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On profinite groups with commutators covered by countably many cosets

Eloisa Detomi, Marta Morigi, and Pavel Shumyatsky

Abstract. Let w be a group-word. Suppose that the set of all w-values in a profinite group G is contained in a union of countably many cosets of subgroups. We are concerned with the question to what extent the structure of the verbal subgroup $w(G)$ depends on the properties of the subgroups. We prove the following theorem.

Let $\mathcal{C}$ be a class of groups closed under taking subgroups, quotients, and such that in any group the product of finitely many normal $\mathcal{C}$-subgroups is again a $\mathcal{C}$-subgroup. If w is a multilinear commutator and G is a profinite group such that the set of all w-values is contained in a union of countably many cosets g_iG_i, where each G_i is in $\mathcal{C}$, then the verbal subgroup $w(G)$ is virtually-$\mathcal{C}$.

This strengthens several known results.

1. Introduction

A covering of a group G is a family $\{S_i\}_{i\in I}$ of subsets of G such that $G = \bigcup_{i\in I} S_i$. The famous result of B.H. Neumann states that if $\{S_i\}$ is a finite covering of G by cosets of subgroups, then G is actually covered by the cosets S_i corresponding to subgroups of finite index in G [**9**]. Therefore whenever a group G is covered by finitely many cosets of subgroups it is natural to expect that some structural information about G can be deduced from the properties of the subgroups. In other words, the general question is to what extent properties of the covering subgroups impact the structure of G.

In recent years some "verbal" variations of these questions became a subject of research activity. Given a group-word $w = w(x_1, \dots, x_n)$, we think of it primarily as a function of n variables defined on any given group G. We denote by $w(G)$ the verbal subgroup of G generated by

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the values of w. When the set of all w-values in a group G is contained in a union of finitely many subgroups (or cosets of subgroups) we wish to know whether the properties of the covering subgroups have impact on the structure of the verbal subgroup $w(G)$. The present article deals with the situation when G is a profinite group.

In the context of profinite groups all the usual concepts of group theory are interpreted topologically. In particular, by a subgroup of a profinite group we mean a closed subgroup. A subgroup is said to be generated by a set S if it is topologically generated by S. Thus, the verbal subgroup $w(G)$ in a profinite group G is a minimal closed subgroup containing the set of w-values. One important tool for dealing with the "covering" problems in profinite groups is the classical Baire's category theorem (cf [**10**, p. 200]): If a locally compact Hausdorff space is a union of countably many closed subsets, then at least one of the subsets has non-empty interior. It follows that if a profinite group is covered by countably many cosets of subgroups, then at least one of the subgroups is open. Thus, in the case of profinite groups we can successfully deal with problems on countable coverings rather than just finite ones.

The reader can consult the articles [**2, 3, 4, 6, 7, 12**] for results on countable coverings of word-values by subgroups. One of the results obtained in [**7**] is that if w is a multilinear commutator and G is a profinite group, then $w(G)$ is finite-by-nilpotent if and only if the set of w-values in G is covered by countably many finite-by-nilpotent subgroups (see Section 2 for the definition of multilinear commutator). It is easy to see that the above result is no longer true if the set of w-values in G is covered by countably many cosets of finite-by-nilpotent subgroups. This can be exemplified by any profinite group G having $w(G)$ virtually nilpotent but not finite-by-nilpotent. In the present article we study groups in which the set of w-values is covered by countably many cosets of $\mathcal{C}$-subgroups, where $\mathcal{C}$ is a class of groups closed under taking subgroups, quotients, and such that in any group the product of finitely many normal $\mathcal{C}$-subgroups is again a $\mathcal{C}$-subgroup.

Our main result is as follows.

THEOREM 1.1. *Let $\mathcal{C}$ be a class of groups closed under taking subgroups, quotients, and such that in any group the product of finitely many normal $\mathcal{C}$-subgroups is again a $\mathcal{C}$-subgroup. Let w be a multilinear commutator word. The verbal subgroup $w(G)$ of a profinite group G is virtually-$\mathcal{C}$ if and only if the set of w-values in G is covered by countably many cosets of $\mathcal{C}$-subgroups.*

We note that many natural classes of groups have the properties as the class $\mathcal{C}$ in the above theorem. For instance, $\mathcal{C}$ can be the class of nilpotent, pronilpotent, locally nilpotent, or soluble groups. Further examples include torsion groups and groups of finite rank. It is been known for sometime that if w is a multilinear commutator and a profinite group G has countably many soluble subgroups whose union contains all w-values, then $w(G)$ is virtually soluble [**1**, Theorem 7]. If G has countably many torsion subgroups (or subgroups of finite rank) whose union contains all w-values, then $w(G)$ is torsion (or of finite rank) [**4**]. Obviously, Theorem 1.1 extends these results. Moreover, in the case where $\mathcal{C}$ is the class of all finite groups, we obtain that the set of w-values in a profinite group G is countable if and only if $w(G)$ is finite. This was one of the main results in [**5**, Theorem 1.1].

