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Weyl Solutions and J-selfadjointness for Dirac Operators

B. Malcolm Brown

Cardiff School of Computer Science and Informatics, Cardiff University, Queen's Buildings, 5 The Parade, Cardiff CF24 3AA, UK
Malcolm.Brown@cs.cardiff.ac.uk

Martin Klaus

Mathematics Department, Virginia Tech, Blacksburg, VA 24061-0123
klaus@math.vt.edu

Mark Malamud

Peoples' Friendship University of Russia (RUDN University), 6 Miklukho-Maklaya St, Moscow, 117198, Russian Federation, malamud3m@gmail.com

Vadim Mogilevskii

Poltava V.G. Korolenko National Pedagogical University, Ostrogradski Street 2, 36000 Poltava, Ukraine
vadim.mogilevskii@gmail.com

Ian Wood

School of Mathematics, Statistics and Actuarial Science, University of Kent, Sibson Building, Canterbury CT2 7FS, UK
i.wood@kent.ac.uk

Abstract

We consider a non-selfadjoint Dirac-type differential expression

$$D(Q)y := J_n \frac{dy}{dx} + Q(x)y, \qquad (0.1)$$

with a non-selfadjoint potential matrix $Q \in L^1_{\mathrm{loc}}(\mathcal{I}, \mathbb{C}^{n \times n})$ and a signature matrix $J_n = -J_n^{-1} = -J_n^* \in \mathbb{C}^{n \times n}$. Here $\mathcal{I}$ denotes either the line $\mathbb{R}$ or the half-line $\mathbb{R}_+$. With this differential expression one associates in $L^2(\mathcal{I}, \mathbb{C}^n)$ the (closed) maximal and minimal operators $D_{\max}(Q)$ and $D_{\min}(Q)$, respectively. One of our main results for the whole line case states that $D_{\max}(Q) = D_{\min}(Q)$ in $L^2(\mathbb{R}, \mathbb{C}^n)$. Moreover, we show that if the minimal operator $D_{\min}(Q)$ in $L^2(\mathbb{R}, \mathbb{C}^n)$ is j-symmetric with respect to an appropriate involution j, then it is j-selfadjoint.

Similar results are valid in the case of the semiaxis $\mathbb{R}_+$. In particular, we show that if $n = 2p$ and the minimal operator $D_{\min}^+(Q)$ in $L^2(\mathbb{R}_+, \mathbb{C}^{2p})$ is

j-symmetric, then there exists a $2p \times p$-Weyl-type matrix solution $\Psi(z, \cdot) \in L^2(\mathbb{R}_+, \mathbb{C}^{2p \times p})$ of the equation $D^+_{\max}(Q)\Psi(z, \cdot) = z\Psi(z, \cdot)$. A similar result is valid for general expression (0.1) whenever there exists a proper extension $\widetilde{A}$ with $\dim(\operatorname{dom} \widetilde{A}/\operatorname{dom} D^+_{\min}(Q)) = p$ and having nonempty resolvent set $\rho(\widetilde{A}) \neq \emptyset$. In particular, it holds if a potential matrix Q has a bounded imaginary part. This leads to the existence of a unique Weyl function for the expression (0.1). The main results are proven by means of a reduction to the self-adjoint case by using the technique of dual pairs of operators.

The differential expression (0.1) is of significance as it appears in the Lax formulation of the vector-valued nonlinear Schrödinger equation.

Keywords: Dirac-type operator, j-selfadjointness, Weyl solution, Weyl function, dual pair of operators
2010 MSC: Primary 34B20, 34L10; Secondary 47B25.

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1. Introduction

Let $\mathcal{I}$ be either the half-line $\mathbb{R}_+ = [0, \infty)$ or the whole line $\mathbb{R}$. The main object of the paper is a study of the general (not necessarily formally self-adjoint) Dirac-type differential expression

$$D(Q)y := J_n \frac{dy}{dx} + Q(x)y, \qquad (1.1)$$

with a potential matrix $Q \in L^1_{\mathrm{loc}}(\mathcal{I}, \mathbb{C}^{n \times n})$ and a signature matrix $J_n = -J_n^{-1} = -J_n^* \in \mathbb{C}^{n \times n}$. One may associate with this differential expression in $L^2(\mathcal{I}, \mathbb{C}^n)$ the (closed) maximal and minimal operators $D_{\max}(Q)$ and $D_{\min}(Q)$, respectively. The maximal operator $D_{\max}(Q)$ is defined by

$$\operatorname{dom} D_{\max}(Q) = \{y \in L^2(\mathcal{I}, \mathbb{C}^n) : y \in AC_{\mathrm{loc}}(\mathcal{I}, \mathbb{C}^n), \ D(Q)y \in L^2(\mathcal{I}, \mathbb{C}^n)\},$$

$$D_{\max}(Q)y = D(Q)y, \quad y \in \operatorname{dom} D_{\max}(Q).$$

The minimal operator $D_{\min}(Q)$ is the closure of the operator $D'_{\min}(Q)$, where $D'_{\min}(Q)$ is the restriction of $D_{\max}(Q)$ to the set of functions $y \in \operatorname{dom} D_{\max}(Q)$ with compact support (in the case of the half-line we mean that $\operatorname{supp} y \subset (0, \infty)$). Clearly, both the operators $D_{\max}(Q)$ and $D_{\min}(Q)$ are densely defined.

Assume that $\mathcal{I} = \mathbb{R}$ and that the expression (1.1) is formally selfadjoint, that is $Q(x) = Q^*(x)$ (a.e. on $\mathbb{R}$). Then according to [23, Sect. 8.6] and [22, Theorem 3.2] we have $D_{\min}(Q) = D_{\max}(Q)(= D_{\min}(Q)^*)$, i.e., the minimal operator is self-adjoint and coincides with the maximal one.

To distinguish between operators acting on the half-line $\mathbb{R}_+$ from those acting on the line $\mathbb{R}$, we write $D^+_{\max}(Q)$ and $D^+_{\min}(Q)$ instead of $D_{\max}(Q)$ and $D_{\min}(Q)$, respectively. With this notation, in the case of the half-line the analogue of our previous result reads as follows: the minimal operator $D^+_{\min}(Q)$ is in the limit-point case at infinity, i.e. the deficiency indices $n_\pm(D^+_{\min}(Q))$ of $D^+_{\min}(Q)$ are the minimal possible:

$$n_\pm(D^+_{\min}(Q)) = \dim \ker(J_n \pm iI) =: \kappa_\pm(J_n) =: \kappa_\pm, \qquad (1.2)$$

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(see [22, Theorem 5.2]).

We now reformulate this statement in terms of a subspace $\Theta \subseteq \mathbb{C}^n$ and define the operators $D_\Theta^+(Q)$ and $(D_\Theta^+)'(Q)$ to be the restrictions of $D_{\max}^+(Q)$ to the domains

$$\operatorname{dom} D_\Theta^+(Q) = \{y \in \operatorname{dom} D_{\max}^+(Q) : y(0) \in \Theta\},$$

and

$$\operatorname{dom} (D_\Theta^+)'(Q) = \{y \in \operatorname{dom} D_\Theta^+(Q) : \ \operatorname{supp} y \subset [0, \infty) \ \text{and is compact}\},$$

respectively. Further we denote by $D_{\Theta,0}^+(Q)$ the closure of $(D_\Theta^+)'(Q)$.

With this notation one easily gets that the minimality condition (1.2) is equivalent to

$$D_{\Theta,0}^+(Q) = D_\Theta^+(Q) \quad \text{for any subspace} \quad \Theta \subseteq \mathbb{C}^n, \tag{1.3}$$

i.e. the coincidence of the minimal and maximal operators.

In the present paper we study a general Dirac-type expression (1.1) with arbitrary complex-valued $Q \in L^1_{\mathrm{loc}}(\mathcal{I}, \mathbb{C}^{n \times n})$ via the abstract setting of dual pairs and j-selfadjoint operators. One of our main results reads as follows.

Theorem 1.1. *For a general Dirac-type expression* (1.1) *on* $\mathbb{R}$ *with potential* $Q \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^{n \times n})$ *the following equality holds:*

$$D_{\min}(Q) = D_{\max}(Q). \tag{1.4}$$

In other words equality (1.4) means that the functions from $\operatorname{dom} D_{\max}(Q)$ with compact support are dense in $\operatorname{dom} D_{\max}(Q)$ equipped with the graph norm.

By using Theorem 1.1 we obtain a criterion for j-selfadjointness of the minimal operator $D_{\min}(Q)$ in $L^2(\mathbb{R}, \mathbb{C}^n)$ with an appropriate conjugation j (see Theorem 3.2). The latter result generalizes results by Gesztesy, Cascaval, and Clark from [6, Theorem 4] and [10] (see also [7, 9]). Namely, assuming that $n = 2p$ they proved j-selfadjointness of the j-symmetric Dirac operator $D_{\min}(Q)$ for the particular case when

$$J_n = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}, \quad Q = -i \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} \quad \text{and} \quad V = V^T \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^{p \times p}). \tag{1.5}$$

Note in this connection that such Dirac-type operators naturally appear in the Lax representation of the known $p \times p$ matrix valued AKNS systems (the focusing NLS equation).

Moreover our above result, i.e. relation (1.4), covers the so-called vector NLS equation mentioned as an unsolved problem at the end of [6], when

$$V = \begin{pmatrix} q_1 & 0 & \dots & 0 \\ q_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ q_n & 0 & \dots & 0 \end{pmatrix}, \qquad q_j \in L^1_{\mathrm{loc}}(\mathbb{R}), \quad j \in \{1, \dots, n\}. \tag{1.6}$$

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It naturally appears in the Lax representation of the vector NLS equation (cf. [8]),

$$i\mathbf{q}_t + \frac{1}{2}\mathbf{q}_{xx} + \|\mathbf{q}\|^2\mathbf{q} = 0, \quad \mathbf{q} = (q_1, \ldots, q_n)^\top, \quad \|\mathbf{q}\|^2 = \mathbf{q}^*\mathbf{q} = \sum |q_j|^2.$$

Turning now to the problem on the semiaxis $\mathbb{R}_+$ we first note that the counterpart of relation (1.3) remains valid in the non-selfadjoint case, i.e. $D_{\Theta,\min}^+ = D_{\Theta,\max}^+$ (see Theorem 3.7). This result is applied to prove the following counterpart on the deficiency indices (see (1.2)) of $D_{\min}^+$ and existence of the Weyl-type matrix solution to the homogeneous equation

$$D(Q)y = J_n\frac{dy}{dx} + Q(x)y = zy, \quad x \in \mathbb{R}_+. \tag{1.7}$$

Theorem 1.2. *Assume that $n = 2p$. Then:*

(i) a proper extension $\widetilde{A} = D_\Theta(Q) \in \mathrm{Ext}\,\{D_{\min}^+(Q), D_{\min}^+(Q^*)\}$ is quasi-selfadjoint, i.e.

$$\dim\left(\mathrm{dom}\,\widetilde{A}/\mathrm{dom}\,D_{\min}^+(Q)\right) = \dim\left(\mathrm{dom}\,\widetilde{A}^*/\mathrm{dom}\,D_{\min}^+(Q^*)\right)\ (=p), \tag{1.8}$$

if and only if $\dim\Theta = p$;

(ii) If the resolvent set of some quasi-selfadjoint extension $\widetilde{A} := D_\Theta^+(Q)$ is nonempty, $\rho(\widetilde{A}) \neq \emptyset$, then

$$\dim\ker\left(D_{\max}^+(Q) - z\right) = \dim\ker\left(D_{\max}^+(Q^*) - \overline{z}\right) = p \quad \text{for any} \quad z \in \rho(\widetilde{A}). \tag{1.9}$$

Moreover, there exists a $2p \times p$-Weyl-type matrix solution of equation (1.7), $\Psi_+(z,\cdot) \in L^2(\mathbb{R}_+, \mathbb{C}^{2p\times p})$, which can be chosen to be holomorphic in $z \in \rho(\widetilde{A})$.

In the case of j-symmetric Dirac operator this result can be improved.

Theorem 1.3. *Let the minimal Dirac operator $D_{\min}^+(Q)$ in $L^2(\mathbb{R}_+, \mathbb{C}^{2p})$ be j-symmetric with respect to an appropriate conjugation j. Assume in addition that the field of regularity $\widehat{\rho}(D_{\min}^+(Q)) \neq \emptyset$. Then*

$$Def\,D_{\min}^+(Q) := \dim\ker\left(D_{\max}^+(Q) - z\right) = p \quad \text{for any} \quad z \in \widehat{\rho}(D_{\min}^+(Q)).$$

Moreover, for any $z \in \widehat{\rho}(D_{\min}^+(Q))$ there exists a $2p \times p$-Weyl-type matrix solution $\Psi_+(z,\cdot) \in L^2(\mathbb{R}_+, \mathbb{C}^{2p\times p})$ of equation (1.7) which can be chosen to be holomorphic in z.

This theorem generalizes the corresponding result from [7, Theorem 5.4] (see also [6] and [10]) where it is proved for $p = 1$ by another method.

Moreover, assuming that the imaginary part $\mathrm{Im}\,Q$, of the potential matrix Q is bounded, i.e. $\alpha_Q \leq \mathrm{Im}\,Q(x) \leq \beta_Q$, $x \in \mathbb{R}_+$, we show (see Proposition 4.11) that

$$\dim\mathfrak{N}_\lambda(D_{\min}^+(Q^*)) = \kappa_+, \qquad \dim\mathfrak{N}_{\overline{\lambda}}(D_{\min}^+(Q)) = \kappa_-, \qquad \mathrm{Im}\,\lambda > \beta_Q,$$

$$\dim \mathfrak{N}_\lambda(D^+_{\min}(Q^*)) = \kappa_-, \qquad \dim \mathfrak{N}_{\overline{\lambda}}(D^+_{\min}(Q)) = \kappa_+, \qquad \operatorname{Im}\lambda < \alpha_Q. \quad (1.10)$$

Here $\mathfrak{N}_\lambda(D^+_{\min}(Q^*)) = \ker\left((D^+_{\max}(Q) - \lambda\right)$. This result is a counterpart of the minimality conditions (1.2) in the non-selfadjoint case.

In the case $n = 2p$ and $\kappa_\pm = p$ this result implies the existence of the $2p \times p$ –Weyl-Titchmarsh-type matrix solution $\Psi(z, \cdot) \in L^2(\mathbb{R}_+, \mathbb{C}^{2p \times p})$ of the equation $D^+_{\max}(Q)\Psi(z, \cdot) = z\Psi(z, \cdot)$ which complements Theorem 1.3.

In turn, we apply this result to prove the uniqueness of the Weyl function for the Dirac operator on the semiaxis (see Proposition 4.15). This result complements a recent result from [15] where it is proved for a skew-symmetric potential matrix Q by extending the classical Weyl limit point – limit circle procedure.

The paper is organized as follows. Section 2 introduces the abstract setting that we will use: dual pairs of operators A and B and their proper extensions. Dual pairs of operators and their extensions have been studied by many authors, see e.g. [13, 17] for the symmetric case and [3, 4, 5, 24, 26, 27, 28] for the non-symmetric case. The main idea of this paper consists of reducing the problem to the study of the properties of the symmetric operator S given by

$$S = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}. \quad (1.11)$$

This allows us to relate the deficiency indices of S to those of the dual pair and to study j-symmetric operators A and their extensions via S. In particular, we prove here Proposition 2.3 containing an analog of the von Neumann formulas.

In turn, using this result we obtain the results of Race [30] and Zhikhar [34] by treating extensions of a densely defined closed j-symmetric operator A as proper extensions of the dual pair $\{A, jAj\}$.

In Section 3 we apply the abstract results to non-selfadjoint Dirac-type operators in $L^2(\mathbb{R}, \mathbb{C}^{2n})$ to prove Theorem 1.1, as well as j-selfadjointness of the j-symmetric minimal Dirac-type operator $D_{\min}(Q)$. Moreover, we prove a counterpart of Theorem 1.1 for $L^2(\mathbb{R}_+, \mathbb{C}^{2n})$ (see Theorem 3.7) and the results of Theorem 1.2.

In Section 4.1 we introduce a boundary triple for a dual pair of minimal Dirac operators $\{D^+_{\min}(Q), D^+_{\min}(Q^*)\}$ and investigate the corresponding Weyl function. In Section 4.2 we prove Theorem 1.3 on existence of the Weyl $2p \times p$-matrix solution for j-symmetric operators.

In Section 4.3 we investigate the Dirac-type operator on the half-line assuming a potential matrix Q with a bounded imaginary part. In particular, we prove the equalities (1.10) for such operators, meaning the minimality of deficiency indices in respective half-planes. Moreover, we extract from this property the uniqueness of the Weyl function for Dirac-type operators with J_n satisfying $\kappa_\pm = p$ (see Proposition 4.15). This result complements the results from [15] (see also [31]) and Proposition 4.11. We also obtain certain additional properties of the Weyl function for such Dirac-type operators. In this connection we also mention the paper [18] where certain properties of Dirac-type operators were investigated in the framework of boundary triples and the corresponding Weyl functions.

2. Dual pairs and j-symmetric operators

In this section we introduce several abstract results for dual pairs of operators. These will be used to study specific problems for Dirac operators in the rest of the paper.

2.1. Dual pairs and symmetric operators.

Definition 2.1. *(i) A pair of linear operators A and B in a Hilbert space $\mathfrak{H}$ forms a dual pair (or adjoint pair) $\{A, B\}$ if*

$$(Af, g) = (f, Bg), \qquad f \in \operatorname{dom} A, \qquad g \in \operatorname{dom} B. \qquad (2.1)$$

If A and B are closable densely defined operators, then (2.1) is equivalent to each of the following relations

$$A \subseteq B^* \quad and \quad B \subseteq A^*. \qquad (2.2)$$

(ii) A densely defined operator $\widetilde{A}$ is called a proper extension of a dual pair $\{A, B\}$ if $A \subset \widetilde{A} \subset B^*$. The set of all proper extensions of a dual pair $\{A, B\}$ is denoted by $\operatorname{Ext} \{A, B\}$.

