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Random walks in a one-dimensional Lévy random environment

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Abstract

We consider a generalization of a one-dimensional stochastic process known in the physical literature as Lévy-Lorentz gas. The process describes the motion of a particle on the real line in the presence of a random array of marked points, whose nearest-neighbor distances are i.i.d. and long-tailed (with finite mean but possibly infinite variance). The motion is a continuous-time, constant-speed interpolation of a symmetric random walk on the marked points. We first study the quenched random walk on the point process, proving the CLT and the convergence of all the accordingly rescaled moments. Then we derive the quenched and annealed CLTs for the continuous-time process.

Mathematics Subject Classification (2010): 60G50, 60F05 (82C41, 60G55).

1 Introduction

For as long as they have existed, random walks have been used as models for a wide range of transport processes in fields as diverse as physics, chemistry and biology.

For a homogeneous random walk on a lattice, under the hypothesis of finite variance of the distribution of jumps, classical results include the central limit theorem

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(CLT), the functional CLT (a.k.a. invariance principle), and normal diffusion, defined as an asymptotically linear time-dependence of the variance of the walker's position.

While the success of homogeneous random walks in capturing the main features of transport in regular media is nowadays apparent, in many interesting situations the walker moves in a complex and/or disordered environment. In such cases, correlations induced by spatial inhomogeneities can have a strong impact on the transport properties, which cannot be simulated by a simple homogenous model [17, 26, 27]. This led (already 40 years ago) to the definition of a class of processes called random walks in random environment (RWRE), where the transition probabilities are themselves random functions of space (cf. [31] for a review). This rich class of walks is typically studied from two different viewpoints: that of the *quenched* processes, where one focuses on the dynamics for a typical fixed environment, and that of the *annealed* (or *averaged*) processes, where the interest is on the effect of averaging over the environments.

On a related note, recent years have witnessed a growing interest around anomalous diffusive processes, where the variance of a moving particle has a super- or sub-linear growth in time. In the physical literature, such anomalous behavior has been observed in many systems: Lorentz gases with infinite horizon, rotating flows, intermittent dynamical systems, etc. [28].

Several models have been put forth to describe such situations. Undoubtedly, the simplest among them are the homogeneous random walks whose transition probabilities have an infinite second moment (and possibly an infinite first moment too) [16]. Especially in the physical literature, they are sometimes dubbed *Lévy flights*. Though Lévy flights easily break normal diffusion, their defining feature is also their most serious drawback, in that the variance of the walker's position is infinite at *all* times, failing to reproduce the superlinear time-dependence that is typical of many systems of interest, such as those mentioned earlier. More realistic models are then considered, called *Lévy walks*: here the jumps are still picked from a long-tailed distribution but the walker needs a certain time to complete a jump (typically a time proportional to the length of the jump, implying constant speed) [12, 30].

Not much work has been done on systems that combine long-tailed jumps and disordered media. To the authors' knowledge, the first such examples are the Lévy flights perturbed by random drift fields introduced in [15]. In this case the cause of the anomalous diffusion is the distribution of the jumps. Two more recent models are those of [3] and [25]; though rather different from one another, both systems are defined by a "normal" (meaning, simple, standard) dynamics on an "anomalous" environment, which forces long jumps and is therefore responsible for the anomalous behavior. In this sense, the models are representative of the many physical situations (human mobility, epidemics, network routing, etc.) in which anomalous diffusion is caused by the complexity of an underlying network (such as a small-world network). The system presented in [3], called by the authors *Lévy-Lorentz gas*, is the starting point of our investigation; we will come back to it momentarily. The only examples

of long-tailed random walks in random environment these authors have found in the rigorous mathematical literature are the long-range walks on point processes studied in [5, 11, 22].

A surge of interest in this topic has lately come from the physics of materials, since a new glassy material has been devised, through which light exhibits anomalous properties that can be experimentally controlled [4]. The design of this so-called *Lévy glass* suggests an interpretation of the motion of light in it by way of a Lévy walk in a disordered environment, as studied in [1, 2, 7, 6, 8] (with varying degrees of approximation). These papers focus on the annealed versions of the models, and no rigorous proof is given.

Inspired by the above models, the system we study here is a generalization of the Lévy-Lorentz gas mentioned earlier. A random array of points, called *targets*, is given on the real line, such that the distances between two neighboring targets are i.i.d. with finite mean; they are, however, allowed to have infinite variance, which is the interesting case here. A particle moves with unit speed between the targets, driven by a random walk that is independent of all the rest. More in detail, we assume that the origin is always a target and that the particle starts from there. A random integer ξ is drawn from a given distribution, upon which the particle starts to travel towards the ξ^{th} target. When the target has been reached the procedure repeats from there.

This is therefore a continuous-time random walk, whose trajectories have long inertial segments due to a random environment, which is why we speak of a *random walk in a Lévy random environment*. These are our results: for the (discrete-time) random walk on the point process, we prove the quenched CLT and the convergence of all the normally rescaled moments to those of a suitable Gaussian. Then, by comparison, we derive the quenched CLT for the continuous-time process. These results imply the annealed CLT for both the continuous- and discrete-time walks.

The paper is organized as follows. In Section 2.1 we give the precise definitions of all the processes associated with our random walk. In Section 2.2 we present our main results, whose proofs are found in Section 3, although some technical results are gathered in Appendix A. Section 3.2 presents a construction that is also of independent interest: a dynamical system describing the annealed process from the point of view of the particle.

2 Model and main results

2.1 Definition of the model

We start by defining the following marked point process on $\mathbb{R}$: let $\zeta := (\zeta_j,\, j \in \mathbb{Z})$ be a sequence of i.i.d. positive random variables with finite mean μ, and define the variables ω_k, $k \in \mathbb{Z}$, via

$$\omega_0 := 0\,, \qquad \omega_k := \omega_{k-1} + \zeta_k\,. \tag{2.1}$$

The process $\omega := (\omega_k,\, k \in \mathbb{Z})$ will be also referred to as the *environment*, and the single points ω_k as the *targets*. We denote the set of all possible environments by $\Omega_{\rm en}$, and the law just defined on it by P.

We are particularly interested in long-tailed ζ_j, with infinite variance, distributed for instance in the basin of attraction of an α-stable distribution, with $\alpha \in (1,2)$. Environments of this type are usually called *Lévy environments* in the physical literature [3, 7, 1].

