $\frac{B_\chi^m}{m}$ are in a sense p-adically continuous functions on m. More precisely: There is one and only one p-adically continuous function $L_p(s, \chi)$ defined on $\mathbb{Z}_p$ such that (for χ even and $p > 2$):

$$L_p(1-m, \chi) = -\frac{B_\chi^m}{m}(1 - \chi(p)p^{m-1}), \tag{1.9}$$

for negative integers $1-m$ with $(p-1) \mid m$. These numbers $1-m$ are dense in $\mathbb{Z}_p$. But it turns out that $L_p(s, \chi)$ is holomorphic in a region larger than $\mathbb{Z}_p$, at least if $\chi \neq 1$, whereas for $\chi = 1$ there is one pole for $s = 1$. Based on these p-adic L-functions Leopoldt considered for any abelian number field K the corresponding p-adic zeta function $\zeta_{K,p}(s)$ as a product of $L_p(s, \chi)$ over all characters. He arrived at the p-adic class number formula which is an analogue of the analytic class number formula given in (1.4).

One related problem is also to prove the non-vanishing of the *p-adic regulator* of a number field K which appears in the p-adic class number formula. This regulator $R_{K,p}$ is obtained if one replaces the ordinary logarithms in the classical regulator with p-adic logarithms. The non-vanishing of $R_{K,p}$ means that the p-adic rank of the topological closure of the image under a suitable diagonal embedding of the group of units of K equals the ordinary rank - this is now known as *Leopoldt's conjecture.* In [Bru67], A. Brumer with the help of a reduction of J. Ax (1937-2006) in [Ax65] proved Leopoldt's conjecture for arbitrary abelian extensions of $\mathbb{Q}$ or an imaginary quadratic base field.

[8] One can show that this is well-defined and it only depends on the character.

Iwasawa's growth formula. One of the first number-theoretic results of K. Iwasawa (1917-1998) is concerned with the growth of certain class numbers in [Iwa59a]. Indeed, let F be a finite extension of $\mathbb{Q}$ and fix, from now on, for simplicity an odd prime p. Then a $\mathbb{Z}_p$-extension is a Galois extension F_∞ of F such that $\Gamma := \mathrm{Gal}(F_\infty/F) \cong \mathbb{Z}_p$. Furthermore, we set $\Gamma_n := \Gamma^{p^n}$ as well as $F_n := F_\infty^{\Gamma_n}$ and we easily see that F_n/F is a cyclic extension. Let now e_n be the largest natural number such that $p^{e_n} \mid h_{F_n}$, where h_{F_n} is the class number of F_n. Then Iwasawa proved the existence of λ, μ, ν for sufficiently large n such that $e_n = \lambda n + \mu p^n + \nu$. The main tool in the proof is the usage of the compact $\mathbb{Z}_p$-module $X = \mathrm{Gal}(L_\infty/F_\infty)$, where L_n is the p-Hilbert class field of F_n, i.e. the maximal abelian p-extension unramified at all primes, and $L_\infty := \bigcup_n L_n$. In 1959, J. P. Serre realised that one can view X as a module over the ring $\Lambda = \mathbb{Z}_p[[T]]$ and then derived the Iwasawa growth formula from the structure theorem for Λ-modules. The latter asserts that for a finitely-generated torsion Λ-module M there is a homomorphism $M \to \bigoplus_{i=1}^t \Lambda/(f_i(T)^{a_i})$, where $f_i(T)$ are irreducible elements of Λ, with finite kernel and cokernel[9]. Then we can define the following invariants of M:

$$f_M(T) = \prod_{i=1}^{t} f_i(T)^{a_i}, \quad \lambda(M) = \deg(f_M(T)), \quad \mu(M) = \max\{m \in \mathbb{N}_0 : p^m \mid f_M(T)\}, \tag{1.10}$$

where $f_M(T)$ is called *characteristic polynomial*. Now we have $\lambda(X) = \lambda$ and $\mu(X) = \mu$, where λ and μ are the same as in the growth formula above.

Herbrand's Theorem. Until now we have only encountered the cardinality of the class group, but also the structure is interesting. J. Herbrand (1908-1931) proved in [Her32] that for $\mathbb{Q}(\zeta_p)$, $2 \le i, j \le p-2$, $i + j \equiv 1 \bmod (p-1)$ and i odd, we have that if $A^{\omega^i} \neq 0$, then $p \mid B_j$, where A is the p-primary subgroup of $Cl(\mathbb{Q}(\zeta_p))$, B_j are the Bernoulli numbers and ω is defined below.

The Iwasawa Main Conjecture for cyclotomic fields. [10] In [Iwa59b], Iwasawa continued by studying the extension $F_\infty := \mathbb{Q}(\zeta_{p^\infty})$. He defined M_∞ as the maximal abelian extension of F_∞ which is pro-p and such that only the primes lying over p are ramified, and set $Y := \mathrm{Gal}(M_\infty/F_\infty)$. Now Galois theory gives a natural decomposition $\mathrm{Gal}(F_\infty/\mathbb{Q}) = \Delta \times \Gamma$ and we can define the Teichmüller character as $\omega : \Delta \to \mu_{p-1} \subset \mathbb{Z}_p^\times$ given by the action of Δ on μ_{p^∞}. Moreover, for any $\mathbb{Z}_p$-module N on which Δ acts we get a decomposition

$$N = \bigoplus_{k=0}^{p-2} N^{\omega^k} \quad \text{with } N^{\omega^k} := \{a \in N : \delta(a) = \omega^k(\delta)a \ \forall \delta \in \Delta\}. \tag{1.11}$$

Iwasawa showed that X^{ω^i} for odd i has no non-zero finite Λ-submodules and that for all k, Y^{ω^k} has no non-zero, finite Λ-submodules.

Let now $\mathbb{Q}_\infty$ be the unique subfield of $\mathbb{Q}(\zeta_{p^\infty})$ such that $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \cong \mathbb{Z}_p$. Then for any number field F the extension $F_\infty := F \cdot \mathbb{Q}_\infty$ over F is called *cyclotomic $\mathbb{Z}_p$-extension.* Iwasawa conjectured that for such extensions the μ-invariant is always zero[11]. This general conjecture is still open, but B. Ferrero and L. Washington in [FW79] proved it for $F/\mathbb{Q}$ being an abelian extension and later W. Sinnott in [Sin84] found another proof of this result by different methods.

[9] The value of t, $f_i(T)$ and a_i are uniquely determined by M, up to the order. Moreover, $f_i(T)$ can be chosen as a polynomial.

[10] This section is based on [Gre01] and we again assume that p is an odd prime.

[11] Meaning X corresponding to F_∞/F has μ-invariant zero.

In [Iwa64], Iwasawa went on to study the structure of class groups of $F_n := \mathbb{Q}(\zeta_{p^n})$. He defined

$$\theta_n^{(i)} = \frac{-1}{p^{n+1}} \sum_{a=1}^{p^{n+1}} a\omega^{-i}(a) \langle \sigma_a \rangle^{-1} \quad \text{and} \quad \theta^{(i)} := \varprojlim_n \theta_n^{(i)} \in \Lambda = \mathbb{Z}_p[[\Gamma]], \tag{1.12}$$

where $\sigma_a \in \mathrm{Gal}(F_n/\mathbb{Q})$ is determined by $\sigma_a(\zeta_{p^{n+1}}) = \zeta_{p^{n+1}}^a$, $\langle \sigma_a \rangle$ is the projection to $\mathrm{Gal}(F_n/F)$ in the decomposition $\mathrm{Gal}(F_n/\mathbb{Q}) = \Delta \times \mathrm{Gal}(F_n/F)$ and $\omega^{-i}(a)$ is determined by the projection of σ_a to Δ, regarding ω^{-i} as character of that group. It follows from Stickelberger's theorem that $\theta_n^{(i)}$ annihilates $A_n^{\omega^i}$. Iwasawa proved, under a *cyclicity hypothesis*, that for i odd and $3 \le i \le p-2$, we have

$$X^{\omega^i} \cong \Lambda/(\theta^{(i)}) \text{ as } \Lambda\text{-modules}. \tag{1.13}$$

For the proof we identify Λ with $\mathbb{Z}_p[[T]]$ and therefore $\theta^{(i)}$ with a power series $g_i(T)$. Moreover, we set $f_i(T) := f_{X^{\omega^i}}(T)$. As $\theta^{(i)}$ annihilates X^{ω^i}, we have $g_i(T) \in (f_i(T))$. Now it remains to show that $f_i(T)/g_i(T) \in \Lambda^\times$, which Iwasawa did under the above-mentioned cyclicity hypothesis. With the help of some computations with Iwasawa invariants one can reduce proving $(f_i(T)) = (g_i(T))$ to show that $g_i(T) \mid f_i(T)$ even without the cyclicity hypothesis, which Iwasawa did in Chapter 7 in [Iwa72].

Let κ be the restriction of the cyclotomic character to Γ and define for $s \in \mathbb{Z}_p$ a continuous homomorphism κ^s by $\kappa(\gamma)^s$ for $\gamma \in \Gamma$ and then extend it to a continuous $\mathbb{Z}_p$-algebra homomorphism $\varphi_s : \Lambda \to \mathbb{Z}_p$. Then Iwasawa proved in [Iwa69] that for j even with $2 \le j \le p-3$ we have $L_p(s, \omega^j) = \varphi_s(\theta^{(i)})$ for all $s \in \mathbb{Z}_p$. Or, equivalently, $g_i(T)$ satisfies the following interpolation property:

$$g_i(\kappa(\gamma_0)^{1-m} - 1) = -(1 - p^{m-1})\frac{1}{m} B_m \tag{1.14}$$

for all $m \ge 1$ such that $m \equiv j \bmod (p-1)$, where γ_0 is a generator of Γ. We see that the interpolation property determines $g_i(T)$ uniquely. Now we can state a version of the *Iwasawa Main Conjecture*, abbreviated by IMC, for cyclotomic fields: For each i odd, $3 \le i \le p-3$ we have

$$(f_i(T)) = (g_i(T)) \text{ as ideals of } \Lambda. \tag{1.15}$$

Let $\dot{g}_i(T) := g_i(\kappa(\gamma_0)(1+T)^{-1} - 1)$, U_n denote the group of units in the completion $(F_n)_{\mathfrak{p}_n}$, where $\mathfrak{p}_n$ is the unique prime of F_n above p, and $\overline{E}_n$ resp. $\overline{C}_n$ the closure of the units E_n resp. cyclotomic units[12] C_n of F_n in U_n. Then we can set $\mathcal{X} := \varprojlim \overline{E}_n/\overline{C}_n$, $\mathcal{Y} := \varprojlim U_n/\overline{C}_n$ and $\mathcal{Z} := \varprojlim U_n/\overline{E}_n$ and Iwasawa showed that there is an exact sequence

$$0 \to \mathcal{X}^{\omega^j} \to \mathcal{Y}^{\omega^j} \to \mathcal{Z}^{\omega^j} \to 0 \tag{1.16}$$

of finitely generated torsion Λ-modules and that for even j, $2 \le j \le p-3$, there is an Λ-isomorphism

$$\mathcal{Y}^{\omega^j} \cong \Lambda/(\dot{g}_i(T)), \text{ where } i + j \equiv 1 \bmod (p-1). \tag{1.17}$$

One can give another formulation of the IMC, namely that for even j, $2 \le j \le p-3$, the characteristic ideals of X^{ω^j} and $\mathcal{X}^{\omega^j}$ are equal.

[12]For $n \not\equiv 2 \bmod 4$ let $V_{\mathbb{Q}(\zeta_n)}$ be the multiplicative group generated by $\{\pm\zeta_n, 1 - \zeta_n^a : 1 \le a \le n-1\}$, then the cyclotomic units of $\mathbb{Q}(\zeta_n)$ are $C_{\mathbb{Q}(\zeta_n)} := V_{\mathbb{Q}(\zeta_n)} \cap \mathcal{O}_{\mathbb{Q}(\zeta_n)}^\times$.

Going beyond Iwasawa. The Iwasawa Main Conjecture can also be stated in a more general setting where F is a finite abelian extension of $\mathbb{Q}$ or a totally real field and $F_\infty = F\mathbb{Q}_\infty$. The necessary p-adic L-functions were constructed by P. Deligne and K. Ribet in [DR80] using Hilbert modular forms and by D. Barsky in [Bar78] and P. Cassou-Noguès in [CN79] using explicit formulas of T. Shintani (1943-1980).

In 1976 [Rib76], Ribet proved the converse of Herbrand's theorem mentioned above, namely that for $2 \leq i, j \leq p-2$, i odd and $i+j \equiv 1 \bmod (p-1)$ it holds: If $p \mid B_j$ then $\mathrm{Gal}(L_0/F_0)^{\omega^i} \neq 0$. Building on ideas of the proof of Ribet the IMC for cyclotomic fields was proven by B. Mazur and A. Wiles in [MW84] and the IMC for totally real base fields was proved by Wiles in [Wil90] using the theory of modular forms.[13]

Elliptic curves and the conjecture of Birch and Swinnerton-Dyer. There is a point of view of number-theoretic problems we have not mentioned yet: the geometric perspective. We will mainly focus on elliptic curves here, which are implicitly already contained in the work of Diophantus.

A modern definition of an elliptic curve over a number field K would read: E is a projective curve of genus 1 with a specific base point on the curve. We denote by $E(K)$ the set of points over K. It turns out that this is an abelian group and H. Poincaré (1854-1912) in [Poi01] defined the rank of $E(K)$ as the minimal number of generators of $E(\mathbb{Q})$, which was not known to be finite at that time. This was only shown 20 years later in [Mor22] by L. Mordell (1888-1972) and then extended and simplified by Weil: For an elliptic curve E over K, $E(K)$ is a finitely generated abelian group, i.e. $E(K) \cong \mathbb{Z}^r \oplus E(K)_{tors}$, where $E(K)_{tors}$ is a finite abelian group.[14] So we have a well-defined rank r of E.

Let Δ be the discriminant of the elliptic curve and define the integer a_p by the equation $|E(\mathbb{F}_p)| = p + 1 - a_p$, where $E(\mathbb{F}_p)$ is the number of solutions of the defining equation of E in $\mathbb{F}_p$ plus the origin. Then we can define an incomplete Hasse-Weil L-function for an elliptic curve $E/\mathbb{Q}$ by setting

$$L(E,s) := \prod_{p \nmid 2\Delta} (1 - a_p p^{-s} + p^{1-2s})^{-1}, \tag{1.18}$$

where the product converges for the real part at least 3/2. H. Hasse (1898-1979) conjectured that as a complex function in s it has a holomorphic continuation to $\mathbb{C}$. This was only shown as a consequence of the modularity theorem proved by C. Breuil, B. Conrad, F. Diamond and R. Taylor in [BCDT01].[15] Now the conjecture of B. Birch and P. Swinnerton-Dyer (BSD conjecture)[16] based on [BSD63] and [BSD65] predicts that the Taylor expansion of $L(E,s)$ at $s=1$ has the form

$$L(E,s) = c(s-1)^r + \text{ higher order terms, with } c \neq 0 \text{ and } r = \mathrm{rank}(E(\mathbb{Q})), \tag{1.19}$$

which can be shortly stated as $\mathrm{ord}_{s=1}(L(E,s)) = \mathrm{rank}_{\mathbb{Z}}(E(\mathbb{Q}))$. Now we can compare this to the Dedekind ζ-function, where we have $\mathrm{ord}_{s=0}(\zeta_K(s)) = \mathrm{rank}_{\mathbb{Z}}(\mathcal{O}_K^\times)$. In [Tat68], J. Tate stated the rank-BSD conjecture in the more general setting of abelian varieties over a number field K, where it says that the rank of the group of K-rational points of an abelian variety A is the order of the zero of an incomplete L-function at $s=1$. This statement uses a generalization of the Hasse-Weil L-function given by Serre [Ser65] or A. Grothendieck (1928-2014), where each of

[13] The approach uses 2-dimensional p-adic representations associated to Hilbert modular forms.

[14] Weil extended this result also to abelian varieties.

[15] For an important class of elliptic curves, the elliptic curves with complex multiplication, which we will discuss below, this was already known beforehand.

[16] This is sometimes also called rank part of the BSD conjecture or weak BSD conjecture.

them defines an L-function for arithmetic schemes[17]. We want to focus on the special case of an elliptic curve over $\mathbb{Q}$ on a more refined conjecture given also in [Tat68].

In order to do that, one has to define Euler product factors for so-called *bad primes*, which are those dividing 2Δ. Although we skip a description of them here, we assume that from now on $L(E,s)$ is a *complete* Hasse-Weil L-function over $E/\mathbb{Q}$. As we have seen above, Dedekind was not only able to compute the order of the zero at $s=0$, but also gave a description of the leading term of the Taylor expansion at $s=1$ in terms of arithmetic invariants (cf. Equation (1.4)). Now in our special case, Tate's conjecture mentioned above, which can be seen as a refinement of the BSD conjecture, predicts that the leading term $L^*(E,1)$ of the complete Hasse-Weil L-function $E/\mathbb{Q}$ is:

$$L^*(E,1) = \frac{\Omega_E \cdot R_E \cdot \#(\text{Ш}(E/\mathbb{Q})) \cdot \prod_{p|\Delta} c_{E,p}}{\#(E(\mathbb{Q})_{tors})^2} \tag{1.20}$$

where Ω_E is the period $\Omega = \int_{E(\mathbb{R})} \frac{dx}{|2y+a_1+a_3|} \in \mathbb{R}$ for the normal form[18] of E, $c_{E,p}$ are small positive integers that measure the reduction of E at p, the regulator R_E measures the complexity of a minimal set of generators of $E(\mathbb{Q})$, and $\text{Ш}(E/\mathbb{Q})$ measures the failure of the Hasse principle. In order to have a well-defined conjecture one has to assume that $\text{Ш}(E/\mathbb{Q})$ is a finite group.

We also want to define an important class of elliptic curves: For an elliptic curve over $\mathbb{C}$ the endomorphism ring $\mathrm{End}(E)$ can now either be isomorphic to $\mathbb{Z}$ or an order $\mathcal{O}$ in an imaginary quadratic number field k. If $\mathrm{End}(E) \cong \mathcal{O}$, we say that E has *complex multiplication.*

Why imaginary quadratic number fields? It is not hard to understand why 19th century mathematicians like Gauss, Kummer and Kronecker were drawn especially to the theory of cyclotomic fields. The beauty and simplicity of the results those mathematicians could obtain for cyclotomic fields is today as mesmerizing as it was then. But the theory served also as a blueprint of what could be true in other situations. We have already mentioned that Gauss foresaw the theory of complex multiplication by extrapolating from what he knew about the cyclotomic theory and 'Kronecker's Jugendtraum' also falls in this category. These two instances are evidence for the notion that looking at abelian extensions of imaginary quadratic number fields is the obvious next thing to do after proving a result for cyclotomic fields. But one might ask why. The most compelling reason is simply that it often works as we will see again below.

Elliptic units. So we want to look at another success story of this principle. There is the classical result (e.g. Theorem 4.9 in [Was97]) that for an even non-trivial Dirichlet character χ with conductor f we have

$$L(1,\chi) = \frac{-\tau(\chi)}{f} \sum_{a=1}^{f} \overline{\chi}(a) \log|1-\zeta_f^a|, \tag{1.21}$$

where $\tau(\chi)$ is a Gauss sum. What is the corresponding result in the case of abelian extensions of imaginary quadratic number fields? After preliminary results of Fueter in [Fue10], a complete answer was given by C. Meyer (1919-2011) in [Mey57], as he succeeded in expressing $L(1,\chi)$ for primitive ray/ring class characters.[19] The main ingredient in the computations are the Kronecker limit formulas, of which one instance is already mentioned in (1.5).

Probably a better known reference for these results are the lecture notes of C.L. Siegel (1869-1981) ([Sie65]), where he wanted to introduce the students to 'some of the important and beautiful

[17] a scheme of finite type over $\mathbb{Z}$

[18] i.e. $E: y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$

[19] Meyer also obtained similar results for abelian extensions of real quadratic number fields in [Mey57].

ideas which were developed by L. Kronecker and E. Hecke'. They contain an explicit description of the value of $L(1,\chi)$ in the imaginary quadratic case as well as many other results, e.g. for abelian extensions of real quadratic fields, where Hecke did some pioneering work. Based on the content of these lectures K. Ramachandra (1933-2011) in [Ram64] constructed what is now called *Ramachandra invariants*, which he used to express also the value of $L(1,\chi)$ using again Kronecker's limit formulas as main input in the proof. He did even more: he showed that his invariants are algebraic, described when they are units and determined their Galois action. He also used them to construct a subgroup of the global unit group of a class field with finite index which he could give explicitly in terms of the class numbers of the class field and the base field and some other arithmetic invariants.

In [Rob73], G. Robert picked up the topic again and constructed, with the same classical modular functions, the invariants for each element of the ray class group $Cl(\mathfrak{f})$, which he called *elliptic units*. He showed their relation to the Ramachandra invariants and that they satisfy similar properties. With these units he also constructed a subgroup of finite index of $\mathcal{O}_{k(\mathfrak{f})}^{\times}$ and computed this index.

BSD conjecture and elliptic units. Although these are certainly interesting results on their own, they seem to help solving only a very particular problem. This changed when J. Coates and Wiles in [CW77] established a link between the BSD conjecture and elliptic units as defined by Robert. They showed that if an elliptic curve E/F has complex multiplication by $\mathcal{O}_k$, where k is imaginary quadratic with class number one, we have: $E(F)$ is infinite implies that $L(E/F,1)=0$ if $F=\mathbb{Q}$ or $F=k$. The problem can be reduced to showing that a certain number $L^*(1)$ (treated as an 'elliptic Bernoulli number') is divisible by infinitely many prime ideals $\mathfrak{p}$ of k. In order to do that they showed that $L^*(1)$ is divisible by a prime of degree 1 if and only if $\mathfrak{p}$ is irregular in an appropriate sense. This notion of irregularity arises from local properties of the elliptic units of Robert.

In giving analogues of theorems of Iwasawa theory in cyclotomic fields elliptic units are also recognized as being useful. In [CW78], Coates and Wiles gave an elliptic analogue of a result of Iwasawa which described the quotient of local units modulo cyclotomic units in terms of p-adic L-functions (cf. (1.17)): Let k be an imaginary quadratic number field with class number one, and E an elliptic curve over k with CM by $\mathcal{O}_k$, ψ a Größencharacter of E over k, $p\mathcal{O}_k=\mathfrak{p}\overline{\mathfrak{p}}$ with $\mathfrak{p}\neq\overline{\mathfrak{p}}$ and p is not anomalous for E and not in a certain set S. Then for $(p-1)\nmid i$ we have

$$\varprojlim_n (U_n/\overline{C}_n)^{(i)} \cong \mathbb{Z}_p[[T]]/(G_i(T)), \tag{1.22}$$

where we first have to introduce some notation in order to understand this theorem: $G_i(T)$ is a power series related to the Hecke L-series for ψ. For the definition of U_n we fix one of the prime factors $\mathfrak{p}$, a uniformizer π coming from $\mathfrak{p}$ via the Größencharacter, let $E_{\pi^{n+1}}$ be the kernel of the endomorphism of π^{n+1} and set $F_n := k(E_{\pi^{n+1}})$. Then $\mathfrak{p}$ is totally ramified in F_n and we denote the unique prime ideal above $\mathfrak{p}$ by $\mathfrak{p}_n$, so we can define U_n as the local units of the completion of F_n at $\mathfrak{p}_n$ which are congruent to 1 mod $\mathfrak{p}_n$. $\overline{C}_n$ is the closure of Robert's group of elliptic units C_n in U_n with respect to the $\mathfrak{p}_n$-adic topology and $(U_n/\overline{C}_n)^{(i)}$ denotes the eigenspace of $U_n/\overline{C}_n$ on which $\mathrm{Gal}(F_0/k)$ acts via χ^i, where χ is the canonical character of $\mathrm{Gal}(F_0/k)$ on E_π.

Prelude to Euler Systems. Usually the main desire of a mathematician is to prove new results. But often finding a different proof of a known theorem can also induce striking developments. One instance of this is certainly the proof of F. Thaine [Tha88] of a result which could also be deduced from the IMC for cyclotomic fields proved by Mazur-Wiles in [MW84]. The result we are talking about is the following: Let F be a real abelian extension of $\mathbb{Q}$ of degree prime to p and $G := \mathrm{Gal}(F/\mathbb{Q})$. Let E be the group of global units of F, C be the subgroup of cyclotomic units, and A be the p-Sylow subgroup of the ideal class group of F. If $\theta \in \mathbb{Z}[G]$ annihilates the p-Sylow subgroup of E/C, then 2θ annihilates A. The method of Thaine used to prove this theorem was also independently found by V. Kolyvagin [Kol88], who applied it at first when studying Selmer groups of modular elliptic curves using Heegner points.

Already in 1987 a paper of K. Rubin ([Rub87a]) was published which contained an extension of the method of Thaine to the case of abelian extensions of imaginary quadratic number fields. Now cyclotomic units were replaced with elliptic units and there was the additional condition that the abelian extension F of the imaginary quadratic number field k had to contain the Hilbert class field of k. In fact, Rubin defined special units of F and used them to construct elements of $\mathbb{Z}[\mathrm{Gal}(F/K)]$ which annihilate certain subquotients of the ideal class group of F. Cyclotomic units and elliptic units are examples of such special units.

Why do we care about the method of proof for these annihilation results? Because now an almost magical thing happens: Most of the things discussed so far suddenly fit together. Rubin in [Rub87b] used the techniques developed in [CW77] and [CW78] to obtain results for the BSD conjecture and the ideal class annihilators arising from elliptic units to prove the following: For an elliptic curve E over an imaginary quadratic number field k he proved that the Tate-Shafarevich group under certain conditions is finite or the $\mathfrak{p}$-part is trivial and that for an elliptic curve over $\mathbb{Q}$ with CM it holds: If $\mathrm{rank}_{\mathbb{Z}}(E(\mathbb{Q})) \geq 2$, then $\mathrm{ord}_{s=1} L(E,s) \geq 2$.

One-variable main conjecture. At this point nobody will be surprised to find out that there is also a generalization of the IMC for cyclotomic fields to abelian extensions over an imaginary quadratic number field k.[20] Let M_∞ be the maximal abelian p-extension of $F_\infty := \bigcup_n F_n$ which is unramified outside the primes above $\mathfrak{p}$ and $Y := \mathrm{Gal}(M_\infty/F_\infty)$, with $Y^{(i)}$ the eigenspace of Y on which $\mathrm{Gal}(F_0/k)$ acts via χ^i. For split primes in k it was already mentioned in [CW78] that for $(p-1) \nmid i$ the assertion

$$Y^{(i)} \text{ and } \varprojlim_n (U_n/\overline{C}_n)^{(i)} \text{ have the same characteristic ideal,} \tag{1.23}$$

could be true, later called *one-variable main conjecture*, and that the case $i \equiv 1 \bmod (p-1)$ would 'have deep consequences for the study of the arithmetic of elliptic curves'. It is also worth recalling (1.13) and (1.17) for cyclotomic fields at this point. This conjecture, (1.23), was proved by Rubin, under some hypotheses, in [Rub91] by controlling the size of certain class groups and using the techniques described above. Also for non-split primes, a formulation of the conjecture was given and proved under more restrictive hypotheses. These results again had applications to the arithmetic of elliptic curves with CM, i.e. results surrounding the BSD conjecture.

Introduction of Euler Systems. So the main input to all these new results is the ability to give an upper bound to the size of ideal classes of cyclotomic fields and Selmer groups of certain elliptic curves. An Euler system was then defined in [Kol90] as a collection of certain Galois cohomology classes satisfying conditions like a norm compatibility. The cyclotomic units, elliptic units and Heegner points mentioned above are all examples of such an Euler system. After preliminary work of Kolyvagin in [Kol90], Rubin in [Rub00], as well as K. Kato and B. Perrin-Riou independently developed then an abstract cohomological machinery which uses an axiomatically defined Euler system as an input and produces upper bounds for the sizes of appropriate Selmer groups as an output.

[20] We resume here the notation of **BSD conjecture and elliptic units**.

Coleman power series and applications. Picking up on a technique introduced in Theorem 5 in [CW78] and [Wil78], Coleman showed in [Col79] the following: Let K be a local field with local parameter π, H be a complete unramified extension, $\mathcal{F}$ a Lubin-Tate formal group, W_n the n-division values and $H_n := H(W_n)$. Then for each $\alpha \in H_n$ there exists an $f_\alpha \in \mathcal{O}_H((T))^\times$ such that $\varphi^i f_\alpha(\omega_n) = N_{n,i}(\alpha)$, where ω_n is a generator of W_n as an $\mathcal{O}_K$-module, $N_{n,i} : H_n \to H_i$ the norm, and φ the Frobenius for H over K. Such a series for a norm-coherent sequence $(\alpha_n)_n$ satisfies a uniqueness property and is called a *Coleman power series*.

In [dS87], E. deShalit generalized Lubin-Tate theory and the theory of Coleman power series to relative extensions in the context of abelian extensions of an imaginary quadratic field. He used these theories to construct p-adic L-functions and proved a functional equation and an analogue of 'Kronecker's second limit formula' for these p-adic L-functions. Then he applied his results to the one-variable main conjecture and the BSD conjecture.

Inspired by [Tha88], D. Solomon in [Sol92] constructed cyclotomic p-units and computed their valuation using the theory of Coleman power series. He applied this result to a 'weak analogue' of Stickelberger's theorem for real abelian fields.

In the meantime there had also been some new developments in the basic theory of elliptic units. Robert succeeded in constructing a function ψ that is a twelfth root of the function φ, which is used to define elliptic units. Now using elliptic units defined via this function ψ and the theory of Coleman series for relative Lubin-Tate extensions W. Bley constructed in [Ble04] elliptic $\mathfrak{p}$-units and computed their valuation in the situation for split primes, under certain hypotheses, in analogy to the result of Solomon.

Stark's conjecture. As we have seen so far, Gauss's D.A. had a major influence on the algebraic number theory of the 19th century. It was possible to embed large parts of his results into a more general framework. One of the more elusive questions coming from D.A. was certainly the class number one problem, namely the question of how many quadratic number fields have class number one. For imaginary quadratic number fields Gauss already suspected he had a complete list of them.

In the 1960s H. Stark gave the first accepted proof ([Sta67]) of this fact and based on the methods used he had the idea that it maybe was possible to evaluate a general Artin L-function at $s = 1$. He later realized that looking at $s = 0$ is simpler and tried to find a theoretical description of $L'(0, \chi)$ for an L-function with a first order zero at $s = 0$ with the help of numerical computations. In 1970, he published his first 'very vague conjectures' which were later tersely presented by Tate (in [Tat84]) giving a Galois-equivariant conjectural link between the values at $s = 0$ of the first non-vanishing derivative of the S-imprimitive Artin L-function $L_{K/k,S}(s, \chi)$ associated to a Galois extension of number fields K/k and certain $\mathbb{Q}[\mathrm{Gal}(K/k)]$-module invariants of the group U_S of S-units in K, where S is a set of places of K satisfying certain conditions. Siegel and Ramachandra proved the instances of the conjecture for imaginary quadratic number fields via complex multiplication and the applications of the Kronecker limit formulas.

Now building on the work of Siegel and Ramachandra, Stark developed an integral refinement of his conjecture for abelian extensions K/k and abelian S-imprimitive L-functions with order at most 1 at $s = 0$ and under additional hypothesis on S: There exists a special v-unit ϵ, called *Stark unit*, such that

$$L'_{K/k,S}(0, \chi) = \frac{-1}{w_K} \sum_{\sigma \in G} \chi(\sigma) \log |\epsilon^\sigma|_w \text{ for each character } \chi \text{ of } G, \tag{1.24}$$

where v is a prime which splits completely in K and w a place above v. Stark proved the cases of $k = \mathbb{Q}$ or k imaginary quadratic in [Sta80]. It has to be noted that a proof of the integral refinement would have far reaching applications to Hilbert's 12th problem and already in the 1980's B. Gross ([Gro81], [Gro88]) developed a refinement of Stark's integral conjecture.

Rubin-Stark conjecture and beyond. Finding Euler systems is generally a difficult task, so it is quite remarkable that Stark's integral conjecture is a source for them. Motivated by this observation Rubin generalized Stark's integral conjecture to the case of abelian extensions of number fields K/k and their S-imprimitive L-functions of order $r \geq 0$ under certain conditions on S. This conjecture is now called the *Rubin-Stark conjecture*. A conjectural Gross-type refinement of Rubin-Stark conjecture was also found by work of Tate, Gross, D. Burns, C. Greither and C. Popescu. This has led to a *Gross-Rubin-Stark conjecture*, which implies the Rubin-Stark conjecture and Gross conjectures and predicts a subtle link between special values of derivatives of global and p-adic L-functions.

Inspired by work of Gross in [Gro88], H. Darmon in [Dar95] formulated a refined class number formula which relates cyclotomic units to certain algebraic regulators in a very particular situation. After proving the non-2-part of this conjecture using Kolyvagin systems, which were developed from Euler systems, in [MR16] Mazur and Rubin generalized Darmon's conjecture and proved certain cases of it. The same conjecture was independently found by T. Sano and so is known as *Mazur-Rubin-Sano conjecture*.

Deligne's and Beilinson's conjecture(s). One of the main themes so far is the interpretation of values of L-series at integers by arithmetical objects. A conjecture of Deligne in [Del79] brings some order to several results presented so far. He conjecturally describes the irrational part of the L-values as determinants of a matrix whose coefficients are up to factor of $2\pi i$ periods at a 'critical integer' for different L-functions.

In a more abstract setting Beilinson's conjectures ([Bei84]) link the leading coefficients at integral arguments of L-functions of algebraic varieties over number fields to the global arithmetical geometry of these varieties. In particular, the leading term should be equal to a value related to a certain regulator up to a rational factor. This should be compared to the leading term of the Dedekind zeta function at $s = 0$ and the covolume of the image of the Dirichlet regulator map. Beilinson's conjectures also deal with the orders of vanishing of quite general L-functions and regulators using the rank and the covolume of motivic cohomology in a very abstract setting.

Tamagawa Number Conjecture and its equivariant refinements. If one would have to summarize most of the number theory presented so far, one way of doing it would be to say that L-functions are related to arithmetic invariants. Prominent instances of the phenomenon we have seen so far are the analytic class number formula (1.4), the refinement of the BSD conjecture (1.20) or the Iwasawa Main Conjecture (1.15).

Although we are far away from fully understanding all these results and conjectures there is an incessant quest for a conjectural framework in a more general setting probably driven by a popular view in mathematics that everything is as we expect it to be. We already encountered the conjectures of Beilinson and Deligne which express the values at integer points of L-functions of smooth projective varieties over number fields in terms of periods and regulator integrals. But these conjectures only determine the special values of the L-functions up to a non-zero rational number. The logical next step was done by S. Bloch and Kato in [BK90], in which they generalized the refinement of the BSD conjecture to L-functions for arbitrary smooth projective varieties over number fields. They were therefore removing the $\mathbb{Q}^\times$ ambiguity, which culminated in a conjecture now known as *Tamagawa Number Conjecture (TNC)* or *Bloch-Kato conjecture for special values of L-functions*. It is rather unsurprising that the first partial results Bloch and

Kato showed concern the Riemann zeta function, e.g. they show the TNC up to a power of 2 for the Tate motive $\mathbb{Q}(r)$ and r even as well as the elliptic curves with complex multiplication since these are the classic test cases for conjectures in arithmetic geometry.

Kato in [Kat93a] refined the theory by defining for a variety X over a number field K and a finite abelian extension L/K the $\mathrm{Gal}(L/K) \ni \sigma$-part of the corresponding L-function and relating special values of such partial L-functions to the $\mathrm{Gal}(L/K)$-module structure on the étale cohomology of $\mathrm{Spec}(\mathcal{O}_L)$ with coefficients in an étale sheaf coming from X. For the situation $K = \mathbb{Q}$, $X = \mathbb{Q}(r)$ and L being a cyclotomic extension of $\mathbb{Q}$ it can be shown that this conjecture is equivalent to the IMC for cyclotomic fields. So Bloch and Kato described the value at zero of L-functions attached to motives with negative weight. By using perfect complexes and their determinants Kato and, independently, J.M. Fontaine (1944-2019) and Perrin-Riou ([FPR94], [Fon92]) also took into account the action of the variety under consideration. This approach via perfect complexes was then used by Burns and M. Flach in [BF96] to define invariants which measure the Galois module structure of the various cohomology groups arising from a motive M over a number field, which admits the action of a finite abelian Galois group. At the end of the 1996 paper they gave a formulation of the *equivariant Tamagawa Number Conjecture* (eTNC) with abelian coefficients, which is Conjecture 4 in [BF96]. In [BF01] Burns and Flach also gave a formulation of the eTNC with non-commutative coefficients and the work of Kato from [Kat93b] on p-adic zeta functions was also generalized, e.g. by T. Fukaya and Kato [FK06], who dealt with the non-abelian situation, too.

The abelian number fields case. So we now have surveyed a massive conjectural framework but learned few about results so far except the testing cases around which the general conjectures are built. But there are some proven instances of the eTNC, the most prominent one being the case of Tate motives of weight ≤ 0 for abelian extensions over $\mathbb{Q}$, i.e. the cyclotomic case proved by Burns and Greither in [BG03] (and independently by A. Huber/G. Kings in [HK03] plus work of M. Witte in [Wit06]). This result can, in a way, be seen as a (probably tentative) peak of the study of cyclotomic fields initiated by Gauss in D.A. approximately 200 years ago. The beauty of this result lies in the fact that it uses a lot of knowledge about cyclotomic fields we have collected over the years. First of all, one can use the Theorem of Kronecker-Weber to reduce to cyclotomic fields and Stark's conjecture for the rationality part of the conjecture. It used the computation of the evaluation of the Dirichlet L-function at $s = 1$, and the functional equation to get the leading term at $s = 0$. The conjecture for a cyclotomic Iwasawa tower is then proved by using the IMC for cyclotomic fields, the vanishing of the μ-invariant for abelian extensions over $\mathbb{Q}$ and a reduction to the localization at height one prime ideals of an Iwasawa algebra. From this result one descents to the finite level of interest with techniques described by J. Nekovář. This descent procedure is quite delicate and uses the result of Solomon on cyclotomic p-units mentioned above as well as a result of Ferrero and R. Greenberg ([FG79]) on the first derivative of a p-adic L-function. Maybe the best way of describing the proof of this result is due to Nekovář who wrote at the end of his review of the paper: 'This is what Iwasawa theory should look like in the new millennium!'.

eTNC implies ... What makes the eTNC such a grand conjecture is that it subsumes a lot of independently developed conjectures in algebraic number theory. For the motive $M = \mathbb{Q}(r)$ it implies or generalizes Stark's conjecture [Tat84], the Rubin-Stark conjecture [Rub96] and its refinements, Popescu's conjecture in [Pop02], the Mazur-Rubin-Sano conjecture of [MR16] and [BKS16], the strong Stark conjecture of Chinburg from [Chi83], the '$\Omega(3)$' conjecture of T. Chinburg from [Chi83] and [Chi85], the Lifted Root Number Conjecture of K. Gruenberg (1928-2007), J. Ritter and A. Weiss [GRW99], and many more. Proofs for this implications can be found in for example in [Bur10], [Bur07], [BKS16]. It certainly also generalizes the analytic class number formula as alluded above and for $M = h^1(E)(1)$, the twisted motive associated to an elliptic curve E, the eTNC (in fact already the TNC) implies the refinement of the BSD conjecture. A proof of this is given in [Kin11]. The list given is only focussing on results discussed above or very near to them by plugging in two classical motives with abelian coefficients. So one may assume this is only the tip of the iceberg.

1.2 The thesis in a nutshell

1.2.1 What is the objective?

The goal of this thesis is to work towards proving the p-part of the eTNC for Tate motives at $s = 0$ over abelian extensions of imaginary quadratic number fields k in which the prime p is non-split in k.

1.2.2 What is the motivation?

First of all one could look at it historically as above: Gauss treated in his Disquisitiones Arithmeticae the antecedent of cyclotomic theory and then somehow predicted that the theory of complex multiplication, intimately related to imaginary quadratic number fields, should exist. Kronecker dreamt about generalising the Kronecker-Weber Theorem to imaginary quadratic number fields and Takagi proved Kronecker's Jugendtraum. Mazur and Wiles proved the IMC for cyclotomic fields, later Rubin and Greither gave an Euler system proof of this result which also worked for the one-variable main conjecture of Coates and Wiles over imaginary number quadratic fields and split primes. Then Rubin found a formulation of the main conjecture for all primes and proved large parts of it under some additional hypotheses. The last step in our historical treatment is the proof of the eTNC for 'untwisted' (and negatively twisted) Tate motives over abelian extensions of $\mathbb{Q}$ by Burns and Greither. So we have seen in many instances that what can be done for cyclotomic fields can also be done in a similar way for abelian extensions of imaginary quadratic number fields. From this perspective the logical next step is to look at the eTNC for Tate motives at $s = 0$ over abelian extensions of imaginary quadratic number fields.

Another motivation, admittedly, is the hope to be able to prove something new, because analogues to the main ingredients of the proof of Burns and Greither exist for abelian extensions of imaginary quadratic number fields: There is the computation of the leading term of the abelian L-function in the imaginary quadratic case via Kronecker's limit formulas, Stark's conjecture and Leopoldt's conjecture are valid for this case, the suitable main conjecture holds under some hypotheses, we have an explicit class field theory and we know a lot about elliptic units which are replacing cyclotomic units here. Moreover, there is an equivariant main conjecture proved under a vanishing assumption by Johnson-Leung and Kings in [JLK11].

1.2.3 What was the starting point?

The idea of proving the eTNC for Tate motives at $s = 0$ via the approach of Burns and Greither was already started by Bley in [Ble06]. He considered the case where we have an imaginary quadratic number field k and an odd prime p which splits in k. For this situation he showed the p-part eTNC for Tate motives at $s = 0$ over abelian extensions of k under the additional hypothesis that $p \nmid 2h_k$. A main new ingredient in the proof was the main result of [Ble04] on the construction of an elliptic $\mathfrak{p}$-unit and its valuation, which is an analogue of the main result in [Sol92]. The proof also uses the corresponding main conjecture and the vanishing of a μ-invariant in the corresponding situation to prove an Iwasawa-theoretic version of the eTNC, what we will call *Limit Conjecture*, for split primes as an intermediate step. As mentioned above, a very similar assertion to this Limit Conjecture was given in [JLK11] for all primes p and a proof for this conjecture was given assuming a certain vanishing result specified in Conjecture 6.3.13.

1.2.4 What are the main new results?

The first new result is an analogue of the main results of [Sol92] and [Ble04] for abelian extensions L of imaginary quadratic number fields k and *non-split* primes in k under the hypotheses that $\mathfrak{p}$, the prime ideal above the rational prime p in k, splits completely in L and $p \nmid h_k$ as well as $p > 3$ (in some special cases we have also results for $p = 2, 3$). This is joint work with my advisor Werner Bley accepted for publication in the Proceedings of the Iwasawa conference 2017 appearing in the series 'Advanced Studies in Pure Mathematics' and both authors have contributed equally to this work.

Then we use this result to prove some special cases for *non-split* primes of the Iwasawa-theoretic conjecture of Mazur, Rubin and Sano (Conjecture 5.1.14), which are Theorem 5.2.2 and Theorem 5.2.3.

Finally we apply the first result to make descent computations, as in [Fla04] and [Ble06] from Limit Conjectures to the p-part of the eTNC for Tate motives at $s = 0$ over abelian extensions of an imaginary quadratic number field k with $p \nmid 2h_k$ and p *inert* in k. In order to do that we have to use an assumption for explicitly computing the cohomology of the complex in question after localizing at certain height one prime ideals. We call this assumption Condition (F) , which is a finiteness condition similar (but slightly stronger) then the homonymous assumption in [BKS17](for details see Section 6.3.4).

1.2.5 How did we get there?

A common theme in Iwasawa theory and related topics for abelian extensions over imaginary quadratic number fields could be summarized as follows: 'Your world is easier, when your primes are split'. What makes the split situation pleasant is for example that if we choose a prime $\mathfrak{p}$ above p in k, there is a unique $\mathbb{Z}_p$-extension which is unramified outside the primes above $\mathfrak{p}$ and in which each finite subextension is cyclic. In the non-split case one gets a canonical $\mathbb{Z}_p^2$-extension, which is also unramified outside of $\mathfrak{p}$ but contains infinitely many $\mathbb{Z}_p$-extensions.

So the first idea was to find at least an assertion which could generalize the main result of [Ble04] to non-split primes. It was not clear at all what that should be, because there is a cyclic canonical extension to work with in the split case and the construction of the elliptic $\mathfrak{p}$-unit heavily relies on the classical Hilbert's Theorem 90 which requires a cyclic extension. However, we simply chose two generators of the $\mathbb{Z}_p^2$-extension and constructed for each of the corresponding $\mathbb{Z}_p$-extensions an elliptic $\mathfrak{p}$-unit. Then we conjectured in analogy to the split case that the valuations at primes above p together should give the valuation of the p-adic logarithm of the elliptic unit at a base level L.

In order to obtain some confidence in the conjecture, we computed for some examples both sides of our conjecture modulo p^n for $p = 3, 5$ and $n = 1, 2, 3$. The main input was an algorithmic implementation of the work of Robert [Rob92] in MAGMA with which we computed the approximate values and then the valuation of elliptic $\mathfrak{p}$-units as well as the p-logarithm of the elliptic unit at the base level. We also tried to get to higher levels n by focussing on ring class extensions with varying success. In all the cases where runtime and memory space would allow it, the conjecture was corroborated. Moreover, we had some weak theoretical results for our conjecture and for a ring class version of it.

Proving this result seemed out of reach mostly because the proof in the split case used the theory of relative Lubin-Tate groups and in the non-split case one would get a Lubin-Tate group of height two, compared to an group of height one in the split case. It became clear that simply translating the known proof is not an option, because we did not see a way of reproducing a main ingredient in the proof of [Sol92] and [Ble04], namely the comparison of two Coleman power series associated to two norm coherent sequences derived from the assertion of the conjecture. But there was hope because after showing an equality containing the two power series, only the constant term of them was used later on. So we thought it might be easier to compute only the constant term, which is not at all a canonical strategy because one normally prefers to deal with a power series not some evaluation of it.

But then we found an article of T. Seiriki in arXiv ([Sei17]) which claimed to reprove a result on Euler systems for function fields by linking the constant term of a Coleman power series to a pairing which is defined by taking the valuation of an element constructed by Hilbert's theorem 90 in a local setting. We soon realized that if this result was correct, it would probably give us what we needed. In trying to understand Seiriki's work we reproved the needed parts and then succeeded in proving our main result on the valuation of an elliptic $\mathfrak{p}$-unit in the non-split case.

Now we could tackle the problem which justified putting so much effort in the result mentioned above: the descent computation of the eTNC for abelian extensions of imaginary quadratic number fields. At the very early stages of this dissertation project, at the Iwasawa conference 2015, we learned about the progress Burns, Kurihara and Sano made towards the descent computations. They showed for an arbitrary abelian extension L/K of number fields that if the extension is contained in a certain extension of $\mathbb{Z}_p$-rank one, they could do the descent computations for this general situation under several assumptions. The two major assumptions were a Limit-Conjecture-type result, which they call *higher-rank main conjecture of Iwasawa theory*, and the Iwasawa-theoretic MRS conjecture. They showed the usefulness of their result by reproving the eTNC for Tate motives at $s = 0$ over abelian fields and remarked that similar arguments can be given to reprove the main result of [Ble06].

Therefore, we decided to develop the results in two directions knowing that they are not without intersection. First of all we tried which results we can show for the Iwasawa-theoretic MRS conjecture using our new result about elliptic $\mathfrak{p}$-units in this case. Then we tried to show what results could be obtained from translating the descent process from [Fla04] and [Ble06] directly to the non-split case, where we decided - for the sake of clarity - to present only the inert case in this dissertation. The experts will certainly object at this point that the descent in [BKS17] is only a streamlined and more general version of the other descent procedures. But the author thinks it is somewhat unnecessary to deal with Rubin-Stark elements in a case where we have concrete realizations of them at our disposal.

1.2.6 Overview chapter by chapter

In Chapter 2 we introduce the basic objects we are going to work with: elliptic functions, elliptic units and abelian L-functions. We compare different definitions of elliptic units and as conclusion we recall the proof for expressing the leading terms of the L-function at $s = 0$ in terms of the elliptic units we are going to use, mainly following [Lan73]. This chapter contains no new results, but we hope the reader will find some benefit in a survey of all important results and definitions we are using.

In Chapter 3 we present a pairing which is defined via the valuation of an element constructed with Hilbert's Theorem 90 and we recall the theory of relative Lubin-Tate groups and Coleman power series. Then we prove Theorem 3.4.1 which links this pairing to the constant term of a corresponding Coleman power series. This chapter relies on ideas of [Sei17] and the content of this chapter is almost verbatim contained in the joint work [BH18] with Bley.

In Chapter 4 we give a version of the main theorems of [Sol92] and [Ble04] for non-split primes in the imaginary quadratic situation. We prove this new result with help of the assertions of Chapter 3 and reprove the two older results with this new technique. Here the non-split result is also contained in joint work [BH18] with Bley .

In Chapter 5 we introduce the reader to a conjecture of Mazur, Rubin and Sano, and an Iwasawa-theoretic version of it. We use the main result of Chapter 4 to obtain new results for the Iwasawa-theoretic MRS for non-split primes. This draws from ideas in the proof of Theorem 4.10 in [BKS17].

Chapter 6 is concerned with the eTNC for Tate motives over abelian extensions of imaginary quadratic number fields. We introduce the eTNC and the different versions of the Limit Conjecture and study relations between them. We survey the descent computations of [BKS17] and show what can be accomplished with the main result of Chapter 5 and Theorem 5.2 of [BKS17]. Following this, we state our descent result as Theorem 6.5.1 and present the necessary computations. As the computation is quite involved we start with a summary and a description of what we are going to do. After some preliminaries we subdivide the computations into three parts, where the non-trivial zeroes case is the most important one, because in order to be able to finish the computations we need Theorem 4.1.16 of Chapter 4.

1.3 Notation

The following notations will be used throughout this thesis, more specific notation will be introduced as soon as we need it. Let k be a number field, $\mathcal{O}_k$ its ring of integers and h_k its class number. For a place v of k we denote by $\mathcal{N}(v)$ the cardinality of the residue class field of v. We denote a set of places of k which lie above the infinite place ∞ in $\mathbb{Q}$ (resp. a prime p) by $S_\infty(k)$ (resp. $S_p(k)$). For a Galois extension L/k, the set of places of k that ramify in L is denoted by $S_{ram}(L/k)$ and for any set S of places of k, we denote by S_L or $S(L)$ the set of places of L which lie above the places of S. Let L/k be an abelian extension with Galois group G. For a place v of k, we denote the decomposition group by G_v or D_v and the inertia group by I_v. If v is unramified in L, the Frobenius automorphism is denoted by Fr_v or σ_v. We denote by $\mathfrak{f}_L$ the conductor of L and if $\mathfrak{c}$ is an integral ideal relatively prime to the conductor $\mathfrak{f}_L$, then we write $\sigma(\mathfrak{c})$ or $(\mathfrak{c}, L/k)$ for the associated Artin automorphism.

For an ideal $\mathfrak{f}$ in $\mathcal{O}_k$ we denote by $Cl(\mathfrak{f})$ the ray class group modulo $\mathfrak{f}$, by $k(\mathfrak{f})$ the ray class field of conductor $\mathfrak{f}$ and by $k(1)$ the Hilbert class field. Moreover, we denote by $w(\mathfrak{f})$ the number of roots of unity congruent to 1 modulo $\mathfrak{f}$, where $w(1)$ equals the number w_k of roots of unity in k. As we will mention imaginary quadratic number fields constantly we abbreviate them sometimes to 'i.q. fields', 'i.q. number fields' or 'imaginary quadratic fields'.

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Chapter 2

Elliptic units and L-functions

As we have seen in the introduction, the main goal of this dissertation is to work towards proving an assertion about the non-zero leading term of an equivariant L-function using special units called elliptic units. The main objective of this chapter is to carefully define elliptic units and L-functions and present classical results about the relation between these two objects sometimes called *Kronecker's first (resp. second) limit formula*. Our interest in this results stems from the fact that it is the most important analytical input in our computations later on. It is certainly no coincidence that the two cases where we have results of this sort (i.e. the cyclotomic and elliptic case) are also the cases where we have the most impressive results for the eTNC for Tate motives.

This chapter is structured as follows: In the first section we define classical elliptic functions, some of them already known by Dedekind and Weber. Then we use them to define elliptic units, where we introduce several variants contained in the literature and compare them.

The second section serves as a brief introduction to the theory of L-functions used in this manuscript. We define the classical abelian L-function, Artin L-functions and conclude the section by presenting the functional equation for the leading terms at $s = 0$ and $s = 1$, respectively.

The third section first presents the Kronecker limit formulas and then uses them to prove relations between L-functions and elliptic units. Indeed, we compute $L(1, \chi)$ with the help of the Kronecker limit formulas and then use the functional equation in order to get an expression of the leading term $L^*(0, \chi)$ using elliptic units. This section ends with the description of an approach that does not require to compute the value of the L-function at $s = 1$ first. This is done in [Sta80] and hinted at in [Kat04].

We want to stress that nothing in this chapter is new and that it is based on several beautiful treatments of this topic like [Lan73], [Mey57], [Sta80], [Sie65] and [Ram64]. One justification for presenting these classical results in depth is that, although there exist several treatments of these topics, it is sometimes not easy to pin down the arguments used in proving the assertions.

A similar survey has been done by Flach as a part of [Fla09] and our survey can be seen as an expanded version of that. For the reader well-versed in these topics it should be enough to look at Definition 2.1.24 and Theorem 2.1.27, i.e. the definition of elliptic units and their norm relation, as well as Proposition 2.3.9 and Corollaries 2.3.5 and 2.3.7, i.e. the expression of the leading term of the L-function at $s = 0$ in terms of these elliptic units, in order to be able to continue with the next chapters.

Notation In this chapter we denote by $\mathbb{H}$ the upper half plane of $\mathbb{C}$. Furthermore, in order to efficiently write down product expansions of elliptic functions we set $q_z = e^{2\pi i z}$ for $z \in \mathbb{C}$. Let L be a lattice in $\mathbb{C}$. If ω_1, ω_2 is a basis of the lattice L over $\mathbb{Z}$ we write $L = [\omega_1, \omega_2]$. We say that a lattice L (resp. an elliptic curve E) has complex multiplication if the endomorphism ring of $\mathbb{C}/L$ (resp. of E) is isomorphic to an order in the ring of integers of an imaginary quadratic field k.

Sometimes we sum over sets containing a 'zero-element', so in order to get a sum that has the chance to be convergent we exclude them from the summation and denote this by $\sum'$ for the summation over all the elements without this zero-element. For example, if we sum over $(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ we would write $\sum'_{(m,n)}$. The same applies to product symbols.

2.1 Elliptic functions and elliptic units

2.1.1 Elliptic functions

Recall that a lattice in $\mathbb{C}$ is a free subgroup of rank 2 over $\mathbb{Z}$, which generates $\mathbb{C}$ over $\mathbb{R}$. Let $L = [\omega_1, \omega_2]$ be a lattice. Unless otherwise specified, we always assume $\omega_2/\omega_1 \in \mathbb{H}$, which is the convention of [Rob73], [Rob90] and [Ram64] but different from [Lan73]. We first summarize some classical definitions.

Definition 2.1.1. a) The *Weierstrass $\wp$-function* is defined by

$$\wp(z,L) = \frac{1}{z^2} + \sum_{\omega \in L}{}' \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

b) We define the *Weierstrass σ-function*, which has zeros of order 1 at all lattice points, by the Weierstrass product

$$\sigma(z,L) = z \prod_{\omega \in L}{}' \left(1 - \frac{z}{\omega}\right) e^{z/\omega + \frac{1}{2}(z/\omega)^2}.$$

c) Formally taking the logarithmic derivative yields the *Weierstrass ζ-function*

$$\zeta(z,L) = \frac{\sigma'(z,L)}{\sigma(z,L)}.$$

d) There is also a constant η_ω such that

$$\zeta(z+\omega, L) = \zeta(z,L) + \eta_\omega \text{ for all } z \in \mathbb{C}.$$

If $L = [\omega_1, \omega_2]$, then one uses the notation $\eta_1 = \eta_{\omega_1}$ and $\eta_2 = \eta_{\omega_2}$.

e) We define the *Dedekind η-function* by

$$\eta(z) = q_z^{1/24} \prod_{n=1}^{\infty} (1 - q_z^n).$$

f) We define the *Delta function* by $\Delta(z) = (2\pi)^{12} \eta^{24}(z)$.

g) For $t \in \mathbb{C}$ and $z \in \mathbb{H}$ we set

$$\theta_1(t,z) = 2 q_z^{1/8} \sin(\pi t) \prod_{n=1}^{\infty} (1 - q_z^n)(1 - 2q_z^n \cos(2\pi t) + q_z^{2n}).$$

Remark 2.1.2. a) The Dedekind η-function is holomorphic on $\mathbb{H}$.

b) By setting $\Delta(L) := \omega_1^{-12}\Delta(\omega_2/\omega_1)$ we get a definition of the Delta function independent of the choice of the basis of L.

c) The function $t \mapsto \theta_1(t; w)$ is entire, the zeroes are points of $[1, w]$ each of order 1 and we have

$$\frac{d}{dt}\theta_1(t; w)_{|t=0} = 2\pi\eta^3(w).$$

d) For $z \in \mathbb{H}$, Ramachandra defines in [Ram64]

$$\Phi_0\left(\begin{pmatrix} v \\ u \end{pmatrix}, z\right) = e^{-\pi i u(v-uz)}\frac{\theta_1(v-uz, z)}{\eta(z)}.$$

Definition 2.1.3. Let L and $\underline{L}$ be two lattices satisfying $L \subseteq \underline{L}$ and the index $N := [\underline{L} : L]$ is odd. Then we define the *Klein function* $K(z; L, \underline{L})$ as the product

$$K(z; L, \underline{L}) = \prod_{(\mu)_L \in T} (\wp(z, L) - \wp(\mu, L))^{-1},$$

where the class $(\mu)_L$ of μ modulo L runs through the finite set T of $(N-1)/2$ torsion points of the torus $\mathbb{C}/L$ such that $T \cup (-T) = (\underline{L}/L) \setminus \{(0)_L\}$.

In order to be able to compare different constructions properly, we also recall the definition of the classical φ-function. The following definition stems from [Rob73].

Definition 2.1.4. We define the following functions:

a)

$$\vartheta(t; \omega_1, \omega_2) = \exp\left(\frac{\pi t^2 \overline{\omega_1}}{2\omega_1 a(L)}\right)\frac{\theta_1(t/\omega_1, \omega_2/\omega_1)}{\eta(\omega_2/\omega_1)} \text{ with } a(L) = \frac{\omega_2\overline{\omega_1} - \omega_1\overline{\omega_2}}{2i},$$

b)

$$\varphi(t; \omega_1, \omega_2) = \exp\left(\frac{-H(t,t)}{2}\right)\vartheta(t; \omega_1, \omega_2) \text{ with } H(t, t') = \frac{\pi \bar{t} t'}{a(L)},$$

c)

$$\eta^{(2)}(\omega_1, \omega_2) = \frac{2\pi}{w_1}\eta^2(\omega_2/\omega_1).$$

Remark 2.1.5. a) The function $t \mapsto \vartheta(t; \omega_1, \omega_2)$ is entire, the zeros are the points of the lattice $L = [\omega_1, \omega_2]$ each of order one. It is the unique reduced theta function with divisor (0) relative to $\mathbb{C}/L$ such that

$$\frac{d}{dt}\vartheta(t; \omega_1, \omega_2)_{|t=0} = \frac{2\pi\eta^2(\omega_2/\omega_1)}{\omega_1}.$$

b) The function $t \mapsto \vartheta^{12}(t; \omega_1, \omega_2)$ is the unique reduced theta function with divisor $12(0)$ relative to $\mathbb{C}/L$ such that

$$\lim_{t\to 0}\frac{\vartheta^{12}(t; \omega_1, \omega_2)}{t^{12}} = \Delta(L).$$

c) The functions $t \mapsto \vartheta^{12}(t;\omega_1,\omega_2)$ and $t \mapsto \varphi^{12}(t;\omega_1,\omega_2)$ are independent of the choice of basis of L and it makes sense to write $\vartheta^{12}(t;L)$ and $\varphi^{12}(t;L)$, respectively. Moreover, let $f > 0$ be an integer. If $t \in \mathbb{C}$ is such that $ft \in L$, then $\varphi^{12f}(t;L) = \varphi^{12f}(t;\omega_1,\omega_2)$ only depends on the class t modulo L. This is Lemma 2 in [Rob73, §1].

d) For $a \in \mathbb{C}^\times$ Lemma 1 in [Rob73, §1] shows that

$$\vartheta(at;a\omega_1,a\omega_2) = \vartheta(t;\omega_1,\omega_2) \text{ and } \varphi(at;a\omega_1,a\omega_2) = \varphi(t;\omega_1,\omega_2).$$

e) One can check that

$$\varphi(t;\omega_1,\omega_2) = i q_{\omega_2/\omega_1}^{1/12} q_{t/\omega_1}^{-1/2}(1-q_{t/\omega_1}) e^{\frac{\pi t(\overline{\omega_1}t-\omega_1\bar{t})}{2\omega_1 a(L)}} \prod_{n=1}^{\infty}\left(1-q_{t/\omega_1}q_{\omega_2/\omega_1}^n\right)\left(1-q_{t/\omega_1}^{-1}q_{\omega_2/\omega_1}^n\right),$$

which is the definition of the φ-function given in [Rob90] in our notation.

f) If we set

$$K(z,L) = \frac{\varphi(z;\omega_1,\omega_2)}{\eta^{(2)}(\omega_1,\omega_2)},$$

then as a function in t, $K(t,L)$ is independent of the choice of basis of L.

g) Assume we are in the situation as in Definition 2.1.3. Then on page 237 in [Rob90] it is shown that

$$K(z;L,\underline{L}) = \frac{K(z,L)^{[\underline{L}:L]}}{K(z,\underline{L})}.$$

Elliptic functions in [Lan73] Let $a = (a_1,a_2) \in \mathbb{Q}^2$ not both integers. Then we say that N is a *precise denominator* of a, if it is the least common multiple of the denominators of a_1 and a_2.

Definition 2.1.6. a) We define

$$\tilde{\varphi}(z;\omega_1,\omega_2) = e^{-\frac{1}{2}\eta_1\omega_1(z/\omega_1)^2} \cdot q_{z/\omega_1}^{\frac{1}{2}} \cdot \sigma(z,L).$$

b) We define the function

$$f(z;\omega_1,\omega_2) = \frac{2\pi i}{\omega_1} \cdot q_{z/\omega_1}^{-\frac{1}{2}} \cdot \eta^2(\omega_2/\omega_1) \cdot \tilde{\varphi}(z;\omega_1,\omega_2)$$

and for $L = [1,\tau]$ we set $f(z;\tau) := f(z;1,\tau)$

c) For $u, v \in \mathbb{R}$ we define

$$\theta(u,v;\tau) = \theta\left(\begin{pmatrix} u \\ v \end{pmatrix};\tau\right) = f(u - v\tau;\tau)e^{\pi i v(v\tau-u)}.$$

d) Let $(a_1,a_2) \in \mathbb{Q}^2$ not both integers, abbreviated by $a = (a_1,a_2)$ and suppose N is the precise denominator of a. We define *Siegel functions of primitive level* N by

$$H(a;\tau) = H\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix};\tau\right) = H_a(\tau) = \theta(a;\tau)^{12N}.$$

Remark 2.1.7. By Theorem 4' in [Lan73] we have the following q-expansion:

$$\tilde{\varphi}(z;\omega_1,\omega_2) = \frac{\omega_1}{2\pi i}(q_{z/\omega_1}-1)\prod_{n=1}^{\infty}\frac{(1-q_{\omega_2/\omega_1}^n q_{z/\omega_1})(1-q_{\omega_2/\omega_1}^n q_{z/\omega_1}^{-1})}{(1-q_{\omega_2/\omega_1}^n)^2}.$$

Work of Robert in [Rob90] The main result of this section is needed later on to define our elliptic units is Theorem 2.1.11. As in [Rob90], for a lattice $L = [w_1, w_2]$ we are going to call a basis satisfying $w_2/w_1 \in \mathbb{H}$ a *positively oriented basis* and we write now often ω for a basis of a lattice L.

Definition/Lemma 2.1.8. [Rob90, p. 236]

a) If $M \in \mathrm{SL}_2(\mathbb{Z})$ and $\widetilde{\omega} = (\widetilde{w_1}, \widetilde{w_2})$ is a basis of L with the identity

$$\begin{pmatrix} \widetilde{w_2} \\ \widetilde{w_1} \end{pmatrix} = M \begin{pmatrix} w_2 \\ w_1 \end{pmatrix},$$

there exists a 12-th root of unity $\rho^{(2)}(M)$ such that $\eta^{(2)}(\widetilde{w_1}, \widetilde{w_2}) = \rho^{(2)}(M)\eta^{(2)}(w_1, w_2)$.

b) Let L and $\underline{L}$ be lattices such that $L \subset \underline{L}$. For each of these choose a positively oriented basis $\omega = (w_1, w_2)$ and $\underline{\omega} = (\underline{w_1}, \underline{w_2})$, respectively. The matrix identity

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = B \begin{pmatrix} \underline{w_1} \\ \underline{w_2} \end{pmatrix}$$

defines a matrix $B = \beta(\omega, \underline{\omega}) \in M_2^{>0}(\mathbb{Z})$ satisfying $\det(\beta(\omega, \underline{\omega})) = [\underline{L} : L]$.

Definition 2.1.9. [Rob90, p. 237] Assume we are in the situation as in Definition/Lemma 2.1.8 b). Then we define F_ω^B as the function

$$F_\omega^B : z \longmapsto \frac{\varphi(z; \omega)^{\det(B)}}{\varphi(z; \underline{\omega})}.$$

Lemma 2.1.10. *[Rob90, p. 237] Assume we are in the situation as in Definition/Lemma 2.1.8 b) and that* $\det(B)$ *is odd. Then we have the equality*

$$F_\omega^B(z) = \frac{\eta^{(2)}(\omega)^{\det(B)}}{\eta^{(2)}(\underline{\omega})} K(z; L, \underline{L}).$$

Theorem 2.1.11. *[Rob90, Théorème principal, (15), p. 238] Let* L *and* $\underline{L}$ *be two complex lattices with bases* ω *and* $\underline{\omega}$ *(pos. oriented) such that* $L \subset \underline{L}$ *and the index* $[\underline{L} : L]$ *is prime to* 6. *Then there exists a* $\rho(\omega, \underline{\omega}) \in \mathbb{C}^\times$ *which has the following three properties:*

a) The quotient

$$\delta(L, \underline{L}) := \rho(\omega, \underline{\omega})^{-1} \frac{\eta^{(2)}(\omega)^{[\underline{L}:L]}}{\eta^{(2)}(\underline{\omega})}$$

is independent of the choice of bases.

b) The holomorphic function

$$\tilde{F}(z; L, \underline{L}) := \rho(\omega, \underline{\omega})^{-1} F_\omega^B(z)$$

with $B = \beta(\omega, \underline{\omega})$ *does not depend on the choice of bases.*

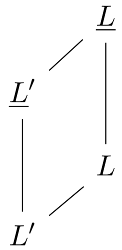
c) Let $L, \underline{L}, \underline{L}', L'$ *be four lattices satisfying the following conditions:*

i) the inclusions $L' \subset L \subset \underline{L}$, $L' \subset \underline{L}' \subset \underline{L}$;

ii) the isomorphisms $\underline{L}/L \cong \underline{L}'/L'$, $L/L' \cong \underline{L}/\underline{L}'$;

iii) $\underline{L}' \cap L = L'$;

which can be illustrated by the following diagram:



Furthermore, let $\{z_j\}$, $1 \leq j \leq [L : L']$, *be a system of representatives of the quotient* L/L'. *Then we have the distribution relation*

$$\tilde{F}(z; L, \underline{L}) = \prod_{j=1}^{[L:L']} \tilde{F}(z + z_j, L', \underline{L}').$$

Remark 2.1.12. a) The number $\rho(\omega, \underline{\omega})$ is unique and a 12-th root of unity.

b) Let $N := [\underline{L} : L]$ be prime to 6. Then we have

$$\tilde{F}(z; L, \underline{L})^{12} = \frac{\Delta(L)^N}{\Delta(\underline{L}) K(z; L, \underline{L})^{12}} \text{ and } \delta(L, \underline{L})^{12} = \frac{\Delta(L)^N}{\Delta(\underline{L})}$$

with $K(z; L, \underline{L})$ from Definition 2.1.3 as well as $\Delta(L)$ and $\Delta(\underline{L})$ from Remark 2.1.2 b). This follows from Lemma 2.1.10 and Corollaire 1 on page 218 in [Rob92], respectively.

Elliptic functions in [dS87] Let $L = [\omega_1, \omega_2]$ be a lattice of $\mathbb{C}$ with $\tau = \omega_2/\omega_1 \in \mathbb{H}$. In [dS87] de Shalit defines

$$\eta(z, L) := \frac{\omega_2\eta_1 - \omega_1\eta_2}{2\pi i A(L)}\bar{z} + \frac{\overline{\omega_1}\eta_2 - \overline{\omega_2}\eta_1}{2\pi i A(L)} z$$

with $A(L) := (2\pi i)^{-1}(\omega_2\overline{\omega_1} - \overline{\omega_2}\omega_1) = \pi^{-1}a(L)$, which is an invariant of the lattice, as well as $\eta_1 := \omega_1 \sum_n \sum_m' (m\omega_1 + n\omega_2)^{-2}$ and $\eta_2 := \omega_2 \sum_m \sum_n' (m\omega_1 + n\omega_2)^{-2}$. Using this he defines what he calls the *fundamental theta function*

$$\tilde{\theta}(z, L) := \Delta(L) \cdot e^{-6\eta(z,L)z} \cdot \sigma(z, L)^{12},$$

where one uses Definition 2.1.1 b) and Remark 2.1.2 b).

With the help of [Wei76, Ch. IV §3 (15)] one can show for $\tau = \frac{\omega_2}{\omega_1}$ and $L = [1, \tau]$:

$$\tilde{\theta}(z; L) = e^{6A(L)^{-1}z(z-\bar{z})} \cdot q_\tau \cdot (q_z^{\frac{1}{2}} - q_z^{\frac{-1}{2}})^{12} \cdot \prod_{n=1}^{\infty} \left((1 - q_\tau^n q_z)(1 - q_\tau^n q_z^{-1})\right)^{12}$$

$$\tilde{\theta}(z; L) = \varphi^{12}(z; 1, \tau), \tag{2.1}$$

with the φ-function defined in Definition 2.1.4 b).

Definition 2.1.13. Let L be a lattice with complex multiplication by $\mathcal{O}_k$. Let $\mathfrak{a}$ be an integral ideal of k. The function

$$\Theta(z; L, \mathfrak{a}) = \frac{\tilde{\theta}(z, L)^{\mathcal{N}(\mathfrak{a})}}{\tilde{\theta}(z, \mathfrak{a}^{-1}L)}$$

is an elliptic function with respect to L.

Approach of Kato in [Kat04] Kato proves in [Kat04] the following proposition.

Proposition 2.1.14. *[Kat04, Prop. 1.3 i)] Let E be an elliptic curve over a scheme S. Let c be an integer which is prime to 6. Then there exists a unique element ${}_c\theta_E \in \mathcal{O}(E \setminus {}_cE)^\times$ satisfying the following conditions*

i) ${}_c\theta_E$ has the divisor $c^2(0) - {}_cE$, where (0) denotes the zero section of E regarded as the Cartier divisor on E and ${}_cE = \ker(c : E \to E)$, where c is the multiplication by c, is also regarded as a Cartier divisor on E.

ii) $N_a({}_c\theta_E) = {}_c\theta_E$ for any integer a which is prime to c, where N_a is the norm map from $\mathcal{O}(E \setminus {}_{ac}E)^\times$ to $\mathcal{O}(E \setminus {}_cE)^\times$ associated to the pull back homomorphism from $\mathcal{O}(E \setminus {}_{ac}E)$ to $\mathcal{O}(E \setminus {}_cE)$ by the multiplication $a : E \setminus {}_{ac}E \to E \setminus {}_cE$.

For $\tau \in \mathbb{H}$ and $z \in \mathbb{C} \setminus c^{-1}(\mathbb{Z}\tau + \mathbb{Z})$, let ${}_c\theta(\tau, z)$ be the value at z of ${}_c\theta$ of the elliptic curve $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ over $\mathbb{C}$. Then

$$ {}_c\theta(\tau, z) = q_\tau^{\frac{1}{12}(c^2-1)}(-q_z)^{\frac{1}{2}(c-c^2)} \cdot \gamma_{q_\tau}(q_z)^{c^2}\gamma_{q_\tau}(q_z^c)^{-1}, $$

where again we set $q_x := e^{2\pi i x}$ and $\gamma_{q_\tau}(q_z) = \prod_{n\geq 0}(1 - q_\tau^n q_z)\prod_{n\geq 1}(1 - q_\tau^n q_z^{-1})$.