A few words about the tools employed in the proof of Theorem 1.1. Rather specific combinatorial techniques for handling multilinear commutator words were developed in [**8, 4, 6**]. The present article is based on further refinements of those techniques. It seems that any attempt to prove a result of similar nature for words that are not multilinear commutator words would require a different approach.

2. Preliminary results

Throughout, we use the same symbol to denote a group-theoretical property and the class of groups with that property. If $\mathcal{C}$ is a class of groups, a virtually-$\mathcal{C}$ group is a group with a normal $\mathcal{C}$-subgroup of finite index. The class of virtually-$\mathcal{C}$ groups will be denoted by $\mathcal{CF}$.

Let $\mathcal{C}$ be a class of groups closed under taking subgroups, quotients, and such that in any group the product of finitely many normal $\mathcal{C}$-subgroups is again a $\mathcal{C}$-subgroup. For instance $\mathcal{C}$ is the class of nilpotent, soluble, or finite groups. The next two lemmas are analogues of Lemma 2.2 of [**7**] and Lemma 2.6 of [**6**], respectively. Therefore we omit their proofs.

Lemma 2.1. *In any group a product of finitely many normal $\mathcal{CF}$-subgroups is again in $\mathcal{CF}$.*

If A is a subset of a group G, we write $\langle A\rangle$ for the subgroup generated by A. If B is another subset, we denote by A^B the set $\{a^b \mid a \in A \text{ and } b \in B\}$.

Lemma 2.2. *Let L be a subgroup of a profinite group G such that the normalizer $N_G(L)$ is open.*

(1) *If L is finite, then $\langle L^G\rangle$ is finite.*

(2) *If L is in $\mathcal{C}$ and H is a normal open subgroup of G contained in $N_G(L)$, then $\langle (L \cap H)^G \rangle$ is in $\mathcal{C}$.*

Throughout this section $w = w(x_1, \ldots, x_n)$ is a multilinear commutator. Multilinear commutators are words which are obtained by nesting commutators, but using always different variables. More formally, the word $w(x) = x$ in one variable is a multilinear commutator; if u and v are multilinear commutators involving different variables then the word $w = [u, v]$ is a multilinear commutator, and all multilinear commutators are obtained in this way.

An important family of multilinear commutators is formed by so-called derived words δ_k, on 2^k variables, defined recursively by

$$\delta_0 = x_1, \qquad \delta_k = [\delta_{k-1}(x_1, \ldots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1}, \ldots, x_{2^k})].$$

Of course $\delta_k(G) = G^{(k)}$ is the k-th term of the derived series of G.

We recall the following well-known result (see for example [**11**, Lemma 4.1]).

Lemma 2.3. *Let G be a group and let w be a multilinear commutator on n variables. Then each δ_n-value is a w-value.*

If $A_1, \ldots, A_n$ are subsets of a group G, we write

$$\mathcal{X}_w(A_1, \ldots, A_n)$$

to denote the set of all w-values $w(a_1, \ldots, a_n)$ with $a_i \in A_i$. Moreover, we write $w(A_1, \ldots, A_n)$ for the subgroup $\langle \mathcal{X}_w(A_1, \ldots, A_n) \rangle$. Note that if every A_i is a normal subgroup of G, then $w(A_1, \ldots, A_n)$ is normal in G.

Let I be a subset of $\{1, \ldots, n\}$. Suppose that we have a family $A_{i_1}, \ldots, A_{i_s}$ of subsets of G with indices running over I and another family $B_{l_1}, \ldots, B_{l_t}$ of subsets with indices running over $\{1, \ldots, n\} \setminus I$. We write

$$w_I(A_i; B_l)$$

for $w(X_1, \ldots, X_n)$, where $X_k = A_k$ if $k \in I$, and $X_k = B_k$ otherwise. On the other hand, whenever $a_i \in A_i$ for $i \in I$ and $b_l \in B_l$ for $l \in \{1, \ldots, n\} \setminus I$, the symbol $w_I(a_i; b_l)$ stands for the element $w(x_1, \ldots, x_n)$, where $x_k = a_k$ if $k \in I$, and $x_k = b_k$ otherwise.

