

# [[CP]] Completely positive divisibility does not mean Markovianity

Simon Milz,<sup>1,\*</sup> M. S. Kim,<sup>2,3</sup> Felix A. Pollock,<sup>1</sup> and Kavan Modi<sup>1</sup>

<sup>1</sup>*School of Physics and Astronomy, Monash University, Clayton, Victoria 3800, Australia*

<sup>2</sup>*QOLS, Blackett Laboratory, Imperial College London, London SW7 2AZ, United Kingdom*

<sup>3</sup>*Korea Institute for Advanced Study, 02455, Seoul, Korea*

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In the classical domain, it is well-known that divisibility does not imply that a stochastic process is Markovian. However, for quantum processes, divisibility is often considered to be synonymous with Markovianity. We show that completely positive (CP) divisible quantum processes can still involve non-Markovian temporal correlations, that we then fully classify using the recently developed process tensor formalism, which generalizes the theory of stochastic processes to the quantum domain.

No system is fully isolated from its surroundings. This is especially true for quantum processes, where along with the surrounding environment, the act of observation can also disturb the system [1]. The field of open system dynamics attempts to develop methods that describe the dynamics of systems, quantum and classical, away from isolation [2]. These tools become crucial in analyzing a whole host of problems, from strong coupling thermodynamics [3] to error correction in quantum technologies [4]. An important consideration for describing open dynamics is the size and length of memory that the surroundings possess about the system's past [5, 6]. In general the future states of the system depend non-trivially on its own past, leading to complex joint measurement statistics in time [7, 8]. A process where the environment has *no* memory is called Markovian, and the complexity of describing such dynamics scales only as the Hilbert space dimension of the system [9], while the complexity of a non-Markovian process can scale exponentially in the number of times considered [10, 11].

The notion of Markovianity plays an important role in fields ranging far beyond the physical sciences. This is both for the fact that many processes in nature appear to be approximated sufficiently well by memoryless dynamics, and the computational and simulation intricacy that arises once memory effects are taken into account [2, 12]. As experimental control over complex quantum systems becomes increasingly sophisticated, the ability to directly determine whether a Markovian description is applicable is becoming ever more important [13–17]. **Consequently, in recent years, a large body of work dedicated to the description and characterization of memory effects in quantum systems has arisen [18–21].** However, strictly testing for the presence of memory effects is an intractable task in general [10], and a zoo of non-Markovianity witnesses for quantum processes has emerged in the past two decades [22–36], also see [21] for a recent review.

**The concept that underpins *all* of these witnesses is completely positive (CP) divisibility, a frequently**

**used proxy for Markovianity.** While experimentally accessible [37–40], the lack of a clear, quantifiable link between CP divisibility and Markovianity casts the former's implications for potential memory effects, and the interpretation of all the memory witnesses derived thereof, into doubt. In this Letter, we first demonstrate that, *a priori*, there are inequivalent experimental definitions of CP divisibility. After clearing up these ambiguities and laying out the connection between these different notions, we close the fundamental gap in the understanding of CP divisibility and comprehensively derive its quantitative relationship to Markovianity. **[[ In this Letter, we demonstrate *a priori*, completely positive (CP) divisibility, the concept underpinning the majority of these witnesses, is ambiguously defined when it comes to experimental implementation. After clearing up these ambiguities, we show the quantitative relationship between Markovianity and CP divisibility.]]** While their **[[inequivalence]]** difference has been pointed out before, *e.g.*, see Ref. [9, 41], our results yield both a quantifiable delineation between them, as well as a comprehensive characterization of the temporal correlations CP divisibility is sensitive to. This, in turn, provides a meaningful way forward for experimentalists looking to definitively characterize noise in their devices **by means of witnesses based on CP divisibility.** To motivate the relation of Markovianity and divisibility we first briefly review them in the context of classical processes.

**Markovianity and divisibility**—Mathematically, a classical process is called Markovian if the current state conditionally only depends on the last one, and not the whole history:

$$\mathbb{P}(x_n, t_n | x_{n-1}, t_{n-1}; \dots; x_0, t_0) = \mathbb{P}(x_n, t_n | x_{n-1}, t_{n-1}). \quad (1)$$

A generalization of this condition to quantum theory has recently been achieved [9].