Let k be an imaginary quadratic number field and fix an embedding $k \to \mathbb{C}$.

Definition 2.1.15. [Kat04, p. 251] Let $\mathfrak{m}$ be a non-zero ideal of $\mathcal{O}_k$ such that $\mathcal{O}_k^\times \to (\mathcal{O}_k/\mathfrak{m})^\times$ is injective. For a field k' over k, by a *CM-pair with modulus* $\mathfrak{m}$ over k' we mean a pair (E, α), where E is an elliptic curve over k' endowed with an isomorphism $\mathcal{O}_k \cong \mathrm{End}(E)$ such that the composite map

$$ \mathcal{O}_k \xrightarrow{\cong} \mathrm{End}(E) \to \mathrm{End}_{k'}(\mathrm{Lie}(E)) \cong k' $$

coincides with the inclusion map, and α is a torsion point in $E(k')$ such that the annihilator of α in $\mathcal{O}_k$ coincides with $\mathfrak{m}$.

In Section 15.4 in [Kat04] Kato makes the following construction: Let k' be a field over k and let E be an elliptic curve over k' such that $\mathrm{End}(E) \cong \mathcal{O}_k$. We normalize this isomorphism in such a way that the composite map

$$ \mathcal{O}_k \xrightarrow{\cong} \mathrm{End}(E) \to \mathrm{End}_{k'}(\mathrm{Lie}(E)) \cong k' $$

is the inclusion map. He argues that, similar as in the proof of Proposition 2.1.14, for an ideal $\mathfrak{a}$ of $\mathcal{O}_k$ which is prime to 6, there is a unique element ${}_\mathfrak{a}\theta_E \in \mathcal{O}(E \setminus E[\mathfrak{a}])^\times$ having the following properties:

a) $\mathrm{div}({}_\mathfrak{a}\theta_E) = \mathcal{N}(\mathfrak{a})(0) - \sum_{P\in E[\mathfrak{a}]}(P)$.

b) $N_b({}_\mathfrak{a}\theta_E) = {}_\mathfrak{a}\theta_E$ for any integer b which is prime to $\mathfrak{a}$. Furthermore, if $\mathfrak{a} = (c)$ for an integer c, we have ${}_\mathfrak{a}\theta_E = {}_c\theta$ for a suitable E.

Proposition 2.1.16. *[Kat04, p. 253] Let $\mathfrak{m}$ be a non-zero ideal of $\mathcal{O}_k$ such that $\mathcal{O}_k^\times \to (\mathcal{O}_k/\mathfrak{m})^\times$ is injective, and let (E, α) be a CM-pair of modulus $\mathfrak{m}$ over $k(\mathfrak{m})$. Then for ideals $\mathfrak{a}, \mathfrak{b}$ which are prime to $6\mathfrak{m}$ we have*

$$ ({}_\mathfrak{b}\theta_E(\alpha))^{\mathcal{N}(\mathfrak{a})}\sigma_\mathfrak{a}({}_\mathfrak{b}\theta_E(\alpha))^{-1} = ({}_\mathfrak{a}\theta_E(\alpha))^{\mathcal{N}(\mathfrak{b})}\sigma_\mathfrak{b}({}_\mathfrak{a}\theta_E(\alpha))^{-1} $$

Now in his survey [Fla09] Flach formulates a lemma in an even more general setting than above. Part iv) of the following lemma is of special interest to us because it gives a direct link between the treatments of Robert in [Rob90] and of Kato in [Kat04] summarized above.

Lemma 2.1.17. *[Fla09, Lemma 2.1] Let E/S be an elliptic curve over a base scheme S and $c : E \to \tilde{E}$ an S-isogeny of degree prime to 6. Then there is a unique function*

$$ {}_c\Theta_{E/S} \in \Gamma(E \setminus \ker(c), \mathcal{O}^\times) $$

satisfying

i) $\mathrm{div}({}_c\Theta_{E/S}) = \deg(c) \cdot (0) - \sum_{P \in \ker(c)} (P)$

ii) For any morphism $g : S' \to S$ we have $g_E^*({}_c\Theta_{E/S}) = {}_{c'}\Theta_{E'/S}$ where $g_E : E' := E \times_S S' \to E$ and c' is a base change of c.

iii) For any S-isogeny $b : E \to E'$ of degree prime to $\deg(c)$ we have $b_*({}_c\Theta_{E/S}) = {}_{c'}\Theta_{E'/S}$, where b_* is the norm map associated to the finite flat morphism

$$ E \setminus \ker(c) \to E' \setminus \ker(c'). $$

Here c' is the isogeny $E' \to E'/b(\ker(c))$.

iv) For $S = \mathrm{Spec}(\mathbb{C})$, the elliptic curve $E = \mathbb{C}/L$ and $c : \mathbb{C}/L \to \mathbb{C}/\tilde{L}$ for lattices $L \subseteq \tilde{L}$ we have

$$ {}_c\Theta_{E/S}(z) = \tilde{F}(z; L, \tilde{L}), $$

with $\tilde{F}(z; L, \tilde{L})$ from Theorem 2.1.11.

2.1.2 Elliptic units

We shall give here a short survey of different definitions of elliptic units in the literature. We refrain from stating all interesting properties they satisfy and only go into detail for the elliptic units defined in [Ble04] because these are the ones we are working with later on. In this section we assume that k is an imaginary quadratic field with discriminant d_k. Furthermore, let $\mathfrak{d}$ be the different of k over $\mathbb{Q}$.

Invariant defined in [Ram64] Here we present the construction of Ramachandra given in [Ram64] but only for the special case we are interested in (i.e. for the case $\mathfrak{g} = (1)$ and $\mathfrak{b}_0 = (1)$ in the notation of [Ram64]).

For a positive integer f and $(v, u) \in \mathbb{Q}^2$ not both integers and having f as a common denominator we define

$$ \Phi\left(\begin{pmatrix} v \\ u \end{pmatrix}, z\right) = \Phi_0\left(\begin{pmatrix} v \\ u \end{pmatrix}, z\right)^{12f}, $$

with Φ_0 from Remark 2.1.2 d).

Fix an integral ideal $\mathfrak{f}$ of k and choose a $\gamma \in k$ such that $\gamma\mathfrak{d}$ has *exact denominator* $\mathfrak{f}$, i.e. such that $\gamma\mathfrak{d}\mathfrak{f}$ is integral and $(\gamma\mathfrak{d}\mathfrak{f}, \mathfrak{f}) = (1)$. The existence of such a γ follows for example from an application of the strong approximation theorem. Let $\mathfrak{b}$ be an ideal of $\mathcal{O}_k$ with $\mathfrak{b} = [b_1, b_2]$ such that $\frac{b_2}{b_1} \in \mathbb{H}$. Then we define

$$ \Phi_\gamma(\mathfrak{b}) = \begin{cases} \Phi\left(\begin{pmatrix} \mathrm{Tr}(\gamma b_2) \\ \mathrm{Tr}(\gamma b_1) \end{pmatrix}, b_2/b_1\right) & \text{if } \mathfrak{f} \nmid \mathfrak{b}, \\ |(2\pi)^{-12f}(\mathcal{N}(\mathfrak{b}))^{6f}\Delta(\mathfrak{b})^f| & \text{if } \mathfrak{f} \mid \mathfrak{b}. \end{cases} $$

Now choose an ideal $\mathfrak{b}_1$ coprime to $\mathfrak{f}$ in the ray class inverse to that of $\gamma\mathfrak{d}\mathfrak{f}$ mod $\mathfrak{f}$.

For $C \in Cl(\mathfrak{f})$ we can then define

$$\Phi_{\mathfrak{f},(1)}(C) := \Phi_\gamma(\mathfrak{b}_1\mathfrak{c}) \text{ for } \mathfrak{c} \in C.$$

Ramachandra shows in [Ram64] that $\Phi_{\mathfrak{f},(1)}(C)$ is

a) independent of the choice of γ and $\mathfrak{b}_1$ as well as

b) an invariant of the class $C \in Cl(\mathfrak{f})$.

Ramachandra invariant in [Lan73] Recall that $\mathfrak{d}^{-1} := \{\lambda \in k : \mathrm{Tr}(\lambda\mathcal{O}_k) \subset \mathbb{Z}\}$. Fix a non-zero integral ideal $\mathfrak{f}$ of k with $\mathfrak{f} \neq (1)$.

Definition 2.1.18. Let $\mathfrak{h}$ be a fractional ideal of $\mathcal{O}_k$. If $\mathfrak{h} = [z_1, z_2]$ with $z_2/z_1 \in \mathbb{H}$, we put

$$H(\mathfrak{h}) = H\left(\begin{pmatrix}\mathrm{Tr}(z_2)\\ \mathrm{Tr}(z_1)\end{pmatrix}; z_2/z_1\right),$$

using Definition 2.1.6 d). It can be shown that this definition is independent of the chosen basis of $\mathfrak{h}$.

Furthermore, let $\mathfrak{b}$ be an integral ideal which is prime to $\mathfrak{f}$. We define the *Ramachandra invariant* as

$$\Phi_{\mathfrak{f}}(\mathfrak{b}) = H(\mathfrak{b}\mathfrak{d}^{-1}\mathfrak{f}^{-1}).$$

Remark 2.1.19. It is easy to see that if N is the smallest positive integer contained in $\mathfrak{f}$ and $\mathfrak{h} = \mathfrak{b}\mathfrak{d}^{-1}\mathfrak{f}^{-1} = [z_1, z_2]$, then N is also the precise denominator of the pair

$$(u, v) = (\mathrm{Tr}(z_2), \mathrm{Tr}(z_1)).$$

A lemma in [Lan73, Ch. 19, §3] shows now that the value $\Phi_{\mathfrak{f}}(\mathfrak{b})$ depends only on the ray class of $\mathfrak{b}$ modulo $\mathfrak{f}$, so it makes sense to denote the Ramachandra invariant by $\Phi_{\mathfrak{f}}(C)$, for $C \in Cl(\mathfrak{f})$ containing $\mathfrak{b}$.

For the case $\mathfrak{f} = (1)$ and $C \in Cl(1)$ we set

$$\Phi_{(1)}(C) := |(2\pi)^{-12}\mathcal{N}(\mathfrak{c})^6\Delta(\mathfrak{c})| \text{ for a } \mathfrak{c} \in C,$$

which is also independent of the choice of $\mathfrak{c}$. In [Lan73] this invariant is denoted by $g(\mathfrak{c}) = g(C)$.

As the names are already suggesting, the Ramachandra invariant of [Lan73] and the invariant of Ramachandra coincide in the cases we have treated.

Lemma 2.1.20. *Let $\mathfrak{f}$ be a non-zero integral ideal in $\mathcal{O}_k$ and $C \in Cl(\mathfrak{f})$. Then we have*

$$\Phi_{\mathfrak{f},(1)}(C) = \Phi_{\mathfrak{f}}(C).$$

Proof. For $\mathfrak{f} = (1)$ this follows directly from the definitions. In the case $\mathfrak{f} \neq (1)$, for $\Phi_{\mathfrak{f},(1)}(C)$ we have to choose a γ such that $\gamma\mathfrak{d}\mathfrak{f}$ has exact denominator $\mathfrak{f}$ and then an integral ideal $\mathfrak{b}_1$ in $[\gamma^{-1}\mathfrak{d}^{-1}\mathfrak{f}^{-1}]_{\mathfrak{f}} \in Cl(\mathfrak{f})$, where we use $[\,\cdot\,]_{\mathfrak{f}}$ as notation for a class in $Cl(\mathfrak{f})$. Now take an integral ideal $\mathfrak{b} \in C$ and choose a basis of the integral ideal $\mathfrak{c} := \mathfrak{b}\mathfrak{b}_1 = [c_1, c_2]$ such that $c_2/c_1 \in \mathbb{H}$ so we get by the definitions

$$\Phi_{\mathfrak{f},(1)}(C) = \Phi_0\left(\begin{pmatrix}\mathrm{Tr}(\gamma c_2)\\ \mathrm{Tr}(\gamma c_1)\end{pmatrix}, \tfrac{c_2}{c_1}\right)^{12f} = e^{-12f\pi i \mathrm{Tr}(\gamma c_1)(\mathrm{Tr}(\gamma c_2) - \mathrm{Tr}(\gamma c_1)\frac{c_2}{c_1})}\frac{\theta_1(\mathrm{Tr}(\gamma c_2) - \mathrm{Tr}(\gamma c_1)\frac{c_2}{c_1}, \frac{c_2}{c_1})^{12f}}{\eta(\frac{c_2}{c_1})^{12f}},$$

where f is the smallest integer contained in $\mathfrak{f}$.

For the other side we take the same $\mathfrak{b} \in C$ as above, and we get $\Phi_{\mathfrak{f}}(C) = \Phi_{\mathfrak{f}}(\mathfrak{b}) = H(\mathfrak{b}\mathfrak{d}^{-1}\mathfrak{f}^{-1})$. We see that $[\mathfrak{c}]_{\mathfrak{f}} = [\mathfrak{b}\mathfrak{b}_0]_{\mathfrak{f}} = [\mathfrak{b}\gamma^{-1}\mathfrak{d}^{-1}\mathfrak{f}^{-1}]_{\mathfrak{f}}$ by definition. But we also have $[\gamma\mathfrak{c}]_{\mathfrak{f}} = [\mathfrak{b}\mathfrak{d}^{-1}\mathfrak{f}^{-1}]_{\mathfrak{f}}$, so by changing $\mathfrak{b} \in C$ above, if necessary, we can get $\mathfrak{b}\mathfrak{d}^{-1}\mathfrak{f}^{-1} = [\gamma c_2, \gamma c_1]$. With the Definition 2.1.18 and Definition 2.1.6 we then obtain

$$\Phi_{\mathfrak{f}}(C) = H\left(\begin{pmatrix}\mathrm{Tr}(c_2)\\ \mathrm{Tr}(c_1)\end{pmatrix}, c_2/c_1\right) = f\left(\mathrm{Tr}(c_2) - \mathrm{Tr}(c_1)\frac{c_2}{c_1}, \frac{c_2}{c_1}\right)^{12f} e^{12f\pi i \mathrm{Tr}(c_1)(\mathrm{Tr}(c_1)\frac{c_2}{c_1} - \mathrm{Tr}(c_2))}.$$

Now with Definitions 2.1.1 and 2.1.6 as well as Remark 2.1.7 we can compute for $L = [\omega_1, \omega_2]$

$$\frac{\theta_1(z/\omega_1, \omega_2/\omega_1)}{\eta(\omega_2/\omega_1)} = -i \cdot f(z; \omega_1, \omega_2).$$

Considering that u and v are switched in the definitions of θ and Φ_0 as well as that $(-i)^{12f} = 1$, we obtain the result. □

Elliptic units of Robert in [Rob73] Recall that k is an imaginary quadratic field and let $\mathfrak{f}$ be a non-zero ideal of $\mathcal{O}_k$.

Let $f \in \mathbb{N}$ be the smallest integer contained in $\mathfrak{f}$. We set

$$A(\mathfrak{f}) := \{(t, \mathfrak{b}) \mid t \in \mathbb{C} \text{ and } \mathfrak{b} \text{ s. t. } \mathfrak{f} = \{\alpha \in \mathcal{O}_k \mid \alpha t \in \mathfrak{b}\} = \mathcal{O}_k \cap t^{-1}\mathfrak{b}\}.$$

We say that $(t, \mathfrak{b})$ and $(t', \mathfrak{b}')$ are equivalent if there exists a $u \in k^\times$ such that (t'/ut) is in the ray modulo $\mathfrak{f}$ and $\mathfrak{b}' = u\mathfrak{b}$. There exists a $v \in \mathcal{O}_k^\times$ such that $t' \equiv vut \bmod \mathfrak{b}'$ and one obtains for two equivalent pairs $(t, \mathfrak{b})$ and $(t', \mathfrak{b}')$ in $A(\mathfrak{f})$ ([Rob73, §2.2, Lemma 3])

$$\varphi^{12f}(t; \mathfrak{b}) = \varphi^{12f}(t'; \mathfrak{b}'),$$

where we use the φ-function defined in Definition 2.1.4 b) and these expressions are well-defined because of Remark 2.1.5 c).

For each pair $(t, \mathfrak{b}) \in A(\mathfrak{f})$, the ideal $t\mathfrak{f}\mathfrak{b}^{-1}$ is an integral ideal prime to $\mathfrak{f}$; letting $C_{(t,\mathfrak{b})} \in Cl(\mathfrak{f})$ be the class of $t\mathfrak{f}\mathfrak{b}^{-1}$ we get the following ([Rob73, §2.2, Lemma 4]): the map $(t, \mathfrak{b}) \mapsto C_{(t,\mathfrak{b})}$ defines an isomorphism of groups between $A(\mathfrak{f})/\sim$ and $Cl(\mathfrak{f})$. So one can make the following

Definition 2.1.21. a) If $\mathfrak{f} \neq (1)$, then we set for each class $C \in Cl(\mathfrak{f})$:

$$\varphi_{\mathfrak{f}}(C) := \varphi^{12f}(t; \mathfrak{b}), \text{ where } (t, \mathfrak{b}) \in A(\mathfrak{f}) \text{ such that } C_{(t,\mathfrak{b})} = C.$$

b) If $\mathfrak{f} = (1)$, then we set for each class $C \in Cl(1)$

$$\varphi_{(1)}(C) = |(2\pi)^{-12}\mathcal{N}(\mathfrak{b})^6\Delta(\mathfrak{b})|, \text{ where } \mathfrak{b} \in C^{-1}.$$

So with the above, we have seen that $\varphi_{\mathfrak{f}}(C)$ does only depend on $\mathfrak{f}$ and the class $C \in Cl(\mathfrak{f})$.

Remark 2.1.22. For $C \in Cl(1)$ we have $\varphi_{(1)}(C) = \Phi_{(1)}(C)$, which follows from Proposition 2 in [Rob73, §2] and Lemma 2.1.20.

We collect some properties of these invariants. Assume $\mathfrak{f} \neq (1)$ and $C, C' \in Cl(\mathfrak{f})$. Then we have by Theorem 1 in [Rob73, §2.3]

a) $\varphi_{\mathfrak{f}}(C) \in \mathcal{O}_{k(\mathfrak{f})}$.

b) $\varphi_{\mathfrak{f}}(C)/\varphi_{\mathfrak{f}}(C') \in \mathcal{O}_{k(\mathfrak{f})}^\times$.

c) $\varphi_{\mathfrak{f}}(C)^{\sigma(C')} = \varphi_{\mathfrak{f}}(CC')$.

Siegel units in the Hilbert class field in [dS87] For any ideal $\mathfrak{a}$ of $\mathcal{O}_k$ we define

$$u(\mathfrak{a}) := \frac{\Delta(\mathcal{O}_k)}{\Delta(\mathfrak{a}^{-1})}$$

and it is a classical result that we have: $u(\mathfrak{a}) \in k(1)$, $u(\mathfrak{a}\mathfrak{b}) = u(\mathfrak{a})^{\sigma_\mathfrak{b}} \cdot u(\mathfrak{b})$ and $(u(\mathfrak{a})) = \mathfrak{a}^{-12}\mathcal{O}_{k(1)}$.

Elliptic units in [Ble04]

Definition 2.1.23. Let $L \subseteq \mathbb{C}$ denote a $\mathbb{Z}$-lattice of rank 2 with complex multiplication by $\mathcal{O}_k$. For any integral $\mathcal{O}_k$-ideal $\mathfrak{a}$ satisfying $(\mathcal{N}(\mathfrak{a}), 6) = 1$ we define a meromorphic function

$$\psi(z; L, \mathfrak{a}) := \tilde{F}(z; L, \mathfrak{a}^{-1}L), \quad z \in \mathbb{C},$$

where $\tilde{F}$ is defined in Theorem 2.1.11 b).

The following object is the main object we are going to use in the following chapters.

Definition 2.1.24. Let $\mathfrak{f}$ be an integral ideal of $\mathcal{O}_k$ and $\mathfrak{a}$ be an integral ideal of $\mathcal{O}_k$ satisfying $(\mathcal{N}(\mathfrak{a}), 6\mathfrak{f}) = 1$, then we call $\psi(1; \mathfrak{f}, \mathfrak{a})$ an *elliptic unit.*

Remark 2.1.25. *The function ψ is periodic with respect to L and satisfies the relation*

$$\psi(\lambda z; \lambda L, \mathfrak{a}) = \psi(z; L, \mathfrak{a}) \text{ for all } \lambda \in \mathbb{C}^\times.$$

Proposition 2.1.26. *[Ble04, Prop. 2.2] Let $\mathfrak{m}$ be an integral $\mathcal{O}_k$-ideal such that $(\mathfrak{m}, \mathfrak{a}) = 1$ and let $\tau \in \mathbb{C}$ denote a primitive $\mathfrak{m}$-division point of $\mathbb{C}/L$. Let $\mathfrak{c}$ denote an integral $\mathcal{O}_k$-ideal such that $(\mathfrak{m}, \mathfrak{c}) = 1$. Then:*

a) $\psi(\tau; L, \mathfrak{a}) \in k(\mathfrak{m})$.

b) $\psi(\tau; L, \mathfrak{a})^{\sigma(\mathfrak{c})} = \psi(\tau; \mathfrak{c}^{-1}L, \mathfrak{a})$.

Theorem 2.1.27. *[Ble04, Thm. 2.3]*

a) Let $\mathfrak{f}$ denote a non-trivial integral ideal of k and let $\mathfrak{p}$ be any prime ideal of k. Suppose that $(\mathfrak{a}, 6\mathfrak{f}\mathfrak{p}) = 1$. If $\tau \in \mathbb{C}$ denotes a primitive $\mathfrak{f}\mathfrak{p}$-division point of $\mathbb{C}/\mathfrak{f}\mathfrak{p}$, then

$$\mathcal{N}_{k(\mathfrak{f}\mathfrak{p})/k(\mathfrak{f})}\left(\psi(\tau; \mathfrak{f}\mathfrak{p}, \mathfrak{a})^{w(\mathfrak{f})/w(\mathfrak{f}\mathfrak{p})}\right) = \begin{cases} \psi(\tau; \mathfrak{f}, \mathfrak{a}), & \text{if } \mathfrak{p} \mid \mathfrak{f}, \\ \psi(\tau; \mathfrak{f}, \mathfrak{a})^{1-\sigma(\mathfrak{p})^{-1}}, & \text{if } \mathfrak{p} \nmid \mathfrak{f}. \end{cases}$$

b) Let $\mathfrak{p}$ be a prime ideal such that $(\mathfrak{a}, 6\mathfrak{p}) = 1$ and let $\tau \in \mathbb{C}$ denote a primitive $\mathfrak{p}$-division point of $\mathbb{C}/\mathfrak{p}$. Then

$$\mathcal{N}_{k(\mathfrak{p})/k(1)}\left(\psi(\tau; \mathfrak{p}, \mathfrak{a})^{w(1)/w(\mathfrak{p})}\right) = \frac{\delta(\mathcal{O}_k, \mathfrak{a}^{-1})}{\delta(\mathfrak{p}, \mathfrak{a}^{-1}\mathfrak{p})},$$

where δ is the function of lattices defined in Theorem 2.1.11 a).

Remark 2.1.28. We also want to discuss the Galois action on an object appearing in Theorem 2.1.27 b). So assume the same conditions as in Theorem 2.1.27 b). Furthermore, let $\mathfrak{c}$ be an integral ideal coprime to $6\mathfrak{a}\mathfrak{p}$. Then we get with Equation (?) [sic] on p. 226 in [Rob92], the fact that $s \mapsto \Lambda(s, L)$ (where Λ is defined (loc. cit.) as a function with s an idele and L a suitable lattice) only depends on the commensurability class of L and a translation between class field theory based on ideles and based on ideals that

$$\frac{\delta(\mathcal{O}_k, \mathfrak{a}^{-1})^{\sigma(\mathfrak{c})}}{\delta(\mathfrak{p}, \mathfrak{p}\mathfrak{a}^{-1})} = \frac{\delta(\mathfrak{c}^{-1}, \mathfrak{a}^{-1}\mathfrak{c}^{-1})}{\delta(\mathfrak{p}\mathfrak{c}^{-1}, \mathfrak{a}^{-1}\mathfrak{p}\mathfrak{c}^{-1})}. \tag{2.2}$$

Moreover, for a fixed $\mathfrak{a}$ we will sometimes shorten the notation to

$$\delta_{\mathfrak{p}} := \frac{\delta(\mathcal{O}_k, \mathfrak{a}^{-1})}{\delta(\mathfrak{p}, \mathfrak{p}\mathfrak{a}^{-1})}. \tag{2.3}$$

Using now (2.2) we can see that for another prime ideal $\mathfrak{q}$ satisfying the conditions we obtain

$$\delta_{\mathfrak{p}}^{1-\sigma(\mathfrak{q})^{-1}} = \delta_{\mathfrak{q}}^{1-\sigma(\mathfrak{p})^{-1}}. \tag{2.4}$$

Proposition 2.1.29. *Let $\mathfrak{f}$ denote a non-trivial integral ideal of k and let $\mathfrak{p}$ be any prime ideal of k. Moreover, let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be two integral ideals such that $(\mathfrak{a}_1, 6\mathfrak{f}\mathfrak{p}) = 1 = (\mathfrak{a}_2, 6\mathfrak{f}\mathfrak{p})$. If $\tau \in \mathbb{C}$ denotes a primitive $\mathfrak{f}\mathfrak{p}$-division point of $\mathbb{C}/\mathfrak{f}\mathfrak{p}$, then*

$$\psi(\tau, \mathfrak{p}\mathfrak{f}, \mathfrak{a}_1)^{\sigma(\mathfrak{a}_2)} = \psi(\tau; \mathfrak{a}_2^{-1}\mathfrak{f}\mathfrak{p}, \mathfrak{a}_1) = \psi(\tau; \mathfrak{f}\mathfrak{p}, \mathfrak{a}_1\mathfrak{a}_2)\psi(\tau; \mathfrak{f}\mathfrak{p}, \mathfrak{a}_2)^{-\mathcal{N}\mathfrak{a}_1}.$$

and

$$\psi(\tau; \mathfrak{f}\mathfrak{p}, \mathfrak{a}_1)^{\sigma(\mathfrak{a}_2)-\mathcal{N}(\mathfrak{a}_2)} = \psi(\tau; \mathfrak{f}\mathfrak{p}, \mathfrak{a}_2)^{\sigma(\mathfrak{a}_1)-\mathcal{N}(\mathfrak{a}_1)}.$$

Proof. The proof of the first part works analogously to the proof of Proposition 2.4 ii), Chp. II in [dS87]. Then for the second part we use the first part to obtain:

$$\begin{aligned}\psi(\tau; \mathfrak{f}\mathfrak{p}, \mathfrak{a}_1)^{\sigma(\mathfrak{a}_2)} \cdot \psi(\tau; \mathfrak{f}\mathfrak{p}, \mathfrak{a}_2)^{\mathcal{N}(\mathfrak{a}_1)} &= \psi(\tau; \mathfrak{f}\mathfrak{p}, \mathfrak{a}_1\mathfrak{a}_2)\\ \psi(\tau, \mathfrak{f}\mathfrak{p}, \mathfrak{a}_2)^{\sigma(\mathfrak{a}_1)} \cdot \psi(\tau; \mathfrak{f}\mathfrak{p}, \mathfrak{a}_1)^{\mathcal{N}(\mathfrak{a}_2)} &= \psi(\tau; \mathfrak{f}\mathfrak{p}, \mathfrak{a}_2\mathfrak{a}_1)\end{aligned}$$

which implies the second assertion. □

Proposition 2.1.30. *[Ble04, Thm. 2.4] Let $\mathfrak{f}$ be a non-trivial integral $\mathcal{O}_k$-ideal and let $\tau \in \mathbb{C}$ be a primitive $\mathfrak{f}$-division point of $\mathbb{C}/L$.*

a) If $\mathfrak{f}$ is composite, then

$$\psi(\tau; L, \mathfrak{a}) \in \mathcal{O}_{k(\mathfrak{f})}^{\times}.$$

b) If $\mathfrak{f}$ is a prime power, then

$$\psi(\tau; L, \mathfrak{a})\mathcal{O}_{k(\mathfrak{f})} = \left(\mathfrak{p}\mathcal{O}_{k(\mathfrak{f})}\right)^{(\mathcal{N}(\mathfrak{a})-1)/\Phi(\mathfrak{p}^n)},$$

where Φ is the Euler Φ-function.

Cyclotomic numbers It is instructive to compare the assertions presented for the elliptic case with the classical results of the cyclotomic theory.

Let $m \in \mathbb{N}$ and let μ_m be the group of m-th roots of unity. For each m we fix a generator ζ_m of μ_m, which should satisfy

$$\zeta_{dm}^d = \zeta_m \text{ for all } m, d \in \mathbb{N}$$

Let $m \in \mathbb{N}$ and p be a rational prime and write $\mathbb{Q}(m)$ for $\mathbb{Q}(\mu_m)$. With [Sol92, Lemma 2.1]

$$\mathcal{N}_{\mathbb{Q}(mp)/\mathbb{Q}(m)}(1-\zeta_{mp}) = \begin{cases} 1-\zeta_m & \text{if } p \mid m, \\ (1-\zeta_m)^{1-\sigma(p)^{-1}} & \text{if } p \nmid m. \end{cases}$$

Furthermore, we have

$$N_{\mathbb{Q}(p)/\mathbb{Q}}(1-\zeta_p) = p \text{ if } p \text{ is odd, also } \mathcal{N}_{\mathbb{Q}(4)/\mathbb{Q}}(1-\zeta_4) = 2,$$

and it is easy to show that (see [Sol92, Cor. 2.1]) if m is not a prime power, we have

$$1-\zeta_m \in \mathcal{O}_{\mathbb{Q}(m)}^\times.$$

Comparing these results now with Theorem 2.1.27 and Proposition 2.1.30 one has to recognize the striking similarity to the elliptic situation.

Elliptic units in [Kat04] and [JLK11] Recall that in Definition 2.1.15 we already defined a CM-pair and discussed some properties surrounding them. In order to be able to give the definition of elliptic units in [Kat04] and [JLK11] we summarize some additional properties of CM-pairs in the following remark.

Remark 2.1.31. [Kat04, pp. 251/252]

a) If (E, α) and (E', α') are isomorphic CM-pairs of the same modulus $\mathfrak{m}$ over k', the isomorphism $(E, \alpha) \to (E', \alpha')$ is unique by the injectivity of $\mathcal{O}_k^\times \to (\mathcal{O}_k/\mathfrak{m})^\times$

b) There exists a CM-pair of modulus $\mathfrak{m}$ over $k(\mathfrak{m})$ which is isomorphic to $(\mathbb{C}/\mathfrak{m}, 1 \bmod \mathfrak{m})$ over $\mathbb{C}$. This CM-pair is unique up to isomorphism. We call this CM-pair of modulus $\mathfrak{m}$ over $k(\mathfrak{m})$ the *canonical CM-pair over* $k(\mathfrak{m})$.

c) Let k' be a field over k and let (E, α) be a CM-pair of modulus $\mathfrak{m}$ over k'. Then there exists a unique homomorphism $k(\mathfrak{m}) \to k'$ for which (E, α) is obtained from the canonical CM-pair over $k(\mathfrak{m})$ by base change.

d) Let k' be a finite abelian extension of k, let $\mathfrak{a}$ be a non-zero prime ideal of $\mathcal{O}_k$ whose prime divisors are unramified in k' and let $\sigma_\mathfrak{a} = (\mathfrak{a}, k'/k) \in \mathrm{Gal}(k'/k)$.

   Let (E, α) be a CM-pair of modulus $\mathfrak{m}$ over k' and let $(E^{\sigma_\mathfrak{a}}, \sigma_\mathfrak{a}(\alpha))$ be the CM-pair of modulus $\mathfrak{m}$ over k' obtained from (E, α) by base change $\sigma_\mathfrak{a} : k' \to k'$. Then $(E^{\sigma_\mathfrak{a}}, \sigma_\mathfrak{a}(\alpha))$ is isomorphic to $(E/E[\mathfrak{a}], \alpha \bmod E[\mathfrak{a}])$ where $E[\mathfrak{a}]$ is the part of E which is annihilated by $\mathfrak{a}$.

   We will denote the unique isomorphism as

$$\eta_\mathfrak{a} : (E/E[\mathfrak{a}], \alpha \bmod E[\mathfrak{a}]) \to (E^{\sigma_\mathfrak{a}}, \sigma_\mathfrak{a}(\alpha)).$$

Definition 2.1.32. Let $\mathfrak{m}$ be a non-zero ideal of $\mathcal{O}_k$ such that $\mathcal{O}_k^\times \to (\mathcal{O}_k/\mathfrak{m})^\times$ is injective, let (E, α) be the canonical CM-pair over $k(\mathfrak{m})$, and let $\mathfrak{a}$ be an ideal of $\mathcal{O}_k$ which is prime to $6\mathfrak{m}$. Then we define the following element:

$$ {}_{\mathfrak{a}}z_{\mathfrak{m}} := {}_{\mathfrak{a}}\theta_E(\alpha)^{-1} \in k(\mathfrak{m})^\times. $$

On page 253 in [Kat04] Kato argues that the following properties hold:

a) The element ${}_{\mathfrak{a}}\theta_E(\alpha)$ is a $\mathfrak{q}$-unit for any $\mathfrak{q}$ of $\mathcal{O}_k$ which is prime to $\mathfrak{m}$, and is a unit if $\mathfrak{m}$ has at least two prime divisors.

b) Let $\mathfrak{m}$ be a non-zero ideal of $\mathcal{O}_k$ and fix a prime p. Then for $n \geq 1$ such that $\mathcal{O}_k^\times \to (\mathcal{O}_k/p^n)^\times$ is injective and for an ideal $\mathfrak{a}$ of $\mathcal{O}_k$ which is prime to $6p\mathfrak{m}$, we have

$$ \mathcal{N}_{k(p^{n+1}\mathfrak{m})/k(p^n\mathfrak{m})}({}_{\mathfrak{a}}z_{p^{n+1}\mathfrak{m}}) = {}_{\mathfrak{a}}z_{p^n\mathfrak{m}}. $$

Definition 2.1.33. (cf. [JLK11, Def. 3.2]) Fix a prime p and let $\mathfrak{m}$ be a non-zero ideal of $\mathcal{O}_k$. Then for $n \geq 1$ such that $\mathcal{O}_k^\times \to (\mathcal{O}_k/p^n)^\times$ is injective and for an ideal $\mathfrak{a}$ of $\mathcal{O}_k$ which is prime to $6p\mathfrak{m}$ we define

$$ \zeta_{\mathfrak{m}} := {}_{\mathfrak{a}}\zeta_{\mathfrak{m}} := \mathcal{N}_{k(p^n\mathfrak{m})/k(\mathfrak{m})}({}_{\mathfrak{a}}\theta_E(\alpha)^{-1}) \in k(\mathfrak{m})^\times $$

and if $k \subset F \subset k(\mathfrak{m})$ has conductor $\mathfrak{m}$, we set

$$ \zeta_F := \mathcal{N}_{k(\mathfrak{m})/F}(\zeta_{\mathfrak{m}}). $$

Comparison of the different definitions Let k be an imaginary quadratic field and $\mathfrak{f}$ be a non-zero integral ideal in $\mathcal{O}_k$. For $C \in Cl(\mathfrak{f})$ we have already shown in Lemma 2.1.20 that

$$ \Phi_{\mathfrak{f},(1)}(C) = \Phi_{\mathfrak{f}}(C), $$

i.e. the construction of Ramachandra in [Ram64] and Lang [Lan73] coincide. Robert shows in Proposition 2 [Rob73, §2] that

$$ \varphi_{\mathfrak{f}}(C) = \overline{\Phi_{\mathfrak{f},(1)}(C)}. \tag{2.5} $$

Let $\mathfrak{a}$ be an ideal coprime to $6\mathfrak{f}$. Then we have by Definition 2.1.13 and (2.1) above

$$ \Theta(z; \mathfrak{f}, \mathfrak{a}) = \frac{\tilde{\theta}(z; \mathfrak{f})^{\mathcal{N}(\mathfrak{a})}}{\tilde{\theta}(z; \mathfrak{f}\mathfrak{a}^{-1})} = \frac{\varphi^{12\mathcal{N}(\mathfrak{a})}(z; \mathfrak{f})}{\varphi^{12}(z; \mathfrak{f}\mathfrak{a}^{-1})} $$

Since ρ in Theorem 2.1.11 is a 12-th root of unity, we have by Definition 2.1.9

$$ \psi^{12}(z; \mathfrak{f}, \mathfrak{a}) = \frac{\varphi^{12\mathcal{N}(\mathfrak{a})}(z; \mathfrak{f})}{\varphi^{12}(z; \mathfrak{f}\mathfrak{a}^{-1})}. \tag{2.6} $$

So in particular we have $\psi(1; \mathfrak{f}, \mathfrak{a})^{12} = \Theta(1; \mathfrak{f}, \mathfrak{a})$.

Assume $\mathfrak{f} \neq (1)$. For a given $C \in Cl(\mathfrak{f})$ and $\mathfrak{c} \in C$ coprime to $\mathfrak{f}$ one can always choose the tuple $(1, \mathfrak{f}\mathfrak{c}^{-1}) \in A(\mathfrak{f})$ because $\mathfrak{f} = \mathfrak{f}\mathfrak{c}^{-1} \cap \mathcal{O}_k$. So we obtain

$$ \varphi_{\mathfrak{f}}(C) = \varphi^{12f}(1; \mathfrak{f}\mathfrak{c}^{-1}) \tag{2.7} $$

and also

$$ \Theta(1; \mathfrak{f}, \mathfrak{a})^f = \psi^{12f}(1; \mathfrak{f}, \mathfrak{a}) = \frac{\varphi^{12f\mathcal{N}(\mathfrak{a})}(z; \mathfrak{f})}{\varphi^{12f}(z; \mathfrak{f}\mathfrak{a}^{-1})} = \varphi_{\mathfrak{f}}(1)^{\mathcal{N}(\mathfrak{a}) - \sigma(\mathfrak{a})}. $$

Let $\mathfrak{a}$ be integral ideals of $\mathcal{O}_k$ with $(\mathfrak{a}, 6\mathfrak{f}) = 1$. With Lemma 2.1.17 iv) we have

$$\psi(1; \mathfrak{m}, \mathfrak{a}) = {}_c\Theta_{E/S}(1) \tag{2.8}$$

for $E = \mathbb{C}/\mathfrak{f}$ and the map $c : \mathbb{C}/\mathfrak{f} \to \mathbb{C}/\mathfrak{f}\mathfrak{a}^{-1}$. For the canonical CM-pair $(\mathbb{C}/\mathfrak{f}, 1 \bmod \mathfrak{f})$ and for the $\mathfrak{f}$ where both are defined we have

$$\zeta_{\mathfrak{f}} = \psi(1; \mathfrak{f}, \mathfrak{a})^{-1} \tag{2.9}$$

with $\zeta_{\mathfrak{f}}$ from Definition 2.1.33.

2.2 L-functions

We briefly recall some properties of basic L-functions. Our treatment here is based on [Tat84] and [Tat11].

2.2.1 Abelian L-functions

Let k be, for the moment, a general number field and let χ be a (generalized) Dirichlet character. Then $L(s, \chi)$ is a complex function, defined for $\mathrm{Re}(s) > 1$ by

$$L(s, \chi) = \prod_{\mathfrak{p}} \left(1 - \chi(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-s}\right)^{-1} = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{\mathcal{N}(\mathfrak{a})^s},$$

where the product is taken over all nonzero prime ideals and the sum is taken over all non-zero integral ideals $\mathfrak{a}$.

This function can be analytically continued to $\mathbb{C}$ if $\chi \neq 1$. For $\chi = 1$, the zeta function $\zeta_k(s) := L(s, 1)$ has an analytic continuation to the complex plane except for a simple pole at $s = 1$. By abuse of notation the analytic (resp. meromorphic) continuation will also be denoted by $L(s, \chi)$ and $\zeta_k(s)$, respectively.

For each infinite place v of k, using the the well-known Γ-function we define

$$\gamma_v(s, \chi) := \begin{cases} \Gamma(\frac{s}{2}) & \text{if } v \text{ is real and } v \nmid \mathfrak{f}_\chi, \\ \Gamma(\frac{s+1}{2}) & \text{if } v \text{ is real and } v \mid \mathfrak{f}_\chi, \\ \Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}) & \text{if } v \text{ is complex.} \end{cases}$$

Then we can set $\Lambda(s, \chi) := \prod_{v|\infty} \gamma_v(s, \chi) L(s, \chi)$.

The function $\xi(s, \chi) := A_\chi^{s/2} \Lambda(s, \chi)$ satisfies the *functional equation*

$$\xi(1 - s, \chi) = W_\chi \xi(s, \overline{\chi}) \quad \text{ with } A_\chi = \frac{|d_k| \mathcal{N}(\mathfrak{f}_\chi)}{\pi^{[k:\mathbb{Q}]}},$$

and a $W_\chi \in \mathbb{C}^\times$ which is called *Artin root number* and satisfies $|W_\chi| = 1$ and $W_{\overline{\chi}} = \overline{W}_\chi$. We also have the *analytic class number formula*, which asserts that the Dedekind zeta function $\zeta_k(s)$ has a simple pole at $s = 1$ with residue

$$\frac{2^{r_1}(2\pi)^{r_2} h_k R_k}{\sqrt{|d_k|} w_k},$$

where R_k is the regulator of k, w_k the number of roots of unity contained in k, r_1 the number of real places and r_2 the number of complex places of k. Using the functional equation and the analytic class number formula we get that the Taylor expansion of $\zeta_k(s)$ at $s = 0$ is

$$\zeta_k(s) = \frac{-h_k R_k}{w_k} s^{r_1 + r_2 - 1} + \dots$$

Let K/k be an abelian extension of number fields with Galois group G. Fix a non-empty set S of places of k, containing $S_{ram}(K/k)$ and S_∞. If v is unramified in K/k, we denote by σ_v the Frobenius automorphism at v in G.

Definition 2.2.1. The *S-imprimitive L-function* $L_S(s,\chi)$ associated to $\chi \in \widehat{G}$ is defined as the meromorphic extension to the whole complex plane of the holomorphic function given by the product

$$\prod_{v \notin S} (1 - \chi(\sigma_v)\mathcal{N}(v)^{-s})^{-1} \quad \text{for Re(s)} > 1.$$

Remark 2.2.2. a) If $\chi \neq 1$, then the L-function $L_S(s,\chi)$ is holomorphic everywhere on $\mathbb{C}$.

b) If $\chi = 1$, then $\zeta_{k,S}(s) := L_S(s,1)$ is holomorphic outside $s = 1$ and has a simple pole of order 1 at $s = 1$.

c) For $G = \mathrm{Gal}(k(\mathfrak{f})/k)$ and a proper character χ modulo $\mathfrak{f}$, we have

$$L_{S_\infty \cup S_{ram}}(s,\chi) = L(s,\chi).$$

We let $r_S(\chi)$ denote the order of vanishing of $L_S(s,\chi)$ at $s = 0$ and $L_S^*(0,\chi)$ the leading term in the Taylor expansion at $s = 0$.

By [Tat84, Chap. I, Prop 3.4] we know that

$$r_S(\chi) = \begin{cases} |\{v \in S : \chi(G_v) = 1\}| & \text{if } \chi \neq 1, \\ |S| - 1 & \text{if } \chi = 1. \end{cases}$$

Let T now be a set of places disjoint to S, then we can define the *S-imprimitive T-modified L-function* as

$$L_{k,S,T}(s,\chi) := \prod_{v \in T} (1 - \chi(\sigma_v)\mathcal{N}(v)^{1-s}) \cdot L_S(s,\chi).$$

Special notation in [Lan73] Fix an imaginary quadratic number field k. Lang defines in [Lan73] the following two L-functions, where we have to assume that $\mathrm{Re}(s) > 1$. They trivially coincide with (analytic resp. meromorphic continuations of) the L-functions from above on the defined region.

Let $\mathcal{O}$ be an order in $\mathcal{O}_k$ with $[\mathcal{O}_k : \mathcal{O}] = c$. A *proper $\mathcal{O}$-ideal* is an ideal $\mathfrak{a}$ of $\mathcal{O}$ for which it holds that $\mathcal{O} = \{\beta \in k : \beta\mathfrak{a} \subset \mathfrak{a}\}$. We collect some well-known properties of proper $\mathcal{O}$-ideals:

a) Principal ideals are proper.

b) For a fractional ideal $\mathfrak{a}$, $\mathfrak{a}$ is proper if and only if $\mathfrak{a}$ is invertible.

c) All ideals in the maximal order are proper.

d) An irreducible proper $\mathcal{O}$-ideal prime to the conductor is a prime ideal.

We denote by $I_\mathcal{O}$ the group of proper $\mathcal{O}$-ideals and by $P_\mathcal{O}$ the group of principal fractional $\mathcal{O}$-ideals. Then the group of proper $\mathcal{O}$-ideal classes is defined by $G_\mathcal{O} = I_\mathcal{O}/P_\mathcal{O}$. Let $J_\mathcal{O}$ be the group of invertible $\mathcal{O}$-ideals, $I_k(c)$ the group of fractional ideals of $\mathcal{O}_k$ coprime to c and

$$P_\mathbb{Z}(c) := \{(\alpha) : \alpha \in k, \alpha \equiv a \bmod c\mathcal{O}_k \text{ for some } a \in \mathbb{Z} \text{ with } (a,c) = 1\}.$$

With these definitions one has the following isomorphisms:

$$\mathrm{Pic}(\mathcal{O}) := J_\mathcal{O}/P_\mathcal{O} \cong G_\mathcal{O} \cong I_k(c)/P_\mathbb{Z}(c).$$

For $\mathcal{O} = \mathcal{O}_k$ we get directly from the definitions: $G_{\mathcal{O}_k} = Cl_k \cong \mathrm{Gal}(k(1)/k)$.

Remark 2.2.3. Unfortunately, after submitting this thesis, the author became aware of a mistake made in Chapter 21 of [Lan73] concerning L-functions of orders which is sometimes cited below. But everything is still true for the maximal order and this is the case which we are going to use in our applications. So we decided to keep the paragraph above about proper ideals but assume from now on that we work with maximal orders. This means when we ask for an ideal to be proper this is automatically fulfilled. This is also the resolution chosen by Lang in the second edition [Lan87].

We assume from now on that $\mathcal{O}$ is the maximal order $\mathcal{O}_k$.

Let A be a proper $\mathcal{O}$-ideal class. Then we define the zeta function $\zeta(s, A) = \sum_{\mathfrak{a}} \mathcal{N}(\mathfrak{a})^{-s}$ by taking the sum over all proper $\mathcal{O}$-ideals in the class. Let χ be a character of the proper $\mathcal{O}$-ideal class group $G_{\mathcal{O}}$. We define the L-series

$$L_{\mathcal{O}}(s, \chi) = \sum_{A} \chi(A)\zeta(s, A) = \prod_{\mathfrak{b}} \left(1 - \frac{\chi(\mathfrak{b})}{\mathcal{N}(\mathfrak{b})^s}\right)^{-1} \tag{2.10}$$

with the product taken over all proper irreducible $\mathcal{O}$-ideals.

Let k be an imaginary quadratic number field, $\mathfrak{g}$ an integral ideal of $\mathcal{O}_k$ and χ a character of $Cl(\mathfrak{g})$. Then we set

$$L_{\mathfrak{g}}(s, \chi) = \sum_{(\mathfrak{a}, \mathfrak{g})=1} \frac{\chi(\mathfrak{a})}{\mathcal{N}(\mathfrak{a})^s}, \tag{2.11}$$

where the sum is taken over all the ideal coprime to $\mathfrak{g}$.

2.2.2 Artin L-functions

This section and the next section are based on Chapter 0 and Chapter 1 in [Tat84], where the reader can find some details and further references for the assertions stated here.

Let K/k be a finite Galois extension with Galois group G. Let $\chi : G \to \mathbb{C}$ be the character of a complex representation $G \to \mathrm{GL}(V)$. For each finite place $\mathfrak{P}$ of K, the element $\sigma_{\mathfrak{P}}$ of $G_{\mathfrak{P}}/I_{\mathfrak{P}}$ acts on $V^{I_{\mathfrak{P}}}$. So we can set for $\mathrm{Re}(s) > 1$:

$$L(s, V) := \prod_{\mathfrak{p}} \left(\det(1 - \sigma_{\mathfrak{P}}\mathcal{N}(\mathfrak{p})^{-s} \mid V^{I_{\mathfrak{P}}})\right)^{-1},$$

where the product is taken over all the finite places $\mathfrak{p}$ of k, and $\mathfrak{P}$ is an arbitrarily chosen place of K above $\mathfrak{p}$. Furthermore, let S be a finite set of places of k containing S_∞. The *S-imprimitive Artin L-function* is defined for $Re(s) > 1$ as

$$L_S(s, \chi) = \prod_{v \notin S} \left(\det(1 - \sigma_w \mathcal{N}(v)^{-s} \mid V^{I_w})\right)^{-1},$$

where the product ranges over all the places v of k not contained in S and w is again an arbitrary place above v in K. It follows directly from the definitions that $L(s, V) = L_{S_\infty}(s, V)$.

It can be shown that the definition is independent of these choices. Furthermore, it can be shown that $L_S(s, V)$ remains unchanged if we replace V with an isomorphic representation, so it is also well-defined to write $L_S(s, \chi)$ instead of $L_S(s, V)$. If $\chi(1) = 1$ the Artin L-function coincides with the primitive abelian L-function.

In order to avoid confusion, we will also include the Galois extension K/k in the notation by writing $L_{K/k,S}(s, \chi)$, if necessary.

2.2.3 Leading terms at $s = 0$ and $s = 1$

Let K/k be a Galois extension of number fields with Galois group $G := \mathrm{Gal}(K/k)$. Given an S-imprimitive Artin L-function $L_S(s,\chi)$ we can write in the neighbourhood of $s = 0$ the expansion:

$$L_S(s,\chi) = c_S(\chi)s^{r_S(\chi)} + \ldots$$

where $r_S(\chi)$ is the order of vanishing at $s = 0$ and $c_S(\chi)$ denotes the leading term of the Laurent expansion. This is the notation of [Tat84], but we will often use the alternative notation $L_S^*(0,\chi)$ for the leading term. If $S = S_\infty$ we reduce the notation to $c(\chi)$ or alternatively $L^*(0,\chi)$.

For the first non-zero term of the Laurent expansion of $L(s,\chi)$ at $s = 1$ we write $c_1(\chi)$ or alternatively $L^*(1,\chi)$. So for a character of $G = \mathrm{Gal}(K/k)$ we have Equation (6.8) in [Tat84]

$$\frac{c_1(\chi)}{c(\overline{\chi})} = (-1)^{r_1(\chi)} \cdot 2^{r_2\chi(1)+a_1(\chi)} \frac{(\pi i)^{(a_2(\chi)+r_2\chi(1))}}{\tau(\chi) i^{r_2\chi(1)} \sqrt{|d_k|}^{\chi(1)}}$$

where $\tau(\chi)$ is the Gauss sum defined on page 19 in [Tat84] and

$$a_1(\chi) = \sum_{v \text{ real}} \dim(V^{G_w}), \qquad a_2(\chi) = \sum_{v \text{ real}} codim(V^{G_w}),$$

$$r_1 = \# \text{ of real places}, \qquad r_2 = \# \text{ of complex places},$$

$$r_1(\chi) = \text{order of vanishing of } L(s,\chi) \text{ at } s = 1.$$

Now let k be an imaginary quadratic field, $\mathfrak{f}$ an ideal in $\mathcal{O}_k$ and χ a non-trivial ray class character modulo $\mathfrak{f}$. For this situation we obtain the following equality which we are going to use later:

$$\frac{c_1(\chi)}{c(\overline{\chi})} = \frac{2\pi i}{\tau(\chi)\sqrt{d_k}}. \tag{2.12}$$

In contrast, for the trivial character we get

$$\frac{c_1(\chi)}{c(\overline{\chi})} = \frac{-2\pi i}{\tau(\chi)\sqrt{d_k}}. \tag{2.13}$$

2.3 Kronecker's limit formulas and applications

2.3.1 Kronecker's limit formulas

In this section we present results which can be traced back to Kronecker (cf. [Kro29a] and [Kro29b]). Later we will use them to prove the relation of the elliptic units and values of L-functions.

Kronecker's first limit formula Let $\tau = x + iy \in \mathbb{H}$ and set

$$E(\tau,s) := \sum_{(m,n)}{}' \frac{y^s}{|m\tau + n|^{2s}}, \text{ for } \mathrm{Re}(s) > 1.$$

This (Eisenstein) series has a simple pole at $s = 1$ with residue π and is holomorphic everywhere else on the complex plane. We are interested in the constant term at $s = 1$. Kronecker proved the following

Theorem 2.3.1. *Let $\eta(\tau)$ be as defined in Definition 2.1.1 e) and γ be the Euler constant. Then we have*

$$E(\tau,s) = \frac{\pi}{s-1} + 2\pi(\gamma - \log(2) - \log(\sqrt{y})|\eta(\tau)|^2) + O(s-1).$$

Kronecker's second limit formula Let u, v be real numbers which are not both integers and $\tau = x + iy \in \mathbb{H}$. We define

$$E_{u,v}(\tau, s) = \sum_{(m,n)}{}' e^{2\pi i(mu+nv)} \frac{y^s}{|m\tau + n|^{2s}}, \qquad \text{for } \mathrm{Re}(s) > 1.$$

Theorem 2.3.2. *The function $E_{u,v}(\tau, s)$ can be continued to an entire function of s, and one has*

$$E_{u,v}(\tau, 1) = -\pi \log |(f(u - v\tau; \tau) q_\tau^{v^2/2})|^2$$

with f defined in Definition 2.1.6 b) and $q_\tau = e^{2\pi i\tau}$.

A proof of both theorems can be found in Chapter 20, §4 (resp. §5) in [Lan73] in fairly modern language and also in [Mey57] in a more classical language.

Putting now together Definition 2.1.6 d) of $H\left(\begin{pmatrix} u \\ v \end{pmatrix}; \tau\right)$ and Theorem 2.3.2, an easy computation gives us the following

Corollary 2.3.3. *Let $u, v \in \mathbb{Q}$ with exact denominator $N > 1$ and $\tau \in \mathbb{H}$. Then we obtain*

$$E_{u,v}(\tau, 1) = \frac{-\pi}{6N} \log \left| H\left(\begin{pmatrix} u \\ v \end{pmatrix}; \tau\right) \right|.$$

2.3.2 An application of Kronecker's first limit formula

In this section we follow the treatment contained in Chapter 21 in [Lan73] respectively [Lan87] (cf. Remark 2.2.3).

Let k be an imaginary quadratic field and $\mathcal{O}$ the maximal order in k. Then for a proper $\mathcal{O}$-ideal class $A \in Cl(\mathcal{O})$ we set $\zeta(s, A) := \sum_{\mathfrak{a}} \mathcal{N}(\mathfrak{a})^{-s}$, where the sum ranges over all non-zero proper $\mathcal{O}$-ideals in the class A and $\mathcal{N}(\mathfrak{a})$ is the unique positive integer which generates $\mathfrak{a}\bar{\mathfrak{a}}$.

Fix $\mathfrak{b} \in A^{-1}$, then $\mathfrak{a}\mathfrak{b} = (\xi_{\mathfrak{a}})$ is principal. This way we get with $\mathfrak{a} \mapsto \xi_{\mathfrak{a}}$ a bijection between the proper $\mathcal{O}$-ideals in A and the $\mathcal{O}$-equivalence classes of the elements of $\mathfrak{b}$, where two elements are equivalent if their quotient is a unit in $\mathcal{O}$.

We assume $\mathfrak{b} = [1, \tau]$ because any proper $\mathcal{O}$-lattice is equivalent to a lattice of this type. We know that $\mathcal{N}(\mathfrak{a})\mathcal{N}(\mathfrak{b}) = \mathcal{N}((\xi_{\mathfrak{a}}))$ so

$$\zeta(s, A) = \frac{\mathcal{N}(\mathfrak{b})^s}{w(\mathcal{O})} \sum_{\xi \in \mathfrak{b}} \mathcal{N}(\xi)^{-s} = \frac{\mathcal{N}(\mathfrak{b})^s}{w(\mathcal{O})} \sum_{(m,n)}{}' |m\tau + n|^{-2s}$$

For the discriminant D we compute

$$D(\mathfrak{b}) = \begin{cases} \det\left(\begin{pmatrix} 1 & \tau \\ 1 & \overline{\tau} \end{pmatrix}\right)^2 = -(2y)^2 & \text{if } \tau = x + iy \in \mathbb{H}, \\ (\mathcal{N}(\mathfrak{b}))^2 D(\mathcal{O}) & \text{always.} \end{cases}$$

So we obtain

$$\zeta(s, A) = \frac{1}{w(\mathcal{O})} \left(\frac{2}{\sqrt{|d_{\mathcal{O}}|}}\right)^s \sum_{(m,n)}{}' \frac{y^s}{|m\tau + n|^{2s}},$$

where we denote by $|d_{\mathcal{O}}|$ the absolute value of the discriminant of $\mathcal{O}$ with $d_{\mathcal{O}} := D(\mathcal{O})$.

We set $g(\mathfrak{b}) := |(2\pi)^{-12}\mathcal{N}(\mathfrak{b})^6\Delta(\mathfrak{b})|$ (cf. Definition 2.1.21). This is an invariant of the equivalence class of $\mathfrak{b}$, so we can write $g(B)$ for $g(\mathfrak{b})$, where B is the proper $\mathcal{O}$-lattice class of $\mathfrak{b}$. By using the beginning of the exponential series of $\left(\frac{2}{\sqrt{|d_\mathcal{O}|}}\right)^{s-1}$ we obtain

$$\zeta(s,A) = \frac{1}{w(\mathcal{O})}\frac{2\pi}{\sqrt{|d_\mathcal{O}|}}\left(\frac{1}{s-1} + 2\gamma - \log(|d_\mathcal{O}|) + \frac{1}{6}\log(g(A^{-1}))\right) + O(s-1).$$

For a non-trivial character χ of $G_\mathcal{O}$, the sum over all $\chi(A)$ for the proper $\mathcal{O}$-ideals classes A is zero, so all the terms not depending on A vanish and we obtain the following theorem

Theorem 2.3.4. *[Lan73, Ch. 21, §1, Thm. 1] Let χ be a non-trivial character of $G_\mathcal{O}$ for the maximal order $\mathcal{O}$. Then*

$$L_\mathcal{O}(1,\chi) = -\frac{\pi}{3w(\mathcal{O})\sqrt{|d_\mathcal{O}|}}\sum_{A\in G_\mathcal{O}}\chi(A)\log(g(A^{-1})),$$

If χ is the trivial character, then

$$L_\mathcal{O}(s,1) = \frac{2\pi h_\mathcal{O}}{w(\mathcal{O})\sqrt{|d_\mathcal{O}|}}\frac{1}{s-1} + \cdots,$$

where $h_\mathcal{O}$ is the order of $G_\mathcal{O}$

Recall that we have $G_\mathcal{O} = Cl(1)$, $\varphi_{(1)}(C) = g(C)$ for a class $C \in Cl(1)$ and that for a character χ modulo (1) we get $\tau(\chi) = 1$. Then with Theorem 2.3.4 and (2.13) we can show

Corollary 2.3.5. *Let $\chi = 1$ be the trivial character. Then we obtain*

$$L^*(0,1) = \frac{-h_k}{w(1)}.$$

Proposition 2.3.6. *Let χ be a non-trivial character of $Cl(1)$. Then we have*

$$L^*(0,\chi^{-1}) = \frac{-1}{6w(1)}\sum_{C\in Cl(1)}\chi(C)\log\left(\varphi_{(1)}(C^{-1})\right).$$

Proof. From (2.12) we obtain

$$L^*(1,\chi) = L^*(0,\chi^{-1})\frac{2\pi i}{\tau(\chi)\sqrt{d_k}}$$

and therefore with Theorem 2.3.4 for $G_{\mathcal{O}_k} = Cl(1)$ we obtain

$$L^*(0,\chi^{-1}) = \frac{-1}{6w(1)}\sum_{C\in Cl(1)}\chi(C)\log(\varphi_{(1)}(C^{-1})).$$

□

Corollary 2.3.7. *Let χ be a character of $Cl(1)$. For a fixed prime ideal $\mathfrak{p}$ choose an auxiliary integral ideal $\mathfrak{a}$ such that $(\mathfrak{a}, 6\mathfrak{p}) = 1$. Then we have*

$$\left(1-\chi(\mathfrak{p})^{-1}\right)L^*(0,\chi^{-1}) = \frac{-2}{w(1)}\frac{1}{\mathcal{N}(\mathfrak{a})-\chi(\mathfrak{a})}\sum_{C\in Cl(1)}\chi(C)\log\left|\left(\frac{\delta(\mathcal{O}_k,\mathfrak{a}^{-1})}{\delta(\mathfrak{p},\mathfrak{p}\mathfrak{a}^{-1})}\right)^{\sigma(C)}\right|,$$

where here we used the absolute value $|z| = (z\cdot\overline{z})^{1/2}$ for $z\in\mathbb{C}$.

Proof. We can compute

$$
\begin{aligned}
&12 \cdot \sum_{C \in Cl(1)} \log \left| \left(\frac{\delta(\mathcal{O}_k, \mathfrak{a}^{-1})}{\delta(\mathfrak{p}, \mathfrak{p}\mathfrak{a}^{-1})} \right)^{\sigma(C)} \right| \chi(C) = \sum_{C \in Cl(1)} \log \left| \frac{\Delta(\mathfrak{c}^{-1})^{\mathcal{N}(\mathfrak{a})} \Delta(\mathfrak{a}^{-1}\mathfrak{c}^{-1}\mathfrak{p})}{\Delta(\mathfrak{a}^{-1}\mathfrak{c}^{-1}) \Delta(\mathfrak{c}^{-1}\mathfrak{p})^{\mathcal{N}(\mathfrak{a})}} \right| \chi(C) \\
&= \mathcal{N}(\mathfrak{a}) \sum_{C \in Cl(1)} \log \left(\varphi_{(1)}(\mathfrak{c}) \right) \chi(C) + \sum_{C \in Cl(1)} \log \left(\varphi_{(1)}(\mathfrak{a}\mathfrak{c}\mathfrak{p}^{-1}) \right) \chi(C) \\
&- \mathcal{N}(\mathfrak{a}) \sum_{C \in Cl(1)} \log \left(\varphi_{(1)}(\mathfrak{p}^{-1}\mathfrak{c}) \right) \chi(C) - \sum_{C \in Cl(1)} \log \left(\varphi_{(1)}(\mathfrak{a}\mathfrak{c}) \right) \chi(C) \\
&= (\mathcal{N}(\mathfrak{a}) - \chi(\mathfrak{a}))(1 - \chi(\mathfrak{p})^{-1}) \sum_{C \in Cl(1)} \chi(C) \log \left(\varphi_{(1)}(C^{-1}) \right)
\end{aligned}
$$

where for the first equality we have used Remark 2.1.12 b) and after choosing an integral $\mathfrak{c}$ in each $C \in Cl(\mathfrak{f})$ we use Proposition 3 in [Rob73, §3.1], for the second the fact that $\sum_{\mathfrak{c} \in Cl(1)} D \cdot \chi(\mathfrak{c}) = 0$ for a constant D and for the third that $\varphi_{(1)}(C)$ for a class $C \in Cl(1)$ is a class invariant. So with Theorem 2.3.4 we obtain the result. □

2.3.3 An application of Kronecker's second limit formula

Let k be an imaginary quadratic number field and $\mathfrak{f}$ be a non-zero ideal of $\mathcal{O}_k$. In this section we follow the treatment contained in Chapter 21 in [Lan73].

In order to express the value of a L-series at $s = 1$ in this case we again use, among other things, the concept of a Gauss sum. For the convenience of the reader we recall the definition and some facts used later on.

Recall that $\mathfrak{d}^{-1} := \{\lambda \in k \mid \mathrm{Tr}(\lambda\mathcal{O}_k) \subseteq \mathbb{Z}\}$ and that $\mathrm{Tr}(\lambda\mathcal{O}_k) \subset \mathbb{Z}$ if and only if $e^{2\pi i \mathrm{Tr}(\lambda\mathcal{O}_k)} = 1$. Let $\gamma \in k^\times$ be such that $(\gamma\mathfrak{f}\mathfrak{d}, \mathfrak{f}) = 1$, that $\gamma\mathfrak{d}$ has exact denominator $\mathfrak{f}$. Then we define for a character χ modulo $\mathfrak{f}$ a Gauss sum

$$
T_\gamma(\chi, \alpha) = \sum_{x \bmod \mathfrak{f}} \chi(x) e^{2\pi i \mathrm{Tr}(x\alpha\gamma)}, \tag{2.14}
$$

which has the following properties:

(G1) If χ is a character modulo $\mathfrak{f}$ and λ is prime to $\mathfrak{f}$, then

$$
T_\gamma(\chi, \alpha\lambda) = \overline{\chi}(\lambda) T_\gamma(\chi, \alpha).
$$

(G2) If χ is induced by a ray class character, one has that $\frac{\overline{\chi}(\gamma\mathfrak{d}\mathfrak{f})}{T_\gamma(\chi,1)}$ is independent of the choice of γ.

(G3) For a proper character χ modulo $\mathfrak{f}$ and $\alpha \in \mathcal{O}_k$ we have

$$
|T_\gamma(\chi, \alpha)| = \begin{cases} 0 & \text{if } \alpha \text{ is not prime to } \mathfrak{f}, \\ \sqrt{\mathcal{N}(\mathfrak{f})} & \text{if } \alpha \text{ is prime to } \mathfrak{f}. \end{cases}
$$

A proof of these properties can be found in Chapter 21 of [Lan73] and we want to stress that for property (G3) we use that we are working with a proper character.

Now let $A \in Cl_k$ and $\mathfrak{b}_A$ be an ideal in A^{-1} prime to $\mathfrak{f}$. Then we have for each $\mathfrak{a} \in A$ that $\mathfrak{a}\mathfrak{b}_A = (\xi_A)$ for some ξ_A and we obtain a bijection between the elements of A prime to $\mathfrak{f}$ and the non-zero principal subideals of $\mathfrak{b}_A$ coprime to $\mathfrak{f}$ via the map $\mathfrak{a} \mapsto (\xi_\mathfrak{a})$.

Recall the definition of $L_{\mathfrak{f}}(s,\chi)$ in (2.11). Now from $\mathcal{N}(\mathfrak{a}\mathfrak{b}_A)=\mathcal{N}((\xi_A))$ and by denoting the set of non-zero elements of $\mathfrak{b}_A$ coprime to $\mathfrak{f}$ by $\mathfrak{b}_A(\mathfrak{f})$ we obtain

$$L_{\mathfrak{f}}(s,\chi)=\frac{1}{w(\mathfrak{f})}\sum_{A\in Cl(\mathfrak{f})}\mathcal{N}(\mathfrak{b}_A)^s\overline{\chi}(\mathfrak{b}_A)\sum_{\xi\in\mathfrak{b}_A(\mathfrak{f})}\frac{\chi(\xi)}{\mathcal{N}(\xi)^s}.$$

Now with the help of properties (G1) and (G3), we obtain for a proper character χ of $Cl(\mathfrak{f})$

$$L_{\mathfrak{f}}(s,\chi)=\frac{1}{w(\mathfrak{f})T_\gamma(\overline{\chi},1)}\sum_{A\in Cl(\mathfrak{f})}\overline{\chi}(b_A)\mathcal{N}(b_A)^s{\sum_{\xi\in\mathfrak{b}_A}}' e^{2\pi i\mathrm{Tr}(\xi\gamma)}\frac{1}{\mathcal{N}(\xi)^s}, \tag{2.15}$$

where we have chosen γ such that $\gamma\mathfrak{f}\mathfrak{d}$ is integral and prime to $\mathfrak{f}$. Details of the proof of this assertion can be found in [Lan73, Ch. 22, §2, Lemma 1].
Now let A be a ray class of $Cl(\mathfrak{f})$ and $\mathfrak{b}$ an ideal of A prime to $\mathfrak{f}$. Then we define

$$E_{\mathfrak{f}}(A,s):=\mathcal{N}(\mathfrak{b}\mathfrak{d}^{-1}\mathfrak{f}^{-1})^s\sum_{\lambda\in\mathfrak{b}\mathfrak{d}^{-1}\mathfrak{f}^{-1}} e^{2\pi i\mathrm{Tr}(\lambda)}\mathcal{N}(\lambda)^{-s}.$$

With this definition we obtain for a proper character χ of $Cl(\mathfrak{f})$ that

$$L_{\mathfrak{f}}(s,\chi)=\frac{\chi(\gamma\mathfrak{d}\mathfrak{f})}{w(\mathfrak{f})T_\gamma(\overline{\chi},1)}\sum_{A\in Cl(\mathfrak{f})}\overline{\chi}(A)E_{\mathfrak{f}}(A,s).$$

This equality is Theorem 1 in [Lan73, Ch. 22, §2] and a detailed proof can be found there.

Let $\mathfrak{b}$ be an ideal in A prime to $\mathfrak{f}$ and let $\mathfrak{b}\mathfrak{d}^{-1}\mathfrak{f}^{-1}=[z_1,z_2]$ with $\tau_A=z_2/z_1=x+iy\in\mathbb{H}$. The non-zero elements $\lambda\in\mathfrak{b}\mathfrak{d}^{-1}\mathfrak{f}^{-1}$ can be written as mz_1+nz_2 with $(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}$. Moreover, we have

$$D(\mathcal{O}_k)\mathcal{N}(\mathfrak{b}\mathfrak{d}^{-1}\mathfrak{f}^{-1})=D(\mathfrak{b}\mathfrak{d}^{-1}\mathfrak{f}^{-1})=\det\left(\begin{pmatrix} z_2 & \overline{z_2}\\ z_1 & \overline{z_1}\end{pmatrix}\right)^2=-\mathcal{N}(z_1^2)(2y)^2$$

so we get $\mathcal{N}(z_1)=\frac{\mathcal{N}(\mathfrak{b}\mathfrak{d}^{-1}\mathfrak{f}^{-1})\sqrt{|d_k|}}{2y}$ and hence $\mathcal{N}(\lambda)=\mathcal{N}(z_1)|m+n\tau_A|^2$. It follows that

$$E_{\mathfrak{f}}(A,s)=\frac{2^s}{|d_k|^{s/2}}E_{u,v}(\tau_A,s)\quad\text{with } u:=\mathrm{Tr}(z_2)\text{ and } v:=\mathrm{Tr}(z_1).$$

If f is the smallest integer contained in $\mathfrak{f}$, we obtain with Kronecker's second limit formula (to be precise with Corollary 2.3.3) that

$$E_{\mathfrak{f}}(A,1)=\frac{-2\pi}{6f\sqrt{|d_k|}}\log|H(\mathfrak{b}\mathfrak{d}^{-1}\mathfrak{f}^{-1})|=\frac{-2\pi}{6f\sqrt{|d_k|}}\log|\Phi_{\mathfrak{f}}(A)|$$

and therefore the following

Theorem 2.3.8. *[Lan73, Ch. 22, §2, Thm. 2] Let $\mathfrak{f}\neq(1)$ be an ideal of $\mathcal{O}_k$. If χ is a proper character of $Cl(\mathfrak{f})$, then*

$$L_{\mathfrak{f}}(1,\chi)=\frac{-2\pi\chi(\gamma\mathfrak{d}\mathfrak{f})}{w(\mathfrak{f})T_\gamma(\overline{\chi},1)\sqrt{|d_k|}6f}\sum_{A\in Cl(\mathfrak{f})}\overline{\chi}(A)\log|\Phi_{\mathfrak{f}}(A)|.$$

Proposition 2.3.9. *Let $\mathfrak{f} \neq (1)$ be a non-zero ideal of $\mathcal{O}_k$ and χ a proper character of $Cl(\mathfrak{f})$. Then we obtain*

$$L^*(0, \chi^{-1}) = \frac{-2}{(\mathcal{N}(\mathfrak{a}) - \chi(\mathfrak{a})) \cdot w(\mathfrak{f})} \sum_{C \in Cl(\mathfrak{f})} \chi^{-1}(C) \log |\psi(1; \mathfrak{f}, \mathfrak{a})^{\sigma(C)}|,$$

where we use the absolute value $|c| = (c \cdot \overline{c})^{1/2}$ and an integral ideal $\mathfrak{a}$ coprime to $6\mathfrak{f}$.