Our interest is in studying extensions in $\operatorname{Ext} \{A, B\}$. As a first step, we will investigate the relation between $\operatorname{dom} A$ and $\operatorname{dom} B^*$ and $\operatorname{dom} B$ and $\operatorname{dom} A^*$, respectively.

To any closable densely defined operator A we associate a Hilbert space $\mathfrak{H}_{+A}$ by equipping $\operatorname{dom} A^*$ with an inner product

$$(f, g)_{+A} = (f, g) + (A^* f, A^* g) \qquad f, g \in \operatorname{dom} A^*. \qquad (2.3)$$

By $\oplus_A$ we denote the orthogonal sum in $\mathfrak{H}_{+A}$. For simplicity of notation, when no confusion is possible, we will omit specifying the operator and let $\mathfrak{H}_+ := \mathfrak{H}_{+A}$ with inner product $(f, g)_+ := (f, g)_{+A}$.

Remark 2.2. *Let $\{A, B\}$ be a dual pair. Then, clearly $\operatorname{dom} B$ is a (closed) subspace of $\mathfrak{H}_{+A}$ if and only if B is closed.*

Proposition 2.3. *Let $\{A, B\}$ be a dual pair of densely defined closed operators in $\mathfrak{H}$. Then the following orthogonal decompositions hold*

$$\operatorname{dom} A^* = \operatorname{dom} B \oplus_A \ker (I + B^* A^*), \qquad (2.4)$$

$$\operatorname{dom} B^* = \operatorname{dom} A \oplus_B \ker (I + A^* B^*). \qquad (2.5)$$

In particular, $A^* = B$ (or, equivalently, $A = B^*$) if and only if

$$\ker (I + B^* A^*) = \{0\} \ \big(\text{or, equivalently, } \ker (I + A^* B^*) = \{0\}\big). \qquad (2.6)$$

Remark 2.4. *As in this proposition, throughout the paper, many results will consist of two statements, the second one being the 'adjoint' of the first. In these cases, we will usually omit the proof of the second statement, as it will follow analogously to the first.*

Proof: By Remark 2.2, $\text{dom } B$ is a closed subspace of $\mathfrak{H}_+$. Let g be orthogonal to $\text{dom } B$ in $\mathfrak{H}_+$. Then for any $f \in \text{dom } B$

$$0 = (f, g)_+ = (A^*f, A^*g) + (f, g) = (Bf, A^*g) + (f, g). \tag{2.7}$$

Hence, $A^*g \in \text{dom } B^*$ and $B^*A^*g = -g$, or $g \in \ker (I + B^*A^*)$. Conversely, if $g \in \ker (I + B^*A^*)$, then by (2.7) we have $g \perp \text{dom } B$. Thus (2.4) is valid. $\square$

Corollary 2.5. *Let $\{A, B\}$ be a dual pair of densely defined operators in $\mathfrak{H}$. Define*

$$n(A^*, B) := \dim (\text{dom } A^*/\text{dom } B) \quad and \quad n(B^*, A) := \dim (\text{dom } B^*/\text{dom } A). \tag{2.8}$$

Then

$$n(A^*, B) = n(B^*, A). \tag{2.9}$$

Proof: The simple equalities

$$A^*(I + B^*A^*) = (I + A^*B^*)A^* \quad \text{and} \quad B^*(I + A^*B^*) = (I + B^*A^*)B^*,$$

together with the fact that $A^*f \neq 0$ for $f \in \ker (I + B^*A^*)$ and $B^*g \neq 0$ for $g \in \ker (I + A^*B^*)$ imply that

$$\dim \ker (I + B^*A^*) = \dim \ker (I + A^*B^*). \tag{2.10}$$

By combining this with (2.4) and (2.5) we get the required result. $\square$

Formulas (2.4) and (2.5) may be considered as generalizations of the J. von Neumann formula known in the extension theory of symmetric operators. In fact, the latter is easily implied by (2.4) if $B = A$.

For the next statement, for λ in the field of regularity $\widehat{\rho}(A)$ of A, we use the common notation $\mathfrak{N}_\lambda(A) = \ker (A^* - \lambda)$ for the defect subspace of the operator A, omitting the parameter A when no confusion can arise.

Corollary 2.6. *Let A be a closed densely defined symmetric operator in $\mathfrak{H}$. Then*

$$\text{dom } A^* = \text{dom } A \oplus_A \mathfrak{N}_i \oplus_A \mathfrak{N}_{-i}. \tag{2.11}$$

Proof: Note that, since A is symmetric, the pair $\{A, A\}$ is a dual pair of operators in $\mathfrak{H}$. Now formula (2.4) becomes

$$\text{dom } A^* = \text{dom } A \oplus_A \ker (I + A^{*2}) = \text{dom } A \oplus_A \ker (A^* + iI) \oplus_A \ker (A^* - iI). \tag{2.12}$$

Here, the last equality follows from $u = -\frac{(A^*-i)u - (A^*+i)u}{2i}$ for any $u \in D(A^*)$. $\square$
We now introduce a symmetric operator S associated with the dual pair $\{A, B\}$ which allows us to extend many results from the symmetric case to our more general setting.

Proposition 2.7. *Let A and B form a dual pair of closed densely defined operators in $\mathfrak{H}$. Then*

(i) the operator

$$S = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \quad with \quad \mathrm{dom}\,(S) = \mathrm{dom}\,(B) \times \mathrm{dom}\,(A) \qquad (2.13)$$

is a symmetric operator in $\mathfrak{H}^2 = \mathfrak{H} \oplus \mathfrak{H}$ with equal deficiency indices,

$$n_+(S) = n_-(S) = n(A^*, B) = n(B^*, A). \qquad (2.14)$$

(ii) S is selfadjoint if and only if

$$A = B^*. \qquad (2.15)$$

Proof: (i) Note that the property of S being symmetric,

$$S = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \subset \begin{pmatrix} 0 & B^* \\ A^* & 0 \end{pmatrix} = S^*, \qquad (2.16)$$

is equivalent to saying that the operators A and B form a dual pair of operators.

It is easily seen that

$$\ker\,(S^* \pm i) = \{f:\ f = \{f_1, f_2\}, \quad f_1 = \pm i B^* f_2, \quad f_2 \in \ker\,(I + A^* B^*)\}. \quad (2.17)$$

Note that $B^* f_2 \neq 0$ for any $f_2 \in \ker\,(I + A^* B^*)$. Therefore B^* maps $\mathcal{H}_0 = \ker\,(I + A^* B^*)$ isomorphically onto $B^* \mathcal{H}_0$.

Hence

$$n_\pm(S) = \dim \ker\,(S^* \mp i) = \dim \ker\,(I + A^* B^*) \qquad (2.18)$$

and the statement follows from the proof of Corollary 2.5.

(ii) This statement is implied immediately by (i). $\qquad \square$

Remark 2.8. *(i) The decompositions (2.4) and (2.5) can also be derived from Proposition 2.7.*

(ii) In [29] operators of the form (2.13) are studied. An upper bound for n_+ and n_- is given in [29, Lemma 2.1], but equality is not shown there.

(iii) Note that for any $\widetilde{A} \in \mathrm{Ext}\,\{A, B\}$, the operator $T = \begin{pmatrix} 0 & \widetilde{A} \\ \widetilde{A}^* & 0 \end{pmatrix}$ is a selfadjoint extension of S. Hence the equality $n_+(S) = n_-(S)$ is immediate.

2.2. Correct dual pairs and quasi-selfadjoint extensions.

The following new definition will play a central role in the paper.

Definition 2.9. *Let $\{A, B\}$ be a dual pair of closed densely defined operators in $\mathfrak{H}$.*

(i) We will call an extension $\widetilde{A} \in \mathrm{Ext}\,\{A, B\}$ quasi-selfadjoint (or quasi-hermitian) if

$$\dim\,(\mathrm{dom}\,\widetilde{A}/\mathrm{dom}\,A) = \dim\,(\mathrm{dom}\,\widetilde{A}^*/\mathrm{dom}\,B). \qquad (2.19)$$

(ii) A a dual pair $\{A, B\}$ will be called a correct dual pair if it admits a quasi-selfadjoint extension.

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The following proposition gives necessary conditions for a dual pair $\{A, B\}$ to be a correct dual pair. In particular, it shows that not any dual pair is correct.

Proposition 2.10. *Let $\{A, B\}$ be a dual pair of operators in $\mathfrak{H}$ and suppose $\widetilde{A} \in \operatorname{Ext}\{A, B\}$.*

(i) The following identities hold

$$
\begin{aligned}
n(A^*, B) &= \dim\left(\operatorname{dom} B^*/\operatorname{dom}\widetilde{A}\right) + \dim\left(\operatorname{dom} A^*/\operatorname{dom}\widetilde{A}^*\right) &\text{(2.20)} \\
&= \dim\left(\operatorname{dom}\widetilde{A}/\operatorname{dom} A\right) + \dim\left(\operatorname{dom}\widetilde{A}^*/\operatorname{dom} B\right).
\end{aligned}
$$

(ii) If $\widetilde{A}$ is a quasi-selfadjoint extension of $\{A, B\}$, then $n(A^*, B)$ is even and

$$
\begin{aligned}
n(A^*, B)/2 &= \dim\left(\operatorname{dom} B^*/\operatorname{dom}\widetilde{A}\right) = \dim\left(\operatorname{dom} A^*/\operatorname{dom}\widetilde{A}^*\right) &\text{(2.21)} \\
&= \dim\left(\operatorname{dom}\widetilde{A}/\operatorname{dom} A\right) = \dim\left(\operatorname{dom}\widetilde{A}^*/\operatorname{dom} B\right).
\end{aligned}
$$

Proof: (i) Introducing the operator $T = \begin{pmatrix} 0 & \widetilde{A} \\ \widetilde{A}^* & 0 \end{pmatrix}$ we note that

$$
S = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \subset T = \begin{pmatrix} 0 & \widetilde{A} \\ \widetilde{A}^* & 0 \end{pmatrix} = T^* \subset \begin{pmatrix} 0 & B^* \\ A^* & 0 \end{pmatrix} = S^*, \qquad \text{(2.22)}
$$

so T is a selfadjoint extension of S. Set

$$
n_l = \dim\left(\operatorname{dom} B^*/\operatorname{dom}\widetilde{A}\right) + \dim\left(\operatorname{dom} A^*/\operatorname{dom}\widetilde{A}^*\right)
$$

and

$$
n_r = \dim\left(\operatorname{dom}\widetilde{A}/\operatorname{dom} A\right) + \dim\left(\operatorname{dom}\widetilde{A}^*/\operatorname{dom} B\right).
$$

It follows from (2.22) that

$$
\dim\left(\operatorname{dom} T/\operatorname{dom} S\right) = n_r, \qquad \text{and} \qquad \dim\left(\operatorname{dom} S^*/\operatorname{dom} T\right) = n_l. \qquad \text{(2.23)}
$$

Since T is a selfadjoint extension of S, then J. von Neumann's formulas yield

$$
n_l = n_r = n_\pm(S). \qquad \text{(2.24)}
$$

Combining (2.24) with (2.14) we get the required result.

(ii) Now let $\widetilde{A}$ be a quasi-selfadjoint extension of $\{A, B\}$, that is (2.19) holds. It is clear that

$$
n(A^*, B) = \dim\left(\operatorname{dom} A^*/\operatorname{dom}\widetilde{A}^*\right) + \dim\left(\operatorname{dom}\widetilde{A}^*/\operatorname{dom} B\right) \qquad \text{(2.25)}
$$

and

$$
n(B^*, A) = \dim\left(\operatorname{dom} B^*/\operatorname{dom}\widetilde{A}\right) + \dim\left(\operatorname{dom}\widetilde{A}/\operatorname{dom} A\right). \qquad \text{(2.26)}
$$

Combining these formulas with (2.19) and (2.9) we get

$$
\dim\left(\operatorname{dom} A^*/\operatorname{dom}\widetilde{A}^*\right) = \dim\left(\operatorname{dom} B^*/\operatorname{dom}\widetilde{A}\right) \qquad \text{(2.27)}
$$

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It follows from (2.20), (2.27) and (2.24) that

$$n_l = 2 \dim (\operatorname{dom} B^*/\operatorname{dom} \widetilde{A}) = 2 \dim (\operatorname{dom} A^*/\operatorname{dom} \widetilde{A}^*) = n(A^*, B). \quad (2.28)$$

The remaining equalities in (2.21) are now implied by combining (2.25), (2.26) and (2.28). □ Next we present a criterion for an extension $\widetilde{A} \in \operatorname{Ext}\{A, B\}$ to be quasi-selfadjoint.

Proposition 2.11. *Let $\{A, B\}$ be a dual pair in $\mathfrak{H}$. Suppose that for some extension $\widetilde{A} \in Ext\{A, B\}$ the following condition holds*

$$\dim (\operatorname{dom} B^*/\operatorname{dom} \widetilde{A}) = \dim (\operatorname{dom} \widetilde{A}/\operatorname{dom} A). \quad (2.29)$$

Then $\widetilde{A}$ is a quasi-selfadjoint extension of $\{A, B\}$.

Proof: It is clear from (2.22)-(2.24) that

$$\dim (\operatorname{dom} B^*/\operatorname{dom} \widetilde{A}) + \dim (\operatorname{dom} A^*/\operatorname{dom} \widetilde{A}^*) = n_\pm(S) \quad (2.30)$$

and

$$\dim (\operatorname{dom} \widetilde{A}/\operatorname{dom} A) + \dim (\operatorname{dom} \widetilde{A}^*/\operatorname{dom} B) = n_\pm(S) \quad (2.31)$$

Combining (2.30) and (2.31) with (2.29) we get

$$\dim (\operatorname{dom} A^*/\operatorname{dom} \widetilde{A}^*) = \dim (\operatorname{dom} \widetilde{A}^*/\operatorname{dom} B). \quad (2.32)$$

On the other hand, combining (2.29) with (2.26) we obtain

$$2 \dim (\operatorname{dom} \widetilde{A}/\operatorname{dom} A) = n(B^*, A) = n_\pm(S), \quad (2.33)$$

and combining (2.32) with (2.25) we have

$$2 \dim (\operatorname{dom} \widetilde{A}^*/\operatorname{dom} B) = n(A^*, B) = n_\pm(S). \quad (2.34)$$

Comparing (2.33) with (2.34) we get the required result. □

2.3. Defect numbers of dual pairs and correct dual pairs.

Here, we introduce a concept of defect numbers of a dual pair and establish their connection with deficiency indices of a symmetric operator S. We first recall a standard definition.

Definition 2.12. *(i) The field of regularity $\widehat{\rho}(A)$ of a closed linear operator A is the set of $\lambda \in \mathbb{C}$ such that for some $\varepsilon > 0$*

$$\|Af - \lambda f\| \geq \varepsilon \|f\|, \quad f \in \operatorname{dom} A. \quad (2.35)$$

(ii) For a dual pair $\{A, B\}$ we let $\widehat{\rho}\{A, B\} := \{\lambda \in \mathbb{C} : \lambda \in \widehat{\rho}(A) \text{ and } \overline{\lambda} \in \widehat{\rho}(B)\}$.

Clearly, $\rho(\widetilde{A}) \subset \widehat{\rho}\{A, B\}$ for each proper extension $\widetilde{A} \in \operatorname{Ext}\{A, B\}$. We continue with a simple and well-known lemma.

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Lemma 2.13. *Let $\{A, B\}$ be a dual pair and $\widetilde{A} \in \mathrm{Ext}\,\{A, B\}$ a proper extension with nonempty resolvent set $\rho(\widetilde{A})$. Then for any $\lambda_0 \in \rho(\widetilde{A})$ the following direct decompositions hold*

$$\mathrm{dom}\,\widetilde{A} = \mathrm{dom}\,A \dot{+} (\widetilde{A} - \lambda_0)^{-1} \mathfrak{N}_{\overline{\lambda}_0}(A), \tag{2.36}$$

$$\mathrm{dom}\,\widetilde{A}^* = \mathrm{dom}\,B \dot{+} (\widetilde{A}^* - \overline{\lambda}_0)^{-1} \mathfrak{N}_{\lambda_0}(B). \tag{2.37}$$

Proof: We prove (2.36). Since $\lambda_0 \in \rho(\widetilde{A})$, we have $\lambda_0 \in \widehat{\rho}(A)$ and thus $\mathrm{ran}\,(A - \lambda_0)$ is closed. Therefore, $\mathfrak{H} = \mathrm{ran}\,(A - \lambda_0) \oplus \mathfrak{N}_{\overline{\lambda}_0}(A)$. Applying $(\widetilde{A} - \lambda_0)^{-1}$ to this equality gives the desired result. $\qquad\square$

Corollary 2.14. *If there exists an extension $\widetilde{A} \in \mathrm{Ext}\,\{A, B\}$ with $\rho(\widetilde{A}) \neq \emptyset$, then*

$$n(A) := \dim \mathfrak{N}_{\overline{\lambda}}(A) = const, \qquad \lambda \in \rho(\widetilde{A}), \tag{2.38}$$

$$n(B) := \dim \mathfrak{N}_{\lambda}(B) = const, \qquad \overline{\lambda} \in \rho(\widetilde{A}^*). \tag{2.39}$$

Proof: The proof is immediate from (2.36) and (2.37) since

$$\dim \mathfrak{N}_{\overline{\lambda}_0}(A) = \dim\,(\mathrm{dom}\,\widetilde{A}/\mathrm{dom}\,A), \qquad \lambda_0 \in \rho(\widetilde{A}), \tag{2.40}$$

and

$$\dim \mathfrak{N}_{\lambda_0}(B) = \dim\,(\mathrm{dom}\,\widetilde{A}^*/\mathrm{dom}\,B), \qquad \lambda_0 \in \rho(\widetilde{A}). \tag{2.41}$$

$$\square$$

Definition 2.15. *The numbers $n(A)$ and $n(B)$ are called the defect numbers of a dual pair $\{A, B\}$.*

We emphasize that $n(A)$ and $n(B)$ do not depend on the choice of $\widetilde{A} \in \mathrm{Ext}\,\{A, B\}$ with $\rho(\widetilde{A}) \neq \emptyset$.