Lemma 2.4. *Assume that G is a group and $A_1, \ldots, A_n, H$ are normal subgroups of G. Let $a_i \in A_i$ and $h_i \in H \cap A_i$ for every $i = 1, \ldots, n$. Let $j \in \{1, \ldots, n\}$ and set $I = \{1, \ldots, n\} \setminus \{j\}$. Then there exists an element*

$$x \in \mathcal{X}_w(a_1(H \cap A_1), \ldots, a_n(H \cap A_n))$$

such that

$$w(a_1h_1, \dots, a_nh_n) = x \cdot w_I(a_ih_i; h_j).$$

Proof. The proof is by induction on the number of variables n appearing in w. If $n = 1$ then $w(a_1h_1) = a_1h_1$ and the statement is trivially true.

So assume that $n \geq 2$ and let $w = [w_1, w_2]$ where w_1, w_1 are multilinear commutators in s and $n - s$ variables, respectively. Write

$$y = w(a_1h_1, \dots, a_nh_n) = [y_1, y_2]$$

where $y_1 = w_1(a_1h_1, \dots, a_sh_s)$, and $y_2 = w_2(a_{s+1}h_{s+1}, \dots, a_nh_n)$.

Assume also that $j > s$. Then by induction $y_2 = xh$, where $x \in \mathcal{X}_{w_2}(a_{s+1}(H \cap A_{s+1}), \dots, a_n(H \cap A_n))$ and

$$h = w_2(a_{s+1}h_{s+1}, \dots, a_{j-1}h_{j-1}, h_j, a_{j+1}h_{j+1}, \dots, a_nh_n).$$

So

$$y = [y_1, y_2] = [y_1, xh] = [y_1, h][y_1, x]^h = [y_1, x]^{h[y_1,h]^{-1}}[y_1, h].$$

Since $\tilde{h} = h[y_1, h]^{-1} \in H$ and $a_i \in A_i$, clearly $[a_i, \tilde{h}] \in H \cap A_i$ and so

$$(a_i\tilde{h}_i)^{\tilde{h}} = a_i[a_i, \tilde{h}]h_i^{\tilde{h}} \in a_i(H \cap A_i),$$

for every $\tilde{h}_i \in H \cap A_i$ and every i. As $x = w_2(a_{s+1}\tilde{h}_{s+1}, \dots, a_n\tilde{h}_n)$, for some $\tilde{h}_i \in H \cap A_i$, it follows that

$$\begin{aligned} [y_1, x]^{\tilde{h}} &= w(a_1h_1, \dots, a_sh_s, a_{s+1}\tilde{h}_{s+1}, \dots, a_n\tilde{h}_n)^{\tilde{h}} \\ &= w((a_1h_1)^{\tilde{h}}, \dots, (a_sh_s)^{\tilde{h}}, (a_{s+1}\tilde{h}_{s+1})^{\tilde{h}}, \dots, (a_n\tilde{h}_n)^{\tilde{h}}) \end{aligned}$$

belongs to $\mathcal{X}_w(a_1(H \cap A_1), \dots, a_n(H \cap A_n))$, as desired.

The case $1 \leq j \leq s$ is similar. By induction $y_1 = xh$, where

$$h = w_1(a_1h_1, \dots, a_{j-1}h_{j-1}, h_j, a_{j+1}h_{j+1}, \dots, a_sh_s)$$

and $x \in \mathcal{X}_{w_1}(a_1(H \cap A_1), \dots, a_s(H \cap A_s))$. So

$$y = [y_1, y_2] = [xh, y_2] = [x, y_2]^h[h, y_2].$$

Note that $h \in H$ and $a_i \in A_i$, therefore, as above, $(a_i\tilde{h}_i)^h \in a_i(A_i \cap H)$ for every $\tilde{h}_i \in H \cap A_i$ and every i. So

$$[x, y_2]^h \in \mathcal{X}_w(a_1(H \cap A_1), \dots, a_n(H \cap A_n))$$

and the result follows. □

Lemma 2.5. *Let $H, A_1, \dots, A_n$ be normal subgroups of a group G. Let V be a subgroup of G and $g \in G$. Assume that for some elements $a_i \in A_i$, the following holds:*

$$\mathcal{X}_w(a_1(H \cap A_1), \dots, a_n(H \cap A_n)) \subseteq gV.$$

Let I be a proper subset of $\{1, \ldots, n\}$. Then

$$w_I(a_i(H \cap A_i); H \cap A_l) \leq V.$$

PROOF. The proof is by induction on $n - |I|$, so first assume that $I = \{1, \ldots, n\} \setminus \{j\}$ for some index j.

We will write for short $H_i = H \cap A_i$, for every $i = 1, \ldots, n$.