Proof. Recall that from (2.7) it follows for a $C \in Cl(\mathfrak{f})$:

$$\varphi_{\mathfrak{f}}(C) = \varphi^{12f}(\mathfrak{f}\mathfrak{c}^{-1}),$$

where $\mathfrak{c}$ is a representative in C, so we can compute

$$\begin{aligned} \log \left| \left(\frac{\varphi^{12\mathcal{N}(\mathfrak{a})}(1; \mathfrak{f})}{\varphi^{12}(1, \mathfrak{f})^{\sigma(\mathfrak{a})}} \right)^{\sigma(\mathfrak{c})} \right| e_\chi &= (\mathcal{N}(\mathfrak{a}) - \chi(\mathfrak{a})) \log |\varphi^{12}(1; \mathfrak{f})^{\sigma(\mathfrak{c})}| e_\chi \\ &= (\mathcal{N}(\mathfrak{a}) - \chi(\mathfrak{a})) \frac{1}{f} \log |\varphi^{12f}(1, \mathfrak{f}\mathfrak{c}^{-1})| e_\chi \\ &= (\mathcal{N}(\mathfrak{a}) - \chi(\mathfrak{a})) \frac{1}{f} \log |\varphi_{\mathfrak{f}}(C)| e_\chi \end{aligned}$$

where for the Galois group G isomorphic to $Cl(\mathfrak{f})$ we set $e_\chi := \frac{1}{|G|} \sum_{g \in G} \chi(g) g^{-1}$ and f denotes the smallest positive integer contained in $\mathfrak{f}$.

With (2.6) we obtain

$$\frac{1}{12} \sum_{C \in Cl(\mathfrak{f})} \chi^{-1}(C) \log |\psi(1; \mathfrak{f}, \mathfrak{a})^{12\sigma(C)}| = \frac{1}{12} \sum_{C \in Cl(\mathfrak{f})} \chi^{-1}(C) \log \left| \left(\frac{\varphi^{12\mathcal{N}(\mathfrak{a})}(1; \mathfrak{f})}{\varphi^{12}(1; \mathfrak{f})^{\sigma(\mathfrak{a})}} \right)^{\sigma(C)} \right|.$$

Because of (2.5) we have $|\Phi_{\mathfrak{f}}(C)| = |\varphi_{\mathfrak{f}}(C)|$. Now with Theorem 2.3.8, (2.12) and the properties of Gauss sums (to see this one has to compare our definition with the one given on p. 19 in [Tat84]) we obtain

$$L^*(0, \chi^{-1}) = \frac{-2}{(\mathcal{N}(\mathfrak{a}) - \chi(\mathfrak{a})) \cdot w(\mathfrak{f})} \sum_{C \in Cl(\mathfrak{f})} \chi^{-1}(C) \log |\psi(1; \mathfrak{f}, \mathfrak{a})^{\sigma(C)}|.$$

□

2.3.4 Other strategy for proving these results

There is another strategy for obtaining the main results of the last two sections. Instead of computing the value of the L-function at $s = 1$ and then using a functional equation, one can try to express the leading term of the L-function at $s = 0$ in terms of Eisenstein series and then use facts about these series.

This strategy is used for example by Stark in [Sta80] in order to prove Theorem 2 (loc. cit.) which is a different variant of our Proposition 2.3.9 for $\mathfrak{f} \neq (1)$.

Similarly, Kato in (3.8.2) of Section 3 in [Kat04] claims that

$$\log(|{}_c\theta_E|) = c^2 E^{(0)}(\tau, z) - E^{(0)}(\tau, cz),$$

where he used the notation defined above and

$$E(k,\tau,z,s)=\sum_{(m,n)}(z+m\tau+n)^{-k}|z+m\tau+n|^{-s} \text{ as well as } E^{(0)}(\tau,z):=\lim_{s\to 0}s^{-1}E(0,\tau,z,s).$$

Unfortunately, there is however no proof for this result in [Kat04], he only mentions at the beginning of Section 3 that all the proofs can be found in [KM85], [Kat76] and [Wei76].

We assume now that $\mathfrak{g}$ is an ideal of $\mathcal{O}_k$ such that $\mathcal{O}_k^\times\to(\mathcal{O}_k/\mathfrak{g})^\times$ is injective or, equivalently, that $w(\mathfrak{g})=1$. Let (E,α) be the canonical CM-pair over $k(\mathfrak{g})$, $\mathfrak{a}$ an ideal prime to $6\mathfrak{f}$ and $G:=\mathrm{Gal}(k(\mathfrak{g})/k)$. In this situation Kato states in Chapter 15 on page 253 in [Kat04] that for any homomorphism $\chi:G\to\mathbb{C}^\times$ we have:

$$\sum_{h\in G}\chi(h)\log|{}_{\mathfrak{a}}z_{\mathfrak{g}}^h|=(\mathcal{N}(\mathfrak{a})-\chi(\sigma_{\mathfrak{a}})^{-1})\lim_{s\to 0}s^{-1}L_{\mathfrak{g}}(s,\chi) \text{ for } {}_{\mathfrak{a}}z_{\mathfrak{g}}={}_{\mathfrak{a}}\theta_E^{-1}(\alpha).$$

He continues saying that this is deduced from (3.8.2) in [Kat04] by taking a suitable element of k as τ in (3.8.2). As we were not able find a proof of (3.8.2) in the given literature, we have chosen to present the deduction of the applications of the limit formulas as we have done above.

In order to get our version from the assertion above, we have to recall (2.8), (2.9) and that Kato sets ${}_{\mathfrak{a}}z_{\mathfrak{g}}:={}_{\mathfrak{a}}\theta_E^{-1}(\alpha)$ with (E,α) being the canonical CM-pair over $k(\mathfrak{g})$, whereas Flach sets ${}_{\mathfrak{a}}z_{\mathfrak{g}}:=\psi(1;\mathfrak{g},\mathfrak{a})={}_c\Theta_{E/S}(1)$ with $S=\mathrm{Spec}(\mathbb{C})$, $L=\mathfrak{g}$, $\tilde{L}=\mathfrak{g}\mathfrak{a}^{-1}$, $E=\mathbb{C}/L$ and $c:\mathbb{C}/L\to\mathbb{C}/\tilde{L}$.

Chapter 3

Computing the constant term of a Coleman power series

In this chapter we use a recent result of T. Seiriki [Sei17] who describes the constant term of a Coleman power series in terms of a pairing which is defined in the local setting of Lubin-Tate extensions in a way similar to Solomon's construction in [Sol92]. We have to slightly adapt Seiriki's result for the setting of relative Lubin-Tate extensions.

For the reader's convenience we give a self-contained proof of Seiriki's result. We follow his strategy but adapt some of his arguments and fill in some details whenever we feel that this is necessary. The content of this chapter is, up to small modifications and corrections, a part of [BH18], which is joint work with my advisor Werner Bley and both authors have contributed equally to this work. In a little more detail this chapter contains the following:

In Section 3.1 we define a pairing in the local setting, using Hilbert's Theorem 90 and the valuation map, show some properties for this pairing and prove Proposition 3.1.5 which can be viewed as an assertion generalizing Hilbert's Theorem 90 under restrictive conditions. In Section 3.2 we recall some facts about relative Lubin-Tate groups and Coleman power series. In Section 3.3 we collect some auxiliary results we need in order to prove the main theorem of this chapter. In Section 3.4 we prove Theorem 3.4.1 which allows us to relate a Coleman power series to the pairing defined in Section 3.1, which is a theorem of T. Seiriki in the case of classical Lubin-Tate groups. From this theorem we derive Corollary 3.4.5 which we are going to use in the next chapter.

3.1 Definition and basic properties of Seiriki's pairing

Let K be a p-adic number field and L/K a finite abelian Galois extension with Galois group G unless specified otherwise. Let v_K be the normalized valuation of K (i.e., $v_K(K^\times) = \mathbb{Z}$). We put $U_{L/K} := \ker(\mathcal{N}_{L/K})$ and write $\widehat{G} := \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$ for the group of characters of G in this chapter.

Definition 3.1.1. For a character $\chi \in \widehat{G}$ we set $K_\chi := L^{\ker(\chi)}$. Then $G_\chi := \mathrm{Gal}(K_\chi/K)$ is a cyclic group whose order is denoted by d_χ (so $d_\chi = \mathrm{ord}(\chi)$). Let $\sigma \in G$ be such that $\chi(\sigma) = 1/d_\chi + \mathbb{Z}$. For each element $u \in U_{L/K}$ we define $u_\chi := \mathcal{N}_{L/K_\chi}(u)$ and observe that $\mathcal{N}_{K_\chi/K}(u_\chi) = 1$. Therefore, by Hilbert's Theorem 90, there exists an element $b_\chi \in K_\chi^\times$ such that

$$u_\chi = b_\chi^{\sigma-1}$$

which is unique up to elements of $K^\times$. We then define

$$\begin{aligned}(\cdot,\cdot)_{L/K} : U_{L/K} \times \widehat{G} &\longrightarrow \mathbb{Q}/\mathbb{Z},\\ (u,\chi) &\mapsto v_K(b_\chi) + \mathbb{Z}.\end{aligned}$$

The pairing is obviously multiplicative in the first variable. Moreover, from the very definition we obtain

$$(u,\chi)_{L/K} = (\mathcal{N}_{L/K_\chi}(u),\chi)_{K_\chi/K}.$$

The proof of multiplicativity in the second variable is more involved. In the following we give an expanded and corrected version of Seiriki's proof of [Sei17, Prop. 2.2].

For a finite extension L/K we write $f_{L/K}$ for the degree of the residue class field extension.

Lemma 3.1.2. *For $\alpha \in L^\times$, $\tau \in G$ and $\chi \in \widehat{G}$ one has*

$$\left(\frac{\tau(\alpha)}{\alpha},\chi\right)_{L/K} = f_{L/K} v_L(\alpha)\chi(\tau) = [L:K] v_K(\alpha)\chi(\tau).$$

Proof. By our definitions $\langle \sigma|_{K_\chi}\rangle = G_\chi$, so that we may fix $s \in \mathbb{Z}_{>0}$ such that $\sigma^s|_{K_\chi} = \tau|_{K_\chi}$.

If we set $\alpha_\chi := \mathcal{N}_{L/K_\chi}(\alpha)$, then

$$\mathcal{N}_{L/K_\chi}\left(\frac{\tau(\alpha)}{\alpha}\right) = \frac{\tau(\alpha_\chi)}{\alpha_\chi} = \frac{\sigma^s(\alpha_\chi)}{\alpha_\chi} = \prod_{j=0}^{s-1} \frac{\sigma(\sigma^j(\alpha_\chi))}{\sigma^j(\alpha_\chi)}$$

and we obtain from the definition of the pairing and multiplicativity in the first variable

$$\left(\frac{\tau(\alpha)}{\alpha},\chi\right)_{L/K} = \sum_{j=0}^{s-1}\left(\frac{\sigma(\sigma^j(\alpha_\chi))}{\sigma^j(\alpha_\chi)},\chi\right)_{K_\chi/K} = \sum_{j=0}^{s-1} v_K(\sigma^j(\alpha_\chi)) = s v_K(\alpha_\chi).$$

Let π_L be a uniformizing element in L and set $\pi_\chi := N_{L/K_\chi}(\pi_L)$. Then $v_{K_\chi}(\pi_\chi) = f_{L/K_\chi}$. We write $\alpha = \pi_L^a \beta$ with $a = v_L(\alpha)$ and $\beta \in \mathcal{O}_L^\times$. Then

$$v_K(\alpha_\chi) = a v_K(\pi_\chi) = a f_{L/K_\chi}\frac{1}{e_{K_\chi/K}} = a f_{L/K}\frac{1}{e_{K_\chi/K} f_{K_\chi/K}} = a f_{L/K}\frac{1}{d_\chi}.$$

Hence we obtain

$$\left(\frac{\tau(\alpha)}{\alpha},\chi\right)_{L/K} = v_L(\alpha) f_{L/K}\frac{s}{d_\chi}$$

and noting $\chi(\tau) = \chi(\sigma^s) = \frac{s}{d_\chi} + \mathbb{Z}$ the first equality of the lemma follows. The second equality is then immediate from $v_L = e_{L/K} v_K$ and $e_{L/K} f_{L/K} = [L:K]$. $\square$

Proposition 3.1.3. *Assume that L/K is a totally ramified finite abelian extension. Then for any $u \in U_{L/K}$ and any $\chi,\chi' \in \widehat{G}$ one has*

$$(u,\chi\chi')_{L/K} = (u,\chi)_{L/K} + (u,\chi')_{L/K}.$$

Proof. Let K_{nr} denote the maximal unramified extension of K. Then $L_{\mathrm{nr}} := LK_{\mathrm{nr}}$ is the maximal unramified extension of L and we may identify $\mathrm{Gal}(L_{\mathrm{nr}}/K_{\mathrm{nr}})$ with G via restriction. By [Ser79, Ch. V.4, Prop. 7] the norm map $\mathcal{N}_{L_{\mathrm{nr}}/K_{\mathrm{nr}}} \colon L_{\mathrm{nr}}^\times \longrightarrow K_{\mathrm{nr}}^\times$ is surjective. By [Ser79, Ch. X.7, Prop. 11] the G-module $L_{\mathrm{nr}}^\times$ is cohomologically trivial, in particular, $\widehat{H}^{-1}(G, L_{\mathrm{nr}}^\times) = 0$.

Hence there exist elements $\alpha_1, \ldots, \alpha_r \in L_{\mathrm{nr}}^\times$ and automorphisms $\sigma_1, \ldots, \sigma_r \in G$ such that $u = \prod_{i=1}^r \frac{\sigma_i(\alpha_i)}{\alpha_i}$. Set $L' := L(\alpha_1, \ldots, \alpha_r)$ and $K' := L' \cap K_{\mathrm{nr}}$. Then

$$(u, \chi)_{L/K} = (u, \chi)_{L'/K'} = \sum_{i=1}^r \left(\frac{\sigma_i(\alpha_i)}{\alpha_i}, \chi \right)_{L'/K'}$$

and Lemma 3.1.2 implies $(u, \chi)_{L/K} = \sum_{i=1}^r v_{L'}(\alpha_i)\chi(\sigma_i)$. Replacing χ by χ' and $\chi\chi'$, respectively, we obtain similar expressions for $(u, \chi')_{L/K}$ and $(u, \chi\chi')_{L/K}$ and the result follows. □

Lemma 3.1.4. *Let L/K be a totally ramified finite abelian extension of degree n. Let $\{u_1, \ldots, u_r\} \subseteq \mathcal{O}_K^\times$ be a finite set of units in K, let K'/K be the unramified extension of degree n and put $L' := K'L$. Then there exist units $u_1', \ldots, u_r' \in \mathcal{O}_{L'}^\times$ such that $u_i = \mathcal{N}_{L'/K'}(u_i')$ for $i = 1, \ldots, r$.*

Proof. It suffices to prove the lemma for $r = 1$. In this case we can simply follow the proof of [Sei17, Lemma 2.4]. We briefly recall the argument.

We write $G_0(L/K)$ and $G_0(L'/K')$ for the inertia subgroups of L/K and L'/K', respectively. Then

$$G_0(L'/K') \xleftarrow{\simeq} \mathcal{O}_{K'}^\times/\mathcal{N}_{L'/K'}(\mathcal{O}_{L'}^\times) \xrightarrow{\mathcal{N}_{K'/K}} \mathcal{O}_K^\times/\mathcal{N}_{L/K}(\mathcal{O}_L^\times) \xrightarrow{\simeq} G_0(L/K),$$

where the left and the right isomorphisms are induced by the reciprocity maps of local class field theory. Furthermore, the middle map is surjective by [Ser79, Ch. V.2], hence it is actually an isomorphism of groups of order n. Since $\mathcal{N}_{K'/K}(u) = u^n \in \mathcal{N}_{L/K}(\mathcal{O}_L^\times)$ we deduce $u \in \mathcal{N}_{L'/K'}(\mathcal{O}_{L'}^\times)$. □

Proposition 3.1.5. *Let L/K be a totally ramified finite abelian extension. Assume that $u \in U_{L/K}$ satisfies $(u, \chi)_{L/K} = 0$ for all characters $\chi \in \widehat{G}$. Then there exist*

a) a finite unramified extension K' of K,

b) an integer r,

c) units $\beta_1', \ldots, \beta_r' \in \mathcal{O}_{L'}^\times$ with $L' := LK'$ and

d) $\sigma_1', \ldots, \sigma_r' \in \mathrm{Gal}(L'/K')$

such that

$$u = \prod_{i=1}^r (\beta_i')^{\sigma_i' - 1}.$$

Proof. We follow the proof of [Sei17, Lemma 2.5] and prove the proposition by induction on the number of generators of G. If G is trivial, the claim is clear. For a non-trivial group G we write $G = \tilde{G} \times H$ with a cyclic subgroup H and apply the inductive hypothesis to the extension M/K where $M := L^H$. To that end we put $u_M := \mathcal{N}_{L/M}(u)$. Then $u_M \in U_{M/K}$ and we first note that $(u_M, \psi)_{M/K} = 0$ for all $\psi \in \widehat{G/H}$ because by the very definition of the pairing we have $(u, \chi)_{L/K} = (u_M, \chi)_{M/K}$ for all $\chi \in \widehat{G}$ with $H \subseteq \ker(\chi)$.

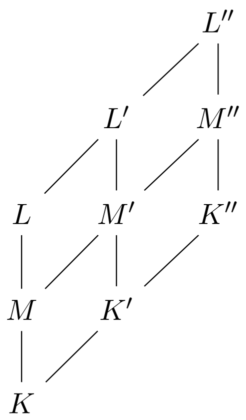
By induction we obtain a finite unramified extension K'/K, an integer r', units $\beta'_1, \ldots, \beta'_{r'}$ in $\mathcal{O}_{M'}^{\times}$ (where $M' = MK'$) and automorphisms $\sigma'_1, \ldots, \sigma'_{r'} \in \mathrm{Gal}(M'/K')$ such that

$$u_M = \mathcal{N}_{L/M}(u) = \prod_{i=1}^{r'} (\beta'_i)^{\sigma'_i - 1}. \tag{3.1}$$

By applying Lemma 3.1.4 to $\beta'_1, \ldots, \beta'_{r'}$ and the extension L'/M' we obtain an unramified extension M''/M', elements $\beta''_1, \ldots, \beta''_{r'} \in \mathcal{O}_{L''}^{\times}$ (where $L'' = L'M''$) such that

$$\beta'_i = \mathcal{N}_{L''/M''}(\beta''_i) \tag{3.2}$$

With K''/K' denoting the unramified extension of degree $[M'' : M']$ we have the following diagram



As a consequence, restriction induces a canonical epimorphism $\mathrm{Gal}(L''/K'') \twoheadrightarrow \mathrm{Gal}(M'/K')$ and we may choose lifts $\sigma''_1, \ldots, \sigma''_{r'} \in \mathrm{Gal}(L''/K'')$ of the elements $\sigma'_1, \ldots, \sigma'_{r'} \in \mathrm{Gal}(M'/K')$. We set

$$u'' := u \cdot \left(\prod_{i=1}^{r'} (\beta''_i)^{\sigma''_i - 1} \right)^{-1} \in \mathcal{O}_{L''}^{\times}.$$

Then a straightforward computation using (3.1) and (3.2) shows that $u'' \in U_{L''/M''}$. We let τ denote a generator of H and apply Hilbert's Theorem 90 to obtain an element $b'' \in (L'')^{\times}$ such that $u'' = (b'')^{\tau - 1}$, and hence

$$u = (b'')^{\tau - 1} \cdot \prod_{i=1}^{r'} (\beta''_i)^{\sigma''_i - 1}.$$

Since we can adapt b'' by elements of $(M'')^{\times}$, this proves the proposition, provided that we can show the following claim.

Claim. *There exists an element $a'' \in (M'')^{\times}$ such that $a''b'' \in \mathcal{O}_{L''}^{\times}$.*

For the proof of the claim we first note that

$$(u'', \psi)_{L''/M''} = 0 \text{ for all } \psi \in \mathrm{Gal}(\widehat{L''/M''}). \tag{3.3}$$

Indeed, if we define $\tilde{\psi} \in \mathrm{Gal}(\widehat{L''/K''})$ by $\tilde{\psi}|_H = \psi$ and $\tilde{\psi}|_{\tilde{G}} = 1$, then Lemma 3.1.6 below shows that $(u'', \psi)_{L''/M''} = (u'', \tilde{\psi})_{L''/K''}$. Furthermore,

$$\begin{aligned}(u'', \tilde{\psi})_{L''/K''} &= (u, \tilde{\psi})_{L''/K''} - \sum_{i=1}^{r'} \left(\frac{\sigma_i''(\beta_i'')}{\beta_i''}, \tilde{\psi} \right)_{L''/K''} \\ &= (u, \tilde{\psi})_{L/K} - \sum_{i=1}^{r'} \left(\frac{\sigma_i''(\beta_i'')}{\beta_i''}, \tilde{\psi} \right)_{L''/K''} \\ &= 0,\end{aligned}$$

where the second equality holds because K''/K is unramified and the last equality follows from $(u, \tilde{\psi})_{L/K} = 0$ (by assumption) and Lemma 3.1.2.

We are finally ready to prove the above claim.
Let $\chi \in \mathrm{Gal}(L''/M'')$ be defined by $\chi(\tau) = \frac{1}{[L'':M'']} + \mathbb{Z}$. We write $e_{L''/M''}$ for the ramification index of L''/M''. By (3.3) and the definition of the pairing we get

$$0 = (u'', \psi)_{L''/M''} = v_{M''}(b'') + \mathbb{Z} = \frac{1}{e_{L''/M''}} v_{L''}(b'') + \mathbb{Z}$$

in $\frac{1}{e_{L''/M''}}\mathbb{Z}/\mathbb{Z}$ and this implies that $e_{L''/M''}$ divides $v_{L''}(b'')$. This, in turn, guarantees the existence of a'' as in the above claim. □

Lemma 3.1.6. *Let L/K be a totally ramified finite abelian extension. Suppose that $G = \tilde{G} \times H$ with a cyclic subgroup H. Set $M := L^H$ and $\tilde{M} := L^{\tilde{G}}$. For $\psi \in \widehat{H}$ we define $\tilde{\psi} \in \widehat{G}$ by $\tilde{\psi}_{|H} = \psi$ and $\tilde{\psi}_{|\tilde{G}} = 1$. Then $(w, \tilde{\psi})_{L/K} = (w, \psi)_{L/M}$ for all $w \in U_{L/M}$ and all $\psi \in \widehat{H}$.*

Proof. Let $\langle \tau \rangle = H$ and define $\chi \in \widehat{H}$ by $\chi(\tau) = \frac{1}{|H|} + \mathbb{Z}$. By Proposition 3.1.3 it suffices to show that $(w, \tilde{\chi})_{L/K} = (w, \chi)_{L/M}$ for all $w \in U_{L/M}$.

Let $\beta \in L^\times$ such that $\frac{\tau(\beta)}{\beta} = w$. Then $(w, \chi)_{L/M} = v_M(\beta)$ in $\mathbb{Q}/\mathbb{Z}$. Since $\ker(\tilde{\chi}) = \tilde{G}$ we have $K_{\tilde{\chi}} = \tilde{M}$ and

$$w_{\tilde{\chi}} = \mathcal{N}_{L/\tilde{M}}(w) = \frac{\tau(\mathcal{N}_{L/\tilde{M}}(\beta))}{\mathcal{N}_{L/\tilde{M}}(\beta)}.$$

Therefore, by definition of the pairing, $(w, \tilde{\chi})_{L/K} = v_K(\mathcal{N}_{L/\tilde{M}}(\beta))$ and a straightforward computation with valuations shows that $v_K(\mathcal{N}_{L/\tilde{M}}(\beta)) = v_M(\beta)$. □

Later we will need the following definition.

Definition 3.1.7. Let H be a local field and H_∞/H an infinite abelian extension. Let $(u_N)_N \in \varprojlim_N N^\times$, where N/H varies over the finite subextensions of H_∞/H, be a norm-coherent sequence with $\mathcal{N}_{N/H}(u_N) = 1$. Furthermore, let χ be a character of finite order of $\mathrm{Gal}(H_\infty/H)$. Choose N such that $H_\chi \subset N$. Then we set

$$u_\chi := \mathcal{N}_{N/H_\chi}(u_N)$$

and define a pairing for the extension H_∞/H by

$$(u, \chi)_{H_\infty/H} := (u_\chi, \chi)_{H_\chi/H}.$$

It is easy to see that multiplicativity in both variables follows from the finite dimensional case.

3.2 Relative Lubin-Tate Groups and Coleman power series

In this section we introduce the notion of relative Lubin-Tate formal groups and also recall some results of the theory of Coleman power series. All results presented here can be found in [dS87].

Let H be a finite extension of $\mathbb{Q}_p$, let $\mathcal{O}_H$ and $\mathfrak{p}_H$ be its valuation ring and valuation ideal, respectively. We let q denote the cardinality of the residue class field $\mathcal{O}_H/\mathfrak{p}_H$. We fix an integer $d > 0$ and let H' be the unramified extension of H of degree d. We write $\varphi \in \mathrm{Gal}(H'/H)$ for the arithmetic Frobenius element.

We write $\mathcal{O}_{H'}$ and $\mathfrak{p}_{H'}$ for the valuation ring and valuation ideal in H' and fix an element $\xi \in H^\times$ with $v_H(\xi) = d$. We set

$$\mathcal{F}_\xi := \{f \in \mathcal{O}_{H'}[[T]] : f \equiv \pi' T (\mathrm{mod}\deg 2),\ \mathcal{N}_{H'/H}(\pi') = \xi,\ f \equiv T^q (\mathrm{mod} \mathfrak{p}_{H'})\}$$

and recall from [dS87, Ch. I, Thm. 1.3] that for each $f \in \mathcal{F}_\xi$ there exists a unique one-dimensional commutative formal group law $F_f \in \mathcal{O}_{H'}[[X, Y]]$ satisfying $F_f^\varphi \circ f = f \circ F_f$. We call F_f a *relative* Lubin-Tate group (relative to the extension H'/H). The case $d = 1$ corresponds to classical Lubin-Tate formal groups.

Let $f, g \in \mathcal{F}_\xi$ with $f = \pi_1 T + \ldots$ and $g = \pi_2 T + \ldots$. For an element $a \in \mathcal{O}_{H'}$ such that $a^{\varphi-1} = \pi_2/\pi_1$ there is a power series $[a]_{f,g} \in \mathcal{O}_{H'}[[T]]$ uniquely determined by the properties of [dS87, Ch. I, Prop. 1.5]. If $f = g$ we write $[a]_f$ in place of $[a]_{f,f}$ and note that the map $\mathcal{O}_H \longrightarrow \mathrm{End}(F_f), a \mapsto [a]_f$, is an injective ring homomorphism.

Let H^c be the algebraic closure of H and write $\mathfrak{p}_{H^c}$ for its valuation ideal. Then we get an $\mathcal{O}_H$-module structure on $\mathfrak{p}_{H^c}$ by setting

$$x +_f y := F_f(x, y) \quad \text{and} \quad a \cdot x := [a]_f(x)$$

for $x, y \in \mathfrak{p}_{H^c}$ and $a \in \mathcal{O}_H$.

For an integer $n \geq 0$ and $f \in \mathcal{F}_\xi$ we set $f^{(n)} := \varphi^{n-1}(f) \circ \cdots \circ \varphi(f) \circ f$. Let π be a prime element of $\mathcal{O}_H$. We define

$$W_f^n := \{\omega \in \mathfrak{p}_{H^c} \mid [\pi^n]_f(\omega) = 0\} = \{\omega \in \mathfrak{p}_{H^c} \mid f^{(n)}(\omega) = 0\}$$

and call W_f^n the *group of division points of level* n of F_f. We also set $\widetilde{W}_f^n := W_f^n \setminus W_f^{n-1}$ and $W_f = \bigcup_n W_f^n$. So W_f is the subgroup of all torsion points of F_f.

We fix $f \in \mathcal{F}_\xi$ and set $H'_n := H'(W_f^{n+1})$. Note that H'_n does not depend on the choice of $f \in \mathcal{F}_\xi$. It is a totally ramified finite abelian extension of H' of degree $(q-1)q^n$. Any $\omega_{n+1} \in \widetilde{W}_f^{n+1}$ generates H'_n over H' and, in addition, is a prime element in $\mathcal{O}_{H'_n}$. For the ring of integers in H'_n we obtain $\mathcal{O}_{H'_n} = \mathcal{O}_{H'}[\omega_{n+1}]$ for each $\omega_{n+1} \in \widetilde{W}_f^{n+1}$. The reciprocity map rec_H induces a group isomorphism $\left(\mathcal{O}_H/\mathfrak{p}^{n+1}\right)^\times \xrightarrow{\cong} \mathrm{Gal}(H'_n/H'), u \mapsto \sigma_u$ with $\sigma_u(\omega) = [u^{-1}]_f(\omega)$ for all $\omega \in W_f^{n+1}$, see [dS87, Ch. I, Prop. 1.8].

In the following we introduce the *Coleman norm operator* and recall some of its properties. Let $R = \mathcal{O}_{H'}[[T]]$ be the ring of power series with coefficients in $\mathcal{O}_{H'}$. By [dS87, Ch. I, Prop. 2.1] there exists a unique multiplicative operator $\mathcal{N} = \mathcal{N}_f \colon R \to R$ such that

$$\mathcal{N}h \circ f = \prod_{\omega \in W_f^1} h(T +_f \omega)$$

for all $h \in R$.

Proposition 3.2.1. *([dS87, Ch. I, Prop. 2.1]) The Coleman norm operator has the following properties:*

a) $\mathcal{N}h \equiv h^{\varphi} (\mathrm{mod}\ \mathfrak{p}_{H'})$.

b) $\mathcal{N}_{\varphi(f)} = \varphi \circ \mathcal{N}_f \circ \varphi^{-1}$.

c) Let $\mathcal{N}_f^{(i)} := \mathcal{N}_{\varphi^{i-1}(f)} \circ \cdots \circ \mathcal{N}_{\varphi(f)} \circ \mathcal{N}_f$. *Then*

$$\left((\mathcal{N}_f^{(i)} h) \circ f^{(i)} \right) (T) = \prod_{\omega \in W_f^i} h(T +_f \omega).$$

d) If $h \in R$, $h \equiv 1 \bmod (\mathfrak{p}_{H'})^i$ *for* $i \geq 1$, *then* $\mathcal{N}_f h \equiv 1 \left(\mathrm{mod} (\mathfrak{p}_{H'})^{i+1} \right)$.

Let $\alpha = (\alpha_n) \in \varprojlim_n (H'_n)^\times$ be a norm-coherent sequence. We fix $\omega_i \in \widetilde{W}^i_{\varphi^{-i}(f)}$ such that $(\varphi^{-i}(f))(\omega_i) = \omega_{i-1}$. There is a unique integer $\nu(\alpha)$ such that $\alpha_n \mathcal{O}_{H'_n} = \mathfrak{p}_{H'_n}^{\nu(\alpha)}$ for all $n \geq 0$. By [dS87, Ch. I, Thm. 2.2] there exists a unique power series $Col_\alpha \in T^{\nu(\alpha)} \cdot \mathcal{O}_{H'}[[T]]^\times$ such that

$$(\varphi^{-(i+1)} Col_\alpha)(\omega_{i+1}) = \alpha_i \tag{3.4}$$

for all $i \geq 0$. The power series Col_α is called the *Coleman power series* associated to α. We recall that by [dS87, Ch. I, Cor. 2.3 (i)] Coleman power series are multiplicative in α, i.e.

$$Col_{\alpha\alpha'} = Col_\alpha \cdot Col_{\alpha'}$$

for norm-coherent sequences $\alpha, \alpha' \in \varprojlim_n (H'_n)^\times$.

Remark 3.2.2. If we fix $\omega_n \in \widetilde{W}^n_{\varphi^{-n}(f)}$ such that $(\varphi^{-n}(f))(\omega_n) = \omega_{n-1}$ for $1 \leq n < \infty$, then we call $\omega = (\omega_n)_{n \geq 0}$ a *generator of the Tate module of* F_f. Note that each ω_n is a division point on $F_{\varphi^{-n}(f)} = F_f^{\varphi^{-n}}$.

We set $H'_\infty := \bigcup_n H'_n$ and let $u = (u_n) \in \varprojlim_n \mathcal{O}_{H'_n}^\times$ be a norm-coherent sequence of units. For later reference we recall the following lemma.

Lemma 3.2.3. *For* $\sigma \in \mathcal{G} := \mathrm{Gal}(H'_\infty / H)$, *there exits a unique isomorphism* $h : F_f \simeq F_{\sigma(f)}$ *such that* $h(\omega) = \sigma(\omega)$ *for all* $\omega \in W_f$. *This* h *is of the form* $[\kappa(\sigma)]_{f,\sigma(f)}$ *for a unique* $\kappa(\sigma) \in \mathcal{O}_{H'}^\times$, *and* $\kappa(\sigma)^{\varphi - 1} = f'(0)^{\sigma - 1}$. *The map* $\kappa : \mathcal{G} \to \mathcal{O}_{H'}^\times$ *is a 1-cocycle, i.e.* $\kappa(\tau\sigma) = \kappa(\sigma)^\tau \cdot \kappa(\tau)$ *for all* $\sigma, \tau \in \mathcal{G}$, *and* Col_u *and* $Col_{\sigma(u)}$ *are related by*

$$Col_{\sigma(u)} = Col_u^\sigma \circ [\kappa(\sigma)]_{f,\sigma(f)}.$$

Proof. This is a generalization of [dS87, Ch. I, Cor. 2.3] or (15) on page 21 in [dS87, Ch. I.3]. □

3.3 Auxiliary results

We continue to use the notation introduced in the previous section.

We fix $f \in \mathcal{F}_\xi$. For this section we fix an integer $m \in \mathbb{Z}_{>0}$ but usually suppress m in our notations. Let $\pi := \pi_H$ be a uniformizing element and define

$$R := R_{H'} := \frac{\mathcal{O}_{H'}[[T]]}{([\pi^{m+1}]_f)} = \frac{\mathcal{O}_{H'}[[T]]}{\left(f^{(m+1)}\right)}.$$

We write $\iota := \iota_{H'}$ for the injective ring homomorphism

$$\begin{aligned} \iota_{H'}\colon R &\longrightarrow \mathcal{O}_{H'} \oplus \bigoplus_{l=0}^{m} \mathcal{O}_{H'_l}, \\ \bar{g} &\mapsto \left(g(0), \left((\varphi^{-(l+1)}g)(\omega_{l+1})\right)_{l=0,\dots,m}\right). \end{aligned}$$

We let $r := r_{H'}\colon R \longrightarrow \mathcal{O}_{H'_m}$ be the composite of ι and the projection to the last component.

If F/H' is a finite unramified extension, we set $R_F := R \otimes_{\mathcal{O}_{H'}} \mathcal{O}_F$ and $F_l := FH'_l$ and note that $\mathcal{O}_{F_l} = \mathcal{O}_F\mathcal{O}_{H'_l}$. Let $\iota_F\colon R_F \longrightarrow \mathcal{O}_F \oplus \bigoplus_{l=0}^{m} \mathcal{O}_{F_l}$ and $r_F\colon R_F \longrightarrow \mathcal{O}_{F_m}$ denote the base change of ι and r along $\mathcal{O}_F$ over $\mathcal{O}_{H'}$. Since $\mathcal{O}_{F_m} = \mathcal{O}_F[\omega_{m+1}]$ the ring homomorphism r_F is surjective. In addition, as it is actually a homomorphism of local rings, we conclude that $r_F\colon R_F^\times \longrightarrow \mathcal{O}_{F_m}^\times$ is surjective as well.

The Galois group $\mathrm{Gal}(F_\infty/F)$ naturally acts on $\mathcal{O}_F \oplus \bigoplus_{l=0}^{m} \mathcal{O}_{F_l}$. The following lemma shows that we can transport this action to R_F via ι_F.

Lemma 3.3.1. *For $\bar{g} \in R_F$ and $\sigma \in \mathrm{Gal}(F_\infty/F)$ the element $\sigma(\iota_F(\bar{g}))$ is contained in the image $\iota_F(R_F)$.*

Proof. We identify $\mathrm{Gal}(F_\infty/F)$ and $\mathrm{Gal}(H'_\infty/H')$ via restriction and let $u \in \mathcal{O}_H^\times$ such that $\sigma = \mathrm{rec}_H(u^{-1})$. Recall that ω_{l+1} is a torsion point for $F_{\varphi^{-(l+1)}(f)} = F_f^{\varphi^{-(l+1)}}$ and hence,

$$\sigma(\omega_{l+1}) = \mathrm{rec}_H(u^{-1})(\omega_{l+1}) = [u]_f^{\varphi^{-(l+1)}}(\omega_{l+1}) \tag{3.5}$$

by [dS87, Ch. I, Prop. 1.8].

We thus obtain

$$\begin{aligned} \sigma(\iota_F(\bar{g})) &= \left(\sigma(g(0)), \left(\sigma((\varphi^{-(l+1)}g)(\omega_{l+1}))\right)_{l=0,\dots,m}\right) \\ &= \left(g(0), \left((\varphi^{-(l+1)}g)([u]_f^{\varphi^{-(l+1)}}(\omega_{l+1}))\right)_{l=0,\dots,m}\right) \\ &= \left(g(0), \left((\varphi^{-(l+1)}(g \circ [u]_f))(\omega_{l+1})\right)_{l=0,\dots,m}\right) \\ &= \iota_F\left(\overline{g \circ [u]_f}\right), \end{aligned}$$

where we use (3.5) for the second equality. □

For $\bar{g} \in R_F$ we define $\mathcal{N}_{R_F/\mathcal{O}_F}(\bar{g})$ to be the norm of the $\mathcal{O}_F$-linear endomorphism of R_F given by multiplication by $\bar{g}$.

Lemma 3.3.2. *Let $\bar{g} \in R_F$. Then:*

a)

$$\mathcal{N}_{R_F/\mathcal{O}_F}(\bar{g}) = g(0) \prod_{l=0}^{m} \left(\mathcal{N}_{F_l/F}((\varphi^{-(l+1)}g)(\omega_{l+1})\right)^{\varphi^{l+1}}.$$

b)

$$\mathcal{N}_{R_F/\mathcal{O}_F}(\bar{g}) = \prod_{\omega \in W_f^{m+1}} g(\omega).$$

Proof. For the proof of a) we first define a modified $\mathcal{O}_F$-module structure on $\mathcal{O}_F \oplus \bigoplus_{l=0}^m \mathcal{O}_{F_l}$ by

$$a * (\beta_{-1}, \beta_0, \dots, \beta_m) := (a\beta_{-1}, \varphi^{-1}(a)\beta_0, \dots, \varphi^{-(m+1)}(a)\beta_m)$$

for $a, \beta_{-1} \in \mathcal{O}_F$ and $\beta_l \in F_l$, $l \in \{0, \dots, m\}$. With respect to this $\mathcal{O}_F$-module structure ι_F is a homomorphism of $\mathcal{O}_F$-modules.

In the same way we define a new $\mathcal{O}_F$-module structure on each of the fields F_l for $l \in \{0, \dots, m\}$. Explicitly, $a * \alpha := \varphi^{-(l+1)}(a)\alpha$ for $a \in \mathcal{O}_F$ and $\alpha \in F_l$.

We fix $l \in \{0, \dots, m\}$ and let $\alpha_1, \dots, \alpha_s$ with $s := [F_l : F]$ denote an $\mathcal{O}_F$-basis of $\mathcal{O}_{F_l}$ with respect to the usual $\mathcal{O}_F$-module structure given by multiplication. Then, for $\beta \in \mathcal{O}_{F_l}$ and $i \in \{1, \dots, s\}$, there exist elements $a_{ij} \in \mathcal{O}_F$ such that

$$\beta\alpha_i = \sum_{j=1}^{s} a_{ij}\alpha_j = \sum_{j=1}^{s} \varphi^{l+1}(a_{ij}) * \alpha_j. \tag{3.6}$$

Multiplication by β is $\mathcal{O}_F$-linear with respect to both $\mathcal{O}_F$-module structures on $\mathcal{O}_{F_l}$ and we write $\mathcal{N}_{F_l/F}$, respectively $\mathcal{N}^*_{F_l/F}$, for the induced norm maps. Then (3.6) implies

$$\mathcal{N}_{F_l/F}(\beta)^{\varphi^{l+1}} = \det\left((a_{ij})_{i,j=1,\dots,s}\right)^{\varphi^{l+1}} = \mathcal{N}^*_{F_l/F}(\beta).$$

Hence part a) of the lemma follows from

$$\mathcal{N}_{R_F/\mathcal{O}_F}(\bar{g}) = g(0) \prod_{l=0}^{m} \left(\mathcal{N}^*_{F_l/F}((\varphi^{-(l+1)}g)(\omega_{l+1}))\right),$$

which in turn is immediate from $\mathcal{O}_F$-linearity of ι_F with respect to the modified $\mathcal{O}_F$-module structure.

In order to prove b) we fix an element $\tau \in \mathrm{Gal}(F_\infty/H)$ such that $\tau|_F = \varphi$. Then we obtain from a)

$$\mathcal{N}_{R_F/\mathcal{O}_F}(\bar{g}) = g(0) \prod_{l=0}^{m} \left(\mathcal{N}_{F_l/F}((\varphi^{-(l+1)}g)(\omega_{l+1})\right)^{\tau^{l+1}} = g(0) \prod_{l=0}^{m} \mathcal{N}_{F_l/F}(g(\omega_{l+1}^{\tau^{l+1}})).$$

Since $\mathrm{Gal}(F_l/F)$ acts simply transitive on $\widetilde{W_f^{l+1}}$ the result easily follows. □

3.4 Seiriki's theorem on the constant term of a Coleman power series

We let F_f be a Lubin-Tate formal group relative to the unramified extension H'/H and resume the notations of Section 3.2. Recall that $H'_n = H'(W_f^{n+1})$ for $n \geq 0$. We also set $H'_\infty := \bigcup_{n\geq 0} H'_n$. If χ is a character of finite order of $\mathrm{Gal}(H'_\infty/H')$ we set $H'_\chi := (H'_\infty)^{\ker(\chi)}$. For a norm-coherent sequence $u = (u_n)_{n\geq 0} \in \varprojlim_n \mathcal{O}^\times_{H'_n}$ we set $u_{H'} := \mathcal{N}_{H'_n/H'}(u_n)$ for any $n \geq 0$.

Theorem 3.4.1. *Let χ be a character of finite order of $\mathrm{Gal}(H'_\infty/H')$ and let $u = (u_n)_{n\geq 0}$ in $\varprojlim_n \mathcal{O}^\times_{H'_n}$ be a norm-coherent sequence with $u_{H'} = 1$. In addition, we assume $Col_u(0) \in \mathcal{O}^\times_H$. Then*

$$(u, \chi)_{H'_\infty/H'} = -\chi(\mathrm{rec}_H(Col_u(0))). \tag{3.7}$$

Remark 3.4.2. In the case $H = H'$ this is essentially Corollary 2.8 in [Sei17], but there is a minus missing in the statement of this corollary.

Remark 3.4.3. a) The right hand side does not depend on the choice $\omega = (\omega_n)_{n\geq 1}$ of a generator of the Tate module. Indeed, if $\omega' = (\omega'_n)_{n\geq 1}$ is another such generator, then there is a unique $\sigma \in \mathrm{Gal}(H'_\infty/H')$ such that $\sigma(\omega_n) = \omega'_n$ for all $n \geq 1$. By local class field theory there exists a unique $v \in \mathcal{O}_{H'}^\times$ with $\mathrm{rec}_{H'}(v) = \sigma$. Then $\omega'_n = \sigma(\omega_n) = [v^{-1}]_f^{\varphi^{-n}}(\omega_n)$ (since ω_n is a torsion point of $F_{\varphi^{-n}(f)}$).

If Col'_u denotes the Coleman power series with respect to ω', then $Col'_u = Col_u \circ [v]_f$ and thus $Col'_u(0) = Col_u(0)$.

b) Without loss of generality we may assume that the sequence $\omega = (\omega_n)_{n\geq 1}$ is norm-coherent. To show this we apply [Ble04, Lemma 4.1] and proceed as follows: We fix a norm-coherent sequence $\beta = (\beta_n)_{n\geq 1}$ of prime elements of H'_n and let $Col_\beta \in T\mathcal{O}'[[T]]$ be the associated Coleman power series. Let $Col_\beta^{-1} \in T\mathcal{O}'[[T]]$ be such that $Col_\beta \circ Col_\beta^{-1} = T$. If we set $f' := Col_\beta^\varphi \circ f \circ Col_\beta^{-1}$, then $f' \in \mathcal{F}_\xi$ and the proof of [Ble04, Lemma 4.1 b)] shows that β is a generator of the Tate module for $F_{f'}$. With respect to $F_{f'}$ and β the Coleman power series associated to u is equal to $Col_u \circ Col_\beta^{-1}$ and $(Col_u \circ Col_\beta^{-1})(0) = Col_u(0)$, so that we may prove the theorem for f replaced by f' and ω replaced by β.

Proof. We fix $m > 0$ and note that it suffices to prove the theorem for an arbitrary character χ of $\mathrm{Gal}(H'_m/H')$. We write $\mathrm{Gal}(H'_m/H')$ as a direct product of cyclic subgroups,

$$\mathrm{Gal}(H'_m/H') = G_1 \times \ldots \times G_s,$$

with $s \geq 1$ and for each $i \in \{1, \ldots, s\}$ we set $U_i := \prod_{j\neq i} G_j$ with the subscript j ranging over $\{1, \ldots, s\}$. For each j we fix a generator σ_j of G_j and define a character $\chi_j \in \widehat{G}_j$ by $\chi_j(\sigma_j) = \frac{1}{|G_j|} + \mathbb{Z}$. Then the characters $\chi_1, \ldots, \chi_s$ generate the group of characters of $\mathrm{Gal}(H'_m/H')$.

Claim 1: For $j = 1, \ldots, s$ there exist units $b_{u,\chi_j} \in \mathcal{O}_H^\times$ such that

$$(u_{\chi_j}, \chi_j)_{H'_{\chi_j}/H'} = \chi_j(\mathrm{rec}_H(b_{u,\chi_j}^{-1})) \text{ and } \mathrm{rec}_H(b_{u,\chi_j}) \in G_j.$$

For the proof of Claim 1 we fix $a_j \in \mathbb{Z}_{>0}$ such that $(u_{\chi_j}, \chi_j)_{H'_{\chi_j}/H'} = \frac{a_j}{|G_j|} + \mathbb{Z}$. Since rec_H induces an isomorphism $\left(\mathcal{O}_H/\mathfrak{p}_H^{m+1}\right)^\times \simeq \mathrm{Gal}(H'_m/H')$ there exists $b_{u,\chi_j} \in \mathcal{O}_H^\times$ such that $\mathrm{rec}_H(b_{u,\chi_j}^{-1}) = \sigma_j^{a_j}$. Claim 1 is now immediate from $\chi_j(\sigma_j) = \frac{1}{|G_j|} + \mathbb{Z}$.

By Remark 3.4.3 we may, without loss of generality, assume that the generator $\omega = (\omega_n)_{n\geq 1}$ of the Tate module is norm coherent. We set $b_u := \prod_{j=1}^s b_{u,\chi_j} \in \mathcal{O}_H^\times$ and define

$$u'_n := \frac{\mathrm{rec}_H(b_u^{-1})(\omega_{n+1})}{\omega_{n+1}}.$$

Then $u' := (u'_n)_{n\geq 0}$ is a norm coherent sequence of units in $\varprojlim_n \mathcal{O}_{H'_n}^\times$ and it is clear that $\mathcal{N}_{H'_n/H'}(u'_n) = 1$.

Claim 2: $Col_{u'}(T) = \frac{[b_u]_f(T)}{T}$, and thus $Col_{u'}(0) = b_u$.

For the proof of Claim 2 we recall that ω_{n+1} is a torsion point of $F_{\varphi^{-(n+1)}(f)} = F_f^{\varphi^{-(n+1)}}$. We have $\mathrm{rec}_H(b_u^{-1})(\omega_{n+1}) = [b_u]_{\varphi^{-(n+1)}(f)}(\omega_{n+1}) = [b_u]_f^{\varphi^{-(n+1)}}(\omega_{n+1})$. Put $g := \frac{[b_u]_f(T)}{T}$ and observe that

$$(\varphi^{-(n+1)}(g))(\omega_{n+1}) = \frac{[b_u]_f^{\varphi^{-(n+1)}}(\omega_{n+1})}{\omega_{n+1}} = \frac{\mathrm{rec}_H(b_u^{-1})(\omega_{n+1})}{\omega_{n+1}} = u'_n,$$

so that $g = \frac{[b_u]_f(T)}{T}$ satisfies the defining equality (3.4) for all $n \geq 0$.

We now set $u'' := u/u'$ and obtain from Claim 2 that $Col_{u''}(0) \cdot b_u = Col_u(0)$. In particular, since we assume that $Col_u(0)$ is contained in $\mathcal{O}_H^\times$, it follows that $Col_{u''}(0) \in \mathcal{O}_H^\times$.

The right hand side of (3.7) is obviously multiplicative in the character, for the left hand side this is shown in Proposition 3.1.3. Thus, it suffices to prove the theorem for each of the characters χ_i, $i = 1, \ldots, s$.

We henceforth fix $i \in \{1, \ldots, s\}$. We observe that by Claim 1 $\chi_i(\mathrm{rec}_H(b_u)) = \chi_i(\mathrm{rec}_H(b_{u,\chi_i}))$ and $(u, \chi_i)_{H'_\infty/H'} = -\chi_i(\mathrm{rec}_H(b_{u,\chi_i}))$. It is now easy to see that for $\chi := \chi_i$ the equality (3.7) is equivalent to $\chi(\mathrm{rec}_H(Col_{u''}(0))) = 0$. Since χ_i is a character of $\mathrm{Gal}(H'_m/H')$ and rec_H induces an isomorphism $\left(\mathcal{O}_H/\mathfrak{p}_H^{m+1}\right)^\times \simeq \mathrm{Gal}(H'_m/H')$ it thus suffices to show that

$$Col_{u''}(0) \equiv 1 \pmod{\mathfrak{p}_H^{m+1}}. \tag{3.8}$$

Claim 3: $(u''_m, \psi)_{H'_m/H'} = 0$ for all characters ψ of $\mathrm{Gal}(H'_m/H')$.

Because of the multiplicativity result of Proposition 3.1.3 it suffices to show that for $j = 1, \ldots, s$ we have $(u''_m, \chi_j)_{H'_m/H'} = 0$. We fix j and compute

$$(u''_m, \chi_j)_{H'_m/H'} = (u_m, \chi_j)_{H'_m/H'} - (u'_m, \chi_j)_{H'_m/H'} = \frac{a_j}{|G_j|} + \mathbb{Z} - (u'_m, \chi_j)_{H'_m/H'}$$

with $a_j \in \mathbb{Z}_{>0}$ as in the proof of Claim 1. Hence it suffices to show that the equality $(u'_m, \chi_j)_{H'_m/H'} = \frac{a_j}{|G_j|} + \mathbb{Z}$ is valid.

We set $\eta_{m+1} := \mathcal{N}_{H'_m/H'_{\chi_j}}(\omega_{m+1})$ and note that $H'_{\chi_j} = (H'_m)^{U_j}$. By Claim 1 and its proof this implies

$$\begin{aligned}
\mathcal{N}_{H'_m/H'_{\chi_j}}(u'_m) &= \frac{\mathrm{rec}_H(b_u^{-1})(\eta_{m+1})}{\eta_{m+1}} = \frac{\left(\prod_{l=1}^s \mathrm{rec}_H(b_{u,\chi_l}^{-1})\right)(\eta_{m+1})}{\eta_{m+1}} \\
&= \frac{\mathrm{rec}_H(b_{u,\chi_j}^{-1})(\eta_{m+1})}{\eta_{m+1}} = \frac{\sigma_j^{a_j}(\eta_{m+1})}{\eta_{m+1}}.
\end{aligned}$$

By definition of the pairing and Lemma 3.1.2 we conclude further

$$\begin{aligned}
\left(u'_m, \chi_j\right)_{H'_m/H'} &= \left(\frac{\sigma_j^{a_j}(\eta_{m+1})}{\eta_{m+1}}, \chi_j\right)_{H'_{\chi_j}/H'} \\
&= v_{H'_{\chi_j}}(\eta_{m+1})\chi_j(\sigma_j^{a_j}) + \mathbb{Z} = \chi_j(\sigma_j^{a_j}) + \mathbb{Z} = \frac{a_j}{|G_j|} + \mathbb{Z},
\end{aligned}$$

as required.

The following two claims now conclude the proof of (3.8), and hence also the proof of Theorem 3.4.1. We will use the notation and results of Section 3.3.

Claim 4: There exists a finite unramified extension F/H' and an element $\widetilde{u''_m} \in R_F^\times$ such that $r_F(\widetilde{u''_m}) = u''_m$ and $\mathcal{N}_{R_F/\mathcal{O}_F}(\widetilde{u''_m}) = 1$.

Claim 5: For any $\bar{x} \in R_F^\times$ with $r_F(\bar{x}) = 1$ one has $\mathcal{N}_{R_F/\mathcal{O}_F}(\bar{x}) \equiv 1 \pmod{\mathfrak{p}_F^{m+1}}$.

Indeed, by Lemma 3.3.2, the defining equality (3.4) for Coleman power series and the fact that $\mathcal{N}_{F_l/F}(u''_l) = \mathcal{N}_{H'_l/H'}(u''_l) = 1$ for all l we have

$$\begin{aligned}
\mathcal{N}_{R_F/\mathcal{O}_F}(\overline{Col_{u''}}) &= Col_{u''}(0) \prod_{l=0}^{m} \left(\mathcal{N}_{F_l/F}\left((\varphi^{-(l+1)}Col_{u''})(\omega_{l+1})\right)\right)^{\varphi^{l+1}} \\
&= Col_{u''}(0) \prod_{l=0}^{m} \left(\mathcal{N}_{F_l/F}(u''_l)\right)^{\varphi^{l+1}} \\
&= Col_{u''}(0).
\end{aligned}$$

Since $r_F(\overline{Col_{u''}}) = u''_m = r_F(\widetilde{u''_m})$ we thus conclude from Claims 4 and 5 that

$$Col_{u''}(0) \equiv 1 \pmod{\mathfrak{p}_F^{m+1}}.$$

Moreover, with $Col_{u''}(0) \in \mathcal{O}_H$ and $\mathfrak{p}_F^{m+1} \cap \mathcal{O}_H = \mathfrak{p}_H^{m+1}$ (since F/H is unramified) the equality of (3.8) follows.

For the proof of Claim 4 we first note that by Claim 3 all assumptions of Proposition 3.1.5 for u''_m and H'_m/H' are satisfied. Hence we conclude that there exists a finite unramified extension F/H', an integer $r > 0$, units $\beta_1, \ldots, \beta_r \in \mathcal{O}_{F_m}^\times$ and automorphisms $\sigma_1, \ldots, \sigma_r \in \mathrm{Gal}(F_m/F)$ such that

$$u''_m = \prod_{j=1}^{r} \frac{\sigma_j(\beta_j)}{\beta_j}.$$

Since we know from Section 3.3 that $r_F\colon R_F^\times \longrightarrow \mathcal{O}_F^\times$ is surjective, we can choose elements $\widetilde{\beta_j} \in R_F^\times$ such that $r_F(\widetilde{\beta_j}) = \beta_j$. Recall also from Lemma 3.3.1 that we have a natural action of $\mathrm{Gal}(F_m/F)$ on R_F and set

$$\widetilde{u''_m} := \prod_{j=1}^{r} \frac{\sigma_j(\widetilde{\beta_j})}{\widetilde{\beta_j}} \in R_F^\times.$$

So $\widetilde{u''_m}$ is a unit in R_F which by construction and Lemma 3.3.2 a) satisfies $\mathcal{N}_{R_F/\mathcal{O}_F}(\widetilde{u''_m}) = 1$.

It finally remains to prove Claim 5. If $r_F(\bar{x}) = 1$ for a power series $x \in \mathcal{O}_F[[T]]$, then it is straightforward to see that $x(\omega) = 1$ for all torsion points $\omega \in \widetilde{W_f^{m+1}}$.

We set $h := \frac{f^{(m+1)}}{f^{(m)}} = \frac{(\varphi^m f)(f^{(m)})}{f^{(m)}}$. Then $h = \tilde{\pi} + h_1(f^{(m)})$ with a power series $h_1 \in T\mathcal{O}_{H'}[[T]]$ and $\tilde{\pi} := \varphi^m(\pi')$, a uniformizing element in H'. The set of zeroes of h is given by $\widetilde{W_f^{m+1}}$, so that a straightforward application of the Weierstrass preparation theorem shows that h divides $x - 1$. We write $x = 1 + hg$ with a power series $g \in \mathcal{O}_F[[T]]$.

By part a) of Lemma 3.3.2 we obtain

$$\begin{aligned} \mathcal{N}_{R_F/\mathcal{O}_F}(\bar{x}) &= x(0) \prod_{l=0}^{m} \mathcal{N}_{F_l/F}\left((\varphi^{-(l+1)}x)(\omega_{l+1})\right)^{\varphi^{l+1}} \\ &= x(0) \prod_{l=0}^{m-1} \mathcal{N}_{F_l/F}\left((\varphi^{-(l+1)}x)(\omega_{l+1})\right)^{\varphi^{l+1}} \end{aligned}$$

where the second equality holds because of $(\varphi^{-(m+1)}x)(\omega_{m+1}) = r_F(\bar{x}) = 1$. As in the proof of part b) of Lemma 3.3.2 we derive

$$\mathcal{N}_{R_F/\mathcal{O}_F}(\bar{x}) = \prod_{\omega \in W_f^m} x(\omega) = \prod_{\omega \in W_f^m} (1 + g(\omega)h(\omega)).$$

Since $h = \tilde{\pi} + h_1(f^{(m)})$ and $f^{(m)}(\omega) = 0$ for all $\omega \in W_f^m$ we further deduce

$$\mathcal{N}_{R_F/\mathcal{O}_F}(\bar{x}) = \prod_{\omega \in W_f^m} (1 + g(\omega)\tilde{\pi}).$$

Set $j(T) := 1 + g(T)\tilde{\pi} \in \mathcal{O}_F[[T]]$. We note that $f \in \mathcal{F}_{\xi_F}$ with $\xi_F := \xi^{[F:H']}$ and with respect to the unramified extension F/H, so that the formal group F_f can also be considered as a Lubin-Tate extension relative to F/H. By Proposition 3.2.1 c) we therefore obtain

$$\left((\mathcal{N}_f^{(m)} j) \circ f^{(m)}\right)(T) = \prod_{\omega \in W_f^m} j(T +_f \omega).$$

As a consequence

$$\mathcal{N}_f^{(m)}(j)(0) = \prod_{\omega \in W_f^m} j(\omega) = \mathcal{N}_{R_F/\mathcal{O}_F}(\bar{x}).$$

Moreover, we have $j \equiv 1 \bmod \mathfrak{p}_F$. Applying Proposition 3.2.1 d) inductively we obtain $\mathcal{N}_f^{(m)}(j) \equiv 1 \bmod \mathfrak{p}_F^{m+1}$ and hence $\mathcal{N}_{R_F/\mathcal{O}_F}(\bar{x}) \equiv 1 \bmod \mathfrak{p}_F^{m+1}$. □

For the proofs of Theorems 4.1.14, 4.1.15 and 4.1.16 we will need a variant of Theorem 3.4.1. Let $u = (u_n)_{n\geq 0} \in \varprojlim_n \mathcal{O}_{H'_n}^\times$ be a norm-coherent sequence. The proof of [Ble04, Lemma 4.2] shows that $\mathcal{N}_{H'_n/H}(u_n) = 1$ for all $n \geq 0$. We fix a set of representatives $\{\tau_1, \ldots, \tau_d\}$ of $\mathrm{Gal}(H'_\infty/H)$ modulo $\mathrm{Gal}(H'_\infty/H')$ and define for all $n \geq 0$

$$w_n := \prod_{i=1}^d \tau_i(u_n). \tag{3.9}$$

Note that w_n depends on the choice of the set $\{\tau_1, \ldots, \tau_d\}$, however, we will suppress this dependency in our notation.

Lemma 3.4.4. *For the elements w_n constructed above, we obtain*

a) $\mathcal{N}_{H'_m/H'_n}(w_m) = w_n$ for $m \geq n \geq 0$.

b) $\mathcal{N}_{H'_n/H'}(w_n) = 1$ for all $n \geq 0$.

c) $Col_w(0) = \mathcal{N}_{H'/H}(Col_u(0))$

Proof. The proofs of a) and b) are immediate from the definitions. Since

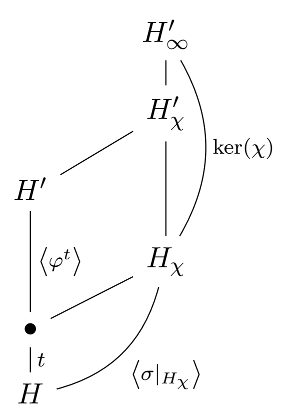
$$\varphi^{-(j+1)}\left(\prod_{i=1}^d Col_{\tau_i(u)}\right)(\omega_{j+1}) = \prod_{i=1}^d \tau_i(u_j) = w_j$$

for all $j \geq 0$ we know by (3.4) that $\prod_{i=1}^d Col_{\tau_i(u)} = Col_w$. By Lemma 3.2.3 we have $Col_{\tau_i(u)} = (Col_u^{\tau_i}) \circ [\kappa(\tau_i)]_{f,\tau_i(f)}$, and as a consequence $Col_{\tau_i(u)}(0) = \tau_i(Col_u(0))$ and so the result of c) obviously follows. □

Corollary 3.4.5. *Let $u = (u_n)_{n\geq 0} \in \varprojlim_n \mathcal{O}_{H'_n}^\times$ be a norm-coherent sequence of units. Then we have for each character χ of $\mathrm{Gal}(H'_\infty/H)$ of finite order*

$$(u, \chi)_{H'_\infty/H} = -\chi(\mathrm{rec}_H(\mathcal{N}_{H'/H}(Col_u(0)))).$$

Proof. We set $H_\chi := (H'_\infty)^{\ker(\chi)}$ and $H'_\chi := (H'_\infty)^{\ker(\chi)\cap \mathrm{Gal}(H'_\infty/H')}$. This is summarized in the following diagram of fields.



Choose an element $\sigma \in \mathrm{Gal}(H'_\infty/H)$ such that $\chi(\sigma) = \frac{1}{[H_\chi:H]} + \mathbb{Z}$. We set $t := [H_\chi \cap H' : H]$ and fix $\tau \in \mathrm{Gal}(H'_\infty/H)$ such that $\tau|_{H_\chi} = 1$ and $\tau|_{H'} = \varphi^t$. Recall that $d = [H' : H]$. It is easy to see that the set $\{\sigma^i\tau^j | 0 \leq i < t, 0 \leq j < d/t\}$ constitutes a set of representatives of $\mathrm{Gal}(H'_\infty/H)$ modulo $\mathrm{Gal}(H'_\infty/H')$. For all $n \geq 0$ we define w_n as in (3.9) with respect to this set of representatives.

Then $\chi(\sigma^t) = \frac{1}{[H'_\chi:H']} + \mathbb{Z}$. Let n be large enough so that $H'_\chi \subseteq H'_n$. By Lemma 3.4.4 b) and Hilbert's Theorem 90 there exists an element $\beta \in (H'_\chi)^\times$ such that $\beta^{\sigma^t - 1} = \mathcal{N}_{H'_n/H'_\chi}(w_n)$. By the definition of the pairing we derive

$$(w, \chi)_{H'_\infty/H'} = v_{H'}(\beta) + \mathbb{Z}. \tag{3.10}$$

If $\tilde{\beta} \in H_\chi^\times$ is such that $\tilde{\beta}^{\sigma-1} = \mathcal{N}_{H'_n/H_\chi}(u_n)$, then

$$(u, \chi)_{H'_\infty/H} = v_H(\tilde{\beta}) + \mathbb{Z}. \tag{3.11}$$

Now we compute

$$\begin{aligned} \tilde{\beta}^{\sigma^t - 1} &= \tilde{\beta}^{(\sigma-1)(1+\sigma+\ldots+\sigma^{t-1})} \\ &= \mathcal{N}_{H'_n/H_\chi}(u_n)^{(1+\sigma+\ldots+\sigma^{t-1})} \\ &= \left(\prod_{j=0}^{d/t-1} \tau^j \left(\mathcal{N}_{H'_n/H'_\chi}(u_n) \right) \right)^{(1+\sigma+\ldots+\sigma^{t-1})} \\ &= \mathcal{N}_{H'_n/H'_\chi} \left(\prod_{j=0}^{d/t-1} \prod_{i=0}^{t-1} \sigma^i \tau^j (u_n) \right) \\ &= \mathcal{N}_{H'_n/H'_\chi}(w_n). \end{aligned}$$

It follows that $\beta/\tilde{\beta} \in (H')^\times$ and hence $v_{H'}(\beta) \equiv v_{H'}(\tilde{\beta}) \pmod{\mathbb{Z}}$. Since H'/H is unramified we obtain furthermore $v_{H'}(\tilde{\beta}) = v_H(\tilde{\beta})$, which combined with (3.10) and (3.11) leads to $(w, \chi)_{H'_\infty/H'} = (u, \chi)_{H'_\infty/H}$. The result is now immediate from Theorem 3.4.1 applied for the norm-coherent sequence w together with Lemma 3.4.4, part c). □

Chapter 4

Construction of $\mathfrak{p}$-units and computation of their valuation

The objective of this chapter is to reprove the main results of Solomon [Sol92, Thm. 2.1] and [Ble04, Thm. 3.4] with the tools developed in Chapter 3 and to present a proof of the main theorem of [BH18], which is one of the main results of this thesis. The latter result was obtained in joint work with my advisor Werner Bley and so parts of this chapter are based on our article [BH18].

In a little more detail this chapter contains the following: In Section 4.1 we construct cyclotomic resp. elliptic $\mathfrak{p}$-units with the help of Hilbert's Theorem 90 and formulate theorems which describe the valuation above $\mathfrak{p}$ in terms of the p-adic logarithm of cyclotomic resp. elliptic units. In Section 4.2 and Section 4.3 we reprove the main theorems of [Sol92] and [Ble04] as well as Theorem 4.1.16 with the help of Corollary 3.4.5 from Chapter 3.

4.1 Formulation of the main theorems

Let k be $\mathbb{Q}$ or an imaginary quadratic field, where the first case is called the *cyclotomic case* and the second case is called the *elliptic case*.

Let L denote a finite abelian extension of k. We fix a prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$ above a rational prime p in the elliptic case and set $\mathfrak{p} = (p)$ in the cyclotomic case. We write H for the completion $k_{\mathfrak{p}}$ of k at $\mathfrak{p}$.

The elliptic case is subdivided in two sub-cases: we call the case, where the rational prime p splits in k the *split case* and the case where p is inert or ramifies in k is called the *non-split case*. The main hypotheses we are going to assume in the theorems of this chapter are as follows:

Hypotheses.

(H1) $\mathfrak{p}$ splits completely in L.

(H2) p does not divide the class number of k.

Remark 4.1.1. Hypothesis (H1) is crucial when constructing the '$\mathfrak{p}$-units'. But in the main application of Solomon [Sol92, Thm. 2.1] resp. of [Ble04, Thm. 3.4], namely the descent computation in the proof of the eTNC in the cyclotomic case (see [BG03] and [Fla04]) resp. the elliptic split case (see [Ble06]), it turns out that they constitute no obstruction to proving an unconditional result because in these cases the assertions are used in a situation where (H1) is naturally satisfied. We will see in Chapters 5 and 6 that the same is true for the elliptic non-split case.

It is obvious that in the cyclotomic case the condition (H2) is automatically fulfilled. In the elliptic case condition (H2) is actually needed, but we are convinced that it should be possible to weaken this restriction or to avoid it entirely.

Setting in the cyclotomic case Let s be the smallest integer such that $s > 1/(p-1)$. So for $p \geq 3$ we have $s = 1$ and for $p = 2$ we have $s = 2$. This condition comes from the convergence of the p-adic logarithm (for details see for example [Coh00, Ch. 4.3]).

As condition (H2) is always fulfilled in the cyclotomic case the only hypothesis in this case is (H1).

Remark 4.1.2. Let $m \in \mathbb{N}$ and $\mathfrak{m} = (m)$. By abuse of notation we denote in the cyclotomic case the field $\mathbb{Q}(\mu_m)$ by $\mathbb{Q}(\mathfrak{m})$ in order to be able to present the following more smoothly and then it is trivial to translate the notation of [Sol92] to ours. This notation can be justified by the conventions used in [Neu99]. In our usual notation one would have $\mathbb{Q}(\mathfrak{m}\infty) = \mathbb{Q}(\mu_m)$ (cf. Proposition 6.7 in [Neu99, Ch. VI] and Satz 7.10 in [Neu11]).

In the elliptic case we have to be slightly more restrictive.

Setting in the elliptic case Here the integer s is defined to be the smallest integer such that $s > e/(p-1)$ with e denoting the ramification index of p in $k/\mathbb{Q}$. So for $p > 3$ we have $s = 1$ and here is an overview for $p = 2$ or 3:

p	split	inert	ramified
2	$s = 2$	$s = 2$	$s = 3$
3	$s = 1$	$s = 1$	$s = 2$

For primes $p > 3$ we will have no further assumptions on $\mathfrak{p}$ and k besides hypotheses (H1) and (H2). However, for $p = 2$ or $p = 3$ we need to impose the following conditions:

- If $p = 2$ we assume that either a2) or b2) holds:
  - a2) p is split in k.
  - b2) p is ramified in k and $H = \mathbb{Q}_2(\zeta_4)$, where ζ_4 is a primitive fourth root of unity in $\mathbb{Q}_2^c$.
- If $p = 3$ we assume that either a3), b3) or c3) holds:
  - a3) p is split in k.
  - b3) p is inert in k.
  - c3) p is ramified in k and $H = \mathbb{Q}_3(\zeta_3)$, where ζ_3 is a primitive third root of unity in $\mathbb{Q}_3^c$.

Remark 4.1.3. One can see that for $p = 2$ in the non-split case the conditions are quite restrictive. The same is true for $p = 3$ when p ramifies.

Now we continue to treat all cases at once. Let $\mathfrak{f}_L$ be the conductor of L and fix an integral ideal $\mathfrak{f}$ of $\mathcal{O}_k$ such that $\mathfrak{f}_L \mid \mathfrak{f}$, $\mathfrak{p} \nmid \mathfrak{f}$ and $w(\mathfrak{f}) = 1$. We set

$$F := k(\mathfrak{f}), \quad k(\mathfrak{p}^\infty) := \bigcup_{n\geq 0} k(\mathfrak{p}^n) \quad \text{and} \quad K_\infty := \bigcup_{n\geq 0} k(\mathfrak{f}\mathfrak{p}^{s+n}).$$

We write T for the torsion subgroup of $\mathrm{Gal}(k(\mathfrak{p}^\infty)/k)$ and let $k_\infty := k(\mathfrak{p}^\infty)^T$ be the fixed field of T.

Then k_∞/k is a $\mathbb{Z}_p^d$-extension with $d = 1$ in the cyclotomic case and in the split case and $d = 2$ in the non-split case. By (H1) $\mathfrak{p}$ is unramified in F, and thus (H2) implies $F \cap k_\infty = k$.

We now investigate the extension K_∞/F. We set $F_\infty := Fk_\infty$ and $L_\infty := Lk_\infty$. Since $\mathrm{Gal}(K_\infty/F_\infty)$ is finite and $\mathrm{Gal}(F_\infty/F)$ is torsion-free we see that $\mathrm{Gal}(K_\infty/F_\infty)$ is the torsion subgroup of $\mathrm{Gal}(K_\infty/F)$. By class field theory we obtain

$$\mathrm{Gal}(K_\infty/F) \cong \varprojlim_n (\mathcal{O}_k/\mathfrak{p}^n)^\times \simeq \mathcal{O}_H^\times = \mu'_H \times (1 + \mathfrak{p}_H)$$

with μ'_H denoting the roots of unity of order $\mathcal{N}(\mathfrak{p}) - 1$ in H. With our definition of s the p-adic exponential $\exp_p$ converges on $\mathfrak{p}_H^s$, so that $1 + \mathfrak{p}_H^s$ is torsion-free.

Lemma 4.1.4. *With s as above we have $K_0 = k(\mathfrak{f}\mathfrak{p}^s)$. Then $K_0 \cap F_\infty = F$ and $K_0F_\infty = K_\infty$.*

Proof. In each case one can show that

$$|(1 + \mathfrak{p}_H)_{tors}| = \left| \frac{1 + \mathfrak{p}_H}{1 + \mathfrak{p}_H^s} \right|,$$

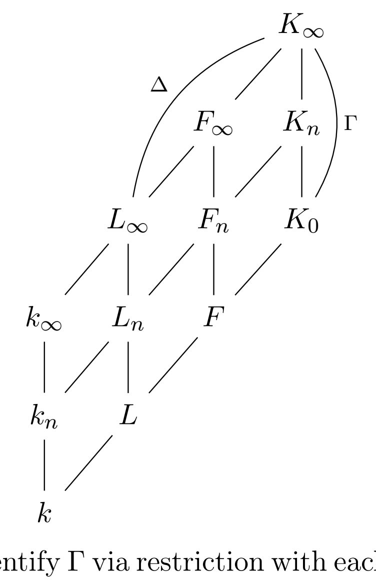
where we write $(1 + \mathfrak{p}_H)_{tors}$ for the subgroup of torsion elements of $1 + \mathfrak{p}_H$. For $s = 1$ this is trivial. In the cyclotomic case for $p = 2$ recall that we have $|(1 + \mathfrak{p}_H)_{tors}| = 2$ and that μ'_H is trivial.
For the special cases in the elliptic setting, we have, for example, if $p = 2$ is ramified and $H = \mathbb{Q}_2(\zeta_4)$, then $|(1 + \mathfrak{p}_H)/(1 + \mathfrak{p}_H^s)| = 4$ and $(1 + \mathfrak{p}_H)_{tors} = \langle \zeta_4 \rangle$. The other special cases can be treated similarly.

