THE SYMPLECTIC ORIGIN OF CONFORMAL AND MINKOWSKI SUPERSPACES

R. Fioresi[†], E. Latini[⋆]

[†] *Dipartimento di Matematica, Università di Bologna*
Piazza di Porta S. Donato, 5. 40126 Bologna. Italy.
e-mail: rita.fioresi@UniBo.it

[⋆] *Institut für Mathematik*
Universität Zürich-Irchel,
Winterthurerstrasse 190, CH-8057 Zürich, Switzerland
e-mail: emanuele.latini@math.uzh.ch

ABSTRACT. Supermanifolds provide a very natural ground to understand and handle supersymmetry from a geometric point of view; supersymmetry in $d = 3, 4, 6$ and 10 dimensions is also deeply related to the normed division algebras.

In this paper we want to show the link between the conformal group and certain types of symplectic transformations over division algebras. Inspired by this observation we then propose a new realization of the real form of the 4 dimensional conformal and Minkowski superspaces we obtain, respectively, as a Lagrangian supermanifold over the twistor superspace $\mathbb{C}^{4|1}$ and a big cell inside it. The beauty of this approach is that it naturally generalizes to the 6 dimensional case (and possibly also to the 10 dimensional one) thus providing an elegant and uniform characterization of the conformal superspaces.

Keywords: Supergeometry, division algebras, conformal geometry, super Yang-Mills and supergravity.

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1. Introduction

Supersymmetry (SUSY) is a part of the modern approach to the theory of elementary particles; supersymmetric string theory offers in fact the most promising model, so far, for the unification of all the elementary forces in a manner compatible with quantum theory and general relativity.
SUSY can be naturally treated by packing all physical fields, taking values in the 4d Minkowsli space $\mathbf{M}^{3,1}$, into a unique object, *i.e.* the superfield, that is assumed to take values in the 4d Minkowski superspace $\mathbf{M}^{3,1|1}$, on which one has locally commuting and anticommuting coordinates.

For mathematicians supersymmetry, the concept of superspace and supermanifolds were inspiring and gave a new look at geometry, both differential and algebraic. In geometry, in fact, "objects" are built out of local pieces: the most general of such object is a superspace, and the symmetries of such an object are then supersymmetries which are described by supergroups. The functor of points originally introduced by Grothendieck to study algebraic geometry, is now an essential tool to recover the geometric nature of supergroups and superspaces, which is otherwise difficult to grasp through the sheaf theoretic approach. It is actually and surprisingly the point of view physicists took at the very beginning of this theory, when points of a superspace were understood with the use of grassmann algebras, which are nothing but superalgebras over a superspace consisting of a point (see [1]). This suggestive point of view was later fully explained and justified by Shvarts and Voronov in [2, 3] and then linked to the theory of functor of points a la Grothendieck in [4, 5]. Later Manin in [6, 7] took full advantage of the machinery of the functor of points and applied it to the theory of superspaces and superschemes in particular to develop the theory of superflags and supergrassmannians, which is of particular interest to us. In the present work however, we shall make an effort to leave the full machinery of functor of points on the background, though employing its power in the description of the T-points of a supergroup or a superspace. Hence we shall rely for the results on the works [8] and [9], where all of the foundations of the theory are fully explained together with their physical significance.

Another relevant invariance principle in physics is given by the conformal symmetry; many physical systems, as those for massless particles, enjoy this symmetry and one may then imagine that there are regimes where conformal invariance is restored. In conformal geometry physics is described by equivalence classes of metrics so that all equations are manifestly locally Weyl invariant.

Minkowski space time, by the way, is not enough to support conformal symmetry, and it needs to be compactified by adding points at infinity to obtain a space endowed with a natural action of the conformal group. This is evident from the Dirac cone construction in which one considers the space of light like rays in $\mathbf{M}^{d,2}$ (known as the conformal space or conformal sphere) and the compactified Minkowski space $\overline{\mathbf{M}}^{d-1,1}$ is then realized as one particular section of the cone. The above flat model for conformal geometry was generalized in [10] by Fefferman and Graham who replaced $\mathbf{M}^{d,2}$ by a $d+2$ dimensional manifold equipped with a metric which admits a hypersurface orthogonal homothety. Note that there is a natural interpretation of this picture as curved Cartan geometry; moreover the Cartan approach naturally leads to a Weyl covariant differential calculus, known as tractor calculus, originally constructed in [11] (see also [12, 13] for a physics oriented review) and then generalized to all parabolic geometries in [14]. It can be understood as the equivalent of the superfield formalism for conformal invariance.

In this paper we aim to point out that the conformal space in $3, 4, 6$ or 10 dimensions may be also understood as a certain Lagrangian manifold over the four normed division algebras $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$; we will use the notation $n := \dim \mathbb{K} = 1, 2, 4$ and 8 and denote by $\mathbb{K}'$ the split version of any division algebra. The relationship between SUSY, sipertwistors and $\mathbb{K}$ is also a recurring theme [15, 16, 17, 18]. For example nonabelian Yang-Mills theories are supersymmetric only if the dimension of the Minkowski spacetime is $d = 3, 4, 6$ or 10 (and the same is true for the Green-Schwarz superstring). In this case SUSY relies on the vanishing of a certain trilinear expression that it turns to be strictly related to the existence of the four normed division algebras in $d - 2$ dimensions [19]. Recently in [20] Duff and collaborators used normed division algebras to give a descr iption of supergravity by "tensoring" super Yang-Mills multiplets[1]. The main argument the authors used is the observation that the entries of second row of the 2×2 half split magic square [22, 23, 24]

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{C}'$	$\mathfrak{so}(2,1)$	$\mathfrak{so}(3,1)$	$\mathfrak{so}(5,1)$	$\mathfrak{so}(9,1)$

can be naturally represented as $\mathfrak{sl}_2(\mathbb{K})$ producing then the Lie algebras isomorphisms

$$\mathfrak{sl}_2(\mathbb{K}) = \mathfrak{so}(n+1,1)\,.$$

We believe that understanding the relation between SUSY and normed division algebras from a supergeometric point of view could also give a fundamental contribution to the study of the quantum properties of supergravity.

