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Asynchronous Distributed Optimization via Randomized Dual Proximal Gradient

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Abstract—In this paper we consider distributed optimization problems in which the cost function is separable, i.e., a sum of possibly non-smooth functions all sharing a common variable, and can be split into a strongly convex term and a convex one. The second term is typically used to encode constraints or to regularize the solution. We propose a class of distributed optimization algorithms based on proximal gradient methods applied to the dual problem. We show that, by choosing suitable primal variable copies, the dual problem is itself separable when written in terms of conjugate functions, and the dual variables can be stacked into non-overlapping blocks associated to the computing nodes. We first show that a weighted proximal gradient on the dual function leads to a synchronous distributed algorithm with local dual proximal gradient updates at each node. Then, as main paper contribution, we develop asynchronous versions of the algorithm in which the node updates are triggered by local timers without any global iteration counter. The algorithms are shown to be proper randomized block-coordinate proximal gradient updates on the dual function.

I. INTRODUCTION

Several estimation, learning, decision and control problems arising in cyber-physical networks involve the distributed solution of a constrained optimization problem. Typically, computing processors have only a partial knowledge of the problem (e.g., a portion of the cost function or a subset of the constraints) and need to cooperate in order to compute a global solution of the whole problem. A key challenge when designing distributed optimization algorithms in peer-to-peer networks is that the communication among the nodes is time-varying and possibly asynchronous.

Early references on distributed optimization algorithms involved primal and dual subgradient methods and Alternating Direction Method of Multipliers (ADMM), designed for synchronous communication protocols over fixed graphs. More recently time-varying versions of these algorithmic ideas have been proposed to cope with more realistic peer-to-peer network scenarios. A Newton-Raphson consensus strategy is proposed in [1] to solve unconstrained, convex optimization problems under asynchronous, symmetric gossip communications. In [2] a primal, synchronous algorithm, called EXTRA, is proposed to solve smooth, unconstrained optimization problems. In [3] the authors propose accelerated distributed gradient methods for unconstrained optimization problems over symmetric, time-varying networks connected on average. In order to deal

with time-varying and directed graph topologies, in [4] a push-sum algorithm for average consensus is combined with a primal subgradient method in order to solve unconstrained convex optimization problems. Paper [5] extends this algorithm to online distributed optimization over time-varying, directed networks. In [6] a novel class of continuous-time, gradient-based distributed algorithms is proposed both for fixed and time-varying graphs and conditions for exponential convergence are provided. A distributed (primal) proximal-gradient method is proposed in [7] to solve (over time-varying, balanced communication graphs) optimization problems with a separable cost function including local differentiable components and a common non-differentiable term. In [8] experiments of a dual averaging algorithm are run for separable problems with a common constraint on time-varying and directed networks.

For general constrained convex optimization problems, in [9] the authors propose a distributed random projection algorithm, that can be used by multiple agents connected over a (balanced) time-varying network. In [10] the author proposes (primal) randomized block-coordinate descent methods for minimizing multi-agent convex optimization problems with linearly coupled constraints over networks. In [11] an asynchronous ADMM-based distributed method is proposed for a separable, constrained optimization problem. The algorithm is shown to converge at the rate $O(1/t)$ (being t the iteration counter). In [12] the ADMM approach is proposed for a more general framework, thus yielding a continuum of algorithms ranging from a fully centralized to a fully distributed. In [13] a method, called ADMM+, is proposed to solve separable, convex optimization problems with a cost function written as the sum of a smooth and a non-smooth term.

Successive block-coordinate updates are proposed in [14], [15] to solve separable optimization problems in a parallel big-data setting. Another class of algorithms exploits the exchange of active constraints among the network nodes to solve constrained optimization problems [16]. This idea has been combined with dual decomposition and cutting-plane methods to solve robust convex optimization problems via polyhedral approximations [17]. These algorithms work under asynchronous, directed and unreliable communication.

The contribution of the paper is twofold. First, for a fixed graph topology, we develop a distributed optimization algorithm (based on a centralized dual proximal gradient idea introduced in [18]) to minimize a separable strongly convex cost function. The proposed distributed algorithm is based on a proper choice of primal constraints (suitably separating the graph-induced and node-local constraints), that gives rise to a dual problem with a separable structure when expressed in

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terms of local conjugate functions. Thus, a proximal gradient applied to such a dual problem turns out to be a distributed algorithm where each node updates: (i) its primal variable through a local minimization and (ii) its dual variables through a suitable local proximal gradient step. The algorithm inherits the convergence properties of the centralized one and exhibits an $O(1/t)$ rate of convergence in objective value. We point out that the algorithm can be accelerated through a Nesterov's scheme [19], obtaining an $O(1/t^2)$ convergence rate.

Second, as main contribution, we propose an asynchronous algorithm for a symmetric *event-triggered* communication protocol. In this communication set-up, a node is in idle mode until its local timer triggers. When in idle, it continuously collects messages from neighboring nodes that are awake and, if needed, updates a primal variable. When the local timer triggers, it updates local primal and dual variables and broadcasts them to neighboring nodes. Under mild assumptions on the local timers, the whole algorithm results into a uniform random choice of one active node per iteration. Using this property and showing that the dual variables can be stacked into separate blocks, we are able to prove that the distributed algorithm corresponds to a block-coordinate proximal gradient, as the one proposed in [20], performed on the dual problem. Specifically, we are able to show that the dual variables handled by a single node represent a single block-variable, and the local update at each triggered node turns out to be a block-coordinate proximal gradient step (in which each node has its own local step-size). The result is that the algorithm inherits the convergence properties of the block-coordinate proximal gradient in [20].

An important property of the distributed algorithm is that it can solve fairly general optimization problems including both composite cost functions and local constraints. A key distinctive feature of the algorithm analysis is the combination of duality theory, coordinate-descent methods, and properties of the proximal operator when applied to conjugate functions.

To summarize, our algorithms compare to the literature in the following way. Works in [1], [3]–[6] do not handle constrained optimization and use different methodologies. In [7], [9], [10] primal approaches are used. Also, local constraints and regularization terms cannot be handled simultaneously. In [7] a proximal operator is used, but only to handle a common, non-smooth cost function (known by all agents) directly on the primal problem. The algorithm in [10] uses a coordinate-descent idea similar to the one we use in this paper, but it works directly on the primal problem, does not handle local constraints and does not make use of proximal operators. In this paper we propose a flexible dual approach to take into account both local constraints and regularization terms. The problem set-up in [11]–[13] is similar to the one considered in this paper. Differently from our approach, which is a dual method, ADMM-based algorithms are proposed in those references. This difference results in different algorithms as well as different requirements on the cost functions. Moreover, compared to these algorithms we are able to use constant, local step-sizes (which can be locally computed at the beginning of the iterations) so that the algorithm can be run asynchronously and without any coordination step. On this regard, we propose

an algorithmic formulation of the asynchronous protocol that explicitly relies on local timers and does not need any global iteration counter in the update laws.

The paper is organized as follows. In Section II we set-up the optimization problem, while in Section III we derive an equivalent dual problem amenable for distributed computation. The distributed algorithms are introduced in Section IV and analyzed in Section V. Finally, in Section VI we highlight the flexibility of the proposed algorithms by showing important optimization scenarios that can be addressed, and corroborate the discussion with numerical computations.

Notation: Given a closed, nonempty convex set X , the indicator function of X is defined as $I_X(x) = 0$ if $x \in X$ and $I_X(x) = +\infty$ otherwise.

Let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$, its conjugate function $\varphi^* : \mathbb{R}^d \rightarrow \mathbb{R}$ is defined as $\varphi^*(y) := \sup_x \{y^\top x - \varphi(x)\}$. Let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ be a closed, proper, convex function and α a positive scalar, the proximal operator $\mathbf{prox}_{\alpha\varphi} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is defined by $\mathbf{prox}_{\alpha\varphi}(v) := \operatorname{argmin}_x \{\varphi(x) + \frac{1}{2\alpha}\|x - v\|^2\}$. We also introduce a generalized version of the proximal operator. Given a positive definite matrix $W \in \mathbb{R}^{d \times d}$, we define

$$\mathbf{prox}_{W,\varphi}(v) := \operatorname{argmin}_x \left\{ \varphi(x) + \frac{1}{2} \|x - v\|_{W^{-1}}^2 \right\}.$$

II. PROBLEM SET-UP AND NETWORK STRUCTURE

We consider the following optimization problem

$$\min_x \sum_{i=1}^n \left(f_i(x) + g_i(x) \right), \quad (1)$$

where $f_i : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ are proper, closed and strongly convex extended real-valued functions with strong convexity parameter $\sigma_i > 0$ and $g_i : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ are proper, closed and convex extended real-valued functions.

Note that the split of f_i and g_i may be non-unique and depend on the problem structure. Intuitively, on f_i an easy minimization step can be performed (e.g., by division free operations, [21]), while g_i has an easy expression of its proximal operator.