It is then easily shown that $(1 + \mathfrak{p}_H^s) \times (1 + \mathfrak{p}_H)_{tors} = 1 + \mathfrak{p}_H$. In summary, we have a direct product decomposition

$$\mathrm{Gal}(K_\infty/F) \cong \mu'_H \times (1 + \mathfrak{p}_H)_{tors} \times (1 + \mathfrak{p}_H^s)$$

and the lemma follows because the fixed field of $1 + \mathfrak{p}_H^s$ (resp. $\mu'_H \times (1 + \mathfrak{p}_H)_{tors}$) is $k(\mathfrak{f}\mathfrak{p}^s) = K_0$ (resp. F_∞). □

In each case we thus obtain the following diagram of fields (with K_n, F_n, L_n and k_n defined below)



We set $\Gamma := \mathrm{Gal}(K_\infty/K_0)$ and identify Γ via restriction with each of the Galois groups $\mathrm{Gal}(F_\infty/F)$, $\mathrm{Gal}(L_\infty/L)$ and $\mathrm{Gal}(k_\infty/k)$. We let K_n, F_n, L_n and k_n denote the fixed field of Γ^{p^n} of $K_\infty, F_\infty, L_\infty$ and k_∞, respectively.

Units in the cyclotomic case Let $m \in \mathbb{N}$ and let μ_m be the group of m-th roots of unity, for each m we fix a primitive root of unity ζ_m of μ_m satisfying

$$\zeta_{dm}^d = \zeta_m \text{ for all } m, d \in \mathbb{N}.$$

Let $m \in \mathbb{N}$, p be a rational prime and set $\mathfrak{m} = (m)$ as well as $\mathfrak{p} = (p)$. Recall that we have with [Sol92, Lemma 2.1]

$$\mathcal{N}_{k(\mathfrak{m}\mathfrak{p})/k(\mathfrak{m})}(1 - \zeta_{mp}) = \begin{cases} 1 - \zeta_m & \text{if } \mathfrak{p} \mid \mathfrak{m}, \\ (1 - \zeta_m)^{1-\sigma_{\mathfrak{p}}^{-1}} & \text{if } \mathfrak{p} \nmid \mathfrak{m}. \end{cases} \tag{4.1}$$

It is easy to show that [Sol92, Cor. 2.1] if m is not a prime power we have

$$1 - \zeta_m \in \mathcal{O}_{k(\mathfrak{m})}^{\times}.$$

Recall that f is the conductor of L and so we have $k \subseteq L \subseteq k(f) \subseteq K_0$ and we set

$$\varepsilon_n^{cyc} := \mathcal{N}_{k(\mathfrak{f}\mathfrak{p}^{n+s})/L_n}(1 - \zeta_{fp^{n+s}}) \in L_n^{\times},$$

where it is clear from the above that ε_n^{cyc} is a unit.

Since Γ has $\mathbb{Z}_p$-rank one, we can choose a topological generator γ of Γ such that it corresponds to a $c \in 1 + p^s\mathbb{Z}_p \setminus 1 + p^{s+1}\mathbb{Z}_p$ via $1 + p^s\mathbb{Z}_p \to \Gamma, a \mapsto (\gamma_a : \zeta \mapsto \zeta^a$ for all $\zeta \in \mu_{p^\infty})$ inducing an isomorphism $\mathbb{Z}_p \to \Gamma, a \mapsto \gamma^a$. By abuse of notation we also write γ for each of the restrictions of γ to K_n, F_n, L_n or k_n.

Units in the elliptic case Recall that $\mathfrak{f}$ is an ideal coprime to $\mathfrak{p}$. We fix an auxiliary integral ideal $\mathfrak{a}$ of $\mathcal{O}_k$ with $(\mathfrak{a}, 6\mathfrak{f}\mathfrak{p}) = 1$. The main difference to the cyclotomic case is, that we have to exchange the cyclotomic elements with elliptic elements from Chapter 2.

Units in the split case So by using Definition 2.1.24 we can set

$$\varepsilon_n^{split} := \mathcal{N}_{K_n/L_n}(\psi(1; \mathfrak{f}\mathfrak{p}^{s+n}, \mathfrak{a})) \in L_n^{\times}.$$

Since p splits in k, we have that $1+\mathfrak{p}^s\mathcal{O}_H \cong 1+p^s\mathbb{Z}_p$ and we can identify $\Gamma := \mathrm{Gal}(K_\infty/K_0) \cong \mathbb{Z}_p$ with $1 + p^s\mathbb{Z}_p$ via $a \mapsto \gamma_a$, where γ_a is uniquely determined by $(\gamma_a)_{|K_n} = \sigma(a_n)$ with $a_n \in \mathcal{O}_k$ such that

$$a_n \equiv 1 \bmod \mathfrak{f},$$
$$a_n \equiv a \bmod \mathfrak{p}^s.$$

So as in [Ble04, Sec. 3] after fixing a $c \in 1 + p^s\mathbb{Z}_p \setminus 1 + p^{s+1}\mathbb{Z}_p$ we get a topological generator $\gamma := \gamma_c$ of Γ.

But here we want to choose a topological generator γ first, which then corresponds to a c as described above. By abuse of notation we also write γ for each of the restrictions of γ to K_n, F_n, L_n or k_n.

Units in the non-split case We fix topological generators γ_1 and γ_2 of Γ and by abuse of notation we also write γ_i, $i = 1, 2$, for each of the restrictions of γ_i to K_n, F_n, L_n or k_n. We write e for the ramification index of p in $k/\mathbb{Q}$. It is not difficult to see that $K_n = k(\mathfrak{f}\mathfrak{p}^{s+en})$ for all $n \geq 0$.

Definition 4.1.5. Let $i \in \{1,2\}$ and set $\Delta := \mathrm{Gal}(K_\infty/L_\infty) \simeq \mathrm{Gal}(K_0/L_0)$. We define

$$K_{i,n} := K_n^{\langle\gamma_i\rangle} \quad L_{i,n} := K_{i,n}^{\Delta}$$

and set

$$\varepsilon_{i,n}^{ns} := \begin{cases} \mathcal{N}_{K_n/L_{i,n}}(\psi(1;\mathfrak{f}\mathfrak{p}^{s+n},\mathfrak{a})), & \text{if } p \text{ is inert in } k/\mathbb{Q}, \\ \mathcal{N}_{K_n/L_{i,n}}(\psi(1;\mathfrak{f}\mathfrak{p}^{s+2n},\mathfrak{a})), & \text{if } p \text{ is ramified in } k/\mathbb{Q}. \end{cases}$$

Remark 4.1.6. The groups $\mathrm{Gal}(K_{i,n}/K_0) \cong \mathrm{Gal}(L_{i,n}/L)$ are cyclic groups of order p^n generated by the image of γ_j where $j \in \{1,2\}, j \neq i$.

So in order to get an overview of the elliptic non-split situation, we can look at the following diagram of fields.

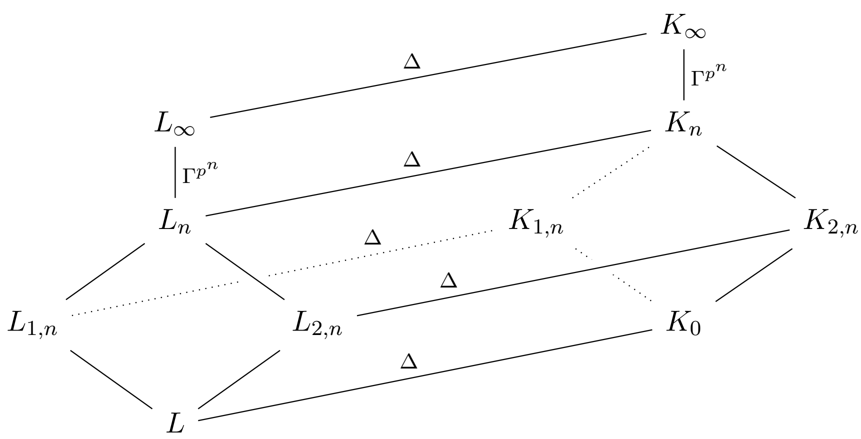


Figure 4.1: Situation elliptic non-split case

Now we will treat all three cases at once since the arguments used for proving the results go through without changes. The reader has to keep in mind that by abuse of notation the number fields and Galois groups are different in each case although they are abbreviated with the same letters.

By the norm relations in (4.1) and in Theorem 2.1.27 together with condition (H1) (because it implies $\sigma_{\mathfrak{p}}|_L = 1$) we obtain the following lemma

Lemma 4.1.7. *cf. [Sol92, Lemma 2.2] For $m, n \in \mathbb{N}_0$ with $m \geq n \geq 0$:*

a)

$$\mathcal{N}_{L_m/L_n}(\varepsilon_m^{cyc}) = \varepsilon_n^{cyc} \text{ and } \mathcal{N}_{L_m/L}(\varepsilon^{cyc}) = 1$$

b)

$$\mathcal{N}_{L_m/L_n}(\varepsilon_m^{split}) = \varepsilon_n^{split} \text{ and } \mathcal{N}_{L_m/L}(\varepsilon^{split}) = 1$$

c) For $i \in \{1,2\}$

$$\mathcal{N}_{L_{i,m}/L_{i,n}}(\varepsilon_{i,m}^{ns}) = \varepsilon_{i,m}^{ns} \quad \text{and} \quad \mathcal{N}_{L_{i,n}/L}(\varepsilon_{i,n}^{ns}) = 1,$$

According to Hilbert's Theorem 90 there exist(s)...

a) ... a unique element $(\beta^{cyc})_n \in L_n^\times/L^\times$ such that

$$(\beta^{cyc})_n^{\gamma-1} = \varepsilon_n^{cyc}.$$

b) ... a unique element $(\beta^{split})_n \in L_n^\times/L^\times$ such that

$$(\beta^{split})_n^{\gamma-1} = \varepsilon_n^{split}.$$

c) ... for $i,j \in \{1,2\}$ with $i \neq j$ unique elements $(\beta^{ns})_{i,n} \in (L_{i,n})^\times/L^\times$ such that

$$(\beta^{ns})_{i,n}^{\gamma_j-1} = \varepsilon_{i,n}^{ns}.$$

Remark 4.1.8. We have to keep in mind, that all the β's constructed above depend on $\mathfrak{f}$, on the choice of generators of Γ and in the elliptic cases also on the choice of the auxiliary ideal $\mathfrak{a}$.

Definition 4.1.9. We define

a)

$$\kappa_n^{cyc} = \kappa_n^{cyc}(L,\gamma,\mathfrak{f}) := \mathcal{N}_{L_n/L}((\beta^{cyc})_n) \in L^\times/\left(L^\times\right)^{p^n}.$$

b)

$$\kappa_n^{split} = \kappa_n^{split}(L,\gamma,\mathfrak{f},\mathfrak{a}) := \mathcal{N}_{L_n/L}((\beta^{split})_n) \in L^\times/\left(L^\times\right)^{p^n}.$$

c) For $i \in \{1,2\}$

$$\kappa_{i,n}^{ns} = \kappa_{i,n}^{ns}(L,\gamma_1,\gamma_2,\mathfrak{f},\mathfrak{a}) := \mathcal{N}_{L_{i,n}/L}((\beta^{ns})_{i,n}) \in L^\times/\left(L^\times\right)^{p^n}.$$

For a prime $\mathfrak{Q}$ of a number field N we write $v_{\mathfrak{Q}}\colon N^\times \longrightarrow \mathbb{Z}$ for the normalized valuation map.

Lemma 4.1.10. *Let $\mathfrak{Q}$ be a prime ideal of L relatively prime to $\mathfrak{p}$. Then*

a)

$$v_{\mathfrak{Q}}(\kappa_n^{cyc}) \equiv 0 \mod p^n\mathbb{Z}.$$

b)

$$v_{\mathfrak{Q}}(\kappa_n^{split}) \equiv 0 \mod p^n\mathbb{Z}.$$

c)

$$v_{\mathfrak{Q}}(\kappa_{i,n}^{ns}) \equiv 0 \mod p^n\mathbb{Z}.$$

Proof. We give here the proof of part c) which is similar to the proof of parts a) and b) cf. [Sol92, Prop. 2.2]. Let $b_n \in L_{i,n}^\times$ be a representative of $(\beta^{ns})_{i,n}$, $\gamma_n := \gamma_{i,n}$ a generator of $\mathrm{Gal}(L_{i,n}/L)$ and set $c_n := \mathcal{N}_{L_{i,n}/L}(b)$. Then c_n is a representative of $\kappa_{i,n}^{ns}$. If we set $D_n := \sum_{k=1}^{p^n-1} k\gamma_n^k$, an easy computation shows that in $\mathbb{Z}[\mathrm{Gal}(L_{i,n}/L)]$ we have

$$(\gamma_n - 1)D_n = p^n - \mathcal{N}_{L_{i,n}/L}. \tag{4.2}$$

Applying the operator D_n to $b_n^{\gamma_n-1} = \varepsilon_{i,n}^{ns}$ we obtain from (4.2) the equality

$$c_n = \mathcal{N}_{L_{i,n}/L}(b_n) = b_n^{p^n}/\epsilon_{i,n}^{D_n}.$$

Let now $\mathfrak{Q}_n$ be an ideal in $L_{i,n}$ above $\mathfrak{Q}$. Since $\mathfrak{Q}$ does not divide $\mathfrak{p}$ the ideal $\mathfrak{Q}$ is unramified in $L_{i,n}$, and hence $v_{\mathfrak{Q}}(c_n) = v_{\mathfrak{Q}_n}(c_n)$. Furthermore, since $c_n \in \kappa_{i,n}^{ns}$

$$v_{\mathfrak{Q}}(\kappa_{i,n}^{ns}) \equiv v_{\mathfrak{Q}}(c_n) \pmod{p^n\mathbb{Z}}$$

and

$$v_{\mathfrak{Q}}(\kappa_{i,n}^{ns}) \equiv -v_{\mathfrak{Q}_n}((\varepsilon_{i,n}^{ns})^{D_n}) \equiv 0 \pmod{p^n\mathbb{Z}},$$

where the last congruence follows because $\varepsilon_{i,n}^{ns}$ is a unit.

□

As in [Sol92, Lemma 2.3] it is not difficult to see that if $m \geq n \geq 0$ then the natural quotient map

$$L^\times / \left(L^\times\right)^{p^m} \to L^\times / \left(L^\times\right)^{p^n} \tag{4.3}$$

takes κ_m^{cyc} to κ_n^{cyc}, resp. κ_m^{split} to κ_n^{split}, resp. $\kappa_{i,m}^{ns}$ to $\kappa_{i,n}^{ns}$.

Definition 4.1.11. a)

$$\kappa^{cyc} := \kappa^{cyc}(L,\gamma,\mathfrak{f}) := (\kappa_{i,n}^{cyc})_{n=0}^{\infty} \in \varprojlim_n L^\times / \left(L^\times\right)^{p^n}.$$

b)

$$\kappa^{split} := \kappa^{split}(L,\gamma,\mathfrak{f},\mathfrak{a}) := (\kappa_{i,n}^{split})_{n=0}^{\infty} \in \varprojlim_n L^\times / \left(L^\times\right)^{p^n}.$$

c) For $i \in \{1,2\}$ we define

$$\kappa_i^{ns} := \kappa_i^{ns}(L,\gamma_1,\gamma_2,\mathfrak{f},\mathfrak{a}) := (\kappa_{i,n}^{ns})_{n=0}^{\infty} \in \varprojlim_n L^\times / \left(L^\times\right)^{p^n}.$$

Recall that $\mathcal{O}_{L,S_\mathfrak{p}}^\times$ denotes the group of $S_\mathfrak{p}$-units of L with $S_\mathfrak{p} = S_\mathfrak{p}(L)$ denoting the set of prime ideals of L above $\mathfrak{p}$. We then have a natural injection

$$\mathcal{O}_{L,S_\mathfrak{p}}^\times \otimes_\mathbb{Z} \mathbb{Z}_p \simeq \varprojlim_n \mathcal{O}_{L,S_\mathfrak{p}}^\times \Big/ \left(\mathcal{O}_{L,S_\mathfrak{p}}^\times\right)^{p^n} \longrightarrow \varprojlim_n L^\times / \left(L^\times\right)^{p^n}.$$

Proposition 4.1.12. *a)* $\kappa^{cyc} \in \mathcal{O}_{L,S_\mathfrak{p}}^\times \otimes_\mathbb{Z} \mathbb{Z}_p$

b) $\kappa^{split} \in \mathcal{O}_{L,S_\mathfrak{p}}^\times \otimes_\mathbb{Z} \mathbb{Z}_p$

c) For $i \in \{1,2\}$ *we have* $\kappa_i^{ns} \in \mathcal{O}_{L,S_\mathfrak{p}}^\times \otimes_\mathbb{Z} \mathbb{Z}_p$

Proof. We give here the proof of part c) which is similar to the proof of parts a) and b) cf. [Sol92, Prop. 2.3]. We fix i and set $\kappa := \kappa_i$ and $\kappa_n := \kappa_{i,n}$. We recall that $\kappa_n \in L^\times/(L^\times)^{p^n}$ and note that it suffices to show $\kappa_n \cap \mathcal{O}_{L,S_\mathfrak{p}}^\times \neq \emptyset$ for all $n \geq 0$. Let m be the order of the $S_\mathfrak{p}$-class group $Cl_{L,S_\mathfrak{p}}$ and recall that

$$Cl_{L,S_\mathfrak{p}} \simeq I_L/\langle \mathfrak{P}_1,\ldots,\mathfrak{P}_r, P_L\rangle,$$

where I_L denotes the group of fractional ideals of L, P_L the subgroup of principal ideals and $\mathfrak{P}_1,\ldots,\mathfrak{P}_r$ the prime ideals of L lying over $\mathfrak{p}$. Write $m = p^t m'$ with a natural number t and $p \nmid m'$ and choose $c \in \kappa_{n+t}$. By Lemma 4.1.10 we obtain

$$(c) = \mathfrak{P}_1^{a_1} \cdots \mathfrak{P}_r^{a_r} \mathfrak{a}^{p^{n+t}}$$

with integers $a_1, \ldots, a_r$ and a fractional ideal $\mathfrak{a}$ which is coprime to $\mathfrak{p}$. Hence $\mathfrak{a}^{p^{n+t}}$ is an element of $\langle \mathfrak{P}_1, \ldots, \mathfrak{P}_r, P_L \rangle$. Clearly $\mathfrak{a}^m \in \langle \mathfrak{P}_1, \ldots, \mathfrak{P}_r, P_L \rangle$ and hence $\mathfrak{a}^{p^t} \in \langle \mathfrak{P}_1, \ldots, \mathfrak{P}_r, P_L \rangle$. Therefore, there exists an element $x \in L^\times$ and integers $b_1, \ldots, b_r$ such that

$$\mathfrak{a}^{p^t} = \mathfrak{P}_1^{b_1} \cdots \mathfrak{P}_r^{b_r}(x).$$

We conclude that

$$(c) = \mathfrak{P}_1^{a_1+p^n b_1} \cdots \mathfrak{P}_r^{a_r+p^n b_r} \left(x^{p^n}\right).$$

It follows that $cx^{-p^n} \in \mathcal{O}_{L,S_\mathfrak{p}}^\times$ and since the natural quotient map in (4.3) takes κ_{n+t} to κ_n we obtain $cx^{-p^n} \in \kappa_n$. □

In order to be able to formulate the main theorems of this chapter we need some additional notation.
For each prime $\mathfrak{P}$ of L above $\mathfrak{p}$ the valuation map $v_\mathfrak{P}\colon L^\times \longrightarrow \mathbb{Z}$ induces a natural homomorphism, also denoted by $v_\mathfrak{P}$,

$$v_\mathfrak{P}\colon \varprojlim_n L^\times / \left(L^\times\right)^{p^n} \to \mathbb{Z}_p.$$

The restriction of $v_\mathfrak{P}$ to $\mathcal{O}_{L,S_\mathfrak{p}}^\times \otimes_\mathbb{Z} \mathbb{Z}_p$ obviously coincides with the $\mathbb{Z}_p$-linear extension of

$$v_\mathfrak{P}\colon \mathcal{O}_{L,S_\mathfrak{p}}^\times \to \mathbb{Z}.$$

By our assumption (H1) each prime $\mathfrak{P} \in S_\mathfrak{p}$ corresponds to a unique embedding

$$j_\mathfrak{P}\colon L \hookrightarrow k_\mathfrak{p}.$$

Definition 4.1.13. We define the *cyclotomic* (resp. *elliptic*) *character* as the map

$$\chi_{\mathrm{cyc}} : \Gamma \to 1 + \mathfrak{p}_H^s \;\text{ resp. }\; \chi_{\mathrm{ell}} : \Gamma \to 1 + \mathfrak{p}_H^s$$

induced by the inverse of the global Artin map.

Now we have an isomorphism given by the following composition

$$\Gamma \xrightarrow[\chi_{\mathrm{ell}}]{\chi_{\mathrm{cyc}}} 1 + \mathfrak{p}_H^s \xrightarrow{\log_p} \mathfrak{p}_H^s \cong \begin{cases} \mathbb{Z}_p & \text{in the cyclotomic and split case} \\ \mathbb{Z}_p^2 & \text{in the non-split case.} \end{cases} \tag{4.4}$$

So we can set

$$\omega_{cyc} := \log_p(\chi_{\mathrm{cyc}}(\gamma)) \qquad \omega_{split} := \log_p(\chi_{\mathrm{ell}}(\gamma)) \qquad \omega_i := \log_p(\chi_{\mathrm{ell}}(\gamma_i)) \text{ for } i \in \{1,2\}$$

In the elliptic non-split case we obtain a $\mathbb{Z}_p$-basis ω_1, ω_2 of $\mathfrak{p}_H^s$. The set $\{\omega_1, \omega_2\}$ is also a $\mathbb{Q}_p$-basis of H and we write $\pi_{\omega_i}\colon H \longrightarrow \mathbb{Q}_p$ for the projection maps. Explicitly, we have for each $\alpha \in H$ the equality $\alpha = \pi_{\omega_1}(\alpha)\omega_1 + \pi_{\omega_2}(\alpha)\omega_2$.

First of all, we now want to restate the main theorems of [Sol92] and [Ble04].

Theorem 4.1.14. *[Sol92, cf. Thm. 2.1] Let p be a rational prime and let L be a finite abelian extension of $\mathbb{Q}$ in which p splits completely, i.e. condition (H1). Let $\mathfrak{P}$ be a prime ideal in L above p. Then we have*

$$v_\mathfrak{P}(\kappa^{cyc}) = \frac{1}{\omega_{cyc}} \log_p \left(j_\mathfrak{P}(\mathcal{N}_{\mathbb{Q}(f)/L}(1-\zeta_f))\right) \text{ in } \mathbb{Z}_p.$$

Let k now be an imaginary quadratic number field.

Theorem 4.1.15. *[Ble04, cf. Thm. 3.4] Let p be a prime which splits in $k/\mathbb{Q}$ and assume* (H1) *and* (H2). *Then for each prime ideal $\mathfrak{P}$ in L above $\mathfrak{p}$ we have*

$$v_{\mathfrak{P}}(\kappa^{split}) = \frac{1}{\omega_{split}} \log_p \left(j_{\mathfrak{P}}(\mathcal{N}_{k(\mathfrak{f})/L}(\psi(1;\mathfrak{f},\mathfrak{a}))) \right) \text{ in } \mathbb{Z}_p.$$

And we can also formulate our main theorem of this chapter:

Theorem 4.1.16. *Let p be a prime which does not split in $k/\mathbb{Q}$ and assume* (H1) *and* (H2). *If $p = 2$ (resp. $p = 3$) we assume in addition that either b2) (resp. b3) or c3)) holds. Then for each prime ideal $\mathfrak{P}$ in L above $\mathfrak{p}$ we have*

$$v_{\mathfrak{P}}(\kappa_j^{ns}) = \pi_{\omega_i} \left(\log_p(j_{\mathfrak{P}}(\mathcal{N}_{k(\mathfrak{f})/L}(\psi(1;\mathfrak{f},\mathfrak{a})))) \right) \text{ in } \mathbb{Z}_p,$$

where $i, j \in \{1,2\}$ with $i \neq j$.

Remark 4.1.17. Theorem 4.1.16 is equivalent to

$$\log_p(j_{\mathfrak{P}}(\mathcal{N}_{k(\mathfrak{f})/L}(\psi(1;\mathfrak{f},\mathfrak{a})))) = v_{\mathfrak{P}}(\kappa_2^{ns})\omega_1 + v_{\mathfrak{P}}(\kappa_1^{ns})\omega_2.$$

It is often convenient to work on finite levels.

Remark 4.1.18. Theorems 4.1.14, 4.1.15 and 4.1.16 are valid, if and only if for all $n \geq 1$

a)

$$v_{\mathfrak{P}}(\kappa_n^{cyc}) \equiv \frac{1}{\omega_{cyc}} \log_p \left(j_{\mathfrak{P}}(\mathcal{N}_{\mathbb{Q}(f)/L}(1-\zeta_f)) \right) \bmod p^n$$

b)

$$v_{\mathfrak{P}}(\kappa_n^{split}) \equiv \frac{1}{\omega_{split}} \log_p \left(j_{\mathfrak{P}}(\mathcal{N}_{k(\mathfrak{f})/L}(\psi(1;\mathfrak{f},\mathfrak{a}))) \right) \bmod p^n$$

c) and for $i, j \in \{1,2\}$ with $i \neq j$:

$$v_{\mathfrak{P}}(\kappa_{j,n}^{ns}) \equiv \pi_{\omega_i} \left(\log_p \left(j_{\mathfrak{P}}(\mathcal{N}_{k(\mathfrak{f})/L}(\psi(1;\mathfrak{f},\mathfrak{a}))) \right) \right) \bmod p^n.$$

Before we present the proofs of the theorems above we want to introduce some additional notation:

Let $\mathfrak{P}$ be a prime of F above $\mathfrak{p}$. Let $\iota : \mathbb{Q}^c \hookrightarrow \mathbb{Q}_p^c$ be a field embedding defining $\mathfrak{P}$. Via ι we view elements of $\mathbb{Q}^c$ as elements of $\mathbb{Q}_p^c$, but we sometimes omit ι in our notation. Furthermore, for any finite extension M/k we write $\widetilde{M}$ for the completion of $\iota(M)$.

4.2 Proof of the cyclotomic case (Theorem 4.1.14)

We fix a rational prime p and we denote by $\mathfrak{p}$ the ideal (p). Recall that L is an abelian extension of $\mathbb{Q}$ in which $\mathfrak{p}$ splits completely, i.e. (H1) holds. Let $\mathfrak{f}$ be the conductor of L and because of (H1) we have that $\mathfrak{p} \nmid \mathfrak{f}$. Recall that in the cyclotomic situation we use the class field theoretical conventions of Neukirch's book [Neu99] (see also Remark 4.1.2).

So in this case $F = \mathbb{Q}(\mathfrak{f})$ and we denote a prime ideal of L above $\mathfrak{p}$ by $\mathfrak{P}$. It is well-known that $F(\mu_{p^n})$ is an abelian extension of F and even of $\mathbb{Q}$ for all $n \in \mathbb{N}$. Let $\widehat{\mathbb{G}}_m$ be the formal group law $F(X,Y) = (1+X)(1+Y) - 1$. Then $\widehat{\mathbb{G}}_m$ can be defined over the completion $\mathcal{O}_{\widetilde{F}}$ of $\mathcal{O}_{F,\mathfrak{P}}$. It is easy to show that $\widehat{\mathbb{G}}_m$ is a relative Lubin-Tate group of height one with respect to the unramified extension $\widetilde{F}/\widetilde{k}$. From [Neu99, Ch. VI, Prop 6.7] we can deduce that $F(\mu_{p^n}) = \mathbb{Q}(\mathfrak{f}\mathfrak{p}^n)$ for all $n \geq 0$. We now set $H := k_\mathfrak{p} = \widetilde{k} = \widetilde{L} = \mathbb{Q}_p$, $H' := \widetilde{F}$ and we resume the notation of Chapter 3. In particular, $\widehat{\mathbb{G}}_m$ is a formal Lubin-Tate group relative to the unramified extension H'/H. Similar as in Chapter 3 we denote by $W^n = \{\zeta - 1 \mid \zeta \in \mathbb{Q}_p^c,\ \zeta^{p^n} = 1\}$ the n-division points of the multiplicative group $\widehat{\mathbb{G}}_m$. Then class field theory implies that

$$\widetilde{\mathbb{Q}(\mathfrak{f}\mathfrak{p}^{n+1})} = H'(W^{n+1}) = H'_n \text{ for } n \geq 0.$$

We set $u_n := \iota(1 - \zeta_{fp^{n+1}})$ for $n \geq 0$ and get a norm-coherent sequence

$$u := (u_n)_{n=0}^{\infty} \in \varprojlim_n \mathcal{O}_{H'_n}^{\times}$$

with an associated Coleman power series $Col_u \in \mathcal{O}_{H'}[[T]]$ depending on a choice of a generator $\omega = (\omega_n)_{n\geq 0}$ of the Tate module of $\widehat{\mathbb{G}}_m$. In particular, we note that as described in [Sol92, Ch. 3] there are several choices of a sequence $(\rho_i)_{i=0}^{\infty}$ of elements of $(\mathbb{Q}_p^c)^{\times}$ such that ρ_i is a primitive p^{s+i}-th root of unity, such that $\rho_{i+1}^p = \rho_i$ for all i, where $\pi_i = \rho_i - 1$ is a uniformizer of $\widetilde{\mathbb{Q}_p(\mathfrak{p}^{s+i})}$. On pages 343 and 344 in [Sol92] it is shown how to choose $(\rho_i)_{i=0}^{\infty}$ in a way that the unique $Col_u \in \mathbb{Z}_p[[T]]$ satisfying

$$Col_u(\rho_i - 1) = u_i$$

can be explicitly determined. So these computations show that there is a choice of $\omega = (\omega_n)_{n\geq 0}$ such that we have

$$Col_u(0) = j_{\mathfrak{P}}(\mathcal{N}_{\mathbb{Q}(\mathfrak{f})/L}(1 - \zeta_f)). \tag{4.5}$$

Now in order to prove Theorem 4.1.14 we will fix $n \geq 0$ and show the congruence a) of Remark 4.1.18.

We first consider the special case where L is the full decomposition field of $\mathfrak{p}$ in F/k. Additionally we have

$$H'_\infty = \widetilde{K_\infty} \text{ and } \widetilde{K_n} = H'_{s-1+n},$$

where we recall that we have $s = 1$ (resp. $s = 2$) for $p \neq 2$ resp. $(p = 2)$.

By a slight abuse of notation we write Γ (resp. Δ) for the Galois group of H'_∞/H'_{s-1} (resp. $H'_\infty/\widetilde{L_\infty}$). Furthermore, let π be a uniformizer of $\mathfrak{p}_H$ and we have $\omega'_{cyc} := \frac{1}{\pi^s}\omega_{cyc} \in \mathbb{Z}_p^{\times}$ since it comes from a generator in Γ. Then we can define a character $\chi_n\colon \mathrm{Gal}(H'_\infty/H) \longrightarrow \mathbb{Q}/\mathbb{Z}$ by

$$\begin{aligned}\chi_n \quad : \quad & \mathrm{Gal}(H'_\infty/H) = \Delta \times \Gamma \twoheadrightarrow \Gamma \xrightarrow{\chi_{\mathrm{cyc}}} 1 + \mathfrak{p}_H^s\mathcal{O}_H \twoheadrightarrow \frac{1 + \mathfrak{p}_H^s\mathcal{O}_H}{1 + \mathfrak{p}_H^{s+n}\mathcal{O}_H} \\ & \xrightarrow{\log_p} \frac{\mathfrak{p}_H^s\mathcal{O}_H}{\mathfrak{p}_H^{s+n}\mathcal{O}_H} \xrightarrow{1/\pi^s} \frac{\mathcal{O}_H}{\mathfrak{p}_H^n\mathcal{O}_H} \xrightarrow{1/\omega'_{cyc}} \mathbb{Z}_p/p^n\mathbb{Z}_p \xrightarrow{\cong} \frac{1}{p^n}\mathbb{Z}/\mathbb{Z},\end{aligned}$$

where the factor ω'_{cyc} is a unit in $\mathbb{Z}_p$ and is necessary to make the definition independent of the choice of γ.

By construction $\ker(\chi_n) = \mathrm{Gal}(H'_\infty/\widetilde{L_n})$ and $\chi_n(\gamma) = \frac{1}{[\widetilde{L_n}:H]} + \mathbb{Z} = \frac{1}{p^n} + \mathbb{Z}$.

Remark 4.2.1. We emphasize that χ_{cyc} arises as the inverse of the projective limit of global Artin isomorphisms $\frac{1+fp^s\mathbb{Z}}{1+fp^{s+n}\mathbb{Z}} \longrightarrow \mathrm{Gal}(\mathbb{Q}(\mathfrak{f}\mathfrak{p}^{s+n})/\mathbb{Q}(\mathfrak{f}\mathfrak{p}^s))$. The computations in case (γ) of [Gra03, Ch. II.4.4.3] show that the composite

$$1 + p^s\mathbb{Z}_p \xrightarrow{\mathrm{rec}_H} \Gamma \xrightarrow{\chi_{\mathrm{cyc}}} 1 + p^s\mathbb{Z}_p$$

is given by $\alpha \mapsto \alpha^{-1}$.

By Corollary 3.4.5 we obtain the following equality in $\mathbb{Q}/\mathbb{Z}$

$$(u,\chi_n)_{H'_\infty/H} = -\chi_n(\mathrm{rec}_H(\mathcal{N}_{H'/H}(Col_u(0)))).$$

We first compute the right hand side of this equality. By construction of χ_n, (4.5) and Remark 4.2.1 we get

$$\begin{aligned} -\chi_n(\mathrm{rec}_H(\mathcal{N}_{H'/H}(Col_u(0)))) &= \frac{1}{p^n}\left(\frac{1}{\omega_{cyc}}\log_p(\mathcal{N}_{H'/H}(Col_u(0)))\right) \\ &= \frac{1}{p^n}\left(\frac{1}{\omega_{cyc}}\log_p(\iota(\mathcal{N}_{F/L}(1-\zeta_f)))\right). \end{aligned} \tag{4.6}$$

For the computation of the left hand side we note that $\iota(({\beta^{cyc}})_n)^{\gamma-1} = \iota(\varepsilon_n^{cyc}) = \mathcal{N}_{\widetilde{K_n}/\widetilde{L_n}}(u_{s-1+n})$, so that by Definition 3.1.1 we obtain

$$(u,\chi_n)_{H'_\infty/H} = \frac{1}{p^n} v_{\widetilde{L_n}}(\iota(({\beta^{cyc}})_n)). \tag{4.7}$$

Combining (4.6) and (4.7) we derive congruence a) of Remark 4.1.18. This concludes the proof of Theorem 4.1.14 in the case that L is the full decomposition field of $\mathfrak{p}$ in F/k.

The general case follows from the special case precisely in the same way as in [Ble04, Sec. 4.3].

4.3 Proofs of the elliptic cases (Theorems 4.1.15 and 4.1.16)

Recall that $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_k$ over a rational prime p. Moreover, recall that $F = k(\mathfrak{f})$ with an ideal $\mathfrak{f}$ such that $\mathfrak{f}_L \mid \mathfrak{f}$, $w(\mathfrak{f}) = 1$ and $\mathfrak{p} \nmid \mathfrak{f}$. By [dS87, Ch. II, Lemma 1.4] there exists an elliptic curve E defined over F with complex multiplication by $\mathcal{O}_k$ and such that $F(E_{tor})$ is an abelian extension of k. The associated Größencharacter is of the form $\psi_{E/F} = \varphi \circ \mathcal{N}_{F/k}$ with a Größencharacter φ of k of infinity type $(1,0)$ and conductor $\mathfrak{f}$. Note that E has good reduction at all primes of F above $\mathfrak{p}$.

Since E has good reduction at $\mathfrak{P}$, we may and will fix a Weierstraßmodel over the localization $\mathcal{O}_{F,\mathfrak{P}}$ of $\mathcal{O}_F$ at $\mathfrak{P}$ such that the associated discriminant Δ_E is a unit in $\mathcal{O}_{F,\mathfrak{P}}$. Replacing E by one of its conjugates, if necessary, we may assume that the period lattice associated with the standard invariant differential of our fixed Weierstraßmodel is given by $\Omega\mathfrak{f}$ with $\Omega \in \mathbb{C}^\times$.

Let $\hat{E}$ be the one-parameter formal group law of E with respect to the parameter $t = -2x/y$. Then $\hat{E}$ is defined over the completion $\mathcal{O}_{\tilde{F}}$ of $\mathcal{O}_{F,\mathfrak{P}}$. By [dS87, Ch. II, Lemma 1.10] this formal group $\hat{E}$ is a relative Lubin-Tate group of height one in the split case resp. of height two in the non-split case with respect to the unramified extension $\widetilde{F}/\widetilde{k}$.

For any integral ideal $\mathfrak{c}$ of k we write $E[\mathfrak{c}]$ for the subgroup of $E(\mathbb{Q}^c)$ annihilated by all elements $\alpha \in \mathfrak{c}$. From [dS87, Ch. II, Prop. 1.6, Prop. 1.9 (i)] we deduce that $F(E[\mathfrak{p}^n]) = k(\mathfrak{f}\mathfrak{p}^n)$ for all $n \geq 0$.

We set $H := \widetilde{k} = \widetilde{L}$, $H' := \widetilde{F}$ and resume the notation of Section 3.2. In particular, $\hat{E}$ is a Lubin-Tate formal group relative to the unramified extension H'/H. Similarly as in Section 3.2 we let $W^n(\hat{E})$ denote the group of division points of level n in $\hat{E}$. Then [dS87, Ch. II, Prop. 1.6, Prop. 1.8] implies that $\widetilde{k(\mathfrak{f}\mathfrak{p}^{n+1})} = H'(W^{n+1}(\hat{E})) = H'_n$ for $n \geq 0$.

We set $u_n := \iota(\psi(1;\mathfrak{f}\mathfrak{p}^{n+1},\mathfrak{a}))$ for $n \geq 0$ and get a norm-coherent sequence $u := (u_n)_{n=0}^\infty$ which is an element of $\varprojlim_n \mathcal{O}_{H'_n}^\times$ with an associated Coleman power series $Col_u \in \mathcal{O}_{H'}[[T]]$ depending on a choice of a generator $\omega = (\omega_n)_{n\geq 0}$ of the Tate module of $\hat{E}$.

As explained in [dS87, Ch. II.4.4] or [Ble04, Sec. 4.3] one can choose a generator of the Tate module of $\hat{E}$ such that Proposition 4.3.1 below holds. Note that in [dS87, Ch. II.4] it is assumed that p is split in $k/\mathbb{Q}$. However, one can show that this assumption is not needed for proving the following result.

Proposition 4.3.1. *Let $P(z) \in \mathbb{C}[[z]]$ be the Taylor series expansion of $\psi(\Omega - z; \Omega\mathfrak{f}, \mathfrak{a})$ at $z = 0$. Let $\lambda_{\hat{E}}$ denote the formal logarithm associated with $\hat{E}$ normalized such that $\lambda'_{\hat{E}}(0) = 1$. Then $P(z) \in F[[z]] \subseteq H'[[z]]$, and moreover:*

a) $P(\lambda_{\hat{E}}(T)) \in \mathcal{O}_{H'}[[T]]$.

b) $Col_u(T) = P(\lambda_{\hat{E}}(T))$.

c) $Col_u(0) = \iota(\psi(1; \mathfrak{f}, \mathfrak{a}))$.

In order to prove Theorems 4.1.15 and 4.1.16 we will fix $n \geq 0$ and show the congruences b) and c) of Remark 4.1.18. We first consider the special case where L is the full decomposition field of $\mathfrak{p}$ in F/k. Let e denote the ramification index of $H/\mathbb{Q}_p$. We have

$$\widetilde{k} = \widetilde{L} = H, \quad \widetilde{F} = H', \quad \widetilde{K_\infty} = H'_\infty \text{ and } \widetilde{K_n} = H'_{s-1+en}.$$

Remark 4.3.2. We emphasize that χ_{ell} arises as the inverse of the projective limit of global Artin isomorphisms $\frac{1+\mathfrak{f}\mathfrak{p}^s}{1+\mathfrak{f}\mathfrak{p}^{s+n}} \longrightarrow \mathrm{Gal}(k(\mathfrak{f}\mathfrak{p}^{s+n})/k(\mathfrak{f}\mathfrak{p}^s))$. The computations in case (γ) of [Gra03, Ch. II.4.4.3] show that the composite

$$1 + \mathfrak{p}_H^s \mathcal{O}_H \xrightarrow{\mathrm{rec}_H} \Gamma \xrightarrow{\chi_{\mathrm{ell}}} 1 + \mathfrak{p}_H^s \mathcal{O}_H$$

is given by $\alpha \mapsto \alpha^{-1}$.

By a slight abuse of notation we write Γ (resp. Δ) for the Galois group of H'_∞/H'_{s-1} (resp. $H'_\infty/\widetilde{L_\infty}$).

Character in the split case Let π be a uniformizer of $\mathfrak{p}_H$ and we have $\omega'_{split} := \frac{1}{\pi^s}\omega_{split} \in \mathbb{Z}_p^\times$ since the element comes form a generator of Γ. Then we can define a character χ_n by

$$\begin{aligned} \chi_n \quad : \quad & \mathrm{Gal}(H'_\infty/H) = \Delta \times \Gamma \twoheadrightarrow \Gamma \xrightarrow{\chi_{\mathrm{ell}}} 1 + \mathfrak{p}_H^s \mathcal{O}_H \twoheadrightarrow \frac{1 + \mathfrak{p}_H^s \mathcal{O}_H}{1 + \mathfrak{p}_H^{s+n} \mathcal{O}_H} \\ & \xrightarrow{\log_p} \frac{\mathfrak{p}_H^s \mathcal{O}_H}{\mathfrak{p}_H^{s+n} \mathcal{O}_H} \xrightarrow{1/\pi^s} \frac{\mathcal{O}_H}{\mathfrak{p}_H^n \mathcal{O}_H} \xrightarrow{1/\omega'_{split}} \mathbb{Z}_p/p^n\mathbb{Z}_p \xrightarrow{\cong} \frac{1}{p^n}\mathbb{Z}/\mathbb{Z}, \end{aligned}$$

where the factor ω'_{split} is a unit in $\mathbb{Z}_p$ and is necessary to make the definition independent of the choice of γ.

By construction $\ker(\chi_n) = \mathrm{Gal}(H'_\infty/\widetilde{L_n})$ and

$$\chi_n(\gamma) = \frac{1}{[\widetilde{L_n} : H]} + \mathbb{Z} = \frac{1}{p^n} + \mathbb{Z}.$$

Character in the non-split case For $i \in \{1, 2\}$ we define a character χ_n by

$$\begin{aligned} \chi_{i,n} \quad : \quad & \mathrm{Gal}(H'_\infty/H) = \Delta \times \Gamma \twoheadrightarrow \Gamma \xrightarrow{\chi_{\mathrm{ell}}} 1 + \mathfrak{p}_H^s \mathcal{O}_H \twoheadrightarrow \\ & \frac{1 + \mathfrak{p}_H^s \mathcal{O}_H}{1 + \mathfrak{p}_H^{s+en} \mathcal{O}_H} \xrightarrow{\log_p} \frac{\mathfrak{p}_H^s \mathcal{O}_H}{\mathfrak{p}_H^{s+en} \mathcal{O}_H} \xrightarrow{\pi_{\omega_j}} \mathbb{Z}_p/p^n\mathbb{Z}_p \cong \frac{1}{p^n}\mathbb{Z}/\mathbb{Z}. \end{aligned}$$

By construction $\ker(\chi_{i,n}) = \mathrm{Gal}(H'_\infty/\widetilde{L_{i,n}})$ and $\chi_{i,n}(\gamma_j) = \frac{1}{[\widetilde{L_{i,n}} : H]} + \mathbb{Z} = \frac{1}{p^n} + \mathbb{Z}$.

Rest of the proof for Theorems 4.1.15 and 4.1.16 By Corollary 3.4.5 we obtain the following equality in $\mathbb{Q}/\mathbb{Z}$

$$(u, \chi_n)_{H'_\infty/H} = -\chi_n(\mathrm{rec}_H(\mathcal{N}_{H'/H}(Col_u(0)))).$$

in the split case, resp.

$$(u, \chi_{i,n})_{H'_\infty/H} = -\chi_{i,n}(\mathrm{rec}_H(\mathcal{N}_{H'/H}(Col_u(0)))).$$

in the non-split case.

We first compute the right hand side of this equality. By construction of χ_n resp. $\chi_{i,n}$, Proposition 4.3.1 and Remark 4.3.2 we get in the split case:

$$\begin{aligned} -\chi_n(\mathrm{rec}_H(\mathcal{N}_{H'/H}(Col_u(0)))) &= \frac{1}{p^n}\frac{1}{\omega_{split}}\left(\log_p(\mathcal{N}_{H'/H}(Col_u(0)))\right) \\ &= \frac{1}{p^n}\frac{1}{\omega_{split}}\left(\log_p(\iota(\mathcal{N}_{F/L}(\psi(1;\mathfrak{f},\mathfrak{a}))))\right). \end{aligned} \tag{4.8}$$

resp. in the non-split case:

$$\begin{aligned} -\chi_{i,n}(\mathrm{rec}_H(\mathcal{N}_{H'/H}(Col_u(0)))) &= \frac{1}{p^n}\pi_{\omega_j}\left(\log_p(\mathcal{N}_{H'/H}(Col_u(0)))\right) \\ &= \frac{1}{p^n}\pi_{\omega_j}\left(\log_p(\iota(\mathcal{N}_{F/L}(\psi(1;\mathfrak{f},\mathfrak{a}))))\right). \end{aligned} \tag{4.9}$$

For the computation of the left hand side we note that

$$\iota((\beta^{split})_n)^{\gamma-1} = \iota(\varepsilon_n^{split}) = \mathcal{N}_{\widetilde{K_n}/\widetilde{L_n}}(u_{s-1+n}),$$

resp. in the non-split case

$$\iota((\beta^{ns})_{i,n})^{\gamma_j-1} = \iota(\varepsilon_{i,n}^{ns}) = \mathcal{N}_{\widetilde{K_n}/\widetilde{L_{i,n}}}(u_{s-1+en}),$$

so that by Definition 3.1.1 we obtain in the split case

$$(u, \chi_n)_{H'_\infty/H} = \frac{1}{p^n}v_{\widetilde{L_n}}(\iota((\beta^{split})_n)). \tag{4.10}$$

resp. in the non-split case

$$(u, \chi_{i,n})_{H'_\infty/H} = \frac{1}{p^n}v_{\widetilde{L_{i,n}}}(\iota((\beta^{ns})_{i,n})). \tag{4.11}$$

Combining in the split case (4.8) and (4.10) resp. in the non-split case (4.9) and (4.11) we derive the congruences b) and c) of Remark 4.1.18. This concludes the proofs of Theorems 4.1.15 and 4.1.16 in the case that L is the full decomposition field of $\mathfrak{p}$ in F/k.

The general case in the split situation is proved in [Ble04, Sec. 4.3]. For the non-split case we can give a similar argument:

Indeed, let D denote the decomposition subfield of F/k with respect to $\mathfrak{p}$. By our hypothesis (H1) we have $L \subseteq D$. For a fixed prime $\mathfrak{P}$ of L we write $\mathfrak{P}_1, \ldots, \mathfrak{P}_r$, $r = [D : L]$, for the primes of D lying over $\mathfrak{P}$. Let $\{g_1 = \mathrm{id}, g_2, \ldots, g_r\}$ denote a set of representatives of $\mathrm{Gal}(F/L)$ modulo $\mathrm{Gal}(F/D)$ such that $\mathfrak{P}_\ell = \mathfrak{P}_1^{g_\ell}$, $\ell = 1, \ldots, r$. One can verify that

$$\kappa_{i,n}^{ns}(L) \equiv \mathcal{N}_{D/L}(\kappa_{i,n}^{ns}(D)) \bmod (L^\times)^{p^n},$$

where $\kappa_{i,n}^{ns}(L) := \kappa_{i,n}^{ns}(L, \gamma_1, \gamma_2, \mathfrak{f}, \mathfrak{a})$ and $\kappa_{i,n}^{ns}(D) := \kappa_{i,n}^{ns}(D, \gamma_1, \gamma_2, \mathfrak{f}, \mathfrak{a})$. So one obtains

$$v_{\mathfrak{P}}(\kappa_{i,n}^{ns}(L)) \equiv v_{\mathfrak{P}}(\mathcal{N}_{D/L}(\kappa_{i,n}^{ns}(D))) = \sum_{\ell=1}^{r} v_{\mathfrak{P}_\ell}(\kappa_{i,n}^{ns}(D)) \bmod p^n\mathbb{Z}.$$

Without loss of generality we may assume that the fixed embedding ι defines the prime $\mathfrak{P}_1$ of D. Then $j_{\mathfrak{P}_\ell} := \iota \circ g_\ell^{-1}$ defines $\mathfrak{P}_\ell$ for $\ell = 1, \ldots, r$. We obtain

$$\begin{aligned}
\pi_{\omega_j}\left(\log_p(\iota(\mathcal{N}_{F/L}(\psi(1;\mathfrak{f},\mathfrak{a}))))\right) &= \pi_{\omega_j}\left(\log_p(\iota(\mathcal{N}_{D/L}(\mathcal{N}_{F/D}(\psi(1;\mathfrak{f},\mathfrak{a})))))\right) \\
&= \pi_{\omega_j}\left(\log_p(\iota(\prod_{\ell=1}^{r} \mathcal{N}_{F/D}(\psi(1;\mathfrak{f},\mathfrak{a}))^{g_\ell}))\right) \\
&= \pi_{\omega_j}\left(\sum_{\ell=1}^{r} \log_p(j_{\mathfrak{P}_1}(\mathcal{N}_{F/D}(\psi(1;\mathfrak{f},\mathfrak{a}))^{g_\ell}))\right) \\
&= \sum_{\ell=1}^{r} \pi_{\omega_j}\left(\log_p(j_{\mathfrak{P}_\ell}(N_{F/D}(\psi(1;\mathfrak{f},\mathfrak{a}))))\right).
\end{aligned}$$

Hence, the general case follows immediately from the special case.

Chapter 5

Application to the Iwasawa-theoretic MRS conjecture

The main reference for this chapter is the work of Burns, Kurihara and Sano [BKS17], which was preceded by [BKS16], [San14] and [MR16]. The main purpose of this chapter is to prove an Iwasawa-theoretic version of a conjecture of Mazur, Rubin and Sano for imaginary quadratic base fields. The possibility to do this was already hinted at in [BKS17, Rem. 5.10] for the case where the rational prime p splits in the imaginary quadratic field k. We now write down a proof for the split case and show some new results for non-split primes (in k) by using the main results of Chapter 4.

In the first section we present the framework in order to be able to formulate a conjecture of Mazur, Rubin and Sano, which we will call *MRS conjecture* from now on, for finite extensions as well as the Iwasawa-theoretic version. Moreover, we summarize some useful properties of the Iwasawa-theoretic assertion which have been shown in [BKS16] and [BKS17].

Following the ideas presented in [BKS17] for abelian fields, we obtain in the second section results for imaginary quadratic base fields k and odd primes p with $p \nmid h_k$ for cases where p splits in k as well as where p does not split in k.

5.1 Rubin-Stark conjecture and MRS conjecture

Let L/k be an abelian extension of a number field k with Galois group G and we denote by $\widehat{G}$ the group of homomorphisms $G \to \mathbb{C}^\times$ of finite order. Let S be a finite set of places of k which contains $S_\infty(k) \cup S_{ram}(L/k)$. We fix a labelling $S := \{v_0, \ldots, v_n\}$. We order the places so that $v_1, \ldots, v_r$ split completely in L and that there are no other places with this property in S. So we assume that at least one place does not split completely in L. We put $V := \{v_1, \ldots, v_r\}$. Moreover, let $S(L)$ or S_L be the set of places of L which lie above the places in S and $\mathcal{O}_{L,S}$ be the ring of $S(L)$-integers of L. For any set W of places of k, we put

$$Y_{L,W} := \bigoplus_{w \in W_L} \mathbb{Z}w \text{ and } X_{L,W} := \left\{ \sum_{w \in W_L} a_w w \in Y_{L,W} : \sum_{w \in W_L} a_w = 0 \right\}.$$

Here we denote by $L_{k,S}(s, \chi)$ the usual S-imprimitive L-function for $\chi \in \widehat{G}$, i.e.

$$L_{k,S}(s,\chi) := \prod_{v \notin S} \left(1 - \chi(\mathrm{Fr}_v)\mathcal{N}(v)^{-s}\right)^{-1},$$

where $\mathrm{Fr}_v \in G$ is the Frobenius of the (unramified) prime v and we put

$$r_S(\chi) := \mathrm{ord}_{s=0} L_{k,S}(s,\chi).$$

We know (by the Dirichlet unit theorem) that the homomorphism of $\mathbb{R}[G]$-modules

$$\lambda_{L,S} : \mathbb{R}\mathcal{O}_{L,S}^{\times} \to \mathbb{R}X_{L,S}$$
$$x \mapsto -\sum_{w \in S_L} \log |x|_w \cdot w$$

is an isomorphism. By [Tat84, Chap. I, Prop 3.4] we know that

$$r_S(\chi) = \dim_{\mathbb{C}}(e_\chi \mathbb{C}\mathcal{O}_{L,S}^{\times}) = \begin{cases} |\{v \in S : \chi(G_v) = 1\}| & \text{if } \chi \neq 1, \\ |S| - 1 & \text{if } \chi = 1, \end{cases}$$

where $e_\chi := \frac{1}{|G|}\sum_{g \in G} \chi(g) g^{-1}$. So we see that $r \leq r_S(\chi)$.

Moreover, for $\chi \in \widehat{G}$ we set $L_\chi := L^{\ker(\chi)}$ and $G_\chi := \mathrm{Gal}(L_\chi/k)$. Take $V_{\chi,S} \subset S$ so that all $v \in V_{\chi,S}$ split completely in L_χ (i.e. $\chi(G_v) = 1$) and $|V_{\chi,S}| = r_S(\chi)$. Note that if $\chi \neq 1$, we have

$$V_{\chi,S} = \{v \in S : \chi(G_v) = 1\}.$$

Let T be a finite set of places of k which is disjoint from S. Recall that the S-truncated T-modified L-function is defined by

$$L_{k,S,T}(s,\chi) := \left(\prod_{v \in T} 1 - \chi(\mathrm{Fr}_v)\mathcal{N}(v)^{1-s}\right) L_{k,S}(s,\chi).$$

The (S,T)-unit group of L is defined by

$$\mathcal{O}_{L,S,T}^{\times} := \ker(\mathcal{O}_{L,S}^{\times} \to \bigoplus_{w \in T_L} \kappa(w)^{\times}),$$

where $\kappa(w)$ is here the residue class field of L at w. Note that $\mathcal{O}_{L,S,T}^{\times}$ is a subgroup of $\mathcal{O}_{L,S}^{\times}$ of finite index and we have

$$r \leq r_S(\chi) = \mathrm{ord}_{s=0} L_{k,S,T}(s,\chi) = \dim_{\mathbb{C}}(e_\chi \mathbb{C}\mathcal{O}_{L,S,T}^{\times}).$$

We put

$$L_{k,S,T}^{(r)} := \lim_{s \to 0} s^{-r} L_{k,S,T}(s,\chi)$$

and

$$\theta_{L/k,S,T}^{(r)} := \sum_{\chi \in \widehat{G}} L_{k,S,T}^{(r)}(0,\chi^{-1}) e_\chi \in \mathbb{R}[G]. \tag{5.1}$$

Definition 5.1.1. Let

$$\tilde{\lambda}_{L,S} : \mathbb{R} \bigwedge_{\mathbb{Z}[G]}^{r} \mathcal{O}_{L,S,T}^{\times} \to \mathbb{R} \bigwedge_{\mathbb{Z}[G]}^{r} X_{L,S}$$

be the isomorphism induced by $\lambda_{L,S}$. Then we define the (r-th order) *Rubin-Stark element*

$$\varepsilon_{L/k,S,T}^{V} \in \mathbb{R} \bigwedge_{\mathbb{Z}[G]}^{r} \mathcal{O}_{L,S,T}^{\times}$$

such that

$$\theta_{L/k,S,T}^{(r)} \cdot (w_1 - w_0) \wedge \ldots \wedge (w_r - w_0) = \tilde{\lambda}_{L,S}(\varepsilon_{L/k,S,T}^{V}) \text{ in } \mathbb{R} \bigwedge_{\mathbb{Z}[G]}^{r} X_{L,S}.$$

Recall that $Y_{L,S}^* = \mathrm{Hom}_{\mathbb{Z}[G]}(Y_{L,S}, \mathbb{Z}[G])$. There is a pairing

$$\bigwedge_{\mathbb{Z}[G]}^{r} X_{L,S} \times \bigwedge_{\mathbb{Z}[G]}^{r} Y_{L,S}^* \to \mathbb{Z}[G].$$

If $\eta \in \bigwedge_{\mathbb{Z}[G]}^r Y_{L,S}^*$ we can define a map

$$R_\eta : \bigwedge_{\mathbb{Z}[G]}^{r} \mathcal{O}_{L,S,T}^{\times} \xrightarrow{\tilde{\lambda}_{L,S}} \mathbb{R} \bigwedge_{\mathbb{Z}[G]}^{r} X_{L,S} \xrightarrow{\eta} \mathbb{R}[G].$$

If $w \in S_L$ define $w^* \in Y_{L,S}^*$ by

$$w^*(w') = \sum_{gw=w'} g \text{ for } w' \in S_L.$$

Remark 5.1.2. Recall that in [Rub96] the absolute value $|\ \ |_w : L_w \to \mathbb{R}^+ \cup \{0\}$ is normalized so that

$$|\alpha|_w = \begin{cases} \pm\alpha & \text{if } L_w = \mathbb{R}, \\ \alpha\bar{\alpha} & \text{if } L_w = \mathbb{C}, \\ \mathcal{N}(w)^{-\mathrm{ord}_w(\alpha)} & \text{if } L_w \text{ is non-archimedean.} \end{cases}$$

Lemma 5.1.3. *[Rub96, Lemma 2.2] If $u_1, \ldots, u_r \in \mathcal{O}_{L,S,T}^{\times}$, $w_1, \ldots, w_r \in S_L$ and $\eta = w_1^* \wedge \ldots \wedge w_r^*$, then*

$$R_\eta(u_1 \wedge \ldots \wedge u_r) = \det\left(\sum_{g\in G} \log|u_i^g|_{w_j} g^{-1}\right)_{i,j}.$$

From now on we assume that $\mathcal{O}_{L,S,T}^{\times}$ is $\mathbb{Z}$-free (T can always be chosen such that this condition is fulfilled).

Definition 5.1.4. We define the *Rubin lattice* as

$$\bigcap_{\mathbb{Z}[G]}^{r} \mathcal{O}_{L,S,T}^{\times} := \left\{ a \in \mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^{r} \mathcal{O}_{L,S,T}^{\times} : \Phi(a) \in \mathbb{Z}[G] \text{ for all } \Phi \in \bigwedge_{\mathbb{Z}[G]}^{r} \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{O}_{L,S,T}^{\times}, \mathbb{Z}[G]) \right\}.$$

The 'classical' Rubin-Stark conjecture asserts:

Conjecture 5.1.5. $\mathrm{RS}(L/k, S, T, V)$

$$\varepsilon_{L/k,S,T}^{V} \in \bigcap_{\mathbb{Z}[G]}^{r} \mathcal{O}_{L,S,T}^{\times}.$$

Remark 5.1.6. The Rubin-Stark conjecture $\mathrm{RS}(L/k, S, T, V)$ is known to hold in the following cases:

a) The case $r = 0$: This is Theorem 3.3 in [Rub96] which is essentially the work of Deligne and Ribet in [DR80].

b) The case $L = k$: This is Theorem 3.2 in [Rub96].

c) The case $[L:k]=2$: This is Corollary 3.5 in [Rub96].

d) The case L is an abelian extension over $\mathbb{Q}$: This is part of Theorem A in [Bur07], where it is shown that the conjecture is implied by the leading term conjecture which is known by [BG03] and [Fla11].

e) If there exists an imaginary quadratic number field F with $h_F=1$ such that $F \subseteq k$, L/F a finite abelian extension and $[L:k]$ is both odd and divisible only by primes which split completely in $F/\mathbb{Q}$. This is also part of Theorem A in [Bur07] which uses in this case the work of [Ble06] to show the leading term conjecture.

f) The case $r=1$ and k an imaginary quadratic number field: This follows from Proposition 2.5 in [Rub96] and Proposition 3.9 in [Tat84, Ch. IV], which is based on [Sta80].

g) For a large class of multi-quadratic extensions L/k and $r=1$: This was done in [DST03].

Now we define the 'p-part' of the Rubin-Stark conjecture. For that we set $U_{L,S,T} := \mathbb{Z}_p\mathcal{O}^{\times}_{L,S,T}$ and fix an isomorphism $\mathbb{C} \cong \mathbb{C}_p$. From this we regard

$$\varepsilon^V_{L/k,S,T} \in \mathbb{C}_p \bigwedge^r_{\mathbb{Z}_p[G]} U_{L,S,T}.$$

We define

$$\bigcap^r_{\mathbb{Z}_p[G]} U_{L,S,T} := \left\{ a \in \mathbb{Q}_p \bigwedge^r_{\mathbb{Z}_p[G]} U_{L,S,T} : \Phi(a) \in \mathbb{Z}_p[G] \text{ for all } \Phi \in \bigwedge^r_{\mathbb{Z}_p[G]} \mathrm{Hom}_{\mathbb{Z}_p[G]}(U_{L,S,T}, \mathbb{Z}_p[G]) \right\}$$

and note that there is a natural isomorphism $\mathbb{Z}_p \bigcap^r_{\mathbb{Z}[G]} \mathcal{O}^{\times}_{L,S,T} \cong \bigcap^r_{\mathbb{Z}_p[G]} U_{L,S,T}$. So one can formulate what in [BKS17] is called the p-component version of the Rubin-Stark conjecture

Conjecture 5.1.7. $\mathrm{RS}(L/k,S,T,V)_p$

$$\varepsilon^V_{L/k,S,T} \in \bigcap^r_{\mathbb{Z}_p[G]} U_{L,S,T}.$$

Remark 5.1.8. The classical Rubin-Stark conjecture $\mathrm{RS}(L/k,S,T,V)$ implies the p-component version $\mathrm{RS}(L/k,S,T,V)_p$ for all primes p.

Remark 5.1.9. From now on we will sometimes suppress $\mathbb{Z}_p[G]$ from the notation $\bigcap^r_{\mathbb{Z}_p[G]}$ if it is clear form the context.

Let k be a number field and M_∞/k be a Galois extension such that $\mathcal{G} := \mathrm{Gal}(M_\infty/k) \cong \Delta \times \Gamma$, where Δ is a finite abelian group and $\Gamma \cong \mathbb{Z}_p$. With the isomorphism fixed above we identify $\widehat{\Delta}$ with $\mathrm{Hom}_{\mathbb{Z}}(\Delta, \overline{\mathbb{Q}}_p^{\times})$.

Furthermore, we set

$M := M_\infty^{\Gamma}$	$k_\infty := M_\infty^{\Delta}$
M_n : n-th layer of M_∞/M	$k_n :=$ n-th layer of k_∞/k
$\mathcal{G}_n := \mathrm{Gal}(M_n/k)$.	

and for $\chi \in \widehat{\mathcal{G}}$ we also set

$$\begin{array}{ll}
L_\chi := M_\infty^{\ker(\chi)} & L_{\chi,\infty} := L_\chi \cdot k_\infty \\
L_{\chi,n} := n\text{-th layer of } L_{\chi,\infty}/L_\chi & \\
\mathcal{G}_\chi := \mathrm{Gal}(L_{\chi,\infty}/k) & \mathcal{G}_{\chi,n} := \mathrm{Gal}(L_{\chi,n}/k) \\
\Gamma_\chi := \mathrm{Gal}(L_{\chi,\infty}/L_\chi) & \Gamma_{\chi,n} := \mathrm{Gal}(L_{\chi,n}/L_\chi) \\
V_\chi := \{v \in S : v \text{ splits completely in } L_{\chi,\infty}\} & r_\chi := |V_\chi|.
\end{array}$$

Let S be a finite set of places of k which contains $S_\infty(k) \cup S_{ram}(M_\infty/k) \cup S_p(k)$, where $S_p(k)$ is the set of primes of k above p, and T a finite set of places which is disjoint from S.

Now we fix a character $\chi \in \widehat{\mathcal{G}}$ and we take a proper subset $V' \subset S$ so that all $v \in V'$ split completely in L_χ and that $V_\chi \subset V'$ and we put $r' := |V'|$. To simplify we set: $L_n := L_{\chi,n}$, $L := L_\chi$, $\mathcal{G}_n := \mathcal{G}_{\chi,n}$, $G := G_\chi = \mathrm{Gal}(L_\chi/k)$, $\Gamma_n := \Gamma_{\chi,n}$, $V := V_\chi$, $r := r_\chi$ and $e := r' - r$. Let $I(\Gamma_n)$ be the augmentation ideal of $\mathbb{Z}_p[\Gamma_n]$ and I_n be the kernel of the natural map $\mathbb{Z}_p[\mathcal{G}_n] \to \mathbb{Z}_p[G]$.

Remark 5.1.10. From [San14, Lemma 2.11] we know that there exists a canonical injection

$$\bigcap^r U_{L,S,T} \hookrightarrow \bigcap^r U_{L_n,S,T}$$

which induces the injection

$$\nu_n : \left(\bigcap^r U_{L,S,T}\right) \otimes_{\mathbb{Z}_p} I(\Gamma_n)^e/I(\Gamma_n)^{e+1} \hookrightarrow \left(\bigcap^r U_{L_n,S,T}\right) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^{e+1}.$$

Remark 5.1.11. For $v \in V' \setminus V$ we denote by

$$\mathrm{rec}_w : L^\times \to \Gamma_n$$

the local reciprocity map at w (a place above v) and so we can define a $\mathbb{Z}[G]$-homomorphism

$$\begin{aligned}
\mathrm{Rec}_w : L^\times &\longrightarrow I_n/I_n^2 \\
a &\longmapsto \sum_{g \in G} (\mathrm{rec}_w(a^g) - 1)\, g^{-1}
\end{aligned}$$

It is shown in [San14, Prop. 2.7] that $\bigwedge_{v \in V' \setminus V} \mathrm{Rec}_w$ induces a homomorphism

$$\mathrm{Rec}_n : \bigcap^{r'} U_{L,S,T} \to \bigcap^r U_{L,S,T} \otimes_{\mathbb{Z}_p} I(\Gamma_n)^e/I(\Gamma_n)^{e+1}.$$

Since $\varprojlim_n I(\Gamma_n)^e/I(\Gamma_n)^{e+1} \cong \mathbb{Z}_p$ the map

$$\varprojlim_n \mathrm{Rec}_n : \bigcap^{r'} U_{L,S,T} \to \bigcap^r U_{L,S,T} \otimes_{\mathbb{Z}_p} \varprojlim_n I(\Gamma_n)^e/I(\Gamma_n)^{e+1}$$

uniquely extends to give a $\mathbb{C}_p$-linear map

$$\mathrm{Rec}_\infty : \mathbb{C}_p \bigwedge^{r'} U_{L,S,T} \to \mathbb{C}_p \left(\bigwedge^r U_{L,S,T} \otimes_{\mathbb{Z}_p} \varprojlim_n I(\Gamma_n)^e/I(\Gamma_n)^{e+1}\right).$$

Definition 5.1.12. We define

$$\begin{array}{rcl}\mathcal{N}_n : \bigcap^r U_{L_n,S,T} & \longrightarrow & \bigcap^r U_{L_n,S,T}\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^{e+1} \\ a & \longmapsto & \sum_{\gamma\in\Gamma_n}\gamma(a)\otimes\gamma^{-1}.\end{array}$$

Then there are the following two conjectures as stated in [BKS17]:

Conjecture 5.1.13. $\mathrm{MRS}(L_n/L/k,S,T,V,V')_p$
Assume Conjectures $\mathrm{RS}(L_n/k,S,T,V)_p$ *and* $\mathrm{RS}(L/k,S,T,V')_p$*. Then we have*

$$\mathcal{N}_n(\varepsilon^V_{L_n/k,S,T}) = (-1)^{re}\,\nu_n\left(\mathrm{Rec}_n\left(\varepsilon^{V'}_{L/k,S,T}\right)\right) \tag{5.2}$$

which is an equality in $\bigcap^r U_{L_n,S,T}\otimes\mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^{e+1}$.

Conjecture 5.1.14. $\mathrm{MRS}(M_\infty/k,S,T,\chi,V')_p$
Assume that Conjecture $\mathrm{RS}(L_n/k,S,T,V)_p$ *is valid for all* n*. Then there exists a (unique)*

$$\xi = (\xi_n)_n \in \bigcap^r U_{L,S,T}\otimes_{\mathbb{Z}_p}\varprojlim_n I(\Gamma_n)^e/I(\Gamma_n)^{e+1}$$

such that

$$\nu_n(\xi_n) = \mathcal{N}_n(\varepsilon^V_{L_n/k,S,T}) \tag{5.3}$$

for all n *and that*

$$e_\chi\xi = (-1)^{re}e_\chi\mathrm{Rec}_\infty(\varepsilon^{V'}_{L/k,S,T}) \text{ in } \mathbb{C}_p\left(\bigwedge^r U_{L,S,T}\otimes_{\mathbb{Z}_p}\varprojlim_n I(\Gamma_n)^e/I(\Gamma_n)^{e+1}\right) \tag{5.4}$$

In [BKS17] there is a summary of some properties of Conjecture 5.1.14, which we want to recall:

Proposition 5.1.15. *[BKS17, Prop. 4.4, Cor. 4.5 and 4.6]*

a) If $V = V'$*, then* $\mathrm{MRS}(M_\infty/k,S,T,\chi,V')$ *holds.*

b) If $V\subset V''\subset V'$*, then* $\mathrm{MRS}(M_\infty/k,S,T,\chi,V')$ *implies* $\mathrm{MRS}(M_\infty/k,S,T,\chi,V'')$.

c) Suppose that $\chi(G_v)=1$ *for all* $v\in S$ *and* $|V'|=|S|-1$*. Then, for any* $V''\subset S$ *with* $V\subset V''$ *and* $|V''|=|S|-1$*,* $\mathrm{MRS}(M_\infty/k,S,T,\chi,V')$ *and* $\mathrm{MRS}(M_\infty/k,S,T,\chi,V'')$ *are equivalent.*

d) If $v\in V'\setminus V$ *is a finite place which is unramified in* L_∞*, then* $\mathrm{MRS}(M_\infty/k,S\setminus\{v\},T,\chi,V'\setminus\{v\})$ *implies* $\mathrm{MRS}(M_\infty/k,S,T,\chi,V')$.

e) If $|V'|\neq|S|-1$ *and* $v\in S\setminus V'$ *is a finite place which is unramified in* M_∞*, then* $\mathrm{MRS}(M_\infty/k,S\setminus\{v\},T,\chi,V')$ *implies* $\mathrm{MRS}(M_\infty/k,S,T,\chi,V')$.

f) If every place v *in* $V'\setminus V$ *is both non-archimedean and unramified in* M_∞*, then* $\mathrm{MRS}(M_\infty/k,S,T,\chi,V')$ *holds.*

g) If $\chi(G_\mathfrak{p})\neq 1$ *for all* $\mathfrak{p}\in S_p(k)$ *which ramify in* $L_{\chi,\infty}$*, then* $\mathrm{MRS}(M_\infty/k,S,T,\chi,V')$ *holds,*

Remark 5.1.16. The validity of $\mathrm{MRS}(L_n/L/k,S,T,V,V')_p$ for all n implies the validity of $\mathrm{MRS}(M_\infty/k,S,T,\chi,V')_p$.

5.2 Iwasawa-theoretic MRS conjecture for imaginary quadratic base fields

Let k be an imaginary quadratic field, p be an odd rational prime and $\mathfrak{p}$ a prime ideal of $\mathcal{O}_k$ above p. For simplicity we also exclude $p = 3$ in the case where p is ramified in k. We assume for this section the following hypothesis:

(H2) $p \nmid h_k$.