Importantly, Markovianity is a logical requirement of conditional independence of a system's future and its past. As such, it is a statement about *multi-time* correlations. From Eq. (1) it is clear that determining if a process is Markovian requires an exponentially large set of conditions to be satisfied. A simpler

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\* [simon.milz@monash.edu](mailto:simon.milz@monash.edu)

criterion that follows from Markovianity, but is not sufficient to define it, is *divisibility*. This requires the conditional probabilities, for any *three* times  $t > s > r$ , to factorize according to the Chapman-Kolmogorov equation:  $\mathbb{P}(x, t|z, r) = \sum_y \mathbb{P}(x, t|y, s)\mathbb{P}(y, s|z, r)$  for all states  $x, y, z$ , where each  $\mathbb{P}$  is a probability distribution. For quantum processes, the natural generalization of a conditional probability distribution (with a single argument) is a completely positive map, i.e., one which preserves the positivity of even correlated density operators on which it acts, and the Chapman-Kolmogorov equation generalizes to the condition for CP divisibility [42]:

**Definition 1** (CP divisibility). *A quantum dynamical process of a system on an interval  $[0, \mathcal{T}]$  is CP divisible if (i) the dynamical map from  $r$  to  $t$  acting on the system of interest can be broken up at  $s$  such that*

$$\Phi_{t:r} = \Phi_{t:s} \circ \Phi_{s:r} \quad \forall \quad \mathcal{T} \geq t \geq s \geq r \geq 0, \quad (2)$$

and (ii) each map  $\Phi_{x:y}$  is completely positive.

Intuitively, the connection between this definition and Markovianity is that CP-maps describe dynamics without initial system-environment correlations [2], and the composition rule in Eq. (2) suggests that the dynamics between intermediate times are independent of the past. Together, these properties could be taken to imply the absence memory effects. While mathematically well-defined [43], *a priori*, the operational meaning of the family of maps  $\{\Phi_{t:s}\}$  is not clear. That is, in an experimental setting, what exact *quantum process tomography* procedure [44] is required to determine whether a process is CP-divisible?

There are (at least) two non-equivalent ways to address this question. **[, each physically justified in its own right.]** In what follows, we first motivate and define the two types of CP divisibility and show their non-equivalence. We then give a full characterization of the non-Markovian temporal correlations that may hide in a divisible process, thus providing a clear connection and delineation between Markovianity and CP divisibility. Throughout this Letter, we will only consider systems with finite Hilbert-space dimension  $d$ .

**CP divisibility by inversion.**— Consider an experimental setup where one is allowed to prepare any desired state at the initial time, i.e.,  $r = 0$  and perform measurements on the system at any later time  $s$ . Within these experimental constraints, using the standard method of quantum process tomography, one can construct a family of maps  $\lambda_0 := \{\Lambda_{s:0}\}$  that describe the dynamics from time  $r = 0$  to time  $s$ , see Fig. 1(a). Under the assumption that all the maps of this family are invertible, we obtain the following definition, which is the one that **[is most frequently used in experimental settings]** most frequently appears in the literature [22, 45, 46]:

**Definition 2** (iCP divisibility). *A process is CP*

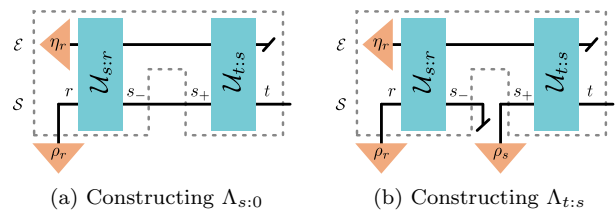


Figure 1. **Circuits for checking iCP and oCP divisibility.** (a) To construct the maps  $\Lambda_{s:0}$  and  $\Lambda_{t:0}$  we may set  $r = 0$  and measure the system at  $s$  or  $t$  respectively. (b) To construct  $\Lambda_{t:s}$ , the system is discarded at  $s_-$  and a fresh state is fed in at  $s_+$ . The dotted line encapsulates the three-time process tensor  $T_{t:s:r}$ .