Remark II.1 (Min-max problems). *Our setup does not require f_i to be differentiable, thus one can also consider each strongly convex function f_i given by $f_i(x) := \max_{j \in \{1, \dots, m_i\}} f_{ij}(x)$, $m_i \in \mathbb{N}$, where $\{f_{ij}(x) \mid j \in \{1, \dots, m_i\}\}$ is a nonempty collection of strongly convex functions.* \square

Since we will work on the dual of problem (1), we introduce the next standard assumption which guarantees that the dual is feasible and equivalent to the primal (strong duality).

Assumption II.2 (Constraint qualification). *The intersection of the relative interior of $\operatorname{dom} \sum_{i=1}^n f_i$ and the relative interior of $\operatorname{dom} \sum_{i=1}^n g_i$ is non-empty.* \square

We want this optimization problem to be solved in a distributed way by a network of peer processors without a central coordinator. Each processor has a local memory, a local computation capability and can exchange information with neighboring nodes. We assume that the communication can occur among nodes that are neighbors in a given fixed,

undirected and connected graph $\mathcal{G} = (\{1, \dots, n\}, \mathcal{E})$, where $\mathcal{E} \subseteq \{1, \dots, n\} \times \{1, \dots, n\}$ is the set of edges. That is, the edge (i, j) models the fact that node i and j can exchange information. We denote by \mathcal{N}_i the set of *neighbors* of node i in the fixed graph \mathcal{G} , i.e., $\mathcal{N}_i := \{j \in \{1, \dots, n\} \mid (i, j) \in \mathcal{E}\}$, and by $|\mathcal{N}_i|$ its cardinality.

In Section IV we will propose distributed algorithms for three communication protocols. Namely, we will consider a synchronous protocol (in which neighboring nodes in the graph communicate according to a common clock), and two asynchronous ones, respectively node-based and edge-based (in which nodes become active according to local timers).

To exploit the sparsity of the graph we introduce copies of x and their coherence (consensus) constraint, so that the optimization problem (1) can be equivalently rewritten as

$$\begin{aligned} \min_{x_1, \dots, x_n} \sum_{i=1}^n \left(f_i(x_i) + g_i(x_i) \right) \\ \text{subj. to } x_i = x_j \quad \forall (i, j) \in \mathcal{E} \end{aligned} \quad (2)$$

with $x_i \in \mathbb{R}^d$ for all $i \in \{1, \dots, n\}$. The connectedness of \mathcal{G} guarantees the equivalence.

Since we propose distributed dual algorithms, next we derive the dual problem and characterize its properties.

III. DUAL PROBLEM DERIVATION

We derive the dual version of the problem that will allow us to design our distributed dual proximal gradient algorithms. To obtain the desired separable structure of the dual problem, we set-up an equivalent formulation of problem (2) by adding new variables z_i , $i \in \{1, \dots, n\}$, i.e.,

$$\begin{aligned} \min_{\substack{x_1, \dots, x_n \\ z_1, \dots, z_n}} \sum_{i=1}^n \left(f_i(x_i) + g_i(z_i) \right) \\ \text{subj. to } x_i = x_j \quad \forall (i, j) \in \mathcal{E} \\ x_i = z_i \quad \forall i \in \{1, \dots, n\}. \end{aligned} \quad (3)$$

Let $\mathbf{x} = [x_1^\top \dots x_n^\top]^\top$ and $\mathbf{z} = [z_1^\top \dots z_n^\top]^\top$, the Lagrangian of primal problem (3) is given by

$$\begin{aligned} L(\mathbf{x}, \mathbf{z}, \Lambda, \mu) &= \sum_{i=1}^n \left(f_i(x_i) + g_i(z_i) \right. \\ &\quad \left. + \sum_{j \in \mathcal{N}_i} \left(\lambda_i^j \right)^\top (x_i - x_j) + \mu_i^\top (x_i - z_i) \right) \\ &= \sum_{i=1}^n \left(f_i(x_i) + \sum_{j \in \mathcal{N}_i} \left(\lambda_i^j \right)^\top (x_i - x_j) + \mu_i^\top x_i \right. \\ &\quad \left. + g_i(z_i) - \mu_i^\top z_i \right), \end{aligned}$$

where Λ and μ are respectively the vectors of the Lagrange multipliers $\lambda_i^j \in \mathbb{R}^d$, $(i, j) \in \mathcal{E}$, and $\mu_i \in \mathbb{R}^d$, $i \in \{1, \dots, n\}$, and in the last line we have separated the terms in \mathbf{x} and \mathbf{z} . Since \mathcal{G} is undirected, the Lagrangian can be rearranged as

$$\begin{aligned} L(\mathbf{x}, \mathbf{z}, \Lambda, \mu) &= \sum_{i=1}^n \left(f_i(x_i) + x_i^\top \left(\sum_{j \in \mathcal{N}_i} (\lambda_i^j - \lambda_j^i) + \mu_i \right) \right. \\ &\quad \left. + g_i(z_i) - z_i^\top \mu_i \right). \end{aligned}$$

The dual function is

$$\begin{aligned} q(\Lambda, \mu) &:= \min_{\mathbf{x}, \mathbf{z}} L(\mathbf{x}, \mathbf{z}, \Lambda, \mu) \\ &= \min_{\mathbf{x}} \sum_{i=1}^n \left(f_i(x_i) + x_i^\top \left(\sum_{j \in \mathcal{N}_i} (\lambda_i^j - \lambda_j^i) + \mu_i \right) \right) \\ &\quad + \min_{\mathbf{z}} \sum_{i=1}^n \left(g_i(z_i) - z_i^\top \mu_i \right) \\ &= \sum_{i=1}^n \min_{x_i} \left(f_i(x_i) + x_i^\top \left(\sum_{j \in \mathcal{N}_i} (\lambda_i^j - \lambda_j^i) + \mu_i \right) \right) \\ &\quad + \sum_{i=1}^n \min_{z_i} \left(g_i(z_i) - z_i^\top \mu_i \right), \end{aligned}$$

where we have used the separability of the Lagrangian with respect to each x_i and each z_i . Then, by using the definition of conjugate function (given in the Notation paragraph), the dual function can be expressed as

$$q(\Lambda, \mu) = \sum_{i=1}^n \left(-f_i^* \left(- \sum_{j \in \mathcal{N}_i} (\lambda_i^j - \lambda_j^i) - \mu_i \right) - g_i^*(\mu_i) \right).$$

The dual problem of (3) consists of maximizing the dual function with respect to dual variables Λ and μ , i.e.,

$$\max_{\Lambda, \mu} \sum_{i=1}^n \left(-f_i^* \left(- \sum_{j \in \mathcal{N}_i} (\lambda_i^j - \lambda_j^i) - \mu_i \right) - g_i^*(\mu_i) \right). \quad (4)$$

By Assumption II.2 the dual problem (4) is feasible and strong duality holds, so that (4) can be equivalently solved to get a solution of (3).

In order to have a more compact notation for problem (4), we stack the dual variables as $y = [y_1^\top \dots y_n^\top]^\top$, where

$$y_i = \begin{bmatrix} \Lambda_i \\ \mu_i \end{bmatrix} \in \mathbb{R}^{d|\mathcal{N}_i|+d} \quad (5)$$

with $\Lambda_i \in \mathbb{R}^{d|\mathcal{N}_i|}$ a vector whose block-component associated to neighbor j is $\lambda_i^j \in \mathbb{R}^d$. Thus, changing sign to the cost function, dual problem (4) can be restated as

$$\min_y \Gamma(y) = F^*(y) + G^*(y), \quad (6)$$

where

$$F^*(y) := \sum_{i=1}^n f_i^* \left(- \sum_{j \in \mathcal{N}_i} (\lambda_i^j - \lambda_j^i) - \mu_i \right), \quad G^*(y) := \sum_{i=1}^n g_i^*(\mu_i).$$

IV. DISTRIBUTED DUAL PROXIMAL ALGORITHMS

In this section we derive the distributed optimization algorithms based on dual proximal gradient methods.

A. Distributed Dual Proximal Gradient (DDPG)

We begin by deriving a synchronous algorithm on a fixed graph. We assume that all the nodes share a common clock. At each time instant $t \in \mathbb{N}$, every node communicates with its neighbors in the graph $\mathcal{G} = (\{1, \dots, n\}, \mathcal{E})$ (defined in Section II) and updates its local variables.

First, we provide an informal description of the distributed optimization algorithm. Each node $i \in \{1, \dots, n\}$ stores a

set of local dual variables λ_j^i , $j \in \mathcal{N}_i$, and μ_i , updated through a local proximal gradient step, and a primal variable x_i^* , updated through a local minimization. Each node uses a properly chosen, *local* step-size α_i for the proximal gradient step. Then, the updated primal and dual values are exchanged with the neighboring nodes. The local dual variables at node i are initialized as λ_{i0}^j , $j \in \mathcal{N}_i$, and μ_{i0} . A pseudo-code of the local update at each node of the distributed algorithm is given in Algorithm 1.