Consider $w(g_1, \ldots, g_n)$, where $g_i \in a_i H_i$ for every $i \neq j$ and $g_j \in H_j$. By Lemma 2.4 we have

$$w(g_1, \ldots, g_{j-1}, a_j g_j, g_{j+1}, \ldots, g_n) = x w(g_1, \ldots, g_n),$$

for some $x \in \mathcal{X}_w(a_1 H_1, \ldots, a_n H_n) \subseteq gV$. As

$$w(g_1, \ldots, a_j g_j, \ldots, g_n) \in \mathcal{X}_w(a_1 H_1, \ldots, a_n H_n),$$

it follows that $w(g_1, \ldots, g_n) \in V$. Since V is subgroup, we deduce that $w_I(a_i H_i; H_l) \leq V$ and this concludes the case $|I| = n - 1$.

Now assume that $|I| \leq n - 2$ and let $I^* = I \cup \{j\}$ for some $j \notin I$. Consider $w(g_1, \ldots, g_n)$, where $g_i \in H_i$ for every $i \in I$ and $g_i \in a_i H_i$ for every $i \notin I$. Then the element $w(g_1, \ldots, g_{j-1}, a_j g_j, g_{j+1}, \ldots, g_n)$ belongs to $w_{I*}(a_i H_i; H_l)$. By Lemma 2.4 we have

$$w(g_1, \ldots, g_{j-1}, a_j g_j, g_{j+1}, \ldots, g_n) = x w(g_1, \ldots, g_n),$$

for some

$$x \in \mathcal{X}_w(g_1 H_1, \ldots, g_{j-1} H_{j-1}, a_j H_j, g_{j+1} H_{j+1}, \ldots, g_n H_n).$$

In particular $x \in w_{I*}(a_i H_i; H_l)$. Since, by induction, $w_{I*}(a_i H_i; H_l) \leq V$, it follows that $w(g_1, \ldots, g_n) \in V$, as we wanted. The proof is complete. □

By applying the previous lemma with $I = \emptyset$ and $A_i = G$ for each i, we obtain the following corollary.

COROLLARY 2.6. *Let G be a group, H and V subgroups of G, and $g \in G$. Assume that H is normal and*

$$\mathcal{X}_w(a_1 H, \ldots, a_n H) \subseteq gV$$

for some elements $a_1, \ldots, a_n \in G$. Then $w(H) \subseteq V$.

The next lemma is Lemma 4.1 in [**6**].

LEMMA 2.7. *Let $A_1, \ldots, A_n$ and H be normal subgroups of a group G. Let I be a subset of $\{1, \ldots, n\}$. Assume that for every proper subset J of I*

$$w_J(A_i; H \cap A_l) = 1.$$

Suppose we are given elements $g_i \in A_i$ with $i \in I$ and elements $h_k \in H \cap A_k$ with $k \in \{1, \dots, n\}$. Then we have

$$w_I(g_i h_i; h_l) = w_I(g_i; h_l).$$

We will now introduce some more notation to handle some particular properties of multilinear commutators. We denote by $\mathbf{I}$ the set of n-tuples $(i_1, \dots, i_n)$, where all entries i_k are non-negative integers. We will view $\mathbf{I}$ as a partially ordered set with the partial order given by the rule that

$$(i_1, \dots, i_n) \le (j_1, \dots, j_n)$$

if and only if $i_1 \le j_1, \dots, i_n \le j_n$.

Given $\mathbf{i} = (i_1, \dots, i_n) \in \mathbf{I}$, we write

$$w(\mathbf{i}) = w(G^{(i_1)}, \dots, G^{(i_n)})$$

for the subgroup generated by the w-values $w(g_1, \dots, g_n)$ with $g_j \in G^{(i_j)}$. Further, let

$$w(\mathbf{i}^+) = \prod w(\mathbf{j}),$$

where the product is taken over all $\mathbf{j} \in \mathbf{I}$ such that $\mathbf{j} > \mathbf{i}$.

Lemma 2.8. [**4**, Corollary 6] *Let $w = w(x_1, \dots, x_n)$ be a multilinear commutator and let $\mathbf{i} \in \mathbf{I}$. If $w(\mathbf{i}^+) = 1$, then $w(\mathbf{i})$ is abelian.*

The following lemma is Proposition 7 in [**4**].

Lemma 2.9. *Let $\mathbf{i} = (i_1, \dots, i_n) \in \mathbf{I}$ and suppose that $w(\mathbf{i}^+) = 1$. If $a_j \in G^{(i_j)}$ for $j = 1, \dots, n$, and $b_s \in G^{(i_s)}$ then*

$$w(a_1, \dots, a_{s-1}, b_s a_s, a_{s+1}, \dots, a_k)$$
$$= w(\tilde{a}_1, \dots, \tilde{a}_{s-1}, b_s, \tilde{a}_{s+1}, \dots, \tilde{a}_k) w(a_1, \dots, a_{s-1}, a_s, a_{s+1}, \dots, a_k),$$

where $\tilde{a}_j$ is a conjugate of a_j and moreover $\tilde{a}_j = a_j$ if $i_j \le i_s$.