*divisible by inversion (iCP-divisible) if for any two maps  $\Lambda_{s:0}, \Lambda_{t:0} \in \lambda_0$  with  $\mathcal{T} \geq t > s \geq 0$  the map*

$$\Phi_{t:s} := \Lambda_{t:0} \circ \Lambda_{s:0}^{-1} \quad (3)$$

*is completely positive.*

Here, we have chosen a convention where experimentally accessible maps are denoted by  $\Lambda$ . Notably, if all elements of  $\lambda_0$  are invertible then each  $\Phi_{t:s}$  constructed according to Eq. (3) is well-defined, and can be obtained computationally  $\lambda_0$ .

**Operational divisibility.**— While iCP divisibility is well-defined and can be checked experimentally, it leaves the operational meaning of the inferred maps  $\Phi_{t:s}$  open [47]; particularly, **[this map does]** these maps do not necessarily relate to anything that could actually be measured at intermediate times. **Additionally, due [its] to their non-operational definition, it is not possible to straightforwardly characterize the memory effects that iCP divisibility is blind to.** It is therefore desirable to provide a more operationally motivated definition of CP divisibility based on experimentally reconstructed maps alone.

To this end, let us consider a scenario where an experimenter has the ability to manipulate the system at *any* time  $s \in [0, \mathcal{T}]$ , which we split infinitesimally into  $s_-$  and  $s_+$  as shown in Fig. 1(b). At time  $s_-$  the system is discarded and, at  $s_+$ , replaced with a fresh one in state  $\rho_s$ . Subsequently, the experimenter measures the system at time  $t$ . With this procedure they can experimentally reconstruct maps  $\lambda := \{\Lambda_{t:s}\}$  as

$$\Lambda_{t:s}[\rho_s] = \text{tr}_{\mathcal{E}} [\mathcal{U}_{t:s}(\rho_s \otimes \eta_s)], \quad (4)$$

where  $\eta_s$  is the reduced state of the environment at time  $s$  and  $\mathcal{U}_{t:s}(x_s) := U_{t:s} x_s U_{t:s}^\dagger = x_t$  is the unitary system-environment map. We can thus define oCP divisibility:

**Definition 3** (oCP divisibility). *A process is operationally CP-divisible (oCP-divisible), if for any  $\mathcal{T} \geq t > s > r \geq 0$*

$$\Lambda_{t:r} = \Lambda_{t:s} \circ \Lambda_{s:r} \quad (5)$$

holds, where the maps above belong to set  $\lambda$  and are defined in Eq. (4).

Importantly, complete positivity of the respective maps is guaranteed by construction, as system-environment correlations are discarded for the reconstruction procedure of  $\Lambda_{t:s}$ . Formally, Eq. (5) is the same as Eq. (2), but with the important distinction that here each map has a clear operational meaning.

Still, there is a level of ambiguity in the procedure for constructing the intermediate maps  $\Lambda_{t:s}$ . In principle, they could depend on preparations at any previous time  $r$ ; such a dependence would imply non-Markovianity [48]. In order for oCP divisibility to be well-defined, Def. 3 implicitly requires that the intermediate maps are independent of any earlier state preparations. Specifically, if there are at least two different states  $\rho_r$  and  $\tilde{\rho}_r$ , such that the corresponding maps  $\Lambda_{t:s}$  and  $\tilde{\Lambda}_{t:s}$  differ, then oCP divisibility is not uniquely defined. Independence of the map  $\Lambda_{t:s}$  from earlier preparations is a non-signalling condition [49–51], as we show formally in the Supplemental Material.

Importantly, this non-signalling requirement is a *conditional* one; for oCP-divisible dynamics, it is necessary that there is no signalling from  $r$  to  $t$  given that the system state was discarded at  $s_-$ . An equivalent way to think about this condition is the following: consider an experiment where one part of a pure entangled state  $\rho_{rr'}$  is fed into the process at time  $r$ . At time  $s_-$  the system is discarded and a fresh state prepared at  $s_+$ , and the experimenter looks for correlation in the resulting state  $\rho_{tr'}$  at time  $t$ . If  $\rho_{tr'} \neq \rho_t \otimes \rho_{r'}$  then we have conditional signalling from  $r$  to  $t$ .

This requirement of conditional non-signalling is reminiscent of the concept of *no information back-flow* attributed to CP-divisible processes [23]. Here, however, in contrast to the increase of trace distance between trajectories, signalling is a genuine multi-time statement.