Algorithm 1 Distributed Dual Proximal Gradient (DDPG)

Processor states: x_i^* , λ_j^i for all $j \in \mathcal{N}_i$ and μ_i

Initialization: $\lambda_j^i(0) = \lambda_{i0}^j$ for all $j \in \mathcal{N}_i$, $\mu_i(0) = \mu_{i0}$

$$x_i^*(0) = \operatorname{argmin}_{x_i} \left\{ x_i^\top \left(\sum_{j \in \mathcal{N}_i} (\lambda_{i0}^j - \lambda_{j0}^i) + \mu_{i0} \right) + f_i(x_i) \right\}$$

Evolution:

FOR: $t = 1, 2, \dots$ DO

receive $x_j^*(t-1)$ for each $j \in \mathcal{N}_i$, update

$$\lambda_j^i(t) = \lambda_j^i(t-1) + \alpha_i [x_j^*(t-1) - x_j^*(t-1)]$$

and update

$$\begin{aligned} \tilde{\mu}_i &= \mu_i(t-1) + \alpha_i x_i^*(t-1) \\ \mu_i(t) &= \tilde{\mu}_i - \alpha_i \operatorname{prox}_{\frac{1}{\alpha_i} g_i} \left(\frac{\tilde{\mu}_i}{\alpha_i} \right) \end{aligned}$$

receive $\lambda_j^i(t)$ for each $j \in \mathcal{N}_i$ and compute

$$x_i^*(t) = \operatorname{argmin}_{x_i} \left\{ x_i^\top \left(\sum_{j \in \mathcal{N}_i} (\lambda_j^i(t) - \lambda_j^i(t)) + \mu_i(t) \right) + f_i(x_i) \right\}$$

Remark IV.1. In order to start the algorithm, a preliminary communication step is needed in which each node i receives from each neighbor j its λ_{j0}^i (to compute $x_i^*(0)$) and the convexity parameter σ_j of f_j (to set α_i , as it will be clear from the analysis in Section V-B). This step can be avoided if the nodes agree to set $\lambda_{i0}^j = 0$ and know a bound for α_i . Also, it is worth noting that, differently from other algorithms, in general $\lambda_j^i(t) \neq -\lambda_j^i(t)$. \square

We point out once more that to run the DDPG algorithm the nodes need to have a common clock. Also, it is worth noting that, as it will be clear from the analysis in Section V-B, to set the local step-size each node needs to know the number of nodes, n , in the network. In the next sections we present two asynchronous distributed algorithms which overcome these limitations.

B. Asynchronous DDPG (A-DDPG)

Next, we propose a node-based asynchronous algorithm. We consider an asynchronous protocol where each node has its own concept of time defined by a local timer, which randomly and independently of the other nodes triggers when to awake itself. Between two triggering events the node is in an *idle* mode, i.e., it continuously receives messages from neighboring nodes and, if needed, runs some auxiliary computation. When

a trigger occurs, it switches into an *awake* mode in which it updates its local (primal and dual) variables and transmits the updated information to its neighbors.

Formally, the triggering process is modeled by means of a local clock $\tau_i \in \mathbb{R}_{\geq 0}$ and a randomly generated waiting time T_i . As long as $\tau_i < T_i$ the node is in idle mode. When $\tau_i = T_i$ the node switches to the awake mode and, after running the local computation, resets $\tau_i = 0$ and draws a new realization of the random variable T_i . We make the following assumption on the local waiting times T_i .

Assumption IV.2 (Exponential i.i.d. local timers). *The waiting times between consecutive triggering events are i.i.d. random variables with same exponential distribution.* \square

When a node i wakes up, it updates its local dual variables λ_j^i , $j \in \mathcal{N}_i$ and μ_i by a local proximal gradient step, and its primal variable x_i^* through a local minimization. The *local* step-size of the proximal gradient step for node i is denoted by α_i . In order to highlight the difference between updated and old variables at node i during the ‘‘awake’’ phase, we denote the updated ones as λ_j^{i+} and μ_i^+ respectively. When a node i is in idle it receives messages from awake neighbors. If a dual variable λ_j^i is received it computes an updated value of x_i^* and broadcasts it to its neighbors. It is worth noting that, being the algorithm asynchronous, there is no common clock as in the synchronous version.

Algorithm 2 Asynchronous DDPG (A-DDPG)

Processor states: x_i^* , λ_j^i for all $j \in \mathcal{N}_i$ and μ_i

Initialization: $\lambda_j^i = \lambda_{i0}^j$ for all $j \in \mathcal{N}_i$, $\mu_i = \mu_{i0}$ and $x_i^* = \operatorname{argmin}_{x_i} \left\{ x_i^\top \left(\sum_{j \in \mathcal{N}_i} (\lambda_{i0}^j - \lambda_{j0}^i) + \mu_{i0} \right) + f_i(x_i) \right\}$ set $\tau_i = 0$ and get a realization T_i

Evolution:

IDLE :

WHILE: $\tau_i < T_i$ DO:

receive x_j^* and/or λ_j^i from each $j \in \mathcal{N}_i$.

IF: λ_j^i is received THEN: compute and broadcast

$$x_i^* = \operatorname{argmin}_{x_i} \left\{ x_i^\top \left(\sum_{k \in \mathcal{N}_i} (\lambda_k^i - \lambda_k^i) + \mu_i \right) + f_i(x_i) \right\}$$

go to **AWAKE**.

AWAKE :

update and broadcast $\lambda_j^{i+} = \lambda_j^i + \alpha_i (x_i^* - x_j^*)$, $\forall j \in \mathcal{N}_i$
update

$$\begin{aligned} \tilde{\mu}_i &= \mu_i + \alpha_i x_i^* \\ \mu_i^+ &= \tilde{\mu}_i - \alpha_i \operatorname{prox}_{\frac{1}{\alpha_i} g_i} \left(\frac{\tilde{\mu}_i}{\alpha_i} \right) \end{aligned}$$

compute and broadcast

$$x_i^* = \operatorname{argmin}_{x_i} \left\{ x_i^\top \left(\sum_{j \in \mathcal{N}_i} (\lambda_j^{i+} - \lambda_j^i) + \mu_i^+ \right) + f_i(x_i) \right\}$$

set $\tau_i = 0$, get a new realization T_i and go to **IDLE**.

C. Edge-based asynchronous algorithm

In this section we present a variant of the A-DDPG in which an edge becomes active uniformly at random, rather than a node. In other words, we assume that timers are associated to edges, rather than to nodes, that is a waiting time T_{ij} is extracted for edge (i, j) .

The processor states and their initialization stay the same except for the timers. Notice that here we are assuming that nodes i and j have a common waiting time T_{ij} and, consistently, a local timer τ_{ij} . Each waiting time T_{ij} satisfies Assumption IV.2.

In Algorithm 3 we report (only) the evolution for this modified scenario. In this edge-based algorithm the dual variable μ_i (associated to the constraint $x_i = z_i$) cannot be updated every time an edge (i, j) , $j \in \mathcal{N}_i$, becomes active (otherwise it would be updated more often than the variables λ_i^j , $j \in \mathcal{N}_i$). Thus, we identify one special neighbor $j_{\mu_i} \in \mathcal{N}_i$ and update μ_i only when the edge (i, j_{μ_i}) is active.

Algorithm 3 Edge-Based Formulation of Algorithm 2 (evolution)

IDLE: WHILE $\tau_{ij} < T_{ij}$ DO: nothing
go to **AWAKE**.

AWAKE:

send x_i^* to j and receive x_j^* from j

update and send to j , $\lambda_i^{j+} = \lambda_i^j + \alpha_i(x_i^* - x_j^*)$

IF: $j = j_{\mu_i}$ THEN: update $\mu_i^+ = \text{prox}_{\alpha_i g_i^*}(\mu_i + \alpha_i x_i^*)$
compute

$$x_i^* = \underset{x_i}{\text{argmin}} \left\{ x_i^\top \left(\sum_{j \in \mathcal{N}_i} (\lambda_i^{j+} - \lambda_i^j) + \mu_i^+ \right) + f_i(x_i) \right\}$$

set $\tau_{ij} = 0$, deal on a new realization T_{ij} and go to **IDLE**.

V. ANALYSIS OF THE DISTRIBUTED ALGORITHMS

To start the analysis we first introduce an extended version of (centralized) proximal gradient methods.

A. Weighted Proximal Gradient Methods

Consider the following optimization problem

$$\min_{y \in \mathbb{R}^N} \Gamma(y) := \Phi(y) + \Psi(y), \quad (7)$$

where $\Phi : \mathbb{R}^N \rightarrow \mathbb{R}$ and $\Psi : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ are convex functions.

We decompose the decision variable as $y = [y_1^\top \dots y_n^\top]^\top$ and, consistently, we decompose the space \mathbb{R}^N into n subspaces as follows. Let $U \in \mathbb{R}^{N \times N}$ be a column permutation of the $N \times N$ identity matrix and, further, let $U = [U_1 \ U_2 \ \dots \ U_n]$ be a decomposition of U into n submatrices, with $U_i \in \mathbb{R}^{N \times N_i}$ and $\sum_i N_i = N$. Thus, any vector $y \in \mathbb{R}^N$ can be uniquely written as $y = \sum_i U_i y_i$ and, viceversa, $y_i = U_i^\top y$.