In a series of papers [9, 25] the complex 4 dimensional Minkowski (super)space was realized as the big cell inside a complex flag (super)manifold where the conformal group $\mathrm{SL}_4(\mathbb{C})$ acts naturally while the real Minkowski

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[1]In literature there are in fact many attempts to understand the quantum theory of gravity through the idea of "Gravity is the square of Yang-Mills theories" idea supported by the fact that (super)gravity scattering amplitudes can be obtained from those of (super)Yang-Mills [21].

(super)space was then obtained as a suitable real form of the complex version. We observe that this approach essentially coincides with the Dirac cone construction; we plan then to take advantages of this observation in a future project where we will try to study supergeometry within the Cartan approach.

In [26], the authors constructed the compactified Minkowsky 3d superspace $\overline{\mathbf{M}}^{2,1}$ and its supersymmetric extension, as a Lagrangian manifold over the twistor space $\mathbb{R}^4$; this relies to the isomorphism $\mathrm{Spin}(3,2) \simeq \mathrm{SP}_4(\mathbb{R})$. This result can be nicely linked with the following observation: the third line of the Freudenthal-Tits 2×2 half split magic square

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{H}'$	$\mathfrak{so}(3,2)$	$\mathfrak{so}(4,2)$	$\mathfrak{so}(6,2)$	$\mathfrak{so}(10,2)$

can be reinterpreted by noting the following isomorphism

$$\widetilde{\mathfrak{sp}}_4(\mathbb{K}) = \mathfrak{so}(n+2,2)$$

with $\widetilde{\mathfrak{sp}}_4(\mathbb{K})$ being the Sudbery symplectic algebra, where, with respect to the traditional definition, the transpose is replaced by hermitian conjugation. Recently, in [27], it was also proposed a Lie group version of the half split 2×2 magic square (see also [28, 29, 30] for further details and the relation with exceptional Lie algebras and groups).

Inspired by these observations we then study in details a symplectic characterization of the 4 dimensional (compactified and real) Minkowski space and superspace respectively. We argue that this approach can be also extended to 6 and possibly 10 dimensions[2] producing thus an uniform description of $\mathbf{M}^{n+1,1|1}$ explicitly involving the four normed division algebras.

More in details, while in [7] supergrassmannians and superflags are mainly understood as complex objects and are constructed by themselves, in [9, 25] these objects come together with the supergroups describing their supersymmetries, because this is one of the main reasons of their physical significance. Furthermore, in [9, 25] real forms of both the 4d Minkowski and conformal superspaces are introduced through suitable involutions, which are compatible with the natural supersymmetric action of the Poincaré and conformal supergroups. In the present work, we shall leave the complex structure on the background and actually show that we can directly obtain the real forms of the 4d Minkowski and conformal superspaces together with their symmetry supergroups, without ever worrying about the complex field. In fact, it is very remarkable that abandoning the complex picture, one can very quickly obtain the real 4d conformal and Minkowski superspaces as Lagrangian supermanifold and its big cell respectively, without the need to go to the superflag, which is undoubtedly a less manageable geometric object, given its many defining relations (twistor relations).

The quantization of spacetime is an intriguing task that has been tackled from many point of views in literature. In [32, 33, 34] the authors studied

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[2]While the generalization to the 6 dimensional case is straightforward, further effort are needed when $d = 10$ due to the notorious problem of constructing the superconformal algebra in dimension bigger than 6 [31]; we plan to tackle this problem in a future project.

the quantum deformation for the complex (chiral) Minkowski and conformal superspaces based on the general machinery developed in [35, 36] for flag varieties. The more direct approach to the real Minkowski and conformal superspace we propose in this paper compares, immediately to the one described in [7, 9, 25] and actually we obtain the very same equations the Poincaré supergroup that we find in the literature, for example in [9, 25], but with far less effort. This opens the possibility to proceed further in the theory and possibly construct a quantum deformation of both real Minkowski and conformal superspaces.

The organization of the paper is as follows:

In Section 2 we review how finite Lorentz transformations of vectors in $n + 2$ dimensional Minkowski space can be characterized by means of "unit determinant" matrices over division algebras; this relation was originally presented in [37]. In Section 3 we discuss the Lie group version of the third row of 2×2 magic square [27]; in particular we show that certain type of symplectic transformations induce an $O(n + 2, 2)$ rotation. In Section 4 we prove in details how the real form of the four dimensional Minkowski and conformal space can be obtained as a Lagrangian manifold containing the twistor space $\mathbb{C}^4$, while in Section 5 we extend this construction to the super case.