In order to define our continuous-time process, we need to introduce two intermediate random walks (RWs). Let $\mathbb{Z}^{\pm}$ be the positive/negative integers, and $\mathbb{N}$ the non-negative integers. Take $\xi := (\xi_i,\, i \in \mathbb{Z}^+)$ to be a sequence of i.i.d. $\mathbb{Z}$-valued random variables, with density $p := (p_k,\, k \in \mathbb{Z})$, where $p_k = p_{-k}$ (symmetry condition), $p_{k+1} \le p_k$, $\forall k \ge 0$ (half-monotonicity), and such that

$$v_p := \sum_k k^2 p_k \in (0,\infty)\,. \tag{2.2}$$

Denote by $S := (S_n,\, n \in \mathbb{N})$ the RW with increments provided by the ξ_i, that is

$$S_0 := 0\,, \qquad S_n := \sum_{i=1}^{n} \xi_i\,, \quad \text{for } n \ge 1. \tag{2.3}$$

This is called the *underlying random walk*. It is defined on the probability space $(\mathbb{Z}^{\mathbb{N}}, Q)$, endowed with the σ-algebra generated by cylinder functions.

The second RW is defined, for each environment $\omega \in \Omega_{\rm en}$, as

$$Y_n \equiv Y_n^{\omega} := \omega_{S_n}\,, \quad \text{for } n \in \mathbb{N}\,. \tag{2.4}$$

In rough terms, $Y := (Y_n,\, n \in \mathbb{N})$ performs the same jumps as S, but on the points of ω. We call it the *random walk on the point process*. The associated probability space is $(\omega^{\mathbb{N}}, Q_\omega)$, where Q_ω is the probability induced on $\omega^{\mathbb{N}}$ by Q via (2.4) (more precisely, Q_ω is defined on the σ-algebra generated by cylinder functions). In particular, for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$,

$$Q_\omega(Y_n = \omega_k) = Q(S_n = k)\,. \tag{2.5}$$

Once we fix the environment and the realization of the dynamics, that is, for any given pair $(S,\omega) \in (\mathbb{Z}^{\mathbb{N}}, \Omega_{\rm en})$, we can define the sequence of *collision times* $\tau(n) \equiv \tau(n; S, \omega)$ via

$$\tau(0) := 0\,, \qquad \tau(n) := \sum_{k=1}^{n} |\omega_{S_k} - \omega_{S_{k-1}}|\,, \quad \text{for } n \ge 1\,. \tag{2.6}$$

Notice that, since the length of the $n^{\rm th}$ jump of the walk is given by $|\omega_{S_n} - \omega_{S_{n-1}}|$, $\tau(n)$ represents the global length of the trajectory up to time n.

Finally, the process we are interested in is the continuous-time process $X(t) \equiv X^{\omega}(t)$ defined by

$$X(t) := Y_n + \mathrm{sgn}(\xi_{n+1})(t - \tau(n))\,, \quad \text{for } t \in [\tau(n), \tau(n+1))\,, \tag{2.7}$$

where sgn is the sign function. In other words, $X := (X(t)\,, t \in [0,\infty))$ is the process whose trajectories interpolate those of the walk Y and whose speed is 1 (save at collision times). In light of the discussion made in the introduction, we describe the above as a continuous-time random walk on a Lévy random environment.

Remark 2.1 Notice that $\xi_{n+1} = 0 \Leftrightarrow S_{n+1} = S_n \Leftrightarrow \tau(n+1) = \tau(n)$. Therefore (2.7) is never used in the case $\xi_{n+1} = 0$, which makes the definition of sgn(0) irrelevant there. More importantly, the self-jumps of the underlying RW (namely, $S_{n+1} = S_n$) are simply not seen by the process $X(t)$. This implies that we can remove any lazy component of $S = (S_n)$ by redefining

$$p'_0 := 0\,, \qquad p'_j := \frac{p_j}{\sum_{k\neq 0} p_k}, \quad \text{for } j \neq 0. \tag{2.8}$$

(Notice that $\sum_{k\neq 0} p_k > 0$, because $v_p > 0$.) In particular, the case where S is a simple symmetric RW, called Lévy-Lorentz gas in [3], is included in our results.

Indicate with $\mathcal{C} := C([0,\infty);\mathbb{R})$ the space of all continuous paths from $[0,\infty)$ to $\mathbb{R}$, endowed with the Skorokhod topology. We denote by P_ω the *quenched law* of X, which is the probability induced by Q on $\mathcal{C}$ by the definitions (2.4)-(2.7).

Finally, we use $\mathbb{P}$ for the *annealed law* of the process, defined on the space $\mathcal{C}\times\Omega_{\text{en}}$ by

$$\mathbb{P}(G \times F) = \int_F P_\omega(G) P(d\omega)\,. \tag{2.9}$$

This is the law that describes the entire randomness of the system.

2.2 Main results

In order to state our main results we need to name a few parameters pertaining to the underlying RW S. Let

$$M := \sum_{k\in\mathbb{Z}} |k|\, p_k = 2\sum_{k=1}^{\infty} k\, p_k \tag{2.10}$$

be its mean absolute jump, and denote

$$\bar{q} := \sup\left\{ q \geq 0 \;\middle|\; \sum_k |k|^q\, p_k < \infty \right\}. \tag{2.11}$$

By our initial assumptions, $\bar{q} \geq 2$. The following is standard:

Proposition 2.2 *S verifies the standard CLT, namely,*

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}} \stackrel{d}{=} \mathcal{N}(0, v_p). \tag{2.12}$$

Also, denoting by E_Q the expectation w.r.t. Q (the law of S),

$$m_q := \lim_{n\to\infty} \frac{E_Q(|S_n|^q)}{n^{q/2}} \tag{2.13}$$

exists at least for all $q \geq 0$, $q \neq \bar{q}$. For $q \in [0, \bar{q})$, it is finite and equals the q^{th} absolute moment of $\mathcal{N}(0, v_p)$; for $q \in (\bar{q}, \infty)$, it is infinite.

For the sake of completeness, we give the proof of Proposition 2.2 at the beginning of the next section. Observe that, if $\bar{q} > 2$, the proposition says that $(S_n/\sqrt{n})$ converges weakly to a suitable Gaussian, together with all the moments of order ≤ 2. If $\bar{q} = 2$, the proposition does not ensure that the second moment converges, but its proof guarantees that $E_Q(|S_n|^2)/n$ is bounded above and below (this follows from (3.1)-(3.2) below and the fact that $E_Q(|\xi_1|^2) = v_p < \infty$ by hypothesis). We describe this situation by saying that the underlying RW is *totally diffusive*.

The purpose of this paper is to show that the random walk on the point process Y is also totally diffusive and its continuous-time interpolation X verifies the CLT, both in the quenched sense, i.e., in a fixed environment, for almost every environment. Recalling that μ denotes the mean of the random variables ζ_i, these are our results:

Theorem 2.3 *For P-a.e. $\omega \in \Omega_{\rm en}$,*

$$\lim_{n\to\infty} \frac{Y_n}{\sqrt{n}} \stackrel{d}{=} \mathcal{N}(0, \mu^2\, v_p)\,. \tag{2.14}$$

The convergence is in distribution, relative to the law P_ω on $\mathcal{C}$.