We let problem (7) satisfy the following assumptions.

Assumption V.1 (Block Lipschitz continuity of $\nabla\Phi$). *The gradient of Φ is block coordinate-wise Lipschitz continuous*

with positive constants L_1, \dots, L_n . That is, for all $y \in \mathbb{R}^N$ and $s_i \in \mathbb{R}^{N_i}$ it holds

$$\|\nabla_i \Phi(y + U_i s_i) - \nabla_i \Phi(y)\| \leq L_i \|s_i\|,$$

where $\nabla_i \Phi(y)$ is the i -th block component of $\nabla \Phi(y)$. \square

Assumption V.2 (Separability of Ψ). *The function Ψ is block-separable, i.e., it can be decomposed as $\Psi(y) = \sum_{i=1}^n \psi_i(y_i)$, with each $\psi_i : \mathbb{R}^{N_i} \rightarrow \mathbb{R} \cup \{+\infty\}$ a proper, closed convex extended real-valued function.* \square

Assumption V.3 (Feasibility). *The set of minimizers of problem (7) is non-empty.* \square

1) *Deterministic descent:* We first show how the standard proximal gradient algorithm can be generalized by using a weighted proximal operator.

Following the same line of proof as in [18], we can prove that a generalized proximal gradient iteration, given by

$$\begin{aligned} y(t+1) &= \text{prox}_{W, \Psi} \left(y(t) - W \nabla \Phi(y(t)) \right) \\ &= \underset{y}{\text{argmin}} \left\{ \Psi(y) + \frac{1}{2} \left\| y - \left(y(t) - W \nabla \Phi(y(t)) \right) \right\|_{W^{-1}}^2 \right\}, \end{aligned} \quad (8)$$

converges in objective value to the optimal solution of (7) with rate $O(1/t)$.

In order to extend the proof of [18, Theorem 3.1] we need to use a generalized version of [18, Lemma 2.1]. To this end we can use a result by Nesterov, given in [22], which is recalled in the following lemma for completeness.

Lemma V.4 (Generalized Descent Lemma). *Let Assumption V.1 hold, then for all $s \in \mathbb{R}^N$*

$$\Phi(y+s) \leq \Phi(y) + s^\top \nabla \Phi(y) + \frac{1}{2} \|s\|_{W^{-1}}^2,$$

where $W := \text{diag}(w_1, \dots, w_n)$ satisfies $w_i \leq \frac{1}{nL_i}$ for all $i \in \{1, \dots, n\}$. \square

Tighter conditions than the one given above can be found in [14] and in [23].

Theorem V.5. *Let Assumption V.1 and V.3 hold and let $\{y(t)\}$ be the sequence generated by iteration (8) applied to problem (7). Then for any $t \geq 1$*

$$\Gamma(y(t)) - \Gamma(y^*) \leq \frac{\|y_0 - y^*\|_{W^{-1}}^2}{2t},$$

where $W := \text{diag}(w_1, \dots, w_n)$ with $w_i \leq \frac{1}{nL_i}$, y_0 is the initial condition and y^* is any minimizer of problem (7).

Proof. The theorem is proven by following the same arguments as in [18, Theorem 3.1], but using Lemma V.4 in place of [18, Lemma 2.1]. \square

2) *Randomized block coordinate descent:* Next, we present a randomized version of the weighted proximal gradient, proposed in [20, Algorithm 2] as Uniform Coordinate Descent for Composite functions (UCDC) algorithm.

The convergence result for UCDC is given in [20, Theorem 5], here reported for completeness.

Algorithm 4 UCDC

Initialization: $y(0) = y_0$
 FOR: $t = 0, 1, 2, \dots$ DO
 choose $i_t \in \{1, \dots, n\}$ with probability $\frac{1}{n}$
 compute

$$T^{(i_t)}(y(t)) = \operatorname{argmin}_{s \in \mathbb{R}^{N_{i_t}}} \left\{ V_{i_t}(y(t), s) \right\} \quad (9a)$$

where

$$V_{i_t}(y, s) := \nabla_{i_t} \Phi(y)^\top s + \frac{L_{i_t}}{2} \|s\|^2 + \psi_{i_t}(y_{i_t} + s) \quad (9b)$$

update

$$y(t+1) = y(t) + U_{i_t} T^{(i_t)}(y(t)). \quad (10)$$

Theorem V.6 ([20, Theorem 5]). *Let Assumptions V.1, V.2 and V.3 hold. Let Γ^* denote the optimal cost of problem (7). Then, for any $\varepsilon \in (0, \Gamma(y_0) - \Gamma^*)$, there exists $\bar{t}(\varepsilon, \rho) > 0$ such that if $y(t)$ is the random sequence generated by UCDC (Algorithm 4) applied to problem (7), then for all $t \geq \bar{t}$ it holds that*

$$\Pr \left(\Gamma(y(t)) - \Gamma^* \leq \varepsilon \right) \geq 1 - \rho,$$

where $y_0 \in \mathbb{R}^N$ is the initial condition and $\rho \in (0, 1)$ is the target confidence. \square

B. Analysis of the synchronous algorithm

We start this section by recalling some useful properties of conjugate functions that will be useful for the convergence analysis of the proposed distributed algorithms.

Lemma V.7 ([24], [25]). *Let φ be a closed, strictly convex function and φ^* its conjugate function. Then*

$$\nabla \varphi^*(y) = \operatorname{argmax}_x \{y^\top x - \varphi(x)\} = \operatorname{argmin}_x \{\varphi(x) - y^\top x\}.$$

Moreover, if φ is strongly convex with convexity parameter σ , then $\nabla \varphi^*$ is Lipschitz continuous with Lipschitz constant given by $\frac{1}{\sigma}$. \square

In the next lemma we establish some important properties of problem (6) that will be useful to analyze the proposed distributed dual proximal algorithms.

Lemma V.8. *Let $\Phi(y) := F^*(y)$ and $\Psi(y) := G^*(y)$ consistently with the notation of problem (7) in Appendix V-A. Problem (6) satisfies Assumption V.1 (block Lipschitz continuity of $\nabla \Phi$), with (block) Lipschitz constants given by*

$$L_i = \sqrt{\frac{1}{\sigma_i^2} + \sum_{j \in \mathcal{N}_i} \left(\frac{1}{\sigma_i} + \frac{1}{\sigma_j} \right)^2}, \quad i \in \{1, \dots, n\},$$

Assumption V.2 (separability of Ψ) and Assumption V.3 (feasibility). \square

Proof. The proof is split into blocks, one for each assumption.

Block Lipschitz continuity of ∇F^* : We show that the gradient of F^* is block coordinate-wise Lipschitz continuous. The i -th block component of ∇F^* is

$$\nabla_i F^*(y) = \begin{bmatrix} \nabla_{\Lambda_i} F^*(y) \\ \nabla_{\mu_i} F^*(y) \end{bmatrix},$$

where the block-component of $\nabla_{\Lambda_i} F^*$ associated to neighbor j is given by $\nabla_{\lambda_j^i} F^*$ and is equal to

$$\begin{aligned} \nabla_{\lambda_j^i} F^*(y) &= \nabla f_i^* \left(- \sum_{k \in \mathcal{N}_i} \left(\lambda_i^k - \lambda_i^i \right) - \mu_i \right) \\ &\quad - \nabla f_j^* \left(- \sum_{k \in \mathcal{N}_j} \left(\lambda_j^k - \lambda_j^j \right) - \mu_j \right). \end{aligned}$$

By Lemma V.7 both ∇f_i^* and ∇f_j^* are Lipschitz continuous with Lipschitz constants $\frac{1}{\sigma_i}$ and $\frac{1}{\sigma_j}$ respectively, thus also $\nabla_{\lambda_j^i} F^*$ is Lipschitz continuous with constant $L_{ij} = \frac{1}{\sigma_i} + \frac{1}{\sigma_j}$. By using the (Euclidean) 2-norm, we have that $\nabla_{\Lambda_i} F^*(y)$ is Lipschitz continuous with constant $\sqrt{\sum_{j \in \mathcal{N}_i} L_{ij}^2}$. Similarly, the gradient of F^* with respect to μ_i is

$$\nabla_{\mu_i} F^*(y) = \nabla f_i^* \left(- \sum_{k \in \mathcal{N}_i} \left(\lambda_i^k - \lambda_i^i \right) - \mu_i \right)$$

and is Lipschitz continuous with constant $\frac{1}{\sigma_i}$. Finally, we conclude that $\nabla_i F^*(y)$ is Lipschitz continuous with constant

$$L_i = \sqrt{\frac{1}{\sigma_i^2} + \sum_{j \in \mathcal{N}_i} L_{ij}^2} = \sqrt{\frac{1}{\sigma_i^2} + \sum_{j \in \mathcal{N}_i} \left(\frac{1}{\sigma_i} + \frac{1}{\sigma_j} \right)^2}.$$

Separability of G^* : By definition $G^*(y) = \sum_{i=1}^n g_i^*(\mu_i)$, where μ_i is a component of the block y_i . Thus, denoting $G_i^*(y_i) := g_i^*(\mu_i)$, it follows immediately $G^*(y) = \sum_{i=1}^n g_i^*(\mu_i) = \sum_{i=1}^n G_i^*(y_i)$.