Note that, by definition, $\lambda_0 \in \widehat{\rho}(A, B)$ if there exists $\varepsilon > 0$ such that

$$\|Af - \lambda_0 f\| \geq \varepsilon \|f\|, \quad f \in \mathrm{dom}\,A \quad \text{and} \quad \|Bg - \overline{\lambda}_0 g\| \geq \varepsilon \|g\|, \quad g \in \mathrm{dom}\,B. \tag{2.42}$$

We set $\mathbb{D}(\lambda_0; \varepsilon) := \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \varepsilon\}$. One easily proves that $\mathbb{D}(\lambda_0; \varepsilon) \subset \widehat{\rho}(A, B)$.

The next result which can be found in [26] and [25, Proposition 2.10] proves that there is a proper extension preserving the 'gap in the spectrum'.

Proposition 2.16. *Let $\{A, B\}$ be a dual pair of densely defined operators in $\mathfrak{H}$. Suppose that for some $\lambda_0 \in \mathbb{C}$ we have*

$$\|Af - \lambda_0 f\| \geq \varepsilon \|f\|, \ f \in \mathrm{dom}\,(A) \quad \text{and} \quad \|Bg - \lambda_0 g\| \geq \varepsilon \|g\|, \ g \in \mathrm{dom}\,(B). \tag{2.43}$$

Then there exists an extension $\widetilde{A} \in \mathrm{Ext}\,\{A, B\}$ preserving the gap, i.e. such that

$$\mathbb{D}(\lambda_0; \varepsilon) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \varepsilon\} \subset \rho(\widetilde{A}) \tag{2.44}$$

and moreover, the norm-preserving estimate holds

$$\|\widetilde{A}f - \lambda_0 f\| \geq \varepsilon \|f\|, \quad f \in \text{dom}\,(\widetilde{A}). \tag{2.45}$$

In particular, there exists a proper extension $\widetilde{A} \in \text{Ext}\,\{A, B\}$ such that $\lambda_0 \in \rho(\widetilde{A})$.

Remark 2.17. *We emphasize that estimate (2.45) implies inclusion (2.44) but not vice versa. These conditions are indeed equivalent for a selfadjoint operator $\widetilde{A} = \widetilde{A}^*$.*

Note also that in [26] and [25, Proposition 2.10] estimate (2.45) was proved but was not stated explicitly. In fact, this estimate is useful in the sequel.

We have the following decompositions of the domains of A^* and B^*.

Lemma 2.18. *Let $\{A, B\}$ be a dual pair of densely defined closed operators in $\mathfrak{H}$. Suppose that $\lambda_0 \in \widehat{\rho}\{A, B\}$, and let $\widetilde{A}$ be a proper extension of the dual pair with $\lambda_0 \in \rho(\widetilde{A})$. Then the following direct decompositions hold*

$$\text{dom}\, B^* = \text{dom}\, A \dotplus (\widetilde{A} - \lambda_0)^{-1} \mathfrak{N}_{\overline{\lambda}_0}(A) \dotplus \mathfrak{N}_{\lambda_0}(B), \tag{2.46}$$

$$\text{dom}\, A^* = \text{dom}\, B \dotplus (\widetilde{A}^* - \overline{\lambda}_0)^{-1} \mathfrak{N}_{\lambda_0}(B) \dotplus \mathfrak{N}_{\overline{\lambda}_0}(A). \tag{2.47}$$

Proof: We prove (2.46). Clearly, we have

$$\text{dom}\, B^* \supseteq \text{dom}\, A + (\widetilde{A} - \lambda_0)^{-1} \mathfrak{N}_{\overline{\lambda}_0}(A) + \mathfrak{N}_{\lambda_0}(B).$$

On the other hand, directness of the sum is easy to show and for any $u \in \text{dom}\,(B^*)$ we have

$$u \;=\; u - (\widetilde{A} - \lambda_0)^{-1}(B^* - \lambda_0)u + (\widetilde{A} - \lambda_0)^{-1}(B^* - \lambda_0)u$$

Since $\lambda_0 \in \widehat{\rho}(A)$, we have that ran $(A - \lambda_0)$ is closed and so $\mathfrak{H} = \text{ran}\,(A - \lambda_0) \oplus \mathfrak{N}_{\overline{\lambda}_0}(A)$ and

$$(B^* - \lambda_0)u = [(A - \lambda_0)g + h]$$

for some $g \in D(A)$ and $h \in \mathfrak{N}_{\overline{\lambda}_0}(A)$. Thus,

$$\begin{aligned} u &= u - (\widetilde{A} - \lambda_0)^{-1}(B^* - \lambda_0)u + (\widetilde{A} - \lambda_0)^{-1}\left[(A - \lambda_0)g + h\right] \\ &= \left[u - (\widetilde{A} - \lambda_0)^{-1}(B^* - \lambda_0)u\right] + g + (\widetilde{A} - \lambda_0)^{-1}h. \end{aligned}$$

Here, the first term on the right lies in $\mathfrak{N}_{\lambda_0}(B)$, g is in dom (A) and we have $(\widetilde{A} - \lambda_0)^{-1}h \in (\widetilde{A} - \lambda_0)^{-1}\mathfrak{N}_{\overline{\lambda}_0}(A)$, proving the decomposition. $\square$ This result directly leads to some identities for deficiency indices.

Corollary 2.19. *Additionally to the conditions of Lemma 2.18, let S be an operator of the form (2.13). Then*

$$n_\pm(S) = \dim \mathfrak{N}_{\overline{\lambda}_0}(A) + \dim \mathfrak{N}_{\lambda_0}(B). \tag{2.48}$$

Proof: It follows from (2.14) and (2.46) that

$$n(B^*, A) = \dim\left(\operatorname{dom} B^*/\operatorname{dom} A\right) = \dim \mathfrak{N}_{\overline{\lambda}_0}(A) + \dim \mathfrak{N}_{\lambda_0}(B), \qquad (2.49)$$

Comparing (2.49) with (2.14) we get the required result. $\qquad\square$

Corollary 2.20. *Additionally to the conditions of Lemma 2.18 suppose that*

$$\dim \mathfrak{N}_{\overline{\lambda}_0}(A) = \dim \mathfrak{N}_{\lambda_0}(B). \qquad (2.50)$$

Then

$$n_\pm(S) = 2 \dim \mathfrak{N}_{\overline{\lambda}_0}(A) = 2 \dim \mathfrak{N}_{\lambda_0}(B). \qquad (2.51)$$

2.4. j-symmetric and j-selfadjoint operators.

In this subsection, we study j-symmetric operators. Making use of the theory of dual pairs we are able to show various results on deficiency indices for such operators. We start with a definition. Recall that j is a conjugation operator in $\mathfrak{H}$, if j is an anti-linear operator satisfying

$$(ju, v)_{\mathfrak{H}} = (jv, u)_{\mathfrak{H}}, \quad u, v \in \mathfrak{H} \qquad \text{and} \qquad j^2 = I. \qquad (2.52)$$

In particular,

$$(ju, jv)_{\mathfrak{H}} = (v, u)_{\mathfrak{H}}, \qquad u, v \in \mathfrak{H}. \qquad (2.53)$$

Definition 2.21. *(i) A densely defined linear operator A in $\mathfrak{H}$ is called j-symmetric if*

$$A \subseteq jA^*j \qquad (\text{or equivalently, if } jAj \subseteq A^*). \qquad (2.54)$$

(ii) The operator A is called j-selfadjoint if

$$A = jA^*j \qquad (\text{or equivalently, if } jAj = A^*). \qquad (2.55)$$

The following two results were originally proved by Race [30]. For the sake of completeness we present proofs here deducing them from Proposition 2.3.

Proposition 2.22. *Let A be a densely defined closed j-symmetric operator in $\mathfrak{H}$. Then the following decompositions hold*

$$\operatorname{dom} A^* = \operatorname{dom}\left(jAj\right) \oplus_A \ker\left(I + (jA^*)^2\right) \qquad (2.56)$$

$$\operatorname{dom}\left(jA^*j\right) = \operatorname{dom} A \oplus_{jAj} \ker\left(I + (A^*j)^2\right) \qquad (2.57)$$

In particular, a j-symmetric operator A is j-selfadjoint if and only if

$$\ker\left(I + (jA^*)^2\right) = \{0\} \qquad (\text{or equivalently, if } \ker\left(I + (A^*j)^2\right) = \{0\}). \quad (2.58)$$

Proof: Let $B = jAj$. Then $\{A, B\}$ is a dual pair of closed densely defined linear operators. Noting that $B^* = jA^*j$ and applying Proposition 2.3 we get

$$\operatorname{dom} A^* = \operatorname{dom}\left(jAj\right) \oplus_A \ker\left(I + jA^*jA^*\right) = \operatorname{dom}\left(jAj\right) \oplus_A \ker\left(I + (jA^*)^2\right). \qquad (2.59)$$

Formula (2.58) is implied by (2.6). $\qquad\square$

Proposition 2.23. *Let A be a closed j-symmetric operator in $\mathfrak{H}$ and $\widetilde{A}$ any j-selfadjoint extension of A. Then*

$$\dim\left(\operatorname{dom}(jA^*j)/\operatorname{dom}\widetilde{A}\right) = \dim\left(\operatorname{dom}\widetilde{A}/\operatorname{dom}A\right). \qquad (2.60)$$

Proof: Setting $B = jAj$ we obtain a dual pair $\{A, B\}$. It is clear that

$$S = \begin{pmatrix} 0 & A \\ jAj & 0 \end{pmatrix} \subset T = \begin{pmatrix} 0 & \widetilde{A} \\ \widetilde{A}^* & 0 \end{pmatrix} = T^* \subset \begin{pmatrix} 0 & jA^*j \\ A^* & 0 \end{pmatrix} = S^*. \qquad (2.61)$$

Since $\widetilde{A}^* = j\widetilde{A}j$, $j\operatorname{dom}A^* = \operatorname{dom}(jA^*j)$ and

$$j\operatorname{dom}\widetilde{A} = \operatorname{dom}(j\widetilde{A}j) = \operatorname{dom}\widetilde{A}^*, \ \ j\operatorname{dom}A = \operatorname{dom}(jAj), \qquad (2.62)$$

we get

$$\dim\left(\operatorname{dom}(jA^*j)/\operatorname{dom}\widetilde{A}\right) = \dim\left(\operatorname{dom}A^*/\operatorname{dom}\widetilde{A}^*\right). \qquad (2.63)$$

It follows that

$$\dim\left(\operatorname{dom}S^*/\operatorname{dom}T\right) = 2\dim\left(\operatorname{dom}(jA^*j)/\operatorname{dom}\widetilde{A}\right), \qquad (2.64)$$

$$\dim\left(\operatorname{dom}T/\operatorname{dom}S\right) = 2\dim\left(\operatorname{dom}\widetilde{A}/\operatorname{dom}A\right). \qquad (2.65)$$

Since T is a selfadjoint extension of S, then by J.von Neumann's formulas

$$\dim\left(\operatorname{dom}S^*/\operatorname{dom}T\right) = \dim\left(\operatorname{dom}T/\operatorname{dom}S\right). \qquad (2.66)$$

Combining (2.64), (2.65) and (2.66) we get the required result. $\qquad\square$

Next we complement Proposition 2.23 by the following simple statement.

Proposition 2.24. *Let A be a closed j-symmetric operator in $\mathfrak{H}$. Then any j-selfadjoint extension $\widetilde{A}$ of A is a quasi-selfadjoint extension of the dual pair $\{A, jAj\}$.*

Proof: Since $\operatorname{dom}(jAj) = j\operatorname{dom}A$ and $\widetilde{A}^* = j\widetilde{A}j$, taking (2.62) into account, we obtain that

$$\begin{aligned}
\dim\left(\operatorname{dom}\widetilde{A}^*/\operatorname{dom}(jAj)\right) &= \dim\left(\operatorname{dom}(j\widetilde{A}j)/\operatorname{dom}(jAj)\right) \qquad (2.67) \\
&= \dim\left(\operatorname{dom}\widetilde{A}/\operatorname{dom}A\right).
\end{aligned}$$

By Definition 2.9 this means that $\widetilde{A}$ is quasi-selfadjoint. $\qquad\square$

Lemma 2.25. *Let A be a densely defined closed j-symmetric operator in $\mathfrak{H}$. Then:*

(i) *the operator*

$$S = \begin{pmatrix} 0 & A \\ jAj & 0 \end{pmatrix} \qquad (2.68)$$

is a closed symmetric operator in $\mathfrak{H}^2 = \mathfrak{H} \oplus \mathfrak{H}$ and $n_+(S) = n_-(S)$;

(ii) *S is selfadjoint if and only if A is j-selfadjoint.*

Proof: Setting $B := jAj$, we note that by Definition 2.9, we have $A \subset B^* = jA^*j$ and $B \subset A^*$. Thus, $\{A, B\}$ forms a dual pair of densely defined closed operators in $\mathfrak{H}$. It remains to apply Proposition 2.7 to obtain the desired result. $\square$

Corollary 2.26. *Let A be a j-symmetric operator in $\mathfrak{H}$, with a nonempty field of regularity $\widehat{\rho}(A)$ and let S be the operator of the form (2.68). Then the following relations hold*

$$2n(A) := 2Def(A) := 2\dim \mathfrak{N}_{\overline{\lambda}_0}(A) = n_{\pm}(S), \qquad for\ any \quad \lambda_0 \in \widehat{\rho}(A). \tag{2.69}$$

Proof: Let $\lambda_0 \in \widehat{\rho}(A)$. Note that $\overline{\lambda}_0 \in \widehat{\rho}(B)$, since for any $f \in \operatorname{dom} A$

$$\|Bjf - \overline{\lambda}_0 jf\| = \|j(Af - \lambda_0 f)\| = \|Af - \lambda_0 f\|. \tag{2.70}$$

We next show that equality (2.50) is satisfied for any $\lambda_0 \in \widehat{\rho}(A)$. Indeed, for any $f \in \ker(A^* - \overline{\lambda}_0)$ we have with $g = jf$,

$$B^* g = jA^* jg = jA^* f = j\overline{\lambda}_0 f = \lambda_0 jf = \lambda_0 g, \tag{2.71}$$

that is $g \in \ker(B^* - \lambda_0)$ and $j\ker(A^* - \overline{\lambda}_0) = \ker(B^* - \lambda_0)$. Hence $\dim \mathfrak{N}_{\overline{\lambda}_0}(A) = \dim \mathfrak{N}_{\lambda_0}(B)$ Now (2.69) is implied by Corollary 2.20. $\square$ The next result is similar to Proposition 2.16 and shows that there is a gap preserving j-selfadjoint extension.

Proposition 2.27. *Let A be a j-symmetric operator in $\mathfrak{H}$. Suppose that*

$$\|Af - \lambda_0 f\| \geq \varepsilon\|f\|, \qquad f \in \operatorname{dom} A, \tag{2.72}$$

in particular, $\mathbb{D}(\lambda_0; \varepsilon) \subset \widehat{\rho}(A)$. Then there exists a j-selfadjoint extension $\widetilde{A}$ of A such that

$$\|\widetilde{A}f - \lambda_0 f\| \geq \varepsilon\|f\|, \quad f \in \operatorname{dom} \widetilde{A}. \tag{2.73}$$

In particular, the j-selfadjoint extension $\widetilde{A}$ preserves the gap $\mathbb{D}(\lambda_0; \varepsilon)$, that is $\mathbb{D}(\lambda_0; \varepsilon) \subset \rho(\widetilde{A})$.