Theorem 2.4 *Let E_ω denote the expectation w.r.t. the measure Q_ω. If $q \in [0, \bar{q})$, then*

$$\lim_{n\to\infty} \frac{E_\omega(|Y_n|^q)}{n^{q/2}} = \mu^q m_q\,. \tag{2.15}$$

If also $q \in 2\mathbb{N} + 1$, then

$$\lim_{n\to\infty} \frac{E_\omega(Y_n^q)}{n^{q/2}} = 0\,. \tag{2.16}$$

Both statements hold for P-a.e. $\omega \in \Omega_{\rm en}$.

Theorem 2.5 *For P-a.e. $\omega \in \Omega_{\rm en}$,*

$$\lim_{t\to\infty} \frac{X(t)}{\sqrt{t}} \stackrel{d}{=} \mathcal{N}\Big(0, \frac{\mu}{M} v_p\Big)\,. \tag{2.17}$$

The convergence is in distribution, relative to the law P_ω on $\mathcal{C}$.

Remark 2.6 In view of Remark 2.1, let us observe that Theorem 2.5 must not depend of the choice of p_0, the lazy component of S. However, as per definitions (2.2) and (2.10), v_p and M do. On the other hand, if we define v_p' and M' by using (p_k') in lieu of (p_k) in (2.2), (2.10), it is immediate to check that $v_p'/M' = v_p/M$.

The quenched CLTs easily imply the annealed CLTs:

Corollary 2.7 *The limits (2.14) and (2.17) hold for the annealed processes as well, that is, w.r.t. the measure* $\mathbb{P}$ *on* $\mathcal{C} \times \Omega_{\rm en}$.

Theorems 2.3 and 2.4 provide a complete characterization of the quenched process Y. As for the physically more relevant process X, it is an open question whether a similar scaling for the quenched moments holds. In the annealed case, heuristic arguments and numerical simulations suggest that the second moment does not always grow linearly in time. In particular [7], if the distribution of the distance between targets behaves like $dP(\zeta_0 \le z)/dz \sim z^{-1-\alpha}$, for $z \to \infty$, the second moment is expect to scale like

$$\mathbb{E}\big(X(t)^2\big) \sim \begin{cases} t^{5/2-\alpha}\,, & 1 \le \alpha \le 3/2\,; \\ t\,, & \alpha > 3/2\,. \end{cases} \tag{2.18}$$

3 Proofs

In this section we prove our results. The most elaborate proof, that of Theorem 2.5, requires a representation of the annealed process as a dynamical system 'from the point of view of the particle'. We present this system in Section 3.2. The technical lemmas which offer little insight on the flow of the proofs are given in the Appendix.

We start with the standard results about the underlying RW.

Proof of Proposition 2.2. The CLT is a well-known result for a finite-variance RW. The convergence of the rescaled moments is in general not so well-known. For this reason we provide a short proof, though other references may be found in the literature (e.g., [29]).

Denoting $\chi_n := \big(\sum_{i=1}^n \xi_i^2\big)^{1/2}$, Burkholder's inequality [9, Thm. 3.2] states that, for all $q > 1$, there exist constants $C_q > c_q > 0$ such that

$$c_q \|\chi_n\|_q \le \|S_n\|_q \le C_q \|\chi_n\|_q\,, \tag{3.1}$$

where $\|\cdot\|_q$ denotes the L^q-norm w.r.t. Q. For $1 < q < \bar{q}$, $E_Q(|\xi_i|^q) := \sum_k |k|^q\, p_k < \infty$. By (3.1), $E_Q(|S_n|^q) = \|S_n\|_q^q$ is asymptotic to

$$\|\chi_n\|_q^q = \left\| \sum_{i=1}^n \xi_i^2 \right\|_{q/2}^{q/2} \le \left(\sum_{i=1}^n \big\|\xi_i^2\big\|_{q/2} \right)^{q/2} = n^{q/2} E_Q(|\xi_1|^q)\,. \tag{3.2}$$

This implies that the $q^{\rm th}$ absolute moment of $(S_n/\sqrt{n})$ is bounded above in n. Since the process converges weakly to $\mathcal{N}(0, v_p)$, a simple argument [14, Ex. 2.5] shows that, $\forall q' \in [0, q)$, the $(q')^{\rm th}$ absolute moment of $(S_n/\sqrt{n})$ converges to the $(q')^{\rm th}$ absolute moment of $\mathcal{N}(0, v_p)$. Since q was arbitrary, the conclusion holds for all $q \in [0, \bar{q})$.

On the other hand, when $q \in (\bar{q}, \infty)$, $E_Q(|S_n|^q)$ behaves like

$$\left\| \sum_{i=1}^{n} \xi_i^2 \right\|_{q/2}^{q/2} \geq E_Q\left(\sum_{i=1}^{n} |\xi_i|^q \right) = \infty \,, \tag{3.3}$$

having used the inequality $(a+b)^{q/2} \geq a^{q/2} + b^{q/2}$ (holding for $a, b \geq 0$ and $q/2 \geq 1$) and the fact that $E_Q(|\xi_i|^q) = \infty$. Q.E.D.

3.1 The random walk on the point process

Proof of Theorem 2.3. This proof follows that of Thm. 1.13 of [5]. By the definition (2.1) of ω, we have

$$\omega_n = \begin{cases} \sum_{k=1}^{n} \zeta_k \,, & n > 0 \,; \\ 0 \,, & n = 0 \,; \\ -\sum_{k=n+1}^{0} \zeta_k \,, & n < 0 \,. \end{cases} \tag{3.4}$$

The strong law of large numbers on (ζ_k) can be expressed as follows: fixed $b \in \mathbb{R}$, for P-a.e. $\omega \in \Omega_{\rm en}$,

$$\lim_{j \to \infty} \frac{\omega_{[bj]}}{j} = b\mu \,, \tag{3.5}$$

where $[r]$ denotes the integer part of $r \in \mathbb{R}$.

Since $Y_n = \omega_{S_n}$, we get that, for $a \in \mathbb{R}$, $\varepsilon > 0$ and P-a.e. $\omega \in \Omega_{\rm en}$,

$$\begin{aligned} \lim_{n\to\infty} Q_\omega\left(\frac{Y_n}{\sqrt{n}} \leq a \right) &\leq \lim_{n\to\infty} \left[Q\left(\frac{\omega_{S_n}}{\sqrt{n}} \leq a \,, \frac{S_n}{\sqrt{n}} > \frac{a}{\mu} + \varepsilon \right) + Q\left(\frac{S_n}{\sqrt{n}} \leq \frac{a}{\mu} + \varepsilon \right) \right] \\ &\leq \lim_{n\to\infty} \left[Q\left(\frac{\omega_{[(\frac{a}{\mu}+\varepsilon)\sqrt{n}]}}{\sqrt{n}} \leq a \right) + Q\left(\frac{S_n}{\sqrt{n}} \leq \frac{a}{\mu} + \varepsilon \right) \right] \\ &= \Phi\left(\frac{a}{\mu\sqrt{v_p}} + \varepsilon' \right) \,, \end{aligned} \tag{3.6}$$

where $\varepsilon' := \varepsilon/\sqrt{v_p}$, Φ is the distribution function of the standard normal, and we have used (3.5) and (2.12). Analogously,

$$\lim_{n\to\infty} Q_\omega\left(\frac{Y_n}{\sqrt{n}} \leq a \right) \geq \Phi\left(\frac{a}{\mu\sqrt{v_p}} - \varepsilon' \right) \,, \tag{3.7}$$

and altogether one gets the desired convergence. Q.E.D.