Recall that if p splits in k, we call this situation the *split case* and if p is non-split in k, we call this situation the *non-split case*.

Let $\mathfrak{f}$ be a non-zero integral ideal in $\mathcal{O}_k$ coprime to $\mathfrak{p}$. Furthermore, let $K'_\infty := k(\mathfrak{f}\mathfrak{p}^\infty)$ and $\mathcal{G}' := \mathrm{Gal}(K'_\infty/k)$. Now we have

$$\Gamma' := \varprojlim_n \mathrm{Gal}(k(\mathfrak{f}\mathfrak{p}^{n+1})/k(\mathfrak{f}\mathfrak{p})) \cong \varprojlim_n \frac{1+\mathfrak{p}\mathcal{O}_k}{1+\mathfrak{p}^{n+1}\mathcal{O}_k} \cong \begin{cases} \mathbb{Z}_p & \text{in the split case,} \\ \mathbb{Z}_p^2 & \text{in the non-split case.} \end{cases}$$

We have $\mathcal{G}' \cong \Gamma \times \Delta$, where $\Delta \cong \mathrm{Gal}(K_0/k)$ and $\Gamma = \mathbb{Z}_p^d$. If K_∞ is a $\mathbb{Z}_p$-extension of K_0 in K'_∞ then we set $k_\infty := K_\infty^\Delta$

For a finite character χ of $\mathcal{G}'$ we set:

$$\begin{array}{lll} L := L_\chi := K_\infty^{\ker(\chi)} & L_\infty := L_{\chi,\infty} := L \cdot k_\infty & L_n := L_{\chi,n} \text{ the } n\text{-th layer of } L_\infty/L \\ \mathcal{G}_n := \mathrm{Gal}(L_n/k) & & \Gamma_n := \mathrm{Gal}(L_n/L) \end{array}$$

Moreover, we set S to be a finite set of places of k, which contains $S_\infty(k)$, $S_{ram}(K_\infty/k)$ and $S_p(k)$ and T to be a finite set of places of k disjoint from S and such that $U_{K_\infty/k,S,T}$ is a free module.

The goal of this section is to prove the following theorems, which are analogues of Theorem 4.10 in [BKS17] for imaginary quadratic fields:

Theorem 5.2.1. *Let p be an odd prime that splits in k and assume $p \nmid h_k$. Then $\mathrm{MRS}(K'_\infty/k, S, T, \chi, V')_p$ holds for each finite character of $\mathcal{G}'$.*

Theorem 5.2.2. *Let p be an odd inert prime in k or a ramified prime in k with $p \neq 2, 3$ and assume that $p \nmid h_k$. Let χ be a finite character of $\mathcal{G}'$ and let K_∞ be a $\mathbb{Z}_p$-extension of K_0 in K'_∞ such that $L_\chi \subseteq K_\infty$. If no finite prime in S splits completely in $L_{\chi,\infty}/k$ and Assumption* (DimAss) *(Assumption 5.2.15) holds, then* $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')_p$ *holds.*

Theorem 5.2.3. *Let p be an odd prime that is inert in k and assume $p \nmid h_k$. Let χ be a finite character of $\mathcal{G}'$ and let K_∞ be a $\mathbb{Z}_p$-extension of K_0 in K'_∞ such that $L_\chi \subseteq K_\infty$. Assume that no finite prime in S splits completely in $L_{\chi,\infty}/k$. Then $\mathrm{MRS}(K_\infty/k, S, T, \chi, V')_p$ holds for all but one explicitly defined $\mathbb{Z}_p$-extension determined by the Rubin-Stark element at L_χ.*

Remark 5.2.4. We want to recall that for $r = |V| = 1$ and k an imaginary quadratic number field $\mathrm{RS}(L_n/k, S, T, V)_p$ holds for all n because of Remark 5.1.6 f) and Remark 5.1.8.

Reduction of the problem With Proposition 5.1.15 b) we can assume that V' is maximal, i.e.

$$r' = \min\{|\{v \in S : \chi(G_v) = 1\}|, |S| - 1\}$$

and with Proposition 5.1.15 g) we can assume $\chi(G_{\mathfrak{p}}) = 1$.

If $\chi = 1$, then we have $r' = |S| - 1$ and so we can assume that $\mathfrak{p} \notin V'$ by Proposition 5.1.15 c). In this case every prime of $v \in V' \setminus V$ is unramified in L_∞, so the assertion follows from Proposition 5.1.15 f).

For simplicity we assume that there is more than one ramified place in $S \setminus \{\mathfrak{p}\}$ that does not split completely in L.

So in total we may assume, with Proposition 5.1.15 d) and e), that

$$S = \{\infty\} \cup \{\mathfrak{p}\} \cup \Sigma \text{ and } V' = \{\infty, \mathfrak{p}\},$$

where Σ is the subset of $S \setminus \{\mathfrak{p}\}$ of ramified places that do not split completely in L and $V := \{\infty\}$ as well as $\chi(G_{\mathfrak{p}}) = 1$.

Remark 5.2.5. We have for a $\mathbb{Z}_p[G]$-lattice M that $\bigcap^r M = \bigwedge^r M$ if $r \leq 1$ with [Rub96, Prop. 1.2 ii)]. Now after the reductions above we can make the following simplifications:

a) The map ν_n defined with the help of [San14, Lemma 2.11] reduces to

$$U_{L,S,T} \otimes_{\mathbb{Z}_p} I(\Gamma_n)/I(\Gamma_n)^2 \hookrightarrow U_{L_n,S,T} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^2$$

which is induced by the inclusion $U_{L,S,T} \hookrightarrow U_{L_n,S,T}$

b) Now we use [San14, Prop. 2.7] with $G := \mathcal{G}_n$, $H := \Gamma_n$ and $G/H = \mathrm{Gal}(L/k)$. Then we have that $M := U_{L,S,T}$ is a $\mathrm{Gal}(L/k)$-lattice, $r' = 2$ and $\Phi := \mathrm{Rec}_{w|_{U_{L,S,T}}}$ and we get the map

$$\begin{aligned} \mathrm{Rec}_n : \bigcap^2 U_{L,S,T} &\to U_{L,S,T} \otimes_{\mathbb{Z}_p} I(\Gamma_n)/I(\Gamma_n)^2 \\ m_1 \wedge m_2 &\mapsto m_2 \otimes \Phi(m_1) - m_1 \otimes \Phi(m_2). \end{aligned}$$

Lemma 5.2.6. *[San14, Prop 3.5] Let $\Omega(k,T)$ be a set of triples (M,S,V), where M is a finite abelian extension of k, S is an admissible set of places and V is a subset of S where $v \in V$ splits completely in M and $|S| \geq |V| + 1$. Let $(M,S,V), (M',S',V) \in \Omega(k,T)$ and suppose that $M \subset M'$ and $S \subset S'$. Then we have*

$$\mathcal{N}^r_{M'/M}(\varepsilon^V_{M'/k,S',T}) = \left(\prod_{v \in S' \setminus S} 1 - \mathrm{Fr}_v^{-1}\right) \varepsilon^V_{M/k,S,T}$$

where $\mathcal{N}_{M'/M} = \mathcal{N}_{\mathrm{Gal}(M'/M)}$ and if $r = 0$, then we regard $\mathcal{N}^r_{M'/M}$ as the natural map $\mathbb{Z}[\mathrm{Gal}(M'/k)] \to \mathbb{Z}[\mathrm{Gal}(M/k)]$.

Elliptic units and Rubin-Stark elements Here we show that for k imaginary quadratic and $r = |V| = 1$ elliptic units can be used to describe the Rubin-Stark elements explicitly therefore connecting the results of Chapter 4 with this concepts developed earlier in this chapter. This is well-known since the Rubin-Stark conjecture coincides with the abelian Stark conjecture in rank one and in this situation it was proved using the properties of elliptic units already in [Sta80] and [Tat84]. Our treatment contains more details and also considers the 'T-modification'. It is instructional for the reader to compare this to the cyclotomic situation treated in [Pop11, Sec. 4.2].

Lemma 5.2.7. *Let $\mathfrak{m}$ be a non-zero integral ideal of $\mathcal{O}_k$. Then there are infinitely many integral ideals $\mathfrak{a}$ of $\mathcal{O}_k$ such that*

i) $(\mathfrak{a}, 6\mathfrak{m}) = 1$,

ii) $\sigma_{\mathfrak{a}} = (\mathfrak{a}, k(\mathfrak{m})/k) = \mathrm{id}$.

Proof. With the Čebotarev density theorem (e.g. [Neu99, Ch. VII, Thm. 13.4]) we know that there are infinitely many prime ideals of k such that $(\mathfrak{a}, k(\mathfrak{m})/k) = \mathrm{id}$. Excluding all the prime ideals dividing $6\mathfrak{m}$ we find infinitely many prime ideals such that i) and ii) are fulfilled. □

Definition 5.2.8. Let $\mathfrak{f}$ be an integral ideal in $\mathcal{O}_k$, $\mathfrak{q}$ be a prime ideal and let $\mathfrak{a}$ be an integral ideal coprime to $6\mathfrak{f}$. For each character χ of $Cl(\mathfrak{f})$, we set

$$S(\chi, \mathfrak{f}) := \begin{cases} \sum_{C \in Cl(\mathfrak{f})} \overline{\chi}(C) \log |\psi(1; \mathfrak{f}, \mathfrak{a})^{\sigma(C)}| & \text{if } \mathfrak{f} \neq (1), \\ \sum_{C \in Cl(1)} \chi(C) \log |\delta_{\mathfrak{q}}^{\sigma(C)}| & \text{if } \mathfrak{f} = (1), \end{cases}$$

where we recall that $\delta_{\mathfrak{q}}$ for a fixed $\mathfrak{a}$ and a fixed prime ideal $\mathfrak{q}$ is defined in Remark 2.1.28.

Lemma 5.2.9. *Let $\mathfrak{f}$ be a non-zero integral ideal in $\mathcal{O}_k$, χ a character of $\mathrm{Gal}(k(\mathfrak{f})/k)$, $\mathfrak{f}_\chi$ the conductor of χ and χ_p the primitive character associated to χ. Then we have*

$$c_{\mathfrak{p}} \cdot w(\mathfrak{f}_\chi) \cdot S(\chi, \mathfrak{f}) = w(\mathfrak{f}) \cdot \prod_{\mathfrak{q} | \mathfrak{f}, \mathfrak{q} \nmid \mathfrak{f}_\chi} (1 - \chi_p^{-1}(\mathfrak{q})) \cdot S(\chi_p, \mathfrak{f}_\chi),$$

where $c_{\mathfrak{p}} := 1$ if $\mathfrak{f}_\chi \neq (1)$ or $\mathfrak{f} = (1)$, and $c_{\mathfrak{p}} := 1 - \chi_p^{-1}(\mathfrak{p})$ if $\mathfrak{f}_\chi = (1)$ and $\mathfrak{f} \neq (1)$.

Proof. In order to shorten the notation we set $L := k(\mathfrak{f})$, $L' := k(\mathfrak{f}_\chi)$, $G := \mathrm{Gal}(L/k)$, $H := \mathrm{Gal}(L/L')$ and $G' := \mathrm{Gal}(L'/k)$. So with the definitions above we have that χ_p is a primitive character of G'. So we have $Cl(\mathfrak{f}) \cong G$ and $Cl(\mathfrak{f}_\chi) \cong G'$. For $\mathfrak{f} = \mathfrak{f}_\chi = (1)$ the equality is trivially correct. Hence, we assume $\mathfrak{f} \neq (1)$. We compute:

$$\begin{aligned} S(\chi, \mathfrak{f}) &= \sum_{g \in G} \chi^{-1}(g) \log |\psi(1; \mathfrak{f}, \mathfrak{a})^g| \\ &= \sum_{h \in H} \sum_{g' \in G'} \chi^{-1}(g'h) \log \left|\psi(1; \mathfrak{f}, \mathfrak{a})^{hg'}\right| \\ &= \sum_{h \in H} \sum_{g' \in G'} \chi^{-1}(g') \log \left|\psi(1; \mathfrak{f}, \mathfrak{a})^{hg'}\right| \\ &= \sum_{g' \in G'} \chi^{-1}(g') \log \left|\mathcal{N}_{L/L'}(\psi(1; \mathfrak{f}, \mathfrak{a}))^{g'}\right| \end{aligned}$$

Assuming $\mathfrak{f}_\chi \neq (1)$ and using Theorem 2.1.27 a) we obtain:

$$S(\chi,\mathfrak{f}) = \frac{w(\mathfrak{f})}{w(\mathfrak{f}_\chi)} \cdot \prod_{\mathfrak{q}|\mathfrak{f},\mathfrak{q}\nmid\mathfrak{f}_\chi} (1-\chi_p^{-1}(\mathfrak{q})) \cdot S(\chi_p,\mathfrak{f}_\chi),$$

where the product ranges over all the prime ideals $\mathfrak{q}$ dividing $\mathfrak{f}$ but not $\mathfrak{f}_\chi$.

Assuming $\mathfrak{f}_\chi = (1)$ and using Theorem 2.1.27 a) and b) we obtain:

$$S(\chi,\mathfrak{f}) = \frac{w(\mathfrak{f})}{w(1)} \cdot \prod_{\mathfrak{q}|\mathfrak{f},\mathfrak{q}\neq\mathfrak{b}} (1-\chi_p^{-1}(\mathfrak{q})) \cdot \left(\sum_{g'\in G'} \chi_p^{-1}(g') \log|\delta_{\mathfrak{b}}^{g'}| \right),$$

where the product ranges over all prime ideal $\mathfrak{q}$ dividing $\mathfrak{f}$ with exception of arbitrarily chosen prime ideal $\mathfrak{b}$. From this we obtain by multiplying $(1-\chi_p^{-1}(\mathfrak{p}))$ and recalling Equation (2.4) we obtain:

$$c_{\mathfrak{p}} \cdot S(\chi,\mathfrak{f}) = \frac{w(\mathfrak{f})}{w(1)} \cdot \prod_{\mathfrak{q}|\mathfrak{f}} (1-\chi_p^{-1}(\mathfrak{q})) \cdot S(\chi_p,(1)),$$

where the product ranges over all prime ideals $\mathfrak{q}$ dividing $\mathfrak{f}$ and so we have shown the assertion. □

Lemma 5.2.10. *Let $\mathfrak{f} \neq (1)$ be a non-zero integral ideal of $\mathcal{O}_k$, $G := \mathrm{Gal}(k(\mathfrak{f})/k)$, χ a character of G, $S = S_\infty \cup S_{ram}(k(\mathfrak{f})/k)$. Then we have*

$$L_S^*(0,\chi^{-1}) = \frac{-1}{w(\mathfrak{f})(\mathcal{N}(\mathfrak{a})-\chi(\mathfrak{a}))} \sum_{g\in G} \chi^{-1}(g) \log|\psi(1;\mathfrak{f},\mathfrak{a})^g|^2 .$$

Proof. In order to shorten the notation, for the chosen ideal $\mathfrak{a}$ in $\mathcal{O}_k$ with $(\mathfrak{a}, 6\mathfrak{f}\mathfrak{p}) = 1$ we denote by $c_{\mathfrak{a}}$ the expression $\mathcal{N}(\mathfrak{a}) - \chi(\mathfrak{a}) \in \mathbb{Z}[G]$ and set $G' := \mathrm{Gal}(k(\mathfrak{f}_\chi)/k)$. Furthermore, we define the set $S' = S_\infty \cup S_{ram}(k(\mathfrak{f}_\chi)/k)$.

Since $S' \subseteq S$ we have for the induced primitive character χ_p of $\mathrm{Gal}(k(\mathfrak{f}_\chi)/k)$ and $r \leq r_S(\chi)$:

$$L_S^{(r)}(0,\chi_p) = \prod_{v\in S\setminus S'} (1-\chi_p(v)) \cdot L_{S'}^{(r)}(0,\chi_p). \tag{5.5}$$

For $\mathfrak{f}_\chi \neq (1)$ we get with Proposition 2.3.9:

$$L_{S'}^*(0,\chi_p^{-1}) = \frac{-2}{c_{\mathfrak{a}} \cdot w(\mathfrak{f}_\chi)} \sum_{g\in G'} \chi_p^{-1}(g) \log|\psi(1;\mathfrak{f}_\chi,\mathfrak{a})^g|,$$

and with (5.5) we get

$$L_S^*(0,\chi_p^{-1}) = \frac{-2}{c_{\mathfrak{a}} w(\mathfrak{f}_\chi)} \cdot \prod_{\mathfrak{q}|\mathfrak{f},\mathfrak{q}\nmid\mathfrak{f}_\chi} (1-\chi_p^{-1}(\mathfrak{q})) \cdot S(\chi_p,\mathfrak{f}_\chi).$$

Using Lemma 5.2.9 we obtain

$$L_S^*(0,\chi_p^{-1}) = \frac{-2}{c_{\mathfrak{a}} \cdot w(\mathfrak{f})} \cdot S(\chi,\mathfrak{f})$$

and using the functoriality properties of S-imprimitive Artin L-functions we get the assertion of the lemma for $\mathfrak{f}_\chi \neq (1)$.

Assume that $\mathfrak{f}_\chi = (1)$, then we get with Corollary 2.3.7

$$\left(1 - \chi_p^{-1}(\mathfrak{p})\right) L^*_{S'}(0, \chi_p^{-1}) = \frac{-2}{w(1) \cdot c_{\mathfrak{a}}} \sum_{g \in G'} \chi(g) \log \left|\delta_{\mathfrak{p}}^g\right|,$$

Now with (5.5) we obtain

$$c_{\mathfrak{p}} \cdot L^*_S(0, \chi_p^{-1}) = \frac{-2}{w(1) \cdot c_{\mathfrak{a}}} \cdot \prod_{\mathfrak{q} | \mathfrak{f}} (1 - \chi^{-1}(\mathfrak{q})) S(\chi_p, (1))$$

and with Lemma 5.2.9 we get

$$L^*_S(0, \chi_p^{-1}) = \frac{-2}{w(1) \cdot c_{\mathfrak{a}}} \cdot S(\chi, \mathfrak{f})$$

and with the same argument as in the previous case we have proven the lemma. □

Lemma 5.2.11. *Let $\mathfrak{f}$ be a non-zero integral ideal of $\mathcal{O}_k$, $G := \mathrm{Gal}(k(\mathfrak{f})/k)$, $S = S_\infty \cup S_{ram}(k(\mathfrak{f})/k)$, $V = \{\infty\}$ and $\mathfrak{a}$ be an integral ideal from Lemma 5.2.7.*

a) For $\mathfrak{p} \mid \mathfrak{f}$ there is an admissible set T and an element $C(T, \mathfrak{a}, \mathfrak{f}) \in \mathbb{Z}_p[G]$ such that

$$\varepsilon^V_{k(\mathfrak{f})/k,S,T} = \psi(1; \mathfrak{f}, \mathfrak{a})^{C(T,\mathfrak{a},\mathfrak{f})}.$$

b) For $\mathfrak{f} = (1)$ and $S' = S \cup \{\mathfrak{p}\}$ there is an admissible set T and an element $C(T, \mathfrak{a}, (1)) \in \mathbb{Z}_p[G]$ such that

$$\varepsilon^V_{k(1)/k,S',T} = \delta_{\mathfrak{p}}^{C(T,\mathfrak{a},(1))}.$$

Proof. First of all, it is easy to see that

$$L^*_{S,T}(0, \chi) = \prod_{v \in T} (1 - \chi(v)\mathcal{N}(v)) L^*_S(0, \chi). \tag{5.6}$$

Recall that we are in a case with $r = 1$. We fix a $v \in S$ and a $w \in S_L$ above that. Then with Definition 5.1.1 and Lemma 5.1.3 the Rubin-Stark element $\varepsilon^V_{k(\mathfrak{f})/k,S,T}$ is defined by

$$w^*(\tilde{\lambda}_{k(\mathfrak{f}),S}(\varepsilon^V_{k(\mathfrak{f})/k,S,T})) = \theta^{(1)}_{k(\mathfrak{f})/k,S,T}(0)$$

This is equivalent to

$$e_\chi w^*(\tilde{\lambda}_{k(\mathfrak{f}),S}(\varepsilon^V_{k(\mathfrak{f})/k,S,T})) = L^*_{S,T}(0, \chi^{-1}) e_\chi \tag{5.7}$$

for all $\chi \in \widehat{G}$.

Let T be an admissible set which contains $\mathfrak{a}$ and χ be a non-trivial character.

For the case where $w(\mathfrak{f}) = 1$ we set

$$C(T, \mathfrak{a}, \mathfrak{f}) := \prod_{v \in T \setminus \{\mathfrak{a}\}} (1 - \mathcal{N}(v)\sigma_v) \in \mathbb{Z}[G].$$

For $w(\mathfrak{f}) \neq 1$ we have to multiply by $w(\mathfrak{f})^{-1}$, but the product is in $\mathbb{Z}_p[G]$ because we assumed $p > 3$ and we know that $w(\mathfrak{f}) \mid 6$ in the imaginary quadratic case.

In part a) we assume for simplicity that $w(\mathfrak{f}) = 1$, so we can compute with Lemma 5.1.3

$$\begin{aligned} e_\chi w^*(\tilde{\lambda}_{k(\mathfrak{f}),S}(\psi_n^{C(T,\mathfrak{a},\mathfrak{f})})) &= e_\chi \sum_{g\in G} \log \left| g(\psi(1;\mathfrak{f},\mathfrak{a})^{C(T,\mathfrak{a},\mathfrak{f})}) \right|^2 g^{-1} \\ &= \sum_{g\in G} \chi^{-1}(g) \log \left| g(\psi(1;\mathfrak{f},\mathfrak{a})^{C(T,\mathfrak{a},\mathfrak{f})}) \right|^2 e_\chi \\ &= \prod_{v\in T\setminus\{\mathfrak{a}\}} (1-\mathcal{N}(v)\chi(v)) \sum_{g\in G} \chi^{-1}(g) \log |\psi(1;\mathfrak{f},\mathfrak{a})^g|^2 e_\chi. \end{aligned}$$

Moreover, with (5.6) and with Lemma 5.2.10 we obtain

$$\begin{aligned} e_\chi L^*_{S,T}(0,\chi^{-1}) &= \frac{-(1-\chi(\mathfrak{a})\mathcal{N}(\mathfrak{a}))}{(\mathcal{N}(\mathfrak{a})-\chi(\mathfrak{a}))} \prod_{v\in T\setminus\{\mathfrak{a}\}} (1-\chi(v)\mathcal{N}(v)) \cdot \left(\sum_{g\in G} \chi^{-1}(g) \log |\psi(1;\mathfrak{f},\mathfrak{a})^g|^2 \right) e_\chi \\ &= \prod_{v\in T\setminus\{\mathfrak{a}\}} (1-\chi(v)\mathcal{N}(v)) \cdot \left(\sum_{g\in G} \chi^{-1}(g) \log |\psi(1;\mathfrak{f},\mathfrak{a})^g|^2 \right) e_\chi. \end{aligned}$$

Now for the trivial character $\chi = 1$, we get the result at once for $|S| > 3$ since $\zeta^{(1)}_{k,S}(0) = 0$ in this case and by the norm relation of Theorem 2.1.27 a) we have on the left side of (5.7) also a zero. For $|S| = 2$ we have a non-zero right hand side of Equation (5.7) given by Corollary 2.3.5 and Lemma 2.1 in [Tat84, Ch. 1]. On the other side we use the norm relation of Theorem 2.1.27 b) and then checking the equality boils down to computing $\mathcal{N}_{k(1)/k}(\delta_\mathfrak{p})$. For case b) one makes the same computations as above with the difference that we replace $\psi(1;\mathfrak{f},\mathfrak{a})$ by $\delta_\mathfrak{p}$. In this case it is unavoidable to have coefficients in $\mathbb{Z}_p[G]$ since $w(1) = 2, 4$ or 6.

□

So from Lemma 5.2.6 we get immediately

Corollary 5.2.12. *For L_n defined at the beginning of the section let $\mathfrak{f}_{L_n}$ and $\mathfrak{f}_L$ be the conductor of L_n and L, respectively. Then we have*

$$\varepsilon^V_{L_n/k,S,T} = \mathcal{N}_{k(\mathfrak{f}_{L_n})/L_n}(\varepsilon^V_{k(\mathfrak{f}_{L_n})/k,S,T}),$$

where $S = \{\infty, \mathfrak{p}\} \cup S_{ram}(L/k)$ because of the construction of L_n. Moreover, we have

$$\varepsilon^V_{L/k,S\setminus\{\mathfrak{p}\},T} = \mathcal{N}_{k(\mathfrak{f}_L)/L}\left(\varepsilon^V_{k(\mathfrak{f}_L)/L,S\setminus\{\mathfrak{p}\},T}\right).$$

From now on, we will use the notation $\varepsilon^V_{L_n} := \varepsilon^V_{L_n/k,S,T}$ and $\varepsilon^V_L := \varepsilon^V_{L/k,S\setminus\{\mathfrak{p}\},T}$, respectively.

As in Chapter 4 we obtain

Lemma 5.2.13. *Let $s \geq n \geq 1$. Then we have*

a) $\mathcal{N}_{L_s/L_n}(\varepsilon^V_{L_s}) = \varepsilon^V_{L_n}$.

b) $\mathcal{N}_{L_n/L}(\varepsilon^V_{L_n}) = 1$.

c) There exists a $\beta_n \in L_n^\times/L^\times$ such that

$$\beta_n^{\gamma-1} = \varepsilon^V_{L_n}.$$

Moreover, we set $\kappa_n := \mathcal{N}_{L_n/L}(\beta_n) \in L^\times/(L^\times)^{p^n}$.

Proof. For part a) and b) we can use Lemma 5.2.6, where for b) we use additionally the assumption that $\mathfrak{p}$ splits completely in L. Part c) follows directly from Hilbert's Theorem 90. □

Now we set $\Gamma_n := \mathrm{Gal}(L_n/L)$ and by abuse of notation we denote by γ a generator of Γ_n.

Lemma 5.2.14. *The equality (5.3) of Conjecture 5.1.14 holds in our situation, i.e.*

$$\mathcal{N}_n(\varepsilon_{L_n}^V) = \nu_n(\xi_n)$$

with $\xi_n := \kappa_n \otimes (\gamma - 1)$, *where* κ_n *is defined in Lemma 5.2.13.*

Proof. We compute

$$\begin{aligned}
\mathcal{N}_n(\varepsilon_{L_n}^V) &= \sum_{\tau\in\Gamma_n} \tau(\varepsilon_{L_n}^V)\otimes\tau^{-1} \\
&= \sum_{\tau\in\Gamma_n} \tau(\varepsilon_{L_n}^V)\otimes(\tau^{-1}-1) + \left(\sum_{\tau\in\Gamma_n} \tau(\varepsilon_{L_n}^V)\right)\otimes 1 \\
&= \sum_{\tau\in\Gamma_n} \tau(\varepsilon_{L_n}^V)\otimes(\tau^{-1}-1),
\end{aligned}$$

where the last equality holds because Lemma 5.2.13 b), i.e. $\mathcal{N}_{L_n/L}(\varepsilon_{L_n}^V) = 1$. Then we compute by using (4.2) for the Kolyvagin operator D_γ:

$$\begin{aligned}
\mathcal{N}_n(\varepsilon_{L_n}^V) &= \sum_{i=0}^{p^n-1} \gamma^i(\varepsilon_{L_n}^V)\otimes(\gamma^{-i}-1) \\
&= -\sum_{i=0}^{p^n-1} \gamma^i(\varepsilon_{L_n}^V)\otimes i(\gamma-1) \\
&= -\sum_{i=0}^{p^n-1} i\gamma^i(\varepsilon_{L_n}^V)\otimes(\gamma-1) \\
&= -\sum_{i=0}^{p^n-1} i\gamma^i(\gamma-1)(\beta_n)\otimes(\gamma-1) \\
&= -D_\gamma(\gamma-1)(\beta_n)\otimes(\gamma-1) \\
&= (\mathcal{N}_{L_n/L} - p^n)(\beta_n)\otimes(\gamma-1) \\
&= \kappa_n\otimes(\gamma-1) - \beta_n\otimes p^n(\gamma-1) \\
&= \kappa_n\otimes(\gamma-1) = \xi_n.
\end{aligned}$$

□

So as in Proposition 4.1.12, we obtain $\kappa := (\kappa_n)_{n=0}^{\infty} \in U_{L,S,T}$.

Assumption 5.2.15. (DimAss) The set

$$\{e_\chi \varepsilon_L^V, e_\chi \kappa\}$$

is a $\mathbb{C}_p$-basis of $e_\chi \mathbb{C}_p U_{L,S}$.

Before we proceed we fix a prime w of L above $\mathfrak{p}$ and if we are in a local setting we embed the global elements via the embedding corresponding to w without explicitly mentioning it. Furthermore, we define the map

$$\begin{aligned} \mathrm{Ord}_w : L^\times &\to \mathbb{Z}_p[G] \\ a &\mapsto \sum_{\sigma \in G} \mathrm{ord}_w(\sigma(a))\sigma^{-1}. \end{aligned}$$

Proposition 5.2.16. *With Assumption 5.2.15 we obtain*

$$\mathrm{Rec}_n(e_\chi \varepsilon^{V'}_{L/k,S,T}) = -e_\chi \kappa \otimes (\gamma - 1) \in \mathbb{C}_p \left(U_{L,S,T} \otimes I(\Gamma_n)/I(\Gamma_n)^2 \right).$$

Proof. By abuse of notation we will denote by Ord_w also the isomorphism

$$\begin{aligned} \mathrm{Ord}_w \colon e_\chi \mathbb{C}_p \bigwedge^2 U_{L,S} &\longrightarrow e_\chi \mathbb{C}_p U_{L,S}, \\ e_\chi u_1 \wedge u_2 &\longmapsto \chi(\mathrm{Ord}_w(u_1)) e_\chi u_2 - \chi(\mathrm{Ord}_w(u_2)) e_\chi u_1, \end{aligned}$$

which is the map denoted by $\Phi_{V',V}$ in [San14]. By [San14, Prop. 3.6] or [Rub96, Prop. 5.2] we have

$$\mathrm{Ord}_w(e_\chi \varepsilon^{V'}_{L/k,S,T}) = -e_\chi \varepsilon^V_L.$$

Now we know from Assumption 5.2.15 that $\{e_\chi \varepsilon^V_L, e_\chi \kappa\}$ is a $\mathbb{C}_p$-basis of $e_\chi \mathbb{C}_p U_{L,S}$. On the other hand we compute

$$\begin{aligned} \mathrm{Ord}_w(e_\chi \varepsilon^V_L \wedge \kappa) &= \chi(\mathrm{Ord}_w(\varepsilon^V_L)) e_\chi \kappa - \chi(\mathrm{Ord}_w(\kappa)) e_\chi \varepsilon^V_L \\ &= -\chi(\mathrm{Ord}_w(\kappa)) e_\chi \varepsilon^V_L, \end{aligned}$$

where the last equality holds because ε^V_L is a unit at w. We conclude that

$$e_\chi \varepsilon^{V'}_{L/k,S,T} = \frac{1}{\chi(\mathrm{Ord}_w(\kappa))} e_\chi(\varepsilon^V_L \wedge \kappa).$$

We obtain that $\mathrm{Rec}_w(\kappa)$ vanishes because $\mathrm{rec}_w(\sigma(\kappa_n)) = \mathrm{rec}_w(\sigma(\mathcal{N}_{L_n/L}(\beta_n))) = 1$ and using the same convention for Rec_w as for Ord_w above we can get

$$\begin{aligned} \mathrm{Rec}_w(e_\chi(\varepsilon^V_L \wedge \kappa)) &= \sum_{\sigma \in G} \left(\mathrm{rec}_w(\sigma(\varepsilon^V_L)) - 1 \right) \sigma^{-1} e_\chi \kappa - \sum_{\sigma \in G} \left(\mathrm{rec}_w(\sigma(\kappa)) - 1 \right) \sigma^{-1} e_\chi \varepsilon^V_L \\ &= \sum_{\sigma \in G} \left(\mathrm{rec}_w(\sigma(\varepsilon^V_L)) - 1 \right) \sigma^{-1} e_\chi \kappa \\ &= \chi(\mathrm{Rec}_w(\varepsilon^V_L)) e_\chi \kappa. \end{aligned}$$

In summary we have

$$\begin{aligned} \mathrm{Rec}_w(e_\chi \varepsilon^{V'}_{L/k,S,T}) &= \frac{1}{\chi(\mathrm{Ord}_w(\kappa))} \mathrm{Rec}_w(e_\chi(\varepsilon^V_L \wedge \kappa)) \\ &= \frac{1}{\chi(\mathrm{Ord}_w(\kappa))} \chi(\mathrm{Rec}_w(\varepsilon^V_L)) e_\chi \kappa. \end{aligned}$$

So it remains to compute $\mathrm{Rec}_w(\varepsilon^V_L) \in I_n/I_n^2$, where

$$0 \longrightarrow I_n \longrightarrow \mathbb{Z}_p[\mathcal{G}_n] \longrightarrow \mathbb{Z}_p[G] \longrightarrow 0.$$

We recall from [San14, (3)] that

$$\begin{array}{rcl} \mathbb{Z}_p[G]\otimes_{\mathbb{Z}_p} I(\Gamma_n)/I(\Gamma_n)^2 & \xrightarrow{\simeq} & I_n/I_n^2, \\ \sigma\otimes\bar{a} & \longmapsto & \overline{\tilde{\sigma}a}, \end{array}$$

Now depending on the case we are in we use Lemma 5.2.18 or Lemma 5.2.19 below to compute the following:

$$\begin{aligned} \mathrm{Rec}_w(\varepsilon_L^V) &= \sum_{\sigma\in G}(\mathrm{rec}_w(\sigma(\varepsilon_L^V))-1)\sigma^{-1} \\ &= \sum_{\sigma\in G}\sigma^{-1}\otimes(\gamma^{-\mathrm{ord}_w(\sigma(\kappa))}-1) \\ &= \sum_{\sigma\in G}-\mathrm{ord}_w(\sigma(\kappa))\sigma^{-1}\otimes(\gamma-1) \\ &= -\mathrm{Ord}_w(\kappa)\otimes(\gamma-1). \end{aligned}$$

So in total we get

$$\mathrm{Rec}_w(e_\chi\varepsilon_{L/k,S,T}^{V'}) = -e_\chi\kappa\otimes(\gamma-1).$$

and with Remark 5.2.5 b) and the isomorphism above we obtain the wanted equality:

$$\mathrm{Rec}_n(e_\chi\varepsilon_{L/k,S,T}^{V'}) = -e_\chi\kappa\otimes(\gamma-1).$$

□

Before presenting the Lemmas 5.2.18 and 5.2.19 as well as showing in which cases Assumption 5.2.15 holds, we want to show how to finish the proof of the main theorems of this section assuming the other results. We set $\xi := \kappa\otimes(\gamma-1)$ and so we have $\xi = \varprojlim_n \xi_n$. With this notation we obtain from Proposition 5.2.16 that

$$e_\chi\xi = -e_\chi\mathrm{Rec}_\infty(\varepsilon_{L/k,S,T}^{V'})$$

and therefore we finished the proof of Theorems 5.2.1, 5.2.2 and 5.2.3 assuming we can show the assertions mentioned above.

Computations with the local reciprocity map Recall that with our reductions we are in a case where $\chi(G_\mathfrak{p}) = 1$, i.e. $\mathfrak{p}\nmid\mathfrak{f}_\chi$ and therefore $L := L_\chi \subseteq K_0$.

Remark 5.2.17. We note that $L_w = k_\mathfrak{p}$. In Lemmas 5.2.18 and 5.2.19, in order to be formally correct, we always have to substitute in the computations, when necessary, $\sigma(\varepsilon_L^V)$ by $\delta\in 1+\mathfrak{p}\mathcal{O}_{k_\mathfrak{p}}$ with $\sigma(\varepsilon_L^V)\equiv\delta \bmod \ker(\mathrm{rec}_w)$. But for this δ one can show that $\log_p(\sigma(\varepsilon_L^V)\delta^{-1}) = 0$, so in the following computations, by abuse of notation, we will continue writing $\sigma(\varepsilon_L^V)$.

The split case

Lemma 5.2.18. *For the situation of Theorem 5.2.1 we have that* $\mathrm{rec}_w(\sigma(\varepsilon_L^V)) = \gamma^{-\mathrm{ord}_w(\sigma(\kappa))}$.

Proof. We also denote by $\mathrm{rec}_w : L^\times \to \mathrm{Gal}(L_\infty/L)$ the local reciprocity map. So we have the following equivalences:

$$\mathrm{rec}_w(\sigma(\varepsilon_L^V)) = \gamma^{s(\sigma)} \Leftrightarrow \sigma(\varepsilon_L^V)^{-1} = \chi_{\mathrm{ell}}(\gamma^{s(\sigma)}) \Leftrightarrow -\frac{1}{\log_p(\chi_{\mathrm{ell}}(\gamma))}\log_p(\sigma(\varepsilon_L^V)) = s(\sigma),$$

where $s(\sigma)$ is the exponent corresponding to γ.

Now we set $\omega := \log_p(\chi_{\mathrm{ell}}(\gamma))$. Now Theorem 4.1.15 shows that

$$\frac{1}{\omega}\log_p(\sigma(\varepsilon_L^V)) = \mathrm{ord}_w(\sigma(\kappa))$$

so that we have $s(\sigma) = -\mathrm{ord}_w(\sigma(\kappa))$ and we get $\mathrm{rec}_w(\sigma(\varepsilon_L^V)) = \gamma^{s(\sigma)} = \gamma^{-\mathrm{ord}_w(\sigma(\kappa))}$. □

The non-split case Let $\gamma_1, \gamma_2 \in \Gamma'$ be such that $K_\infty = (K'_\infty)^{\langle\gamma_1\rangle} =: K_{1,\infty}$, $\gamma_2 \mid_{K_\infty} = \gamma$ and $\langle\gamma_1, \gamma_2\rangle = \Gamma'$. With this we set

$$\begin{array}{ccc} k'_\infty = (K'_\infty)^\Delta, & k_\infty := k_{1,\infty} := K_\infty^\Delta, & k_{2,\infty} := K_{2,\infty}^\Delta, \\ L'_\infty := L \cdot k'_\infty, & L_\infty := L_{1,\infty} := L \cdot k_{1,\infty}, & L_{2,\infty} := L \cdot k_{2,\infty}. \end{array}$$

Moreover, we set $\omega_i := \log_p(\chi_{\mathrm{ell}}(\gamma_i))$ and, a little bit different than before, we denote by κ_{γ_2} or κ_γ what we up to now called κ being the element constructed in K_∞ and by κ_{γ_1} the analogous element in $K_{2,\infty}$.

By Theorem 4.1.16 and Remark 4.1.17 as well as Lemma 5.2.11 we obtain

$$\log_p(\varepsilon_L^V) = \mathrm{ord}_w(\kappa_{\gamma_2})\omega_2 + \mathrm{ord}_w(\kappa_{\gamma_1})\omega_1, \tag{5.8}$$

Lemma 5.2.19. *For the situation of Theorems 5.2.2 and 5.2.3 we have that*

$$\mathrm{rec}_w(\sigma(\varepsilon_L^V)) = \gamma_2^{-\mathrm{ord}_w(\sigma(\kappa_{\gamma_2}))}.$$

Proof. Recall the local reciprocity map $\mathrm{rec}'_w : L^\times \to \mathrm{Gal}(L'_\infty/L)$. Now considering Remark 5.2.17 we have the following equivalences:

$$\begin{aligned} \mathrm{rec}'_w(\sigma(\varepsilon_L^V)) = \gamma_1^{s_1(\sigma)}\gamma_2^{s_2(\sigma)} &\Leftrightarrow \sigma(\varepsilon_L^V)^{-1} = \chi_{\mathrm{ell}}(\gamma_1^{s_1(\sigma)}\gamma_2^{s_2(\sigma)}) \\ &\Leftrightarrow -\log_p(\sigma(\varepsilon_L^V)) = s_1(\sigma)\log_p(\chi_{\mathrm{ell}}(\gamma_1)) + s_2(\sigma)\log_p(\chi_{\mathrm{ell}}(\gamma_2)), \end{aligned}$$

where $s_1(\sigma)$ and $s_2(\sigma)$ are the exponents corresponding to γ_1 and γ_2. So with Equation (5.8) and considering the commutative diagram

$$\begin{array}{ccc} L^\times & \xrightarrow{\mathrm{rec}'_w} & \mathrm{Gal}(L'_\infty/L) \\ & \searrow^{\mathrm{rec}_w} & \downarrow \\ & & \mathrm{Gal}(L_\infty/L) \end{array}$$

we obtain $\mathrm{rec}_w(\sigma(\varepsilon_L^V)) = \gamma_2^{s_2(\sigma)} = \gamma_2^{-\mathrm{ord}_w(\sigma(\kappa_{\gamma_2}))}$.

□

Assumption (DimAss) **in the split case** The next two arguments are based on Lemma 4.11 [BKS17] and pp. 108-110 in [Ble06]. Recall that we have set

$$U_L := \mathcal{O}_L^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$$
$$U_{L,S} := \mathcal{O}_{L,S}^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$$
$$U_{L_\infty,S} := \varprojlim_n \left(\mathcal{O}_{L_n,S,T}^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p\right)$$
$$\varepsilon_{L_\infty}^V := \varepsilon_{L_\infty/k,S}^V := (\varepsilon_{L_n}^V)_{n=0}^\infty \in U_{L_\infty,S}$$

Let $\Lambda := \mathbb{Z}_p[[\mathcal{G}]]$ and regard $\mathbb{C}_p$ as a Λ-algebra via χ. With the Bockstein map of [BKS17, Sec. 5B] we have that

$$e_\chi \mathbb{C}_p U_{L,S} = e_\chi \mathbb{C}_p U_L \oplus (U_{L_\infty,S} \otimes_\Lambda \mathbb{C}_p)$$

Because $e_\chi \varepsilon_L^V$ is non-zero, this is a basis of $e_\chi \mathbb{C}_p U_L$ because $e_\chi \mathbb{C}_p U_L = e_\chi \mathbb{C}_p U_{L,S\setminus\{\mathfrak{p}\}}$. It is left to show that $e_\chi \kappa$ is a basis of $U_{L_\infty,S} \otimes_\Lambda \mathbb{C}_p$. Exactly as in Lemma 5.9 in [Ble06] we can see that

$$0 \to U_{L_\infty,S} \xrightarrow{\gamma-1} U_{L_\infty,S} \xrightarrow{g} U_{L,S}$$

is an exact sequence, where g is the canonical projection. We have $g(\varepsilon_{L_\infty}^V) = 1$, so there is a unique $\alpha \in U_{L_\infty,S} \otimes \mathbb{Q}_p$ such that

$$(\gamma - 1)\alpha = \varepsilon_{L_\infty}^V$$

Now let $\mathfrak{p}_\chi := \ker(\chi : \Lambda \to \mathbb{C}_p)$ and $\Lambda_{\mathfrak{p}_\chi}$ the localization at the prime ideal $\mathfrak{p}_\chi$. In the split case one can now show with the Iwasawa Main Conjecture proved in Theorem 3.1 in [Ble06] for $p \nmid 2h_k$ that α is a basis of $U_{L_\infty,S} \otimes_\Lambda \Lambda_{\mathfrak{p}_\chi}$ (cf. Lemma 5.5 and p. 110 in [Ble06]). Now by looking at the image of α under the map

$$U_{L_\infty,S} \otimes_\Lambda \Lambda_{\mathfrak{p}_\chi} \xrightarrow{\chi} U_{L_\infty,S} \otimes_\Lambda \mathbb{C}_p \hookrightarrow e_\chi \mathbb{C}_p U_{L,S}$$

we see that it is equal to $e_\chi \kappa \in U_{L_\infty,S} \otimes \mathbb{C}_p$, so we shown that it is a basis, as needed.

Remark 5.2.20. One vital ingredient of the proof of Assumption 5.2.15 in the split case is a one-variable main conjecture. Unfortunately, to the knowledge of the author there are no unconditional proofs of this conjecture in the non-split case, only some partial results by Rubin in [Rub91]. Another result in this direction is the result of Johnson-Leung and Kings in [JLK11] presented in Chapter 6, but no unconditional result for the non-split case can be implied from their assertions because they have to assume a vanishing of a μ-invariant which is open for this case.

The case of Theorem 5.2.3 As the relevant main conjecture is not proven for the non-split cases, we want to present a variant which does not use a main conjecture but is less general.

Recall that we are now in a situation where p is odd and inert in k, and χ is a non-trivial character with $\chi(G_\mathfrak{p}) = 1$.

Lemma 5.2.21. *Assume* $\mathrm{ord}_w(\kappa_{\gamma_2}) \neq 0$ *then we have that*

$$\{e_\chi \varepsilon_L^V, e_\chi \kappa_{\gamma_2}\}$$

is a $\mathbb{C}_p$*-basis of* $e_\chi \mathbb{C}_p U_{L,S}$.

Proof. We set $S' := \{\infty, \mathfrak{p}\}$. First of all with the Dirichlet regulator map and the definition of $Y_{L,S}$ we see that:

$$e_\chi \mathbb{C}_p U_{L,S} \cong e_\chi \mathbb{C}_p X_{L,S} \cong e_\chi \mathbb{C}_p Y_{L,S} \cong e_\chi \mathbb{C}_p \bigoplus_{v\in S} \mathbb{Z}_p[G/G_v]$$
$$\cong e_\chi \mathbb{C}_p \bigoplus_{v\in S'} \mathbb{Z}_p[G/G_v] \cong e_\chi \mathbb{C}_p Y_{L,S'} \cong e_\chi \mathbb{C}_p X_{L,S'} \cong e_\chi \mathbb{C}_p U_{L,S'},$$

where we used that by assumption no prime in $S \setminus \{\infty, \mathfrak{p}\}$ splits completely in L/k.

Now we know that $\dim_{\mathbb{C}_p}(e_\chi \mathbb{C}_p U_{L,S'}) = 2$ and so $e_\chi \mathbb{C}_p U_{L,S}$ has $\mathbb{C}_p$-dimension 2. As $e_\chi \kappa_{\gamma_2}, e_\chi \varepsilon_L^V \neq 0$ it is left to show that $e_\chi \kappa_{\gamma_2}$ and $e_\chi \varepsilon_L^V$ are linearly independent. To show this assume that they are linearly dependent, i.e. that there exists an $a \in \mathbb{C}_p^\times$ such that $e_\chi \kappa_{\gamma_2} = (e_\chi \varepsilon_L^V)^a$. But $\mathrm{ord}_w(\kappa_{\gamma_2}) \neq 0$ by assumption and $\mathrm{ord}_w(\varepsilon_L^V) = 0$, which is a contradiction. □

Now the non-vanishing of $L_{S\setminus\{\mathfrak{p}\}}(1,\chi)$ (by the functional equation linked to the definition of the Rubin-Stark elements) implies that

$$\log_p(\varepsilon_L^V) \neq 0. \tag{5.9}$$

Indeed, because ε_L^V comes from the element $\psi_{k(\mathfrak{f})}$ it is a unit away from $\mathfrak{f}$ (which is coprime to $\mathfrak{p}$), so the embedding of ε_L^V in $k_\mathfrak{p}$ has no factor π^n, where π is a uniformizer in $\mathfrak{p}\mathcal{O}_{k_\mathfrak{p}}$. It also is not some root of unity ζ because that would contradict the non-vanishing. Because $\log_p(a) = 0$ if and only if $a \in \pi^\mathbb{Z} \times \mu$, where μ are the roots of unity, the assertion follows.

In particular, we directly see from Equation (5.9) and Theorem 4.1.15 that this strategy used here in the non-split case can also be used in the split case to show (DimAss) without using the Iwasawa main conjecture.

So from Equation (5.9) we directly get the following

Lemma 5.2.22. *Let γ_a be a fixed element of Γ'. Then the following assertions are equivalent:*

i) $\mathrm{ord}_\mathfrak{P}(\kappa_{\gamma_b}) \neq 0$ *for all γ_b such that $\langle \gamma_b, \gamma_a \rangle = \Gamma'$,*

ii) $\mathrm{ord}_\mathfrak{P}(\kappa_{\gamma_b}) \neq 0$ *for one γ_b with $\langle \gamma_b, \gamma_a \rangle = \Gamma'$,*

iii) $\mathbb{Q}_p \log_p(\varepsilon_L^V) \neq \mathbb{Q}_p \log_p(\chi_{\mathrm{ell}}(\gamma_a))$.

Therefore, all $(K'_\infty)^{\langle \gamma_a \rangle}$ have to be excluded for which we have

$$\mathbb{Q}_p \log_p(\varepsilon_L^V) = \mathbb{Q}_p \log_p(\chi_{\mathrm{ell}}(\gamma_a)).$$

Lemma 5.2.23. *We have*

$$\mathbb{Q}_p \log_p(\chi_{\mathrm{ell}}(\gamma_a)) = \mathbb{Q}_p \log_p(\chi_{\mathrm{ell}}(\gamma'_a)) \Leftrightarrow (K'_\infty)^{\langle \gamma_a \rangle} = (K'_\infty)^{\langle \gamma'_a \rangle}.$$

Proof. For the direction from left to right, let $up^r \log_p(\chi_{\mathrm{ell}}(\gamma_a)) = vp^t \log_p(\chi_{\mathrm{ell}}(\gamma'_a))$ with u, v in $\mathbb{Z}_p^\times$. Without loss of generality we assume $t \geq r$ so we get

$$\log_p(\chi_{\mathrm{ell}}(\gamma_a)) = \frac{v}{u} p^{t-r} \log_p(\chi_{\mathrm{ell}}(\gamma'_a)).$$

If now $d := t - s > 0$ we complete to bases γ_b, γ_a and γ'_b, γ'_a. So with the notation from above we obtain

$$\begin{pmatrix} \omega_b \\ \omega_a \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & \frac{v}{u} p^d \end{pmatrix} \begin{pmatrix} \omega'_b \\ \omega'_a \end{pmatrix},$$

which is a contradiction because both are $\mathbb{Z}_p$-bases of $\mathfrak{p}^s$. Therefore, we obtain

$$\log_p(\chi_{\mathrm{ell}}(\gamma_a)) = \log_p(\chi_{\mathrm{ell}}((\gamma_a')^{u/v}))$$

and so $\langle \gamma_a \rangle = \langle (\gamma_a')^u \rangle = \langle \gamma_a' \rangle$.

For the other direction we assume that $(K_\infty')^{\langle \gamma_a \rangle} = (K_\infty')^{\langle \gamma_a' \rangle}$ which is equivalent to $\gamma_a = (\gamma_a')^u$ with $u \in \mathbb{Z}_p^\times$ and therefore we get

$$\log_p(\chi_{\mathrm{ell}}(\gamma_a)) = u \log_p(\chi_{\mathrm{ell}}(\gamma_a')).$$

□

This finishes the proof of Theorem 5.2.3 and therefore this chapter. □

Chapter 6

Application to the eTNC for abelian extensions of imaginary quadratic fields

In this chapter we want to present applications of the main results of Chapter 4 to the equivariant Tamagawa Number Conjecture. A successful strategy for proving the eTNC for Tate motives over abelian fields was developed in [BG03] and further explained in [Fla04]. It can be summarized as follows: One uses the Iwasawa Main Conjecture for cyclotomic fields (originally proved by Mazur and Wiles [MW84] and later reproved using the Euler-system-method by Rubin and Greither (see e.g. in [Rub00] or [Gre92])) to prove an equivariant version of the Iwasawa Main Conjecture, which we call *Limit Conjecture*. In the proof of the Limit Conjecture a classical result of Ferrero and Washington ([FW79]) concerning the μ-invariant of the cyclotomic $\mathbb{Z}_p$-extension is used. Because of the Theorem of Kronecker-Weber for each absolutely abelian field L there exists a natural number $f \in \mathbb{N}$ such that $L \subseteq \mathbb{Q}(\zeta_f)$ and by functoriality properties of the eTNC it is enough to consider the cyclotomic field. So for a fixed prime p we use the Limit Conjecture for the infinite extension $\mathbb{Q}(\zeta_{fp^\infty})$ over $\mathbb{Q}$.

Now one descends from this Limit Conjecture to the finite extension by 'taking coinvariants'. A main feature of this descent is that it can be done characterwise, where in the cyclotomic case we have to consider odd and even characters. For odd characters a result of Ferrero and Greenberg ([FG79]) is used for the cases where the associated p-adic L-function has trivial zeroes. For even characters, a result of Solomon ([Sol92]), namely Theorem 4.1.14 in Chapter 4, is used. Using all of this an unconditional proof of the eTNC for Tate motives over abelian fields was established after some problems at prime 2 had been resolved in [Fla11].

Using the methods developed for the cyclotomic case Bley (in [Ble06]) was able to show the p-part of the eTNC for 'untwisted' Tate motives over abelian extensions L of an imaginary quadratic field k for all odd p which split in k and $p \nmid h_k$. This was done by using the main conjecture for split primes proved by Bley also in [Ble06] with the help of the Euler-system-method. Then, as in the cyclotomic case, one proves a Limit Conjecture for the tower of fields $\bigcup_n k(\mathfrak{p}^{n+1}\mathfrak{f}_L)$, where $\mathfrak{p}$ is a prime ideal in $\mathcal{O}_k$ over a rational prime p which splits in k. In this proof the vanishing of the μ-invariant of a certain Iwasawa module is used, which holds true in the split case by a result of Gillard [Gil85] for $p \neq 2, 3$ and by a result in [OV16] for $p = 2, 3$. By class field theory and functoriality of the eTNC it is again enough to consider ray class fields.

Once again one can do now the descent computations characterwise, where in this case all the characters behave like even characters in the cyclotomic case. In the descent an analogue of the result of Solomon in the elliptic case for split primes is used. The latter was proved by Bley in [Ble04] and is our Theorem 4.1.15.

In 2011 a paper of Johnson-Leung and Kings was published ([JLK11]) which presents a result about a variant of the Limit Conjecture stated by Flach as Conjecture 8 in [Fla09] for all p in the situation of imaginary quadratic base fields. Moreover, they show that under the assumption that a certain cohomology module vanishes after localization at singular primes (see Section 6.3.3), their variant of the Limit Conjecture holds for all primes p. These Limit Conjectures are analogues of the known Limit Conjectures in the cyclotomic and the split imaginary quadratic case.

Both situations presented above have in common that they use special explicit 'units' (cyclotomic or elliptic) which are related to torsion points of a 'geometric object'. These, in turn, are related to L-functions by classical results computing $L(1, \chi)$ with the help of cyclotomic units or applications of the first and second Kronecker limit formula (Corollary 2.3.7 and Proposition 2.3.9). As we have seen in Chapter 5, these ideas can be generalized to define Rubin-Stark elements.

By using these Rubin-Stark elements Burns, Kurihara and Sano (mainly in [BKS17], but see also [BKS16]) generalized, inspired by the known cases, the descent computations to establish a framework to prove the eTNC for 'untwisted' Tate motives over an abelian extension L/K of number fields supposing several rather strong assumptions.

One of the main insights of [BKS17] was that an Iwasawa-theoretic version of a conjecture of Mazur, Rubin and Sano (which is presented in Chapter 5 as Conjecture 5.1.14) can be seen as the crucial input into the descent, aside from what they call *higher-rank main conjecture of Iwasawa theory*. The other computations generalize rather easily to Rubin-Stark elements, although some additional assumptions have to be imposed. To demonstrate the usefulness of their work, they show the eTNC for Tate motives at $s = 0$ over abelian extensions of $\mathbb{Q}$ by proving the Iwasawa-theoretic MRS conjecture, where the main input is the result of Solomon (Theorem 4.1.14), and then applying their descent machinery. It is also remarked [BKS17, Remark 5.10] that similarly the main result of [Ble06] can be reproved.

The main results of this chapter are as follows:

- We outline a proof of the p-part of the eTNC for Tate motives at $s = 0$ over abelian extensions of k for split primes with $p \nmid h_k$ using the techniques developed in [BKS17]. Moreover, we present the consequences of Theorems 5.2.2 and 5.2.3 when using these techniques.

- We show a result concerning the p-part of the eTNC for inert primes (Theorem 6.5.1) by using the new results obtained in Chapter 4 directly, i.e. that appropriate Limit Conjectures imply the eTNC for Tate motives at $s = 0$ over abelian extensions of an imaginary quadratic number field k assuming a finiteness condition described in Section 6.3.4 and $p \nmid h_k$.

Unfortunately, we are not able to obtain any unconditional results for the p-part of the eTNC for non-split primes. One major obstacle is the lack of a result similar to that of Gillard in [Gil85] on the μ-invariant in the non-split case. A substitute for this is Conjecture 6.3.13 (stated in [JLK11]), which we call (VanishAss). In the split case the result of Gillard implies this conjecture which is Corollary 5.12 in [JLK11].

Another obstruction is that one has to prove that the Limit Conjecture for an appropriate rank two extension (Conjecture 6.3.2) implies the Limit Conjecture for particular $\mathbb{Z}_p$-subextensions (Conjecture 6.3.3). This problem looks approachable, however we are at the moment not able to prove that, so we formulated Conjecture 6.3.5.

The third obstruction is a finiteness condition we are going to call Condition $\mathrm{F}(L_\infty/k)$ for rank one $\mathbb{Z}_p$-extensions L_∞/k as described in Section 6.3.4, which is slightly stronger than the homonymous condition assumed in the main result of [BKS17]. This is used to compute the cohomology modules of the complex in question after localizing at height one prime ideals. It is a sufficient condition which delivers what we need in the computations. Moreover, we have for each character a certain choice for the $\mathbb{Z}_p$-extension we want to work with and this is necessary because Condition $\mathrm{F}(L_\infty/k)$ is known to fail in certain examples. But with a little more work one should be able to replace Condition $\mathrm{F}(L_\infty/k)$ with a weaker hypothesis which is conjecturally always true (see Remark 6.5.3 for more details on this topic).

Overview of this chapter First of all, we recall some prerequisites which are needed to formulate the eTNC for Tate motives. This is largely based on the treatment in [BG03].

Then we will formulate several versions of the eTNC for Tate motives at $s = 0$ over abelian extensions of number fields.

The third section formulates the Limit Conjectures of Johnson-Leung/Kings and of Flach and shows a relation between those two. It formulates Limit Conjectures for some special $\mathbb{Z}_p$-extensions which are used in the next sections. Furthermore, it introduces the higher-rank main conjecture of Iwasawa theory from [BKS17], which is the starting point of the descent in loc. cit.

The fourth section presents the main theorem (Theorem 5.2) of [BKS17] and how it is connected with the main results of Chapter 5.

The last section is concerned with the formulation and the proof of the main result (Theorem 6.5.1) of this chapter.

6.1 Preliminaries

This section will state some basic definitions and results and follows very closely the treatment in [BG03].

Let R be any commutative ring and let $\mathcal{P}(R)$ denote the category of graded invertible R-modules, where a graded invertible R-module is a pair (L, α) consisting of an invertible R-module L and a locally constant function $\alpha : \mathrm{Spec}(R) \to \mathbb{Z}$ and isomorphisms of such. The tensor product is defined as

$$(L, \alpha) \otimes (M, \beta) := (L \otimes_R M, \alpha + \beta)$$

and the unit object is $(R, 0)$. We define $(L, \alpha)^{-1} := (\mathrm{Hom}_R(L, R), -\alpha)$ and we extend this definition to a covariant functor $\mathcal{P}(R) \to \mathcal{P}(R)$ by setting $h^{-1} := \mathrm{Hom}_R(h, R)^{-1}$.

Definition 6.1.1. For a finitely generated projective R-module P we define

$$\mathrm{Det}_R(P) := \left(\bigwedge_R^{\mathrm{rank}_R P} P, \mathrm{rank}_R P \right) \in \mathrm{Ob}(\mathcal{P}(R))$$

and for a bounded complex $P^\bullet$ of such modules we set

$$\mathrm{Det}_R(P^\bullet) := \bigotimes_{i \in \mathbb{Z}} \mathrm{Det}_R^{(-1)^i}(P^i),$$

where the tensor product is of graded invertible R-modules and the sign convention is different from [BG03], but consistent with [Fla09] and [JLK11]. We sometimes write $\mathrm{Det}_R^{-1}(-)$ in place of $\mathrm{Det}_R(-)^{-1}$.

We write $D^p(R)$ for the category of perfect complexes of R-modules and $D^{pis}(R)$ for the subcategory in which the objects are the same but the morphisms are restricted to quasi-isomorphism.

We say that an R-module is *perfect* if the associated complex $X[-1]$ belongs to $D^p(R)$, and for such X we set

$$\mathrm{Det}_R(X) := \mathrm{Det}_R(X[-1]).$$

In the next proposition we recall some standard properties (see [KM76] or [BG03]) of the determinant functor:

Proposition 6.1.2. *a) If R is reduced, then Det_R extends to a functor from $D^{pis}(R)$ to $\mathcal{P}(R)$ in such a way that for every distinguished triangle*

$$C_1 \to C_2 \to C_3 \text{ in } D^p(R)$$

there is an isomorphism in $\mathcal{P}(R)$

$$\mathrm{Det}_R^{-1}(C_1) \otimes \mathrm{Det}_R(C_2) \cong \mathrm{Det}_R(C_3),$$

which is functorial in the triangle.

b) If C is bounded and each cohomology module is perfect, then C belongs to $D^p(R)$ and there is a canonical isomorphism

$$\mathrm{Det}_R(C) \cong \bigotimes_{i\in\mathbb{Z}} \mathrm{Det}_R^{(-1)^i}(H^i(C)).$$

In particular, if C is acyclic, this gives a canonical isomorphism in $\mathcal{P}(R)$

$$\mathrm{Det}_R(C) \cong (R, 0).$$

c) If X is a finitely generated torsion R-module which has projective dimension at most one, then its first Fitting ideal $\mathrm{Fitt}_R(X)$ is an invertible ideal of R and one has

$$\mathrm{Det}_R(X) = (\mathrm{Fitt}_R(X), 0).$$

In particular, if

$$0 \to \cdots \to X^i \to X^{i+1} \to \cdots \to 0$$

is any bounded exact sequence of finitely generated torsion R-modules which are each of projective dimension at most one, then one has an equality

$$\prod_{i\in\mathbb{Z}} \mathrm{Fitt}_R^{(-1)^i}(X^i) = R.$$

Let G be any finite abelian group. For any commutative ring Z we write $x \mapsto x^\#$ for the Z-linear involution of the group ring $Z[G]$ which satisfies $g^\# = g^{-1}$ for each $g \in G$. If X is any (complex of) $Z[G]$-module(s), then we write $X^\#$ for the scalar extension of X with respect to the morphism $x \mapsto x^\#$.

Let Z be any commutative ring. For any finitely generated projective $Z[G]$-module X, resp. object X in $D^p(Z[G])$, we set

$$X^* := \mathrm{Hom}_Z(X, Z), \text{ resp. } X^* := \mathrm{RHom}_Z(X, Z)$$

which we regard as endowed with the contragredient G-action. We observe that if X is a finitely generated projective $Z[G]$-module, resp. an object X in $D^p(Z[G])$, then so is X^*. We also recall that for any $Z[G]$-module X one has a canonical isomorphism

$$X^* \cong \mathrm{Hom}_{Z[G]}(X, Z[G])^\#$$

and that this induces for each object X of $D^p(Z[G])$ a canonical isomorphism

$$\mathrm{Det}_{Z[G]}X^* \cong \mathrm{Det}_{Z[G]}^{-1}X^{\#}.$$

Assume R is regular. For any isomorphism of finitely generated R-modules $V \xrightarrow{\alpha} W$ we let

$$\alpha_{triv} : \mathrm{Det}_R(V) \otimes \mathrm{Det}_R^{-1}(W) \xrightarrow{\cong} (R, 0)$$

denote the isomorphism in $\mathcal{P}(R)$ which is obtained by composing $\mathrm{Det}_R(\alpha) \otimes 1$ with the natural evaluation pairing $\mathrm{eval} : \mathrm{Det}_R(W) \otimes \mathrm{Det}_R^{-1}(W) \xrightarrow{\cong} (R, 0)$.

We now fix an abelian extension L/k of number fields with Galois group G, an odd prime p and a set of places S of k which contains S_∞, $S_{ram}(L/k)$ and $S_p(k)$ and by abuse of notation we write also S for all the places in L lying above the places in S. So we write

$$\pi : \mathrm{Spec}(\mathcal{O}_{L,S}) \to \mathrm{Spec}(\mathcal{O}_{k,S})$$

for the morphism of spectra which is associated with $\mathcal{O}_{k,S} \subseteq \mathcal{O}_{L,S}$.

For any commutative ring Z and any étale sheaf $\mathcal{F}$ on $\mathrm{Spec}(Z)$ we write $R\Gamma(Z, \mathcal{F})$ and $H^i(Z, \mathcal{F})$ instead of

$$R\Gamma(\mathrm{Spec}(Z)_{\acute{e}t}, \mathcal{F}) \text{ and } H^i(R\Gamma(\mathrm{Spec}(Z)_{\acute{e}t}, \mathcal{F})).$$

If $\mathcal{F}$ is any (p-adic) étale sheaf on $\mathrm{Spec}(\mathcal{O}_{k,S})$, then the cohomology with compact support is defined to lie in a canonical distinguished triangle

$$R\Gamma_c(\mathcal{O}_{k,S}, \mathcal{F}) \to R\Gamma(\mathcal{O}_{k,S}, \mathcal{F}) \to \bigoplus_{v \in S} R\Gamma(k_v, \mathcal{F}).$$

We recall that π_* is exact and hence that there is a canonical identification

$$R\Gamma_?(\mathcal{O}_{k,S}, \mathcal{F}_L) \cong R\Gamma_?(\mathcal{O}_{L,S}, \pi^*\mathcal{F}),$$

where $R\Gamma_?(-,-)$ denotes either $R\Gamma(-,-)$ or $R\Gamma_c(-,-)$ and we set $\mathcal{F}_L := \pi_*\pi^*\mathcal{F}$.

Let $\Sigma(k)$ bet the set of embeddings of k into $\mathbb{C}$ and for any $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-module X we write X^+, resp. X^-, for the submodule of X on which the non-trivial element $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts as multiplication by 1, resp. -1.

If T_p is any lisse $\mathbb{Z}_p$-sheaf of $\mathbb{Z}_p[G]$-modules on $\mathrm{Spec}(\mathcal{O}_{k,S})_{\acute{e}t}$ for which each stalk is projective over $\mathbb{Z}_p[G]$, then the complexes

$$R\Gamma_c(\mathcal{O}_{k,S}, T_p), \quad R\Gamma(\mathcal{O}_{k,S}, T_p^*(1))^* \text{ and } \left(\prod_{\Sigma(k)} T_p\right)^+ [0]$$

belong to $D^p(\mathbb{Z}_p[G])$, which is shown in [Fla00, Prop. 5.2].

Artin-Verdier duality on the level of complexes combined with Lemma 16 in [BF98] induces a canonical distinguished triangle in $D^p(\mathbb{Z}_p[G])$:

$$R\Gamma_c(\mathcal{O}_{k,S}, T_p) \to R\Gamma(\mathcal{O}_{k,S}, T_p^*(1))^*[-3] \to \left(\prod_{\Sigma(k)} T_p\right)^+ [0].$$

Proposition 6.1.3. *[BG03, Lemma 3.2]*

a) $R\Gamma(\mathcal{O}_{k,S}, \mathbb{Z}_p(1)_L)$ *is acyclic outside of degrees* 1 *and* 2.

b) There is a canonical isomorphism of $\mathbb{Z}_p[G]$-modules

$$\iota_L^1 : H^1(\mathcal{O}_{k,S}, \mathbb{Z}_p(1)_L) \cong \mathcal{O}_{L,S}^{\times} \otimes \mathbb{Z}_p.$$

c) We also have a short exact sequence

$$0 \to \mathrm{Pic}(\mathcal{O}_{L,S}) \otimes \mathbb{Z}_p \to H^2(\mathcal{O}_{k,S}, \mathbb{Z}_p(1)_L) \to X_{L,S\setminus S_\infty} \otimes \mathbb{Z}_p \to 0.$$

where $X_{L,S\setminus S_\infty}$ is defined in Definition 6.2.2.

Lemma 6.1.4. *[Fla04, Lemma 5.3] Let R be a Noetherian Cohen-Macaulay ring with total ring of fractions $Q(R)$. Suppose R is a finite product of local rings. If I and J are invertible R-submodules of some invertible $Q(R)$-module M, then $I = J$ if and only if $I_\mathfrak{q} = J_\mathfrak{q}$ (inside $M_\mathfrak{q}$) for all height one prime ideals $\mathfrak{q}$ of R.*

6.2 Formulation of the eTNC for Tate motives at $s = 0$

We fix a number field k and let L be an abelian extension of k with Galois group G. We set

$$M = h^0(\mathrm{Spec}(L))(0) \qquad A = \mathbb{Q}[G] \qquad \mathcal{A} = \mathbb{Z}[G],$$

where $h^0(\mathrm{Spec}(L))(0)$ is an *'untwisted' Tate motive* and we will suppress (0) from now on. Furthermore, we set ${}_AM$ as M considered over k with an action of A and A_p (resp. $\mathcal{A}_p$) as $A \otimes \mathbb{Q}_p$ (resp. $\mathcal{A} \otimes \mathbb{Z}_p$).

We define a $G \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-module

$$Y_0(L) := \prod_{\Sigma(L)} \mathbb{Z},$$

where $\Sigma(L)$ is the set of embeddings of L into $\mathbb{C}$ and $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\Sigma(L)$.

Definition 6.2.1. For the *fundamental line* one sets

$$\Xi({}_AM) := \mathrm{Det}_A^{-1}(\mathcal{O}_L^{\times} \otimes \mathbb{Q})^{\#} \otimes \mathrm{Det}_A(\mathbb{Q}) \otimes \mathrm{Det}_A(Y_0(L)^+ \otimes \mathbb{Q})^{\#}.$$

Definition 6.2.2. For any set S of places of k we define

$$Y_{L,S} = Y_S(L) := \bigoplus_{v \in S(L)} \mathbb{Z},$$

and $X_{L,S} = X_S(L)$ to be the kernel of the homomorphism $Y_S \to \mathbb{Z}$ which sends each place w in S to 1. Observe that both $Y_{L,S}$ and $X_{L,S}$ have natural G-actions.

Definition of ${}_A\vartheta_\infty$ The space $Y_0^+(L) \otimes \mathbb{Q}$ identifies naturally with $Y_{L,S_\infty} \otimes \mathbb{Q}$, hence there is a canonical isomorphism

$$\iota_L : \mathrm{Det}_A(\mathbb{Q}) \otimes \mathrm{Det}_A(Y_0^+(L) \otimes \mathbb{Q}) \cong \mathrm{Det}_A(X_{L,S_\infty} \otimes \mathbb{Q}).$$

For each place w in L we let $|\cdot|_w$ denote the absolute value with respect to w. We write

$$\mathrm{R}_L : \mathcal{O}_L^\times \otimes \mathbb{R} \xrightarrow{\cong} X_{L,S_\infty} \otimes \mathbb{R}$$

for the $\mathbb{R}[G]$-equivariant isomorphism which satisfies

$$\mathrm{R}_L(u) = -\sum_{w \in S_\infty(L)} \log |u|_w \cdot w \quad \text{for each } u \in \mathcal{O}_L^\times.$$

Definition 6.2.3. We let

$${}_A\vartheta_\infty : (\mathbb{R}[G], 0) \xrightarrow{\cong} \Xi({}_AM)^\# \otimes \mathbb{R}$$

denote the isomorphism induced by $(\mathrm{R}_L)_{triv}$ and ι_L.

The motivic L-function We set $\widehat{G} := \mathrm{Hom}(G, \mathbb{C}^\times)$. For each set S of places of k which contains S_∞ and each character $\chi \in \widehat{G}$ we let $L_S(s,\chi)$ denote the generalized Dirichlet S-imprimitive L-function defined in Chapter 2.

We write $L_S({}_AM, s)$ for the S-truncated $\mathbb{C}[G]$-valued L-function for the motive ${}_AM$ as defined in [BF96, Def. 2.1] or [Del79]. Then, with respect to the canonical identification $\mathbb{C}[G] \cong \prod_{\widehat{G}} \mathbb{C}$, one has an equality of functions (cf. [BF96, Lemma 2.2])

$$L_S({}_AM, s) = (L_S(s,\chi))_{\chi \in \widehat{G}}.$$

We set $L({}_AM, s) := L_{S_\infty}({}_AM, s)$ and $L(s,\chi) := L_{S_\infty}(s,\chi)$. We also write $L_S^*(0,\chi)$ resp. $L^*({}_AM, 0)$ for the leading term in the Laurent expansion of $L_S(s,\chi)$ resp. $L({}_AM, s)$ at $s = 0$. Moreover, we recall (see [BF96, Def. 2.1]) that

$$L^*({}_AM, 0) \in (A \otimes \mathbb{R})^\times.$$

Let $r({}_AM) := \mathrm{ord}_{s=0}(L({}_AM, s)) \in H^0(\mathrm{Spec}(A \otimes \mathbb{R}), \mathbb{Z})$, i.e. a locally constant function on $\mathrm{Spec}(A \otimes \mathbb{R})$.