Proof: Setting $B := jAj$ we note that $\{A, B\}$ is a dual pair of operators and due to (2.70) $\|Bg - \overline{\lambda}_0 g\| \geq \varepsilon\|g\|$, $g \in \operatorname{dom} B$. By Proposition 2.16 there exists a proper extension $\widetilde{A}_1$ of $\{A, B\}$, obeying (2.73). We set $\mathcal{T} := \varepsilon(\widetilde{A}_1 - \lambda_0)^{-1}$ and further

$$\mathcal{S} := (\mathcal{T} + j\mathcal{T}^* j)/2 = \varepsilon[(\widetilde{A}_1 - \lambda_0)^{-1} + j(\widetilde{A}_1^* - \overline{\lambda}_0)^{-1} j]/2 \tag{2.74}$$

Clearly, $\mathcal{S}$ is contractive and

$$\mathcal{S}^* = \varepsilon[(\widetilde{A}_1^* - \overline{\lambda}_0)^{-1} + j(\widetilde{A}_1 - \lambda_0)^{-1} j]/2, \tag{2.75}$$

so $j\mathcal{S}^*j = \mathcal{S}$ and $\mathcal{S}$ is j-selfadjoint. Moreover, as $\widetilde{A}_1 \in \mathrm{Ext}\,\{A,B\}$, we have $\widetilde{A}_1 \supset A$ and $\widetilde{A}_1^* \supset jAj$. Thus $j\widetilde{A}_1^*j \supset A$. Therefore, $\mathcal{S}$ is an extension of the non-densely defined contraction $\mathcal{T}_1 = \varepsilon(A - \lambda_0)^{-1} : \mathrm{ran}\,(A - \lambda_0) \to \mathfrak{H}$. Altogether we have

$$\mathcal{T}_1 \subset \mathcal{S} = j\mathcal{S}^*j, \qquad \|\mathcal{S}\| \le 1. \tag{2.76}$$

Assume $\mathcal{S}f = 0$ for some f and let $g \in \mathrm{dom}\,A$. Then using that $\mathcal{S}$ extends $\mathcal{T}_1$ we obtain

$$
\begin{aligned}
0 &= (\mathcal{S}f, j(A - \lambda_0)g) = (f, \mathcal{S}^*j(A - \lambda_0)g)\\
&= (f, j\mathcal{S}(A - \lambda_0)g) = (f, \varepsilon jg) = (g, \varepsilon jf).
\end{aligned}
$$

Density of $\mathrm{dom}\,A$ implies that $\varepsilon jf = 0$, so $f = 0$ and $\mathcal{S}$ has trivial kernel. As $\mathcal{S}$ is an injective contraction, it is invertible and we can set $\widetilde{A} = \varepsilon\mathcal{S}^{-1} + \lambda_0$. Then

$$
\begin{aligned}
\widetilde{A}^* &= \varepsilon(\mathcal{S}^{-1})^* + \overline{\lambda_0} = \varepsilon(\mathcal{S}^*)^{-1} + \overline{\lambda_0}\\
&= \varepsilon j\mathcal{S}^{-1}j + \overline{\lambda_0} = j(\varepsilon\mathcal{S}^{-1} + \lambda_0)j = j\widetilde{A}j
\end{aligned}
$$

and $\widetilde{A}$ is j-selfadjoint. Let $g \in \mathrm{dom}\,A$. Then

$$g = \frac{1}{\varepsilon}\mathcal{T}_1(A - \lambda_0)g = \frac{1}{\varepsilon}\mathcal{S}(A - \lambda_0)g \tag{2.77}$$

and $g \in \mathrm{dom}\,\mathcal{S}^{-1}$ with $(\varepsilon\mathcal{S}^{-1} + \lambda_0)g = Ag$, so $\widetilde{A}$ extends A. It satisfies (2.73), as $\mathcal{S}$ is a contraction. $\qquad\square$

Corollary 2.28 ([34]). *Let A be j-symmetric operator in $\mathfrak{H}$ and $\widehat{\rho}(A) \ne \emptyset$. Then:*

(i) For any $\lambda_0 \in \widehat{\rho}(A)$ there exists a j-selfadjoint extension $\widetilde{A}$ of A with $\lambda_0 \in \rho(\widetilde{A})$.

(ii) For any $\lambda_0 \in \rho(\widetilde{A})$ the following direct decompositions hold

$$\mathrm{dom}\,jA^*j = \mathrm{dom}\,A \dotplus (\widetilde{A} - \lambda_0)^{-1}\mathfrak{N}_{\overline{\lambda}_0}(A) \dotplus j\mathfrak{N}_{\overline{\lambda}_0}(A), \tag{2.78}$$

$$\mathrm{dom}\,A^* = \mathrm{dom}\,jAj \dotplus (\widetilde{A}^* - \overline{\lambda}_0)^{-1}j\mathfrak{N}_{\overline{\lambda}_0}(B) \dotplus \mathfrak{N}_{\overline{\lambda}_0}(A). \tag{2.79}$$

Proof: (i) This statement is immediate from Proposition 2.27.

(ii) These formulas are implied by formulas (2.46) and (2.47), respectively, with $B = jAj$ if one takes the relation $j\mathfrak{N}_{\overline{\lambda}_0}(A) = \mathfrak{N}_{\overline{\lambda}_0}(B)$ into account. $\qquad\square$
This result implies one more justification of the existence of the defect number $\mathrm{Def}(A)$ for any j-symmetric operator A (see Corollary 2.26).

Corollary 2.29. *Let A be a j-symmetric operator in $\mathfrak{H}$ with a nonempty field of regularity $\widehat{\rho}(A)$ and let $B = jAj$, so that $\{A,B\}$ is a dual pair of densely defined operators in $\mathfrak{H}$. Then $\widehat{\rho}(A) = \widehat{\rho}\{A,B\}$ and*

$$\mathrm{Def}(A) := \dim \mathfrak{N}_{\overline{\lambda}}(A) = \dim \mathfrak{N}_{\lambda}(B) \qquad \text{for any}\quad \lambda \in \widehat{\rho}\{A,B\}. \tag{2.80}$$

Proof: By Proposition 2.27, for any $\lambda_0 \in \widehat{\rho}(A)$ there exists a j-selfadjoint extension $\widetilde{A}$ satisfying $\lambda_0 \in \rho(\widetilde{A})$. It follows from (2.78) that

$$\dim\left(\operatorname{dom} jA^*j / \operatorname{dom} A\right) = 2\dim \mathfrak{N}_{\overline{\lambda_0}}(A) \quad \text{for any} \quad \lambda_0 \in \widehat{\rho}(A). \tag{2.81}$$

Hence $\dim \mathfrak{N}_{\lambda_0}(A)$ does not depend on $\lambda_0 \in \widehat{\rho}(A)$. This proves (2.80). $\qquad\square$

Remark 2.30. *The existence of a j-selfadjoint extension of a j-symmetric operator has been well-known for a long time. It was proved by Zhikhar [34, Theorems 2,3] for the first time using the Vishik method [32], see e.g. [16, Section 22] and [14, Theorem III.5.8 & Theorem III.5.9]. However, the preservation of the gap (in the sense of (2.73)) seems to be new. It can be considered as a generalization of well known result of Krein [21] which ensures existence of a selfadjoint extension $\widetilde{A}$ preserving a gap of a symmetric operator A with a gap.*

It also complements a result by Brasche et al in [2, Theorem 3.1], which states that there need not be a selfadjoint extension of a symmetric operator that commutes with j.

Note also that Lemma 2.18 and Corollary 2.28 generalize Vishik's results from [32].

3. Non-selfadjoint Dirac-type operators

3.1. Dirac-type expressions and relative operators

We now consider applications of the previous theory to Dirac-type operators, both on the semiaxis and the whole line.

In the following $\mathfrak{H}$, $\mathcal{H}$ are Hilbert spaces, $\boldsymbol{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ is the set of bounded operators defined on $\mathfrak{H}_1$ with values in $\mathfrak{H}_2$ and $\boldsymbol{B}(\mathfrak{H}) := \boldsymbol{B}(\mathfrak{H}, \mathfrak{H})$. An operator $T \in \boldsymbol{B}(\mathbb{C}^m)$ will often be identified with its matrix $T = (T_{ij})_{i,j=1}^m$ in the standard basis of $\mathbb{C}^m$.

Moreover, $AC_{\mathrm{loc}}(\mathcal{I}, \mathbb{C}^m)$ is the set of all $\mathbb{C}^m$-valued vector-functions, defined on an interval $\mathcal{I} \subset \mathbb{R}$ and absolutely continuous on each compact subinterval $[\alpha, \beta] \subset \mathcal{I}$, $L^2(\mathcal{I}, \mathbb{C}^m)$ is the Hilbert space of all Borel measurable functions $f : \mathcal{I} \to \mathbb{C}^m$ with

$$\int_{\mathcal{I}} |f(t)|^2 \, dt < \infty, \quad \text{where } |f(t)|^2 = \sum_{j=1}^m |f_j(t)|^2.$$

$L^1_{\mathrm{loc}}(\mathcal{I}, \boldsymbol{B}(\mathbb{C}^m))$ is the set of all operator-functions $F : \mathcal{I} \to \boldsymbol{B}(\mathbb{C}^m)$ integrable on each compact interval $[a, b] \subset \mathcal{I}$, $L^2(\mathcal{I}, \boldsymbol{B}(\mathbb{C}^{m_1}, \mathbb{C}^{m_2}))$ is the set of all operator-functions $F : \mathcal{I} \to \boldsymbol{B}(\mathbb{C}^{m_1}, \mathbb{C}^{m_2}))$ such that $\int_{\mathcal{I}} |F(t)|^2 \, dt < \infty$ or, equivalently, $F(\cdot)h \in L^2(\mathcal{I}, \mathbb{C}^{m_2})$ for all $h \in \mathbb{C}^{m_1}$. Moreover, for $\alpha \in \mathbb{R}$ we denote by $\mathbb{C}_{\alpha,+}$ and $\mathbb{C}_{\alpha,-}$ the open half-planes $\mathbb{C}_{\alpha,+} = \{z \in \mathbb{C} : \operatorname{Im} z > \alpha\}$ and $\mathbb{C}_{\alpha,-} = \{z \in \mathbb{C} : \operatorname{Im} z < \alpha\}$, respectively.

Let $\mathcal{I} = \langle a, b \rangle$, $-\infty \le a < b \le \infty$, be an interval in $\mathbb{R}$ (each of the endpoints a or b may belong or not belong to $\mathcal{I}$). We recall that we are considering the Dirac-type differential expression

$$D(Q)y := J_n \frac{dy}{dx} + Q(x)y \tag{3.1}$$

on $\mathcal{I}$ with the operator $J_n \in \boldsymbol{B}(\mathbb{C}^n)$ such that $J_n^* = J_n^{-1} = -J_n$ and the operator (matrix) potential $Q(x)$. In the following we suppose (unless otherwise stated) that $Q \in L^1_{\mathrm{loc}}(\mathcal{I}, \boldsymbol{B}(\mathbb{C}^n))$. Expression (3.1) is called formally selfadjoint if $Q(x) = Q^*(x)$ (a.e. on $\mathcal{I}$).

Let $\mathfrak{H} := L^2(\mathcal{I}, \mathbb{C}^n)$ and let $D_{\max}(Q)$ be the maximal operator in $\mathfrak{H}$ defined by

$$\mathrm{dom}\, D_{\max}(Q) = \{y \in \mathfrak{H} : y \in AC_{\mathrm{loc}}(\mathcal{I}, \mathbb{C}^n), D(Q)y \in \mathfrak{H}\}$$

$$D_{\max}(Q)y = D(Q)y, \quad y \in \mathrm{dom}\, D_{\max}(Q).$$

Define the preminimal operator by $D'_{\min}(Q) := D_{\max}(Q) \upharpoonright \mathrm{dom}\, D'_{\min}(Q)$, where $\mathrm{dom}\, D'_{\min}(Q)$ is the set of all $y \in \mathrm{dom}\, D_{\max}(Q)$ such that $\mathrm{supp}\, y$ is compact and $y(a) = 0\,(y(b) = 0)$ if $a \in \mathcal{I}$ (resp. $b \in \mathcal{I}$). It follows by standard techniques (see e.g. [23]) that $D'_{\min}(Q)$ is a densely defined closable operator in $\mathfrak{H}$. The minimal operator $D_{\min}(Q)$ is then defined as the closure of $D'_{\min}(Q)$ and satisfies $(D_{\min}(Q))^* = D_{\max}(Q^*)$. Since obviously $D_{\min}(Q) \subset D_{\max}(Q) = (D_{\min}(Q^*))^*$, it follows that $\{D_{\min}(Q), D_{\min}(Q^*)\}$ forms a dual pair of closed densely defined operators in $\mathfrak{H}$.

3.2. Dirac-type operators on the whole line

We now prove our first main result, Theorem 1.1. **Proof:**[Proof of Theorem 1.1] Let $\widetilde{J}_{2n} \in \boldsymbol{B}(\mathbb{C}^{2n})$ and $\widetilde{Q}(x)(\in \boldsymbol{B}(\mathbb{C}^{2n}))$ be given by

$$\widetilde{J}_{2n} = \begin{pmatrix} 0 & J_n \\ J_n & 0 \end{pmatrix}, \qquad \widetilde{Q}(x) = \begin{pmatrix} 0 & Q(x) \\ Q^*(x) & 0 \end{pmatrix} \tag{3.2}$$

Since obviously $\widetilde{J}_{2n}^* = \widetilde{J}_{2n}^{-1} = -\widetilde{J}_{2n}$ and $\widetilde{Q}^*(x) = \widetilde{Q}(x)$, the equality

$$D(\widetilde{Q})\widetilde{y} = \widetilde{J}_{2n} \frac{d\widetilde{y}}{dx} + \widetilde{Q}(x)\widetilde{y} \tag{3.3}$$

defines a formally selfadjoint Dirac type expression $D(\widetilde{Q})\widetilde{y}$ on $\mathbb{R}$ and therefore according to [22, Theorem 3.2] $D_{\min}(\widetilde{Q}) = (D_{\min}(\widetilde{Q}))^*(= D_{\max}(\widetilde{Q}))$. On the other hand, one can easily verify that

$$D_{\min}(\widetilde{Q}) = \begin{pmatrix} 0 & D_{\min}(Q) \\ D_{\min}(Q^*) & 0 \end{pmatrix}.$$

Hence by Proposition 2.7, (ii) $D_{\min}(Q) = (D_{\min}(Q^*))^* = D_{\max}(Q)$, which proves (1.4). $\square$ In the following we denote by j_n the complex conjugation in $\mathbb{C}^n$, i.e.

$$j_n h = \overline{h} := \{\overline{h}_1, \overline{h}_2, \ldots, \overline{h}_n\}, \qquad h = \{h_1, h_2, \ldots, h_n\} \in \mathbb{C}^n$$

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Let $n = 2p$, $U = \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}$ and let $\mathcal{I} = \langle a, b \rangle$ be an interval in $\mathbb{R}$. Then the equalities

$$\widetilde{j}_n := U j_n (= j_n U) \quad \text{and} \quad (jf)(t) := \widetilde{j}_n f(t), \ \ t \in \mathcal{I}, \qquad f \in L^2(\mathcal{I}, \mathbb{C}^n) \quad (3.4)$$

define conjugations $\widetilde{j}_n$ and j in $\mathbb{C}^n$ and $L^2(\mathcal{I}, \mathbb{C}^n)$ respectively.

In what follows $C^\top$ denotes the matrix transpose to the matrix C.

Proposition 3.1. *Let $n = 2p$, let $\mathcal{I} = \langle a, b \rangle$ be an interval in $\mathbb{R}$ and let $D(Q)$ be the Dirac type expression (3.1) on $\mathcal{I}$ with $Q \in L^1_{\mathrm{loc}}(\mathcal{I}, \boldsymbol{B}(\mathbb{C}^n))$ and*

$$J_n = i \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix} : \mathbb{C}^p \oplus \mathbb{C}^p \to \mathbb{C}^p \oplus \mathbb{C}^p, \qquad (3.5)$$

$$Q(x) = \begin{pmatrix} Q_{11}(x) & Q_{12}(x) \\ Q_{21}(x) & Q_{22}(x) \end{pmatrix} : \mathbb{C}^p \oplus \mathbb{C}^p \to \mathbb{C}^p \oplus \mathbb{C}^p. \qquad (3.6)$$

Assume also that j is a conjugation in $\mathfrak{H} = L^2(\mathcal{I}, \mathbb{C}^n)$ given by (3.4). If

$$Q_{11}^\top(x) = Q_{22}(x), \quad Q_{12}^\top(x) = Q_{12}(x), \quad Q_{21}^\top(x) = Q_{21}(x) \quad (a.e. \ on \ \ \mathcal{I}), \tag{3.7}$$

then the operator $D_{\min}(Q)$ is j-symmetric and

$$j D_{\max}(Q) j = D_{\max}(Q^*), \qquad j D_{\min}(Q) j = D_{\min}(Q^*). \qquad (3.8)$$

Here the operator $Q_{ij}(x)(\in \boldsymbol{B}(\mathbb{C}^p))$ is identified with its matrix in the standard basis of $\mathbb{C}^p$.

Conversely, if $Q \in L^2_{\mathrm{loc}}(\mathcal{I}, \boldsymbol{B}(\mathbb{C}^n))$ and $D_{\min}(Q)$ is j-symmetric, then (3.7) holds.

Proof: Since $j_n = j_p \oplus j_p$, it follows that

$$\begin{aligned}
\widetilde{j}_n Q(x) \widetilde{j}_n &= j_n U Q(x) U j_n = \begin{pmatrix} j_p Q_{22}(x) j_p & j_p Q_{21}(x) j_p \\ j_p Q_{12}(x) j_p & j_p Q_{11}(x) j_p \end{pmatrix} \\
&= \begin{pmatrix} \overline{Q_{22}(x)} & \overline{Q_{21}(x)} \\ \overline{Q_{12}(x)} & \overline{Q_{11}(x)} \end{pmatrix}.
\end{aligned}$$

Therefore (3.7) is equivalent to

$$\widetilde{j}_n Q(x) \widetilde{j}_n = Q^*(x) \ \ (\text{a.e. on } \mathcal{I}). \qquad (3.9)$$

Moreover, one easily checks that

$$\widetilde{j}_n J_n \widetilde{j}_n = J_n. \qquad (3.10)$$

Let (3.7) be satisfied. Then (3.9) and (3.10) hold, which implies the first equality in (3.8). By using this equality one gets

$$D_{\min}(Q) = (D_{\max}(Q^*))^* = (j D_{\max}(Q) j)^* = j (D_{\max}(Q))^* j = j D_{\min}(Q^*) j.$$

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This proves the second equality in (3.8). Since $D_{\min}(Q^*) \subset D_{\max}(Q^*) = (D_{\min}(Q))^*$, it follows from the second equality in (3.8) that $D_{\min}(Q)$ is j-symmetric.

Conversely, let $Q \in L^2_{\mathrm{loc}}(\mathcal{I}, \boldsymbol{B}(\mathbb{C}^n))$ and let the operator $D_{\min}(Q)$ be j-symmetric. It is easily seen that for each compact interval $[\alpha, \beta] \subset \mathcal{I}$ and for $h \in \mathbb{C}^n$ there exists a function $y \in \mathrm{dom}\, D'_{\min}(Q)$ such that $y(x) = \widetilde{j}_n h$, $x \in [\alpha, \beta]$. Clearly, a function $y_0 = jy$ satisfies $jy_0 \in \mathrm{dom}\, D_{\min}(Q)$, $y_0(x) = h$, $x \in [\alpha, \beta]$, and for this function

$$(jD_{\min}(Q)jy_0)(x) = \widetilde{j}_n Q(x)\widetilde{j}_n h, \ \ (D_{\max}(Q^*)y_0)(x) = Q^*(x)h \ \ \text{a.e. on } [\alpha, \beta].$$

This and the inclusion $jD_{\min}(Q)j \subset D_{\max}(Q^*)$ give (3.9), which in turn yields (3.7). $\qquad\square$

Now we are in position to state a criterion for the Dirac-type operator $D_{\min}(Q)$ to be j-selfadjoint on the line $\mathbb{R}$.