Proof of Theorem 2.4. The basic ingredients of the proof are the convergence of the moments of $(S_n/\sqrt{n})$ and the law of large numbers for ω_n, cf. (3.5).

By the latter, for every $\varepsilon > 0$ and P-a.e. ω, there exists $k_0 \equiv k_0(\varepsilon, \omega)$ such that, for all $|k| \geq k_0$,

$$\left| \frac{\omega_k}{k} - \mu \right| < \varepsilon \,. \tag{3.8}$$

In particular there exists $c \equiv c(\omega) > 0$ such that $|\omega_k| \le c|k|$, for all $k \in \mathbb{Z}$. Let us fix a P-typical ω. Recalling that $Y_n = \omega_{S_n}$, we have:

$$\begin{aligned}\frac{E_\omega(|Y_n|^q)}{n^{q/2}} &= E_Q\left(\frac{|\omega_{S_n}|^q}{|S_n|^q}\,\frac{|S_n|^q}{n^{q/2}}\right)\\ &= E_Q\left(1_{\{|S_n|\ge k_0\}}\frac{|\omega_{S_n}|^q}{|S_n|^q}\,\frac{|S_n|^q}{n^{q/2}}\right) + E_Q\left(1_{\{|S_n|<k_0\}}\frac{|\omega_{S_n}|^q}{|S_n|^q}\,\frac{|S_n|^q}{n^{q/2}}\right) \qquad (3.9)\\ &\le (\mu+\varepsilon)^q\,E_Q\left(\frac{|S_n|^q}{n^{q/2}}\right) + c^q\,Q(|S_n|<k_0)^{1/r'}E_Q\left(\frac{|S_n|^{rq}}{n^{rq/2}}\right)^{1/r},\end{aligned}$$

where in the last step we have used Hölder's inequality with $r > 1$ and $1/r' + 1/r = 1$. Now choose r so that $rq < \bar{q}$. By Proposition 2.2,

$$\lim_{n\to\infty} Q\left(|S_n| < k_0\right) = 0\,; \qquad (3.10)$$

$$\lim_{n\to\infty} E_Q\left(\frac{|S_n|^q}{n^{q/2}}\right) = m_q\,; \qquad (3.11)$$

$$\lim_{n\to\infty} E_Q\left(\frac{|S_n|^{rq}}{n^{rq/2}}\right) = m_{rq} < \infty\,. \qquad (3.12)$$

In conclusion,

$$\limsup_{n\to\infty}\ \frac{E_\omega(|Y_n|^q)}{n^{q/2}} \le (\mu+\varepsilon)^q m_q\,. \qquad (3.13)$$

Similarly, for the lower bound,

$$\begin{aligned}E_Q\left(\frac{|\omega_{S_n}|^q}{|S_n|^q}\,\frac{|S_n|^q}{n^{q/2}}\right) &\ge (\mu-\varepsilon)^q\,E_Q\left(1_{\{|S_n|\ge k_0\}}\frac{|S_n|^q}{n^{q/2}}\right)\\ &= (\mu-\varepsilon)^q\,E_Q\left(\frac{|S_n|^q}{n^{q/2}}\right) - (\mu-\varepsilon)^q\,E_Q\left(1_{\{|S_n|<k_0\}}\frac{|S_n|^q}{n^{q/2}}\right) \qquad (3.14)\\ &\ge (\mu-\varepsilon)^q\,E_Q\left(\frac{|S_n|^q}{n^{q/2}}\right) - (\mu-\varepsilon)^q\,Q(|S_n|<k_0)^{1/r'}E_Q\left(\frac{|S_n|^{rq}}{n^{rq/2}}\right)^{1/r},\end{aligned}$$

which, by the same arguments as above, gives

$$\liminf_{n\to\infty}\ \frac{E_\omega(|Y_n|^q)}{n^{q/2}} \ge (\mu-\varepsilon)^q m_q\,. \qquad (3.15)$$

Assertion (2.15) follows from (3.13), (3.15) and the arbitrariness of ε. The limit (2.16) is proved in a similar fashion upon rewriting

$$\frac{E_\omega(Y_n^q)}{n^{q/2}} = \frac{E_\omega\big(1_{\{S_n\ge 0\}}\,|Y_n|^q\big)}{n^{q/2}} - \frac{E_\omega\big(1_{\{S_n<0\}}|\,Y_n|^q\big)}{n^{q/2}}\,. \qquad (3.16)$$

Q.E.D.

3.2 The point-of-view-of-the-particle dynamical system

In this section we introduce a process, or rather a dynamical system, that describes the point of view of the particle for the RW on the point process.

Keeping in mind the definitions and notation of Section 2.1, let $(\mathbb{Z}^{\mathbb{Z}^+}, Q_o)$ denote the probability space of the sequences $\xi = (\xi_i,\, i \in \mathbb{Z}^+)$, namely, Q_o is the probability for $\mathbb{Z}^+$ copies of i.i.d. $\mathbb{Z}$-valued variables with density $p = (p_k,\, k \in \mathbb{Z})$. Indicate with σ_ξ the left shift on this space. Evidently, σ_ξ preserves Q_o and is ergodic. This process is isomorphic to the RW $(S_n,\, n \in \mathbb{N})$, by construction of the latter. When conjugated with the natural isomorphism, σ_ξ acts on $(\mathbb{Z}^{\mathbb{N}}, Q)$ as: $(S_n,\, n \in \mathbb{N}) \mapsto (S_{n+1} - S_1,\, n \in \mathbb{N})$.

Further denote by $((\mathbb{R}^+)^{\mathbb{Z}}, P_o)$ the probability space of the sequences $\zeta := (\zeta_j,\, j \in \mathbb{Z})$, where P_o is the Bernoulli measure based on the variables ζ_j defined in Section 2.1. Indicate with σ_ζ the left shift on this space: σ_ζ is an ergodic automorphism of $((\mathbb{R}^+)^{\mathbb{Z}}, P_o)$. Again, there is a natural isomorphism between $((\mathbb{R}^+)^{\mathbb{Z}}, P_o)$ and $(\Omega_{\rm en}, P)$. Upon conjugation with it, σ_ζ acts on $(\Omega_{\rm en}, P)$ as: $(\omega_k,\, k \in \mathbb{Z}) \mapsto (\omega_{k+1} - \omega_1,\, k \in \mathbb{Z})$. Also, σ_ζ^{-1} acts as: $(\omega_k,\, k \in \mathbb{Z}) \mapsto (\omega_{k-1} - \omega_{-1},\, k \in \mathbb{Z})$.