2. Normed divison algebras and Lorentz transformations

A normed division algebra $\mathbb{K}$ is a real algebra together with a norm $|\bullet|$ such that for all $v, w \in \mathbb{K}$ we have $|vw| = |v||w|$. By a classical result (see Hurwitz [38]) there are only four normed division algebras: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$, with $n := \dim \mathbb{K} = 1, 2, 4, 8$ respectively. Moreover every normed division algebra is equipped with an involutive automorphism $v \mapsto v^*$, *i.e.* the conjugation, such that $v^{**} = v$ and $(vw)^* = w^* v^*$; this leads to a natural decomposition

$$\mathrm{Re}(v) := \frac{v + v^*}{2}, \quad \mathrm{Im}(v) := \frac{v - v^*}{2}$$

that can be used to define the inner product

$$(v, w) := \mathrm{Re}(v\, w^*) = \mathrm{Re}(w\, v^*)$$

and thus also the norm

$$|v| := \sqrt{v^*\, v}\,.$$

which is real. A division algebra element can be written as the linear combination $v = v^i e_i$ with $v^i \in \mathbb{R}$ and $i = 1, .., n$. The first basis element is the real one $e_1 = 1$, while the others are imaginary units $(e_i)^2 = -1$, $i \neq 1$.
In the case $\mathbb{K} = \mathbb{H}$ the multiplication rules for the imaginary units are given by

$$e_3 e_4 = -e_4 e_3 = e_2 \, , \; e_4 e_2 = -e_2 e_4 = e_3 \, , \; e_2 e_3 = -e_3 e_2 = e_4$$

and similar relations, encoded in the so called Fano plane, hold also for octonions (see for example [37] for more details). We will then denote by $\mathrm{M}_n(\mathbb{K})$ the space of $n \times n$ matrices with entries in $\mathbb{K}$ and we say that $A \in \mathrm{M}_n(\mathbb{K})$ is *hermitian* if $(A^*)^t := A^\dagger = A$; the space of $n \times n$ hermitian matrices will be denoted by $\mathrm{H}_n(\mathbb{K})$.

It is useful sometimes to represent any $\mathbb{H}$ valued $n \times n$ matrix as a $\mathbb{C}$ valued $2n \times 2n$ matrix through the following map

$$
\begin{aligned}
\mathrm{Z} \; : \; \mathrm{M}_n(\mathbb{H}) \; &\to \; \mathrm{M}_{2n}(\mathbb{C}) \\
A \; &\mapsto \; \begin{pmatrix} z(A) & -w^*(A) \\ w(A) & z^*(A) \end{pmatrix}
\end{aligned}
$$

(1)

where $z(A)$ and $w(A)$ mean that all the entries v of the matrix A are mapped into

$$z(v) = v_1 + i v_2 \, , \quad w(v) = v_3 - i v_4$$

where $i := e_2$ is the usual imaginary units for complex numbers.

Division algebras are often used in physics to easily handle supersymmetry; the minimal spinorial representations of the $n + 2$ dimensional Lorentz group are, in fact, isomorphic as vector spaces to $\mathbb{K}^2$. This fact relies on the observation that vectors of the $n + 2$ dimensional Minkowski space $\mathbf{M}^{n+1,1}$ can be naturally identified with an element of $\mathrm{H}_2(\mathbb{K})$. Consider $\mathbf{x} = (x_0, \cdots, x_{n+1}) \in \mathbf{M}^{n+1,1}$ and rearrange it as follows

$$\mathcal{X} = \begin{pmatrix} x_0 + x_{n+1} & v^* \\ v & x_0 - x_{n+1} \end{pmatrix} \in \mathrm{H}_2(\mathbb{K})$$

where $v \in \mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$ respectively is constructed with $x_1, \cdots, x_n$; we recognize on the top left and bottom right spot the lightcone directions we will often denote by $x_\pm = x_0 \pm x_{n+1}$. We introduce now the so called *trace reversal matrix* defined as follows

$$\widetilde{\mathcal{X}} = -\begin{pmatrix} x_0 - x_{n+1} & -v^* \\ -v & x_0 + x_{n+1} \end{pmatrix} \in \mathrm{H}_2(\mathbb{K})$$

and we then observe that

$$-\det \mathcal{X} = -x_0^2 + x_{n+1}^2 + |v|^2 = -x_0^2 + \sum_{i=1}^{n+1} x_i^2 := \mathbf{g}(\mathbf{x}, \mathbf{x})$$

or equivalently

$$\mathcal{X}\widetilde{\mathcal{X}} = \widetilde{\mathcal{X}}\mathcal{X} = \mathbf{g}(\mathbf{x}, \mathbf{x}) \, \mathbb{1}$$

with $\mathbf{g}(\bullet, \bullet)$ being the pseudo-Riemannian metric of signature $(n + 1, 1)$ and $\mathbb{1}$ being the identity matrix. Note that for an $\mathbb{H}$ or $\mathbb{O}$ valued matrix we do not have a natural and well defined notion of determinant while if we restrict to the case 2×2 hermitian matrices it can be unambiguously determined by the usual formula.

It is instructive to reformulate the previous result introducing the symplectic matrix

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so that

$$\mathcal{X}^t \epsilon \mathcal{X} = \det \mathcal{X} \, \epsilon \,.$$

We consider now transformations of the form

(2)
$$
\begin{aligned}
H_2(\mathbb{K}) &\rightarrow H_2(\mathbb{K}) \\
\mathcal{X} &\mapsto \lambda \mathcal{X} \lambda^\dagger =: \mathcal{X}', \quad \lambda \in M_2(\mathbb{K})
\end{aligned}
$$

where extra care must be taken with octonions by requiring $(\lambda \mathcal{X})\lambda^\dagger = \lambda(\mathcal{X}\lambda^\dagger)$; we note that $\det \mathcal{X}'$ is again well defined thus if we restrict the matrices λ to those preserving the determinant under (2), they will then induce a Lorentz transformation.