Corollary 2.10. *Assume that $w(\mathbf{i}^+) = 1$ and let $a_j \in G^{(i_j)}$ for $j = 1, \dots, n$. Let l be an integer. Then $w(a_1, \dots, a_n)^l = w(b_1, \dots, b_n)$ for some $b_1, \dots, b_n$ with $b_j \in G^{(i_j)}$.*

Proof. Let i_s be maximal among all i_j's, with $j = 1, \dots, k$. Note that by Lemma 2.9 for every $a_j \in G^{(i_j)}$, where $j = 1, \dots, n$, and every $b_s \in G^{(i_s)}$ we have:

$$w(a_1, \dots, a_{s-1}, b_s a_s, a_{s+1}, \dots, a_k)$$
$$= w(a_1, \dots, a_{s-1}, b_s, a_{s+1}, \dots, a_k) w(a_1, \dots, a_{s-1}, a_s, a_{s+1}, \dots, a_k).$$

It follows that

$$w(a_1, \dots, a_{s-1}, a_s, a_{s+1}, \dots, a_k)^l = w(a_1, \dots, a_{s-1}, a_s^l, a_{s+1}, \dots, a_k)$$

for every integer l. This proves the result. □

Recall that an element of a group G is called an FC-element if it has only finitely many conjugates in G. The next result is Lemma 2.7 in [**7**].

LEMMA 2.11. *Let $G = \langle H, a_1, \ldots, a_s \rangle$ be a profinite group, where H is an open abelian normal subgroup and $a_1, \ldots, a_s$ are FC-elements. Then G' is finite.*

3. Proof of the main theorem

Recall that $\mathcal{C}$ is a class of groups closed under taking subgroups, quotients, and such that in any group the product of finitely many normal $\mathcal{C}$-subgroups is again a $\mathcal{C}$-subgroup.

Throughout this section we will work under the following hypothesis:

HYPOTHESIS 3.1. *Let $w = w(x_1, \ldots, x_n)$ be a multilinear commutator and let G be a profinite group in which the set of w-values is contained in a union of countably many cosets t_iG_i of subgroups G_i, where each $G_i \in \mathcal{C}$.*

LEMMA 3.2. *Assume Hypothesis 3.1. Then G contains an open normal subgroup H such that $w(H)$ is in $\mathcal{C}$.*

PROOF. For each positive integer i consider the set

$$S_i = \{(g_1, \ldots, g_n) \in G \times \cdots \times G \mid w(g_1, \ldots, g_n) \in t_iG_i\}.$$

Note that the sets S_i are closed in $G \times \cdots \times G$ and cover the whole group $G \times \cdots \times G$. By the Baire category theorem at least one of these sets has non-empty interior. Hence, there exist an open normal subgroup H of G, elements $a_1, \ldots a_n \in G$, and an integer j such that $w(a_1H, \ldots, a_nH) \subseteq t_jG_j$. By Corollary 2.6 we have $w(H) \leq G_j$, so the result follows. □

LEMMA 3.3. *Assume Hypothesis 3.1 and let $a \in G$ be a w-value. There exists a normal open subgroup H_a in G such that $[H_a, a]$ is in $\mathcal{C}$.*

PROOF. For each positive integer i let

$$S_i = \{x \in G \mid a^x \in t_iG_i\}.$$

Note that the sets S_i are closed in G and cover the whole group G. By the Baire category theorem at least one of these sets has non-empty interior. Hence, there exist an open normal subgroup H of G, an element $b \in G$, and an integer j such that $a^{hb} \in t_jG_j$ for any $h \in H$. Of course we can assume that $t_j = a^b$, so that $a^{-b}a^{hb} \in G_j$ for every $h \in H$. Thus $a^{-1}a^h \in G_j^b$ for every $h \in H$. Hence, $[a, H] = [H, a] \leq G_j^b$ is in $\mathcal{C}$. □

Recall that $G^{(i)}$ denotes the i-th term of the derived series of a group G.