Set $\Sigma := \mathbb{Z}^{\mathbb{Z}^+} \times (\mathbb{R}^+)^{\mathbb{Z}}$ and $\nu := Q_o \otimes P_o$, and define $T : \Sigma \longrightarrow \Sigma$ via

$$T(\xi, \zeta) := (\sigma_\xi(\xi), \sigma_\zeta^{\xi_1}(\zeta))\,. \tag{3.17}$$

We think of (Σ, ν, T) as a dynamical system. Let us call ξ the *dynamical variable* and ζ the *environmental variable*, or simply the *environment*. Fix an initial condition (ξ, ζ). The first component of the dynamical variable, ξ_1, determines the jump that the underlying RW is about to make, namely $Y_1 = \omega_{\xi_1}$. Applying T translates the environment by the quantity $-Y_1$ (corresponding to $|\xi_1|$ discrete shifts in the direction opposite to the jump), and shifts the dynamical variable, so that the system is ready for the next jump (determined by ξ_2) under the pretense that Y_1 is the origin.

In other words, this dynamical system describes the annealed process from the point of view of the particle (PVP). This is why we call it the *PVP dynamical system*.

Theorem 3.1 *(Σ, ν, T) is measure-preserving and ergodic.*

The proof of this theorem is found in Appendix A.1. The isomorphisms $\xi \leftrightarrow S$ and $\zeta \leftrightarrow \omega$, mentioned earlier, entail that Theorem 3.1 is equivalent to the following:

Corollary 3.2 *The mapping*

$$((S_n,\, \omega_k),\, n \in \mathbb{N},\, k \in \mathbb{Z}) \mapsto ((S_{n+1} - S_1,\, \omega_{k+S_1} - \omega_{S_1}),\, n \in \mathbb{N},\, k \in \mathbb{Z}) \tag{3.18}$$

on $(\mathbb{Z}^{\mathbb{N}} \times \Omega_{\rm en}, Q \otimes P)$ is measure-preserving and ergodic.

The following technical lemma, needed in the proof of the main result, will also be proved in Appendix A.1.

Lemma 3.3 *For $\zeta \in (\mathbb{R}^+)^{\mathbb{Z}}$, set $F_\zeta := \mathbb{Z}^{\mathbb{Z}^+} \times \{\zeta\}$ (in the remainder, any such set will be referred to as an horizontal fiber of Σ). Then,*

$$T(F_\zeta) = \bigcup_{k\in\mathbb{Z}} F_{\sigma_\zeta^k(\zeta)}\,. \tag{3.19}$$

Furthermore (with a minor abuse of notation) indicate with $Q_o(\,\cdot\,|\,F_\zeta)$ the measure on F_ζ induced by Q_o via the identification $F_\zeta \cong \mathbb{Z}^{\mathbb{Z}^+}$. T pushes this measure to

$$T_* Q_o(\,\cdot\,|\,F_\zeta) = \sum_{k\in\mathbb{Z}} p_k\, Q_o(\,\cdot\,|F_{\sigma_\zeta^k(\zeta)})\,. \tag{3.20}$$

3.3 CLT of the Lévy walk

We will prove Theorem 2.5 by controlling the continuous-time walk $(X(t))$ through the discrete-time walk (Y_n). To this goal, it is convenient to introduce a quantity which counts the number of collisions of the process $X(t)$ up to time t. Formally, for every $t \in \mathbb{R}^+$, set

$$n(t) \equiv n(t; S, \omega) := \max\{m \in \mathbb{N} \mid t \geq \tau(m)\}\,. \tag{3.21}$$

This is a sort of inverse function of the collision time $\tau(n)$, defined in (2.6). In point of fact, when $\tau(n)$ is strictly monotonic (which occurs when S has no lazy component, cf. Remark 2.1), $n(t)$ is a suitable piecewise extension of the inverse of $\tau(n)$.

Lemma 3.4 *In view of the definitions (2.6) and (3.21), which depend on $(S,\omega) \in \mathbb{Z}^{\mathbb{N}} \times \Omega_{\rm en}$, we have that, $(Q\otimes P)$-almost surely, equivalently, $\mathbb{P}$-almost surely,*

$$\lim_{n\to\infty} \frac{\tau(n)}{n} = M\mu\,; \tag{3.22}$$

$$\lim_{t\to\infty} \frac{t}{n(t)} = M\mu\,. \tag{3.23}$$

Proof. By (2.6) we see that $\tau(n)$ is the Birkhoff sum of the function

$$g(S,\omega) := |\omega_{S_1} - \omega_{S_0}| = |\omega_{S_1}| = |\omega_{\xi_1}|\,, \tag{3.24}$$

on $\mathbb{Z}^{\mathbb{N}} \times \Omega_{\rm en}$, relative to the dynamics (3.18). So (3.22) follows by Corollary 3.2 and the Birkhoff theorem: for $(Q\otimes P)$-a.e. choice of $(S,\omega) \in \mathbb{Z}^{\mathbb{N}} \times \Omega_{\rm en}$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\tau(n)}{n} &= \int_{\mathbb{Z}^{\mathbb{N}}\times\Omega_{\rm en}} g\, d(Q\otimes P) \\
&= \int_{\mathbb{Z}^{\mathbb{Z}^+}\times\Omega_{\rm en}} |\omega_{\xi_1}|\, d(Q_o\otimes P) \\
&= \sum_{k\in\mathbb{Z}} p_k \int_{\Omega_{\rm en}} |\omega_k|\, dP \\
&= \sum_{k\in\mathbb{Z}} p_k |k| \mu = M\mu\,,
\end{aligned} \tag{3.25}$$

having used some of the notation and arguments given in Section 3.2.

Moreover, since by definition $n(t) \to \infty$, almost surely, as $t \to \infty$, (3.22) implies that

$$\lim_{t\to\infty} \frac{\tau(n(t)+1)}{n(t)} = \lim_{t\to\infty} \frac{\tau(n(t))}{n(t)} = M\mu\,. \tag{3.26}$$

But

$$\lim_{t\to\infty} \left| \frac{t}{n(t)} - \frac{\tau(n(t))}{n(t)} \right| \le \lim_{t\to\infty} \left(\frac{\tau(n(t)+1)}{n(t)} - \frac{\tau(n(t))}{n(t)} \right) = 0\,, \tag{3.27}$$

giving (3.23). Q.E.D.

Lemma 3.5 *For P-a.e. $\omega \in \Omega_{\rm en}$ (equivalently, P_o-a.e. $\zeta \in (\mathbb{R}^+)^{\mathbb{Z}}$) and every $\ell \in \mathbb{Z}$,*

$$\lim_{n\to\infty} E_\omega(\omega_{S_n+\ell} - \omega_{S_n+\ell-1}) = \lim_{n\to\infty} E_\omega(\zeta_{S_n+\ell}) = \mu\,. \tag{3.28}$$

Also, given any positive sequence $\psi_n = o(\sqrt{n})$, $n \to \infty$, the above limits are uniform for $|\ell| \le \psi_n$ (for a fixed ω, or ζ).