For the case of $\mathbb{R}$ or $\mathbb{C}$, this leads to the notorious statement that the special linear groups $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$ double cover the 3 and 4 dimensional Lorentz group (or its connected component to the identity); one is then tempted to generalize this statement and identify the $n + 2$ dimensional Lorentz transformations with unit determinant matrices over $\mathbb{K}$ but when we deal with $\mathbb{H}$ or $\mathbb{O}$ the construction of the special linear group requires further assumption and elucidation since, as we commented previously, we do not have the notion of determinant.

In order to get around this problem, we note that the following relation

$$\det(\lambda \mathcal{X} \lambda^\dagger) = \det(\lambda \lambda^\dagger)\det(\mathcal{X})$$

is satisfied for every $\mathbb{K}$ and then the transformation (2) induces a Lorentz rotation if

(3)
$$\det(\lambda \lambda^\dagger) = 1 \,;$$

this is again unambiguous since $\lambda \lambda^\dagger$ is hermitian.

Focusing on quaternions there is an an easy way to analyze (3) by using the map defined in (1); explicitly, for the case $n = 2$ we get

$$
Z\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \begin{pmatrix} z(a) & z(b) & -w^*(a) & -w^*(b) \\ z(c) & z(d) & -w^*(c) & -w^*(d) \\ w(a) & w(b) & z^*(a) & z^*(b) \\ w(c) & w(d) & z^*(c) & z^*(d) \end{pmatrix}.
$$

Using this formula, it is straightforward to prove that

$$\det(Z[\lambda]) \in \mathbb{R}_+ \cup \{0\} \ \forall \, \lambda \in M_2(\mathbb{H})$$

and that

$$\det(\lambda \lambda^\dagger)^2 = \det(Z[\lambda \lambda^\dagger]) = \det(Z[\lambda])\det(Z[\lambda^\dagger]) \,.$$

We thus solve (3) by requiring $\det(Z[\lambda]) = 1$ and we can then unambiguosly construct the group

$$SL(2, \mathbb{H}) = \{\lambda \in M_2(\mathbb{H}) \, | \, \det(Z[\lambda]) = 1\} \,.$$

When viewed in this way, $SL(2, \mathbb{H})$ has a natural real Lie group structure (Ref. [41] Theorem 2.1.2). It is interesting to observe that those matrices,

through (2), do not produce any parity or time reversal transformation, thus providing a natural double cover of the connected component to the identity of the 6 dimensional Lorentz group.

For the case of octonions, due to the lack of associativity, one must also impose certain compatibility conditions on the matrix λ appearing in (2); this case was originally extensively studied in [37] where it was also observed that not all 10 dimensional Lorentz transformations can be achieved with a single matrix but, instead, to produce all of them one must also consider the product of 2 octonionic valued matrices; this is the case, in fact, of transverse rotations induced by the group $\mathrm{Aut}(\mathbb{O})$; along this line, the group $\mathrm{SL}_2(\mathbb{O})$ was then explicitly characterized by a generating set of matrices.
In conclusion we can then establish the following isomorphism

$$\mathrm{SL}_2(\mathbb{K}) \simeq \mathrm{Spin}(n+1,1)\,.$$

The matrix $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ defined previosuly, have a natural interpretation in terms of Dirac gamma matrices, that we now plan to discuss.
Consider the spinor bundle; S_+ and S_- are both just $\mathbb{K}^2$ as real vector spaces, but they differ as representation of $\mathrm{Spin}(n+1,1)$ (see [15] for more details). We first define how a vector acts on spinors through the gamma matrices in the Weyl base

$$
\begin{aligned}
\gamma: \quad \mathbf{M}^{n+1,1} \quad &\to \quad \mathrm{Hom}(S_+, S_-) \\
\mathbf{x} \quad &\mapsto \quad \mathcal{X} \\
\widetilde{\gamma}: \quad \mathbf{M}^{n+1,1} \quad &\to \quad \mathrm{Hom}(S_-, S_+) \\
\mathbf{x} \quad &\mapsto \quad -\widetilde{\mathcal{X}}\,.
\end{aligned}
$$

In particular, our realization coincides, in 3 and 4 dimensions, with the following standard choices:

$$
\begin{aligned}
\gamma \quad &= \quad (\mathbb{1}, \sigma_1, \sigma_3) \\
\widetilde{\gamma} \quad &= \quad (\mathbb{1}, -\sigma_1, -\sigma_3)
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma \quad &= \quad (\mathbb{1}, \sigma_1, \sigma_2, \sigma_3) \\
\widetilde{\gamma} \quad &= \quad (\mathbb{1}, -\sigma_1, -\sigma_2, -\sigma_3)
\end{aligned}
$$

respectively, with σ_1, σ_2 and σ_3 being the standard Pauli matrices. Dirac matrices can be then easily constructed as

$$\Gamma = \begin{pmatrix} 0 & \gamma \\ \widetilde{\gamma} & 0 \end{pmatrix} \in \mathrm{M}_4(\mathbb{K})\,,$$

and in this base the chiral matrix is diagonal. They generate the map

$$\Gamma : TM \to \mathrm{End}(S_- \oplus S_+)\,.$$

explicitly given by

$$\Gamma(\mathbf{x})(\psi, \lambda) = (-\widetilde{\mathcal{X}}\lambda, \mathcal{X}\psi)$$

with $\psi \in S_-$ and $\lambda \in S_+$. Moreover, by construction, $\Gamma(\mathbf{x})$ satisfies the Clifford algebra relation $\Gamma(\mathbf{x})^2 = \mathbf{g}(\mathbf{x}, \mathbf{x})\mathbb{1}$.