Conditional non-signalling is, for example, satisfied if, between time steps, the system interacts only once with a part of the environment that is discarded afterwards, or if the environment state is constant in time. However, while conditional non-signalling is necessary for oCP divisibility, it is not sufficient, as we show in the Supplemental Material (see also [52]). **While both notions of CP divisibility [[that we introduced]] are well-defined, they differ in their experimental reconstruction, and the meaning of the maps that they comprise[[are made up of]]. Now, before further discussing their relationship to Markovianity[[ relation between Markovianity and divisibility]], we show that they [[indeed]] do not coincide. [[Before we further discuss the relation between Markovianity and divisibility, we first show that the two notions of CP divisibility introduced above do not coincide.]]**

**oCP divisibility  $\neq$  iCP divisibility.**— Despite their superficial resemblance, the relationship between iCP- and oCP-divisible dynamics is *a priori* unclear.

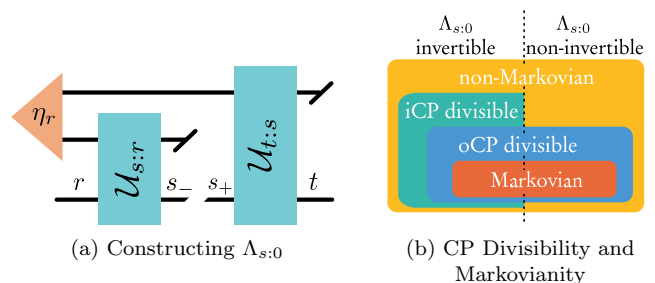


Figure 2. **Divisibility and Markovianity.** (a) The system interacts with one part of the correlated environment state leading to a non-Markovian oCP-divisible process. (b) The hierarchy of sets of processes with varying degrees of temporal correlations.

First, note that iCP divisibility is only defined if all elements of the set  $\lambda_0$  are invertible. This limitation does not apply to oCP divisibility, where any map belonging to the set  $\lambda$  can be non-invertible. Focussing on the invertible case, we find that oCP divisibility implies iCP divisibility by direct application of Eq. (5).

To see that the converse does not hold, we construct an iCP-divisible dynamics that is conditionally signalling, and thus not oCP-divisible. Consider the two circuits in Fig. 1, where both the system and the environment are qubits, and let the initial environment state be maximally mixed, i.e.,  $\eta_r = \mathbb{1}_2/2$ . The system-environment dynamics is given by the partial swap  $U_{s:r} = \exp(-i\omega\mathbf{S}u) = \cos(\omega u)\mathbb{1}_4 - i\sin(\omega u)\mathbf{S}$ , where  $\mathbf{S}|ij\rangle = |ji\rangle$ , and  $u := s - r$ . We show in the Supplemental Material that the resulting dynamics on the system is iCP-divisible for  $\omega t \leq \frac{\pi}{2}$ . On the other hand, if we discard the state of the system at  $s_-$  and insert a fresh state at  $s_+$  we will find that the corresponding state at  $t$  will depend on  $\rho_r$  due to the partial swap. In other words we have signalling, and therefore the process is not oCP-divisible.

Operationally CP-divisible dynamics form a strict subset of iCP-divisible ones, see Fig. 2(b). While the operational requirement is harder to check experimentally, it has a threefold advantage: first the involved maps have a clear-cut operational meaning, and the property of oCP divisibility ties in effortlessly with frameworks tailored for the discussion of non-Markovian quantum processes. Second, the definition of oCP divisibility does not rely on the invertibility of  $\Lambda_{s:0}$  and thus has wider applicability. Lastly, oCP divisibility breaks down for a larger class of memory effects than iCP divisibility, and consequently outperforms it as a witness of non-Markovianity.