Proof. To start with, the first equality of (3.28) follows trivially by the definitions of ω and ζ, so we prove the second one.

Again, we use the machinery of Section 3.2. Set $h(\xi,\zeta) := \zeta_\ell$. By (3.17) and (2.3) we write

$$h \circ T^n(\xi,\zeta) = h(\sigma_\xi^n(\xi), \sigma_\zeta^{S_n}(\zeta)) = \zeta_{S_n+\ell}\,. \tag{3.29}$$

Now, recall the definitions given in the statement of Lemma 3.3. Observe that taking the expectation E_ω is tantamount to integrating over the fiber F_ζ w.r.t. the measure $Q_o(\,\cdot\,|\,F_\zeta)$, where ζ corresponds to ω via (2.1). Hence, with the notation

$$p_j^{(n)} := \sum_{k_1+\cdots+k_n=j} p_{k_1}\cdots p_{k_n} = Q(S_n = j)\,, \tag{3.30}$$

we get

$$\begin{aligned} E_\omega(\zeta_{S_n+\ell}) &= \int_{F_\zeta} (h\circ T^n)\, dQ_o(\,\cdot\,|\,F_\zeta) \\ &= \sum_{k_1,\ldots,k_n} p_{k_1}\cdots p_{k_n} \int h\, dQ_o(\,\cdot\,|\,F_{\sigma_\zeta^{k_1+\cdots+k_n}(\zeta)}) \\ &= \sum_{j\in\mathbb{Z}} p_j^{(n)} \int h\, dQ_o(\,\cdot\,|\,F_{\sigma_\zeta^{j}(\zeta)}) \\ &= \sum_{j\in\mathbb{Z}} p_j^{(n)}\, \zeta_{j+\ell}\,. \end{aligned} \tag{3.31}$$

In the second equality above, we have applied Lemma 3.3 recursively n times: the summation is over $\mathbb{Z}^n$ and each integral is taken over the horizontal fiber specified

by the integration measure. In the fourth equality we have used that h is constant along horizontal fibers.

At this point we want to apply Lemma A.4 of the Appendix with $a_j := \zeta_{j+\ell}$ and $p^{(n)}$ as above. We need to check the hypotheses of the lemma. First off, $p^{(n)}$ verifies condition *(i)* because it is symmetric and half-monotonic (this is, e.g., a consequence of Lemma A.5, as p is symmetric and half-monotonic by assumption). It also verifies condition *(ii)*, because the underlying RW satisfies the CLT.

Remark 3.6 This is the only point in the paper where the half-monotonicity of p is used.

As for the hypothesis on a, we use the ergodicity of the process ζ. Thus, for P_o-a.e. $\zeta \in (\mathbb{R}^+)^{\mathbb{Z}}$,

$$\lim_{k\to\infty} \frac{1}{k}\sum_{j=0}^{k-1} a_j = \lim_{k\to\infty} \frac{1}{k}\sum_{j=-1}^{-k} a_j = \mathbb{E}(\zeta_1) = \mu\,. \tag{3.32}$$

So Lemma A.4 can be applied almost surely in ζ, equivalently in ω. Using the notation of that lemma, (3.31) becomes

$$E_\omega(\zeta_{S_n+\ell}) = \sum_{j\in\mathbb{Z}} p_j^{(n)} a_j = \mathcal{E}_n(a)\,, \tag{3.33}$$

showing that, for P-a.e. $\omega \in \Omega_{\rm en}$, (3.33) converges to (3.32), as $n \to \infty$, which proves the limit (3.28).

Finally, observe that the underlying random walk is strongly aperiodic by hypothesis: this implies (rather easily) that, as $n \to \infty$, $p_{j-\ell}^{(n)} - p_j^{(n)} = o(p_j^{(n)})$, uniformly in $j \in \mathbb{Z}$ and $|\ell| \le \psi_n$. Since (3.33) can be rewritten as $E_\omega(\zeta_{S_n+\ell}) = \sum_j p_{j-\ell}^{(n)} \zeta_j$, its limit is the same as for $\ell = 0$, uniformly for $|\ell| \le \psi_n$. Q.E.D.

We are now ready to prove our main theorem (we will not prove the obvious Corollary 2.7).

Proof of Theorem 2.5. Let us define $\bar n(t) := [t/M\mu]$. Compare $\bar n(t)$ to $n(t)$: for fixed t, the former is a constant while the latter is a random variable on $(\omega^{\mathbb{N}}, Q_\omega)$. For P-a.e. $\omega \in \Omega_{\rm en}$, we have that, Q_ω-almost surely,

$$\lim_{t\to\infty} \frac{n(t) - \bar n(t)}{t} = 0 \tag{3.34}$$

(this follows form Lemma 3.4 and Fubini's Theorem). Moreover, by Theorem 2.3 and the definition of $\bar n(t)$,

$$\lim_{t\to\infty} \frac{Y_{\bar n(t)}}{\sqrt{t}} \stackrel{d}{=} \mathcal{N}\Big(0, \frac{\mu}{M} v_p\Big)\,, \tag{3.35}$$

for P-a.e. ω. Since $X(t)$ always lies between $Y_{n(t)}$ and $Y_{n(t)+1}$, it is easy to see that

$$\begin{aligned}\left|\frac{X(t)}{\sqrt{t}} - \frac{Y_{\bar{n}(t)}}{\sqrt{t}}\right| &\leq \max\left\{\frac{|Y_{n(t)} - Y_{\bar{n}(t)}|}{\sqrt{t}}\,,\, \frac{|Y_{n(t)+1} - Y_{\bar{n}(t)}|}{\sqrt{t}}\right\} \\ &\leq \frac{|Y_{n(t)} - Y_{\bar{n}(t)}|}{\sqrt{t}} + \frac{|Y_{n(t)+1} - Y_{\bar{n}(t)}|}{\sqrt{t}}\,.\end{aligned} \tag{3.36}$$

In light of (3.35), and using Slutzky's Theorem [18, Thm. 13.18], Theorem 2.5 will be proved once we prove that, P-almost surely, the two terms in the second line of (3.36) converge to 0 in distribution, and thus in probability, w.r.t. Q_ω. We will only show the convergence of the first term, the second one being completely analogous.