Proposition 3.4. *Assume Hypothesis 3.1. Then $G^{(2n)}$ is in $\mathcal{CF}$.*

Proof. By Lemma 3.2 there exists an open normal subgroup H such that $w(H)$ is in $\mathcal{C}$. Lemma 2.3 implies that $H^{(n)}$ is in $\mathcal{C}$. Let $K = G^{(n)}$ and $L = K \cap H$. Note that L is open in K. Choose a finite set of δ_n-values $a_1, \ldots, a_s$ such that $K = \langle L, a_1, \ldots, a_s\rangle$ and let $H_{a_1}, \ldots, H_{a_s}$ be normal open subgroups of G such that $[H_{a_j}, a_j]$ is in $\mathcal{C}$ for every j (see Lemma 3.3). Note that for each j the subgroup $[H_{a_j}, a_j]$ is a normal subgroup of H_{a_j} so $\langle [H_{a_j}, a_j]^G\rangle$ is in $\mathcal{C}$. Let $N_1 \le K$ be the subgroup generated by $L^{(n)}$ and the subgroups $\langle [H_{a_j}, a_j]^G\rangle$ for $j = 1, \ldots, s$. Note that N_1 is in $\mathcal{C}$. The images of $a_1, \ldots, a_s$ in the quotient G/N_1 are FC-elements while the image of L in G/L' is abelian. Therefore by Lemma 2.11 the group $KN_1/L'N_1$ has finite derived group. In other words $L'N_1$ has finite index in $K'N_1$. In particular there exist finitely many δ_n-values $b_1, \ldots, b_t$ such that $K'N_1 = \langle L', b_1, \ldots, b_t, N_1\rangle$.

As above, there exist normal open subgroups $H_{b_1}, \ldots, H_{b_t}$ of G such that $\langle [H_{b_j}, b_j]^G\rangle$ is in $\mathcal{C}$ for every j. Let N_2 be the subgroup generated by N_1 and the subgroups $\langle [H_{b_j}, b_j]^G\rangle$ for $j = 1, \ldots, t$. Note that N_2 is in $\mathcal{C}$. Again, $b_1N_2, \ldots, b_tN_2$ are FC-elements in G/N_2 and arguing as before we obtain that $L^{(2)}N_2$ has finite index in $K^{(2)}N_2$. By iterating this argument we get that $L^{(n)}N_n$ has finite index in $K^{(n)}N_n$ for some normal $\mathcal{C}$-subgroup N_n, so $L^{(n)}(K^{(n)} \cap N_n)$ has finite index in $K^{(n)} = G^{(2n)}$. As $L^{(n)} \le H^{(n)}$ is in $\mathcal{C}$ it follows that $G^{(2n)}$ is in $\mathcal{CF}$, as desired. □

Recall the notation introduced in Section 2: whenever I is a subset of $\{1, \ldots, n\}$ and $A_{i_1}, \ldots, A_{i_s}$ and $B_{l_1}, \ldots, B_{l_t}$ are families of subsets of G with indices running over I and $\{1, \ldots, n\} \setminus I$, respectively, we write

$$w_I(A_i; B_l)$$

for the subgroup $w(X_1, \ldots, X_n)$, where $X_k = A_k$ if $k \in I$, and $X_k = B_k$ otherwise. Moreover, whenever $a_i \in A_i$ for $i \in I$ and $b_l \in B_l$ for $l \in \{1, \ldots, n\}\setminus I$, the symbol $w_I(a_i; b_l)$ stands for the element $w(x_1, \ldots, x_n)$, where $x_k = a_k$ if $k \in I$, and $x_k = b_k$ otherwise.

Furthermore, given $\mathbf{i} = (i_1, \ldots, i_n) \in \mathbf{I}$, we write

$$w(\mathbf{i}) = w(G^{(i_1)}, \ldots, G^{(i_n)})$$

for the subgroup generated by the w-values $w(g_1, \ldots, g_n)$ with $g_j \in G^{(i_j)}$ and we set $w(\mathbf{i}^+) = \prod w(\mathbf{j})$, where the product is taken over all $\mathbf{j} \in \mathbf{I}$ such that $\mathbf{j} > \mathbf{i}$.

Lemma 3.5. *Assume Hypothesis 3.1. Let $A_1, \dots, A_n$ be normal subgroups of G and let I be a proper subset of $\{1, \dots, n\}$. Assume that there exist a normal $\mathcal{CF}$-subgroup T of G and an open normal subgroup H such that:*

(*) $w_J(A_i; H \cap A_l) \leq T$ *for every proper subset J of I.*

Then for any given set of elements $\{g_i\}_{i \in I}$, where $g_i \in A_i$, there exist an open normal subgroup U of G, contained in H, and a normal $\mathcal{CF}$-subgroup N of G, containing T, such that

$$w_I(g_i; U \cap A_l) \leq N.$$

Proof. Consider the sets

$$S_j = \{(h_1, \dots, h_n) \mid h_k \in H \cap A_k \text{ and } w_I(g_i h_i; h_l) \in g_j G_j\}.$$