**CP divisibility  $\neq$  Markovianity.**— Even though oCP divisibility is a stricter requirement than iCP divisibility, it does not enforce Markovianity; for clarity, we will show this by means of a discrete time example. For an *ante litteram* continuous example of non-Markovian oCP-divisible dynamics, see [41,

53]. We take inspiration from collision models [54–57] with correlated environment states [58, 59]: Let the environment at  $r = 0$  be in a correlated bipartite state that is uncorrelated with the system. The dynamics  $U_{y:x}$  between any two (of a set of three) times is such that the system only interacts with one part of the environment (denoted by  $x$ ) that is discarded afterwards, see Fig. 2(a). This scenario satisfies the necessary non-signalling condition. Now, if we choose the unitaries  $U_{y:x}$  to be the swap operator  $\mathbf{S}_{sx}$  between the system and part  $x$  of the environment, then we have  $\Lambda_{t:s} = \Lambda_{t:s} \circ \Lambda_{s:r}$ , and the dynamics is oCP-divisible.

However, the process is non-Markovian; suppose the experimenter, instead of discarding it, stores the system state at time  $s_-$ , and inserts a fresh state at  $s_+$ . The dynamics is allowed to continue to  $t$  and that state too is stored. The joint state  $\rho_{st}$  will be correlated even though the states inserted into the process, at times  $r$  and  $s_+$ , were independent. In particular, for the above case the resulting overall state  $\rho_{st}$  is exactly the correlated initial state of the environment. The experimenter could thus detect memory effects between different times from observing the system only, even though the dynamics is oCP-divisible [60].

An oCP-divisible process can be seen as one that is Markovian *on average*. Specifically, consider a multi-time process where an experimenter measures the system at each time, before independently preparing it in a new state; what oCP divisibility implies is that, if all past measurement outcomes are forgotten or averaged over – which is equivalent to discarding the system state before re-preparation – then the future statistics only depend on the current preparation. A quantum Markov process, in contrast, requires that the future statistics only depend on the current preparation for *any* sequence of measurement outcomes [9, 11, 61–63]. We now fully characterize the temporal correlations that can persist in oCP-divisible dynamics, [thus providing a quantifiable connection between the Markovianity – the absence of memory effects – and CP divisibility – the concept underlying the majority of memory witnesses employed in the literature.](#)

**Correlations in divisible processes.**— The four classes of processes illustrated in Fig. 2(b) also have analogues in the classical domain. A classical stochastic process is described by a joint distribution

$$\mathbb{P}(x_n, t_n; \dots; x_0, t_0), \quad (6)$$

over the state of the system at different times, satisfying the Kolmogorov conditions [64, 65]. To check if a given process is Markovian necessitates checking all conditional probabilities given in Eq. (1), which requires the full distribution of Eq. (6). However, to infer the divisibility of a process, by inversion or operationally, requires only the bipartite marginal distributions of Eq. (6):  $\{\mathbb{P}(x_s, t_s, x_0, t_0)\}_{s=1}^n$  and  $\{\mathbb{P}(x_s, t_s, x_r, t_r)\}_{s>r=0}^n$  respectively. Thus we have the same hierarchy as in

Fig. 2(b) for temporal correlations in classical processes.

The quantum generalization of Eq. (6) is a multipartite positive operator  $T_{n:\dots:1:0}$ , called the *process tensor* [10, 11, 66, 67] which satisfies *generalized* Kolmogorov conditions [41, 68]. Analogous to the classical case, the process tensor captures all temporal correlations in quantum processes, including across multiple time steps, in our case three. The probability of observing a sequence of events  $\{x_r, x_s, x_t\}$ , can be computed by contracting the process tensor  $T_{t:s:r}$  with generalized measurement operators  $M_x$

$$\mathbb{P}(x_t, x_s, x_r | \mathcal{J}_t, \mathcal{J}_s, \mathcal{J}_r) = \text{tr}[(M_{x_t} \otimes M_{x_s} \otimes M_{x_r}) T_{t:s:r}]. \quad (7)$$

The last equation is simply a generalization of the Born rule to processes in time [69], where  $\mathcal{J}$  denotes an instrument [70], which is a collection of conditional transformations (CP maps)  $\{M_{x_s}\}$  that update the system after a particular event is observed; these generalize the concept of positive operator valued measure (POVM). By convention, and without loss of generality, each element of Eq. (7) is expressed in terms of Choi state [71, 72].