Applying the Portemanteau Theorem [18, Thm. 13.16], it will suffice to prove that, given an ω for which (3.34) holds, and a bounded Lipschitz function $f : \mathbb{R} \to \mathbb{R}$,

$$\lim_{t\to\infty} E_\omega\left(f\left(\frac{|Y_{n(t)} - Y_{\bar{n}(t)}|}{\sqrt{t}}\right)\right) = f(0)\,. \tag{3.37}$$

So, fix $\varepsilon > 0$. By (3.34), one can find a 'bad' set $B_1 \subset \omega^{\mathbb{N}}$, with $Q_\omega(B_1) \leq \varepsilon/6\|f\|_\infty$, and a function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, with $\lim_{t\to\infty} \phi(t)/t = 0$, such that

$$|n(t) - \bar{n}(t)| \leq \phi(t) \tag{3.38}$$

for all realizations of the dynamics in $\omega^{\mathbb{N}} \setminus B_1$. Moreover, by (2.12), there exist another bad set $B_2 \subset \omega^{\mathbb{N}}$, again with $Q_\omega(B_2) \leq \varepsilon/6\|f\|_\infty$, and a constant $C > 0$ such that, for all realizations in $\omega^{\mathbb{N}} \setminus B_2$,

$$\left|S_{n(t)} - S_{\bar{n}(t)}\right| \leq C\sqrt{|n(t) - \bar{n}(t)|}. \tag{3.39}$$

Altogether, for all realizations in $\omega^{\mathbb{N}} \setminus (B_1 \cup B_2)$,

$$\left|S_{n(t)} - S_{\bar{n}(t)}\right| \leq C\sqrt{\phi(t)}. \tag{3.40}$$

We split the average in the l.h.s. of (3.37) in two parts, restricting it, respectively, to $B_1 \cup B_2$ and its complement $G := \omega^{\mathbb{N}} \setminus (B_1 \cup B_2)$. For the first part, we estimate

$$E_\omega\left(1_{B_1\cup B_2}\left|f\left(\frac{|Y_{n(t)} - Y_{\bar{n}(t)}|}{\sqrt{t}}\right) - f(0)\right|\right) \leq 2\|f\|_\infty\, Q_\omega(B_1 \cup B_2) \leq \frac{2}{3}\varepsilon\,, \tag{3.41}$$

where 1_A denotes the indicator function of $A \subset \omega^{\mathbb{N}}$. For the second part, if c is the Lipschitz constant of f, we write

$$E_\omega\left(1_G\left|f\left(\frac{|Y_{n(t)} - Y_{\bar{n}(t)}|}{\sqrt{t}}\right) - f(0)\right|\right) \leq \frac{c}{\sqrt{t}}\, E_\omega\big(1_G\left|Y_{n(t)} - Y_{\bar{n}(t)}\right|\big)\,. \tag{3.42}$$

By definition of the processes Y, ω, and ζ (cf. Section 2.1),

$$Y_{n(t)} - Y_{\bar{n}(t)} = \omega_{S_{n(t)}} - \omega_{S_{\bar{n}(t)}} = \begin{cases} \sum_{\ell=1}^{S_{n(t)}-S_{\bar{n}(t)}} \zeta_{S_{\bar{n}(t)}+\ell}\,, & \text{if } S_{n(t)} > S_{\bar{n}(t)}\,; \\ 0\,, & \text{if } S_{n(t)} = S_{\bar{n}(t)}\,; \\ -\sum_{\ell=0}^{S_{n(t)}-S_{\bar{n}(t)}+1} \zeta_{S_{\bar{n}(t)}+\ell}\,, & \text{if } S_{n(t)} < S_{\bar{n}(t)}\,. \end{cases} \tag{3.43}$$

Therefore, using also (3.40),

$$\begin{aligned} E_\omega\big(1_G \left|Y_{n(t)} - Y_{\bar{n}(t)}\right|\big) &< E_\omega\left(1_G \sum_{\ell=0}^{S_{n(t)}-S_{\bar{n}(t)}} \zeta_{S_{\bar{n}(t)}+\ell}\right) \\ &\le \Big(C\sqrt{\phi(t)}+1\Big) \sup_{|\ell|\le C\sqrt{\phi(t)}+1} E_\omega\Big(\zeta_{S_{\bar{n}(t)}+\ell}\Big)\,. \end{aligned} \tag{3.44}$$

Since, for $t \to \infty$, $\bar{n}(t) \sim t$ and $\phi(t) = o(t)$, Lemma 3.5 can be applied to the leftmost term of (3.44). Accordingly, (3.42) and (3.44) imply

$$E_\omega\left(1_G \left| f\left(\frac{|Y_{n(t)} - Y_{\bar{n}(t)}|}{\sqrt{t}}\right) - f(0)\right|\right) \le C'\sqrt{\frac{\phi(t)}{t}} \le \frac{\varepsilon}{3}\,, \tag{3.45}$$

for some constant $C' > 0$ and all t large enough. This, together with (3.41), gives (3.37), and concludes the proof of Theorem 2.5. Q.E.D.

A Appendix: Technical lemmas

A.1 Ergodicity of the PVP dynamical system

In this section we give the prove the ergodicity of the PVP dynamical system introduced in Section 3.2, and another related result.

Proof of Theorem 3.1. We follow the same ideas as in [20, 21, 13]. Let us first prove that T preserves ν.

Set $A := B \times C$, where B is an elementary cylinder of $\mathbb{Z}^{\mathbb{Z}^+}$ and C is a measurable subset of $(\mathbb{R}^+)^{\mathbb{Z}}$. It is not hard to see that $T^{-1}(A) = \bigsqcup_{k\in\mathbb{Z}} B_k \times \sigma_\zeta^{-k}(C)$, where

$$B_k := \Big\{(k, \xi_1, \xi_2, \ldots) \in \mathbb{Z}^{\mathbb{Z}^+} \;\Big|\; (\xi_1, \xi_2, \ldots) \in B\Big\}\,. \tag{A.1}$$

By the choice of B and by definition of Q_o, $Q_o(B_k) = p_k Q_o(B)$. Also, by the P_o-invariance of σ_ζ, $P_o(\sigma_\zeta^{-k}(C)) = P_o(C)$. This shows that

$$\nu(T^{-1}(A)) = \sum_{k\in\mathbb{Z}} \nu(B_k \times \sigma_\zeta^{-k}(C)) = \sum_{k\in\mathbb{Z}} p_k\, Q_o(B) P_o(C) = \nu(A). \tag{A.2}$$

This then extends to all measurable sets A, proving our first assertion. For the second assertion we need a lemma.

Lemma A.1 *Every T-invariant set $A \subseteq \Sigma$ is of the form $A = \mathbb{Z}^{\mathbb{Z}^+} \times C$ mod ν (meaning that the equality holds up to ν-null sets), where C is a measurable set of $(\mathbb{R}^+)^{\mathbb{Z}}$.*

Proof of Lemma A.1. We first give some preliminary definitions and results. Let us endow $\mathbb{Z}^{\mathbb{Z}^+}$ with the distance

$$d(\xi, \xi') := \left[\min\left\{n \in \mathbb{Z}^+ \mid \xi_n \neq \xi'_n\right\}\right]^{-1}. \tag{A.3}$$

This is an *ultrametric* distance, namely, $\forall \xi, \xi', \xi'' \in \mathbb{Z}^{\mathbb{Z}^+}$,

$$d(\xi, \xi'') \leq \max\{d(\xi, \xi'), d(\xi', \xi'')\}, \tag{A.4}$$

and its (open) balls are the cylinders

$$\mathcal{B}_\varepsilon(\xi) := \left\{\xi' \in \mathbb{Z}^{\mathbb{Z}^+} \;\middle|\; \xi'_i = \xi_i,\ \forall i = 1, 2, \ldots, [\varepsilon^{-1}]\right\}, \tag{A.5}$$

where, once again, $[\cdot]$ indicates the integer part of a real number. This makes $\mathbb{Z}^{\mathbb{Z}^+}$ a Polish ultrametric space, which is an observation that will soon be useful.