Note that the sets S_j are closed in the group $(H \cap A_1) \times \cdots \times (H \cap A_n)$ and cover the whole group. By the Baire category theorem at least one of these sets has non-empty interior. Hence, there exist an integer r, open subgroups V_k of $H \cap A_k$, and elements $b_k \in H \cap A_k$ for every $k = 1, \dots, n$ such that

$$w_I(g_i b_i v_i; b_l v_l) \in t_r G_r,$$

for every $v_i \in V_i$. Each subgroup V_k is of the form $V_k = U_k \cap H \cap A_k$ where U_k is an open subgroup of G and we can assume that U_k is normal in G. Let $U = U_1 \cap \cdots \cap U_n \cap H$. Note that U is an open normal subgroup of G contained in H. Thus

$$w_I(g_i b_i u_i; b_l u_l) \in t_r G_r,$$

for every $u_i \in U \cap A_i$. Now we apply Lemma 2.5 to pass from the cosets $b_l(U \cap A_l)$ to the subgroups $U \cap A_l$, for every $l \notin I$. It follows from Lemma 2.5 that the subgroup

$$K = w_I(g_i b_i (U \cap A_i); U \cap A_l)$$

is contained in G_r and so it is in $\mathcal{CF}$. Note that $K \leq U$. Since U has finite index in G and normalizes K, by Lemma 2.1, $\langle K^G \rangle$ is in $\mathcal{C}$.

Set $N = T\langle K^G \rangle$ and note that $N \in \mathcal{C}$. Using (*) and the fact that $T \leq N$ and $b_i(U \cap A_i) \subseteq H \cap A_i$, we can apply Lemma 2.7 to the group G/N. Therefore

$$w_I(g_i; U \cap A_l)N = w_I(g_i b_i (U \cap A_i); U \cap A_l)N.$$

Since $w_I(g_i b_i (U \cap A_i); U \cap A_l) \leq N$, we deduce that

$$w_I(g_i; U \cap A_l) \leq N,$$

as desired. ∎

LEMMA 3.6. *Assume Hypothesis 3.1. Let* $A_1, \dots, A_n$ *be normal subgroups of* G *and let* I *be a proper subset of* $\{1, \dots, n\}$*. Assume that there exist a normal* $\mathcal{CF}$*-subgroup* T *of* G *and an open normal subgroup* H *such that:*

(*) $w_J(A_i; H \cap A_l) \le T$ *for every proper subset* J *of* I*.*

Then there exist an open normal subgroup U *of* G*, contained in* H*, and a normal* $\mathcal{CF}$*-subgroup* N *of* G*, containing* T*, such that*

$$w_I(A_i; U \cap A_l) \le N.$$

PROOF. For each $i \in I$ choose a set R_i of coset representatives of $H \cap A_i$ in A_i. Note that all those sets are finite. We apply Lemma 3.5 to each choice of elements $\bar{g} = \{g_i\}_{i \in I}$, with $g_i \in R_i$: let $U_{\bar{g}}$ and $N_{\bar{g}}$ be normal subgroups of G such that $w_I(g_i; U_{\bar{g}} \cap A_l) \le N_{\bar{g}}$. The existence of the subgroups $U_{\bar{g}}$ and $N_{\bar{g}}$ is guaranteed by Lemma 3.5. Remark that there are only a finitely many subgroups $U_{\bar{g}}$ and $N_{\bar{g}}$. Then $U = \cap_{\bar{g}} U_{\bar{g}}$ is a normal open subgroup of G contained in H and $N = \prod_{\bar{g}} N_{\bar{g}}$ is a normal $\mathcal{CF}$-subgroup containing T, such that

$$w_I(g_i; U \cap A_l) \le N$$

for every choice of $g_i \in R_i$. Note that, by condition (*) and Lemma 2.7,

$$w_I(g_i(H \cap A_i); U \cap A_l) = w_I(g_i; U \cap A_l) \le N.$$

Since $A_i = \cup_{g_i \in R_i} g_i(H \cap A_i)$ for every $i \in I$, we conclude that

$$w_I(A_i; U \cap A_l) = \langle \cup_{\bar{g}} w_I(g_i(H \cap A_i); U \cap A_l) \rangle \le N,$$

as desired. □

LEMMA 3.7. *Assume Hypothesis 3.1. Assume that there exist an* n*-tuple* $\mathbf{i} \in \mathbf{I}$*, a normal* $\mathcal{CF}$*-subgroup* T *of* G *and an open normal subgroup* H *such that:*

- $w(\mathbf{i}^+) \le T$.
- $w(H) \le T$.