3. Symplectic realization of the conformal group

In the following we want to extend the analysis of the previous section to the case of the conformal group $\mathrm{SO}(n+2,2)$ and relate it to a certain class of symplectic transformations.

We introduce $\widetilde{\mathbf{M}}^{n+2,2}$ that is a pseudo-Riemannian manifold equipped with the flat metric $\mathbf{G}(\bullet,\bullet)$ with signature $(n+2,2)$, known in the conformal geometry literature as the Fefferman and Graham ambient space [10], or better to say its flat limit. We denote by x_{n+2} and $x_{0'}$ the extra space like and time like directions and we indicate by $\mathbf{X} = (x_{0'}, x_0, x_1, \cdots, x_{n+2})$ a general vector in the ambient space; moreover we name $x_p = x_{0'} + x_{n+2}$ and $x_m = x_{0'} - x_{n+2}$ the new pair of lightcone coordinates.

In analogy with the case of the Lorentz group, we look for a matrix representation of the vector $\mathbf{X}$. To achieve this task our strategy is to use the $n+4$ dimensional Dirac gamma matrices as a guideline; we note that there is a standard method to construct them starting from the $n + 2$ dimensional ones:

$$\Upsilon = \left(\frac{1}{2} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \begin{pmatrix} \gamma & 0 \\ 0 & -\widetilde{\gamma} \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \right)$$

and

$$\widetilde{\Upsilon} = \left(\frac{1}{2} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \begin{pmatrix} \widetilde{\gamma} & 0 \\ 0 & -\gamma \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \right) .$$

Inspired by this result we construct

$$\mathbb{X} = \begin{pmatrix} \mathcal{X} & x_p \mathbb{1} \\ x_m \mathbb{1} & \widetilde{\mathcal{X}} \end{pmatrix} \in \mathrm{M}_4(\mathbb{K}) .$$

We now look for a characterization of the metric in this representation. To this aim we introduce the symplectic form

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

and inspired by the Lie algebra isomorphism

$$\widetilde{\mathfrak{sp}}_4(\mathbb{K}) = \mathfrak{so}(n+2,2)$$

with the Subdery symplectic algebra given by

$$\widetilde{\mathfrak{sp}}_4(\mathbb{K}) = \{X \in \mathrm{M}_4(\mathbb{K}) \,|\, X^\dagger J + JX = J, \ \mathrm{tr}X = 0\}$$

we compute the following:

$$(4) \quad \mathbb{X}^\dagger J \mathbb{X} = \begin{pmatrix} 0 & -x_m x_p \mathbb{1} + \widetilde{\mathcal{X}}\mathcal{X} \\ x_m x_p \mathbb{1} - \widetilde{\mathcal{X}}\mathcal{X} & 0 \end{pmatrix} = \left(\sum_{i=1}^{n+2} x_i^2 - x_{0'}^2 - x_0^2 \right) J .$$

We recognize into the bracket the ambient metric $\mathbf{G}(\mathbf{X},\mathbf{X})$. The main observation is that the relation (4) remains invariant if one transforms now $\mathbb{X}$ as follows

$$\mathbb{X} \to \lambda \mathbb{X} \lambda^\dagger \quad \text{with} \quad \lambda \in \mathrm{M}_4(\mathbb{K}) \text{ so that } \lambda^\dagger J \lambda = J .$$

We conclude then that those types of transformations induce an $\mathrm{O}(n+2,2)$ rotation and, starting from here, one can hope to construct the spinorial

representation of the conformal groups $SO(n+2,2)$ (or their connected component to the identity), *i.e.* the spin group. For the case $\mathbb{K} = \mathbb{R}$ one obtains the notorious result $\mathrm{Spin}(3,2) = \mathrm{Sp}_4(\mathbb{R})$ while when $\mathbb{K} = \mathbb{C}$ we construct

$$\widetilde{\mathrm{Sp}}_4(\mathbb{C}) = \{\lambda \in \mathrm{SL}_4(\mathbb{C}) \,|\, \lambda^\dagger J \lambda = J\}$$

and we use the tilde to emphasize the fact that we are using hermitian conjugation instead of the usual matrix transposition. This group double covers the connected component of the identity of $SO(4,2)$ and we further analyze its properties and its Lie algebra in the next chapter.