Theorem 3.2. *Let $n = 2p$, $D(Q)$ be the Dirac-type expression (3.1) on $\mathbb{R}$ with J_n and $Q(\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}, \boldsymbol{B}(\mathbb{C}^n))$ given by (3.6) and let j be the conjugation in $L^2(\mathbb{R}_+, \mathbb{C}^n)$ given by (3.4). If the relations (3.7) are fulfilled, then the operator $D_{\min}(Q)(= D_{\max}(Q))$ is j-selfadjoint. Conversely, if $Q \in L^2_{\mathrm{loc}}(\mathbb{R}, \boldsymbol{B}(\mathbb{C}^n))$ and $D_{\min}(Q)$ is j-selfadjoint, then (3.7) holds.*

Proof: By Proposition 3.1, conditions (3.7) imply relations (3.8). On the other hand, Theorem 1.1 ensures the equality $(D_{\min}(Q))^* = D_{\max}(Q^*) = D_{\min}(Q^*)$. Combining this relation with the second one in (3.8) yields

$$jD_{\min}(Q)j = D_{\min}(Q^*) = (D_{\min}(Q))^*.$$

This proves the result. $\square$ The following result is immediate from Theorems 1.1 and 3.2.

Corollary 3.3. *In addition to the assumptions of Theorem 3.2 let*

$$Q(x) = i \begin{pmatrix} 0 & -q(x) \\ -q^*(x) & 0 \end{pmatrix}, \tag{3.11}$$

where $q \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^{p \times p})$ and $q(x) = q^\top(x)$ (a.e. on $\mathbb{R}$). Then $D_{\min}(Q) = D_{\max}(Q)$ and the operator $D_{\min}(Q)$ is j-selfadjoint.

Remark 3.4. *Corollary 3.3 was proved by another method in [7, Theorem 3.5] in the scalar case ($p = 1$) and in [6, Theorem 4] for general p. It was also generalized for a wider class of off-diagonal potential matrices Q in [20]. We emphasize however that our method of reduction of the problem to the self-adjoint case is apparently applied to such problems here for the first time.*

3.3. Dirac-type operators on the half-line

In this subsection we assume that the Dirac-type expression (3.1) is defined on the half-line $\mathbb{R}_+ = [0, \infty)$. The corresponding minimal and maximal operators in $\mathfrak{H} := L^2(\mathbb{R}_+, \mathbb{C}^n)$ will be denoted by $D_{\min}(Q) = D^+_{\min}(Q)$ and $D_{\max}(Q) = D^+_{\max}(Q)$.

Proposition 3.5. *For expression* (3.1) *on* $\mathbb{R}_+$ *the (Lagrange) identity*

$$(D_{\max}^+(Q)y, z)_{\mathfrak{H}} - (y, D_{\max}^+(Q^*)z)_{\mathfrak{H}} = -(J_n y(0), z(0))_{\mathbb{C}^n} \tag{3.12}$$

holds for every $y \in \operatorname{dom} D_{\max}^+(Q)$ *and* $z \in \operatorname{dom} D_{\max}^+(Q^*)$.

Proof: Let $\widetilde{J}_{2n}$ and $\widetilde{Q}(x)$ be the same as in (3.2) in the proof of Theorem 1.1, let $D(\widetilde{Q})\widetilde{y}$ be the formally selfadjoint expression (3.3) and let $\widetilde{\mathfrak{H}} := L^2(\mathbb{R}_+, \mathbb{C}^{2n}) = \mathfrak{H} \oplus \mathfrak{H}$. Then according to [22, Theorem 3.2] the following Lagrange identity holds:

$$(D_{\max}^+(\widetilde{Q})\widetilde{y}, \widetilde{z})_{\widetilde{\mathfrak{H}}} - (\widetilde{y}, D_{\max}^+(\widetilde{Q})\widetilde{z})_{\widetilde{\mathfrak{H}}} = -(\widetilde{J}_{2n}\widetilde{y}(0), \widetilde{z}(0)), \quad \widetilde{y}, \widetilde{z} \in \operatorname{dom} D_{\max}^+(\widetilde{Q}). \tag{3.13}$$

Let $y \in \operatorname{dom} D_{\max}^+(Q)$, $z \in \operatorname{dom} D_{\max}^+(Q^*)$ and let $\widetilde{y} = 0 \oplus y(\in \mathfrak{H} \oplus \mathfrak{H})$, $\widetilde{z} = z \oplus 0(\in \mathfrak{H} \oplus \mathfrak{H})$. Clearly, $\widetilde{y}, \widetilde{z} \in AC_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C}^{2n}) \cap \widetilde{\mathfrak{H}}$ and for $x \in \mathbb{R}_+$

$$D(\widetilde{Q})\widetilde{y} = \{J_n y'(x) + Q(x)y(x), 0\}, \quad D(\widetilde{Q})\widetilde{z} = \{0, J_n z'(x) + Q^*(x)y(x)\}.$$

Therefore $D(\widetilde{Q})\widetilde{y} \in \widetilde{\mathfrak{H}}$, $D(\widetilde{Q})\widetilde{z} \in \widetilde{\mathfrak{H}}$ and, consequently, $\widetilde{y}, \widetilde{z} \in \operatorname{dom} D_{\max}^+(\widetilde{Q})$ and $D_{\max}^+(\widetilde{Q})\widetilde{y} = \{D_{\max}^+(Q)y, 0\}$, $D_{\max}^+(\widetilde{Q})\widetilde{z} = \{0, D_{\max}^+(Q^*)z\}$. Hence

$$(D_{\max}^+(\widetilde{Q})\widetilde{y}, \widetilde{z})_{\widetilde{\mathfrak{H}}} - (\widetilde{y}, D_{\max}^+(\widetilde{Q})\widetilde{z})_{\widetilde{\mathfrak{H}}} = (D_{\max}^+(Q)y, z)_{\mathfrak{H}} - (y, D_{\max}^+(Q^*)z)_{\mathfrak{H}}.$$

Moreover,

$$(\widetilde{J}_{2n}\widetilde{y}(0), \widetilde{z}(0))_{\widetilde{\mathfrak{H}}} = (\{J_n y(0), 0\}, \{z(0), 0\})_{\widetilde{\mathfrak{H}}} = (J_n y(0), z(0))_{\mathfrak{H}}$$

and (3.13) yields (3.12). $\qquad\square$

Lemma 3.6. *For any* $h \in \mathbb{C}^n$ *there exists* $y \in \operatorname{dom} D_{\max}^+(Q)$ *with compact support such that* $y(0) = h$.

Proof: In the case of the formally selfadjoint expression (3.1) the statement is well known (see e.g. [22]). In the general case it suffices to apply the known statement to the formally selfadjoint expression (3.3). $\qquad\square$ For a subspace $\Theta \subset \mathbb{C}^n$ we let

$$\Theta^\times = \mathbb{C}^n \ominus J_n \Theta. \tag{3.14}$$

and define the operators $(D_{\Theta,0}^+)'(Q)$ and $D_\Theta^+(Q)$ to be restrictions of $D_{\max}^+(Q)$ to the domains

$$\operatorname{dom}(D_{\Theta,0}^+)'(Q) = \{y \in \operatorname{dom} D_{\max}^+(Q) : y(0) \in \Theta \text{ and } \operatorname{supp} y \text{ is compact}\}$$

and

$$\operatorname{dom} D_\Theta^+(Q) = \{y \in \operatorname{dom} D_{\max}^+(Q) : y(0) \in \Theta\},$$

respectively. Denote also by $D_{\Theta,0}^+(Q)$ the closure of $(D_{\Theta,0}^+)'(Q)$.

Theorem 3.7. *With the above notations the following holds:*

(i) *For any subspace Θ of $\mathbb{C}^n$ the operators $D^+_{\Theta,0}(Q)$ and $D^+_{\Theta}(Q)$ coincide, that is*

$$D^+_{\Theta,0}(Q) = D^+_{\Theta}(Q). \tag{3.15}$$

Moreover,

$$(D^+_{\Theta}(Q))^* = D^+_{\Theta^\times}(Q^*). \tag{3.16}$$

(ii) *The equality*

$$\widetilde{A} = D^+_{\Theta}(Q) \tag{3.17}$$

establishes a bijective correspondence between all subspaces $\Theta \subset \mathbb{C}^n$ and all extensions $\widetilde{A} \in \mathrm{Ext}\,\{D^+_{\min}(Q), D^+_{\min}(Q^*)\}$. In particular, $D^+_{\min}(Q) = D^+_{\Theta_0}(Q)$ with $\Theta_0 = \{0\}$ and hence

$$\mathrm{dom}\, D^+_{\min}(Q) = \{y \in \mathrm{dom}\, D^+_{\max}(Q) : y(0) = 0\}. \tag{3.18}$$

(iii) *The dual pair $\{D^+_{\min}(Q), D^+_{\min}(Q^*)\}$ is correct if and only if $n = 2p$. If this condition is satisfied, then the equality (3.17) gives a bijective correspondence between all subspaces $\Theta \subset \mathbb{C}^{2p}$ with $\dim \Theta = p$ and all quasi-selfadjoint extensions $\widetilde{A} \in \mathrm{Ext}\,\{D^+_{\min}(Q), D^+_{\min}(Q^*)\}$.*

Proof: (i) We first show that

$$(D^+_{\Theta,0}(Q))^* = (D^+_{\Theta}(Q))^* = D^+_{\Theta^\times}(Q^*). \tag{3.19}$$

Since $(D^+_{\min})'(Q) \subset (D^+_{\Theta,0})'(Q) \subset D^+_{\Theta}(Q)$ and $((D^+_{\min})'(Q))^* = (D^+_{\min}(Q))^* = D^+_{\max}(Q^*)$, we have the inclusions $((D^+_{\Theta,0})'(Q))^* \subset D^+_{\max}(Q^*)$ and $(D^+_{\Theta}(Q))^* \subset D^+_{\max}(Q^*)$. This and the Lagrange identity (3.12) yield

$$\mathrm{dom}\,((D^+_{\Theta,0})'(Q))^*$$
$$= \{z \in \mathrm{dom}\, D^+_{\max}(Q^*) : (J_n y(0), z(0)) = 0,\ y \in \mathrm{dom}\,(D^+_{\Theta,0})'(Q)\},$$

$$\mathrm{dom}\,(D^+_{\Theta}(Q))^* = \{z \in \mathrm{dom}\, D^+_{\max}(Q^*) : (J_n y(0), z(0)) = 0,\ y \in \mathrm{dom}\, D^+_{\Theta}(Q)\}.$$

It follows from Lemma 3.6 that

$$\{y(0) : y \in \mathrm{dom}\,(D^+_{\Theta,0})'(Q)\} = \{y(0) : y \in \mathrm{dom}\, D^+_{\Theta}(Q)\} = \Theta.$$

Therefore by (3.14)

$$\mathrm{dom}\,((D^+_{\Theta,0})'(Q))^* = \mathrm{dom}\,(D^+_{\Theta}(Q))^* = \mathrm{dom}\, D^+_{\Theta^\times}(Q^*),$$

which together with the equality $(D'_{\Theta,0}(Q))^* = (D_{\Theta,0}(Q))^*$ yields (3.19). The equalities (3.15) and (3.16) are immediate from (3.19).

(ii) The equalities $D^+_{\min}(Q) = D^+_{\{0\}}(Q)$ and (3.18) directly follow from the definition of $D^+_{\min}(Q)$ and statement (i). Next, for each subspace $\Theta \subseteq \mathbb{C}^n$ the inclusion $D_{\Theta}(Q) \in \mathrm{Ext}\,\{D^+_{\min}(Q), D^+_{\min}(Q^*)\}$ is obvious. Conversely, let $\widetilde{A} \in \mathrm{Ext}\,\{D^+_{\min}(Q), D^+_{\min}(Q^*)\}$ and let

$$\Theta := \{y(0) : y \in \mathrm{dom}\,\widetilde{A}\}. \tag{3.20}$$

Then for any $y \in \operatorname{dom} D_\Theta^+(Q)$ there exists $u \in \operatorname{dom} \widetilde{A}$ such that $y(0) = u(0)$ and hence $v := y - u$ satisfies $v(0) = 0$. Therefore by (3.18) $v \in \operatorname{dom} D_{\min}^+(Q) \subset \operatorname{dom} \widetilde{A}$ and, consequently, $y = u + v \in \operatorname{dom} \widetilde{A}$. Thus $\operatorname{dom} D_\Theta^+(Q) \subset \operatorname{dom} \widetilde{A}$. Since the inverse inclusion $\operatorname{dom} \widetilde{A} \subset \operatorname{dom} D_\Theta^+(Q)$ is obvious, it follows that $\operatorname{dom} \widetilde{A} = \operatorname{dom} D_\Theta^+(Q)$ and hence (3.17) holds with Θ given by (3.20).

(iii) It follows from (3.18) that for each extension $\widetilde{A} = D_\Theta(Q)$ in the set $\operatorname{Ext}\{D_{\min}^+(Q), D_{\min}^+(Q^*)\}$ the equality

$$\dim\left(\operatorname{dom} \widetilde{A}/\operatorname{dom} D_{\min}^+(Q)\right) = \dim \Theta \qquad (3.21)$$

holds. Moreover, by (3.16) one has $\dim\left(\operatorname{dom} \widetilde{A}^*/\operatorname{dom} D_{\min}^+(Q^*)\right) = \dim \Theta^\times$. Combining these facts with the obvious identity $\dim \Theta + \dim \Theta^\times = n$ and statement (ii), one gets the required statement. $\qquad\square$

Corollary 3.8. *With the notation above*

$$n((D_{\min}^+(Q^*))^*, D_{\min}^+(Q))(= \dim\left(\operatorname{dom} D_{\max}^+(Q)/\operatorname{dom} D_{\min}^+(Q)\right)) = n. \quad (3.22)$$

Proof: It follows from Lemma 3.6 that $D_{\max}^+(Q) = D_\Theta(Q)$ with $\Theta = \mathbb{C}^n$. Combining this relation with (3.21) yields (3.22). $\qquad\square$

Remark 3.9. *If $n = 2p$ and*

$$J_n = \begin{pmatrix} 0 & -I_p \\ I_p & 0 \end{pmatrix} : \mathbb{C}^p \oplus \mathbb{C}^p \to \mathbb{C}^p \oplus \mathbb{C}^p, \qquad (3.23)$$

then a subspace $\Theta \in \mathbb{C}^p \oplus \mathbb{C}^p$ is called a linear relation in $\mathbb{C}^p$ and the linear relation

$$\begin{aligned} \Theta^* &= \mathbb{C}^n \ominus J_n \Theta \\ &= \{\{h, h'\} \in \mathbb{C}^p \oplus \mathbb{C}^p : (h', k)_{\mathbb{C}^p} - (h, k')_{\mathbb{C}^p} = 0 \text{ for all } \{k, k'\} \in \Theta\} \end{aligned}$$

is called the adjoint relation of Θ (see e.g. [11]). Clearly, in this case $\Theta^\times = \Theta^*$. Moreover, if the Dirac-type expression (3.1), (3.23) is formally selfadjoint and $\Theta = \Theta^*$, then by Theorem 3.7 we have that $D_\Theta^+(Q)(= D_{\Theta,0}^+(Q))$ is a selfadjoint extension of $D_{\min}^+(Q)$.

Definition 3.10. *A pair of operators (matrices) $C_1, C_2 \in \boldsymbol{B}(\mathbb{C}^p)$ will be called admissible if $\operatorname{rank}(C_1, C_2) = p$.*

Corollary 3.11. *Assume that $n = 2p$, so that functions $y \in \operatorname{dom} D_{\max}^+(Q)$ and $z \in \operatorname{dom} D_{\max}^+(Q^*)$ admit the representation*

$$y(t) = \{y_1(t), y_2(t)\}(\in \mathbb{C}^p \oplus \mathbb{C}^p), \ z(t) = \{z_1(t), z_2(t)\}(\in \mathbb{C}^p \oplus \mathbb{C}^p), \ t \in \mathbb{R}_+. \tag{3.24}$$

Then for each admissible operator pair $C_1, C_2 \in \boldsymbol{B}(\mathbb{C}^p)$ the equalities (the boundary conditions)

$$\operatorname{dom} \widetilde{A} = \{y \in \operatorname{dom} D_{\max}^+(Q) : C_1 y_1(0) + C_2 y_2(0) = 0\}, \qquad (3.25)$$

$$\widetilde{A} = D_{\max}^+(Q) \restriction \operatorname{dom} \widetilde{A} \qquad (3.26)$$

define a quasi-selfadjoint extension $\widetilde{A} \in \operatorname{Ext}\{D_{\min}^+(Q), D_{\min}^+(Q^*)\}$ and conversely for each such extension $\widetilde{A}$ there exists an admissible operator pair C_1, $C_2 \in \boldsymbol{B}(\mathbb{C}^p)$ such that (3.26) holds.