Definition of ${}_A\vartheta_p$ For each prime p, let M_p be the p-adic realization of M. There exists a canonical $\mathbb{Q}_p[G]$-equivariant isomorphism

$${}_A\vartheta_p : \Xi({}_AM)^\# \otimes \mathbb{Q}_p \xrightarrow{\cong} \mathrm{Det}_{A_p}\left(R\Gamma_c(\mathcal{O}_{k,S}, M_p)\right)^\#,$$

which is the isomorphism in (4) in [BF98, Sec. 2], which we will describe in more detail later in the special case we are going to consider. So we can give the formulation of the eTNC for Tate motives at $s = 0$ (cf. [BF96, Conj. 4]):

Conjecture 6.2.4. eTNC$({}_AM, \mathcal{A}, p)$
The following statements hold for the triple $({}_AM, \mathcal{A}, p)$:

Rationality: *One has*

$${}_A\vartheta_\infty(L^*({}_AM, 0)^{-1}) \in \Xi({}_AM) \otimes 1.$$

Integrality: *Let S be any finite set of k which contains S_∞, $S_{ram}(L/k)$ and $S_p(k)$, and let T_p denote any projective $G_\mathbb{Q}$-stable $\mathcal{A}_p$-lattice in M_p. Then one has an equality:*

$${}_A\vartheta_p({}_A\vartheta_\infty(L^*({}_AM, 0)^{-1})) \cdot \mathcal{A}_p = \mathrm{Det}_{\mathcal{A}_p}(R\Gamma_c(\mathcal{O}_{k,S}, T_p)).$$

Remark 6.2.5. a) Rationality and integrality part of Conjecture 6.2.4 are Conjectures 2 and 3 (equivariant version) in [Fla04] in our situation, which can also be found as Conjecture 4 iii) and iv) in [BF96]. The usage of $L^*({}_AM, 0)$ instead of $L^*({}_AM, 0)^{-1}$ in the formulation in [BF96] stems from the fact that a different sign convention is used for the definition of the determinant of a complex.

b) In order to be a well posed conjecture two additional conditions have to be satisfied which are Conjecture 4 i) and ii) in [BF96]:

   i) The function $L({}_AM, s)$ can be analytically continued to $s = 0$, and

   ii) $r({}_AM) = \dim_A(H^1_f(M^*(1))) - \dim_A(H^0_f(M^*(1)))$.

   These are satisfied in the case we are looking at because i) is a classical result and ii) is proved in [Fla04, Thm. 7.1 a)] or in [BF03].

c) There are also more general formulations of the eTNC. One of them also allows non-commutative coefficients. This formulation can be found in [BF01] as Conjecture 4 iii) and iv) and from the arguments there one can see that the formulation specializes to our formulation when considering commutative coefficients.

d) Even for the more general situation we mention in c) it is shown as Theorem 2.2.4 in [Bur01] that Stark's conjecture is equivalent to the rationality part of Conjecture 6.2.4.

e) The equality of the integrality part of Conjecture 6.2.4 behaves functorially with respect to change of extension. If it is valid for a prime p for any given extension L/k, then it is also valid for p for any extension F/E with $k \subseteq E \subseteq F \subseteq L$. This follows directly from Proposition 4.1 in [BF01].

f) Lemma 5 in [BF01] implies that the conjecture is independent of the choice of S and T_p as long as the choice is admissible. Example b) on page 524 in [BF01] guarantees that such a T_p exists in our situation.

g) In Section 4.3 in [BF01] it is argued that the validity of the integrality part of Conjecture 6.2.4 for all p is equivalent to Conjecture iv) of [BF01] in the case in question.

Preparations for explicit computations Recall that L/k is a finite abelian extension of number fields. We set here $S = S_{ram}(L/k) \cup S_\infty(k) \cup S_p(k)$, $S_f := S \setminus S_\infty(k)$. By abuse of notation we denote the set of places above the places of S resp. S_f in an extension of k also by S resp. S_f. We define

$$\Delta(L) := \mathrm{RHom}_{\mathbb{Z}_p}\left(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p), \mathbb{Z}_p\right)[-3] \text{ as in [Ble06].} \tag{6.1}$$

Then the following holds:

a) $\Delta(L)$ is a perfect complex of $\mathcal{A}_p$-modules.

b) There is a natural isomorphism

$$\mathrm{Det}_{\mathcal{A}_p}(\Delta(L)) \cong \mathrm{Det}_{\mathcal{A}_p}(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p))^{\#} \tag{6.2}$$

and with Tate-Poitou duality, the Kummer sequence and ideas from [BF98, Prop. 3.3]

c) one has

$$H^i(\Delta(L)) = 0 \text{ for } i \neq 1, 2,$$

a canonical isomorphism

$$H^1(\Delta(L)) \cong H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1)) \cong \mathcal{O}_{L,S}^\times \otimes \mathbb{Z}_p =: U_{L,S}$$

and a short exact sequence

$$0 \to \mathrm{Pic}(\mathcal{O}_{L,S}) \otimes \mathbb{Z}_p \to H^2(\Delta(L)) \to X_{L,S} \otimes \mathbb{Z}_p \to 0.$$

Definition 6.2.6. [BF98, cf. Lemma 2] For every $\chi \in \widehat{G}$ and every finite place v of k we introduce the following notation: For $x \in \mathbb{Q}^\times$, resp. $a \in A$, let ${}^*x \in A^\times$, resp. $a^* \in A$, be the element such that $\chi({}^*x) = x$, resp. $\chi(a^*) = 0$, if $\chi(G_v) = 1$ and $\chi({}^*x) = 1$, resp. $\chi(a^*) = \chi(a)$ otherwise. Now set

$$\mathcal{E}_v := {}^*|G_v/I_v| \cdot (1 - Fr_v^*)^{-1} \in A^\times, \tag{6.3}$$

where $Fr_v \in \mathbb{Q}[G/I_v] \cong \mathbb{Q}[G]^{I_v} \subset A$ is the Frobenius automorphism at v and $I_v \subseteq G_v \subseteq G$ are the inertia and decomposition group for a place $w \mid v$ in L/k, respectively.

The isomorphism

$${}_A\vartheta_p : \Xi({}_AM)^\# \otimes \mathbb{Q}_p \to \mathrm{Det}_{A_p}(\Delta(L) \otimes \mathbb{Q}_p)$$

is given by the composite

$$\begin{aligned}
&\mathrm{Det}_{A_p}^{-1}(\mathcal{O}_L^\times \otimes_\mathbb{Z} \mathbb{Q}_p) \otimes \mathrm{Det}_{A_p}(X_{S_\infty} \otimes_\mathbb{Z} \mathbb{Q}_p) && (6.4)\\
\xrightarrow{\varphi_1}&\mathrm{Det}_{A_p}^{-1}(\mathcal{O}_{L,S}^\times \otimes_\mathbb{Z} \mathbb{Q}_p) \otimes \mathrm{Det}_{A_p}(X_S \otimes_\mathbb{Z} \mathbb{Q}_p) && (6.5)\\
\xrightarrow{\varphi_2}&\mathrm{Det}_{A_p}^{-1}(\mathcal{O}_{L,S}^\times \otimes_\mathbb{Z} \mathbb{Q}_p) \otimes \mathrm{Det}_{A_p}(X_S \otimes_\mathbb{Z} \mathbb{Q}_p) && (6.6)\\
\xrightarrow{\varphi_3}&\mathrm{Det}_{A_p}(\Delta(L) \otimes \mathbb{Q}_p), && (6.7)
\end{aligned}$$

where we have the following maps: φ_1 is induced by the split short exact sequences

$$0 \to \mathcal{O}_L^\times \otimes \mathbb{Q}_p \to \mathcal{O}_{L,S}^\times \otimes \mathbb{Q}_p \to Y_{L,S_f} \otimes \mathbb{Q}_p \to 0$$
$$0 \to X_{L,S_\infty} \otimes \mathbb{Q}_p \to X_{L,S} \otimes \mathbb{Q}_p \to Y_{L,S_f} \otimes \mathbb{Q}_p \to 0.$$

The isomorphism φ_2 is the multiplication with the Euler factor $\prod_{v \in S_f} \mathcal{E}_v^\# \in A^\times$, where we have

$$\mathcal{E}_v = \sum_{\chi(G_v)=1} |G_v/I_v| e_\chi + \sum_{\chi(G_v)\neq 1} (1 - \chi(f_v)^{-1}) e_\chi, \tag{6.8}$$

where $f_v \in G_v$ denotes a lift of the Frobenius element in G_v/I_v and we use the ring isomorphism $A \cong \prod_{\chi \in \widehat{G}/\sim_\mathbb{Q}} \mathbb{Q}(\chi)$.

The isomorphism φ_3 arises from the explicit description of the cohomology groups of $H^i(\Delta(L))$ for $i = 1, 2$, (6.2) and the canonical isomorphism

$$\mathrm{Det}_{A_p}(\Delta(L) \otimes \mathbb{Q}_p) \cong \bigotimes_{i \in \mathbb{Z}} \mathrm{Det}_{A_p}^{(-1)^i}(H^i(\Delta(L) \otimes \mathbb{Q}_p)).$$

6.2.1 The eTNC for Tate motives at $s = 0$ in [BKS17]

In this subsection we present a formulation of the eTNC for Tate motives at $s = 0$ which is given in [BKS17]. The main motivation for that is that in loc. cit. they show how to descend from a 'higher-rank main conjecture of Iwasawa theory' to this formulation using the MRS conjecture for which we proved special cases in Chapter 5. Moreover, this is a neat way to present the conjecture. We partly adopt the notation of their paper by writing $\mathbb{C}_pM$ and $\mathbb{Z}_pM$ instead of $\mathbb{C}_p \otimes M$ and $\mathbb{Z}_p \otimes M$, respectively, for an appropriate module M.

Let L/k be a finite abelian extension of number fields with Galois group G. Fix a prime p and assume that $S_p(k) \cup S_\infty(k) \cup S_{ram}(L/k) \subset S$ and that T is disjoint from S. Consider the complex

$$C_{L,S} := R\mathrm{Hom}_{\mathbb{Z}_p}(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p), \mathbb{Z}_p)[-2]$$

i.e. the complex is shifted here only by -2 instead of -3 as we did above. A complex $C_{L,S,T}$ which lies in the distinguished triangle

$$C_{L,S,T} \to C_{L,S} \to \bigoplus_{w \in T(L)} \mathbb{Z}_p\kappa(w)^\times[0]$$

can be constructed as in Proposition 2.4 in [BKS16].

Remark 6.2.7. a) $C_{L,S}$ is a perfect complex of $\mathbb{Z}_p[G]$-modules, which is acyclic outside of degree zero and one.

b) We have a canonical isomorphism

$$H^0(C_{L,S,T}) \cong U_{L,S,T} := \mathbb{Z}_p\mathcal{O}^\times_{L,S,T}.$$

c) We have a canonical exact sequence

$$0 \to A_S^T(L) \to H^1(C_{L,S,T}) \to X_{L,S} \to 0,$$

where $A_S^T(L)$ is the p-part of the ray class group of $\mathcal{O}_{L,S}$ with modulus $\prod_{w \in T(L)} w$ and, by abuse of notation, $X_{L,S} := \mathbb{Z}_pX_{L,S}$.

We fix an isomorphism $\mathbb{C} \cong \mathbb{C}_p$ and we regard $\theta^*_{L/k,S,T}(0)$, which is the leading term of the Taylor expansion of the equivariant abelian L-function at $s = 0$ defined in (5.1) in Chapter 5, as an element of $\mathbb{C}_p[G]^\times$.

The *zeta element for* $\mathbb{G}_m$

$$z_{L/k,S,T} \in \mathbb{C}_p\mathrm{Det}_{\mathbb{Z}_p[G]}(C_{L,S,T})$$

is defined to be the element which corresponds to $\theta^*_{L/k,S,T}(0)$

$$\begin{aligned}
\mathbb{C}_p\mathrm{Det}_{\mathbb{Z}_p[G]}(C_{L,S,T}) &\xrightarrow{\cong} \mathrm{Det}_{\mathbb{C}_p[G]}(\mathbb{C}_pU_{L,S,T}) \otimes_{\mathbb{C}_p[G]} \mathrm{Det}^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_pX_{L,S}) \\
&\xrightarrow{\cong} \mathrm{Det}_{\mathbb{C}_p[G]}(\mathbb{C}_pX_{L,S}) \otimes_{\mathbb{C}_p[G]} \mathrm{Det}^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_pX_{L,S}) \\
&\xrightarrow{\cong} \mathbb{C}_p[G],
\end{aligned}$$

where the second isomorphism is induced by the regulator map $\lambda_{L,S}$ and the third isomorphism by the evaluation map.

Then the eTNC for the 'untwisted' Tate motive $h^0(\mathrm{Spec}(L))$ and with coefficients $\mathbb{Z}_p[G]$ can be formulated as follows

Conjecture 6.2.8. $\mathrm{eTNC}(h^0(\mathrm{Spec}(L)), \mathbb{Z}_p[G])$

$$\mathbb{Z}_p[G] \cdot z_{L/k,S,T} = \mathrm{Det}_{\mathbb{Z}_p[G]}(C_{L,S,T}).$$

Remark 6.2.9. One can show that as long as one takes admissible sets S and T, meaning S and T satisfy the conditions specified above, the validity of the conjecture is independent of the choice of S and T.

Take $\chi \in \widehat{G}$ and suppose that $r_S(\chi) < |S|$. We set $L_\chi := L^{\ker(\chi)}$ and $G_\chi := \mathrm{Gal}(L_\chi/k)$. Take $V_{\chi,S} \subset S$, such that all $v \in V_{\chi,S}$ split completely in L_χ (i.e. $\chi(G_v) = 1$) and $|V_{\chi,S}| = r_S(\chi)$. Note that if $\chi \neq 1$, we have $V_{\chi,S} = \{v \in S \mid \chi(G_v) = 1\}$.

As presented in [BKS17] one gets a non-canonical isomorphism

$$e_\chi \left(\mathrm{Det}_{\mathbb{C}_p[G]}(\mathbb{C}_p U_{L,S,T}) \otimes_{\mathbb{C}_p[G]} \mathrm{Det}^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p X_{L,S}) \right) \cong e_\chi \mathbb{C}_p \bigwedge^{r_S(\chi)} U_{L_\chi,S,T}$$

Proposition 6.2.10. *[BKS17, Prop. 2.5]*
Suppose $r_S(\chi) < |S|$ for every $\chi \in \widehat{G}$. Then $\mathrm{eTNC}(h^0(\mathrm{Spec}(L), \mathbb{Z}_p[G])$ holds if and only if there is a $\mathbb{Z}_p[G]$-basis $\mathcal{L}_{L/k,S,T}$ of $\mathrm{Det}_{\mathbb{C}_p[G]}(C_{L,S,T})$ such that, for every $\chi \in \widehat{G}$, the image of $e_\chi \mathcal{L}_{L/k,S,T}$ under the isomorphism

$$e_\chi \mathbb{C}_p \mathrm{Det}_{\mathbb{Z}_p[G]}(C_{L,S,T}) \cong e_\chi \left(\mathrm{Det}_{\mathbb{C}_p[G]}(\mathbb{C}_p U_{L,S,T}) \otimes_{\mathbb{C}_p[G]} \mathrm{Det}^{-1}_{\mathbb{C}_p[G]}(\mathbb{C}_p X_{L,S}) \right) \cong e_\chi \mathbb{C}_p \bigwedge^{r_S(\chi)} U_{L_\chi,S,T}$$

coincides with $e_\chi \epsilon^{V_{\chi,S}}_{L_\chi/k,S,T}$.

6.3 Limit Conjectures

In this section we want to present conjectures which are used as the basis for the descent procedure which can be viewed as IMC-type conjectures. In fact, in the cases where this conjecture is known, the main input are the main conjectures in the appropriate situations.

6.3.1 Limit Conjectures as in [Fla04], [Ble06] and [Fla09]

Let p be an odd rational prime and k be a number field. Furthermore, let F_∞/k be an abelian extension with $\mathbb{Z}_p$-rank d such that there is a finite (abelian) extension F of k with $\mathrm{Gal}(F_\infty/F) \cong \mathbb{Z}_p^d$ and a direct decomposition

$$\mathrm{Gal}(F_\infty/k) \cong \Gamma \times H \text{ with } H \cong \mathrm{Gal}(F/k) \text{ and } \Gamma := \mathrm{Gal}(F_\infty/F) \cong \mathbb{Z}_p^d.$$

Then we denote by

$$\Lambda_{F_\infty/k} := \varprojlim_{k \subseteq_f M \subset F_\infty} \mathbb{Z}_p[\mathrm{Gal}(M/k)] \cong \mathbb{Z}_p[H][[T_1, \dots, T_d]]$$

the completed group ring. The elements $T_i = \gamma_i - 1$ depend on the choice of topological generators $\gamma_1, \dots, \gamma_d$ of Γ.

We fix an embedding $\mathbb{Q}_p^c \hookrightarrow \mathbb{C}$ and identify $\mathrm{Hom}(H, \mathbb{C}^\times)$ with the $\mathbb{Q}_p^c$-valued characters. The total ring of fractions $Q(\Lambda_{F_\infty/k})$ of $\Lambda_{F_\infty/k}$ is the product of fields indexed by the $\mathbb{Q}_p^c$-valued characters of H which are associated with the set of $\mathbb{Q}_p$-irreducible representations of H, i.e.

$$Q(\Lambda_{F_\infty/k}) \cong \prod_{\alpha \in \widehat{H}/\sim_{\mathbb{Q}_p}} Q(\alpha).$$

Moreover, we work in the derived category $\mathcal{D}^p(\Lambda_{F_\infty/k})$ and define

$$\Delta_{F_\infty/k} := \varprojlim_{F\subseteq_f M\subset F_\infty} \Delta(M),$$

where we use the definition given in (6.1). Moreover, we set for an admissible set S

$$U_{F_\infty/k,S} := \varprojlim_{F\subseteq_f M\subset F_\infty} \mathcal{O}_{M,S}^\times \otimes \mathbb{Z}_p,$$

$$P_{F_\infty/k,S} := \varprojlim_{F\subseteq_f M\subset F_\infty} \operatorname{Pic}(\mathcal{O}_{M,S}) \otimes \mathbb{Z}_p,$$

$$X_{F_\infty/k,S} := \varprojlim_{F\subseteq_f M\subset F_\infty} X_S(M) \otimes \mathbb{Z}_p,$$

where the limit over the Picard groups is taken with respect to the norm maps and the transition maps for $X_{F_\infty/k,S}$ are defined by sending each place to its restriction.
From now on we assume that k is an imaginary quadratic field.

As in Proposition 5.1 in [BG03] one can show the following proposition

Proposition 6.3.1. *a) $\Delta_{F_\infty/k}$ is a perfect complex.*

b) $H^i(\Delta_{F_\infty/k}) = 0$ for $i \neq 1, 2$.

c) $H^1(\Delta_{F_\infty/k}) = U_{L_\infty/k,S}$.

d) There exists a short exact sequence

$$0 \to P_{F_\infty/k,S} \to H^2(\Delta_{F_\infty/k}) \to X_{F_\infty/k,S} \to 0.$$

Let $p > 3$ be a rational prime and $\mathfrak{f}$ be a non-zero ideal of $\mathcal{O}_k$ with $w(\mathfrak{f}) = 1$ and $(\mathfrak{f}, p) = 1$. We also assume the hypothesis

(H2) p does not divide the class number of k.

Then we set $K'_n := k(\mathfrak{f}p^{n+1})$ and $K'_\infty := k(\mathfrak{f}p^\infty)$. So we get that

$$\mathcal{G}' := \operatorname{Gal}(K'_\infty/k) \cong \Gamma' \times H \text{ with } H \cong (\mathcal{G}')_{tor} \text{ and } \Gamma' := \operatorname{Gal}(K'_\infty/K'_0) \cong \mathbb{Z}_p^2.$$

Let $\mathfrak{a}$ be an integral ideal prime to $6\mathfrak{f}p$ and we set $c_\mathfrak{a} := \mathcal{N}\mathfrak{a} - \sigma_\mathfrak{a}$. Then we have defined in Definition 2.1.24 the elliptic unit $\psi(1; \mathfrak{f}p^m, \mathfrak{a})$ for $m \in \mathbb{N}_0$. Moreover, let τ be an embedding from $\mathbb{Q}^c \to \mathbb{C}$. Then we set

$$\psi_{\mathfrak{f},\mathfrak{a}} := \{\psi(1; \mathfrak{f}p^{n+1}, \mathfrak{a})\}_{n\geq 0} \in U_{K'_\infty/k,S}, \qquad \tau_{K'_\infty} := \{\tau_{|K'_n}\}_{n\geq 0} \in Y_{K'_\infty/k,S}.$$

As on p. 277 in [Fla09] we obtain

$$\mathcal{L} := c_\mathfrak{a} \cdot (\psi_{\mathfrak{f},\mathfrak{a}}^{-1} \otimes \tau_{K'_\infty})$$

is a $Q(\Lambda_{K'_\infty/k})$-basis of

$$\operatorname{Det}_{\Lambda_{K'_\infty/k}}(\Delta_{K'_\infty/k}) \otimes Q(\Lambda_{K'_\infty/k}).$$

We can now state Conjecture 8 of [Fla09], which we call *Limit Conjecture*[1]:

Conjecture 6.3.2. $\mathrm{LC}(K'_\infty/k, p)$
There is an identity of invertible $\Lambda_{K'_\infty/k}$-submodules

$$\mathcal{L} \cdot \Lambda_{K'_\infty/k} = \operatorname{Det}_{\Lambda_{K'_\infty/k}}(\Delta_{K'_\infty/k})$$

of $\operatorname{Det}_{Q(\Lambda_{K'_\infty/k})}\left(\Delta_{K'_\infty/k} \otimes Q(\Lambda_{K'_\infty/k})\right)$.

[1] in [Fla09] it is called Iwasawa Main Conjecture

Rank one Limit Conjectures Let K'_∞ be as above and let K_∞ be such that $k \subset K_\infty \subset K'_\infty$ and $\Gamma_1 := \mathrm{Gal}(K'_\infty/K_\infty) \cong \mathbb{Z}_p$. Let Γ_2 be a subgroup of $\mathrm{Gal}(K_\infty/k)$ with $\Gamma_2 \cong \mathbb{Z}_p$. Then we set $K_0 := K_\infty^{\Gamma_2}$ and we denote by K_n the intermediate levels of K_∞/K_0. Let $m(n)$ be the smallest natural number such that $K_n \subseteq K'_{m(n)}$. So we can now set:

$$\psi_{\mathfrak{f},\mathfrak{a},K_\infty} := \{\mathcal{N}_{K'_{m(n)}/K_n}\psi(1;\mathfrak{f}p^{m(n)},\mathfrak{a})\}_{n\geq 0} \in U_{K_\infty/k,S}, \qquad \tau_{K_\infty} := \{\tau_{|K_n}\}_{n\geq 0} \in Y_{K_\infty/k,S}.$$

Now similar to the reasoning in the rank two case as well as in [Ble06, p. 94] and in [Fla04, p. 11] we have that

$$\mathcal{L}_{K_\infty} := c_{\mathfrak{a}} \cdot (\psi^{-1}_{\mathfrak{f},\mathfrak{a},K_\infty} \otimes \tau_{K_\infty})$$

is a $Q(\Lambda_{K_\infty/k})$-basis of

$$\mathrm{Det}_{\Lambda_{K_\infty/k}}(\Delta_{K_\infty/k}) \otimes Q(\Lambda_{K_\infty/k}).$$

Then one can state the following conjecture for an 'intermediate level':

Conjecture 6.3.3. $\mathrm{LC}(K_\infty/k,p)$
There is an identity of invertible $\Lambda_{K_\infty/k}$-submodules

$$\mathcal{L}_{K_\infty} \cdot \Lambda_{K_\infty} = \mathrm{Det}_{\Lambda_{K_\infty/k}}(\Delta_{K_\infty/k})$$

of $\mathrm{Det}_{Q(\Lambda_{K_\infty/k})}\left(\Delta_{K_\infty/k} \otimes Q(\Lambda_{K_\infty/k})\right)$.

Remark 6.3.4. Let $p > 3$ be a rational prime which splits completely in k, i.e. $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ with $\mathfrak{p} \neq \overline{\mathfrak{p}}$, and let $\mathfrak{f}$ be an integral ideal of $\mathcal{O}_k$ Furthermore, assume that $p \nmid h_k$. We set $K_\infty := k(\mathfrak{f}\mathfrak{p}^\infty)$ which is a $\mathbb{Z}_p$-extension of K_0 in $K'_\infty := k(\mathfrak{f}p^\infty)$. Then Theorem 5.1 in [Ble06] shows that $\mathrm{LC}(K_\infty/k,p)$ holds.

Descent to an intermediate level On the one hand, as we will see below, Johnson-Leung and Kings are presenting a proof of a variant of Conjecture 6.3.2 in [JLK11] by assuming that certain cohomology groups vanish after localizing at singular prime ideals of height one. On the other hand, the descent formalism developed in [BG03], [Fla04] (and used in [Ble06] for the split i.q. case) is valid for rank one extensions. Therefore, we would be interested in proving the following conjecture:

Conjecture 6.3.5. *The conjecture* $\mathrm{LC}(K'_\infty/k,p)$ *implies* $\mathrm{LC}(K_\infty/k,p)$ *for extensions* K_∞/k *as constructed above.*

6.3.2 Limit Conjecture of Johnson-Leung/Kings in [JLK11]

Now we introduce parts of the notation of [JLK11]. Let p be a prime number and k be an imaginary quadratic field. For a non-zero ideal $\mathfrak{f}$ of $\mathcal{O}_k$ we define $K'_\infty := k(\mathfrak{f}p^\infty)$ and $\mathcal{G}_{\mathfrak{f}} := \mathrm{Gal}(K'_\infty/k)$. We denote by $H \subset \mathcal{G}_{\mathfrak{f}}$ the torsion subgroup of $\mathcal{G}_{\mathfrak{f}}$ and we fix a splitting

$$\mathcal{G}_{\mathfrak{f}} \cong \Gamma' \times H.$$

Let $\Lambda(\Gamma')$ be the Iwasawa algebra of Γ'. Then we know that $\Lambda(\Gamma')$ is non-canonically isomorphic to $\mathbb{Z}_p[[T_1,T_2]]$. Furthermore, let $\Lambda(\mathcal{G}_{\mathfrak{f}})$ be the Iwasawa algebra of $\mathcal{G}_{\mathfrak{f}}$. Then we know that $\Lambda(\mathcal{G}_{\mathfrak{f}})$ is non-canonically isomorphic to $\mathbb{Z}_p[H][[T_1,T_2]]$. Furthermore, we set

$$\Omega := \Lambda(\mathcal{G}_{\mathfrak{f}}) \text{ and } \Omega(1) := \Omega \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$$

and we have

$$H^i(\mathcal{O}_k[1/p\mathfrak{f}], \Omega(1)) = \varprojlim_{k \subset F \subset K'_\infty} H^i(\mathcal{O}_F[1/p\mathfrak{f}], \mathbb{Z}_p(1)).$$

Recall the notation of Chapter 2 and let $\mathfrak{a}$ be an auxiliary ideal which is coprime to $6p\mathfrak{f}$. We have the following notation

$$\zeta(\mathfrak{f}) := c_{\mathfrak{a}}^{-1}\, {}_{\mathfrak{a}}\zeta(\mathfrak{f}) \in H^1(\mathcal{O}_k[1/p\mathfrak{f}], \Omega(1)) \otimes Q(\Omega), \qquad {}_{\mathfrak{a}}\zeta(\mathfrak{f}) := \varprojlim_n ({}_{\mathfrak{a}}\zeta_{\mathfrak{f}p^n}^{-1}) \in H^1(\mathcal{O}_k[1/p\mathfrak{f}], \Omega(1)),$$

$${}_{\mathfrak{a}}\zeta_{\mathfrak{f}} := \mathcal{N}_{k(p^r\mathfrak{f})/k(\mathfrak{f})}\, {}_{\mathfrak{a}}\theta_E(\alpha)^{-1}, \qquad J_\Omega := \mathrm{Ann}_\Omega(\mathbb{Z}_p[H](1)),$$

where here (E, α) is the CM-pair of modulus $p^r\mathfrak{f}$ defined over $k(p^r\mathfrak{f})$ and r is an integer $r \geq 1$ such that $\mathcal{O}_k^\times \to (\mathcal{O}_k/p^r\mathcal{O}_k)^\times$ is injective.

Remark 6.3.6. We have defined ${}_{\mathfrak{a}}\zeta(\mathfrak{f})$ differently from [JLK11] (with an inverse) but this is compensated by the fact that they use the Dirichlet regulator map without a minus sign contrary to [Fla04], [Ble06] and [Fla09].

As we are trying to avoid introducing even more notation we state only the assertion Johnson-Leung and Kings give under 'In particular,...' of their *equivariant main conjecture* in [JLK11] because this is what we are going to use.

Conjecture 6.3.7. $\mathrm{LC}_{\mathrm{JoKi}}(K'_\infty/k, p)$
Let $\kappa_{\mathfrak{f}}$ be the inclusion of perfect complexes as defined on p. 101 in [JLK11]. Then $\kappa_{\mathfrak{f}}$ induces an isomorphism of Ω-modules

$$\mathrm{Det}_\Omega(H^1(\mathcal{O}_k[1/p\mathfrak{f}], \Omega(1))/J_\Omega(\zeta(\mathfrak{f}))) \cong \mathrm{Det}_\Omega(H^2(\mathcal{O}_k[1/p\mathfrak{f}], \Omega(1))).$$

6.3.3 Vanishing assumptions

Definition of the μ-invariant and λ-invariant Let Λ be an Iwasawa algebra for which $\Lambda \cong \mathbb{Z}_p[[T]]$ and let M be a finitely generated torsion Λ-module. Then there exists an elementary torsion module E_M of the form

$$\bigoplus_{i=1}^{s} \Lambda/(p^{n_i}) \oplus \bigoplus_{j=1}^{t} \Lambda/(f_j(T))^{l_j},$$

where f_j is an irreducible and distinguished polynomial, and E_M is pseudo-isomorphic to M. Then one sets

$$\mu(M) := \sum_{i=1}^{s} n_i \text{ and } \lambda(M) := \sum_{j=1}^{t} l_j \cdot \deg(f_j).$$

Vanishing of μ-invariant of class groups for split primes Let k be an imaginary quadratic field, p be an odd split prime in $k/\mathbb{Q}$, $\mathfrak{p}$ a prime ideal above p and $\mathfrak{f}$ be an integral ideal of $\mathcal{O}_k$ with $(\mathfrak{p}, \mathfrak{f}) = 1$ and w.l.o.g. $w(\mathfrak{f}) = 1$.

We set $K_\infty := k(\mathfrak{f}\mathfrak{p}^\infty)$, $K_n := k(\mathfrak{f}\mathfrak{p}^{n+1})$, $\mathcal{G} := \mathrm{Gal}(K_\infty/k)$ and $P_{K_\infty/k} := P_{K_\infty/k,\emptyset}$. Assume that $\mathrm{Gal}(K_\infty/k)$ is isomorphic to $\Gamma \times H$, with $\Gamma \cong \mathbb{Z}_p$ and $H \cong \mathrm{Gal}(K_0/k)$, e.g. this is guaranteed if we are assuming that $p \nmid h_k$.

Let M_n be the maximal abelian pro-p extension of K_n unramified outside the primes above $\mathfrak{p}$ and L_n the maximal unramified abelian p-extension L_n of K_n. With $M_\infty := \bigcup_n M_n$ and $L_\infty := \bigcup_n L_n$ we can define $\mathcal{X}(\mathfrak{f}) := \mathrm{Gal}(M_\infty/K_\infty)$ and $\mathcal{W}(\mathfrak{f}) := \mathrm{Gal}(L_\infty/K_\infty)$, which are topological $\mathbb{Z}_p[[\mathcal{G}]]$-modules. As L_n is the p-Hilbert class field of K_n we obtain $\mathcal{W}(\mathfrak{f}) \cong P_{K_\infty/k}$.

Theorem 6.3.8. *[Gil85, Thm. 3.4], [OV16] Let p be a rational prime. The group $\mathcal{X}(\mathfrak{f})$ has no $\mathbb{Z}_p$-torsion. In particular, it is a finitely generated $\mathbb{Z}_p$-module.*

Corollary 6.3.9. *$P_{K_\infty/k}$ is a finitely generated $\mathbb{Z}_p$-module, i.e. the μ-invariant of $P_{K_\infty/k}$ vanishes.*

Proof. By Theorem 6.3.8 we know that $\mathcal{X}(\mathfrak{f})$ is a finitely generated $\mathbb{Z}_p$-module. Furthermore, we know that $\mathcal{W}(\mathfrak{f}) \cong P_{K_\infty/k}$ is a quotient of $\mathcal{X}(\mathfrak{f})$ by global class field theory, so the result follows. □

μ-invariant and singular prime ideals of height one This part follows closely Section 3C2 in [BKS17]. Fix a prime $p > 3$. Let F_∞/k be an extension such that $\mathcal{G} := \mathrm{Gal}(F_\infty/k)$ is isomorphic to $\Gamma \times H$, with $\Gamma \cong \mathbb{Z}_p$ and H a finite abelian group. Moreover, set $\Lambda := \mathbb{Z}_p[[\mathcal{G}]]$.

Definition 6.3.10. We call a height one prime ideal $\mathfrak{q}$ of Λ regular if $p \notin \mathfrak{q}$ and singular if $p \in \mathfrak{q}$.

If $\mathfrak{q}$ is a regular height one prime ideal of Λ, since we have

$$\Lambda[1/p] = \bigoplus_{\chi \in \widehat{H}/\sim_{\mathbb{Q}_p}} \Lambda_\chi[1/p]$$

with $\Lambda_\chi := \mathbb{Z}_p[\mathrm{im}(\chi)][[\Gamma]]$ we obtain $Q(\Lambda_\mathfrak{q}) = Q(\Lambda_{\chi_\mathfrak{q}})$ for a $\chi_\mathfrak{q} \in \widehat{H}/\sim_{\mathbb{Q}_p}$.
We also have the decomposition

$$\Lambda = \bigoplus_{\chi \in \widehat{H'}/\sim_{\mathbb{Q}_p}} \mathbb{Z}_p[\mathrm{im}(\chi)][H_p][[\Gamma]]$$

where H_p is the p-Sylow subgroup of H, and H' is the unique subgroup of H which is isomorphic to H/H_p. One can show that there is a 1-1 correspondence between all the singular primes of Λ and the set $\widehat{H}'/\sim_{\mathbb{Q}_p}$ and for such a singular $\mathfrak{q}$ we denote such a character by $\chi_\mathfrak{q}$ and obtain that

$$Q(\Lambda_\mathfrak{q}) = \bigoplus_{\substack{\chi \in \widehat{H}/\sim_{\mathbb{Q}_p}, \\ \chi_{|H'} = \chi_\mathfrak{q}}} Q(\Lambda_\chi).$$

For any height one prime ideal $\mathfrak{q}$ of Λ, we define a subset $Y_\mathfrak{q} \subset \widehat{H}/\sim_{\mathbb{Q}_p}$ by

$$Y_\mathfrak{q} := \begin{cases} \{\chi_\mathfrak{q}\} & \text{if } \mathfrak{q} \text{ is regular,} \\ \{\chi \in \widehat{H}/\sim_{\mathbb{Q}_p} : \chi\,|_{H'} = \chi_\mathfrak{q}\} & \text{if } \mathfrak{q} \text{ is singular.} \end{cases}$$

Lemma 6.3.11. *[BKS17, Lemma 3.8] Let M be a finitely generated torsion Λ-module and let $\mathfrak{q}$ be a singular prime ideal. Then the following are equivalent:*

a) The μ-invariant of the $\mathbb{Z}_p[[\Gamma]]$-module $e_{\chi_\mathfrak{q}}M$ vanishes.

b) For any $\chi \in Y_\mathfrak{q}$, the μ-invariant of the $\mathbb{Z}_p[im(\chi)][[\Gamma]]$-module $M \otimes_{\mathbb{Z}_p[H']} \mathbb{Z}_p[im(\chi)]$ vanishes.

c) $M_{\mathfrak{q}} = 0$.

So with Corollary 6.3.9 and Lemma 6.3.11 we obtain in the imaginary quadratic split prime situation from above the following:

Lemma 6.3.12. *The μ-invariant of $P_{K_\infty/k} \otimes_{\mathbb{Z}_p[H']} \mathbb{Z}_p(\chi)$ vanishes for all characters in $Y_{\mathfrak{q}}$ with $\mathfrak{q}$ a singular prime ideal of Λ. In particular, for all singular ideals $\mathfrak{q}$ of Λ we have $P_{K_\infty/k,\mathfrak{q}} = 0$.*

Vanishing of the second cohomology after localization at singular height 1 prime ideals The following conjecture from [JLK11] is an analogue/generalization for Lemma 6.3.12 which makes sense for the split and the non-split case. But only the split case is proven and it uses Corollary 6.3.9 in a crucial way. We use the notation of [JLK11] which we have introduced in Section 6.3.2, in particular k is an imaginary quadratic number field and $S = S_\infty \cup S_{ram}(K'_\infty/k)$.

Conjecture 6.3.13. (VanishAss) *For each height one prime ideal $\mathfrak{q}$ of Ω with $p \in \mathfrak{q}$ we have*

$$H^2(\mathcal{O}_{k,S}, \Omega(1))_{\mathfrak{q}} = 0.$$

We want to sketch the proof of Corollary 5.12 in [JLK11] which states:
Let p be a split prime in k and $\mathfrak{f}$ an integral ideal of $\mathcal{O}_k$. Then (VanishAss) holds, i.e. for any singular height one prime ideal $\mathfrak{q}$ of Ω, we have

$$H^2(\mathcal{O}_{k,S}, \Omega(1))_{\mathfrak{q}} = 0. \tag{6.9}$$

Sketch of the proof of (6.9) Let E be a finite extension of $\mathbb{Q}$, $\mathcal{O} := \mathcal{O}_E$ its ring of integers. In fact, in [JLK11] E is an extension which contains all values of the characters of $\mathrm{Gal}(k(\mathfrak{f})/k)$ and $\mathcal{O}_p := \mathcal{O} \otimes \mathbb{Z}_p$. So we can set $\Omega_{\mathcal{O}} := \Omega \otimes \mathcal{O}_p$.

Recall that $K'_\infty := k(\mathfrak{f}p^\infty)$, $p\mathcal{O}_k = \mathfrak{p}\overline{\mathfrak{p}}$ with $\mathfrak{p} \neq \overline{\mathfrak{p}}$ and $\mathcal{G} := \mathrm{Gal}(K'_\infty/k) = \Gamma' \times H$ with H the torsion subgroup of $\mathcal{G}$. Then we set $K'_0 := (K'_\infty)^{\Gamma'}$.

Let $k_{1,\infty}$ resp. $k_{2,\infty}$ be the $\mathbb{Z}_p$-extension of k which is unramified outside of $\mathfrak{p}$ resp. $\overline{\mathfrak{p}}$. We set $K_{1,\infty} := K'_0 k_{1,\infty}$ resp. $K_{2,\infty} := K'_0 k_{2,\infty}$ and $\mathcal{H} := \mathrm{Gal}(K'_\infty/K_{1,\infty})$.

Then with Corollary 6.3.9 one can show that $H^2(\mathcal{O}_S, \Omega_{\mathcal{O}}(1))$ is a finitely generated $\Lambda(\mathcal{H})$-module, which is Corollary 5.10 in [JLK11]. Then they prove the following lemma, which finishes the proof of (6.9).

Lemma 6.3.14. *[JLK11, Lemma 5.11]. Let M be an $\Omega_{\mathcal{O}}$-module which is finitely generated as $\Lambda_{\mathcal{O}}(\mathcal{H})$-module. Then for any singular height one prime ideal $\mathfrak{q}$ in $\Omega_{\mathcal{O}}$ one has $M_{\mathfrak{q}} = 0$.*

6.3.4 Finiteness assumption - Condition (F)

The terminology *Condition (F)* we adopted here stems from [BKS17], where it is one of their assumptions for their descent result (Theorem 5.2 in [BKS17]). We will see below that this condition can be shown in the split imaginary quadratic case (as is done in [Ble06]). In both instances it is used in order to determine the second cohomology of the complex in more detail after localizing at height one prime ideals (cf. Lemma 5.12 in [BKS17]).

Let M be a number field and M_∞/M be a $\mathbb{Z}_p$-extension with Galois group Γ and we denote the n-th level by M_n. Let Σ be a finite set of places of M. Furthermore, we denote by $P_{M_\infty,\Sigma}$ the projective limit $\varprojlim_{M \subseteq M_n \subseteq M_\infty} \mathrm{Pic}(\mathcal{O}_{M_n,\Sigma}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Condition (F) The module of Γ-coinvariants of $P_{M_\infty,\Sigma}$ is finite.

We abbreviate this condition by $\mathrm{F}(M_\infty/M,\Sigma)$. As we will see below Condition (F) does not hold in general for arbitrary finite sets S, but it is conjectured to always hold if S contains the places in M above p and the infinite places.

Case $\Sigma = S_\infty$ There are some known cases where $\mathrm{F}(M_\infty/M,S_\infty(M))$ holds, where we abbreviate this condition to $\mathrm{F}(M_\infty/M)$ from now on.

i) Let M be a totally real abelian extension of $\mathbb{Q}$ and M_∞ be the cyclotomic $\mathbb{Z}_p$-extension of M. Then $\mathrm{F}(M_\infty/M)$ is shown in [Gre73].

ii) Let M be a totally imaginary abelian extension of $\mathbb{Q}$ and M_∞ be the cyclotomic $\mathbb{Z}_p$-extension of M. Let M^+ be the maximal totally real subfield of M. Assume that no prime above p in M^+ splits completely in M. Then $\mathrm{F}(M_\infty/M)$ is also shown in [Gre73].

iii) Let k be an imaginary quadratic field, $p \geq 5$ be a prime which splits k and $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_k$ above p. Let $\mathfrak{f}$ be a non-zero ideal of $\mathcal{O}_k$ coprime to $\mathfrak{p}$. We set $M_\infty := k(\mathfrak{f}\mathfrak{p}^\infty)$ and $M := k(\mathfrak{f}\mathfrak{p})$. We will show in Lemma 6.3.15 that $\mathrm{F}(M_\infty/M)$ holds.

But there is also an example where $\mathrm{F}(M_\infty/M)$ fails to hold:

i) Let M be an totally imaginary abelian extension of $\mathbb{Q}$ and M_∞ be the cyclotomic $\mathbb{Z}_p$-extension of M. Let M^+ be the maximal totally real subfield of M. Assume that all primes above p in M^+ split completely in M. Then it can be easily seen from the results in [Gre73] that $\mathrm{F}(M_\infty/M)$ does not hold.

Case Σ contains places S_∞ and S_p For the cyclotomic $\mathbb{Z}_p$-extension M_∞/M and $\Sigma = S_p(M) \cup S_\infty(M)$ Condition $\mathrm{F}(M_\infty/M,\Sigma)$ is known as Gross-Kuzmin conjecture. More generally, it is conjectured that $\mathrm{F}(M_\infty/M,\Sigma)$ holds for any $\mathbb{Z}_p$-extension M_∞/M and this assertion is sometimes called generalized Gross-Kuzmin conjecture. In particular, if Σ contains $S_\infty(M)$ and $S_p(M)$, as is often an assumption below, the generalized Gross-Kuzmin conjecture, i.e. $\mathrm{F}(M_\infty/M,S_p(M)\cup S_\infty(M))$, implies $\mathrm{F}(M_\infty/M,\Sigma)$. In the following cases this conjecture is known.

i) It is easily seen that $\mathrm{F}(M_\infty/M)$ implies $\mathrm{F}(M_\infty/M,\Sigma)$. So we get all the cases of the list for $\Sigma = S_\infty(M)$.

ii) Let M be an abelian extension of $\mathbb{Q}$, M_∞/M be the cyclotomic $\mathbb{Z}_p$-extension of M. Then [Gre73] shows that $\mathrm{F}(M_\infty/M,\Sigma)$ holds.

iii) Let M be a number field in which there are at most two primes above p and M_∞/M be the cyclotomic $\mathbb{Z}_p$-extension of M. Then it is shown in [Kle19] that $\mathrm{F}(M_\infty/M,\Sigma)$ holds.

iv) Let M be a totally real field and assume that Leopoldt's conjecture is true for M at p. Then for the cyclotomic $\mathbb{Z}_p$-extension M_∞/M we know by [Kol91] that $\mathrm{F}(M_\infty/M,\Sigma)$ holds.

Lemma 6.3.15. *Let k be an imaginary quadratic field, $p \geq 5$ be a prime which splits k, $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_k$ above p and $\mathfrak{f}$ a non-zero ideal of $\mathcal{O}_k$ coprime to $\mathfrak{p}$. Then $\mathrm{F}(k(\mathfrak{p}^\infty\mathfrak{f})/k(\mathfrak{p}\mathfrak{f}))$ holds.*

Proof. In this proof we are following ideas from the proof of Theorem 1.4 in [Rub88].

Let M_∞ be the maximal abelian p-extension of $\mathcal{K}_\infty := k(\mathfrak{f}\mathfrak{p}^\infty)$ which is unramified outside the primes above $\mathfrak{p}$, M_n be the maximal abelian p-extension of $K_n := k(\mathfrak{f}\mathfrak{p}^{n+1})$ which is unramified outside the primes above $\mathfrak{p}$, L_∞ the maximal abelian extension of $\mathcal{K}_\infty$ inside M_∞ which is everywhere unramified and L the maximal abelian extension of K_0 inside M_∞ which is everywhere unramified. Then by class field theory we get $\mathrm{Gal}(L_\infty/k(\mathfrak{f}\mathfrak{p}^\infty)) \cong P_{\mathcal{K}_\infty}$ and that $\mathrm{Gal}(L/\mathcal{K}_\infty)$ is isomorphic to $(P_{\mathcal{K}_\infty})_\Gamma$ with $\Gamma = \mathrm{Gal}(\mathcal{K}_\infty/K_0)$.

We claim that $\mathrm{rk}_{\mathbb{Z}_p}(\mathrm{Gal}(M_0/K_0)) = 1$. Then if we assume that $\mathrm{Gal}(L/\mathcal{K}_\infty)$ is not finite, because it is finitely generated over $\mathbb{Z}_p$, we obtain that the $\mathrm{rk}_{\mathbb{Z}_p}(\mathrm{Gal}(L/K_0) \geq 1$, but this is a contradiction to $\mathrm{rk}_{\mathbb{Z}_p}(\mathrm{Gal}(M_0/K_0)) = 1$.

So what is left to do is to show that $\mathrm{rk}_{\mathbb{Z}_p}(\mathrm{Gal}(M_0/K_0)) = 1$, where we are using ideas of Section 2 in [Coa83]. Let v be a place of K_n above $\mathfrak{p}$, let $U_{n,v}$ denote the local units which are congruent to 1 modulo v of the completion of K_n at v. Let E_n be the global units of K_n which are congruent to 1 modulo v for each place v in K_n and ι_n be the diagonal embedding of E_n in $\prod_{v|\mathfrak{p}} U_{n,v}$. Then by the Artin map we obtain the exact sequence

$$0 \longrightarrow \left(\prod_{v|\mathfrak{p}} U_{n,v}\right) / \overline{\iota_n(E_n)} \longrightarrow \mathrm{Gal}(M_n/K_n) \longrightarrow A_n \longrightarrow 0, \tag{6.10}$$

where $\overline{\iota_n(E_n)}$ is the $\mathfrak{p}$-adic closure of $\iota_n(E_n)$. Now we can compute that $\mathrm{rk}_{\mathbb{Z}_p}(\prod_{v|\mathfrak{p}} U_{n,v})$ is equal to $[K_n : k]$. By Leopoldt's conjecture, which is valid for abelian extensions of imaginary quadratic fields, we know that $\mathrm{rk}_{\mathbb{Z}_p}(\overline{\iota_n(E_n)}) = [K_n : k] - 1$. As A_n is finite the claim follows now directly by looking at the exact sequence (6.10). □

Recall the situation from [BKS17]. Let $\mathcal{G} := \mathrm{Gal}(F_\infty/k) \cong \Gamma \times H$ with $\Gamma \cong \mathbb{Z}_p$ and H a finite abelian group. Let L be a finite extension in F_∞/k such that $\mathrm{Gal}(F_\infty/L)$ is a $\mathbb{Z}_p$-extension, $G := \mathrm{Gal}(L/k)$. Let Σ be any finite set of places of k.

By abuse of notation we denote the Condition $\mathrm{F}(F_\infty/L)$ (resp. $\mathrm{F}(F_\infty/L, \Sigma)$) in Section 6.5 by $\mathrm{F}(F_\infty/k)$ (resp. $\mathrm{F}(F_\infty/k, \Sigma)$) if the L we are using is clear from the context.

Let $\chi \in \widehat{G}$ and recall that we then set $L_\chi := L^{\ker(\chi)}$, $F_0 := F_\infty^\Gamma$, $k_\infty := F_\infty^H$, $L_{\chi,\infty} := L_\chi k_\infty$ and $\Gamma_\chi := \mathrm{Gal}(L_{\chi,\infty}/L_\chi)$. Let S be an admissible set of places of k, in particular it contains S_∞ and S_p and T a finite set of places disjoint from S. For any intermediate field F in F_∞/k, where F runs over all intermediate fields of F/k that are finite over k we denote by $P_{F,S,T}$ the inverse limit of the p-primary part of the ray class group of $\mathcal{O}_{F,S}$ with modulus $\prod_{w \in T(F)} w$ with respect to the norm maps.

Condition (F) in [BKS17] For every $\chi \in \widehat{G}$, the module of Γ_χ-coinvariants of $P_{L_{\chi,\infty},S,T}$ is finite.

Remark 6.3.16. One can show that Condition (F) in [BKS17] is equivalent to $\mathrm{F}(L_{\chi,\infty}/L_\chi, S)$ for each $\chi \in \widehat{G}$.

6.3.5 Higher-rank main conjecture of Iwasawa theory in [BKS17]

In [BKS17] there are yet several other formulations of IMC-type conjectures which are the starting points of the descent to the $\mathrm{eTNC}(h^0(\mathrm{Spec}(L)), \mathbb{Z}_p[G])$ and we want to present here two of them.

Here k is a general number field. Fix a prime p and an isomorphism $\mathbb{C} \cong \mathbb{C}_p$. Let F_∞/k be a Galois extension with

$$\mathcal{G} := \mathrm{Gal}(F_\infty/k) \cong \Gamma \times H \text{ such that } \Gamma \cong \mathbb{Z}_p \text{ and } H \text{ finite abelian.}$$

Then we set $F := F_\infty^\Gamma$ and $k_\infty := F_\infty^H$ and for each character $\chi \in \widehat{\mathcal{G}}$:

$$\begin{array}{lll} L_\chi := F_\infty^{\ker(\chi)} & L_{\chi,\infty} := L_\chi \cdot k_\infty & L_{\chi,n} := n\text{-th layer of } L_{\chi,\infty}/L_\chi \\ \mathcal{G}_\chi := \mathrm{Gal}(L_{\chi,\infty}/k) & G_\chi := \mathrm{Gal}(L_\chi/k) & \Gamma_\chi := \mathrm{Gal}(L_{\chi,\infty}/L_\chi) \end{array}$$

We let S be a finite set of places of k which contains $S_\infty(k) \cup S_{ram}(F_\infty/k) \cup S_p(k)$ and T be a finite set of places of k which is disjoint from S. We set $V_\chi := \{v \in S : v \text{ splits completely in } L_{\chi,\infty}\}$ and let r_χ be the cardinality of V_χ.

Furthermore, we set $\Lambda := \mathbb{Z}_p[[\mathcal{G}]]$, for $\chi \in \widehat{H}$ we put $\Lambda_\chi := \mathbb{Z}_p[\mathrm{im}\chi][[\Gamma]]$ and for $C_{F_\infty,S,T}$ we take the inverse limit of the complex $C_{M,S,T}$ with respect to norm maps over all intermediate fields M that are finite over k in F_∞/k.

First formulation For any $\chi \in \widehat{\mathcal{G}}$ there is a natural homomorphism

$$\begin{aligned} \lambda_\chi : \mathrm{Det}_\Lambda(\mathrm{C}_{F_\infty,S,T}) &\to \mathrm{Det}_{\mathbb{Z}_p[G_\chi]}(\mathrm{C}_{L_\chi,S,T}) \\ &\hookrightarrow \mathrm{Det}_{\mathbb{C}_p[G_\chi]}(\mathbb{C}_p\mathrm{C}_{L_\chi,S,T}) \\ &\cong \mathrm{Det}_{\mathbb{C}_p[G_\chi]}(\mathbb{C}_p\mathrm{U}_{L_\chi,S,T}) \otimes \mathrm{Det}^{-1}_{\mathbb{C}_p[G_\chi]}(\mathbb{C}_p\mathrm{X}_{L_\chi,S}) \\ &\cong \mathrm{Det}_{\mathbb{C}_p[G_\chi]}(\mathbb{C}_p\mathrm{X}_{L_\chi,S}) \otimes \mathrm{Det}^{-1}_{\mathbb{C}_p[G_\chi]}(\mathbb{C}_p\mathrm{X}_{L_\chi,S}) \\ &\cong \mathbb{C}_p[G_\chi] \\ &\xrightarrow{\chi} \mathbb{C}_p. \end{aligned}$$

Conjecture 6.3.17. (hIMC($F_\infty/k, S, T, p$))
There exists a Λ-basis $\mathcal{L}_{L_\chi,S,T}$ of the module $\mathrm{Det}_\Lambda(\mathrm{C}_{F_\infty,S,T})$ *for which at every $\chi \in \widehat{H}$ and every $\psi \in \widehat{G}_\chi$ for which $r_S(\psi) = r_\chi$ one has*

$$\lambda_\psi(\mathcal{L}_{F_\infty,S,T}) = L_{k,S,T}(0, \psi^{-1}).$$

Remark 6.3.18. The validity of (hIMC) is independent of the choice of T. Therefore, we can choose T such that $\mathrm{U}_{F,S,T}$ is torsion-free.

Remark 6.3.19. The assertion of Conjecture 6.3.17 is valid if and only if there is a Λ-basis $\mathcal{L}_{F_\infty,S,T}$ of $\mathrm{Det}_\Lambda(\mathrm{C}_{F_\infty,S,T})$ for which for every character $\chi \in \widehat{H}$ and every $n \in \mathbb{N}$ we have

$$j^{V_\chi}_{L_{\chi,n}}(\mathcal{L}_{F_\infty,S,T}) = \epsilon^{V_\chi}_{L_{\chi,n},S,T},$$

where we set

$$j^{V_\chi}_{L_{\chi,n}} : \mathrm{Det}_\Lambda(\mathrm{C}_{F_\infty,S,T}) \to \mathrm{Det}_{\mathbb{Z}_p[\mathcal{G}_{\chi,n}]}(C_{L_{\chi,n},S,T}) \xrightarrow{\pi^{V_\chi}_{L_{\chi,n},S,T}} \mathbb{C}_p \bigwedge^{r_\chi} \mathrm{U}_{L_{\chi,n},S,T}.$$

and $\pi^{V_\chi}_{L_{\chi,n},S,T}$ is a map from $\mathrm{Det}_{\mathbb{Z}_p[G]}(C_{L_{\chi,n},S,T})$ to $\mathbb{C}_p \bigwedge^{r_\chi} \mathrm{U}_{L_{\chi,n},S,T}$ as described in Section 2D. in [BKS17].

For each character $\chi \in \widehat{H}$ there is a natural ring homomorphism

$$\mathbb{Z}_p[[\mathcal{G}_\chi]] \xrightarrow{\chi} \Lambda_\chi$$

and we define

$$\bigcap^{r_\chi} \mathrm{U}_{L_{\chi,\infty},S,T} := \varprojlim_n \bigcap^{r_\chi} \mathrm{U}_{L_{\chi,n},S,T}$$

and

$$\epsilon^{V_\chi}_{L_{\chi,\infty},S,T} := \varprojlim_n \epsilon^{V_\chi}_{L_{\chi,n},S,T} \in \bigcap^{r_\chi} U_{L_{\chi,\infty},S,T}$$

Second formulation For this formulation we assume that p is odd and that the following condition holds:

(SplitAss) V_χ contains no finite places for every $\chi \in \widehat{H}$, i.e. no finite place of S splits completely in $L_{\chi,\infty}$.

Remark 6.3.20. Let $\mathfrak{q}$ be a singular prime ideal of Λ. Then V_χ is independent of $\chi \in Y_\mathfrak{q}$. In particular, for any $\chi \in Y_\mathfrak{q}$, then $Q(\Lambda_\mathfrak{q})$-module $U_{F_\infty,S,T} \otimes Q(\Lambda_\mathfrak{q})$ is free of rank r_χ. This is Lemma 3.9 in [BKS17]. So we can set for the height one prime ideal $\mathfrak{q}$ of Λ: $V_\mathfrak{q} := V_\chi$ and $r_\mathfrak{q} := r_\chi$ by choosing some $\chi \in Y_\mathfrak{q}$.

Assume $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ holds for all $\chi \in \widehat{H}$ and n. Then we define

$$\varepsilon^{\mathfrak{q}}_{F_\infty/k,S,T} \in \left(\bigwedge^{r_\mathfrak{q}} U_{F_\infty,S,T}\right) \otimes Q(\Lambda_\mathfrak{q})$$

as the image of

$$\left(\varepsilon^{V_\chi}_{L_{\chi,\infty}/k,S,T}\right)_{\chi \in Y_\mathfrak{q}} \in \bigoplus_{\chi \in Y_\mathfrak{q}} \bigcap^{r_\mathfrak{q}} U_{L_{\chi,\infty},S,T}$$

under the map

$$\bigoplus_{\chi \in Y_\mathfrak{q}} \bigcap^{r_\mathfrak{q}} U_{L_{\chi,\infty},S,T} \to \bigoplus_{\chi \in Y_\mathfrak{q}} \left(\bigcap^{r_\mathfrak{q}} U_{L_{\chi,\infty},S,T}\right) \otimes_{\mathbb{Z}_p[[\mathcal{G}_\chi]]} Q(\Lambda_\chi) = \left(\bigwedge^{r_\mathfrak{q}} U_{F_\infty,S,T}\right) \otimes Q(\Lambda_\mathfrak{q}).$$

Lemma 6.3.21. *[BKS17, Lemma 3.10] Let $\mathfrak{q}$ be a height one prime ideal of Λ. When $\mathfrak{q}$ is singular, assume that μ-invariant of $e_{\chi_\mathfrak{q}} P_{F_\infty,S,T}$ vanishes.*

a) The $\Lambda_\mathfrak{q}$-module $(U_{F_\infty,S,T})_\mathfrak{q}$ is free of rank $r_\mathfrak{q}$.

b) If $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ holds for all $\chi \in \widehat{H}$ and $n \in \mathbb{N}$, then there is an inclusion

$$\varepsilon^{\mathfrak{q}}_{F_\infty/k,S,T} \cdot \Lambda_\mathfrak{q} \subset \left(\bigwedge^{r_\mathfrak{q}} U_{F_\infty,S,T}\right)_\mathfrak{q}$$

Lemma 6.3.22. *[BKS17, Prop. 3.11, Rem. 3.12] Assume that*

i) $\mathrm{RS}(L_{\chi,n}/k, S, T, V_\chi)_p$ holds for all $\chi \in \widehat{H}$ and n large enough, and *ii) for each* $\chi \in \widehat{H}'/\sim_{\mathbb{Q}_p}$ *we have that the* μ*-invariant of* $e_{\chi_{\mathfrak{q}}} P_{F_\infty,S,T}$ *vanishes.*

Then $\mathrm{hIMC}(F_\infty/k,S,T)$ *holds if and only if*

$$\varepsilon^{\mathfrak{q}}_{F_\infty/k,S,T} \cdot \Lambda_{\mathfrak{q}} = \mathrm{Fitt}_{\Lambda_{\mathfrak{q}}}(P_{F_\infty,S,T,\mathfrak{q}})\mathrm{Fitt}_{\Lambda_{\mathfrak{q}}}(X_{F_\infty,S\setminus V_{\mathfrak{q}},\mathfrak{q}}) \left(\bigwedge^{r_{\mathfrak{q}}} U_{F_\infty,S,T}\right)_{\mathfrak{q}}$$

for every height-one prime ideal $\mathfrak{q}$ *of* Λ.

Remark 6.3.23. Let k be an imaginary quadratic field, $p>3$ be a split prime in $k/\mathbb{Q}$, $\mathfrak{p}$ a fixed prime ideal of $\mathcal{O}_k$ over p, $\mathfrak{f}$ an integral ideal of $\mathcal{O}_k$ with $(\mathfrak{f},\mathfrak{p})=1$.

Furthermore, let $S=S_\infty(k)\cup S_{ram}(K_\infty/k)\cup S_p(k)$, set $K_\infty := k(\mathfrak{f}\mathfrak{p}^\infty)$, $\tau := \tau_{K_\infty}$, let $\mathfrak{a}$ be an integral ideal of $\mathcal{O}_k$ coprime to $6\mathfrak{f}\mathfrak{p}$, T a set of primes containing $\mathfrak{a}$ disjoint from S.

Assumption i) in Lemma 6.3.22 holds because of Remark 5.1.6 f) and Remark 5.1.8 as (SplitAss) holds in this case. Assumption ii) holds because of Lemma 6.3.12.

If one transfers the proof of the Limit Conjecture (Theorem 5.1 in [Ble06]) to a T-version of the Limit Conjecture by defining a T-version $\psi^T_{\mathfrak{f},\mathfrak{a}}$ of $\psi_{\mathfrak{f},\mathfrak{a}}$, one obtains for regular height one prime ideals $\mathfrak{q}$

$$c_{\mathfrak{a}} \cdot \mathrm{Fitt}_{\Lambda_{\mathfrak{q}}}(U_{K_\infty,S,T,\mathfrak{q}}/\psi^T_{\mathfrak{f},\mathfrak{a}}\Lambda_{\mathfrak{q}}) = \mathrm{Fitt}_{\Lambda_{\mathfrak{q}}}(P_{K_\infty,S,T,\mathfrak{q}}) \cdot \mathrm{Fitt}_{\Lambda_{\mathfrak{q}}}(Y_{K_\infty,S}/\tau\Lambda_{\mathfrak{q}})$$

and with (SplitAss) we get $V_\chi=\{\infty\}$ and so $Y_{K_\infty,S,\mathfrak{q}}/\tau\Lambda_{\mathfrak{q}} \cong X_{K_\infty,S\setminus V_{\mathfrak{q}},\mathfrak{q}}$ and therefore

$$c_{\mathfrak{a}} \cdot \mathrm{Fitt}_{\Lambda_{\mathfrak{q}}}(U_{K_\infty,S,T,\mathfrak{q}}/\psi^T_{\mathfrak{f},\mathfrak{a}}\Lambda_{\mathfrak{q}}) = \mathrm{Fitt}_{\Lambda_{\mathfrak{q}}}(P_{K_\infty,S,T,\mathfrak{q}}) \cdot \mathrm{Fitt}_{\Lambda_{\mathfrak{q}}}(X_{K_\infty,S\setminus V_{\mathfrak{q}},\mathfrak{q}}).$$

In the case where $\mathfrak{q}$ is a singular prime ideal of height one of Λ, we should obtain

$$H^0(C_{K_\infty,S,T,\mathfrak{q}}) = U_{K_\infty,S,T,\mathfrak{q}} = c_{\mathfrak{a}}^{-1}\psi^T_{\mathfrak{f},\mathfrak{a}} \cdot \Lambda_{\mathfrak{q}} \text{ and } H^1(C_{K_\infty,S,T,\mathfrak{q}}) = \tau\Lambda_{\mathfrak{q}}$$

so $C_{K_\infty,S,T,\mathfrak{q}}$ has perfect cohomology and

$$\mathrm{Fitt}_{\Lambda_{\mathfrak{q}}}(P_{K_\infty,S,T,\mathfrak{q}}) = \Lambda_{\mathfrak{q}} \text{ and } \mathrm{Fitt}_{\Lambda_{\mathfrak{q}}}(X_{K_\infty,S\setminus V_{\mathfrak{q}},\mathfrak{q}}) = \Lambda_{\mathfrak{q}}.$$

So a proof of $\mathrm{hIMC}(K_\infty/k,S,T)$ seems attainable considering Lemma 5.2.11.

6.3.6 A relation between the Limit Conjectures

The main result of Johnson-Leung and Kings in [JLK11] can be stated as follows:

Theorem 6.3.24. *If* (VanishAss) *holds, then* $\mathrm{LC}_{\mathrm{JoKi}}(K'_\infty/k,p)$ *holds. In particular, if* p *splits in* k *we have an unconditional result.*

Remark 6.3.25. As we mentioned in Remark 6.3.6 we normalized the element $\zeta(\mathfrak{f})$ differently as in [JLK11], but as they also use a different Dirichlet regulator map (without a minus sign) our formulation of Conjecture 6.3.7 should still hold with the same proofs as in [JLK11].

So it is natural to ask about the relation between the Limit Conjectures $\mathrm{LC}(K'_\infty/k,p)$ and $\mathrm{LC}_{\mathrm{JoKi}}(K'/k,p)$. The following remark outlines a strategy how one could possibly prove the implication that would be most interesting to us.

Remark 6.3.26. The assertion we are interested in can be formulated as follows: If (VanishAss) holds, then

$$\mathrm{LC}_{\mathrm{JoKi}}(K'_\infty/k,p) \text{ implies } \mathrm{LC}(K'_\infty/k,p)$$

for all the cases where both are defined.

Now working towards a proof of this assertion we let $S' := \{v \text{ places in } k \text{ with } v \mid \mathfrak{f}p\}$ and $S := \{v \text{ places in } k \text{ with } v \mid \mathfrak{f}p\infty\}$. Recall that we have for the cohomology of Δ^∞

a) $H^i(\Delta^\infty) = 0$ for $i \neq 1, 2$.

b)
$$H^1(\Delta^\infty) = U_{K'_\infty,S}.$$

c) We also have a short exact sequence

$$0 \to P_{K'_\infty,S} \to H^2(\Delta^\infty) \to X_{K'_\infty,S} \to 0$$

and similarly for the cohomology of [JLK11].

a)
$$H^1(\mathcal{O}_{k,S}, \Omega(1)) = U_{K'_\infty,S}$$

b)
$$0 \to P_{K'_\infty,S} \to H^2(\mathcal{O}_{k,S}, \Omega(1)) \to X_{K'_\infty,S'} \to 0$$

which is essentially Proposition 5.1 of [BG03] with small adaptations, because the base field is k instead of $\mathbb{Q}$.

As in (22) on p. 96 in [Ble06] we obtain

$$0 \to X^\infty_{\{v|\mathfrak{f}p\}} \to X^\infty_{\{v|\mathfrak{f}p\infty\}} \to Y^\infty_{\{v|\infty\}} \to 0 \tag{6.11}$$

and from the definition of τ it follows that

$$\mathrm{Det}_\Omega(\tau \cdot \Omega) = \mathrm{Det}_\Omega(Y^\infty_{\{v|\infty\}}). \tag{6.12}$$

Combining (6.11) and (6.12) as well as Proposition 6.1.2 we get

$$\mathrm{Det}_\Omega(X_{K'_\infty,S}) = \mathrm{Det}_\Omega(\tau \cdot \Omega) \otimes \mathrm{Det}_\Omega(X_{K'_\infty,S'}). \tag{6.13}$$

Recall that we have for an integral ideal $\mathfrak{a}$ of k prime to $6\mathfrak{f}p$

$${}_{\mathfrak{a}}\zeta_{\mathfrak{f}p^n}^{-1} = \psi(1; \mathfrak{f}p^n, \mathfrak{a}) \tag{6.14}$$

by (2.9) and by our definition of $\zeta(\mathfrak{f}p^n)$ we obtain $\psi_{\mathfrak{f},\mathfrak{a}} = {}_{\mathfrak{a}}\zeta(\mathfrak{f})$.

We also have that $\Omega = \Lambda_{K'_\infty/k}$. As in Lemma 6.6 in [JLK11] one obtains

$$\mathrm{Det}_\Omega(\zeta(\mathfrak{f})\Omega) = c_{\mathfrak{a}}^{-1}\mathrm{Det}_\Omega({}_{\mathfrak{a}}\zeta(\mathfrak{f})\Omega) = \mathrm{Det}_\Omega(J_\Omega\zeta((\mathfrak{f}))). \tag{6.15}$$

With Lemma 6.1.4 it suffices to show the assertion after localising at each $\mathfrak{q}$.

To shorten the notation we set $U^\infty_S := U_{K'_\infty,S}$ and P^∞_S, X^∞_S analogously; also $\mathcal{L} := \mathcal{L}_{K'_\infty}$.

Case: localization at regular prime ideal Let $\mathfrak{q}$ now be a regular prime ideal of height 1. For this case it suffices to show that

$$\mathcal{L} \cdot \Omega_{\mathfrak{q}} = \mathrm{Det}_{\Omega_{\mathfrak{q}}}^{-1}(H^1(\Delta_{\mathfrak{q}}^{\infty})) \otimes \mathrm{Det}_{\Omega_{\mathfrak{q}}}(H^2(\Delta_{\mathfrak{q}}^{\infty}))$$

and therefore that

$$\mathcal{L} \cdot \Omega_{\mathfrak{q}} = \mathrm{Det}_{\Omega_{\mathfrak{q}}}^{-1}(U_{S,\mathfrak{q}}^{\infty}) \otimes \mathrm{Det}_{\Omega_{\mathfrak{q}}}(P_{S,\mathfrak{q}}^{\infty}) \otimes \mathrm{Det}_{\Omega_{\mathfrak{q}}}(X_{S,\mathfrak{q}}^{\infty})$$

and with (6.13) that

$$\mathcal{L} \cdot \Omega_{\mathfrak{q}} = \mathrm{Det}_{\Omega_{\mathfrak{q}}}^{-1}(U_{S,\mathfrak{q}}^{\infty}) \otimes \mathrm{Det}_{\Omega_{\mathfrak{q}}}(P_{S,\mathfrak{q}}^{\infty}) \otimes \mathrm{Det}_{\Omega_{\mathfrak{q}}}(\tau\Omega_{\mathfrak{q}}) \otimes \mathrm{Det}_{\Omega_{\mathfrak{q}}}(X_{S',\mathfrak{q}}^{\infty}).$$

Using the definition of $\mathcal{L}$ it suffices to show that

$$c_{\mathfrak{a}}\psi_{\mathfrak{f},\mathfrak{a}}^{-1}\Omega_{\mathfrak{q}} = \mathrm{Det}_{\Omega_{\mathfrak{q}}}^{-1}(U_{S,\mathfrak{q}}^{\infty}) \otimes \mathrm{Det}_{\Omega_{\mathfrak{q}}}(P_{S,\mathfrak{q}}^{\infty}) \otimes \mathrm{Det}_{\Omega_{\mathfrak{q}}}(X_{S',\mathfrak{q}}^{\infty}). \tag{6.16}$$

So assume now that $\mathrm{LC}_{\mathrm{JoKi}}(K'_{\infty}/k,p)$ holds, i.e.

$$\mathrm{Det}_{\Omega}(H^1(\mathcal{O}_{k,S},\Omega(1))/J_{\Omega}(\zeta(\mathfrak{f}))) = \mathrm{Det}_{\Omega}(H^2(\mathcal{O}_{k,S},\Omega(1)))$$

we can use the computation of the cohomology, Proposition 6.1.2 a) and localize at $\mathfrak{q}$ to get

$$\mathrm{Det}_{\Omega_{\mathfrak{q}}}(U_{S,\mathfrak{q}}^{\infty}/J_{\Omega}(\zeta(\mathfrak{f}))_{\mathfrak{q}}) = \mathrm{Det}_{\Omega_{\mathfrak{q}}}(P_{S,\mathfrak{q}}^{\infty}) \otimes \mathrm{Det}_{\Omega_{\mathfrak{q}}}(X_{S',\mathfrak{q}}^{\infty}).$$

Next we consider the short exact sequence

$$0 \to J_{\Omega}\zeta(\mathfrak{f}) \to U_S^{\infty} \to U_S^{\infty}/(J_{\Omega}\zeta(\mathfrak{f})) \to 0$$

and Proposition 6.1.2 a) to get

$$\mathrm{Det}_{\Omega}(U_S^{\infty}) = \mathrm{Det}_{\Omega}(J_{\Omega}(\zeta(\mathfrak{f}))) \otimes \mathrm{Det}_{\Omega}(U_S^{\infty}/(J_{\Omega}\zeta(\mathfrak{f})))$$

to obtain

$$\mathrm{Det}_{\Omega_{\mathfrak{q}}}^{-1}(J_{\Omega}(\zeta(\mathfrak{f}))_{\mathfrak{q}}) \otimes \mathrm{Det}_{\Omega_{\mathfrak{q}}}(U_{S,\mathfrak{q}}^{\infty}) = \mathrm{Det}_{\Omega_{\mathfrak{q}}}(P_{S,\mathfrak{q}}^{\infty}) \otimes \mathrm{Det}_{\Omega_{\mathfrak{q}}}(X_{S',\mathfrak{q}}^{\infty})$$

and with (6.15) we obtain

$$\mathrm{Det}_{\Omega_{\mathfrak{q}}}(U_{S,\mathfrak{q}}^{\infty}) = c_{\mathfrak{a}}^{-1}\mathrm{Det}_{\Omega_{\mathfrak{q}}}({}_{\mathfrak{a}}\zeta(\mathfrak{f})\Omega_{\mathfrak{q}}) \otimes \mathrm{Det}_{\Omega_{\mathfrak{q}}}(P_{S,\mathfrak{q}}^{\infty}) \otimes \mathrm{Det}_{\Omega_{\mathfrak{q}}}(X_{S',\mathfrak{q}}^{\infty}).$$

as $\psi_{\mathfrak{f},\mathfrak{a}} = {}_{\mathfrak{a}}\zeta(\mathfrak{f})$ so one gets

$$c_{\mathfrak{a}}\psi_{\mathfrak{f},\mathfrak{a}}^{-1}\Omega_{\mathfrak{q}} = \mathrm{Det}_{\Omega_{\mathfrak{q}}}^{-1}(U_{S,\mathfrak{q}}^{\infty}) \otimes \mathrm{Det}_{\Omega_{\mathfrak{q}}}(P_{S,\mathfrak{q}}^{\infty}) \otimes \mathrm{Det}_{\Omega_{\mathfrak{q}}}(X_{S',\mathfrak{q}}^{\infty})$$

as we wished.