Mathematically, the process tensor  $T := T_{t:s:r}$  is an operator on Hilbert spaces  $\mathcal{H}_r \otimes \mathcal{H}_{s_-} \otimes \mathcal{H}_{s_+} \otimes \mathcal{H}_t$ . For both processes in Fig. 1, the process tensor is exactly the same; it is the object within the dotted lines. The difference between the two panels lies entirely in the instrument at  $s$ . The instrument at  $r$  is a preparation with one deterministic element  $M_{x_r} = \rho_r$ , and the instrument at  $t$  is a measurement  $\{M_{x_t} = \Pi_{x_t}\}$ , where the latter are POVM elements. The instrument at  $s$  for Fig. 1(a) deterministically implements the identity channel, which has Choi state  $M_{x_s} = \varphi_{s_{\pm}}^+$ , where  $\varphi_{s_{\pm}}^+ := \sum_{jk}^d |jj\rangle \langle kk|$  and  $s_{\pm} := s_- s_+$ . The instrument at  $s$  for Fig. 1(b) also has a single element:  $M_{x_s} = \mathbb{1} \otimes \rho_s$ , where  $\mathbb{1}$  denotes the trace at  $s_-$  followed by preparation of  $\rho_s$ . For completeness we review the details of the process tensor formalism in the Supplemental Material and only include important details here.

Using Eq. (7) and the details of the instruments, we recover the maps in Eq. (5) from the process tensor. Let  $L_{x:y}$  denote the Choi state of  $\Lambda_{x:y}$ . For an oCP-divisible process, we can show that  $L_{t:r} = \text{tr}_{s_{\pm}}(\varphi_{s_{\pm}}^+ T)$ , while  $L_{s:r} = \text{tr}_{s_+t}[T]/d$  and  $L_{t:s} = \text{tr}_{rs_-}[T]/d$ . With this, we can rephrase oCP divisibility as

$$\text{tr}_s(\varphi_{s_{\pm}}^+ T) = \frac{1}{d^2} \text{tr}_{s_{\pm}} \left[ \text{tr}_{rs_-}(T) \varphi_{s_{\pm}}^+ \text{tr}_{s_+t}(T) \right]. \quad (8)$$

A detailed derivation of above statements is given in the Supplemental Material.

On the other hand, the process tensor formalism leads to an unambiguous quantum Markov condition [9, 11, 61, 63]. A quantum process is said to be Markov iff the Choi state of the corresponding process tensor has the form  $T^{\text{Markov}} = L_{t:s} \otimes L_{s:r}$ ; any deviation from this product form implies detectable non-Markovian correlations. Since Eq. (8) does not force  $T$  to be of Markov form,



oCP-divisible processes are not necessarily memoryless. Specifically, representing  $T = L_{t,s} \otimes L_{s,r} + \chi_{tsr}$ , where the matrix  $\chi$  contains all tripartite non-Markovian correlations and satisfies  $\text{tr}_{s-r}[\chi_{tsr}] = \text{tr}_{ts+}[\chi_{tsr}] = 0$ , we see that Eq. (8) implies  $\text{tr}_{s\pm}(\varphi_{s\pm}^+ \chi_{tsr}) = 0$ , which provides a full classification of non-Markovian temporal correlations that can be present despite the dynamics being oCP-divisible.

**Conclusions.**— In this Letter, we have provided an operationally motivated definition of CP divisibility that is stricter than the frequently used one relying on the invertibility of  $\Lambda_{s,0}$ . We showed that oCP divisibility is closely connected to non-signalling conditions and implies the absence of information flow from the environment to the system. **[[Additionally, we have demonstrated that the sets of oCP-divisible and Markovian dynamics do not coincide.]]** Additionally, **[[Nevertheless,]]** we have shown that oCP divisibility can be interpreted as Markovianity on average, **yet oCP divisible processes can still display non-trivial memory effects.** Finally, by employing the process tensor formalism, we have fully classified the non-Markovian temporal correlations to which the criterion of CP divisibility is blind.

To build near-term quantum technologies will

require effective methods for detecting and addressing non-Markovian noise [73]. We have shed light on divisibility from an operational point of view, which helps us to identify the classes of temporal correlations that may evade regularly used checks for non-Markovianity. However, there are trade-offs between uncovering temporal correlations and the requisite number of experiments that must be performed. Our results enable experimentalists to make informed decisions about investing resources in classifying the non-Markovian noise at hand.

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