The temptation is again to extend this statement to all division algebras. In [27, 28, 29], in fact, the authors realized $SO(n + 2, 2)$ transformation by giving an explicit Clifford algebra description of $SU(2, \mathbb{H}' \otimes \mathbb{K})$ that turns out to be equivalent to the symplectic description as $\widetilde{\mathrm{Sp}}_4(\mathbb{K})$ (see in particular [28] for a characterization of this group in the octonionic case and its connection with exceptional Lie groups); one can in conclusion establish the isomorphism

$$\mathrm{Spin}(n + 2, 2) \cong \widetilde{\mathrm{Sp}}_4(\mathbb{K}).$$

4. The $4d$ Conformal and Minkowski spaces

In this section we would like to reinterpret the conformal space as a Lagrangian manifold and the Minkowski space $\mathbf{M}^{3,1}$ as a suitable big cell (hence dense) inside it. In this way the Lagrangian manifold appears as natural compactification of the Minkowski space $\mathbf{M}^{3,1}$ and we will see that the conformal and Poincaré groups will appear effortlessly in this picture as the symmetry groups for those spaces.

Let us recall that in the previous section, following Sudbery, we constructed the real Lie group $\widetilde{\mathrm{Sp}}_4(\mathbb{C})$; for convenience we report in the following its definition

$$\widetilde{\mathrm{Sp}}_4(\mathbb{C}) = \{\lambda \in \mathrm{SL}_4(\mathbb{C}) \,|\, \lambda^\dagger J \lambda = J\}, \qquad \text{with} \qquad J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

An easy calculation shows that $\lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widetilde{\mathrm{Sp}}_4(\mathbb{C})$ if and only if

$$a^\dagger c = c^\dagger a, \quad b^\dagger d = d^\dagger b, \quad a^\dagger d - c^\dagger b = \mathbb{1}$$

where a, b, c, d are 2×2 matrices.

The Lie algebra of $\widetilde{\mathrm{Sp}}_4(\mathbb{C})$ is explicitly given by

$$\widetilde{\mathfrak{sp}}_4(\mathbb{C}) = \left\{ \begin{pmatrix} x & y \\ z & -x^\dagger \end{pmatrix} \in \mathfrak{sl}_4(\mathbb{C}),\, z = z^\dagger,\, y = y^\dagger,\, x = x^\dagger \right\}.$$

In [41] the conformal space was constructed starting from the complex conformal group $\mathrm{SL}_4(\mathbb{C})$ and looking at involutions giving the real form

$SU(2,2)$[3]. In the following we instead take advantage of the symplectic interpretation of the conformal group to propose a new characterization of the 4 dimensional Minkowski space.

Define now the inner product

$$\langle u, v \rangle := u^\dagger J v \,;$$

we then construct the Lagrangian $\mathcal{L}$, that is, the manifold of totally isotropic subspaces with respect to the above inner product. Since $G = \widetilde{Sp}_4(\mathbb{C})$ acts transitively on $\mathcal{L}$, we have that

$$(5) \qquad \mathcal{L} = G \cdot \langle \mathbf{e}_1, \mathbf{e}_2 \rangle \cong \left\{ \begin{pmatrix} a \\ c \end{pmatrix}, a^\dagger a = c^\dagger a \right\} / GL_2(\mathbb{C}) \,,$$

with $\{\mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_4\}$ the standard basis for $\mathbb{C}^4$. The action of $GL_2(\mathbb{C})$ is needed to take into account the base change of a chosen Lagrangian subspace. As one can readily check:

$$\mathcal{L} = \widetilde{Sp}_4(\mathbb{C})/P$$

where

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & (a^\dagger)^{-1} \end{pmatrix} \right\} \subset \widetilde{Sp}_4(\mathbb{C})$$

is the stabilizer of $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$.

We define the subset $\mathbf{M}^{3,1} \subset \mathcal{L}$ consisting of those elements in $\mathcal{L}$ with a invertible (see eq. (5)) and we write:

$$\mathbf{M}^{3,1} = \left\{ \begin{pmatrix} \mathbb{1} \\ \mathcal{X} \end{pmatrix} \text{ with } \mathcal{X}^\dagger = \mathcal{X} \right\} \,.$$

This expression is obtained from (5) by right multiplying $\begin{pmatrix} a \\ c \end{pmatrix}$ by $a^{-1} \in GL_2(\mathbb{C})$, and the interpretation of the hermitian matrix $\mathcal{X}$ is the one discussed in Section 2. $\mathbf{M}^{3,1}$ is an open dense set in $\mathcal{L}$, which is compact. $\mathbf{M}^{3,1}$ is our model for the *real Minkowski space* and its compactification $\mathcal{L}$ is the model for the *real conformal space*. We now want to justify this terminology by computing the groups that naturally act on these spaces. We first observe that $\mathcal{L}$ carries a natural action of $\widetilde{Sp}_4(\mathbb{C})$, which we identify with the conformal group since it preserves the ambient metric, as we have shown in Section 3.

We now want to compute the subgroup in $\widetilde{Sp}_4(\mathbb{C})$ preserving $\mathbf{M}^{3,1}$. We will see it becomes naturally identified with the real Poincaré group and its action on $\mathbf{M}^{3,1}$ is the correct one, restricting the action of the conformal group $\widetilde{Sp}_4(\mathbb{C})$ on the conformal space $\mathcal{L}$. Let $\lambda \in \widetilde{Sp}_4(\mathbb{C})$ be such that $\lambda \cdot \mathbf{M}^{3,1} = \mathbf{M}^{3,1}$, that is :

$$\lambda \cdot \mathbf{x} = \begin{pmatrix} l & m \\ nl & r \end{pmatrix} \begin{pmatrix} \mathbb{1} \\ \mathcal{X} \end{pmatrix} = \begin{pmatrix} l + m\mathcal{X} \\ nl + r\mathcal{X} \end{pmatrix} \,.$$

---

[3]It is not hard to prove that $\widetilde{Sp}_4(\mathbb{C})$ and $SU(2,2)$ are diffeomorphic globally, and they both double cover the conformal group $SO(4,2)$. We anyhow prefer in this paper to use $\widetilde{Sp}_4(\mathbb{C})$ that suggests a more natural generalization to the higher dimensional cases.