Case: localization at singular prime ideals Let $\mathfrak{q}$ now be a singular height 1 prime ideal. So assume now that $\mathrm{LC}_{\mathrm{JoKi}}(K'_{\infty}/k,p)$ holds, i.e.

$$\mathrm{Det}_{\Omega_{\mathfrak{q}}}(U_{S,\mathfrak{q}}^{\infty}/J_{\Omega_{\mathfrak{q}}}(\zeta(\mathfrak{f}))) = \mathrm{Det}_{\Omega_{\mathfrak{q}}}(H^2(\mathcal{O}_{k,S},\Omega(1))_{\mathfrak{q}}).$$

Since we are also assuming (VanishAss), i.e.

$$H^2(\mathcal{O}_{k,S},\Omega(1))_{\mathfrak{q}} = 0.$$

and using (6.15) we get that

$$H^1(\Delta_{\mathfrak{q}}^\infty) = U_{S,\mathfrak{q}}^\infty = c_{\mathfrak{a}}^{-1}\psi_{\mathfrak{f},\mathfrak{a}}\Omega_{\mathfrak{q}}. \tag{6.17}$$

So we can conclude that $\Delta_{\mathfrak{q}}^\infty$ has perfect cohomology, i.e.

$$\mathrm{Det}_{\Omega_{\mathfrak{q}}}(\Delta_{\mathfrak{q}}^\infty) = \mathrm{Det}_{\Omega_{\mathfrak{q}}}^{-1}(H^1(\Delta_{\mathfrak{q}}^\infty)) \otimes \mathrm{Det}_{\Omega_{\mathfrak{q}}}(H^2(\Delta_{\mathfrak{q}}^\infty)).$$

So one has to check that

$$\mathcal{L} \cdot \Omega_{\mathfrak{q}} = \mathrm{Det}_{\Omega_{\mathfrak{q}}}^{-1}(U_{S,\mathfrak{q}}^\infty) \otimes \mathrm{Det}(P_{S,\mathfrak{q}}^\infty) \otimes \mathrm{Det}_{\Omega_{\mathfrak{q}}}(\tau\Omega_{\mathfrak{q}}) \otimes \mathrm{Det}(X_{S,\mathfrak{q}}^\infty)$$

which is equivalent to checking

$$c_{\mathfrak{a}}\psi_{\mathfrak{f},\mathfrak{a}}^{-1} \cdot \Omega_{\mathfrak{q}} = \mathrm{Det}_{\Omega_{\mathfrak{q}}}^{-1}(U_{S,\mathfrak{q}}^\infty) \otimes \mathrm{Det}_{\Omega_{\mathfrak{q}}}(H^2(\mathcal{O}_{k,S}, \Omega(1))_{\mathfrak{q}})$$

but this holds because of (VanishAss) and (6.17).

This is labelled a sketch because there a still some delicate details, concerning the isomorphism from [JLK11] etc., one has to consider in order to obtain a rigorous proof.

Remark 6.3.27. We give here a proof strategy for: $\mathrm{LC}(K'_\infty/k, p)$ implies $\mathrm{LC}_{\mathrm{JoKi}}(K'_\infty/k, p)$ under certain (rather strong) conditions. In fact, assume $H^i(\Delta_{\mathfrak{q}}^\infty)$ are free $\Omega_{\mathfrak{q}}$-modules for each singular height one prime ideal $\mathfrak{q}$ and all i. Then we can show that

$$\mathrm{LC}(K'_\infty/k, p) \Rightarrow \mathrm{LC}_{\mathrm{JoKi}}(K'_\infty/k, p)$$

for all the cases where both are defined.
For regular prime ideals of height 1 we assume $\mathrm{LC}(K'_\infty/k, p)$ and get (6.16), with the considerations above one obtains again for each regular height 1 prime ideal $\mathfrak{q}$:

$$\mathrm{Det}_{\Omega_{\mathfrak{q}}}(H^1(\mathcal{O}_{k,S}, \Omega(1))/J_\Omega(\zeta(\mathfrak{f})))_{\mathfrak{q}} = \mathrm{Det}_{\Omega}(H^2(\mathcal{O}_{k,S}, \Omega(1))_{\mathfrak{q}}).$$

For singular prime ideals of height 1 we assume that $\mathrm{LC}(K'_\infty/k, p)$ holds and with the assumption that the cohomology modules of $\Delta_{\mathfrak{q}}^\infty$ are free $\Omega_{\mathfrak{q}}$-modules the complex has perfect cohomology, so under these assumptions one can treat this case as the case of regular prime ideals. We note that for the complex $\Delta_{K_\infty/k}^\infty$ in the split case with $K_\infty := k(\mathfrak{f}\mathfrak{p}^\infty)$ it can be shown that the freeness assumption we need is fulfilled.

6.4 Descent to finite levels as in [BKS17]

We continue to use the notation of Section 6.3.5 and we recall that we have an extension F_∞/k with

$$\mathcal{G} := \mathrm{Gal}(F_\infty/k) = \Gamma \times H \text{ with } \Gamma \cong \mathbb{Z}_p \text{ and } H \text{ a finite abelian group.}$$

Let F be a finite extension over k in F_∞ with Galois group $G := \mathrm{Gal}(F/k)$. We set

$$V := V_\chi = \{v \in S : v \text{ splits completely in } L_{\chi,\infty}\} \qquad r := |V|$$
$$V' := V'_\chi := \text{a maximal set s.t. } |V'_\chi| = \min\{|\{v \in S : \chi(G_v) = 1\}|, |S|-1\} \quad r' := \#V', e := r' - r$$

Fix a representative of $C_{F_\infty,S,T}$:

$$\Pi_{F_\infty} \xrightarrow{\psi_\infty} \Pi_{F_\infty},$$

where the first term is in degree zero, and Π_{F_∞} is a free Λ-module with basis $b_1, \dots, b_d$. This representative is chosen so that the natural surjection

$$\Pi_{F_\infty} \to H^1(\mathrm{C}_{F_\infty,S,T}) \to \mathrm{X}_{F_\infty,S}$$

sends b_i to $w_i - w_0$ for every $i \in \{1, \dots, r'\}$.

We define the height one prime ideal $\mathfrak{p} := \ker(\Lambda \xrightarrow{\chi} \mathbb{Q}_p(\chi))$ and the discrete valuation ring $\Lambda_\mathfrak{p}$. We write P for its maximal ideal and we see that χ induces an isomorphism $E := \Lambda_\mathfrak{p}/P \cong \mathbb{Q}_p(\chi)$. We call S and T admissible sets if S contains $S_\infty(k) \cup S_{ram}(F_\infty/k) \cup S_p(k)$ and two places of unequal residue characteristics, respectively. The main result of [BKS17] reads as follows:

Theorem 6.4.1. *[BKS17, Thm. 5.2] Fix a prime p. Let F be a finite extension of k in F_∞ and S and T admissible sets. Assume that the following conditions hold:*

(R) For every $\chi \in \widehat{G}$, we have $r_S(\chi) < |S|$.

(S) No finite place splits completely in k_∞.

(hIMC) *The conjecture* hIMC$(F_\infty/k, S, T)$ *holds.*

(F) For every $\chi \in \widehat{G}$, the module of Γ_χ-coinvariants of $P_{L_{\chi,\infty}/k,S,T}$ is finite.

(MRS) For every $\chi \in \widehat{G}$, the conjecture MRS$(F_\infty/k, S, T, \chi, V')$ *is valid for a maximal set V', so that*

$$|V'| = \min\{|\{v \in S \mid \chi(G_v) = 1\}|, |S|-1\}.$$

Then the conjecture eTNC$(h^0(\mathrm{Spec}(F)), \mathbb{Z}_p[G])$ *holds.*

Overview of the proof of Theorem 5.2 in [BKS17] Fix a character $\chi \in \widehat{G}$. Assuming Condition (F) holds, Lemma 5.12 in [BKS17] computes the cohomology of $\mathrm{C}_{F_\infty,S,T} \otimes \Lambda_\mathfrak{p}$:

$$H^0(\mathrm{C}_{F_\infty,S,T} \otimes \Lambda_\mathfrak{p}) \cong \mathrm{U}_{F_\infty,S,T} \otimes \Lambda_\mathfrak{p}, \qquad H^1(\mathrm{C}_{F_\infty,S,T} \otimes \Lambda_\mathfrak{p}) \cong \mathrm{X}_{F_\infty,S} \otimes \Lambda_\mathfrak{p},$$
$$H^1(\mathrm{C}_{F_\infty,S,T} \otimes \Lambda_\mathfrak{p})_{tors} \cong X_{F_\infty,S\setminus V} \otimes \Lambda_\mathfrak{p}, \qquad H^1(\mathrm{C}_{F_\infty,S,T} \otimes \Lambda_\mathfrak{p})_{tors} \text{ is ann. by } P,$$
$$H^1(\mathrm{C}_{F_\infty,S,T} \otimes \Lambda_\mathfrak{p})/H^1(\mathrm{C}_{F_\infty,S,T} \otimes \Lambda_\mathfrak{p})_{tors} \cong Y_{F_\infty,V} \otimes \Lambda_\mathfrak{p}, \qquad \dim_E(H^1(\mathrm{C}_{F_\infty,S,T} \otimes \Lambda_\mathfrak{p})_{tors}) = e.$$

The *Bockstein map*

$$\begin{aligned}\beta : H^0((\mathrm{C}_{F_\infty,S,T} \otimes \Lambda_\mathfrak{p}) \otimes E) &\to H^1(\mathrm{C}_{F_\infty,S,T} \otimes \Lambda_\mathfrak{p} \otimes_{\Lambda_\mathfrak{p}} P) \\ &= H^1(\mathrm{C}_{F_\infty,S,T} \otimes \Lambda_\mathfrak{p}) \otimes_{\Lambda_\mathfrak{p}} P \\ &\to H^1((\mathrm{C}_{F_\infty,S,T} \otimes \Lambda_\mathfrak{p}) \otimes E) \otimes_E P/P^2\end{aligned}$$

is induced by the exact triangle

$$\mathrm{C}_{F_\infty,S,T} \otimes \Lambda_{\mathfrak{p}} \otimes_{\Lambda_{\mathfrak{p}}} P \to \mathrm{C}_{F_\infty,S,T} \otimes \Lambda_{\mathfrak{p}} \to (\mathrm{C}_{F_\infty,S,T} \otimes \Lambda_{\mathfrak{p}}) \otimes E.$$

The Bockstein map β itself is then induced by the map

$$\begin{aligned} \mathrm{U}_{L_\chi,S,T} &\to X_{L_\chi,S} \otimes I(\Gamma_\chi)/I(\Gamma_\chi)^2 \\ a &\mapsto \sum_{w \in S(L_\chi)} w \otimes (\mathrm{rec}_w(a) - 1) \end{aligned}$$

which is Proposition 5.14 in [BKS17].

For any intermediate field M of F_∞, we denote by $\mathcal{L}_{M/k,S,T}$ the image of the element $\mathcal{L}_{F_\infty/k,S,T}$ of $\mathrm{Det}_\Lambda(C_{L_\chi,S,S,T})$ under the isomorphism

$$\mathbb{Z}_p[\mathrm{Gal}(M/k)] \otimes_\Lambda \mathrm{Det}_\Lambda(C_{F_\infty,S,T}) \cong \mathrm{Det}_{\mathbb{Z}_p[\mathrm{Gal}(M/k)]}(C_{M,S,T}).$$

Furthermore, Proposition 5.16 in [BKS17] and Lemma 5.17 in [BKS17] show that:

a) $e_\chi \mathrm{Rec}_\infty$ is injective.

b)
$$\nu_n^{-1}(\mathcal{N}_n(\pi^V_{L_{\chi,n}}(\mathcal{L}_{L_{\chi,n},S,T}))) = (-1)^{re}\mathrm{Rec}_n\left(\pi^{V'}_{L_\chi/k,S,T}(\mathcal{L}_{L_{\chi,n},S,T})\right)$$

As we have assumed condition (R) it suffices, by Proposition 6.2.10, to show that there exists a $\mathbb{Z}_p[G]$-basis $\mathcal{L}_{F,S,T}$ of $\mathrm{Det}_{\mathbb{Z}_p[G]}(\mathrm{C}_{F,S,T})$ such that the image of $e_\chi\mathcal{L}_{F,S,T}$ under the isomorphism

$$g_\chi : \ e_\chi\mathbb{C}_p\mathrm{Det}_{\mathbb{Z}_p[G]}(\mathrm{C}_{F,S,T}) \cong e_\chi\mathbb{C}_p \bigwedge^{r_S(\chi)} U_{L_\chi,S,T}$$

coincides with $e_\chi\epsilon^{V'}_{L_\chi,S,T}$ with V' places of S which split completely in L_χ, i.e.

$$g_\chi(e_\chi\mathcal{L}_{F,S,T}) = e_\chi\epsilon^{V'}_{L_\chi,S,T}. \tag{6.18}$$

With Remark 6.3.19 we deduce from hIMC$(F_\infty/k,S,T)$

$$j^V_{L_{\chi,n}}(\mathcal{L}_{F_\infty,S,T}) = \epsilon^V_{L_{\chi,n},S,T}$$

so we can write this as

$$\pi^V_{L_{\chi,n}}(\mathcal{L}_{L_{\chi,n},S,T}) = \epsilon^V_{L_{\chi,n},S,T}$$

so with b) from above we obtain

$$\nu_n^{-1}(\mathcal{N}_n(\epsilon^V_{L_{\chi,n}})) = (-1)^{re}\mathrm{Rec}_n(\pi^{V'}_{L_\chi,S,T}(\mathcal{L}_{L_\chi,S,T})).$$

Now we set $\kappa_n := \nu_n^{-1}(\mathcal{N}_n(\epsilon^V_{L_{\chi,n}}))$ and $\kappa := \varprojlim_n \kappa_n = (-1)^{re}\mathrm{Rec}_\infty(\pi^{V'}_{L_\chi,S,T}(\mathcal{L}_{L_\chi,S,T}))$.

Recall that MRS$(F_\infty/k,S,T,\chi,V')$ with $\chi \in \widehat{G}$ implies

$$e_\chi\kappa = (-1)^{re}e_\chi\mathrm{Rec}_\infty(\epsilon^{V'}_{L_\chi/k,S,T})$$

and so this implies

$$(-1)^{re}e_\chi\mathrm{Rec}_\infty(\epsilon^{V'}_{L_\chi/k,S,T}) = (-1)^{re}e_\chi\mathrm{Rec}_\infty(\pi^{V'}_{L_\chi,S,T}(\mathcal{L}_{L_\chi,S,T})).$$

With b) $e_\chi\mathrm{Rec}_\infty$ is injective, so we get

$$\varepsilon^{V'}_{L_\chi,S,T} = \pi^{V'}_{L_\chi,S,T}(\mathcal{L}_{L_\chi,S,T}).$$

So because $\varepsilon^{V'}_{L_\chi,S,T} = \pi^{V'}_{L_\chi,S,T}(\mathcal{L}_{L_\chi,S,T})$ implies $g_\chi(e_\chi\mathcal{L}_{F,S,T}) = e_\chi\epsilon^{V'}_{L_\chi,S,T}$, (6.18) is shown and therefore the theorem follows.

Applications to abelian extensions of imaginary quadratic fields Let k be an imaginary quadratic field and fix a prime $p > 3$. Let $\mathfrak{f}$ be an integral ideal of $\mathcal{O}_k$ with $(\mathfrak{f}, p) = 1$ and $w(\mathfrak{f}) = 1$ and let $\mathfrak{p}$ be a prime ideal above p in $\mathcal{O}_k$. Assume that (H2) holds, i.e. that $p \nmid h_k$. Then we set $K'_\infty := k(\mathfrak{f}p^\infty)$, $K_0 := k(\mathfrak{f}\mathfrak{p})$ and we obtain the direct decomposition

$$\mathcal{G} := \mathrm{Gal}(K'_\infty/k) \cong \Gamma' \times H$$

with $H \cong \mathrm{Gal}(K_0/k)$ and $(K'_\infty)^H \cong \mathbb{Z}_p^2$. Now let K_∞ be a $\mathbb{Z}_p$-extension of K_0. Then there are topological generators γ_1, γ_2 of Γ' such that $(K'_\infty)^{\langle\gamma_1\rangle} = K_\infty$.

Now we want to state the following corollary of Theorem 6.4.1 by combining results from above.

Corollary 6.4.2. *Let $(K_\infty/k, p)$ be as above and let F be a finite extension of k in K_∞ with Galois group $G := \mathrm{Gal}(F/k)$ and assume the following conditions:*

(S) *No finite place splits completely in k_∞.*

(F) *For every $\chi \in \widehat{G}$, the module of Γ_χ-coinvariants of $P_{L_{\chi,\infty}/k,S,T}$ is finite.*

*(*hIMC*)* *The conjecture* hIMC$(K_\infty/k, S, T)$ *holds.*

*(*DimAss*)* *The set $\{e_\chi \varepsilon_L^V, e_\chi \kappa\}$ is a $\mathbb{C}_p$-basis of $e_\chi \mathbb{C}_p U_{L_\chi,S}$ for each character described on page 78, i.e. the ones remaining after the reduction of the problem in the proof of Theorems 5.2.1, 5.2.2 and 5.2.3.*

Then eTNC$(h^0(\mathrm{Spec}(F)), \mathbb{Z}_p[G])$ *holds.*

Proof. Obviously we want to use Theorem 6.4.1. First of all we choose an admissible set S such that (R) holds, i.e. $S \supset \{\infty\} \cup S_p(k) \cup S_{ram}(K_\infty/k)$. According to Theorem 5.2.2 (DimAss) implies that MRS$(K_\infty/k, S, T, \chi, V')$ holds and as we assume the other conditions of Theorem 6.4.1 the result follows. □

Remark 6.4.3. The condition (DimAss) is known in certain cases by Lemma 5.2.21 for non-split primes.

Outline of a 'new' proof strategy of the main result in [Ble06]. Recall that we assume $p \geq 5$ is prime and splits in k with $p \nmid h_k$. Let F be some finite abelian extension of k with Galois group G. Then we want to show that eTNC$(h^0(\mathrm{Spec}(F)), \mathbb{Z}_p[G])$ holds.

First of all by global class field theory we always find an $\mathfrak{f}$ and an integer n such that $F \subseteq k(\mathfrak{f}\mathfrak{p}^n)$ with $\mathfrak{f}$ satisfying the necessary conditions. From the proof of Theorem 5.2.1 we know that Condition (DimAss) holds in this case and Condition (F) by Lemma 6.3.15, Remark 6.3.16 and the well-known properties of the generalized Gross-Kuzmin conjecture. Considering Remarks 6.3.4 and 6.3.23 one should be able to show that hIMC$(K_\infty/k, S, T)$ is valid in this particular situation, and then with Corollary 6.4.2 we would get the result.

6.5 Descent for abelian extensions of imaginary quadratic fields in the inert case

6.5.1 Statement of the main result of this chapter

Theorem 6.5.1. *Let L be a finite abelian extension of an imaginary quadratic field k with Galois group G. Let p be an odd inert prime in k such that $p \nmid h_k$. If for each $\chi \in \widehat{G}/\sim_{\mathbb{Q}_p}$ there is a $\mathbb{Z}_p$-extension L_∞, in which no finite prime dividing $p\mathfrak{f}_L$ splits completely and which satisfies* $\mathrm{LC}(L_\infty/k, p)$ *and Condition* $\mathrm{F}(L_\infty/k)$, *then* $\mathrm{eTNC}(h^0(\mathrm{Spec}(L)), \mathbb{Z}_p[G])$ *holds.*

Remark 6.5.2. a) For split primes $p \geq 3$ with $p \nmid h_k$ we know $\mathrm{eTNC}(h^0(\mathrm{Spec}(L), \mathbb{Z}_p[G])$ by work of Bley in [Ble06]. In fact our treatment relies heavily on the methods used in [Ble06] and can be seen as adaptation of the methods for inert primes. The main new ingredient is Theorem 4.1.16 in Chapter 4.

b) The main result of Chapter 4 also holds for ramified primes $p > 3$ and it is not too hard to adapt the descent computations to the ramified case. But for the sake of clarity of the presentation we do not present these descent computations.

c) For $p = 3$ in the ramified case and especially $p = 2$ the situation is more complicated because we do only have partial results for these primes in Chapter 4 and similar reasons restricting us there, would also create problems in the descent computations.

d) The condition $p \nmid h_k$ should not be necessary, but nevertheless we were unable to remove this condition. It should certainly be possible to relax the condition at least to only assuming $k(1) \cap \mathbb{K} = k$, where $\mathbb{K}$ is the composite of all $\mathbb{Z}_p$-extensions of k.

e) With the functoriality properties of the integrality part of the eTNC (Remark 6.2.5 e)) it is easy to see that the problem can be reduced to considering only ray class fields.

Remark 6.5.3. Condition $\mathrm{F}(L_\infty/k)$ is an assumption based on what is true for the the split imaginary quadratic case (see Lemma 6.3.15). But as we have already seen in Section 6.3.4 it is not true for each cyclotomic $\mathbb{Z}_p$-extension, so it is very likely that it does not hold for each extension as constructed in Corollary 6.5.19. In order to hedge against this possibility we remark that first of all we have a certain choice for each character. Moreover, Condition $\mathrm{F}(L_\infty/k)$ is sufficient for our computations, but what we really use is $\left(P_{L_\infty/k}\right)_{\mathfrak{q}_\chi} = 0$ so that we obtain with (6.26) an exact sequence

$$0 \to U_{L_\infty/k,\mathfrak{q}_\chi} \to U_{L_\infty/k,S,\mathfrak{q}_\chi} \to Y_{L_\infty/k,\{w|\mathfrak{p}\},\mathfrak{q}_\chi} \to 0. \tag{6.19}$$

for each choice of an extension L_∞ corresponding to a character that has trivial zeroes.

But as proposed by one of the referees, with a finer analysis of the computations in the trivial zeroes case one should be able to replace Condition $\mathrm{F}(L_\infty/k)$ with Condition $\mathrm{F}(L_\infty/k, S)$ by introducing an auxiliary module and therefore get a four term exact sequence instead of (6.19). This extra term introduced here should then cancel out in the computations. Evidence that this should work is that [BKS17] also only assumes Condition $\mathrm{F}(L_\infty/k, S)$ for doing essentially the same descent in a more general context.

The rest of this chapter is now devoted the proof of Theorem 6.5.1. We fix an imaginary quadratic field k, an odd rational prime p and a prime ideal $\mathfrak{p}$ above p. Because of Remark 6.5.2 e) we can assume without loss of generality that $L := k(\mathfrak{f})$ for $\mathfrak{f}$ an integral ideal with $w(\mathfrak{f}) = 1$. We denote the Galois group of L/k by G and by $\mathfrak{f}_0$ the part that is coprime to $\mathfrak{p}$. Furthermore, we set

$$M = h^0(\mathrm{Spec}(L)) \qquad A = \mathbb{Q}[G] \qquad \mathcal{A} = \mathbb{Z}[G].$$

The rest of this section is organized as follows:

Sec. 6.5.2 contains the computation of the χ-component of ${}_A\vartheta_\infty(L^*({}_AM,0)^{-1})$.

Sec. 6.5.3 contains the strategy of the descent as well as some important lemmas from [Fla04] which we need for our computation.

Sec. 6.5.4 contains some preparations for the descent computations.

Sec. 6.5.5 treats the no trivial zeroes case.

Sec. 6.5.6 treats the trivial zeroes case, which is the key part.

Sec. 6.5.7 finishes the proof by treating the case of the trivial character.

6.5.2 Computation of the χ-component of ${}_A\vartheta_\infty(L^*({}_AM,0)^{-1})$

For each $\chi \in \widehat{G}/\sim_{\mathbb{Q}}$ we have $e_\chi = \sum_{\eta\in\chi} e_\eta \in \mathbb{Q}[G]$ and we denote by $\mathbb{Q}(\chi)$ the field generated by the values of η for any $\eta \in \chi$. Furthermore, we have $\mathbb{Q}[G] \cong \prod_{\chi\in\widehat{G}/\sim_{\mathbb{Q}}} \mathbb{Q}(\chi)$.

There is a canonical isomorphism

$$\begin{aligned}\Xi({}_AM)^{\#} &\cong \mathrm{Det}_A^{-1}(\mathcal{O}_L^\times \otimes \mathbb{Q}) \otimes \mathrm{Det}_A(X_{L,S_\infty} \otimes \mathbb{Q}) \\ &\cong \prod_\chi \mathrm{Det}_{\mathbb{Q}(\chi)}^{-1}(\mathcal{O}_L^\times \otimes \mathbb{Q}(\chi)) \otimes \mathrm{Det}_{\mathbb{Q}(\chi)}(X_{L,S_\infty} \otimes \mathbb{Q}(\chi))\end{aligned} \tag{6.20}$$

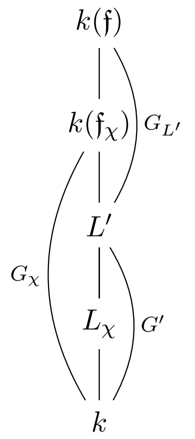
By abuse of notation we denote one of the elements of $\chi \in \widehat{G}/\sim_{\mathbb{Q}}$ also by χ. Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_k$ such that $(\mathfrak{a}, 6p\mathfrak{f}) = 1$, $L_\chi := L^{\ker(\chi)}$ and $\mathfrak{f}_\chi$ the conductor of χ, i.e the 'smallest' ideal such that $L_\chi \subset k(\mathfrak{f}_\chi)$. Let L' be such that $L_\chi \subseteq L' \subseteq k(\mathfrak{f}_\chi)$. Then we set

$$\psi_\chi := \begin{cases} \mathcal{N}_{k(\mathfrak{f}_\chi)/L'}(\psi(1;\mathfrak{f}_\chi,\mathfrak{a})) & \text{if } \mathfrak{f}_\chi \neq (1), \\ \mathcal{N}_{k(1)/L'}\left(\frac{\delta(\mathcal{O}_k,\mathfrak{a}^{-1})}{\delta(\mathfrak{p},\mathfrak{p}\mathfrak{a}^{-1})}\right) & \text{if } \mathfrak{f}_\chi = (1). \end{cases}$$

Proposition 6.5.4. *The χ-component of ${}_A\vartheta_\infty(L^*({}_AM,0)^{-1})$ is equal to*

$$\begin{cases} (\mathcal{N}(\mathfrak{a}) - \chi(\mathfrak{a})) \cdot w(\mathfrak{f}_\chi) \cdot [k(\mathfrak{f}) : L'] \cdot \psi_\chi^{-1} \otimes \tau_{|L'}, & \mathfrak{f}_\chi \neq (1), \\ (\mathcal{N}(\mathfrak{a}) - \chi(\mathfrak{a}))(1 - \chi(\mathfrak{p})^{-1}) \cdot w(1) \cdot [k(\mathfrak{f}) : L'] \cdot \psi_\chi^{-1} \otimes \tau_{|L'}, & \mathfrak{f}_\chi = (1), \chi \neq 1, \\ \frac{-w(1)}{h_k}, & \chi = 1. \end{cases}$$

Proof. Assume first that $\mathfrak{f}_\chi \neq (1)$ and that we are in the situation that can be illustrated as follows:



Then we can compute that

$$\begin{aligned} R(e_\chi \psi_\chi) &= -e_\chi \sum_{w \in S_\infty} \log |\psi_\chi|_w \cdot w = -\sum_{g \in G} \log |\psi_\chi^g| \chi^{-1}(g) e_\chi w \\ &= -\sum_{\tau \in G'} \sum_{\sigma \in G_{L'}} \chi^{-1}(\tau) \log |\psi_\chi^{\sigma\tau}| e_\chi w = -[k(\mathfrak{f}) : L'] \sum_{\tau \in G'} \chi^{-1}(\tau) \log |\psi_\chi^\tau| e_\chi w \end{aligned}$$

Furthermore, using $L_\chi \subseteq L' \subseteq k(\mathfrak{f}_\chi)$ it is easy to see that

$$\sum_{\tau \in G'} \chi^{-1}(\tau) \log |\psi_\chi^\tau| = \sum_{g \in G_\chi} \chi^{-1}(g) \log |\psi(1; \mathfrak{f}_\chi, \mathfrak{a})^g|$$

and from Proposition 2.3.9 we obtain

$$L^*(0, \chi^{-1}) = -\frac{1}{\mathcal{N}(\mathfrak{a}) - \chi(\mathfrak{a})} \frac{1}{w(\mathfrak{f}_\chi)} \sum_{g \in G_\chi} \chi^{-1}(g) \log |\psi(1; \mathfrak{f}_\chi, \mathfrak{a})^g|$$

So we obtain

$$R(e_\chi \psi_\chi) = w(\mathfrak{f}_\chi)(\mathcal{N}(\mathfrak{a}) - \chi(\mathfrak{a}))[k(\mathfrak{f}) : L'] L^*(0, \chi^{-1})$$

and then Definition 6.2.3 implies that

$$e_\chi \, {}_A\vartheta_\infty(L^*({}_AM, 0)^{-1})) = [k(\mathfrak{f}) : L'] w(\mathfrak{f}_\chi)(\mathcal{N}(\mathfrak{a}) - \chi(\mathfrak{a})) e_\chi \psi_\chi^{-1} \otimes \tau_{L'}$$

In the other cases the proofs are similar, where one uses as main input Corollary 2.3.7 resp. Corollary 2.3.5. Moreover, the proof for $\mathfrak{f}_\chi = (1)$ and $\chi \neq 1$ is also treated on page 91 in [Ble06]. □

6.5.3 Strategy of the descent

In this section we want to present the strategy for the descent. In order to do that we start by presenting a key lemma which will be used in a crucial way later.

Definition 6.5.5. Let R be a discrete valuation ring with uniformizer ω. For an R-module Q we put

$$Q_\omega := \{q \in Q : \omega q = 0\} \text{ and } Q/\omega := Q/\omega Q.$$

Lemma 6.5.6. *[Fla04, Lemma 5.7] Let R be a DVR with fraction field $Q(R)$, residue field κ and uniformizer ω. Suppose that:*

i) Δ *is a perfect complex of R-modules.*

ii) $\omega \cdot H^i(\Delta)_{tors} = 0$ *for all* i.

We define free R-modules M^i *by the short exact sequence*

$$0 \to H^i(\Delta)_\omega \to H^i(\Delta) \to M^i \to 0.$$

a) The exact triangle in the derived category of R-modules:

$$\Delta \xrightarrow{\omega} \Delta \to \Delta \otimes^{\mathbb{L}}_R \kappa \to \Delta[1]$$

induces an exact sequence of κ*-vector spaces*

$$0 \to H^i(\Delta)/\omega \to H^i(\Delta \otimes^{\mathbb{L}}_R \kappa) \to H^{i+1}(\Delta)_\omega \to 0.$$

b) There is an isomorphism

$$\begin{aligned}\mathrm{Det}_\kappa(H^i(\Delta \otimes^{\mathbb{L}}_R \kappa)) &\cong \mathrm{Det}_\kappa(H^i(\Delta)/\omega) \otimes_\kappa \mathrm{Det}_\kappa(H^{i+1}(\Delta)_\omega) \\ &\cong \mathrm{Det}_\kappa(H^i(\Delta)_\omega) \otimes_\kappa \mathrm{Det}_\kappa(M^i/\omega) \otimes_\kappa \mathrm{Det}_\kappa(H^{i+1}(\Delta)_\omega).\end{aligned}$$

and hence an isomorphism

$$\phi_\omega : \mathrm{Det}_\kappa(\Delta \otimes^{\mathbb{L}}_R \kappa) \cong \bigotimes_{i\in\mathbb{Z}} \mathrm{Det}_\kappa(M^i/\omega)^{(-1)^i}$$

c) For each i *fix an R-basis* β_i *of* $\mathrm{Det}_R(M^i)$. *Let* $e \in \mathbb{Z}$ *be such that*

$$b_\omega := \omega^e \bigotimes_{i\in\mathbb{Z}} (\beta_i)^{(-1)^i}$$

is an R-basis of

$$\mathrm{Det}_R(\Delta) \subseteq \mathrm{Det}_{Q(R)}(\Delta \otimes_R Q(R)) \cong \bigotimes_{i\in\mathbb{Z}} (\mathrm{Det}_{Q(R)}(M^i \otimes_R Q(R)))^{(-1)^i}.$$

Then the isomorphism

$$\mathrm{Det}_R(\Delta) \otimes_R \kappa \cong \mathrm{Det}_\kappa(\Delta \otimes^{\mathbb{L}}_R \kappa) \xrightarrow{\phi_\omega} \bigotimes_{i\in\mathbb{Z}} \mathrm{Det}_\kappa(M^i/\omega)^{(-1)^i}$$

maps $b_\omega \otimes 1$ *to* $\otimes_{i\in\mathbb{Z}} \overline{\beta}_i^{(-1)^i}$.

Assume the same setting as in Lemma 6.5.6 including the assumptions i) and ii). Moreover, assume that $\Delta \otimes^{\mathbb{L}}_R \kappa$ is concentrated in degrees 1 and 2. Then ϕ_ω from Lemma 6.5.6 is induced by the exact sequence of κ-vector spaces

$$0 \to M^1/\omega \to H^1(\Delta \otimes^{\mathbb{L}}_R \kappa) \xrightarrow{\beta_\omega} H^2(\Delta \otimes^{\mathbb{L}}_R \kappa) \to M^2/\omega \to 0, \tag{6.21}$$

where β_ω, a so called Bockstein map, is the composite

$$H^1(\Delta \otimes^{\mathbb{L}}_R \kappa) \to H^2(\Delta)_\omega \to H^2(\Delta)/\omega \to H^2(\Delta \otimes^{\mathbb{L}}_R \kappa). \tag{6.22}$$

Let N_∞ be either a rank one extension in K'_∞ such that L is in N_∞/k, or K'_∞ itself.

Remark 6.5.7. Set Λ to be $\Lambda_{N_\infty/k}$.

a) There is a ring homomorphism

$$\Lambda \to \mathcal{A}_p \subseteq A_p = \prod_{\chi \in \widehat{G}/\sim_{\mathbb{Q}_p}} \mathbb{Q}_p(\chi).$$

b) There are canonical isomorphisms of complexes

$$\Delta_{N_\infty/k} \otimes^{\mathbb{L}}_{\Lambda} \mathcal{A}_p \cong \Delta(L).$$

c) There are canonical isomorphisms of determinants

$$\mathrm{Det}_\Lambda(\Delta_{N_\infty/k}) \otimes_\Lambda \mathcal{A}_p \cong \mathrm{Det}_{\mathcal{A}_p}(\Delta(L)).$$

We assume from now on that p is inert in k.
Let K'_∞ be as above and let L_∞ be a rank one extension in K'_∞, with $k(\mathfrak{f}) \subset L_\infty$ and

$$\mathcal{G} := \mathrm{Gal}(L_\infty/k) \cong \mathrm{Gal}(L_\infty/L_0) \times \mathrm{Gal}(L_0/K_0) \times \mathrm{Gal}(K_0/k),$$

as constructed below in Corollary 6.5.19.

Remark 6.5.8. We denote a topological generator of $\mathrm{Gal}(L_\infty/L_0)$ by γ. We also sometimes write $\mathfrak{p}$ or simply p for $p\mathcal{O}_k$.

Let $\Lambda := \Lambda_{L_\infty/k}$ and by abuse of notation $\chi : \Lambda \to \mathbb{Q}_p(\chi)$. For this χ we set $\mathfrak{q}_\chi := \ker(\chi)$, $L_\chi := L^{\ker(\chi)}$ and $\kappa_\chi = \mathbb{Q}_p(\chi)$. Then we know that the following assertions hold:

a) $\mathfrak{q}_\chi$ is a regular prime ideal of Λ.

b) $\Lambda_{\mathfrak{q}_\chi}$ is a discrete valuation ring with field of fractions κ_χ.

Remark 6.5.9. Now with the proof of Lemma 5.11 in [BKS17] one can show that for $m := [L : L_0]$ the element γ^{p^m} is a generator of $\mathrm{Gal}(L_\infty/L)$ and that $\omega := 1 - \gamma^{p^m}$ is the uniformizer of $\Lambda_{\mathfrak{q}_\chi}$. Analysing our construction below we then can see that for $\mathfrak{f} := \mathfrak{f}_0\mathfrak{p}^\nu$ this m is equal to $\nu - 1$ if $\nu \geq 1$.

We set for the objects of Lemma 6.5.6: $R := \Lambda_{\mathfrak{q}_\chi}$ and $\Delta := (\Delta_{L_\infty/k})_{\mathfrak{q}_\chi}$ and recall that Δ is a perfect complex.

Definition 6.5.10. For any prime divisor $\mathfrak{l} \mid \mathfrak{f}_0$ we write $I_\mathfrak{l} \subset G_\mathfrak{l} \subset \mathcal{G}$ for the inertia and decomposition subgroups at $\mathfrak{l}$. Let $\mathfrak{l} \mid \mathfrak{f}_0$ then $Fr_\mathfrak{l}$ is a lift of the Frobenius of $G_\mathfrak{l}/I_\mathfrak{l}$ to $G_\mathfrak{l}$.

We view $\psi \in \widehat{\mathrm{Gal}(K_0/k)}$ as a character of $\mathcal{G}$ by inflation and we denote by $\mathfrak{d}$ the (prime to $\mathfrak{p}$)-part of the conductor of ψ. Note that if $\mathfrak{l} \nmid \mathfrak{d}$, i.e. $\psi\,|_{I_\mathfrak{l}} = 1$, then $Fr_\mathfrak{l}$ is a well-defined element in $\Lambda_{\mathfrak{q}_\chi}$.

Moreover, we define for $\mathfrak{l} \mid \mathfrak{f}_0$, $f_\mathfrak{l}$ as the residue degree at $\mathfrak{l}$ of L/k.

Definition 6.5.11. For $\mathfrak{l} \mid \mathfrak{f}_0$ the element $c_{\mathfrak{l},\gamma} \in \mathbb{Z}_p$ is defined by

$$\gamma^{c_{\mathfrak{l},\gamma} \cdot p^m} = \mathrm{Fr}_\mathfrak{l}^{-f_\mathfrak{l}}.$$

For $\mathfrak{p}$ we set

$$c_{\mathfrak{p},\gamma} := \pi_\gamma\left(\frac{1}{p}\log_p\left(\chi_{\mathrm{ell}}(\gamma^{p^m})\right)\right) \in \mathbb{Z}_p,$$

where π_γ is the projection map corresponding to the generator γ via

$$\Gamma \xrightarrow[\cong]{\chi_{\mathrm{ell}}} 1 + p\mathcal{O}_{k_\mathfrak{p}} \xrightarrow[\cong]{\log_p} p\mathcal{O}_{k_\mathfrak{p}} \xrightarrow[\cong]{\frac{1}{p}} \mathcal{O}_{k_\mathfrak{p}} \xrightarrow{\pi_\gamma} \mathbb{Z}_p.$$

Definition 6.5.12. Let K be any finite abelian extension of k. For a place $w \mid \mathfrak{p}$ in K/k and $u \in K$ we write $u_w = j_w(u)$, where

$$j_w : \mathbb{Q}^c \to \mathbb{Q}_p^c.$$

Recall the following lemma which we will use for the datum: $\mathcal{H} = \pi_1^{ét}(\mathrm{Spec}(\mathcal{O}_{L,S}))$, $\Gamma = \mathrm{Gal}(L_\infty/L)$, $\gamma_0 = \gamma^{p^m}$ and $M = \mathbb{Z}_p(1)$.

Lemma 6.5.13. *[Fla04, Lemma 5.9] Let Γ be a free $\mathbb{Z}_p$-module of rank one with generator γ_0 and $\mathcal{H} \to \Gamma$ be a surjection of profinite groups.*

Denote by $\theta \in H^1(\mathcal{H}, \mathbb{Z}_p)$ the unique homomorphism factoring through Γ for which $\theta(\gamma_0) = 1$ and put $\Lambda = \mathbb{Z}_p[[\Gamma]]$.

For any continuous $\mathbb{Z}_p[[G]]$-module M we have an exact triangle in the derived category of Λ-modules

$$R\Gamma(\mathcal{H}, M \otimes \Lambda) \xrightarrow{1-\gamma_0} R\Gamma(\mathcal{H}, M \otimes \Lambda) \to R\Gamma(\mathcal{H}, M \otimes \Lambda) \otimes_\Lambda^{\mathbb{L}} \mathbb{Z}_p \cong R\Gamma(\mathcal{H}, M).$$

Then the Bockstein map

$$\beta^i : H^i(\mathcal{H}, M) \to H^{i+1}(\mathcal{H}, M)$$

arising from the triangle coincides with the cup product $\theta \cup -$.

Lemma 6.5.14. *Let N be a finite extension of $\mathbb{Q}_p$. Then we have the following isomorphisms:*

$$H^1(N, \mathbb{Q}_p(1)) \cong \widetilde{N^\times} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \quad \text{and} \quad H^2(N, \mathbb{Q}_p(1)) \xrightarrow[\cong]{\mathrm{inv}} \mathbb{Q}_p,$$

where inv *is the invariant map and $\widetilde{N^\times}$ is the p-adic completion of $N^\times$.*

Proof. Using the Kummer sequence we obtain

$$H^1(N, \mu_{p^n}) \cong N^\times/(N^\times)^{p^n}$$

and by passing to the inverse limit therefore

$$H^1(N, \mathbb{Z}_p(1)) \cong \widetilde{N^\times}.$$

Now by tensoring with $\mathbb{Q}_p$ we obtain the first result. Also from the Kummer sequence we obtain

$$H^2(N, \mu_{p^n}) = \ker(H^2(N, (N^c)^\times) \xrightarrow{p^n} H^2(N, (N^c)^\times))$$

and so by using the invariant map we obtain $H^2(N, \mu_{p^n}) \cong \frac{1}{p^n}\mathbb{Z}/\mathbb{Z}$. By taking the inverse limit and tensoring with $\mathbb{Q}_p$ our result follows. □

Let N again be a finite extension of $\mathbb{Q}_p$. Then we have for $\xi \in H^1(N, \mathbb{Q}_p(1))$ and $\eta \in \mathrm{Hom}_{cont}(\mathrm{Gal}(N^c/N), \mathbb{Q}_p)$:

$$\mathrm{inv}_N(\xi \cup \eta) = \eta(\mathrm{rec}_N(\xi)) \tag{6.23}$$

Let K be a number field and S be a finite set of places of K. We denote by K_S the maximal extension of K which is unramified outside of S and $G_S := \mathrm{Gal}(K_S/K)$. Then we have that $G_S = \pi_1^{\acute{e}t}(\mathrm{Spec}(\mathcal{O}_{K,S}))$ and $H^1_{\acute{e}t}(K, \mathcal{F}) = H^1(G_S, \mathcal{F})$ for an admissible sheaf $\mathcal{F}$.

Proposition 6.5.15. *cf. [Fla04, Lemma 5.8] Let $S = S_{ram}(L/k) \cup S_\infty \cup S_p(k)$. The Bockstein map β_ω is induced by the map*

$$H^1(\Delta(L)) \otimes \mathbb{Q}_p = \mathcal{O}_{L,S}^\times \to X_S \otimes \mathbb{Q}_p = H^2(\Delta(L)) \otimes \mathbb{Q}_p$$

given by

$$u \mapsto \begin{cases} c_{\mathfrak{l}} \cdot \mathrm{ord}_w(u) & \text{for a place } w \mid \mathfrak{l} \mid \mathfrak{f}_0, \\ c_{\mathfrak{p},\gamma}^{-1} \cdot \pi_\gamma(\frac{1}{p}(\log_p(N_{L_w/k_{\mathfrak{p}}}(u_w)))) & \text{for a place } w \mid \mathfrak{p}, \end{cases}$$

where we read the right hand side componentwise.

Proof. We adapt the arguments of Lemma 5.8 in [Fla04] to our situation.

We have an exact triangle

$$R\Gamma(\mathcal{O}_{k,S}, T_p^*(1)) \to \Delta(L) \to Y_{L,S_\infty} \otimes \mathbb{Z}_p[-2]$$

and by passing to the inverse limit we obtain an exact triangle

$$R\Gamma(\mathcal{O}_{k,S}, T_p^*(1)^\infty) \to \Delta_{L_\infty/k} \to Y_{L_\infty/k}[-2]$$

which induces, after localization at $\mathfrak{q}_\chi$, the following commutative diagram of Bockstein maps:

$$\begin{array}{ccccc} H^1(\mathcal{O}_{L,S}, \mathbb{Z}_p(1)) \otimes \mathbb{Q}_p(\chi) & \longrightarrow & H^1(\Delta(L)) \otimes \mathbb{Q}_p(\chi) & \longrightarrow & 0 \\ \downarrow \beta' & & \downarrow \beta_\omega & & \downarrow \\ H^2(\mathcal{O}_{L,S}, \mathbb{Z}_p(1)) \otimes \mathbb{Q}_p(\chi) & \longrightarrow & H^2(\Delta(L)) \otimes \mathbb{Q}_p(\chi) & \longrightarrow & Y_{S_\infty} \otimes \mathbb{Q}_p(\chi) \end{array}$$

So the image of β_ω has no components at the infinite places. Now by Lemma 6.5.13 the Bockstein map β_ω is induced by the cup product of θ over the field L as defined in Lemma 6.5.13.

The projection formula for cup products now shows that β_ω is induced by the cup product:

$$H^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(1)) = \mathcal{O}_{L,S}^\times \xrightarrow{\theta\cup} X_{S_{ram}\cup\{\mathfrak{p}\}} \otimes \mathbb{Q}_p = H^2(\mathcal{O}_{L,S}, \mathbb{Q}_p(1))$$

For any place w in L we now have a commutative diagram:

$$\begin{array}{ccc} H^1(\mathcal{O}_{L,S}, \mathbb{Q}_p(1)) & \xrightarrow{\theta\cup} & H^2(\mathcal{O}_{L,S}, \mathbb{Q}_p(1)) \\ \downarrow & & \downarrow \\ H^1(\mathcal{O}_{L_w}, \mathbb{Q}_p(1)) & \xrightarrow{\mathrm{res}(\theta)\cup} & H^2(L_w, \mathbb{Q}_p(1)) \end{array}$$

Recall that $H^2(L_w, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p$ by Lemma 6.5.14. We have $L \subset L_\infty \subset L_S$ and we set $G_S := \mathrm{Gal}(L_S/L)$. Let $\theta \in H^1(G_S, \mathbb{Z}_p)$ which factors through $\mathrm{Gal}(L_\infty/L)$ (with generator γ_0) such that γ_0 maps to 1.

We begin by treating the case where $\mathfrak{l} \mid \mathfrak{f}_0$ and $w := w_{\mathfrak{l}}$ is a place above $\mathfrak{l}$:
Let $u \in L_w^\times$. We know that $\mathrm{rec}_w(u) = \mathrm{Fr}_w^{\mathrm{ord}_w(u)}$. Then it follows from (6.23) and the skew commutativity of the cup product that

$$Res_w(\theta) \cup Res_w(u) = -\theta(\mathrm{rec}_w(u)) = \theta(Fr_w^{-1}) \cdot \mathrm{ord}_w(u).$$

Now $\gamma^{c_{\mathfrak{l}} p^m} = Fr_{\mathfrak{l}}^{-f_{\mathfrak{l}}} = Fr_w^{-1}$ with $c_{\mathfrak{l}} \in \mathbb{Z}_p$ and $f_{\mathfrak{l}}$ the residue degree of w in L/k, so we get

$$\theta(Fr_w^{-1}) \cdot \mathrm{ord}_w(u) = \theta(Fr_{\mathfrak{l}}^{-f_{\mathfrak{l}}}) \cdot \mathrm{ord}_w(u) = c_{\mathfrak{l}} \cdot \mathrm{ord}_w(u).$$

The other case we are interested in is when $w \mid \mathfrak{p}$. Now $\theta = c_{\mathfrak{p},\gamma}^{-1} \cdot \pi_\gamma(\frac{1}{p}(\log_p(\chi_{\mathrm{ell}}(\cdot))))$ and by construction we have $\theta(\gamma^{p^m}) = 1$. Then it follows that

$$\begin{aligned} Res_w(\theta) \cup Res_w(u) &= -\theta(\mathrm{rec}_w(u)) = -c_{\mathfrak{p},\gamma}^{-1} \cdot \pi_\gamma \left(\frac{1}{p}(\log_p(\chi_{\mathrm{ell}}(\mathrm{rec}_w(u)))) \right) \\ &= -c_{\mathfrak{p},\gamma}^{-1} \cdot \pi_\gamma \left(\frac{1}{p}(\log_p(\chi_{\mathrm{ell}}(\mathrm{rec}_{\mathfrak{p}}(N_{L_w/k_{\mathfrak{p}}}(u))))) \right) \\ &= c_{\mathfrak{p},\gamma}^{-1} \cdot \pi_\gamma \left(\frac{1}{p}(\log_p(N_{L_w/k_{\mathfrak{p}}}(u))) \right), \end{aligned}$$

where we obtain the first equality with the same arguments as in the other case, the third equality follows from the well-known properties of the reciprocity map and for the fourth equality recall Remark 4.3.2.

□

Let N_∞ be either a rank one extension in K'_∞ such that L is in N_∞/k, or K'_∞ itself. Let α be the isomorphism given by Remark 6.5.7:

$$\alpha : \mathrm{Det}_\Lambda(\Delta_{N_\infty/k}) \otimes \mathcal{A}_p \cong \mathrm{Det}_{\mathcal{A}_p}(\Delta(L)) \subset \mathrm{Det}_{A_p}(\Delta(L) \otimes \mathbb{Q}_p).$$

Assume that we have $\mathrm{LC}(K'_\infty/k, p)$. Then we know that $\alpha(\mathcal{L}_{K'_\infty} \otimes 1)$ is a $\mathcal{A}_p$-generator of $\mathrm{Det}_{\mathcal{A}_p}(\Delta(L))$. So what is left to show is that

$${}_A\vartheta_p({}_A\vartheta_\infty(L^*({}_AM, 0)^{-1})) = \alpha(\mathcal{L}_{K'_\infty} \otimes 1).$$

Now to shorten the notation we set, for the moment, $\Lambda := \Lambda_{K'_\infty/k}$. Furthermore, let $G := \mathrm{Gal}(L/k)$ and let δ be the morphism such that the following diagram

$$\begin{array}{ccc} \mathrm{Det}_{\mathbb{Q}(\Lambda)}(\Delta^\infty \otimes Q(\Lambda)) \otimes A_p & \xrightarrow{\alpha_2} & \mathrm{Det}_{\mathbb{Q}(\Lambda)}(H^\bullet(\Delta^\infty \otimes Q(\Lambda))) \otimes A_p \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \delta} \\ \mathrm{Det}_{A_p}(\Delta(L) \otimes \mathbb{Q}_p) & \xrightarrow{\alpha_3} & \mathrm{Det}_{A_p}((H^\bullet(\Delta(L) \otimes \mathbb{Q}_p)) \end{array}$$

commutes and the maps α_1, α_2, α_3 as well as δ are isomorphisms. Furthermore, we know that $A_p = \prod_{\chi \in \widehat{G}/\sim_{\mathbb{Q}_p}} \mathbb{Q}_p(\chi)$.

Definition 6.5.16. We define the isomorphism

$$\phi_\chi : \mathrm{Det}_{\mathbb{Q}_p(\chi)}(\Delta(L) \otimes_{\mathcal{A}_p} \mathbb{Q}_p(\chi)) \cong \mathrm{Det}^{-1}_{\mathbb{Q}_p(\chi)}(\mathcal{O}_L^\times \otimes_{\mathcal{A}_p} \mathbb{Q}_p(\chi)) \otimes_{\mathbb{Q}_p(\chi)} \mathrm{Det}_{\mathbb{Q}_p(\chi)}(X_{L,S_\infty} \otimes_{\mathcal{A}_p} \mathbb{Q}_p(\chi))$$

as induced by φ_1^{-1}, φ_2^{-1} and φ_3^{-1} from (6.5), (6.6) and (6.7). Moreover, we define the isomorphism

$$\phi'_\chi : \mathrm{Det}_{\mathbb{Q}_p(\chi)}(\Delta(L) \otimes_{\mathcal{A}_p} \mathbb{Q}_p(\chi)) \cong \mathrm{Det}^{-1}_{\mathbb{Q}_p(\chi)}(\mathcal{O}_L^\times \otimes_{\mathcal{A}_p} \mathbb{Q}_p(\chi)) \otimes_{\mathbb{Q}_p(\chi)} \mathrm{Det}_{\mathbb{Q}_p(\chi)}(X_{L,S_\infty} \otimes_{\mathcal{A}_p} \mathbb{Q}_p(\chi))$$

as induced by φ_1^{-1} and φ_3^{-1}.

We have that

$$\begin{aligned}\mathrm{Det}_{A_p}(\Delta(L)\otimes\mathbb{Q}_p) &\longrightarrow \mathrm{Det}_{A_p}(H^\bullet(\Delta(L)\otimes\mathbb{Q}_p))\\ &\longrightarrow \prod_\chi \mathrm{Det}_{\mathbb{Q}_p(\chi)}(H^\bullet(\Delta(L)\otimes\mathbb{Q}_p(\chi)))\\ &\longrightarrow \prod_\chi \mathrm{Det}^{-1}_{\mathbb{Q}_p(\chi)}(\mathcal{O}_L^\times\otimes_{\mathcal{A}_p}\mathbb{Q}_p(\chi))\otimes_{\mathbb{Q}_p(\chi)}\mathrm{Det}_{\mathbb{Q}_p(\chi)}(X_{L,S_\infty}\otimes_{\mathcal{A}_p}\mathbb{Q}_p(\chi))\end{aligned}$$

and with (6.20) and by abuse of notation we obtain:

$$\begin{aligned}{}_A\vartheta_\infty:\mathbb{R}[G] &\longrightarrow \Xi({}_AM)^\#\otimes\mathbb{R}\\ &\longrightarrow \left(\prod_\chi \mathrm{Det}^{-1}_{\mathbb{Q}(\chi)}(\mathcal{O}_L^\times\otimes\mathbb{Q}(\chi))\otimes\mathrm{Det}_{\mathbb{Q}(\chi)}(X_{L,S_\infty}\otimes\mathbb{Q}(\chi))\right)\otimes\mathbb{R}\end{aligned}$$

So because we know the rationality part of the eTNC by Remark 6.5.2 d) and the known validity of Stark's conjecture for abelian extensions of k we get

$${}_A\vartheta_\infty(L^*({}_AM,0)^{-1})\in\prod_\chi \mathrm{Det}^{-1}_{\mathbb{Q}(\chi)}(\mathcal{O}_L^\times\otimes\mathbb{Q}(\chi))\otimes\mathrm{Det}_{\mathbb{Q}(\chi)}(X_{L,S_\infty}\otimes\mathbb{Q}(\chi)).$$

Since we have already computed the χ-component of ${}_A\vartheta_\infty(L^*({}_AM,0)^{-1})$ explicitly in Proposition 6.5.4 it is enough to show that, where we use from now on the reduced notation τ for the embedding at the correct level,

$$\phi_\chi(\delta(\alpha(\mathcal{L}_{K'_\infty}\otimes 1))_\chi)=\begin{cases}(\mathcal{N}(\mathfrak{a})-\chi(\mathfrak{a}))\cdot w(\mathfrak{f}_\chi)\cdot[k(\mathfrak{f}):k(\mathfrak{f}_\chi)]\cdot e_\chi\psi_\chi^{-1}\otimes\tau, & \mathfrak{f}_\chi\neq(1),\\ (\mathcal{N}(\mathfrak{a})-\chi(\mathfrak{a}))\cdot(1-\chi(\mathfrak{p})^{-1})\cdot w(1)\cdot[k(\mathfrak{f}):k(\mathfrak{f}_\chi)]\cdot e_\chi\psi_\chi^{-1}\otimes\tau, & \mathfrak{f}_\chi=(1),\chi\neq 1,\\ \frac{-w(1)}{h_k}, & \chi=1.\end{cases}\tag{6.24}$$

Now in order to be able to use the descent techniques used in [Fla04] and [Ble06] the idea is now to descend in two steps:
Let χ and K'_∞ be as above. For each χ we construct below in Corollary 6.5.19 a rank one extension L_∞ in K'_∞ with $k(\mathfrak{f})\subset L_\infty$ and we use then this extension for the descent to the χ-component.

We set $\Lambda_{L_\infty}:=\Lambda_{L_\infty/k}$ and $\Delta_{L_\infty}:=\Delta_{L_\infty/k}$. Now $\mathrm{LC}(L_\infty/k,p)$ asserts that there is an identity of invertible Λ_{L_∞}-submodules

$$\mathcal{L}_{L_\infty}\cdot\Lambda_{L_\infty}=\mathrm{Det}_\Lambda(\Delta_{L_\infty})$$

of $\mathrm{Det}_{Q(\Lambda)}(\Delta_{L_\infty}\otimes Q(\Lambda))$. Formally, we have now the isomorphisms

$$\Delta_{K'_\infty}\otimes\mathbb{Z}_p[G]\cong\Delta_{L_\infty}\otimes\mathbb{Z}_p[G]$$

so it is enough to show that

$$\phi_\chi(\delta(\alpha(\mathcal{L}_{L_\infty}\otimes 1))_\chi)=\begin{cases}(\mathcal{N}(\mathfrak{a})-\chi(\mathfrak{a}))\cdot w(\mathfrak{f}_\chi)[k(\mathfrak{f}):k(\mathfrak{f}_\chi)]\cdot\psi_\chi^{-1}\otimes e_\chi\tau, & \mathfrak{f}_\chi\neq(1),\\ (\mathcal{N}(\mathfrak{a})-\chi(\mathfrak{a}))(1-\chi(\mathfrak{p})^{-1})\cdot w(1)\cdot[k(\mathfrak{f}):k(\mathfrak{f}_\chi)]\cdot e_\chi\psi_\chi^{-1}\otimes\tau, & \mathfrak{f}_\chi=(1),\chi\neq 1,\\ \frac{-w(1)}{h_k}, & \chi=1.\end{cases}\tag{6.25}$$

Summary of the strategy of the descent Now we can present the strategy for proving (6.25), which is based on the strategy used in [Fla04] and [Ble06].

We again set $\Lambda := \Lambda_{L_\infty/k}$, $\mathcal{L} := \mathcal{L}_{L_\infty}$ and $R := \Lambda_{\mathfrak{q}_\chi}$. So we have to

A) Check the conditions for Lemma 6.5.6.

B) Compute M^1 and M^2 as defined in Lemma 6.5.6.

C) Find an R-basis β_1 and β_2 of $\mathrm{Det}_R(M^1)$ and of $\mathrm{Det}_R(M^2)$, respectively.

D) Compute $\overline{\beta}_1 \in \mathrm{Det}_\kappa(M^1/\omega)$ and $\overline{\beta}_2 \in \mathrm{Det}_\kappa(M^2/\omega)$, respectively.

E) Write $\mathcal{L}$ in $\mathrm{Det}(\Delta^\infty)_{\mathfrak{q}_\chi}$ and express it in the form of

$$B \cdot \omega^e \beta_1^{-1} \otimes \beta_2$$

for an $e \in \mathbb{Z}$, i.e. that $\omega^e \beta_1^{-1} \otimes \beta_2$ is an R-basis of $\mathrm{Det}_R(\Delta^\infty_{\mathfrak{q}_\chi})$.

F) Compute

$$A := \phi'_\chi \circ \phi_\omega^{-1}(\overline{\beta}_1^{-1} \otimes \overline{\beta}_2).$$

Now Lemma 6.5.6 gives us that

$$\phi_\omega(B^{-1}(\mathcal{L} \otimes 1)) = \overline{\beta}_1^{-1} \otimes \overline{\beta}_2$$

and hence

$$\phi'_\chi(\mathcal{L} \otimes 1) = B \cdot A.$$

So the last step is

G) Check if the second equality of the following holds:

$$\phi_\chi(\delta(\mathcal{L} \otimes 1)) = \mathcal{E}_S^{-1} \cdot B \cdot A = \begin{cases} (\mathcal{N}(\mathfrak{a}) - \chi(\mathfrak{a})) \cdot w(\mathfrak{f}_\chi)[k(\mathfrak{f}) : k(\mathfrak{f}_\chi)] \cdot e_\chi(\psi_\chi^{-1} \otimes \tau), & \mathfrak{f}_\chi \neq (1), \\ (\mathcal{N}(\mathfrak{a}) - \chi(\mathfrak{a})) \cdot w_1 \cdot [k(\mathfrak{f}) : k(\mathfrak{f}_\chi)] \cdot e_\chi(\psi_\chi^{-1} \otimes \tau), & \mathfrak{f}_\chi = (1), \chi \neq 1, \\ \frac{-w(1)}{h_k}, & \chi = 1, \end{cases}$$

where $\mathcal{E}_S$ are the Euler factors defined in (6.8).

In order to use this strategy efficiently we divide the problem into three cases according to properties of χ if the other things are fixed.

Case I: $\chi_{|D_\mathfrak{p}} \neq 1$.

Case II: $\chi \neq 1$ and $\chi_{|D_\mathfrak{p}} = 1$.

Case III: χ is the trivial character, i.e. $\chi = 1$.

6.5.4 Preparations for the descent

Recall that $K'_\infty := k(\mathfrak{f}\mathfrak{p}^\infty)$ and $\mathfrak{f} = \mathfrak{f}_0\mathfrak{p}^\nu$ with $(\mathfrak{p}, \mathfrak{f}_0) = 1$. We fix an integral ideal $\mathfrak{a}$ of k such that $(\mathfrak{a}, 6\mathfrak{p}\mathfrak{f}) = 1$, but we suppress this $\mathfrak{a}$ from the notation for an elliptic unit, e.g. we write $\psi(1; \mathfrak{f})$ instead of $\psi(1; \mathfrak{f}, \mathfrak{a})$. Furthermore, we set $L := k(\mathfrak{f})$, $G := \mathrm{Gal}(L/k)$, $L_\chi := L^{\ker(\chi)}$, $L' := k(\mathfrak{f}_\chi)$ for χ from $\widehat{G}/\sim_{\mathbb{Q}_p}$. Without loss of generality we assume that $L \supset K_0$.

Lemma 6.5.17. *For a given prime ideal $\mathfrak{q} \nmid \mathfrak{f}$ there is at most one rank one $\mathbb{Z}_p$-extension Z in K'_∞ with $\mathrm{Gal}(Z/k) \cong \mathrm{Gal}(Z/K_0) \times \mathrm{Gal}(K_0/k)$ such that primes above $\mathfrak{q}$ split completely in Z/K_0.*

Proof. We recall from Proposition 1.9 iii) in [dS87, Ch. 2] that for a prime ideal $\mathfrak{Q}$ above $\mathfrak{q}$ in K_0 the order of the decomposition group of $\mathfrak{Q}$ in $\mathrm{Gal}(K'_n/K_0)$ is asymptotic to cp^n for n large and c being a constant.

Moreover, it is well-known that if a prime ideal $\mathfrak{Q}$ in K_0 is completely split in two different $\mathbb{Z}_p$-extensions of K_0 it is completely split in the compositum.

So assume now that $\mathfrak{Q}$ was completely split in two different $\mathbb{Z}_p$-extensions as described above. Then it would be completely split in the compositum, which is K'_∞, but this is a contradiction to the fact we recalled at the beginning of the proof. So there can be at most one $\mathbb{Z}_p$-extension of the given form.

□

Lemma 6.5.18. *Fix a χ and therefore L_χ. There are infinitely many rank one extensions of k which contain L_χ.*

Proof. Let us choose two topological generators γ_1 and γ_2 of $\Gamma' := \mathrm{Gal}(K'_\infty/K_0)$ and we set $L' := K_0L_\chi$. By using the elementary divisor theorem we can get $t_1, t_2 \in \mathbb{Z}_p$ such that $\mathrm{Gal}(K'_\infty/L') = \langle \gamma_1^{t_1}, \gamma_2^{t_2} \rangle$. Now since $\mathrm{Gal}(L_\chi/k)$ is cyclic, $\mathrm{Gal}(L'/K_0)$ is cyclic, but this implies that w.l.o.g. $t_1 = 1$. So $L' = (K'_\infty)^H$ for $H = \langle \gamma_1, \gamma_2^{t_2} \rangle$. We can also assume that $t := t_2 \neq 1$ because otherwise L_χ is already contained in K_0 and then it is obvious that there are infinitely many rank one extension extensions of k which contain L_χ and are contained in K'_∞. Now since the determinant of

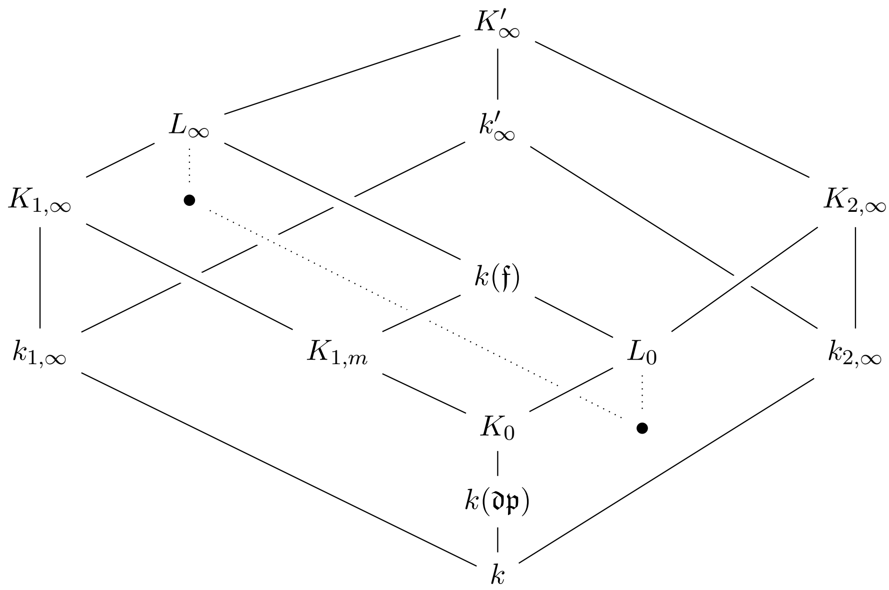
$$\begin{pmatrix} 1 & bt \\ 0 & 1 \end{pmatrix}$$

for all $b \in \mathbb{Z}_p \setminus \{0\}$ is 1, tuples $(\gamma_1\gamma_2^{bt}, \gamma_2)$ also generate Γ' and each fixed field of $\langle \gamma_1\gamma_2^{bt} \rangle \subset H$ is a rank one extension of k. So there are infinitely many of them containing L_χ and that are contained in K'_∞. □

Now combining Lemma 6.5.17 and Lemma 6.5.18 we get the following corollary:

Corollary 6.5.19. *There are $\mathbb{Z}_p$-extensions $K_{1,\infty}$ of K_0 contained in K'_∞ and containing L_χ for which no $\mathfrak{q} \mid \mathfrak{f}_0$ splits completely. Let γ_1 be such that $K_{1,\infty} = (K'_\infty)^{\langle \gamma_1 \rangle}$ for a fixed $K_{1,\infty}$. Then we can set $L_\infty := (K'_\infty)^{\langle \gamma_1^{p^m} \rangle}$, where $m := \nu - 1$ from above. In particular, we have $k(\mathfrak{f}) \subset L_\infty$.*

This can be illustrated as follows, where $L_0 := K_{2,m}$ and the significance of the dotted line will become clear later.



From the construction of $K_{1,\infty}$ and L_∞, using condition $p \nmid h_k$, we have a direct decomposition

$$\mathrm{Gal}(L_\infty/k) \cong \mathrm{Gal}(L_\infty/L_0) \times \mathrm{Gal}(L_0/K_0) \times \mathrm{Gal}(K_0/k),$$

where $\mathrm{Gal}(L_\infty/L_0) \cong \mathbb{Z}_p$ and the n-th level of that extension is denoted by L_n.

Now for this L_∞ we can set the following notation

$$\begin{aligned} S &:= S_{ram}(L/k) \cup S_\infty \cup S_p(k) & S_r &:= S_{ram}(L/k) \setminus \{\mathfrak{p}\} \\ S_p &:= S_p(k) & S_{rp} &:= S_{ram}(L/k) \cup S_p(k) \\ \text{and for } Z \in \{S_r, S, S_p, S_{pr}, \emptyset\} &: & & \\ U_Z^\infty &:= U_{L_\infty, Z} & P_Z^\infty &:= P_{L_\infty, Z} \\ X_Z^\infty &:= X_{L_\infty, Z} & Y_Z^\infty &:= Y_{L_\infty, Z} \\ \mathcal{G} &:= \mathrm{Gal}(L_\infty/k) & \Lambda &:= \mathbb{Z}_p[[\mathcal{G}]] \end{aligned}$$

Remark 6.5.20. a) If nothing else is specified we use the restriction as the transition maps in the projective limit.

b) We want to caution the reader that our definition of S_p does not coincide with the definition in the literature e.g. in [Ble06] or [Fla04].

Now let

$$Y_{S_{rp},\beta}^\infty := \varprojlim_n Y_{S_{rp}}(L_n) \otimes \mathbb{Z}_p$$

with respect to the transition maps

$$Y_{S_{rp}}(L_{n+1}) \xrightarrow{\beta_{n+1/n}} Y_{S_{rp}}(L_n)$$

induced by $w \mapsto f_{w|v} \cdot v$, if $w \in S_{rp}(L_{n+1})$ and $v \in S_{rp}(L_n)$ is the restriction of w and $f_{w|v}$ is the residue degree. So we have an exact sequence of Λ-modules

$$0 \to U^\infty \to U_S^\infty \to Y_{S_{rp},\beta}^\infty \to P^\infty \to P_S^\infty \to 0$$

Let now $\mathfrak{q}$ be a prime divisor of $\mathfrak{f}_0$ and $n_0 \in \mathbb{N}$ such that no further splitting of primes above $\mathfrak{q}$ in $K_{1,\infty}/K_{1,n_0}$ occurs. Such an n_0 exists per construction in our $K_{1,\infty}$ so also in L_∞ (see Lemma 6.5.17). Then for $m \geq n \geq n_0$, $\beta_{m|n}(w)$ is of the form $p^{f(m,n)}w_{|L_n}$ with $f(m,n)$ a function which tends to infinity if m goes to infinity. So $Y^\infty_{S_r,\beta} = 0$ and so we get an exact sequence

$$0 \to U^\infty \to U^\infty_S \to Y^\infty_{S_p,\beta} \to P^\infty \to P^\infty_S \to 0. \tag{6.26}$$

Furthermore, one can get

$$0 \to X^\infty_{S_r} \to X^\infty_S \to Y^\infty_{S_p} \oplus Y^\infty_{S_\infty} \to 0. \tag{6.27}$$

Remark 6.5.21. For the places above $\mathfrak{p}$ and ∞, $\beta_{n+1|n}$ and transition maps induced by the restriction coincide, so we get $Y^\infty_{S_\infty} = Y^\infty_{S_\infty,\beta}$ and $Y^\infty_{S_p} = Y^\infty_{S_p,\beta}$.

Definition 6.5.22. Let $\mathfrak{f}_\chi$ be the conductor of the character χ. In particular, $\mathfrak{f}_\chi \mid \mathfrak{f}$. Recall that we have

$$\mathcal{G} := \mathrm{Gal}(L_\infty/k) \cong H \times J \times \Gamma,$$

where we have $L_0 = K_{2,m}$, and $H := \mathrm{Gal}(K_0/k)$, $J := \mathrm{Gal}(L_0/K_0)$ as well as $\Gamma := \mathrm{Gal}(L_\infty/L_0)$.