Proof: Clearly, for each admissible pair $C_1, C_2 \in \boldsymbol{B}(\mathbb{C}^p)$ the equality

$$\Theta = \{\{h_1, h_2\} \in \mathbb{C}^p \oplus \mathbb{C}^p : C_1 h_1 + C_2 h_2 = 0\} \qquad (3.27)$$

defines a subspace $\Theta \in \mathbb{C}^{2p}$ with dimension $\dim \Theta = p$ and conversely for each such subspace Θ there exists an admissible pair $C_1, C_2 \in \boldsymbol{B}(\mathbb{C}^p)$ such that (3.27) holds. This and Theorem 3.7, (iii) yield the desired statement. $\qquad \square$

Corollary 3.12. *Let $A := D_{\min}^+(Q)$ and $B := D_{\min}^+(Q^*)$. Assume that there exists an extension $\widetilde{A} \in \operatorname{Ext}\{A, B\}$ such that $\rho(\widetilde{A}) \neq \emptyset$ and $\dim \mathfrak{N}_{\overline{\lambda_0}}(A) = \dim \mathfrak{N}_{\lambda_0}(B) =: p$ for some $\lambda_0 \in \rho(\widetilde{A})$. Then:*

(i) *$n = 2p$ and*

$$\dim \mathfrak{N}_{\overline{\lambda}}(A) = \dim \mathfrak{N}_\lambda(B) = p, \quad \lambda \in \rho(\widetilde{A}); \qquad (3.28)$$

(ii) *$\widetilde{A}$ is a quasi-selfadjoint extension of the dual pair $\{A, B\}$ (hence the dual pair is correct) and $\widetilde{A} = D_\Theta(Q)$ with a subspace $\Theta \subset \mathbb{C}^{2p}$ of the dimension $\dim \Theta = p$.*

Proof: (i) Combining of (3.22) with (2.49) gives $n = 2p$. Equality (3.28) is immediate from Corollary 2.14.

(ii) This statement follows from (2.40), (2.41) and Theorem 3.7, (iii). $\qquad \square$

Finally, we summarize our main results on non-selfadjoint Dirac type operators.

Theorem 3.13. *Assume that $n = 2p$. Then:*

(i) the pair $\{D_{\min}^+(Q), D_{\min}^+(Q^*)\}$ forms the correct dual pair;

(ii) a proper extension $\widetilde{A} \in \operatorname{Ext}\{D_{\min}^+(Q), D_{\min}^+(Q^*)\}$ is quasi-selfadjoint if and only if $\widetilde{A} = D_\Theta(Q)$ with Θ being a p-dimensional subspace of $\mathbb{C}^{2p}$, $\dim \Theta = p$;

(iii) Assume in addition that for some subspace $\Theta \subset \mathbb{C}$ with $\dim \Theta = p$ the corresponding (quasi-selfadjoint) extension $\widetilde{A} := D_\Theta^+(Q)$ has the nonempty resolvent set, $\rho(\widetilde{A}) \neq \emptyset$. Then

$$\dim \ker\left(D_{\max}^+(Q) - z\right) = \dim \ker\left(D_{\max}^+(Q^*) - \overline{z}\right) = p \quad \text{for any} \quad z \in \rho(\widetilde{A}). \qquad (3.29)$$

Proof: (i), (ii). It follows from Theorem 3.7, (iii) and (3.21) that $\widetilde{A}$ is a quasi-selfadjoint extension of the dual pair $\{D_{\min}^+(Q), D_{\min}^+(Q^*)\}$ if and only if $\dim\left(\operatorname{dom} \widetilde{A}/\operatorname{dom} D_{\min}^+(Q)\right) = \dim \Theta = p$.

(iii) Since $\dim \Theta = p$, the extension $\widetilde{A} = D_\Theta^+(Q)$ is quasi-selfadjoint, and the required relations (3.29) are implied by identities (2.40), (2.41). $\qquad \square$

4. Weyl solutions and Weyl functions for Dirac-type operators on the half-line

4.1. Boundary triples for dual pairs and their Weyl functions

We first recall the definitions of a boundary triple for a dual pair $\{A, B\}$ of closed densely defined operators A, B in $\mathfrak{H}$ and their respective γ-field and Weyl function.

Definition 4.1. *[24] A collection* $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma^B, \Gamma^A\}$ *consisting of Hilbert spaces* $\mathcal{H}_0$ *and* $\mathcal{H}_1$ *and linear mappings*

$$\Gamma^B = \begin{pmatrix} \Gamma_0^B \\ \Gamma_1^B \end{pmatrix} : \operatorname{dom} B^* \to \mathcal{H}_0 \oplus \mathcal{H}_1 \quad and \quad \Gamma^A = \begin{pmatrix} \Gamma_0^A \\ \Gamma_1^A \end{pmatrix} : \operatorname{dom} A^* \to \mathcal{H}_1 \oplus \mathcal{H}_0$$

is called a boundary triple for $\{A, B\}$ *if the mappings* Γ^B *and* Γ^A *are surjective and the following Green identity holds:*

$$(B^* f, g) - (f, A^* g) = (\Gamma_1^B f, \Gamma_0^A g) - (\Gamma_0^B f, \Gamma_1^A g), \qquad f \in \operatorname{dom} B^*, \ g \in \operatorname{dom} A^*. \tag{4.1}$$

With each boundary triple $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma^B, \Gamma^A\}$ for $\{A, B\}$ one associates an extension $A_0 \in \operatorname{Ext}\{A, B\}$ given by $A_0 = B^* \upharpoonright \ker \Gamma_0^B$. Moreover, if $\rho(A_0) \neq \emptyset$, one associates with Π a γ-field and the corresponding Weyl function.

Definition 4.2. *[28] The operator-functions* $M(\cdot) : \rho(A_0) \to \boldsymbol{B}(\mathcal{H}_0, \mathcal{H}_1)$ *and* $\gamma(\cdot) : \rho(A_0) \to \boldsymbol{B}(\mathcal{H}_0, \mathfrak{H})$ *defined for* $\lambda \in \rho(A_0)$ *by*

$$\Gamma_1^B \upharpoonright \mathfrak{N}_\lambda(B) = M(\lambda)(\Gamma_0^B \upharpoonright \mathfrak{N}_\lambda(B)) \quad and \quad \gamma(\lambda) = (\Gamma_0^B \upharpoonright \mathfrak{N}_\lambda(B))^{-1}. \tag{4.2}$$

are called the Weyl function and the γ-field *of the boundary triple* $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma^B, \Gamma^A\}$ *for* $\{A, B\}$ *respectively.*

Note that Lemma 2.18 ensures the γ-field $\gamma(\cdot)$ and Weyl function are well defined. Indeed, it follows from the decomposition (2.46) that for each $\lambda \in \rho(A_0)$ the mapping $\Gamma_0^B \upharpoonright \mathfrak{N}_\lambda(B)$ is a topological isomorphism of $\mathfrak{N}_\lambda(B)$ onto $\mathcal{H}_0$, hence $\gamma(\cdot)$ is well defined on $\rho(A_0)$. Moreover, it is shown in [28] that $M(\cdot)$ and $\gamma(\cdot)$ are holomorphic in $\rho(A_0)$.

Remark 4.3. *Let S be a closed densely defined symmetric operator in $\mathfrak{H}$. Then a collection* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *consisting of a Hilbert space $\mathcal{H}$ and linear operators* $\Gamma_j : \operatorname{dom} S^* \to \mathcal{H}$, $j \in \{0, 1\}$, *is called a boundary triple for S^* if the operator* $\Gamma = (\Gamma_0, \Gamma_1)^\top$ *is surjective and the abstract Green identity*

$$(S^* f, g) - (f, S^* g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g), \quad f, g \in \operatorname{dom} S^* \tag{4.3}$$

is valid (see e.g. [17]). In this connection note that Definition 4.1 of a boundary triple $\{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma^B, \Gamma^A\}$ for a dual pair $\{A, B\}$ generalizes the above definition of a boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for S^* and coincides with it in the case $A = B =: S$ if additionally $\mathcal{H}_1 = \mathcal{H}_0 =: \mathcal{H}$ and $\Gamma^B = \Gamma^A =: \Gamma$. Observe also that in this case $S_0 = S^* \upharpoonright \ker \Gamma_0$ is a selfadjoint extension of S and the Weyl function $M(\cdot)$ of the triple Π in the sense of Definition 4.2 turns into the Weyl function in the sense of [12].

Lemma 4.4. *Let S be a closed densely defined symmetric operator in $\mathfrak{H}$ with the deficiency indices $n_\pm(S) = \dim \mathfrak{N}_\lambda(S)$, $\lambda \in \mathbb{C}_\pm$, let $T \in \boldsymbol{B}(\mathfrak{H})$ be a selfadjoint operator, let $\alpha = \inf\limits_{f \in \mathfrak{H}} \frac{(Tf,f)}{||f||^2}$ and $\beta = \sup\limits_{f \in \mathfrak{H}} \frac{(Tf,f)}{||f||^2}$ be the lower lower and the upper bounds of T respectively and let*

$$A = S + iT, \qquad B = S - iT.$$

Then:

(i) *We have $\operatorname{dom} A = \operatorname{dom} B = \mathcal{D}$, where $\mathcal{D} := \operatorname{dom} S$. Moreover, for $f \in \mathcal{D}$*

$$\alpha ||f||^2 \le \operatorname{Im}(Af, f) \le \beta ||f||^2, \qquad -\beta ||f||^2 \le \operatorname{Im}(Bf, f) \le -\alpha ||f||^2,$$

and the operators A and B form a dual pair $\{A,B\}$.

(ii) *The inclusion $(\mathbb{C}_{\alpha,-} \cup \mathbb{C}_{\beta,+}) \subset \widehat{\rho}\{A, B\}$ holds and*

$$\dim \mathfrak{N}_\lambda(B) = n_+(S), \qquad \dim \mathfrak{N}_{\overline{\lambda}}(A) = n_-(S), \quad \lambda \in \mathbb{C}_{\beta,+}, \qquad (4.4)$$
$$\dim \mathfrak{N}_\lambda(B) = n_-(S), \qquad \dim \mathfrak{N}_{\overline{\lambda}}(A) = n_+(S), \quad \lambda \in \mathbb{C}_{\alpha,-}. \qquad (4.5)$$

(iii) *The following equalities hold*

$$\operatorname{dom} A^* = \operatorname{dom} B^* = \mathcal{D}_*, \qquad A^* = S^* - iT, \quad B^* = S^* + iT, \qquad (4.6)$$

where $\mathcal{D}_* := \operatorname{dom} S^*$.

Proof: Statements (i) and (iii) are obvious. Statement (ii) is immediate from the known results of the perturbation theory for linear operators (see e.g. [1, Theorem 3.7.1]) □

Theorem 4.5. *With the assumptions of Lemma 4.4 let also $n_+(S) = n_-(S)$, $\mathcal{D}_* = \operatorname{dom} S^*$ and let $\widehat{\Pi} = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for S^* (see Remark 4.3). Then:*

(i) *the collection $\Pi = \{\mathcal{H} \oplus \mathcal{H}, \Gamma^B, \Gamma^A\}$ with operators $\Gamma^B = (\Gamma_0^B, \Gamma_1^B)^\top : \mathcal{D}_* \to \mathcal{H} \oplus \mathcal{H}$ and $\Gamma^A = (\Gamma_0^A, \Gamma_1^A)^\top : \mathcal{D}_* \to \mathcal{H} \oplus \mathcal{H}$ defined by*

$$\Gamma_0^B = \Gamma_0^A = \Gamma_0, \qquad \Gamma_1^B = \Gamma_1^A = \Gamma_1 \qquad (4.7)$$

is a boundary triple for the dual pair $\{A, B\}$. Moreover, the extension $A_0 (= B^* \upharpoonright \ker \Gamma_0^B) \in \operatorname{Ext} \{A, B\}$ satisfies

$$\alpha ||f||^2 \le \operatorname{Im}(A_0 f, f) \le \beta ||f||^2, \ f \in \operatorname{dom} A_0, \text{ and } \mathbb{C}_{\alpha,-} \cup \mathbb{C}_{\beta,+} \subset \rho(A_0); \qquad (4.8)$$

(ii) *the Weyl function $M(\cdot)$ of the triple Π satisfies the identities*

$$M(\lambda) - M^*(\mu) = \gamma^*(\mu)[(\lambda - \overline{\mu})I_\mathfrak{H} - 2iT]\gamma(\lambda), \quad \lambda, \mu \in \rho(A_0) \qquad (4.9)$$
$$\operatorname{Im} M(\lambda) = \gamma^*(\lambda)(\operatorname{Im} \lambda \cdot I_\mathfrak{H} - T)\gamma(\lambda), \quad \lambda \in \rho(A_0), \qquad (4.10)$$

where $\gamma(\lambda)$ is the γ-field of Π. Moreover,

$$\operatorname{Im} M(\lambda) \ge \delta_\lambda I, \quad \lambda \in \mathbb{C}_{\beta,+}; \qquad \operatorname{Im} M(\lambda) \le \delta_\lambda' I, \quad \lambda \in \mathbb{C}_{\alpha,-} \qquad (4.11)$$

with some $\delta_\lambda > 0$ and $\delta_\lambda' < 0$ (depending on λ).

Proof: (i) Since by (4.7) we have $\Gamma^B = \Gamma^A = \Gamma$ and the mapping Γ is surjective, so are the mappings Γ^B and Γ^A. Moreover, using (4.6), (4.7) and the identity (4.3) (for the triple $\widehat{\Pi}$) one gets

$$
\begin{aligned}
(B^* f, g) - (f, A^* g) &= (S^* f, g) + i(Tf, g) - (f, S^* g) - i(f, Tg) \\
&= (S^* f, g) - (f, S^* g) \\
&= (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g) \\
&= (\Gamma_1^B f, \Gamma_0^A g) - (\Gamma_0^B f, \Gamma_1^A g), \quad f, g \in \mathcal{D}_*.
\end{aligned}
$$

Hence Γ^B and Γ^A satisfy (4.1) and, consequently, $\Pi = \{\mathcal{H} \oplus \mathcal{H}, \Gamma^B, \Gamma^A\}$ is a boundary triple for $\{A, B\}$.

Next assume that $S_0 = S^* \upharpoonright \ker \Gamma_0$ is a selfadjoint extension of S corresponding to the triple $\widehat{\Pi}$ for S^*. Then

$$
A_0 = B^* \upharpoonright \ker \Gamma_0^B = (S^* + iT) \upharpoonright \ker \Gamma_0 = S_0 + iT
$$

and Lemma 4.4 with $n_\pm(S_0) = 0$ yields (4.8).

(ii) Since $\mathfrak{N}_\lambda(B) = \{f \in \operatorname{dom} B^* : B^* f = \lambda f\}$, it follows from (4.6) that

$$
\mathfrak{N}_\lambda(B) = \{f \in \mathcal{D}_* : S^* f = \lambda f - iTf\}, \quad \lambda \in \mathbb{C}. \tag{4.12}
$$

Assume that $h \in \mathcal{H}$, $\lambda, \mu \in \rho(A_0)$ and

$$
f_\lambda = \gamma(\lambda) h \in \mathfrak{N}_\lambda(B), \qquad g_\mu = \gamma(\mu) h \in \mathfrak{N}_\mu(B).
$$

Then by (4.12)

$$
S^* f_\lambda = \lambda f_\lambda - iT f_\lambda = (\lambda I - iT)\gamma(\lambda)h, \qquad S^* g_\mu = \mu g_\mu - iT g_\mu = (\mu I - iT)\gamma(\mu)h
$$

and hence

$$
\begin{aligned}
&(S^* f_\lambda, g_\mu) - (f_\lambda, S^* g_\mu) \\
&= ((\lambda I - iT)\gamma(\lambda)h, \gamma(\mu)h) - (\gamma(\lambda)h, (\mu I - iT)\gamma(\mu)h) \\
&= (\gamma^*(\mu)[(\lambda I - iT) - (\overline{\mu} I + iT)]\gamma(\lambda)h, h) \\
&= (\gamma^*(\mu)[(\lambda - \overline{\mu})I - 2iT]\gamma(\lambda)h, h).
\end{aligned}
$$

On the other hand, by the definition of $\gamma(\lambda)$ and $M(\lambda)$ one has

$$
\Gamma_0 f_\lambda = \Gamma_0^B f_\lambda = h, \quad \Gamma_1 f_\lambda = \Gamma_1^B f_\lambda = M(\lambda)h,
$$

$$
\Gamma_0 g_\mu = \Gamma_0^B g_\mu = h, \quad \Gamma_1 g_\mu = \Gamma_1^B g_\mu = M(\mu)h
$$

and, consequently,

$$
\begin{aligned}
(\Gamma_1 f_\lambda, \Gamma_0 g_\mu) - (\Gamma_0 f_\lambda, \Gamma_1 g_\mu) &= (M(\lambda)h, h) - (h, M(\mu)h) \\
&= ((M(\lambda) - M^*(\mu))h, h).
\end{aligned}
$$

Now an application of the identity (4.3) for the triple $\widehat{\Pi}$ to f_λ and g_μ yields

$$
(\gamma^*(\mu)[(\lambda - \overline{\mu})I - 2iT]\gamma(\lambda)h, h) = ((M(\lambda) - M^*(\mu))h, h), \ h \in \mathcal{H}, \ \lambda, \mu \in \rho(A_0).
$$

This implies identity (4.9), which in turn gives (4.10). Finally, (4.11) directly follows from (4.10). $\square$

4.2. The Weyl solution and the Weyl function for Dirac operators

In this subsection we assume that the Dirac-type expression (3.1) is defined on the half-line $\mathbb{R}_+$. First we prove the existence of the Weyl solution on $\mathbb{R}_+$.