Then $w(\mathbf{i})$ *is in* $\mathcal{CF}$*.*

PROOF. Let $\mathbf{i} = (i_1, \dots, i_n)$. We will write for short

$$A_j = G^{(i_j)},$$

for every $j = 1, \dots, n$. It is enough to prove the following statement: for every subset I of $\{1, \dots, n\}$, there exist an open normal subgroup U_I of G contained in H and a normal $\mathcal{CF}$-subgroup N_I containing T such that $w_I(A_i; U_I \cap A_l) \le N_I$.

The proof is by induction on the size k of I. If $k = 0$, then $I = \emptyset$ and

$$w_{\emptyset}(A_i; H \cap A_i) = w(H \cap A_1, \dots, H \cap A_n) \leq w(H) \leq T.$$

So assume $k > 0$. Let $J_1, \dots J_s$ be all the proper subsets of I. By induction, for each $t = 1, \dots, s$ there exist an open normal subgroup U_t of G contained in H and a normal $\mathcal{CF}$-subgroup N_t containing T such that $w_{J_t}(A_i; U_t \cap A_l) \leq N_t$. Let $U = \cap_t U_t$ and $N = \langle N_t | t = 1, \dots, s \rangle$. Then

$$w_J(A_i; U \cap A_l) \leq N$$

for every proper subset J of I.

If $k \neq n$ we can apply Lemma 3.6 to I. We obtain that there exist an open normal subgroup U_I of G contained in H and a normal $\mathcal{CF}$-subgroup N_I containing T such that $w_I(A_i; U_I \cap A_l) \leq N_I$, as desired.

So we are left with the case when $k = n$, and thus, by definition, $w(A_1, \dots, A_n) = w(\mathbf{i})$.

For each $i \in I$ choose a set R_i of coset representatives of $H \cap A_i$ in A_i. Note that all those sets are finite. We pass to the quotient $\bar{G} = G/N$. By Lemma 2.7 for each choice of elements $\bar{g}_1, \dots, \bar{g}_n$ with $\bar{g}_i \in \bar{R}_i$ and for each $\bar{h}_1, \dots, \bar{h}_n \in \bar{U} \cap \bar{A}_i$, we have

$$w(\bar{g}_1 \bar{h}_1, \dots, \bar{g}_n \bar{h}_n) = w(\bar{g}_1, \dots, \bar{g}_n).$$

So the set

$$\mathcal{X}_w(\bar{A}_1, \dots, \bar{A}_n)$$

is finite.

By Lemma 2.10 every power of an element in $\mathcal{X}_w(\bar{A}_1, \dots, \bar{A}_n)$ is again in $\mathcal{X}_w(\bar{A}_1, \dots, \bar{A}_n)$. So every element in $\mathcal{X}_w(\bar{A}_1, \dots, \bar{A}_n)$ has finite order. Therefore $w(\bar{A}_1, \dots, \bar{A}_n)$ is generated by finitely many elements of finite order, and being abelian by Lemma 2.8, it is actually finite. It follows that $w(A_1, \dots, A_n)$ is in $\mathcal{CF}$, as desired. □

We are now ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1 Obviously, if $w(G)$ is in $\mathcal{CF}$ then the set of w-values in G is covered by countably many cosets of $\mathcal{C}$-subgroups. Therefore we only need to show that if the set of w-values is covered by countably many $\mathcal{C}$-subgroups then $w(G)$ is in $\mathcal{CF}$.

Thus, assume that the set of w-values in G is covered by countably many $\mathcal{C}$-subgroups. Proposition 3.4 states that $G^{(2n)}$ is in $\mathcal{CF}$.

Let H be as in Lemma 3.2. Then $w(H)$ is in $\mathcal{C}$. Let $T = G^{(2n)} w(H)$. Then T is in $\mathcal{CF}$ by Lemma 2.1. Since $G^{(2n)} \leq T$ it follows that G/T is soluble.

Thus there exist only finitely many $\mathbf{i} \in \mathbf{I}$ such that $w(\mathbf{i})T/T \neq 1$. By induction on the number of such n-tuples $\mathbf{i}$, we will prove that every subgroup $w(\mathbf{i})$ is in $\mathcal{CF}$.

Choose $\mathbf{i} = (i_1, \dots, i_n) \in \mathbf{I}$ such that $w(\mathbf{i})T/T \neq 1$ while $w(\mathbf{j})T/T = 1$ whenever $\mathbf{i} < \mathbf{j}$. Now we apply Lemma 3.7 and we obtain that $w(\mathbf{i})$ is in $\mathcal{CF}$. Let $N = w(\mathbf{i})T$. Then induction on the number of $\mathbf{j} \in \mathbf{I}$ such that $w(\mathbf{j}) \not\leq N$ leads us to the conclusion that $w(G)$ is in $\mathcal{CF}$. □

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