Feasibility: From Assumption II.2 and the convexity condition on f_i and g_i , strong duality holds, i.e., dual problem (6) is feasible and admits at least a minimizer y^* . \square

Next, we recall how the proximal operators of a function and its conjugate are related.

Lemma V.9 (Moreau decomposition, [26]). *Let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ be a closed, convex function and φ^* its conjugate. Then, $\forall x \in \mathbb{R}^d$, $x = \mathbf{prox}_{\varphi}(x) + \mathbf{prox}_{\varphi^*}(x)$. \square*

Lemma V.10 (Extended Moreau decomposition). *Let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ be a closed, convex function and φ^* its conjugate. Then, for any $x \in \mathbb{R}^d$ and $\alpha > 0$, it holds*

$$x = \mathbf{prox}_{\alpha \varphi}(x) + \alpha \mathbf{prox}_{\frac{1}{\alpha} \varphi^*} \left(\frac{x}{\alpha} \right).$$

Proof. Let $h(x) = \alpha \varphi(x)$, then from the Moreau decomposition in Lemma V.9, it holds $x = \mathbf{prox}_h(x) + \mathbf{prox}_{h^*}(x)$. To prove the result we simply need to compute $\mathbf{prox}_{h^*}(x)$ in terms of φ^* . First, from the definition of conjugate function we obtain $h^*(x) = \alpha \varphi^* \left(\frac{x}{\alpha} \right)$. Then, by using the definition of proximal operator and standard algebraic properties from minimization, it holds true that $\mathbf{prox}_{h^*}(x) = \alpha \mathbf{prox}_{\frac{1}{\alpha} \varphi^*} \left(\frac{x}{\alpha} \right)$, so that the proof follows. \square

The next lemma shows how the (weighted) proximal operator of G^* can be split into local proximal operators that can be independently carried out by each single node.

Lemma V.11. *Let $y = [y_1^\top \dots y_n^\top]^\top \in \mathbb{R}^{n(D+d)}$ where $y_i = [\Lambda_i^\top \mu_i^\top]^\top$ with $\Lambda_i \in \mathbb{R}^D$ and $\mu_i \in \mathbb{R}^d$, $i \in \{1, \dots, n\}$. Let $G^*(y) = \sum_{i=1}^n g_i^*(\mu_i)$, then for a diagonal weight matrix $D_\alpha = \text{diag}(\alpha_1, \dots, \alpha_n) > 0$, the proximal operator $\text{prox}_{D_\alpha, G^*}$ evaluated at y is given by*

$$\text{prox}_{D_\alpha, G^*}(y) = \begin{bmatrix} \Lambda_1 \\ \text{prox}_{\alpha_1 g_1^*}(\mu_1) \\ \vdots \\ \Lambda_n \\ \text{prox}_{\alpha_n g_n^*}(\mu_n) \end{bmatrix}.$$

□

Proof. Let $\eta = [\eta_1^\top \dots \eta_n^\top]^\top \in \mathbb{R}^{n(D+d)}$, with $\eta_i = [u_i^\top v_i^\top]^\top$, $u_i \in \mathbb{R}^D$ and $v_i \in \mathbb{R}^d$, be a variable with the same block structure of $y = [y_1^\top \dots y_n^\top]^\top \in \mathbb{R}^{n(D+d)}$, with $y_i = [\Lambda_i^\top \mu_i^\top]^\top$ (as defined in (5)).

By using the definition of weighted proximal operator and the separability of both G^* and the norm function, we have

$$\begin{aligned} \text{prox}_{D_\alpha, G^*}(y) &:= \underset{\eta \in \mathbb{R}^{n(D+d)}}{\text{argmin}} \left\{ G^*(\eta) + \frac{1}{2} \|\eta - y\|_{D_\alpha^{-1}}^2 \right\} \\ &= \underset{\eta}{\text{argmin}} \left\{ \sum_{i=1}^n \left(g_i^*(v_i) + \frac{1}{2\alpha_i} \|u_i - \Lambda_i\|^2 + \frac{1}{2\alpha_i} \|v_i - \mu_i\|^2 \right) \right\}. \end{aligned}$$

The minimization splits on each component η_i of η , giving

$$\text{prox}_{D_\alpha, G^*}(y) = \begin{bmatrix} \underset{u_1}{\text{argmin}} \|u_1 - \Lambda_1\|^2 \\ \underset{v_1}{\text{argmin}} \left\{ g_1^*(v_1) + \frac{1}{2\alpha_1} \|v_1 - \mu_1\|^2 \right\} \\ \vdots \\ \underset{u_n}{\text{argmin}} \|u_n - \Lambda_n\|^2 \\ \underset{v_n}{\text{argmin}} \left\{ g_n^*(v_n) + \frac{1}{2\alpha_n} \|v_n - \mu_n\|^2 \right\} \end{bmatrix}$$

so that the proof follows from the definition of proximal operator. □

We are ready to show the convergence of the DDPG introduced in Algorithm 1.

Theorem V.12. *For each $i \in \{1, \dots, n\}$, let f_i be a proper, closed and strongly convex extended real-valued function with strong convexity parameter $\sigma_i > 0$, and let g_i be a proper convex extended real-valued function. Suppose that in Algorithm 1 the local step-size α_i is chosen such that $0 < \alpha_i \leq \frac{1}{nL_i}$, with L_i given by*

$$L_i = \sqrt{\frac{1}{\sigma_i^2} + \sum_{j \in \mathcal{N}_i} \left(\frac{1}{\sigma_i} + \frac{1}{\sigma_j} \right)^2}, \quad \forall i \in \{1, \dots, n\}. \quad (11)$$

Then the sequence $y(t) = [y_1(t)^\top \dots y_n(t)^\top]^\top$ generated by the DDPG (Algorithm 1) for all $t \geq 1$ satisfies

$$\Gamma(y(t)) - \Gamma(y^*) \leq \frac{\|y_0 - y^*\|_{D_\alpha^{-1}}^2}{2t},$$

where y^* is any minimizer of (6), $y_0 = [y_1(0)^\top \dots y_n(0)^\top]^\top$ is the initial condition and $D_\alpha := \text{diag}(\alpha_1, \dots, \alpha_n)$.

Proof. To prove the theorem, we proceed in two steps. First, from Lemma V.8, problem (6) satisfies the assumptions of Theorem V.5 and, thus, a (deterministic) weighted proximal gradient solves the problem. Thus, we need to show that DDPG (Algorithm 1) is a weighted proximal gradient scheme.

The weighted proximal gradient algorithm, (8), applied to problem (6), with $W := D_\alpha$, is given by

$$y(t+1) = \text{prox}_{D_\alpha, G^*}(y(t) - D_\alpha \nabla F^*(y(t))). \quad (12)$$

Now, by Lemma V.8, L_i given in (11) is the Lipschitz constant of the i -th block of ∇F^* . Thus, using the hypothesis $\alpha_i \leq \frac{1}{nL_i}$, we can apply Theorem V.5, which ensures convergence with a rate of $O(1/t)$ in objective value.

In order to disclose the distributed update rule, we first split (12) into two steps, i.e.,

$$\tilde{y} = y(t) - D_\alpha \nabla F^*(y(t)) \quad (13a)$$

$$y(t+1) = \text{prox}_{D_\alpha, G^*}(\tilde{y}) \quad (13b)$$

and, then, compute explicitly each component of both the equations. Focusing on (13a) and considering that D_α is diagonal, we can write the i -th block component of \tilde{y} as

$$\tilde{y}_i = \begin{bmatrix} \tilde{\Lambda}_i \\ \tilde{\mu}_i \end{bmatrix} = y_i(t) - \alpha_i \nabla_i F^*(y(t)),$$

where the explicit update of the block-component of $\tilde{\Lambda}_i$ associated to neighbor j is

$$\begin{aligned} \tilde{\lambda}_i^j &= \lambda_i^j(t) - \alpha_i \left. \frac{\partial F^*(y)}{\partial \lambda_i^j} \right|_{y=y(t)} \\ &= \lambda_i^j(t) + \alpha_i \left[\nabla f_i^* \left(- \sum_{k \in \mathcal{N}_i} (\lambda_i^k(t) - \lambda_k^i(t)) - \mu_i(t) \right) \right. \\ &\quad \left. - \nabla f_j^* \left(- \sum_{k \in \mathcal{N}_j} (\lambda_j^k(t) - \lambda_k^j(t)) - \mu_j(t) \right) \right] \end{aligned} \quad (14)$$

and the explicit update of $\tilde{\mu}_i$ is

$$\begin{aligned} \tilde{\mu}_i &= \mu_i(t) - \alpha_i \left. \frac{\partial F^*(y)}{\partial \mu_i} \right|_{y=y(t)} \\ &= \mu_i(t) + \alpha_i \nabla f_i^* \left(- \sum_{k \in \mathcal{N}_i} (\lambda_i^k(t) - \lambda_k^i(t)) - \mu_i(t) \right). \end{aligned} \quad (15)$$