Theorem 4.6. *Suppose that $n = 2p$ and the operator J_n in (3.1) is of the form (3.23). Let $A := D_{\min}^+(Q)$ and $B := D_{\min}^+(Q^*)$, so that $A^* = D_{\max}^+(Q^*)$ and $B^* = D_{\max}^+(Q)$. Moreover, let $C_1, C_2 \in \boldsymbol{B}(\mathbb{C}^p)$ be an admissible operator pair and let $\widetilde{A}$ be the corresponding quasi-selfadjoint extension of the form (3.26). If $\rho(\widetilde{A}) \neq \emptyset$, then the following hold:*

(i) *There exist a pair of operators*

$$X = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} : \mathbb{C}^p \oplus \mathbb{C}^p \to \mathbb{C}^p \oplus \mathbb{C}^p, \tag{4.13}$$

$$Y = \begin{pmatrix} C_1' & C_2' \\ C_3' & C_4' \end{pmatrix} : \mathbb{C}^p \oplus \mathbb{C}^p \to \mathbb{C}^p \oplus \mathbb{C}^p \tag{4.14}$$

such that $Y^* J_n X = J_n$ and for each such a pair the equalities

$$\Gamma_0^B y = C_1 y_1(0) + C_2 y_2(0), \ \ \Gamma_1^B y = C_3 y_1(0) + C_4 y_2(0), \tag{4.15}$$

$$\Gamma_0^A y = C_1' z_1(0) + C_2' z_2(0), \ \ \Gamma_1^A y = C_3' z_1(0) + C_4' z_2(0), \tag{4.16}$$

$$\Gamma^B = (\Gamma_0^B, \Gamma_1^B)^\top, \qquad \Gamma^A = (\Gamma_0^A, \Gamma_1^A)^\top, \tag{4.17}$$

where $y \in \operatorname{dom} B^*$ and $z \in \operatorname{dom} A^*$, define a boundary triple $\Pi = \{\mathbb{C}^p \oplus \mathbb{C}^p, \Gamma^B, \Gamma^A\}$ for $\{A, B\}$ (in (4.15) and (4.16) y_j and z_j are taken from (3.24)). Moreover, for this triple $A_0 (= B^* \upharpoonright \ker \Gamma_0^B) = \widetilde{A}$.

(ii) *For each $\lambda \in \rho(\widetilde{A})$ the homogeneous equation*

$$D(Q)y = \lambda y \tag{4.18}$$

has a unique L^2-operator-valued solution (the Weyl solution)

$$v(\cdot, \lambda) = (v_1(\cdot, \lambda), v_2(\cdot, \lambda))^\top \in L^2(\mathbb{R}_+, \boldsymbol{B}(\mathbb{C}^p, \mathbb{C}^{2p})) \tag{4.19}$$

satisfying

$$C_1 v_1(0, \lambda) + C_2 v_2(0, \lambda) = I_p, \quad \lambda \in \rho(\widetilde{A}). \tag{4.20}$$

Moreover, the solution $v(\cdot, \lambda)$ is holomorphic in $\lambda \in \rho(\widetilde{A})$.

(iii) *The Weyl function $M(\cdot)$ of the triple Π is*

$$M(\lambda) = C_3 v_1(0, \lambda) + C_4 v_2(0, \lambda), \quad \lambda \in \rho(\widetilde{A}). \tag{4.21}$$

If $\varphi(\cdot, \lambda)$ and $\psi(\cdot, \lambda)$ are $\boldsymbol{B}(\mathbb{C}^p, \mathbb{C}^p \oplus \mathbb{C}^p)$-valued solutions of (4.18) satisfying

$$X\varphi(0, \lambda) = \begin{pmatrix} 0 \\ I_p \end{pmatrix}, \qquad X\psi(0, \lambda) = \begin{pmatrix} I_p \\ 0 \end{pmatrix}, \tag{4.22}$$

then the Weyl function $M(\cdot)$ can also be defined as the unique operator-valued function $M(\cdot) : \rho(\widetilde{A}) \to \boldsymbol{B}(\mathbb{C}^p)$ such that

$$\varphi(\cdot, \lambda)M(\lambda) + \psi(\cdot, \lambda) \in L^2(\mathbb{R}_+, \boldsymbol{B}(\mathbb{C}^p, \mathbb{C}^n)), \quad \lambda \in \rho(\widetilde{A}). \tag{4.23}$$

(iv) *If $\lambda \in \rho(\widetilde{A})$ and $\widehat{v}(\cdot, \lambda) = (\widehat{v}_1(\cdot, \lambda), \widehat{v}_2(\cdot, \lambda))^\top \in L^2(\mathbb{R}_+, \boldsymbol{B}(\mathbb{C}^p, \mathbb{C}^{2p}))$ is an operator solution of (4.18), then the operator $C_1 v_1(0, \lambda) + C_2 v_2(0, \lambda)$ is invertible and*

$$M(\lambda) = (C_3 v_1(0, \lambda) + C_4 v_2(0, \lambda))(C_1 v_1(0, \lambda) + C_2 v_2(0, \lambda))^{-1}.$$

Proof: (i) Let $\widetilde{\Gamma}^B = (\widetilde{\Gamma}_0^B, \widetilde{\Gamma}_1^B)^\top$ and $\widetilde{\Gamma}^A = (\widetilde{\Gamma}_0^A, \widetilde{\Gamma}_1^A)^\top$ be operators defined by

$$\widetilde{\Gamma}_0^B y = y_1(0), \quad \widetilde{\Gamma}_1^B y = y_2(0), \quad y \in \text{dom } B^*, \tag{4.24}$$

$$\widetilde{\Gamma}_0^A z = z_1(0), \quad \widetilde{\Gamma}_1^A z = z_2(0), \quad z \in \text{dom } A^*. \tag{4.25}$$

It follows from (3.23) and the Lagrange identity (3.12) that these operators satisfy (4.1). Moreover, by Lemma 3.6 the operators Γ^B and Γ^A are surjective. Therefore the collection $\widetilde{\Pi} = \{\mathbb{C}^p \oplus \mathbb{C}^p, \widetilde{\Gamma}^B, \widetilde{\Gamma}^A\}$ is a boundary triple for $\{A, B\}$.

Since the pair C_1, C_2 is admissible, there exist operators $C_3, C_4 \in \boldsymbol{B}(\mathbb{C}^p)$ and $C_j' \in \boldsymbol{B}(\mathbb{C}^p)$, $j \in \{1, \ldots, 4\}$, such that the equalities (4.13), (4.14) define invertible operators X and Y satisfying $Y^* J X = J$. Moreover, according to [28, Proposition 4.6] for each such pair the equalities

$$\Gamma^B = \begin{pmatrix} \Gamma_0^B \\ \Gamma_1^B \end{pmatrix} := \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} \begin{pmatrix} \widetilde{\Gamma}_0^B \\ \widetilde{\Gamma}_1^B \end{pmatrix}, \quad \Gamma^A = \begin{pmatrix} \Gamma_0^A \\ \Gamma_1^A \end{pmatrix} := \begin{pmatrix} C_1' & C_2' \\ C_3' & C_4' \end{pmatrix} \begin{pmatrix} \widetilde{\Gamma}_0^A \\ \widetilde{\Gamma}_1^A \end{pmatrix}$$

define a boundary triple $\Pi = \{\mathbb{C}^p \oplus \mathbb{C}^p, \Gamma^B, \Gamma^A\}$ for the dual pair $\{A, B\}$ and in view of (4.24), (4.25), the mappings Γ^B and Γ^A are of the form (4.15) – (4.17). Moreover, the equality $A_0 = \widetilde{A}$ is implied by combining (3.26) with (4.15).

(ii) Let $\gamma(\cdot)$ be the γ-field (4.2) of the triple Π. Since $\mathfrak{N}_\lambda(B)$ coincides with the space of all $L^2(\mathbb{R}_+, \mathbb{C}^n)$-solutions of equation (4.18), the equality

$$v(t, \lambda)h = (\gamma(\lambda)h)(t), \qquad h \in \mathbb{C}^p, \quad \lambda \in \rho(\widetilde{A}) \tag{4.26}$$

defines a unique operator solution $v(t, \lambda)(\in \boldsymbol{B}(\mathbb{C}^p, \mathbb{C}^n))$ of (4.18) such that $v(\cdot, \lambda)h \in L^2(\mathbb{R}_+, \mathbb{C}^n)$ and

$$\Gamma_0^B v(\cdot, \lambda)h = h, \quad h \in \mathbb{C}^p, \ \lambda \in \rho(\widetilde{A}). \tag{4.27}$$

Moreover, it follows from (4.15) and the block representation (4.19) of $v(\cdot, \lambda)$ that (4.27) is equivalent to (4.20). The holomorphy of $v(\cdot, \lambda)$ in $\rho(\widetilde{A})$ is immediate from (4.26) since the γ-field $\gamma(\cdot)$ is holomorphic on $\rho(\widetilde{A})$.

(iii) Recall that in accordance with (4.2), $M(\lambda) = \Gamma_1^B \gamma(\lambda)$. Therefore it follows from (4.26) and (4.15) that (4.21) holds. Next, combining (4.20) with (4.21) yields

$$X v(0, \lambda) = \begin{pmatrix} I_p \\ M(\lambda) \end{pmatrix} = X(\varphi(0, \lambda)M(\lambda) + \psi(0, \lambda)).$$

Since X is invertible, one gets $v(t, \lambda) = \varphi(t, \lambda)M(\lambda) + \psi(t, \lambda)$. This proves the second statement in (iii).

(iv) This statement is immediate by combining (ii) with (iii). $\qquad\square$ The following theorem is an extended version of Theorem 1.3 from the Introduction. In particular, we prove here the existence of a holomorphic Weyl solution for a j-symmetric Dirac-type operator on the half-line.

Theorem 4.7. *Let $n = 2p$, let $D(Q)$ be the Dirac-type expression (3.1) on $\mathbb{R}_+$ with J_n and $Q(\cdot)$ given by (3.6), let the relations (3.7) be fulfilled and let j be the conjugation in $L^2(\mathbb{R}_+, \mathbb{C}^n)$ given by (3.4). If $\widehat{\rho}(D^+_{\min}(Q)) \neq \emptyset$, then:*

(i) *The minimal operator $D^+_{\min}(Q)$ is j-symmetric and $\mathrm{Def}(D^+_{\min}(Q)) = p$, i.e., the following equality holds for each $\lambda \in \widehat{\rho}(D^+_{\min}(Q))$ (or, equivalently, for each $\lambda \in \widehat{\rho}\{D^+_{\min}(Q), D^+_{\min}(Q^*)\}$):*

$$\dim \mathfrak{N}_\lambda(D^+_{\min}(Q^*)) = \dim \mathfrak{N}_{\overline{\lambda}}(D^+_{\min}(Q)) = p. \qquad (4.28)$$

(ii) *There exists an operator $C \in \boldsymbol{B}(\mathbb{C}^p, \mathbb{C}^n)$ such that the equalities*

$$\widetilde{A} = D^+_{\max}(Q) \restriction \mathrm{dom}\,\widetilde{A}, \quad \mathrm{dom}\,\widetilde{A} = \{y \in \mathrm{dom}\,D^+_{\max}(Q) : Cy = 0\},$$

define a j-selfadjoint extension $\widetilde{A}$ of $D^+_{\min}(Q)$ with $\rho(\widetilde{A}) \neq \emptyset$ and for each $\lambda \in \rho(\widetilde{A})$ there exists a unique operator-valued solution of (4.18) (the Weyl solution) $\Psi_+(\cdot, \lambda) \in L^2(\mathbb{R}_+, \boldsymbol{B}(\mathbb{C}^p, \mathbb{C}^n))$ satisfying the condition $C\Psi_+(0, \lambda) = I_p$. Moreover, the solution $\Psi_+(\cdot, \lambda)$ is holomorphic in $\rho(\widetilde{A})$.

Proof: (i) As in Proposition 3.1 (see formula (3.8)) the operator $D^+_{\min}(Q)$ is j-symmetric and

$$j(D^+_{\min}(Q))^* j = j D^+_{\max}(Q^*) j = D^+_{\max}(Q).$$

Combining this formula with (2.81) and Corollary 3.8 (see formula (3.22)) gives

$$2\mathrm{Def}D^+_{\min}(Q) = \dim\left(\mathrm{dom}\,(j D^+_{\min}(Q)^* j)/\mathrm{dom}\,D^+_{\min}(Q)\right)$$
$$= \dim\left(\mathrm{dom}\,D^+_{\max}(Q)/\mathrm{dom}\,D^+_{\min}(Q)\right) = n = 2p. \qquad (4.29)$$

This proves the statement.

(ii) According to Proposition 2.27 there exists a j-selfadjoint extension $\widetilde{A}$ of $D^+_{\min}(Q)$ such that $\rho(\widetilde{A}) \neq \emptyset$. Since $\widetilde{A}$ is a quasi-selfadjoint extension, the statement (ii) follows from Corollary 3.11 and Theorem 4.6(ii). $\qquad\square$

Corollary 4.8. *Assume the conditions of Theorem 4.7, $\lambda_0 \in \widehat{\rho}(D^+_{\min}(Q))$ and let $\mathbb{D}(\lambda_0; \varepsilon)$ be a gap for the operator $D^+_{\min}(Q)$. Then there is an operator $C \in \boldsymbol{B}(\mathbb{C}^p, \mathbb{C}^n)$ such that the corresponding Weyl solution $\Psi_+(\cdot, \lambda)$ is holomorphic in $\mathbb{D}(\lambda_0; \varepsilon)$.*

Proof: According to Proposition 2.27 there exists a j-selfadjoint extension $\widetilde{A}$ of $D^+_{\min}(Q)$ preserving the gap $\mathbb{D}(\lambda_0; \varepsilon)$. Starting with this extension it remains to repeat the reasoning of Theorem 4.7(ii). $\qquad\square$

Remark 4.9.   (i) *In the scalar case ($p = 1$) and a Dirac-type expression $D(Q)$ on $\mathbb{R}_+$ with $Q(\cdot)$ of the form (3.11) and arbitrary scalar $q(\cdot)$, Theorem 4.7 was proved by another method in [7] (see also [10]). Note also the paper [20], where equality (4.28) (without a connection with the Weyl type solutions) was proved for a Dirac-type expression (3.1) with a special off-diagonal potential matrix $Q(\cdot)$ such that (3.8) is satisfied.*

(ii) *Clearly, a similar result is valid for the Dirac system on $\mathbb{R}_-$ with a potential matrix $Q \in L^1_{\mathrm{loc}}(\mathbb{R}_-, \mathbb{C}^{n\times n})$, i.e. there exists the unique $\boldsymbol{B}(\mathbb{C}^p, \mathbb{C}^n)$-valued Weyl solution $\Psi_-(\cdot, \lambda) \in L^2(\mathbb{R}_-, \boldsymbol{B}(\mathbb{C}^p, \mathbb{C}^n))$.*
Note also, that in the scalar case ($p = 1$) for a Dirac-type operator on the line with a potential matrix $Q(\in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^{2\times 2}))$ of the form (3.11) a stronger result is proved in [7, Theorem 5.4], namely

$$\sup_{x\in\mathbb{R}} \left[\left(\int_{-\infty}^{x} \|\Psi_-(t, z)\|^2_{\mathbb{C}^2} \right) \cdot \left(\int_x^\infty \|\Psi_+(t, z)\|^2_{\mathbb{C}^2}\, dx \right) \right] < \infty. \qquad (4.30)$$

4.3. The Weyl-function of almost formally selfadjoint Dirac-type operators

The Dirac-type expression (3.1) admits the representation

$$D(Q)y = J_n \frac{dy}{dx} + Q_1(x)y + iQ_2(x)y, \qquad (4.31)$$

where $Q_1(x) = \overline{Q_1(x)} := \operatorname{Re} Q(x)$ and $Q_2(x) = \overline{Q_2(x)} := \operatorname{Im} Q(x)$, $x \in \mathbb{R}_+$.

Definition 4.10. *The Dirac-type expression (4.31) will be called almost formally selfadjoint if the operator-function $Q_2(\cdot)$ is bounded, that is $||Q_2(x)|| \leq C$, $x \in \mathbb{R}_+$, with some $C \in \mathbb{R}$.*

Let the expression (4.31) be almost selfadjoint and let $\alpha(x) = \inf_{h\in\mathbb{C}^n} \frac{(Q_2(x)h,h)}{||h||^2}$ and $\beta(x) = \sup_{h\in\mathbb{C}^n} \frac{(Q_2(x)h,h)}{||h||^2}$ be lower and upper bounds respectively of $Q_2(x)$. In the following we denote by α_Q and β_Q the essential infimum and supremum of $Q_2(x)$ respectively:

$$\alpha_Q = \operatorname{ess} \inf_{x\in\mathbb{R}_+} Q_2(x), \qquad \beta_Q = \operatorname{ess} \sup_{x\in\mathbb{R}_+} Q_2(x).$$

Proposition 4.11. *Let expression (4.31) on $\mathbb{R}_+$ be almost formally selfadjoint, let $\kappa_\pm = \dim \ker(J_n \pm iI_n)$ and let $A := D^+_{\min}(Q)$, $B := D^+_{\min}(Q^*)$. Then $\operatorname{dom} A = \operatorname{dom} B =: \mathcal{D}$, $\operatorname{dom} A^* = \operatorname{dom} B^*$ and for $f \in \mathcal{D}$*

$$\alpha_Q||f||^2 \leq \operatorname{Im}(Af, f) \leq \beta_Q||f||^2, \qquad -\beta_Q||f||^2 \leq \operatorname{Im}(Bf, f) \leq -\alpha_Q||f||^2.$$

Moreover, $(\mathbb{C}_{\alpha_Q,-} \cup \mathbb{C}_{\beta_Q,+}) \subset \widehat{\rho}\{A, B\}$ and

$$\dim \mathfrak{N}_\lambda(B) = \kappa_+, \qquad \dim \mathfrak{N}_{\overline{\lambda}}(A) = \kappa_-, \quad \lambda \in \mathbb{C}_{\beta_Q,+},$$

$$\dim \mathfrak{N}_\lambda(B) = \kappa_-, \qquad \dim \mathfrak{N}_{\overline{\lambda}}(A) = \kappa_+, \quad \lambda \in \mathbb{C}_{\alpha_Q,-}.$$

Proof: Let $S := D^+_{\min}(Q_1)$ be the minimal operator generated in $\mathfrak{H}$ by the formally selfadjoint Dirac-type expression

$$D(Q_1)y = J_n \frac{dy}{dx} + Q_1(x)y \qquad (4.32)$$

and let T be the multiplication operator in $\mathfrak{H}$ defined by

$$(Tf)(x) = Q_2(x)f(x), \quad f(\cdot) \in \mathfrak{H}. \qquad (4.33)$$

Clearly, S is a closed densely defined symmetric operator in $\mathfrak{H}$, and in accordance with [22, Theorem 5.2] $n_\pm(S) = \kappa_\pm$. Moreover, $T = T^* \in \boldsymbol{B}(\mathfrak{H})$ and the lower and upper bounds of T are α_Q and β_Q respectively. Therefore (4.31) yields

$$A = S + iT, \qquad B = S - iT. \qquad (4.34)$$

It remains to apply Lemma 4.4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 4.12. *Assume that $n = 2p$, expression (4.31) on $\mathbb{R}_+$ is almost formally selfadjoint and J_n is of the form (3.23). Then:*

(i) *For each operator $\Phi = \Phi^* \in \boldsymbol{B}(\mathbb{C}^p)$ the equalities (the boundary conditions)*

$$\operatorname{dom} \widetilde{A}_\Phi = \{y \in \operatorname{dom} D^+_{\max}(Q) : \cos\Phi \cdot y_1(0) + \sin\Phi \cdot y_2(0) = 0\},$$

$$\widetilde{A}_\Phi = D^+_{\max}(Q) \upharpoonright \operatorname{dom} \widetilde{A}_\Phi$$

define a quasi-selfadjoint extension $\widetilde{A}_\Phi \in \operatorname{Ext}\{D^+_{\min}(Q), D^+_{\min}(Q^*)\}$ satisfying

$$\alpha_Q ||f||^2 \le \operatorname{Im}(\widetilde{A}_\Phi f, f) \le \beta_Q ||f||^2, \quad f \in \operatorname{dom} \widetilde{A}_\Phi \qquad (4.35)$$

and $(\mathbb{C}_{\alpha_Q,-} \cup \mathbb{C}_{\beta_Q,+}) \subset \rho(\widetilde{A}_\Phi)$.