Hence $l + m\mathcal{X}$ must be invertible for all $\mathcal{X}$. In particular this gives immediately that l must be invertible, so our condition says $\mathbb{1} + q\mathcal{X}$ invertible for all $\mathcal{X}$, with $q = l^{-1}m$. By the conditions defining $\widetilde{\mathrm{Sp}}_4(\mathbb{C})$, we have that n, $r^\dagger m$ are hermitian and so is $r^\dagger - m^\dagger n = l^{-1}$. So $q = l^{-1}m = r^\dagger m - m^\dagger nm$ is also hermitian. If $q \neq 0$ (i.e. $m \neq 0$), then $q^2 > 0$ so it has an eigenvalue $k > 0$. Then $\mathbb{1} + q\mathcal{X}$ is not invertible for $\mathcal{X} = -k^{-1}q$. So we conclude $m = 0$.

We have thus proven that the subgroup $\widehat{P}$ leaving $\mathbf{M}_{3,1}$ invariant is the the transpose of P namely:

$$\widehat{P} = \left\{ \begin{pmatrix} l & 0 \\ nl & (l^\dagger)^{-1} \end{pmatrix} \right\} \subset \widetilde{\mathrm{Sp}}_4(\mathbb{C}).$$

It acts on $\mathbf{M}^{3,1}$ as follows

$$\widehat{P} \quad \times \quad \mathbf{M}^{3,1} \quad \longrightarrow \quad \mathbf{M}^{3,1}$$

$$\begin{pmatrix} l & 0 \\ nl & (l^\dagger)^{-1} \end{pmatrix} \quad , \quad \mathcal{X} \quad \mapsto \quad n + (l^{-1})^\dagger \mathcal{X} l^{-1}$$

and we identify the first term with space time translations, while the second contribution represents both Lorentz rotations and dilations; we observe that this coincide with the group stabilizing a light like ray of the Dirac cone. With an abuse of terminology we call $\widehat{P}$ the *Poincaré group*.

5. The $4d$ Conformal and Minkowski superspaces

In this section we want to generalize the results discussed in the previous section to the supersetting and thus construct the Minkowski superspace $\mathbf{M}^{3,1|1}$ using the super version of $\widetilde{\mathrm{Sp}}_4(\mathbb{C})$, namely $\widetilde{\mathrm{SpO}}(4|1)$, a real form of the symplectic-orthogonal supergroup (for the definition of the complex Osp and its equivalent SpO see [8] Ch. 11; in [40] it is also discussed its connection with susy curves). We shall define this supergroup via its functor of points. The R-points of the general linear supergroup consist of the group of invertible $m + n \times m + n$ matrices with coefficients in the commutative superalgebra R (the diagonal blocks have even coefficients, the off diagonal blocks odd coefficients). We denote such R points with $\mathrm{GL}(m|n)(R)$. The R-points of a (closed) subsupergroup G of $\mathrm{GL}(m|n)$ consist of matrices in $\mathrm{GL}(m|n)(R)$ satisfying certain algebraic condition. We are going to realize $G = \widetilde{\mathrm{SpO}}(4|1)$ the real symplectic-orthogonal supergroup precisely in this way. For all of the relevant definitions and the details we are unable to give here, we invite the reader to consult [8] Ch. 1, 9, 11.

We then define:

$$\widetilde{\mathrm{SpO}}(4|1)(R) = \{\Lambda \in \mathrm{SL}_{4|1}(R) \,|\, \Lambda^\dagger \mathcal{J} \Lambda = \mathcal{J}\}, \qquad \text{with} \qquad \mathcal{J} = \begin{pmatrix} 0 & \mathbb{1} & 0 \\ -\mathbb{1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This gives us effectively a supergroup functor $\widetilde{\mathrm{SpO}}(4|1) : (\mathrm{salg}) \longrightarrow (\mathrm{sets})$, $R \mapsto \widetilde{\mathrm{SpO}}(4|1)(R)$, ((salg) is the category of commutative real superalgebras, (sets) is the category of sets). This functor is *representable*, in other words, there is a Lie supergroup corresponding to it, in the sheaf theoretic approach

(i.e. a superspace locally isomorphic to $\mathbb{R}^{M|N}$). $\widetilde{\mathrm{SpO}}(4|1)$ is a closed subgroup of the complex symplectic-orthogonal supergroup $\widetilde{\mathrm{SpO}}(4|1)$, viewed as a real supergroup. We shall not worry about the definition of our functors on the arrows: such definition comes from the one of $\mathrm{GL}(m|n)$ (see [8] Ch. 1, 11).