Now, denoting

$$x_i^*(t) := \nabla f_i^* \left(- \sum_{k \in \mathcal{N}_i} (\lambda_i^k(t) - \lambda_k^i(t)) - \mu_i(t) \right),$$

from Lemma V.7 it holds

$$x_i^*(t) = \underset{x_i}{\text{argmin}} \left\{ x_i^\top \left(\sum_{k \in \mathcal{N}_i} (\lambda_i^k(t) - \lambda_k^i(t)) + \mu_i(t) \right) + f_i(x_i) \right\}.$$

Thus, we can rewrite (14) and (15) in terms of x_i^* obtaining

$$\begin{aligned} \tilde{\lambda}_i^j &= \lambda_i^j(t) + \alpha_i [x_i^*(t) - x_j^*(t)] \\ \tilde{\mu}_i &= \mu_i(t) + \alpha_i x_i^*(t). \end{aligned}$$

Finally, the last step consists of applying the rule (13b) to \tilde{y} . In order to highlight the distributed update, we rewrite (13b) in a component-wise fashion, i.e.,

$$y(t+1) = \begin{bmatrix} \Lambda_1(t+1) \\ \mu_1(t+1) \\ \vdots \\ \Lambda_n(t+1) \\ \mu_n(t+1) \end{bmatrix} = \mathbf{prox}_{D_\alpha, G^*} \left(\begin{bmatrix} \tilde{\Lambda}_1 \\ \tilde{\mu}_1 \\ \vdots \\ \tilde{\Lambda}_n \\ \tilde{\mu}_n \end{bmatrix} \right),$$

and applying Lemma V.11 and Lemma V.10 with $\varphi_i = g_i^*$, we obtain

$$y(t+1) = \begin{bmatrix} \tilde{\Lambda}_1 \\ \mathbf{prox}_{\alpha_1 g_1^*}(\tilde{\mu}_1) \\ \vdots \\ \tilde{\Lambda}_n \\ \mathbf{prox}_{\alpha_n g_n^*}(\tilde{\mu}_n) \end{bmatrix} = \begin{bmatrix} \tilde{\Lambda}_1 \\ \tilde{\mu}_1 - \alpha_1 \mathbf{prox}_{\frac{1}{\alpha_1} g_1^*} \left(\frac{\tilde{\mu}_1}{\alpha_1} \right) \\ \vdots \\ \tilde{\Lambda}_n \\ \tilde{\mu}_n - \alpha_n \mathbf{prox}_{\frac{1}{\alpha_n} g_n^*} \left(\frac{\tilde{\mu}_n}{\alpha_n} \right) \end{bmatrix},$$

so that the proof follows. \square

Remark V.13 (Nesterov's acceleration). *We can include a Nesterov's extrapolation step in the algorithm, which accelerates the algorithm (further details in [19]), attaining a faster $O(1/t^2)$ convergence rate in objective value. In order to implement the acceleration, each node needs to store a copy of the dual variables at the previous iteration. Thus, the update law in (13) would be changed in the following*

$$\begin{aligned} \tilde{y} &= y(t) - D_\alpha \nabla F^*(y(t)) \\ \hat{y}(t) &= \mathbf{prox}_{D_\alpha, G^*}(\tilde{y}) \\ y(t+1) &= \hat{y}(t) + \theta_t (\hat{y}(t) - \hat{y}(t-1)). \end{aligned}$$

where θ_t represents the Nesterov overshoot parameter. \square

C. Analysis of the node-based asynchronous algorithm

In order to analyze the algorithm we start recalling some properties of i.i.d. exponential random variables. Let $i_t \in \{1, \dots, n\}$, $t = 1, 2, \dots$ be the sequence identifying the generic t -th triggered node. Assumption IV.2 implies that i_t is an i.i.d. uniform process on the alphabet $\{1, \dots, n\}$. Each triggering will induce an iteration of the distributed optimization algorithm, so that t will be a *universal*, discrete time indicating the t -th iteration of the algorithm itself. Thus, from an external, global perspective, the described local asynchronous updates result into an algorithmic evolution in which, at each iteration, only one node wakes up randomly, uniformly and independently from previous iterations. This variable will be used in the statement and in the proof of Theorem V.14. However, we want to stress that this iteration counter does not need to be known by the agents.

Theorem V.14. *For each $i \in \{1, \dots, n\}$, let f_i be a proper, closed and strongly convex extended real-valued function with strong convexity parameter $\sigma_i > 0$, and let g_i be a proper convex extended real-valued function. Suppose that in Algorithm 2 each local step-size α_i is chosen such that $0 < \alpha_i \leq \frac{1}{L_i}$, with*

$$L_i = \sqrt{\frac{1}{\sigma_i^2} + \sum_{j \in \mathcal{N}_i} \left(\frac{1}{\sigma_i} + \frac{1}{\sigma_j} \right)^2}, \quad \forall i \in \{1, \dots, n\}. \quad (16)$$

Then the sequence $y(t) = [y_1(t)^\top \dots y_n(t)^\top]^\top$ generated by the A-DDPG (Algorithm 2) converges in objective value with high probability, i.e., for any $\varepsilon \in (0, \Gamma(y_0))$, where $y_0 = [y_1(0)^\top \dots y_n(0)^\top]^\top$ is the initial condition, and target confidence $0 < \rho < 1$, there exists $\bar{t}(\varepsilon, \rho) > 0$ such that for all $t \geq \bar{t}$ it holds

$$\Pr \left(\Gamma(y(t)) - \Gamma^* \leq \varepsilon \right) \geq 1 - \rho,$$

where Γ^* is the optimal cost of problem (6).

Proof. To prove the theorem, we proceed in two steps. First, we show that we can apply the Uniform Coordinate Descent for Composite functions (Algorithm 4) to solve problem (6). Second, we show that, when applied to this problem, Algorithm 4 gives the iterates of our A-DDPG.

The first part follows immediately by Lemma V.8, which asserts that problem (6) satisfies the assumptions of Theorem V.6, so that Algorithm 4 solves it.

Next, we show that the two algorithms have the same update. First, by Lemma V.8, L_i given in (16) is the Lipschitz constant of the i -th block of ∇F^* . Thus, in the rest of the proof, following [20], we set $\alpha_i = \frac{1}{L_i}$ (the maximum allowable value). Clearly the convergence is preserved if a smaller stepsize is used.

Consistently with the notation in Algorithm 4, let i_t denote the uniform-randomly selected index at iteration t . Thus, $T^{(i_t)}(y(t)) = \operatorname{argmin}_{s_{i_t} \in \mathbb{R}^{N_{i_t}}} \left\{ V_{i_t}(y(t), s_{i_t}) \right\}$ defined in (9) can be written in terms of a proximal gradient update applied to the i_t -th block component of y . In fact, by definition, for our function $\Gamma = F^* + G^*$, we have

$$\begin{aligned} T^{(i_t)}(y(t)) &= \operatorname{argmin}_{s \in \mathbb{R}^{N_{i_t}}} \left\{ \nabla_{i_t} F^*(y(t))^\top s \right. \\ &\quad \left. + \frac{L_{i_t}}{2} \|s\|^2 + g_{i_t}^*(y_{i_t}(t) + s) \right\}. \end{aligned}$$

In order to apply the formal definition of a proximal gradient step, we add a constant term and introduce a change of variable given by $\bar{s} := y_{i_t}(t) + s$, obtaining

$$\begin{aligned} T^{(i_t)}(y(t)) &= -y_{i_t}(t) + \operatorname{argmin}_{\bar{s} \in \mathbb{R}^{N_{i_t}}} \left\{ U_{i_t}^\top F^*(y(t)) \right. \\ &\quad \left. + \nabla_{i_t} F^*(y(t))^\top (\bar{s} - y_{i_t}(t)) + \frac{L_{i_t}}{2} \|\bar{s} - y_{i_t}(t)\|^2 + g_{i_t}^*(\bar{s}) \right\}, \end{aligned}$$

which yields

$$T^{(i_t)}(y(t)) = -y_{i_t}(t) + \mathbf{prox}_{\frac{1}{L_{i_t}} g_{i_t}^*} \left(y_{i_t}(t) - \frac{1}{L_{i_t}} \nabla_{i_t} F^*(y(t)) \right).$$

Thus, update (10) in fact changes only the component y_{i_t} of y , which is updated as

$$\begin{aligned} y_{i_t}(t+1) &= y_{i_t}(t) + T^{(i_t)}(y(t)) \\ &= \mathbf{prox}_{\frac{1}{L_{i_t}} g_{i_t}^*} \left(y_{i_t}(t) - \frac{1}{L_{i_t}} \nabla_{i_t} F^*(y(t)) \right), \end{aligned} \quad (17)$$

while all the other ones remain unchanged, i.e., $y_i(t+1) = y_i(t)$ for all $i \in \{1, \dots, n\}$ with $i \neq i_t$.