For $\psi \in \widehat{H}$ we denote the conductor by f_ψ. Now we know that $\mathfrak{f}_\psi$ has the form $\mathfrak{d}\mathfrak{p}$ or $\mathfrak{d}$ with $\mathfrak{d} \mid \mathfrak{f}_0$. So we can write $\mathfrak{f}_\chi$ as $\mathfrak{d}\mathfrak{p}^{\nu'}$.

We have $\mathfrak{f} = \mathfrak{f}_0\mathfrak{p}^\nu$ with $\nu > 0$ and $(\mathfrak{p}, \mathfrak{f}_0) = 1$. Furthermore we set $\varepsilon(\psi) = 0$ if $\psi \neq 1$ and $\varepsilon(\psi) = 1$ if $\psi = 1$ and we shorten the notation to ε.

We set

$$\eta_{\mathfrak{f}_0} := \{\mathcal{N}_{K'_{m(n)}/L_n}(\psi(1;\mathfrak{f}_0\mathfrak{p}^{m(n)}))\}_{n\geq 0},$$
$$\eta_{\mathfrak{d}} := \{\mathcal{N}_{K'_{m(n)}/L_n}(\psi(1;\mathfrak{d}\mathfrak{p}^{m(n)}))\}_{n\geq 0},$$

where in this definition ψ is the function defined in Definition 2.1.23 and $m(n)$ is the smallest natural number ℓ such that $L_n \subseteq K'_\ell$.

Lemma 6.5.23.

$$\mathrm{Fitt}_{\Lambda_{\mathfrak{q}_\chi}}(X^\infty_{S_r,\mathfrak{q}_\chi}) = T^{-\varepsilon} \prod_{\mathfrak{l}|\mathfrak{f},\mathfrak{l}\nmid\mathfrak{d}} \left(1 - Fr_{\mathfrak{l}}^{-1}\right) \Lambda_{\mathfrak{q}_\chi} = \mathrm{Fitt}_{\Lambda_{\mathfrak{q}_\chi}} \left(\Lambda_{\mathfrak{q}_\chi} T^\varepsilon \eta_{\mathfrak{d}}/\Lambda_{\mathfrak{q}_\chi}\eta_{\mathfrak{f}_0}\right),$$

where the product goes over prime divisors $\mathfrak{l}$.

Proof. We adapt the argument of Lemma 5.5 in [Fla04] to our situation.
First of all, as in part c) of Lemma 6.5.26 we see that $X^\infty_{S_r,\mathfrak{q}_\chi}$ is torsion so the left side is well-defined.
Since we have Corollary 6.5.19 we know that for $\mathfrak{l} \mid \mathfrak{f}_0$, $D_\mathfrak{l}$ has finite index in $\mathcal{G}$ and that $I_\mathfrak{l} \subset D_\mathfrak{l}$ is finite. Moreover, we have a direct decomposition $D_\mathfrak{l} = I_\mathfrak{l} \times \overline{\langle Fr_\mathfrak{l}\rangle}$. Indeed, $I_\mathfrak{l} \cap \overline{\langle Fr_\mathfrak{l}\rangle}$ is trivial because otherwise there would be a $c \in \mathbb{Z}$ such that $Fr_\mathfrak{l}^c \in I_\mathfrak{l}$, which would imply that $Fr_\mathfrak{l}$ has finite order and therefore $|D_\mathfrak{l}| < \infty$. This would be a contradiction to $|\mathcal{G}/D_\mathfrak{l}| < \infty$. Next we know that we have an isomorphism of Λ-modules

$$Y^\infty_{\{v|\mathfrak{l}\}} \cong \mathrm{Ind}^{\mathcal{G}}_{D_\mathfrak{l}} \mathbb{Z}_p \cong \mathbb{Z}_p[\mathcal{G}/D_\mathfrak{l}]$$

and the short exact sequence of Λ-modules

$$0 \to X^\infty_{S_r} \to Y^\infty_{S_r} \to \mathbb{Z}_p \to 0.$$

We localize the sequence now at $\mathfrak{q}_\chi$ to get

$$0 \to X^\infty_{S_r,\mathfrak{q}_\chi} \to Y^\infty_{S_r,\mathfrak{q}_\chi} \to \mathbb{Z}_{p,\mathfrak{q}_\chi} \to 0$$

Now $\mathbb{Z}_{p,\mathfrak{q}_\chi}$ is trivial if $\chi \neq 1$. So in this case it is enough to compute the Fitting ideal of $Y^\infty_{S_r,\mathfrak{q}_\chi}$. If $\psi \mid_{I_\mathfrak{l}} \neq 1$, i.e. $\mathfrak{l} \mid \mathfrak{d}$, we get $Y^\infty_{\{w|\mathfrak{l}\},\mathfrak{q}_\chi} = 0$. Otherwise, in the case $\psi \mid_{I_\mathfrak{l}} = 1$ we use the exact sequence

$$0 \to \Lambda \xrightarrow{1-Fr_\mathfrak{l}^{-1}} \Lambda \to \mathbb{Z}_p[\mathcal{G}/\langle Fr_\mathfrak{l}\rangle] \to 0.$$

But we also have $\mathbb{Z}_p[\mathcal{G}/\langle \mathrm{Fr}_\mathfrak{l}\rangle]_{\mathfrak{q}_\chi} = \mathbb{Z}_p[\mathcal{G}/D_\mathfrak{l}]_{\mathfrak{q}_\chi}$ so we obtain

$$0 \to \Lambda_{\mathfrak{q}_\chi} \xrightarrow{1-Fr_\mathfrak{l}^{-1}} \Lambda_{\mathfrak{q}_\chi} \to Y^\infty_{\{w|\mathfrak{l}\},\mathfrak{q}_\chi} \to 0.$$

Therefore we know that $\mathrm{Fitt}_{\Lambda_{\mathfrak{q}_\chi}}(X^\infty_{\{w|\mathfrak{l}\},\mathfrak{q}_\chi}) = (1 - Fr_\mathfrak{l}^{-1})\Lambda_{\mathfrak{q}_\chi}$ for $\chi \mid_{I_\mathfrak{l}} = 1$ and $\chi \neq 1$. For $\chi = 1$ we obtain from the exact sequence of Λ-modules

$$0 \to \Lambda_{\mathfrak{q}_\chi} \xrightarrow{1-\gamma} \Lambda_{\mathfrak{q}_\chi} \to \mathbb{Z}_{p,\mathfrak{q}_\chi} \to 0$$

the term T^{-1}.

The second equality follows from the fact that $\mathcal{N}_{k(\mathfrak{f}_0\mathfrak{p})/k(\mathfrak{d}\mathfrak{p})}$ is a unit in $\Lambda_{\mathfrak{q}_\chi}$ and so we get with the norm relations for elliptic units

$$\mathcal{N}_{k(\mathfrak{f}_0\mathfrak{p})/k(\mathfrak{d}\mathfrak{p})} \cdot \eta_{\mathfrak{f}_0} = \left(\prod_{\mathfrak{l}|\mathfrak{f},\mathfrak{l}\nmid\mathfrak{d}} \left(1 - Fr_\mathfrak{l}^{-1}\right) \right) \eta_\mathfrak{d}.$$

□

Let N_∞ be a rank one $\mathbb{Z}_p$-extension in K'_∞ with $\mathrm{Gal}(N_\infty/k) \cong \Gamma \times H$ such that $\Gamma \cong \mathbb{Z}_p$ and H a finite abelian group.

With the same proof as of Lemma 5.12 in [BKS17] or on p.104 in [Ble06] we obtain

Lemma 6.5.24. *If Condition* $\mathrm{F}(N_\infty/k, S)$ *holds, then we have* $\left(P_{N_\infty/k,S}\right)_{\mathfrak{q}_\chi} = 0$.

6.5.5 Proof for Case I (no trivial zeroes case)

We assume now that we are in the situation where $\chi_{|D_\mathfrak{p}} \neq 1$.

Lemma 6.5.25. *Assuming* $P^\infty_{S,\mathfrak{q}_\chi} = 0$ *we obtain*

a) $H^1(\Delta^\infty_{\mathfrak{q}_\chi}) = (U^\infty_S)_{\mathfrak{q}_\chi} \cong U^\infty_{\mathfrak{q}_\chi}$.

b) $H^2(\Delta^\infty_{\mathfrak{q}_\chi}) = (X^\infty_S)_{\mathfrak{q}_\chi}$.

Proof. We obtain from Proposition 6.3.1 c) that $H^1(\Delta^\infty_{\mathfrak{q}_\chi}) = (U^\infty_S)_{\mathfrak{q}_\chi}$. Moreover, from (6.26) and Remark 6.5.21 we get the exact sequence

$$0 \to U^\infty_{\mathfrak{q}_\chi} \to U^\infty_{S,\mathfrak{q}_\chi} \to Y^\infty_{S_p,\mathfrak{q}_\chi} \to P^\infty_{\mathfrak{q}_\chi} \to P^\infty_{S,\mathfrak{q}_\chi} \to 0.$$

Now with $\chi_{|D_\mathfrak{p}} \neq 1$ and $Y^\infty_{S_p} = \mathbb{Z}_p[\mathcal{G}/D_\mathfrak{p}]$ we get that $Y^\infty_{S_p} \otimes \Lambda_{\mathfrak{q}_\chi} = 0$ and therefore assertion a).

Assertion b) follows directly from Proposition 6.3.1 and the assumption that $P^\infty_{S,\mathfrak{q}_\chi} = 0$. □

Step A The following lemma is shown (in greater generality) in Lemma 5.12 in [BKS17][2].

Lemma 6.5.26. *Assuming* $P^{\infty}_{S,\mathfrak{q}_\chi} = 0$ *and* $\mathrm{LC}(L_\infty/k,p)$*, it holds that:*

a) $H^1(\Delta^{\infty}_{\mathfrak{q}_\chi})_{tors} = 0$.

b) $H^2(\Delta^{\infty}_{\mathfrak{q}_\chi})_{tors} = (X^{\infty}_{S_{rp}})_{\mathfrak{q}_\chi}$.

c) $\omega \cdot H^2(\Delta^{\infty}_{\mathfrak{q}_\chi})_{tors} = 0$.

In particular, the assumptions for using Lemma 6.5.6 are satisfied.

Proof. Part a) follows from Lemma 6.5.27 b), where we use $\mathrm{LC}(L_\infty/k,p)$ and $P^{\infty}_{S,\mathfrak{q}_\chi} = 0$. For part b), we get with Lemma 6.5.25, where we use $P^{\infty}_{S,\mathfrak{q}_\chi} = 0$, that

$$H^2(\Delta^{\infty}_{\mathfrak{q}_\chi}) = (X^{\infty}_{S})_{\mathfrak{q}_\chi}$$

. Because there is an isomorphism

$$X^{\infty}_{S} \otimes Q(\Lambda_\alpha) \cong Y^{\infty}_{S_\infty} \otimes Q(\Lambda_\alpha)$$

for $\Lambda_\alpha := \mathbb{Z}_p[\mathrm{im}(\alpha)][[T]]$ with $\alpha := \psi \times \eta_1 \in \widehat{H \times J}$ as defined in Definition 6.5.22 and the short exact sequence

$$0 \to X^{\infty}_{S_{rp}} \to X^{\infty}_{S} \to Y^{\infty}_{S_\infty} \to 0$$

we obtain $H^2(\Delta^{\infty}_{\mathfrak{q}_\chi})_{tors} = (X^{\infty}_{S_{rp}})_{\mathfrak{q}_\chi}$. For part c) we notice that $\omega \sim 1 - \gamma^{p^n}$ and $\gamma^{p^n} \in D_{\mathfrak{l}}$ for sufficiently large n and $\mathfrak{l} \mid \mathfrak{f}_0\mathfrak{p}$. It follows that $\omega = 1 - \gamma^{p^m}$ annihilates $X^{\infty}_{S_{rp}}$.

□

Step B and C

Lemma 6.5.27. *Assume Condition* $P^{\infty}_{S,\mathfrak{q}_\chi} = 0$ *holds. Then we obtain that*

a)

$$M^2 = Y^{\infty}_{S_\infty,\mathfrak{q}_\chi} \text{ with } \beta_2 = \tau$$

b) and if we additionally assume that $\mathrm{LC}(L_\infty/k,p)$ *holds, then we obtain*

$$M^1 = U^{\infty}_{\mathfrak{q}_\chi} \text{ with } \beta_1 = \eta_{\mathfrak{d}}.$$

Proof. We prove assertion a) first: There is the short exact sequence of Λ-modules

$$0 \to X^{\infty}_{S_{pr}} \to X^{\infty}_{S} \to Y^{\infty}_{S_\infty} \to 0.$$

By Lemma 6.5.26 b) and Lemma 6.5.25 we therefore get that

$$M^2 = (Y^{\infty}_{S_\infty})_{\mathfrak{q}_\chi} \text{ with } \beta_2 = \tau.$$

Now we prove assertion b): We have that $\mathrm{LC}(L_\infty/k,p)$ holds, i.e.

$$c_{\mathfrak{a}} \cdot (\eta_{\mathfrak{f}_0}^{-1} \otimes \tau)\Lambda = \mathrm{Det}_\Lambda(\Delta_{L_\infty/k}).$$

[2]We remind the reader that there is a shift in the definition of the complex in their work.

Since $\mathfrak{q}_\chi$ is a regular prime ideal of height one, we obtain with Lemma 6.1.4

$$c_{\mathfrak{a}} \cdot (\eta_{\mathfrak{f}_0}^{-1} \otimes \tau)\Lambda_{\mathfrak{q}_\chi} = \mathrm{Det}_{\Lambda_{\mathfrak{q}_\chi}}^{-1}(H^1(\Delta_{\mathfrak{q}_\chi}^\infty)) \otimes \mathrm{Det}_{\Lambda_{\mathfrak{q}_\chi}}(H^2(\Delta_{\mathfrak{q}_\chi}^\infty)).$$

Using Lemma 6.5.25, part a), the sequence in (6.27) and $\left(Y_{S_p}^\infty\right)_{\mathfrak{q}_\chi} = 0$ (because $\chi_{|D_\mathfrak{p}} \neq 1$) we obtain

$$c_{\mathfrak{a}} \cdot \mathrm{Det}_{\Lambda_{\mathfrak{q}_\chi}}(U_{\mathfrak{q}_\chi}^\infty/\Lambda_{\mathfrak{q}_\chi}\eta_{\mathfrak{f}_0}) = \mathrm{Det}_{\Lambda_{\mathfrak{q}_\chi}}(X_{S_r,\mathfrak{q}_\chi}^\infty).$$

Since $U_{\mathfrak{q}_\chi}^\infty$ has rank one, $U_{\mathfrak{q}_\chi}^\infty/\Lambda_{\mathfrak{q}_\chi}\eta_{\mathfrak{f}_0}$ is torsion and moreover, $X_{S_r,\mathfrak{q}_\chi}^\infty$ is torsion so we obtain

$$c_{\mathfrak{a}} \cdot \mathrm{Fitt}_{\Lambda_{\mathfrak{q}_\chi}}(U_{\mathfrak{q}_\chi}^\infty/\Lambda_{\mathfrak{q}_\chi}\eta_{\mathfrak{f}_0}) = \mathrm{Fitt}_{\Lambda_{\mathfrak{q}_\chi}}(X_{S_r,\mathfrak{q}_\chi}^\infty).$$

Now Lemma 6.5.23 asserts that

$$\mathrm{Fitt}_{\Lambda_{\mathfrak{q}_\chi}}\left(\Lambda_{\mathfrak{q}_\chi}T^\varepsilon\eta_{\mathfrak{d}}/\Lambda_{\mathfrak{q}_\chi}\eta_{\mathfrak{f}_0}\right) = T^{-\varepsilon}\prod_{\mathfrak{l}|\mathfrak{f},\mathfrak{l}\nmid\mathfrak{d}}\left(1 - Fr_{\mathfrak{l}}^{-1}\right)\Lambda_{\mathfrak{q}_\chi} = \mathrm{Fitt}_{\Lambda_{\mathfrak{q}_\chi}}(X_{S_r,\mathfrak{q}_\chi}^\infty).$$

So with Lemma 6.5.23 we obtain

$$U_{\mathfrak{q}_\chi}^\infty = c_{\mathfrak{a}}T^\varepsilon\eta_{\mathfrak{d}}\Lambda_{\mathfrak{q}_\chi}.$$

As on p. 104 in [Ble06], $T^\varepsilon = (1-\gamma)^\varepsilon$ and $c_{\mathfrak{a}}$ are units in $\Lambda_{\mathfrak{q}_\chi}$. So we can choose as a $\Lambda_{\mathfrak{q}_\chi}$-basis of $U_{\mathfrak{q}_\chi}^\infty$ the element $\eta_{\mathfrak{d}}$. Therefore, we get $H^1(\Delta_{\mathfrak{q}_\chi}^\infty) = U_{S,\mathfrak{q}_\chi}^\infty \cong U_{\mathfrak{q}_\chi}^\infty = \eta_{\mathfrak{d}}\Lambda_{\mathfrak{q}_\chi}$, so $\beta_1 = \eta_{\mathfrak{d}}$.

□

From here on until the end of this subsection we follow very closely the computations of pp. 105-108 in [Ble06], but we nevertheless include them for the reader's convenience.

Step D

Lemma 6.5.28. *There is an embedding*

$$\iota : U_{\mathfrak{q}_\chi}^\infty/\omega U_{\mathfrak{q}_\chi}^\infty \to \mathcal{O}_{k(\mathfrak{f}),S}^\times \otimes \mathbb{Q}_p(\chi),$$
$$u \mapsto u_m \otimes 1.$$

Proof. First we have that $\omega U_{\mathfrak{q}_\chi}^\infty \subset \ker(\iota)$ because $u_n^{\gamma^{p^m}} = u_n$ for $u_n \in L_n$ with $n \leq m$. Furthermore, $U_{\mathfrak{q}_\chi}^\infty$ is a free rank one $\Lambda_{\mathfrak{q}_\chi}$-module so it is isomorphic to $\Lambda_{\mathfrak{q}_\chi}$ and therefore a homomorphism from $U_{\mathfrak{q}_\chi}^\infty/\omega U_{\mathfrak{q}_\chi}^\infty$ can only be the zero map or injective. But there are obviously elements not in the kernel.

□

Lemma 6.5.29. *For*

$$T_\chi := \begin{cases} (1-\chi^{-1}(\mathfrak{p})) & \mathfrak{f}_\chi \neq (1) \\ 1 & \mathfrak{f}_\chi = (1) \end{cases}$$

we have

$$\overline{\beta}_1 = T_\chi\psi_\chi \otimes [k(\mathfrak{f}) : k(\mathfrak{d}\mathfrak{p}^{\nu'})]^{-1} \text{ and } \overline{\beta}_2 = \tau.$$

Proof. Assume first that $\nu, \nu' > 0$ and recall that $\mathfrak{f}_\chi = \mathfrak{d}\mathfrak{p}^{\nu'}$. Then we get

$$\begin{aligned}\overline{\beta}_1 &= \psi(1, \mathfrak{d}\mathfrak{p}^\nu) \otimes 1 \\ &= N_{k(\mathfrak{d}\mathfrak{p}^\nu)/k(\mathfrak{d}\mathfrak{p}^{\nu'})}(\psi(1, \mathfrak{d}\mathfrak{p}^\nu)) \otimes [k(\mathfrak{d}\mathfrak{p}^\nu) : k(\mathfrak{d}\mathfrak{p}^{\nu'})]^{-1} \\ &= \psi(1; \mathfrak{f}_\chi) \otimes [k(\mathfrak{d}\mathfrak{p}^\nu) : k(\mathfrak{d}\mathfrak{p}^{\nu'})]^{-1}.\end{aligned}$$

Then assume that $\nu > 0$ and $\nu' = 0$. So we have $\mathfrak{f}_\chi = \mathfrak{d}$.

$$\begin{aligned}\overline{\beta}_1 &= \psi(1, \mathfrak{d}\mathfrak{p}^\nu) \otimes 1 \\ &= \mathcal{N}_{k(\mathfrak{d}\mathfrak{p}^\nu)/k(\mathfrak{d})}\psi(1, \mathfrak{d}\mathfrak{p}^\nu) \otimes [k(\mathfrak{d}\mathfrak{p}^\nu) : k(\mathfrak{d})]^{-1} \\ &= (1 - \chi^{-1}(\mathfrak{p}))\psi(1, \mathfrak{d}) \otimes [k(\mathfrak{d}\mathfrak{p}^\nu) : k(\mathfrak{d})]^{-1} \\ &= (1 - \chi^{-1}(\mathfrak{p}))\psi(1, \mathfrak{f}_\chi) \otimes [k(\mathfrak{d}\mathfrak{p}^\nu) : k(\mathfrak{d})]^{-1}.\end{aligned}$$

□

Step E For $\mathfrak{l} \mid \mathfrak{f}_0$ choose a place $w_\mathfrak{l}$ above $\mathfrak{l}$ in $k(\mathfrak{f})/k$. Let $S_\mathfrak{l}$ be the set of places above $\mathfrak{l}$. Then we obtain

$$Y_{S_\mathfrak{l}} \otimes \mathbb{Q}_p(\chi) = \begin{cases} 0, & \chi_{|D_\mathfrak{l}} \neq 1, \\ \mathbb{Q}_p(\chi) \cdot w_\mathfrak{l}, & \chi_{|D_\mathfrak{l}} = 1. \end{cases}$$

We choose for each $\mathfrak{l} \mid \mathfrak{f}_0$ with $\chi_{|D_\mathfrak{l}} = 1$ an element $x_\mathfrak{l} \in L^\times$ such that

$$\mathrm{ord}_{w_\mathfrak{l}}(x_\mathfrak{l}) \neq 0 \text{ and } \mathrm{ord}_w(x_\mathfrak{l}) = 0 \text{ for all } w \neq w_\mathfrak{l},$$

Moreover, we set $J := \{\mathfrak{l} \mid \mathfrak{f}_0 \text{ with } \chi_{|D_\mathfrak{l}} = 1\}$, $x_J := \bigwedge_{\mathfrak{l} \in J} x_\mathfrak{l}$, $w_J := \bigwedge_{\mathfrak{l} \in J} w_\mathfrak{l}$ and

$$\mathrm{val} : \mathbb{Q}_p(\chi)x_\mathfrak{l} \to Y_{S_\mathfrak{l}} \otimes \mathbb{Q}_p(\chi) = \mathbb{Q}_p(\chi) \cdot w_\mathfrak{l}.$$

and we obtain

$$\begin{aligned}\{\overline{\beta}_1\} \cup \{x_\mathfrak{l} : \mathfrak{l} \in J\} &\text{ is a } \mathbb{Q}_p(\chi) \text{ -basis of } H^1(\Delta^\infty_{\mathfrak{q}_\chi} \otimes \mathbb{Q}_p(\chi)) \\ \{\overline{\beta}_2\} \cup \{w_\mathfrak{l} : \mathfrak{l} \in J\} &\text{ is a } \mathbb{Q}_p(\chi)\text{-basis of } H^2(\Delta^\infty_{\mathfrak{q}_\chi} \otimes \mathbb{Q}_p(\chi))\end{aligned}$$

We also get $e = -|J|$ as on p. 107 in [Ble06].

Lemma 6.5.30. *If we read* $\mathcal{L}$ *in* $(\mathrm{Det}_\Lambda(\Delta^\infty))_{\mathfrak{q}_\chi}$*, we can write*

$$\mathcal{L} = \underbrace{c_\mathfrak{a} \cdot d' \cdot \prod_{\mathfrak{l} \in I}(1 - Fr_\mathfrak{l}^{-1})^{-1} \prod_{\mathfrak{l} \in J} \frac{\omega}{1 - Fr_\mathfrak{l}^{-1}}}_{B:=} \cdot (\omega^e \beta_1^{-1} \otimes \beta_2)$$

with $I := \{\mathfrak{l} \mid \mathfrak{f}_0 \text{ such that } \mathfrak{l} \nmid \mathfrak{d} \text{ and } \chi_{|D_\mathfrak{l}} \neq 1\}$ *and* $d' := [k(\mathfrak{f}_0\mathfrak{p}) : k(\mathfrak{d}\mathfrak{p})]$.

Proof.

$$\begin{aligned}\mathcal{L} &= c_\mathfrak{a}(\eta_{\mathfrak{f}_0}^{-1} \otimes \tau) \\ &= c_\mathfrak{a} \cdot [k(\mathfrak{f}_0\mathfrak{p}) : k(\mathfrak{d}\mathfrak{p})] \cdot [\mathcal{N}_{k(\mathfrak{f}_0\mathfrak{p})/k(\mathfrak{d}\mathfrak{p})}]^{-1}(\eta_{\mathfrak{f}_0}^{-1} \otimes \tau) \\ &= c_\mathfrak{a} \cdot [k(\mathfrak{f}_0\mathfrak{p}) : k(\mathfrak{d}\mathfrak{p})] \cdot \prod_{\mathfrak{l} \in I \cup J} (1 - Fr_\mathfrak{l}^{-1})^{-1} \cdot (\eta_\mathfrak{d}^{-1} \otimes \tau) \\ &= c_\mathfrak{a} \cdot [k(\mathfrak{f}_0\mathfrak{p}) : k(\mathfrak{d}\mathfrak{p})] \cdot \prod_{\mathfrak{l} \in I} \cdot (1 - Fr_\mathfrak{l}^{-1})^{-1} \cdot \prod_{\mathfrak{l} \in J} \frac{\omega}{1 - Fr_\mathfrak{l}^{-1}} \cdot (\omega^e \beta_1^{-1} \otimes \beta_2).\end{aligned}$$

□

Step F

Lemma 6.5.31. *It holds that*

$$\phi'_\chi \circ \phi_\omega^{-1}(\overline{\beta}_1^{-1} \otimes \overline{\beta}_2) = \prod_{\mathfrak{l}\in J} c_{\mathfrak{l}} \cdot d \cdot T_\chi^{-1} e_\chi \psi_\chi^{-1} \otimes \tau =: A$$

with $d := [k(\mathfrak{d}\mathfrak{p}^{\nu}) : k(\mathfrak{d}\mathfrak{p}^{\nu'})]$.

Proof.

$$\begin{aligned}\phi'_\chi \circ \phi_\omega^{-1}(\overline{\beta}_1^{-1} \otimes \overline{\beta}_2) &= \phi'_\chi(\overline{\beta}^{-1} \wedge x_J^{-1} \otimes \beta_\omega(x_J) \wedge \overline{\beta}_2)\\ &= \prod_{\mathfrak{l}\in J} c_{\mathfrak{l},\gamma} \cdot \phi'_\chi(\overline{\beta}_1^{-1} \wedge x_J^{-1} \otimes \mathrm{val}(x_J) \wedge \overline{\beta}_2)\\ &= \prod_{\mathfrak{l}\in J} c_{\mathfrak{l},\gamma} \cdot (\overline{\beta}_1^{-1} \otimes \overline{\beta}_2)\\ &= \prod_{\mathfrak{l}\in J} c_{\mathfrak{l},\gamma} \cdot [k(\mathfrak{d}\mathfrak{p}^{\nu}) : k(\mathfrak{d}\mathfrak{p}^{\nu'})] \cdot T_\chi^{-1} e_\chi \psi_\chi^{-1} \otimes \tau,\end{aligned}$$

where the first equality comes from (6.21), the second equality from Proposition 6.5.15 and the last equality uses Lemma 6.5.29. □

Step G

Lemma 6.5.32. *The following two equalities hold:*

a)

$$e_\chi \mathcal{E}_S^{-1} = \prod_{\mathfrak{l}\in J} f_{\mathfrak{l}}^{-1} \cdot \prod_{\mathfrak{l}\in I} (1 - \chi(Fr_{\mathfrak{l}}^{-1})),$$

b)

$$\chi\left(\frac{\omega}{1 - Fr_{\mathfrak{l}}^{-1}}\right) = \frac{f_{\mathfrak{l}}}{c_{\mathfrak{l},\gamma}} \text{ for } \mathfrak{l} \in J.$$

Proof. Part a) follows easily from the fact that for $\mathfrak{l} \in S_{rp}$ we have

$$\mathcal{E}_{\mathfrak{l}} = \sum_{\chi_{|D_{\mathfrak{l}}}=1} f_{\mathfrak{l}} e_\chi + \sum_{\chi_{|D_{\mathfrak{l}}}\neq 1} (1 - \chi(\mathrm{Fr}_{\mathfrak{l}}^{-1})) e_\chi \text{ and } \mathcal{E}_S = \prod_{\mathfrak{l}\in S} \mathcal{E}_{\mathfrak{l}},$$

whereas b) follow from the fact that $\chi_{|D_{\mathfrak{l}}} = 1$ for $\mathfrak{l} \in J$ and that

$$(1 - \mathrm{Fr}_{\mathfrak{l}}^{-1}) \cdot (1 + \ldots + \mathrm{Fr}_{\mathfrak{l}}^{-f_{\mathfrak{l}}+1}) = 1 - \mathrm{Fr}_{\mathfrak{l}}^{-f_{\mathfrak{l}}} = 1 - \gamma^{c_{\mathfrak{l},\gamma} p^m}.$$

□

Lemma 6.5.33. *It holds that*

$$e_\chi \mathcal{E}_S^{-1} \cdot B \cdot A = \begin{cases} c_{\mathfrak{a}} \cdot w(\mathfrak{f}_\chi)[k(\mathfrak{f}) : k(\mathfrak{f}_\chi)] \cdot e_\chi(\psi_\chi^{-1} \otimes \tau) & \mathfrak{f}_\chi \neq (1),\\ c_{\mathfrak{a}} \cdot w(1) \cdot [k(\mathfrak{f}) : k(1)] \cdot T_\chi \cdot e_\chi(\psi_\chi^{-1} \otimes \tau) & \mathfrak{f}_\chi = (1).\end{cases}$$

Recall that by abuse of notation we denote by $c_{\mathfrak{a}}$ the elements $\mathcal{N}(\mathfrak{a}) - \sigma_{\mathfrak{a}}$ and $\mathcal{N}(\mathfrak{a}) - \chi(\mathfrak{a})$. This notation is unfortunately similar to $c_{\mathfrak{l},\gamma}$ and $c_{\mathfrak{p},\gamma}$, which one can see from Definition 6.5.11 is something completely different.

Proof. Now by combining Lemmas 6.5.32 a), 6.5.30 and 6.5.31 we get

$$\begin{aligned} e_\chi \mathcal{E}_S^{-1} \cdot B \cdot A = &\prod_{\mathfrak{l}\in J} f_{\mathfrak{l}}^{-1} \prod_{\mathfrak{l}\in I} (1 - \chi(Fr_{\mathfrak{l}}^{-1})) \\ &\cdot c_{\mathfrak{a}} \cdot d' \cdot \prod_{\mathfrak{l}\in I} (1 - \chi(Fr_{\mathfrak{l}}^{-1}))^{-1} \prod_{\mathfrak{l}\in J} \frac{\omega}{1 - Fr_{\mathfrak{l}}^{-1}} \\ &\cdot \prod_{\mathfrak{l}\in J} c_{\mathfrak{l},\gamma} \cdot d \cdot T_\chi^{-1} e_\chi \psi_\chi^{-1} \otimes \tau \end{aligned}$$

With $\chi(\mathfrak{p}) = e_\chi Fr_{\mathfrak{p}} = 0$ because $\chi_{|D_{\mathfrak{p}}} \neq 1$ and Lemma 6.5.32 b) it remains to show that

$$d \cdot d' = w(\mathfrak{f}_\chi)[k(\mathfrak{f}) : k(\mathfrak{f}_\chi)]$$

but with the help of Theorem 3.1.9 in [Sch10] this is an easy computation.

□

6.5.6 Proof for Case II (trivial zeroes case)

We recall that we have fixed an odd prime p, let $\mathfrak{p}$ be the prime ideal of k above p and let k be an imaginary quadratic field. Also we have assumed that p is inert in k and $p \nmid h_k$. Let $\mathfrak{f}$ be an integral ideal of k with $\mathfrak{f} = \mathfrak{f}_0\mathfrak{p}^\nu$ with $(\mathfrak{f}_0, \mathfrak{p}) = 1$ and without loss of generality $\nu \geq 1$. Let χ be a non-trivial character such that $\chi |_{D_{\mathfrak{p}}} = 1$. We denote by $\mathfrak{f}_\chi$ the conductor of the character χ and we have $\mathfrak{f}_\chi \mid \mathfrak{f}$ by definition and in this case $\mathfrak{p} \nmid \mathfrak{f}_\chi$, because $\chi_{|D_{\mathfrak{p}}} = 1$.

We again follow very closely the computations of pp. 108-114 in [Ble06] and make the appropriate adaptations for the inert case if necessary.

For any subgroup B of $\mathcal{G}$ we define J_B to be the kernel of the canonical map $\mathbb{Z}_p[[\mathcal{G}]] \to \mathbb{Z}_p[[\mathcal{G}/B]]$.

Lemma 6.5.34. *a) We have the isomorphism* $Y^\infty_{S_p,\mathfrak{q}_\chi} \cong \mathbb{Q}_p(\chi)$.

b) It holds that $(\gamma - 1)U^\infty_{S,\mathfrak{q}_\chi} = U^\infty_{\mathfrak{q}_\chi}$.

Proof. For a) we have

$$Y^\infty_{S_p,\mathfrak{q}_\chi} \cong \mathbb{Z}_p[G/D_{\mathfrak{p}}] \otimes_\Lambda \Lambda_{\mathfrak{q}_\chi} \cong \Lambda_{\mathfrak{q}_\chi}/J_{D_{\mathfrak{p}}}\Lambda_{\mathfrak{q}_\chi}.$$

Because we have $\chi_{|D_{\mathfrak{p}}} = 1$ we get $\Gamma \subseteq D_{\mathfrak{p}}$, and therefore one has $\gamma^{p^m} - 1 \sim \gamma - 1$. It follows that $Y^\infty_{S_p,\mathfrak{q}_\chi} \cong \mathbb{Q}_p(\chi)$. Part b) follows from the structure theorem for modules over principal ideal rings. □

Step A As in Case I we get the following lemma.

Lemma 6.5.35. *Assuming* $P^\infty_{S,\mathfrak{q}_\chi} = 0$ *we obtain*

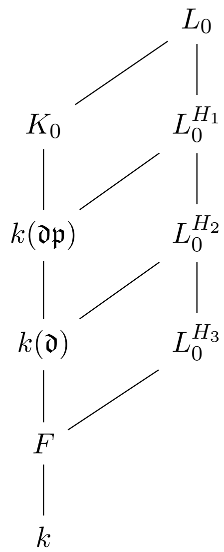
a) $H^1(\Delta^\infty_{\mathfrak{q}_\chi}) = (U^\infty_S)_{\mathfrak{q}_\chi}$ *and* $H^2(\Delta^\infty_{\mathfrak{q}_\chi}) = (X^\infty_S)_{\mathfrak{q}_\chi}$.

b) $H^1(\Delta^\infty_{\mathfrak{q}_\chi})_{tors} = 0$.

c) $H^2(\Delta^\infty_{\mathfrak{q}_\chi})_{tors} = (X^\infty_{S_{pr}})_{\mathfrak{q}_\chi}$ *and* $\omega \cdot H^2(\Delta^\infty_{\mathfrak{q}_\chi})_{tors} = 0$.

In particular, the assumptions for using Lemma 6.5.6 are satisfied.

Step B and C Let F be the decomposition field of $k(\mathfrak{d})/k$ at $\mathfrak{p}$. First of all we set some notation $H := \mathrm{Gal}(K_0/k)$, $H_1 := \mathrm{Gal}(K_0/k(\mathfrak{d}\mathfrak{p}))$, $H_2 := \mathrm{Gal}(K_0/k(\mathfrak{d}))$, $H_3 := \mathrm{Gal}(K_0/F)$, $K_n'' := k(\mathfrak{d}\mathfrak{p}^{n+1})$ and illustrate it in the following diagram of fields.



Since F is the decomposition field of $k(\mathfrak{d})/k$ at $\mathfrak{p}$, we have $k \subseteq F \subset k(\mathfrak{d})$. We set $F_\infty := F \cdot k_{1,\infty}$ and so F_∞/F is also a $\mathbb{Z}_p$-extension with Galois group isomorphic to Γ. We denote by F_n an intermediate level of this $\mathbb{Z}_p$-extension and recall that an extension F_n/F is now a cyclic extension of degree p^n.

Definition 6.5.36. For an admissible set of places S of k we set

$$U^\infty_{k(\mathfrak{f}_\chi),S'} := \varprojlim_n \mathcal{O}^\times_{L_n^{H_1},S'} \otimes \mathbb{Z}_p$$

for some finite set S' of places of k which contains S.

Lemma 6.5.37. *a) The following sequence is exact:*

$$0 \to U^\infty_{k(\mathfrak{f}_\chi),S} \xrightarrow{\gamma-1} U^\infty_{k(\mathfrak{f}_\chi),S} \to \left(U^\infty_{k(\mathfrak{f}_\chi),S}\right)_\Gamma \to 0.$$

b) The canonical map

$$\iota : \left(U^\infty_{k(\mathfrak{f}_\chi),S}\right)_\Gamma \to \mathcal{O}^\times_{L_0^{H_1},S} \otimes \mathbb{Z}_p \text{ is injective.}$$

Proof. We can use exactly the same argument as in Lemma 5.9 in [Ble06]. The only thing we have to think about whether we can show that

$$U^\infty_{k(\mathfrak{f}_\chi),S} \cong U^\infty_{k(\mathfrak{f}_\chi),S_\infty \cup S_p}.$$

There could be finite primes in S which are splitting infinitely. But if necessary we can choose, as we have seen above, a $\mathbb{Z}_p$-extension in which this does not happen. □

As in [Ble06] we choose an auxiliary prime ideal $\mathfrak{b}$ of $\mathcal{O}_k$ such that

$$(\mathfrak{b}, \mathfrak{f}) = 1, \qquad w(\mathfrak{b}) = 1, \qquad \chi(\mathfrak{b}) \neq 1,$$

and so we can set

$$B'_n := k(\mathfrak{d}\mathfrak{b}\mathfrak{p}^{n+1}).$$

Using this ideal we can define a new element

$$\eta := \left\{\mathcal{N}_{B'_n/L_n^{H_1}}\left(\psi(1;\mathfrak{b}\mathfrak{d}\mathfrak{p}^{n+1})\right)\right\}_{n\geq 0}.$$

Lemma 6.5.38. *The following two equalities hold:*

a) $\iota(N_{L_0^{H_1}/F}(\eta)) = N_{L_0^{H_1}/F}(\eta^0)$,

b) $N_{L_0^{H_1}/F}(\eta^0) = 1$.

Proof. With the proof of Lemma 5.9 in [Ble06] we see that ι maps an element to its zero-th component. For b) we compute

$$\begin{aligned} N_{L_0^{H_1}/F}(\eta^0) &= N_{k(\mathfrak{d})/F}\left(N_{L_0^{H_1}/k(\mathfrak{d})}(\mathcal{N}_{B'_0/L_0^{H_1}}(\psi(1;\mathfrak{b}\mathfrak{d}\mathfrak{p})))\right) \\ &= N_{k(\mathfrak{d}\mathfrak{b})/F}\left(\psi(1;\mathfrak{d}\mathfrak{b})^{1-\sigma(\mathfrak{p})^{-1}}\right) \\ &= (1-\sigma_{|F}(\mathfrak{p})^{-1})N_{k(\mathfrak{d}\mathfrak{b})/F}\left(\psi(1;\mathfrak{d}\mathfrak{b})\right) = 1 \end{aligned}$$

where we have used that $\sigma_{|F}(\mathfrak{p})$ is the identity because $\mathfrak{p}$ splits completely in F, and the norm relations. □

Combining now Lemmas 6.5.37 and 6.5.38 we get the following lemma.

Lemma 6.5.39. *There exists an element* $z^\infty \in U^\infty_{k(\mathfrak{f}_\chi),S} \otimes \mathbb{Q}_p$ *such that*

$$(\gamma-1)z^\infty = \frac{1}{[L_0^{H_1}:F]}N_{L_0^{H_1}/F}(\eta).$$

Recall that we have $\eta^n_{\mathfrak{d}} := \mathcal{N}_{(K'_n)^{H_1}/L_n^{H_1}}(\psi(1;\mathfrak{d}\mathfrak{p}^{n+1}))$ so we get

$$\mathcal{N}_{L_n^{H_1}/F_n}(\eta^n) = \mathcal{N}_{B'_n/F_n}(\psi(1;\mathfrak{b}\mathfrak{d}\mathfrak{p}^{n+1})) = (1-Fr_{\mathfrak{b}}^{-1})\mathcal{N}_{L_n^{H_1}/F_n}(\eta^n_{\mathfrak{d}}) \tag{6.28}$$

Lemma 6.5.40. *Assume* $\mathrm{LC}(L_\infty/k,p)$ *and* $P^\infty_{\mathfrak{q}_\chi}=0$ *holds. Then we obtain that*

$$M^1 = U^\infty_{S,\mathfrak{q}_\chi} \text{ with } \beta_1 = z^\infty$$

and

$$M^2 = X^\infty_{S_\infty,\mathfrak{q}_\chi} \text{ with } \beta_2 = \tau.$$

Proof. Similar as in Case I in Lemma 6.5.27 - assuming $\mathrm{LC}(L_\infty/k,p)$ and $P^\infty_{S,\mathfrak{q}_\chi}=0$ (which is implied by $P^\infty_{\mathfrak{q}_\chi}=0$) - we can get that $U^\infty_{\mathfrak{q}_\chi} = \Lambda_{\mathfrak{q}_\chi}\eta_{\mathfrak{d}}$. There is a slight modification necessary, because now we do not have $Y^\infty_{S_p,\mathfrak{q}_\chi}=0$ in Case II and therefore we obtain the following exact sequence by using only $P^\infty_{\mathfrak{q}_\chi}=0$:

$$0\to U^\infty_{\mathfrak{q}_\chi}\to U^\infty_{S,\mathfrak{q}_\chi}\to Y^\infty_{S_p,\mathfrak{q}_\chi}\to 0.$$

But the extra factor in $H^1(\Delta^\infty_{\mathfrak{q}_\chi})$ cancels out with a factor we obtain from the short exact sequence (6.27). Now Lemma 6.5.34 and Lemma 6.5.39 gives us that $U^\infty_{S,\mathfrak{q}_\chi} = \Lambda_{\mathfrak{q}_\chi}z^\infty$ so because we know that $H^1(\Delta^\infty_{\mathfrak{q}_\chi}) = U^\infty_{S,\mathfrak{q}_\chi}$ the first assertion follows. The second assertion is proven exactly as in Case I. □

Step D

Lemma 6.5.41. *We have*

$$\overline{\beta}_1 = z_m \text{ and } \overline{\beta}_2 = \tau.$$

Proof. As in Case I in Lemma 6.5.28 we now again have the injective map

$$M^1/\omega \to \mathcal{O}^\times_{k(\mathfrak{f}),S} \otimes \mathbb{Q}_p(\chi)$$
$$u \mapsto u_m \otimes 1$$

and we recall that $\overline{\beta}_1$ is the image of $\beta_1 \in M^1/\omega$. The result for $\overline{\beta}_2$ is clear. □

Step E

Construction of κ_∞ Now we set

$$\varepsilon_n := \mathcal{N}_{B'_n/F_n}(\psi(1;\mathfrak{b}\mathfrak{d}\mathfrak{p}^{n+1}))$$

which is a norm-compatible system and therefore with Hilbert's Theorem 90 there exists a $\alpha_n \in F_n^\times/F^\times$ (because we have $\varepsilon_0 = 1$ by Lemma 6.5.38 b)) such that

$$\alpha_n^{\gamma-1} = \varepsilon_n.$$

So we can define $\kappa_n := \mathcal{N}_{F_n/F}(\alpha_n)$ and $\kappa_\infty := \{\kappa_n\}_{n=0}^\infty \in \varprojlim_n F^\times/(F^\times)^{p^n}$. Now by definition of F the prime ideal $\mathfrak{p}$ splits completely in F.
Now with Theorem 4.1.16 we obtain in the inert case

$$\operatorname{ord}_w(\kappa_\infty) = \pi_\gamma(\frac{1}{p}\log_p(\mathcal{N}_{k(\mathfrak{d}\mathfrak{b})/F}(\psi(1;\mathfrak{d}\mathfrak{b})))) \tag{6.29}$$

for a place w of F above $\mathfrak{p}$, because multiplication by $1/p$ induces an isomorphism between $p\mathcal{O}_{k_\mathfrak{p}}$ and $\mathcal{O}_{k_\mathfrak{p}}$.

Lemma 6.5.42. *We have the following two equalities:*

a)

$$\alpha_n = \mathcal{N}_{L_n^{H_1}/F_n}(z_n) \text{ in } F_n^\times/F^\times.$$

b)

$$\kappa_\infty = \{\mathcal{N}_{L_0^{H_1}/F}(z_0)\}_{n=0}^\infty \text{ in } \varprojlim_n F^\times/\left(F^\times\right)^{p^n}.$$

Proof. In order to prove assertion a) we only need to look at the defining equations of α_n and z_n.
For b) we can compute

$$\kappa_n = \mathcal{N}_{F_n/F}(\alpha_n) = \mathcal{N}_{F_n/F}(\mathcal{N}_{L_n^{H_1}/F_n}(z_n)) = \mathcal{N}_{L_0^{H_1}/F}(\mathcal{N}_{L_n^{H_1}/L_0^{H_1}}(z_n)) = \mathcal{N}_{L_0^{H_1}/F}(z_0)$$

where we have used the diagram of fields above and a). □

We collect some properties we are going to need in the following lemmas.

Remark 6.5.43. a) Let L/K be an abelian extension of number fields, and $\mathfrak{p}$ a prime ideal of K and assume that there is only one prime ideal $\mathfrak{P}$ above $\mathfrak{p}$. Then we have

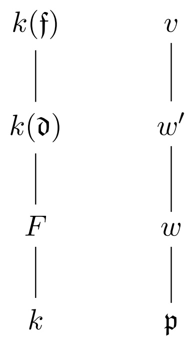
$$f(\mathfrak{P} \mid \mathfrak{p}) \cdot \mathrm{ord}_{\mathfrak{P}}(\alpha) = \mathrm{ord}_{\mathfrak{p}}(\mathcal{N}_{L/K}(\alpha)) \text{ for each } \alpha \in L.$$

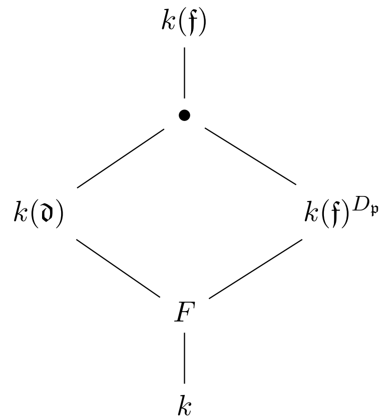
b) Primes above $\mathfrak{p}$ are totally ramified in $k(\mathfrak{d}\mathfrak{p})/k(\mathfrak{d})$, $L_0^{H_1}/k(\mathfrak{d}\mathfrak{p})$ as well as in $L_m^{H_1}/L_0^{H_1}$.

c) Let L/K be an abelian extension of number fields and let $\mathfrak{P}$ be a prime ideal in L over $\mathfrak{p}$ then we have

$$\frac{1}{e(\mathfrak{P} \mid \mathfrak{p})}\mathrm{ord}_{\mathfrak{P}} = \mathrm{ord}_{\mathfrak{p}}.$$

In order to facilitate the computation one should consider the following diagrams of fields.





Lemma 6.5.44. *The following equality holds*

$$\pi_\gamma\left(\frac{1}{p}\log_p(\mathcal{N}_{k(\mathfrak{f})_v/k_\mathfrak{p}}(\mathcal{N}_{k(\mathfrak{d}\mathfrak{b})/k(\mathfrak{d})}(\psi(1;\mathfrak{b}\mathfrak{d}))\right) = f_\mathfrak{p} \cdot \mathrm{ord}_v(\mathcal{N}_{L_0^{H_1}/k(\mathfrak{d})}(z_0)),$$

where $f_\mathfrak{p}$ is the residue degree $f(v \mid \mathfrak{p})$ of $\mathfrak{p}$ in $k(\mathfrak{f})/k$.

Proof. If we look at the right diagram above we obtain:

$$\log_p\left(\mathcal{N}_{k(\mathfrak{f})_v/k_\mathfrak{p}}(\mathcal{N}_{k(\mathfrak{d}\mathfrak{b})/k(\mathfrak{d})}(\psi(1;\mathfrak{b}\mathfrak{d})))\right) = \frac{|D_\mathfrak{p}|}{[k(\mathfrak{d}):F]}\log_p\left(\mathcal{N}_{k(\mathfrak{d})/F}(\mathcal{N}_{k(\mathfrak{d}\mathfrak{b})/k(\mathfrak{d})}(\psi(1;\mathfrak{b}\mathfrak{d})))\right)$$

and then we compute the following

$$\begin{aligned}
\pi_\gamma\left(\frac{1}{p}\log_p(\mathcal{N}_{k(\mathfrak{f})_v/k_\mathfrak{p}}(\mathcal{N}_{k(\mathfrak{d}\mathfrak{b})/k(\mathfrak{d})}(\psi(1;\mathfrak{b}\mathfrak{d}))))\right) &= \frac{|D_\mathfrak{p}|}{[k(\mathfrak{d}):F]}\pi_\gamma\left(\frac{1}{p}\log_p\left(\mathcal{N}_{k(\mathfrak{d})/F}(\mathcal{N}_{k(\mathfrak{d}\mathfrak{b})/k(\mathfrak{d})}(\psi(1;\mathfrak{b}\mathfrak{d})))\right)\right) \\
&= \frac{|D_\mathfrak{p}|}{[k(\mathfrak{d}):F]}\mathrm{ord}_w(\mathcal{N}_{L_0^{H_1}/F}(z_0)) \\
&= \frac{|D_\mathfrak{p}|}{[k(\mathfrak{d}):F]}f(w' \mid w)\mathrm{ord}_{w'}(\mathcal{N}_{L_0^{H_1}/k(\mathfrak{d})}(z_0)) \\
&= |D_\mathfrak{p}|\frac{1}{e(v \mid w')}\mathrm{ord}_v(\mathcal{N}_{L_0^{H_1}/k(\mathfrak{d})}(z_0)) \\
&= f_\mathfrak{p} \cdot \mathrm{ord}_v(\mathcal{N}_{L_0^{H_1}/k(\mathfrak{d})}(z_0)),
\end{aligned}$$

where in the first equality we use the equality from directly above and the second equality follows from (6.29) and Lemma 6.5.42. For the third equality we use that $f(w' \mid w) \cdot \mathrm{ord}_{w'}(\alpha) = \mathrm{ord}_w(\mathcal{N}_{k(\mathfrak{d})/F}(\alpha))$ if there is only one place above w, which it is because F is the decomposition field of $k(\mathfrak{d})/k$ at $\mathfrak{p}$. Moreover, $\mathrm{ord}_w(\alpha) = e(w' \mid w) \cdot \mathrm{ord}_{w'}(\alpha)$ and $e(w' \mid w) = 1$ because $\mathfrak{p} \nmid \mathfrak{d}$. The fourth equality holds because $f(w' \mid w) = [k(\mathfrak{d}) : F]$ because $e(w' \mid w) = 1$. Moreover, $\frac{1}{e(v|w')}\mathrm{ord}_v = \mathrm{ord}_{w'}$. The fifth equality follows because $|D_{\mathfrak{p}}| = e(v \mid \mathfrak{p})\cdot f(v \mid \mathfrak{p}) = e(v \mid w')\cdot f(v \mid \mathfrak{p})$ since $\mathfrak{p} \nmid \mathfrak{d}$. □

Lemma 6.5.45. *In $\Lambda_{\mathfrak{q}_\chi}$ we have*

a) $(\gamma - 1)\beta_1 = (\gamma - 1)z^\infty = \frac{1}{[L_0^{H_1}:F]}\mathcal{N}_{L_0^{H_1}/F}(\eta)$.

b) $\frac{\omega}{1-\gamma} = \sum_{i=0}^{p^m-1} \gamma^i =: T$.

c) $\omega\beta_1 = -T(1-\sigma(\mathfrak{b})^{-1})\eta_{\mathfrak{d}}$.

d) $e = -(|J|+1)$, *with e from Lemma 6.5.6.*

Proof. We get a) by combining Lemmas 6.5.39 and 6.5.40.

For b): Because of Remark 6.5.9 we have $\omega = 1 - \gamma^{p^m}$ and therefore

$$(1-\gamma)\cdot \sum_{i=0}^{p^m-1} \gamma^i = \sum_{i=0}^{p^m-1} \gamma^i - \sum_{i=0}^{p^m-1} \gamma^{i+1} = 1 - \gamma^{p^m} = \omega.$$

For c) we compute:

$$\begin{aligned}
\omega\beta_1 &= (1-\gamma)T\beta_1 \\
&= -T\frac{1}{[L_0^{H_1}:F]}N_{L_0^{H_1}/F}(\eta) \\
&= -T\cdot\frac{1}{[L_0^{H_1}:F]}\cdot(1-\sigma(\mathfrak{b})^{-1})\cdot N_{L_0^{H_1}/F}(\eta_{\mathfrak{d}}) \\
&= -T\cdot(1-\sigma(\mathfrak{b})^{-1})\cdot\eta_{\mathfrak{d}}
\end{aligned}$$

where we have used a) and b). For d) we can use the proof of p. 107 in [Ble06]. □

We define $I := \{\mathfrak{l} \mid \mathfrak{f}_0 \text{ with } \mathfrak{l} \nmid \mathfrak{f}_\chi\}$ and $I' := \{\mathfrak{l} \mid \mathfrak{f}_0 \text{ with } \mathfrak{l} \nmid \mathfrak{f}_\chi \text{ and } \chi_{|D_\mathfrak{l}\neq 1}\}$.

Lemma 6.5.46. *If we read $\mathcal{L}$ in $(\mathrm{Det}_\Lambda(\Delta^\infty))_{\mathfrak{q}_\chi}$ we get*

$$\mathcal{L} = \underbrace{-T\cdot c_{\mathfrak{a}}\cdot[L_0:L_0^{H_1}]\cdot\prod_{\mathfrak{l}\in I'}(1-Fr_{\mathfrak{l}}^{-1})^{-1}\cdot\prod_{\mathfrak{l}\in J}\frac{\omega}{1-Fr_{\mathfrak{l}}^{-1}}}_{B:=}\,\omega^e\beta_1^{-1}\otimes\beta_2.$$

Proof.

$$\begin{aligned}
\mathcal{L} = c_{\mathfrak{a}}\cdot\eta_{\mathfrak{f}_0}^{-1}\otimes\tau &= c_{\mathfrak{a}}\cdot[L_0:L_0^{H_1}]\cdot\left(\mathrm{Tr}_{k(\mathfrak{f}_0\mathfrak{p})/k(\mathfrak{d}\mathfrak{p})}\eta_{\mathfrak{f}_0}\right)^{-1}\otimes\tau \\
&= c_{\mathfrak{a}}\cdot[L_0:L_0^{H_1}]\cdot\prod_{\mathfrak{l}\in I}(1-Fr_{\mathfrak{l}}^{-1})^{-1}\eta_{\mathfrak{d}}^{-1}\otimes\tau \\
&= -T\cdot(1-\sigma(\mathfrak{b})^{-1})\cdot c_{\mathfrak{a}}\cdot[L_0:L_0^{H_1}]\cdot\prod_{\mathfrak{l}\in I}(1-Fr_{\mathfrak{l}}^{-1})^{-1}\omega^{-1}\beta_1^{-1}\otimes\beta_2 \\
&= -T\cdot(1-\sigma(\mathfrak{b})^{-1})\cdot c_{\mathfrak{a}}\cdot[L_0:L_0^{H_1}]\cdot\prod_{\mathfrak{l}\in I'}\frac{1}{1-Fr_{\mathfrak{l}}^{-1}}\cdot\prod_{\mathfrak{l}\in J}\frac{\omega}{1-Fr_{\mathfrak{l}}^{-1}}\omega^e\beta_1^{-1}\otimes\beta_2.
\end{aligned}$$

□

Lemma 6.5.47. *It holds that*

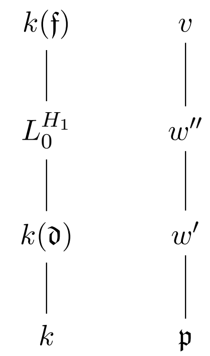
$$\beta_\omega(\mathcal{N}_{k(\mathfrak{d}\mathfrak{b})/k(\mathfrak{d})}\psi(1;\mathfrak{b}\mathfrak{d}))_v = c_{\mathfrak{p},\gamma}^{-1} \cdot f_{\mathfrak{p}} \cdot [L_m^{H_1} : k(\mathfrak{d})]\mathrm{ord}_v(\overline{\beta}_1)$$

with β_ω from Proposition 6.5.15.

Proof.

$$\begin{aligned}\beta_\omega(\mathcal{N}_{k(\mathfrak{d}\mathfrak{b})/k(\mathfrak{d})}\psi(1;\mathfrak{b}\mathfrak{d}))_v &= c_{\mathfrak{p},\gamma}^{-1} \cdot f_{\mathfrak{p}} \cdot \mathrm{ord}_v(\mathcal{N}_{L_0^{H_1}/k(\mathfrak{d})}(z_0)) \\ &= c_{\mathfrak{p},\gamma}^{-1} \cdot f_{\mathfrak{p}} \cdot [L_0^{H_1} : k(\mathfrak{d})] \cdot \mathrm{ord}_v(z_0) \\ &= c_{\mathfrak{p},\gamma}^{-1} \cdot f_{\mathfrak{p}} \cdot [L_m^{H_1} : L_0^{H_1}] \cdot [L_0^{H_1} : k(\mathfrak{d})] \cdot \mathrm{ord}_v(z_m) \\ &= c_{\mathfrak{p},\gamma}^{-1} \cdot f_{\mathfrak{p}} \cdot [L_m^{H_1} : k(\mathfrak{d})] \cdot \mathrm{ord}_v(\overline{\beta}_1).\end{aligned}$$

The first equality follows from Proposition 6.5.15 and Lemma 6.5.44. For the second equality we consider the diagram:



With Remark 6.5.43 c) we obtain

$$\mathrm{ord}_v(z_0) = \mathrm{ord}_{w''}(z_0) \cdot e(v \mid w'') \text{ and } \mathrm{ord}_v(\mathcal{N}_{L_0^{H_1}/k(\mathfrak{d})}(z_0)) = \mathrm{ord}_{w'}(\mathcal{N}_{L_0^{H_1}/k(\mathfrak{d})}(z_0)) \cdot e(v \mid w')$$

and from Remark 6.5.43 a) combined with b) we get

$$\mathrm{ord}_{w''}(z_0) = \mathrm{ord}_{w'}(\mathcal{N}_{L_0^{H_1}/k(\mathfrak{d})}(z_0)).$$

So since $e(w'' \mid w') = [L_0^{H_1} : k(\mathfrak{d})]$ because of Remark 6.5.43 b) we obtain the second equality. Using the fact that $z_0 = \mathcal{N}_{L_m^{H_1}/L_0^{H_1}}(z_m)$ we can prove the third equality analogous to the second equality. The last equality follows from Lemma 6.5.41. □

Step F

Lemma 6.5.48. *We have the following equality*

$$\phi'_\chi \circ \phi_\omega^{-1}(\overline{\beta}_1^{-1} \otimes \overline{\beta}_2) = -c_{\mathfrak{p},\gamma}^{-1} \cdot \prod_{\mathfrak{l}\in J} c_{\mathfrak{l},\gamma} \cdot [L_m^{H_1} : k(\mathfrak{d})] \cdot \left(\mathcal{N}_{k(\mathfrak{d}\mathfrak{b})/k(\mathfrak{d})}\psi(1;\mathfrak{b}\mathfrak{d})\right)^{-1} \otimes \tau =: A$$

with ϕ'_χ from Definition 6.5.16.

Proof. We compute

$$\begin{aligned}
&\phi'_\chi \circ \phi_\omega^{-1}(\bar\beta_1^{-1} \otimes \bar\beta_2) \\
&= \phi'_\chi \left(\bar\beta_1^{-1} \wedge \left(\mathcal{N}_{k(\mathfrak{d}\mathfrak{b})/k(\mathfrak{d})}\psi(1;\mathfrak{b}\mathfrak{d})\right)^{-1} \wedge x_J^{-1} \otimes \beta_\omega(x_J) \wedge \beta_\omega(\mathcal{N}_{k(\mathfrak{d}\mathfrak{b})/k(\mathfrak{d})}\psi(1;\mathfrak{b}\mathfrak{d})) \wedge \bar\beta_2\right) \\
&= -\prod_{\mathfrak{l}\in J} c_{\mathfrak{l},\gamma} \cdot c_{\mathfrak{p},\gamma}^{-1} \cdot f_\mathfrak{p} \cdot [L_m^{H_1} : k(\mathfrak{d})] \left(\left(\mathcal{N}_{k(\mathfrak{d}\mathfrak{b})/k(\mathfrak{d})}\psi(1;\mathfrak{b}\mathfrak{d})\right)^{-1} \otimes \tau\right)
\end{aligned}$$

where we get the first equality from (6.22) and the second equality from Lemma 6.5.47. □

Step G

Lemma 6.5.49. *i) For $\mathfrak{l} \in J$ we have $\chi\left(\frac{\omega}{1-Fr_\mathfrak{l}^{-1}}\right) = \frac{f_\mathfrak{l}}{c_{\mathfrak{l},\gamma}}$.*

ii) It holds that $\chi(T) = c_{\mathfrak{p},\gamma}$.

Proof. For a) we can use the same proof as in Lemma 6.5.32 a). For b) we have

$$c_{\mathfrak{p},\gamma} = \pi_\gamma\left(\frac{1}{p}(\chi_{\mathrm{ell}}(\gamma^{p^m}))\right) = \chi\left(\sum_{i=0}^{p^m-1} \gamma^i\right) = \chi(T).$$

□

Lemma 6.5.50. *We have*

$$e_\chi \mathcal{E}_S^{-1} \cdot B \cdot A = c_\mathfrak{a} \cdot [k(\mathfrak{f}) : k(\mathfrak{d})] \cdot e_\chi(\psi_\chi^{-1} \otimes \tau).$$

Proof. We put everything we have computed so far together (additionally to the lemmas developed for this case, we also use Lemma 6.5.32 from Case I) to obtain:

$$\begin{aligned}
e_\chi \mathcal{E}_{S_p}^{-1} \cdot B \cdot A = &\prod_{\mathfrak{l}\in J\cup\{\mathfrak{p}\}} f_\mathfrak{l}^{-1} \cdot \chi\left(\prod_{\mathfrak{l}\in I'}(1-Fr_\mathfrak{l}^{-1}\right) \\
&- T \cdot (1-\sigma(\mathfrak{b})^{-1}) \cdot c_\mathfrak{a} \cdot [L_0 : L_0^{H_1}] \cdot \prod_{\mathfrak{l}\in I'} \frac{1}{1-Fr_\mathfrak{l}^{-1}} \cdot \prod_{\mathfrak{l}\in J} \frac{\omega}{1-Fr_\mathfrak{l}^{-1}} \\
&- c_{\mathfrak{p},\gamma}^{-1} \cdot \prod_{\mathfrak{l}\in J} c_{\mathfrak{l},\gamma} \cdot [L_m^{H_1} : k(\mathfrak{d})] \cdot \left(\mathcal{N}_{k(\mathfrak{d}\mathfrak{b})/k(\mathfrak{d})}\psi(1;\mathfrak{b}\mathfrak{d})\right)^{-1} \otimes \tau.
\end{aligned}$$

We know from the norm relations and because we have by construction that $w(\mathfrak{b}) = w(\mathfrak{f}) = 1$:

$$\begin{aligned}
\left(\mathcal{N}_{k(\mathfrak{b}\mathfrak{d})/k(\mathfrak{d})}(\psi(1;\mathfrak{b}\mathfrak{d}))\right)^{-1} &= \left(\frac{1}{w(\mathfrak{d})}\mathcal{N}_{k(\mathfrak{b}\mathfrak{d})/k(\mathfrak{d})}(\psi(1;\mathfrak{b}\mathfrak{d})^{w(\mathfrak{d})})\right)^{-1} \\
&= \begin{cases} (1-\sigma(\mathfrak{b})^{-1})^{-1} w(\mathfrak{d})\psi_\chi^{-1} & \text{if } \mathfrak{d} \neq 1, \\ w(1)\left(\frac{\delta(\mathcal{O}_k,\mathfrak{a}^{-1})}{\delta(\mathfrak{b},\mathfrak{a}^{-1}\mathfrak{b})}\right)^{-1} & \text{if } \mathfrak{d} = 1, \end{cases}
\end{aligned}$$

and one has the relation

$$\left(\frac{\delta(\mathcal{O}_k,\mathfrak{a}^{-1})}{\delta(\mathfrak{b},\mathfrak{a}^{-1}\mathfrak{b})}\right)^{1-\sigma(\mathfrak{p})^{-1}} = \left(\frac{\delta(\mathcal{O}_k,\mathfrak{a}^{-1})}{\delta(\mathfrak{p},\mathfrak{a}^{-1}\mathfrak{p})}\right)^{1-\sigma(\mathfrak{b})^{-1}}.$$

Using Lemma 6.5.49 we know that

$$e_\chi \mathcal{E}_S^{-1} \cdot B \cdot A = \begin{cases} w(\mathfrak{d})[L_m^{H_1} : k(\mathfrak{d})] \cdot [L_0 : L_0^{H_1}] \cdot \psi_\chi^{-1} \cdot \otimes \tau & \text{if } \mathfrak{d} \neq (1), \\ w(1)[L_m^{H_1} : k(1)] \cdot [L_0 : L_0^{H_1}] \cdot \psi_\chi^{-1} \otimes \tau & \text{if } \mathfrak{d} = (1). \end{cases}$$

So it remains to show that

$$[k(\mathfrak{f}) : k(\mathfrak{d})] = [L_m^{H_1} : k(\mathfrak{d})] \cdot [L_0 : L_0^{H_1}].$$

We already know that $[k(\mathfrak{f}) : k(\mathfrak{d})] = [k(\mathfrak{d}\mathfrak{p}^\nu) : k(\mathfrak{d})] \cdot [K_0 : k(\mathfrak{d}\mathfrak{p})]$ and that $[L_0 : L_0^{H_1}] = [K_0 : k(\mathfrak{d}\mathfrak{p})]$ so the problem is reduced to showing that

$$[k(\mathfrak{d}\mathfrak{p}^\nu) : k(\mathfrak{d})] = [L_m^{H_1} : k(\mathfrak{d})].$$

But recall that $H_1 := \mathrm{Gal}(k(\mathfrak{f}_0\mathfrak{p})/k(\mathfrak{d}\mathfrak{p})) \cong \mathrm{Gal}(k(\mathfrak{f}_0\mathfrak{p}^m)/k(\mathfrak{d}\mathfrak{p}^m))$ and that by construction we have $L_m = k(\mathfrak{f})$, with $m > 1$, so we can establish the wanted equality. □

6.5.7 Proof for Case III (trivial character case)

Now we want to treat the case of the trivial character $\chi = 1$ of $\mathrm{Gal}(k(\mathfrak{f})/k)$. We again assume that $\mathfrak{p} \mid \mathfrak{f}$ so that we have $\mathfrak{f} = \mathfrak{f}_0\mathfrak{p}^\nu$ with $\nu \geq 1$. Now $\mathfrak{d} = 1()$ and we have that $H^1(\Delta_{\mathfrak{q}_\chi}^\infty) = U_{\mathfrak{q}_\chi}^\infty = \Lambda_{\mathfrak{q}_\chi}\eta_1$ and $H^2(\Delta_{\mathfrak{q}_\chi}^\infty) = X_S^\infty$. We obtain that

$$M^1 = U_{\mathfrak{q}_\chi}^\infty \text{ with } \beta_1 = \eta_1 \text{ and } M^2 = Y_{S_\infty}^\infty \text{ with } \beta_2 = \tau.$$

A lift of $\tau_{|k(\mathfrak{f})} \in M^1/\omega$ is given by $\tau' := \tau - v_\mathfrak{p}$ for a fixed prime ideal $v_\mathfrak{p}$ above $\mathfrak{p}$.
Now $\overline{\beta}_1 = \psi(1; \mathfrak{p}^\nu) \otimes 1 \in \mathcal{O}_{k(\mathfrak{f}),S}^\times \otimes \mathbb{Q}_p$. Furthermore, we can compute with Theorem 2.4 in [Ble06] that:

$$val(\overline{\beta}_1) = \frac{\mathcal{N}(\mathfrak{a}) - 1}{\Phi(\mathfrak{p}^\nu)} \frac{[k(\mathfrak{f}) : k]}{f_\mathfrak{p}} v_\mathfrak{p} \in Y_{S_{rp}}$$

and we have a short exact sequence

$$0 \to X_{S_\infty} \otimes \mathbb{Q}_p \to X_S \otimes \mathbb{Q}_p \to Y_{S_{rp}} \otimes \mathbb{Q}_p \to 0$$

with an explicit splitting given by

$$v \mapsto v - \frac{1}{[k(\mathfrak{f}) : k]} \mathrm{Tr}_{k(\mathfrak{f})/k} \tau_{|k(\mathfrak{f})}$$

and we obtain $val(\overline{\beta}_1) = -\frac{\mathcal{N}(\mathfrak{a})-1}{\Phi(\mathfrak{p}^\nu)} \frac{[k(\mathfrak{f}):k]}{f_\mathfrak{p}} \tau'$. So on the one hand we can compute

$$\begin{aligned} (\phi'_\chi \circ \phi_\omega^{-1})(\overline{\beta}_1^{-1} \otimes \overline{\beta}_2) &= \phi'_\chi(\overline{\beta}_1^{-1} \wedge x_J^{-1} \otimes \beta_\omega(x_J) \wedge \tau') \\ &= -\prod_{\mathfrak{l} \mid \mathfrak{f}_0} c_\mathfrak{l} \cdot \frac{\Phi(\mathfrak{p}^\nu)}{\mathcal{N}(\mathfrak{a}) - 1} \frac{f_\mathfrak{p}}{[k(\mathfrak{f}) : k]} \cdot \phi'_\chi(\overline{\beta}_1^{-1} \wedge x_J^{-1} \otimes val(x_J) \wedge val(\overline{\beta}_1)) \\ &= -\prod_{\mathfrak{l} \mid \mathfrak{f}_0} c_\mathfrak{l} \cdot \frac{\Phi(\mathfrak{p}^\nu)}{\mathcal{N}(\mathfrak{a}) - 1} \frac{f_\mathfrak{p}}{[k(\mathfrak{f}) : k]} \end{aligned}$$

and on the other hand we get as in the previous two cases

$$\begin{aligned} \mathcal{L} \otimes 1 &= c_\mathfrak{a} \cdot \eta_{\mathfrak{f}_0}^{-1} \otimes \tau \\ &= c_\mathfrak{a} \cdot [L_0 : L_0^{H_1}] \cdot \prod_{\mathfrak{l} \mid \mathfrak{f}_0} (1 - Fr_\mathfrak{l}^{-1})^{-1} \eta_1^{-1} \otimes \tau \\ &= c_\mathfrak{a} \cdot [L_0 : L_0^{H_1}] \cdot \prod_{\mathfrak{l} \mid \mathfrak{f}_0} \frac{\omega}{1 - Fr_\mathfrak{l}^{-1}} \omega^e \beta_1^{-1} \otimes \beta_2. \end{aligned}$$

For the Euler factor we obtain in this case $\mathcal{E}_S^{-1} = \prod_{\mathfrak{l}|\mathfrak{f}_0\mathfrak{p}} f_{\mathfrak{l}}^{-1}$. So we can conclude that

$$\begin{aligned}
\mathcal{E}_S^{-1} \cdot \phi'_\chi(\mathcal{L} \otimes 1) &= -\prod_{\mathfrak{l}|\mathfrak{f}_0\mathfrak{p}} f_{\mathfrak{l}}^{-1} c_{\mathfrak{a}} \cdot [L_0 : L_0^{H_1}] \cdot \prod_{\mathfrak{l}|\mathfrak{f}_0} \frac{\omega}{1 - Fr_{\mathfrak{l}}^{-1}} \prod_{\mathfrak{l}|\mathfrak{f}_0} c_{\mathfrak{l}} \frac{\Phi(\mathfrak{p}^\nu)}{\mathcal{N}(\mathfrak{a}) - 1} \frac{f_{\mathfrak{p}}}{[k(\mathfrak{f}) : k]} \\
&= -\frac{[L_0 : L_0^{H_1}] \cdot \Phi(\mathfrak{p}^\nu)}{[k(\mathfrak{f}) : k]} \\
&= -\frac{h_k}{w(1)}.
\end{aligned}$$

So we get the equality from Step G also for this case and this finishes the proof of Theorem 6.5.1.

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Eidesstattliche Versicherung

(Siehe Promotionsordnung vom 12.07.11, § 8, Abs. 2 Pkt. .5.)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

Hofer, Martin Gerhard

Name, Vorname

München, 07.08.2019 — Martin Hofer

Ort, Datum — Unterschrift Doktorand/in

Formular 3.2