If

$$\Lambda = \begin{pmatrix} A & \alpha \\ \beta & u \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \beta = (\beta_1, \beta_2), \quad \alpha = (\alpha_1, \alpha_2)^t,$$

with $\beta_i, \alpha_i \in R^2$, we obtain from the condition $\Lambda^\dagger \mathcal{J} \Lambda = \mathcal{J}$ the following set of equations

$$(6) \qquad \begin{cases} A^\dagger J A + \beta^\dagger \beta = J \\ \\ A^\dagger J \alpha + \beta^\dagger u = 0 \\ \\ \alpha^\dagger J A + u^\dagger \beta = 0 \\ \\ \alpha^\dagger J \alpha + u^\dagger u = 1 \end{cases} \iff \begin{cases} a^\dagger c - c^\dagger a + \beta_1^\dagger \beta_1 = 0 \\ a^\dagger d - c^\dagger b + \beta_1^\dagger \beta_2 = \mathbb{1} \\ b^\dagger c - d^\dagger a + \beta_2^\dagger \beta_1 = -\mathbb{1} \\ b^\dagger d - d^\dagger b + \beta_2^\dagger \beta_2 = 0 \\ -c^\dagger \alpha_1 + a^\dagger \alpha_2 + \beta_1^\dagger u = 0 \\ -d^\dagger \alpha_1 + b^\dagger \alpha_2 + \beta_2^\dagger u = 0 \\ -\alpha_2^\dagger \alpha_1 + \alpha_1^\dagger \alpha_2 + u^\dagger u = 1 \end{cases}$$

Notice that these equations correspond for $\alpha = 0$ and $\beta = 0$ to the ones obtained in Section 4.

We now proceed as we did in Section 4 and look at the supermanifold $\mathcal{L}$ of $2|0$ totally isotropic subspaces. If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathcal{E}\}$ is a basis for $\mathbb{C}^{4|1}$, we define $\mathcal{L}$ as the orbit of $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$. This is a supermanifold (see [8] Proposition 9.1.4). If R is a local superalgebra, we have

$$(7) \qquad \mathcal{L}(R) = \mathrm{G} \cdot \langle \mathcal{E}_1, \mathcal{E}_2 \rangle = \left\{ \begin{pmatrix} a \\ c \\ \beta_1 \end{pmatrix} \middle| a^\dagger c - c^\dagger a + \beta_1^\dagger \beta_1 = 0 \right\} / \mathrm{GL}_2(R).$$

Note that $\mathrm{GL}_2(R)$ accounts as before for possible change of basis. We then look, as in Section 4, to the open subset of $\mathcal{L}$ consisting of those subspaces corresponding to a invertible. We call it $\mathbf{M}^{3,1|1}$, it will be our model for the *Minkowski superspace*, while $\mathcal{L}$ is the compactification of $\mathbf{M}^{3,1|1}$ and it is the *conformal superspace*. By multiplying by a suitable element of $\mathrm{GL}_2(R)$ we have:

$$\mathbf{M}^{3,1|1}(R) = \left\{ \begin{pmatrix} \mathbb{1} \\ \mathcal{Y} \\ \zeta \end{pmatrix} \middle| \mathcal{Y}^\dagger = \mathcal{Y} + \zeta^\dagger \zeta \right\}.$$

Here R is a commutative superalgebra, not necessarily local as before. Notice that $\mathcal{Y} = ca^{-1}$, $\zeta = \beta_1 a^{-1}$ with respect to the expression in (7). Hence the equation is obtained immediately from (7) by setting $a = 1$. This is precisely the condition found in [25]. Notice that here the condition is coming naturally from the context we have chosen, while in [25] the same condition is obtained with more effort, through an involution of the conformal superspace. With this approach we are able to compute directly the real Minkowski superspace without resorting to the superflag.

We now turn and examine the Poincaré supergroup. We want a super-group acting on $\mathbf{M}^{3,1|1}$. We notice that the supergroup functor

$$\widehat{sP}(R) = \left\{ \begin{pmatrix} L & 0 & 0 \\ M & R & R\phi \\ d\chi & 0 & d \end{pmatrix} \right\}$$

leaves $\mathbf{M}^{3,1|1}$ invariant, it is representable (it is a closed subsupergroup of $\widetilde{\mathrm{Sp}}(4|1)$ and its reduced group is the Poincaré group. (We use the notation as in [25] so to make the comparison easier). We take then $\widehat{sP}(R)$ as our definition for the Poincaré supergroup. Applying the equations in (6) to $\widehat{sP}(R)$ we obtain:

$$R = (L^\dagger)^{-1}, \quad \phi = \chi^\dagger, \quad ML^{-1} = (ML^{-1})^\dagger + (L^\dagger)^{-1}\chi^\dagger\chi L^{-1}.$$

Hence

$$\widehat{sP}(R) = \left\{ \begin{pmatrix} L & 0 & 0 \\ M & (L^\dagger)^{-1} & (L^\dagger)^{-1}\chi^\dagger \\ d\chi & 0 & d \end{pmatrix} \right\}$$

which is precisely the Poincare' supergroup as given in [25].

The action on $\mathbf{M}^{3,1|1}$ can then be readily computed, and it yields:

$$\widehat{sP} \quad \times \quad \mathbf{M}^{3,1|1} \quad \longrightarrow \quad \mathbf{M}^{3,1|1}$$

$$\begin{pmatrix} L & 0 & 0 \\ M & (L^\dagger)^{-1} & (\chi L^{-1})^\dagger \\ d\chi & 0 & d \end{pmatrix} \quad , \quad \begin{pmatrix} \mathbb{1} \\ \mathcal{Y} \\ \zeta \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} \mathbb{1} \\ ML^{-1} + (L^\dagger)^{-1}\mathcal{Y}L^{-1} + (\chi L^{-1})^\dagger \zeta L^{-1} \\ d\chi L^{-1} + d\zeta L^{-1} \end{pmatrix}$$

as expected.

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