Following the same steps as in the proof of Theorem V.12, we split the update in (17) into a gradient and a proximal steps. The gradient step is given by

$$\tilde{y}_{i_t} = \begin{bmatrix} \tilde{\Lambda}_{i_t} \\ \tilde{\mu}_{i_t} \end{bmatrix} = y_{i_t}(t) - \frac{1}{L_{i_t}} \nabla_{i_t} F^*(y(t))$$

where $\tilde{\Lambda}_{i_t}$ and $\tilde{\mu}_{i_t}$ are the same as in (14) and (15) respectively. The proximal operator step turns out to be

$$y_{i_t}(t+1) = \begin{bmatrix} \Lambda_{i_t}(t+1) \\ \mu_{i_t}(t+1) \end{bmatrix} = \mathbf{prox}_{\frac{1}{L_{i_t}} g_{i_t}^*} \left(\begin{bmatrix} \tilde{\Lambda}_{i_t} \\ \tilde{\mu}_{i_t} \end{bmatrix} \right).$$

Applying Lemma V.11 on the i_t -th block with $\alpha_{i_t} = 1/L_{i_t}$, we can rewrite (17) as

$$\begin{bmatrix} \Lambda_{i_t}(t+1) \\ \mu_{i_t}(t+1) \end{bmatrix} = \begin{bmatrix} \tilde{\Lambda}_{i_t}(t) \\ \tilde{\mu}_{i_t} - \frac{1}{L_{i_t}} \mathbf{prox}_{L_{i_t} g_{i_t}}(L_{i_t} \tilde{\mu}_{i_t}) \end{bmatrix}, \quad (18)$$

where each component of $\tilde{\Lambda}_{i_t}$ is given by

$$\tilde{\lambda}_{i_t}^j = \lambda_{i_t}^j(t) + \frac{1}{L_{i_t}} [x_{i_t}^*(t) - x_j^*(t)],$$

and

$$\tilde{\mu}_{i_t} = \mu_{i_t}(t) + \frac{1}{L_{i_t}} x_{i_t}^*(t),$$

with

$$x_i^*(t) = \underset{x_i}{\operatorname{argmin}} \left\{ x_i^\top \left(\sum_{k \in \mathcal{N}_i} (\lambda_i^k(t) - \lambda_k^i(t)) + \mu_i(t) \right) + f_i(x_i) \right\}.$$

Here we have used again the property from Lemma V.7

$$\nabla f_i^* \left(- \sum_{k \in \mathcal{N}_i} (\lambda_i^k(t) - \lambda_k^i(t)) - \mu_i(t) \right) = x_i^*(t).$$

Now, from Assumption IV.2 a sequence of nodes i_t , $t = 1, 2, \dots$, becomes active according to a uniform distribution, so that each node triggering can be associated to an iteration of Algorithm 4 given by the update in (18). That is, only a node i_t is active in the network, which performs an update of its dual variables y_{i_t} . In order to perform the local update, the selected node i_t needs to know the most updated information after the last triggering. As regards the neighbors' dual variables λ_j^i , $j \in \mathcal{N}_{i_t}$, they have been broadcast by each $j \in \mathcal{N}_{i_t}$ the last time it has become active. Regarding the primal variables x_j^* , $j \in \mathcal{N}_{i_t}$, the situation is a little more tricky. Indeed, x_j^* , $j \in \mathcal{N}_{i_t}$, may have changed in the past due to either j or one of its neighbors has become active. In both cases j has to broadcast to i_t its updated dual variable (either because it has become active or because it has received, in idle, an updated dual variable from one of its neighbors). \square

Remark V.15. *Differently from the synchronous algorithm, in the asynchronous version nodes do not need to know the number of nodes, n , in order to set their local step-size. In fact, each node i can set its parameter α_i by only knowing the convexity parameters σ_i and σ_j , $j \in \mathcal{N}_i$.* \square

Remark V.16. *If a strongly convex, separable penalty term is added to the dual function $\Gamma = F^* + G^*$, then it becomes strongly convex, so that a stronger result from [20, Theorem 7] applies, i.e., linear convergence with high probability*

is guaranteed. Note that strong convexity of the dual function Γ is obtained if the primal function has Lipschitz continuous gradient, [27, Chapter X, Theorem 4.2.2]. \square

D. Analysis of the edge-based asynchronous algorithm

The convergence of Algorithm 3 relies essentially on the same arguments as in Theorem V.14, but with a different block partition. In fact, we have to split the y variable into $|\mathcal{E}|$ blocks (with $|\mathcal{E}|$ the number of edges of \mathcal{G}). Notice that since the dual variables μ_i are only n , they need to be associated to a subset of edges. Thus, the variable y is split in blocks y_{ij} given by $y_{ij} = [\lambda_i^j \ \lambda_j^i \ \mu_i]$ if $j = j_{\mu_i}$ and $y_{ij} = [\lambda_i^j \ \lambda_j^i]$ otherwise. This is why in the algorithm μ_i is updated only when neighbor j_{μ_i} becomes active.

VI. MOTIVATING OPTIMIZATION SCENARIOS AND NUMERICAL COMPUTATIONS

A. Constrained optimization

As first concrete setup, we consider a separable constrained convex optimization problem

$$\begin{aligned} \min_x \quad & \sum_{i=1}^n h_i(x) \\ \text{subj. to} \quad & x \in \bigcap_{i=1}^n X_i \subseteq \mathbb{R}^d \end{aligned} \quad (19)$$

where each h_i is a strongly convex function and each X_i is a closed, convex set.

This problem structure is widely investigated in distributed and large-scale optimization as shown in the literature review. Notice that, as pointed out in our discussion in the introduction, we assume strong convexity of h_i , but we do not need to assume smoothness.

We can rewrite this problem by transforming the constraints into additional terms in the objective function, by using indicator functions I_{X_i} associated to each X_i ,

$$\min_{x \in \mathbb{R}^d} \sum_{i=1}^n (h_i(x) + I_{X_i}(x)).$$

Since each X_i is a convex set, then I_{X_i} is a convex function. Thus the problem can be mapped in our distributed setup (1) by setting $f_i(x) = h_i(x)$ and $g_i(x) = I_{X_i}(x)$.

Treating the local constraints in this way, we have to perform a local *unconstrained* minimization step when computing x_i^* , while the local feasibility is entrusted to the proximal operator of g_i . In fact, the proximal operator of a convex indicator function reduces to the standard Euclidean projection, i.e.,

$$\mathbf{prox}_{I_X}(v) = \underset{x}{\operatorname{argmin}} \left\{ I_X(x) + \frac{1}{2} \|x - v\|^2 \right\} = \Pi_X(v).$$

Remark VI.1. *When considering quadratic costs h_i , we can benefit greatly from a numerical point of view. In fact, an unconstrained quadratic program can be solved via efficient methods, which often result in division-free algorithms (possibly after some off-line precomputations), and can be implemented in fixed-point arithmetic, see [21] for further details.* \square

An attractive feature of our setup is that one can conveniently decide how to rewrite the local constraints. In the formulation above, we suggested to include the local constraint into g_i . But it is also reasonable to include the constraint into f_i , by consider the indicator function in its definition, i.e., define

$$f_i(x) := \begin{cases} h_i(x) & \text{if } x \in X_i \\ +\infty & \text{otherwise.} \end{cases} \quad (20)$$

and, thus, have g_i identically zero (still convex). This strategy results into an algorithm which is basically an asynchronous distributed dual decomposition algorithm. Notice that with this choice recursive feasibility is obtained provided that the local algorithm solves the minimization in an interior point fashion.

Between these two extreme scenarios one could also consider other possibilities. Indeed, it could be the case that one can benefit from splitting each local constraint X_i into two distinct contributions, i.e., $X_i = Y_i \cup Z_i$. In this way the indicator function of Y_i (e.g., the positive orthant) could be included into f_i , allowing for a simpler constrained local minimization step, while the other constraint could be mapped into the second term as $g_i(x) = I_{Z_i}(x)$.

The choice in (20) leads to a simple observation: leaving the g_i equal to zero seems to be a waste of a degree of freedom that could be differently exploited, e.g., by introducing a regularization term.

B. Regularized and constrained optimization

As highlighted in the previous paragraph, the flexibility of our algorithmic framework allows us to handle, together with local constraints, also a regularization cost through the g_i . Regularize the solution is a useful technique in many applications as sparse design, robust estimation in statistics, support vector machine (SVM) in machine learning, total variation reconstruction in signal processing and geophysics, and compressed sensing. In these problems, the cost f_i is a loss function representing how the predictions based on a theoretical model mismatch the real data. Next, we focus on the most widespread choice for the loss function, which is the least square cost, giving rise to the following optimization problem

$$\min_x \sum_{i=1}^n \|A_i x - b_i\|^2 \quad (21)$$

where A_i are data/regressors and b_i are labels/observations.

A typical challenge arising in regression problems is due to the fact that problem (21) is often ill-posed and standard algorithms easily incur in over-fitting phenomena. A viable technique to prevent over-fitting consists of adding a regularization cost; usual choices are the ℓ_2 -norm, also referred as Tikhonov regularization or ridge regression, or the ℓ_1 -norm, which leads to the so called LASSO (Least Absolute Shrinkage and Selection Operator) problem

$$\min_x \sum_{i=1}^n \|A_i x - b_i\|^2 + \gamma \|x\|_1$$

where γ is a positive scalar.