(ii) *For each $\lambda \in \rho(\widetilde{A}_\Phi)$ there exists a unique operator solution*

$$v_\Phi(t, \lambda) = (v_{1,\Phi}(t,\lambda), v_{2,\Phi}(t,\lambda))^\top : \mathbb{C}^p \to \mathbb{C}^p \oplus \mathbb{C}^p, \quad t \in \mathbb{R}_+ \qquad (4.36)$$

of the homogeneous equation (4.18) such that $v_\Phi(\cdot, \lambda)h \in L^2(\mathbb{R}_+, \mathbb{C}^n)$, $h \in \mathbb{C}^p$, and

$$\cos\Phi \cdot v_{1,\Phi}(0, \lambda) + \sin\Phi \cdot v_{2,\Phi}(0, \lambda) = I_p, \quad \lambda \in \rho(\widetilde{A}_\Phi). \qquad (4.37)$$

Proof: Let $A := D^+_{\min}(Q)$, $B := D^+_{\min}(Q^*)$, let S and T be the same as in the proof of Proposition 4.11 and let $\mathcal{D}_* := \operatorname{dom} S^*$. Then by (4.34) and Lemma 4.4, (iii) the equalities (4.6) hold.

Next, assume that $\Phi = \Phi^* \in \boldsymbol{B}(\mathbb{C}^p)$. Then the operator

$$X = \begin{pmatrix} \cos\Phi & \sin\Phi \\ -\sin\Phi & \cos\Phi \end{pmatrix} : \mathbb{C}^p \oplus \mathbb{C}^p \to \mathbb{C}^p \oplus \mathbb{C}^p \qquad (4.38)$$

satisfies $X^* J_n X = J_n$ and by Theorem 4.6, (i) applied to the Dirac-type expression (4.32) the equalities

$$\Gamma_0 y = \cos \Phi \cdot y_1(0) + \sin \Phi \cdot y_2(0), \quad \Gamma_1 y = -\sin \Phi \cdot y_1(0) + \cos \Phi \cdot y_2(0), \quad y \in \mathcal{D}_* \tag{4.39}$$

define a boundary triple $\widehat{\Pi} = \{\mathbb{C}^p, \Gamma_0, \Gamma_1\}$ for S^*. Moreover, application of Theorem 4.6, (i) to the dual pair $\{A, B\}$ implies that a collection $\Pi = \{\mathbb{C}^p \oplus \mathbb{C}^p, \Gamma^B, \Gamma^A\}$ with operators $\Gamma^B = (\Gamma_0^B, \Gamma_1^B)^\top$ and $\Gamma^A = (\Gamma_0^A, \Gamma_1^A)^\top$ defined by

$$\Gamma_0^B y = \Gamma_0^A y = \cos \Phi \cdot y_1(0) + \sin \Phi \cdot y_2(0), \tag{4.40}$$
$$\Gamma_1^B y = \Gamma_1^A y = -\sin \Phi \cdot y_1(0) + \cos \Phi \cdot y_2(0), \quad y \in \mathcal{D}_* \tag{4.41}$$

is a boundary triple for $\{A, B\}$ and for this triple $A_0(= B^* \upharpoonright \ker \Gamma_0^B) = \widetilde{A}_\Phi$. Since the triples Π and $\widehat{\Pi}$ are connected via (4.7), it follows from Theorem 4.5, (i) that (4.35) holds.

Statement (ii) directly follows from Theorem 4.6, (ii) applied to the extension $\widetilde{A} = \widetilde{A}_\Phi$. $\square$ Let the assumptions of Theorem 4.12 be satisfied and let $\Phi = \Phi^* \in \boldsymbol{B}(\mathbb{C}^p)$.

Definition 4.13. *The operator solution $v(\cdot, \lambda) = v_\Phi(\cdot, \lambda)$, $\lambda \in \rho(\widetilde{A}_\Phi)$, of (4.18) defined in Theorem 4.12 will be called the Weyl solution (with respect to the parameter Φ).*

Definition 4.14. *The operator function $M(\cdot) = M_\Phi(\cdot) : \rho(\widetilde{A}_\Phi) \to \boldsymbol{B}(\mathbb{C}^p)$ defined by*

$$M_\Phi(\lambda) = -\sin \Phi \cdot v_{1,\Phi}(0, \lambda) + \cos \Phi \cdot v_{2,\Phi}(0, \lambda), \quad \lambda \in \rho(\widetilde{A}_\Phi) \tag{4.42}$$

will be called the Weyl function of the almost formally-selfadjoint Dirac-type expression (4.31) on $\mathbb{R}_+$ (with respect to the parameter Φ).

Proposition 4.15. *Assume the hypothesis of Theorem 4.12. Let $\Phi = \Phi^* \in \boldsymbol{B}(\mathbb{C}^p)$ and let $\varphi(\cdot, \lambda)$ and $\psi(\cdot, \lambda)$ be $\boldsymbol{B}(\mathbb{C}^p, \mathbb{C}^p \oplus \mathbb{C}^p)$-valued solutions of (4.18) with the initial values*

$$\varphi(0, \lambda) = \begin{pmatrix} -\sin \Phi \\ \cos \Phi \end{pmatrix}, \qquad \psi(0, \lambda) = \begin{pmatrix} \cos \Phi \\ \sin \Phi \end{pmatrix}. \tag{4.43}$$

Then:

(i) *the Weyl function $M_\Phi(\cdot)$ of expression (4.31) can be defined as a unique operator-function $M_\Phi(\cdot) : \rho(\widetilde{A}_\Phi) \to \boldsymbol{B}(\mathbb{C}^p)$ such that (4.23) holds, i.e.,*

$$\varphi(\cdot, \lambda) M(\lambda) + \psi(\cdot, \lambda) \in L^2(\mathbb{R}_+, \boldsymbol{B}(\mathbb{C}^p, \mathbb{C}^n)), \quad \lambda \in \rho(\widetilde{A}_\Phi);$$

(ii) *$M_\Phi(\cdot)$ is holomorphic in $\rho(\widetilde{A}_\Phi)$ and for $\lambda, \mu \in \rho(\widetilde{A}_\Phi)$ it satisfies the identities*

$$M_\Phi(\lambda) - M_\Phi^*(\mu) = \int_{\mathbb{R}_+} v_\Phi^*(x, \mu)((\lambda - \overline{\mu}) I_n - 2i Q_2(x)) v_\Phi(x, \lambda) \, dx, \tag{4.44}$$

$$\operatorname{Im} M_\Phi(\lambda) = \int_{\mathbb{R}_+} v_\Phi^*(x,\lambda)(\operatorname{Im}\lambda \cdot I_n - Q_2(x))v_\Phi(x,\lambda)\,dx. \qquad (4.45)$$

Moreover,

$$\operatorname{Im} M_\Phi(\lambda) \geq \delta_\lambda I, \quad \lambda \in \mathbb{C}_{\beta_Q,+}; \qquad \operatorname{Im} M_\Phi(\lambda) \leq \delta_\lambda' I, \quad \lambda \in \mathbb{C}_{\alpha_Q,-} \qquad (4.46)$$

with some $\delta_\lambda > 0$ and $\delta_\lambda' < 0$ (depending on λ).

Proof: Let $A := D_{\min}^+(Q)$, $B := D_{\min}^+(Q^*)$ and let X be the operator (4.38). It was shown in the proof of Theorem 4.12 that the operators $\Gamma^B = (\Gamma_0^B, \Gamma_1^B)^\top$ and $\Gamma^A = (\Gamma_0^A, \Gamma_1^A)^\top$ defined by (4.40) and (4.41) form a boundary triple $\Pi = \{\mathbb{C}^p \oplus \mathbb{C}^p, \Gamma^B, \Gamma^A\}$ for $\{A, B\}$. Since by (4.38) the operators C_3 and C_4 in (4.21) are $C_3 = -\sin\Phi$ and $C_4 = \cos\Phi$, it follows from Theorem 4.6, (iii) and (4.42) that the Weyl function $M(\cdot)$ of Π coincides with $M_\Phi(\cdot)$. Moreover,

$$X^{-1} = \begin{pmatrix} \cos\Phi & -\sin\Phi \\ \sin\Phi & \cos\Phi \end{pmatrix}$$

and hence the initial conditions (4.22) are equivalent to (4.43). Now statement (i) follows from Theorem 4.6, (iii).

Next, assume that S and T are the same as in the proof of Proposition 4.11 and let $\widehat{\Pi} = \{\mathbb{C}^p \oplus \mathbb{C}^p, \Gamma_0, \Gamma_1\}$ be the boundary triple (4.39) for S^*. Since A and B admit representation (4.34) and boundary triples Π and $\widehat{\Pi}$ are connected via (4.7), it follows from Theorem 4.5 that identities (4.9) and (4.10) hold with $M(\lambda) = M_\Phi(\lambda)$ and $A_0 = \widetilde{A}_\Phi$. By using (4.26) one can easily verify that

$$\gamma^*(\lambda)f(\cdot) = \int_{\mathbb{R}_+} v^*(t,\lambda)f(t)\,dt, \quad f(\cdot) \in \mathfrak{H}, \ \lambda \in \rho(\widetilde{A}_\Phi).$$

This and (4.26), (4.33) imply that (4.9) and (4.10) can be written in the form (4.44) and (4.45), respectively. Finally, (4.46) is immediate from (4.11) and the equality $M(\lambda) = M_\Phi(\lambda)$, $\lambda \in \mathbb{C}_{\beta_Q,+} \cap \mathbb{C}_{\alpha_Q,-}$. $\qquad\square$

Corollary 4.16. *Let $n = 2p$, and let $D(Q)$ be an almost formally-selfadjoint Dirac-type expression (3.1) (or, equivalently, (4.31)) on $\mathbb{R}_+$ with*

$$J_n = \begin{pmatrix} -iI_p & 0 \\ 0 & iI_p \end{pmatrix} : \mathbb{C}^p \oplus \mathbb{C}^p \to \mathbb{C}^p \oplus \mathbb{C}^p \qquad (4.47)$$

and let $\varphi(\cdot,\lambda)$ and $\psi(\cdot,\lambda)$ be $\boldsymbol{B}(\mathbb{C}^p, \mathbb{C}^p \oplus \mathbb{C}^p)$-valued solution of (4.18) with the initial values

$$\varphi(0,\lambda) = \begin{pmatrix} 0 \\ I_p \end{pmatrix}, \qquad \psi(0,\lambda) = \begin{pmatrix} I_p \\ 0 \end{pmatrix}.$$

Then there exists a unique operator-function $M_s(\cdot) : \mathbb{C}_{\beta_Q,+} \to \boldsymbol{B}(\mathbb{C}^p)$ such that

$$\varphi(\cdot,\lambda)M_s(\lambda) + \psi(\cdot,\lambda) \in L^2(\mathbb{R}_+, \boldsymbol{B}(\mathbb{C}^p, \mathbb{C}^n)), \quad \lambda \in \mathbb{C}_{\beta_Q,+}. \qquad (4.48)$$

Moreover, $M_s(\cdot)$ is holomorphic on $\mathbb{C}_{\beta_Q,+}$ and satisfies $\|M_s(\lambda)\| < 1$, $\lambda \in \mathbb{C}_{\beta_Q,+}$.

Proof: One can easily verify that the equality $U = \frac{1}{\sqrt{2}} \begin{pmatrix} iI_p & I_p \\ -iI_p & I_p \end{pmatrix}$ defines a unitary operator $U \in \boldsymbol{B}(\mathbb{C}^n)$ such that

$$\widehat{J}_n := U^* J_n U = \begin{pmatrix} 0 & -I_p \\ I_p & 0 \end{pmatrix}.$$

Let $\widehat{Q}(x) := U^* Q(x) U = \widehat{Q}_1(x) + i\widehat{Q}_2(x)$ with $\widehat{Q}_1(x) = \operatorname{Re} \widehat{Q}(x)$ and $\widehat{Q}_2(x) = \operatorname{Im} \widehat{Q}(x)$, let

$$D(\widehat{Q})y := \widehat{J}_n \frac{dy}{dx} + \widehat{Q}_1(x)y + i\widehat{Q}_2(x)y, \quad x \in \mathbb{R}_+$$

be the respective almost selfadjoint Dirac-type expression and let $M_0(\cdot)$ be the Weyl function of $D(\widehat{Q})$ (with respect to $\Phi = 0$). Since obviously $\beta_{\widehat{Q}} = \beta_Q$, it follows that $M_0(\cdot)$ is defined on $\mathbb{C}_{\beta_Q,+}$ and by (4.46) the equality (the Cayley transform of $M_0(\lambda)$)

$$M_s(\lambda) = (M_0(\lambda) - iI_p)(M_0(\lambda) + iI_p)^{-1}, \quad \lambda \in \mathbb{C}_{\beta_Q,+}$$

defines a holomorphic operator-function $M_s(\cdot) : \mathbb{C}_{\beta_Q,+} \to \boldsymbol{B}(\mathbb{C}^p)$ such that $\|M_s(\lambda)\| < 1$, $\lambda \in \mathbb{C}_{\beta_Q,+}$. Let $v_0(\cdot,\lambda) \in L^2(\mathbb{R}_+, \boldsymbol{B}(\mathbb{C}^p, \mathbb{C}^n))$ be the Weyl solution of the homogeneous equation $D(\widehat{Q})y = \lambda y$ (with respect to $\Phi = 0$) and let

$$v_s(x,\lambda) = Uv_0(x,\lambda)\sqrt{2}(M_0(\lambda) + iI_p)^{-1}, \quad x \in \mathbb{R}_+, \; \lambda \in \mathbb{C}_{\beta_Q,+}.$$

Clearly, $v_s(\cdot,\lambda)$ is a solution of (4.18) and $v_s(\cdot,\lambda) \in L^2(\mathbb{R}_+, \boldsymbol{B}(\mathbb{C}^p, \mathbb{C}^n))$. Moreover, since $\Phi = 0$, it follows from (4.37) and (4.42) that $v_0(0,\lambda) = (I_p, M_0(\lambda))^\top$ and hence $v_s(0,\lambda) = (I_p, M_s(\lambda))^\top$. Therefore, $v_s(x,\lambda) = \varphi(x,\lambda)M_s(\lambda) + \psi(x,\lambda)$, which implies (4.48). $\square$

Remark 4.17. (i) *Let $D(Q)$ be an almost formally-selfadjoint Dirac-type expression (3.1) with J_n of the form (4.47), let $A := D^+_{\min}(Q)$, $B := D^+_{\min}(Q^*)$ and let $D_* = \operatorname{dom} A^* = \operatorname{dom} B^*$ (see Proposition 4.11). It follows from the Lagrange identity (3.12) and Lemma 3.6 that a collection $\Pi_s = \{\mathbb{C}^p \oplus \mathbb{C}^p, \Gamma^B, \Gamma^A\}$ with operators $\Gamma^B = (\Gamma^B_0, \Gamma^B_1)^\top$ and $\Gamma^A = (\Gamma^A_0, \Gamma^A_1)^\top$ defined by*

$$\Gamma^B_0 y = y_1(0), \quad \Gamma^B_1 y = y_2(0), \quad \Gamma^A_0 y = iy_2(0), \quad \Gamma^A_1 y = iy_1(0), \; y \in \mathcal{D}_*$$

is a boundary triple for $\{A, B\}$ (here $y_j(0)$ are taken from (3.24)). Moreover, one can easily prove that $M_s(\cdot)$ is the Weyl function of Π_s.

(ii) The operator-function $M_s(\cdot)$ coincides with the Weyl function introduced in [15, 31] for special almost formally-selfadjoint Dirac-type expression (3.1), (4.47) with $Q(x)$ of the form

$$Q(x) = i \begin{pmatrix} 0 & q(x) \\ q^*(x) & 0 \end{pmatrix} : \mathbb{C}^p \oplus \mathbb{C}^p \to \mathbb{C}^p \oplus \mathbb{C}^p.$$

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