In some cases (as, e.g., in distributed estimation [28]) one may be interested in having the solution bounded in a given box or leaving in a reduced subspace. This gives rise to the so called *constrained LASSO* problem, see, e.g., [23], [29], [30].

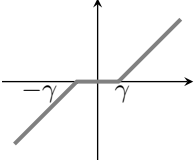
As discussed before, our setup can simultaneously manage a constrained and regularized problem as the constrained lasso. The first way to map the problem in our setup is by defining

$$f_i(x) := \begin{cases} \|A_i x - b_i\|^2 & \text{if } x \in X_i \\ +\infty & \text{otherwise} \end{cases} \quad (22)$$

and setting

$$g_i(x) := \frac{\gamma}{n} \|x\|_1.$$

The proximal operator of the ℓ_1 -norm admits an analytic solution which is well known as soft thresholding operator. When applied to a vector $v \in \mathbb{R}^d$ (with ℓ -th component v_ℓ), it gives a vector in \mathbb{R}^d whose ℓ -th component is

$$\left(\text{prox}_{\gamma \|\cdot\|_1}(v)\right)_\ell = \begin{cases} v_\ell - \gamma, & v_\ell > \gamma \\ 0, & |v_\ell| \leq \gamma \\ v_\ell + \gamma, & v_\ell < -\gamma \end{cases}$$


i.e., it thresholds the components of v which are in modulus greater then γ , see, e.g., [18], [31].

Alternatively, we may include both the constraint X_i and the regularization term into the g_i , obtaining an unconstrained local minimization at each node. This choice is particularly appealing when the constraint X_i is a box, i.e., $X_i = \{v \in \mathbb{R}^d \mid lb_\ell \leq v_\ell \leq ub_\ell \text{ for all } \ell \in \{1, \dots, d\}\}$. In this case the proximal operator of g_i becomes a *saturated* version of the soft-thresholding operator, as depicted in Figure 1.

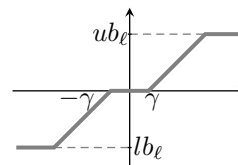


Fig. 1. Saturated soft-thresholding operator.

C. Numerical tests

In this section we provide a numerical example showing the effectiveness of the proposed algorithms.

We test the proposed distributed algorithms on a constrained LASSO optimization problem,

$$\min_{lb \leq x \leq ub} \sum_{i=1}^n \|A_i x - b_i\|^2 + \gamma \|x\|_1,$$

where $x \in \mathbb{R}^3$ is the decision variable, and $A_i \in \mathbb{R}^{150 \times 3}$ and $b_i \in \mathbb{R}^{150}$ represent respectively the data matrix and the labels associated with examples assigned to node i . The inequality $lb \leq x \leq ub$ is meant component-wise.

We randomly generate the LASSO data following the idea suggested in [31]. Each element of A_i is $\sim \mathcal{N}(0, 1)$, and b_i s are generated by perturbing a “true” solution x_{true} (which has around a half nonzero entries) with an additive noise $v \sim \mathcal{N}(0, 10^{-2}I)$. Then the matrix A_i and the vector b_i are normalized with respect to the number of local samples at each node. The box bounds are set to $lb = [-0.8 \ -0.8 \ -0.8]^\top$ and $ub = [0.8 \ 0.8 \ 0.8]^\top$, while the regularization parameter is $\gamma = 0.1$.

To match the problem with our distributed framework, we introduce copies x_i of the decision variable x . Consistently, we define the local functions f_i as the least-square costs in (22), where each X_i is the box defined by lb and ub . We let each g_i be the ℓ_1 -norm regularization term with local parameter γ/n . We initialize to zero the dual variables λ_i^j , $j \in \mathcal{N}_i$, and μ_i for all $i \in \{1, \dots, n\}$, and use as step-sizes $\alpha_i = L_i$, where L_i has the expression in (11), with σ_i being the smallest eigenvalue of $A_i^\top A_i$.

We consider an undirected connected Erdős-Rényi graph \mathcal{G} , with parameter 0.2, connecting $n = 50$ nodes.

We run both the synchronous and the asynchronous algorithms over this underlying graph and we stop them if the difference between the current dual cost and the optimal value drops below the threshold of 10^{-6} .

Figure 2 shows the difference between the dual cost at each iteration t and the optimal value, $\Gamma(y(t)) - \Gamma^*$, in a logarithmic scale. In particular, the rates of convergence of the synchronous (left) and asynchronous (right) algorithms are shown. For the asynchronous algorithm, we normalize the iteration counter t with respect the number of agents n .

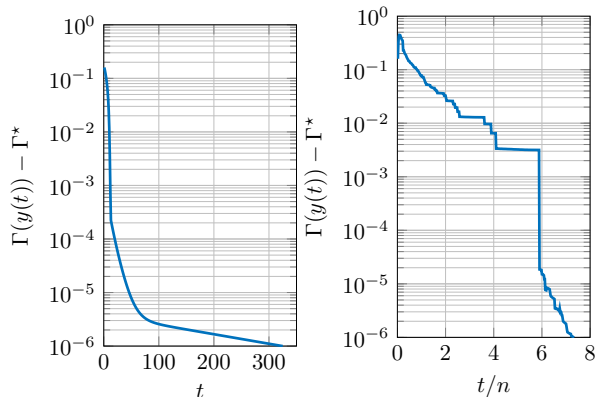


Fig. 2. Evolution of the cost error, in logarithmic scale, for the synchronous (left) and node-based asynchronous (right) distributed algorithms.

Then we concentrate on the asynchronous algorithm. In Figure 3 we plot the evolution of the (three) components of the primal variables, $x_i^*(t)$, $i \in \{1, \dots, n\}$. The horizontal dotted lines represent the optimal solution. It is worth noting that the optimal solution has a first component slightly below the constraint boundary, a second component equal to zero, and a third component on the constraint boundary. This optimal solution can be intuitively explained by the “simultaneous influence” of the box constraints (which restrict the set of admissible values) and the regularization term (which enforces sparsity of the vector x). In the inset the first iterations for a

subset of selected nodes are highlighted, in order to better show the transient, piece-wise constant behavior due to the gossip update and the effect of the box constraint on each component.

Specifically, for the first component, it can be seen how the temporary solution of some agents hits the boundary in some iterations (e.g., one of them hits the boundary from iteration 50 to iteration 100) and then converges to the (feasible) optimal value. The second components are always inside the box constraint and converge to zero, while the third components start inside the box and then in a finite number of iterations hit the boundary.

VII. CONCLUSIONS

In this paper we have proposed a class of distributed optimization algorithms, based on dual proximal gradient methods, to solve constrained optimization problems with non-smooth, separable objective functions. The main idea is to construct a suitable, separable dual problem via a proper choice of primal constraints and solve it through proximal gradient algorithms. Thanks to the separable structure of the dual problem in terms of local conjugate functions, a weighted proximal gradient update results into a distributed algorithm, where each node performs a local minimization on its primal variable, and a local proximal gradient update on its dual variables. As main contribution, two asynchronous, event-triggered distributed algorithms are proposed, in which the local updates at each node are triggered by a local timer, without relying on any global iteration counter. The asynchronous algorithms are shown to be special instances of a randomized, block-coordinate proximal gradient method on the dual problem. The convergence analysis is based on a proper combination of duality theory, coordinate-descent methods, and properties of proximal operators.

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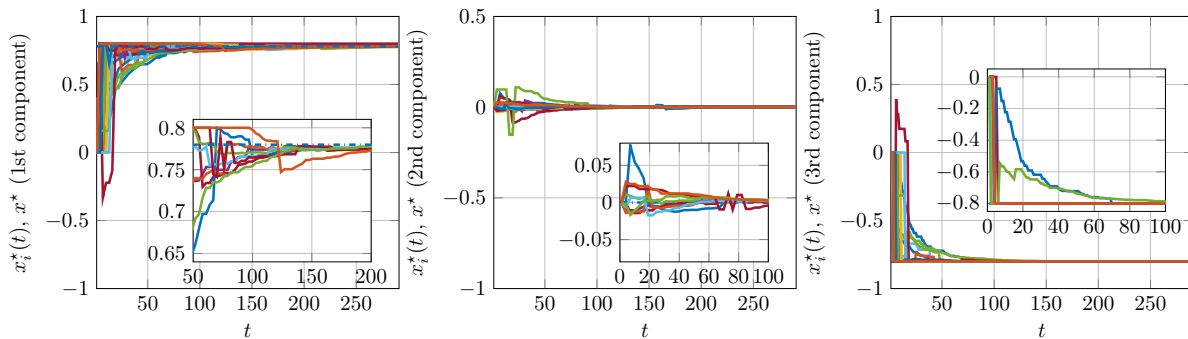


Fig. 3. Evolution of the three components of primal variables $x_i^*(t)$, $i \in \{1, \dots, n\}$, for the node-based A-DDPG.

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