Given $\mathcal{B}_\varepsilon(\xi)$, as in (A.5), and an elementary cylinder $B \subseteq \mathcal{B}_\varepsilon(\xi)$, namely $B = \{\xi' \mid \xi'_i = \xi_i,\ \forall i = 1, 2, \ldots, k\}$, with $k \geq [\varepsilon_{-1}]$, it is easy to see that

$$\frac{Q_o(\sigma_\xi(B))}{Q_o(\sigma_\xi(\mathcal{B}_\varepsilon(\xi)))} = \frac{Q_o(B)}{Q_o(\mathcal{B}_\varepsilon(\xi))}. \tag{A.6}$$

So this holds for any measurable $B \subseteq \mathcal{B}_\varepsilon(\xi)$. If B is not necessarily a subset of $\mathcal{B}_\varepsilon(\xi)$, we can only state that

$$Q_o(\sigma_\xi(B) \,|\, \sigma_\xi(\mathcal{B}_\varepsilon(\xi))) \geq Q_o(B \,|\, \mathcal{B}_\varepsilon(\xi)). \tag{A.7}$$

(This follows from (A.6), replacing B with $B \cap \mathcal{B}_\varepsilon(\xi)$ and using the general inclusion $\sigma_\xi(B \cap \mathcal{B}_\varepsilon(\xi)) \subseteq \sigma_\xi(B) \cap \sigma_\xi(\mathcal{B}_\varepsilon(\xi))$.)

Now, recall the definitions of F_ζ and $Q_o(\cdot \,|\, F_\zeta)$ from the statement of Lemma 3.3. It follows from the above arguments that F_ζ is a Polish ultrametric space endowed with the Borel measure $Q_o(\cdot \,|\, F_\zeta)$. By [24, Prop. 2.10], Lebesgue's Density Theorem holds. In particular, if $\mathcal{B}_\varepsilon(\xi, \zeta)$ denotes the ball of center (ξ, ζ) and radius ε in F_ζ (corresponding to $\mathcal{B}_\varepsilon(\xi) \subset \mathbb{Z}^{\mathbb{Z}^+}$ through the identification $F_\zeta \cong \mathbb{Z}^{\mathbb{Z}^+}$), we have the following:

Lemma A.2 *Let A be a measurable subset of Σ and $\zeta \in (\mathbb{R}^+)^{\mathbb{Z}}$ be such that $A \cap F_\zeta$ is measurable (this happens for P_o-a.e. ζ). Then, a.e. $(\xi, \zeta) \in A \cap F_\zeta$, relative to $Q_o(\cdot \,|\, F_\zeta)$, is a density point of $A \cap F_\zeta$. This means that*

$$\lim_{\varepsilon \to 0^+} Q_o(A \,|\, \mathcal{B}_\varepsilon(\xi, \zeta)) := \lim_{\varepsilon \to 0^+} \frac{Q_o(A \cap \mathcal{B}_\varepsilon(\xi, \zeta) \,|\, F_\zeta)}{Q_o(\mathcal{B}_\varepsilon(\xi, \zeta) \,|\, F_\zeta)} = 1.$$

We finally come to the actual proof of the lemma. Let us first assume $\nu(A) > 0$, otherwise one sets $C := \varnothing$ and the proof is finished. Then, by contradiction, we assume that A is not of the type $A = \mathbb{Z}^{\mathbb{Z}^+} \times C \bmod \nu$, that is, it is not a union of horizontal fibers, modulo null sets. Therefore, for a small enough $\delta > 0$, the set

$$C_\delta := \left\{ \zeta \in (\mathbb{R}^+)^{\mathbb{Z}} \;\middle|\; \delta \le Q_o(A \,|\, F_\zeta) \le 1 - \delta \right\} \tag{A.8}$$

has positive P_o-measure. Set $A' := \bigsqcup_{\zeta \in C_\delta} (A \cap F_\zeta)$. By Fubini, $\nu(A') > 0$.

We claim that we can find a point $(\xi, \zeta) \in A'$ which is both a recurrent point to A' w.r.t. T (i.e.,

$$T^n(\xi, \zeta) \in A' \,, \tag{A.9}$$

for countably many values of n), and a density point of $A \cap F_\zeta$, relative to $Q_o(\,\cdot \,|\, F_\zeta)$. This is true because, by Poincaré's Recurrence Theorem, ν-a.a. points in A' recur to A'. By Fubini and the definition of A', this implies that, for P_o-a.a. $\zeta \in C_\delta$, (ξ, ζ) is a recurrent point (to A'), for Q_o-a.e. $\xi \in A' \cap F_\zeta$. Now, consider a typical ζ in the sense just described and exclude from the recurrent points contained in $A' \cap F_\zeta$ those that are not density points of $A' \cap F_\zeta$. By Lemma A.2 this amounts to excluding a negligible set of points. Any remaining point verifies our claim (in fact, $A' \cap F_\zeta = A \cap F_\zeta$, by definition of A').

Therefore, we can find a large enough n that verifies (A.9) and

$$Q_o(A \,|\, \mathcal{B}_{1/n}(\xi, \zeta)) > 1 - \delta \,. \tag{A.10}$$

Notice that, via (3.17) and (2.3), it is easy to find an expression for the iterates of (ξ, ζ):

$$T^n(\xi, \zeta) := (\sigma_\xi^n(\xi), \sigma_\zeta^{S_n}(\zeta)) \,. \tag{A.11}$$

The above makes it clear that T^n acts on F_ζ by operating n shifts in the dynamical variable and mapping the environment to the new environment $\sigma_\zeta^{S_n}(\zeta)$. But, by (A.5), $\sigma_\xi^n(\mathcal{B}_{1/n}(\xi)) = \mathbb{Z}^{\mathbb{Z}^+}$. Therefore, $T^n(\mathcal{B}_{1/n}(\xi, \zeta)) = F_{\sigma_\zeta^{S_n}(\zeta)}$. On the other hand, using the invariance of A, (A.7) and (A.10), we can write

$$\begin{aligned} Q_o(A \,|\, F_{\sigma_\zeta^{S_n}(\zeta)}) &= Q_o(T^n(A) \,|\, T^n(\mathcal{B}_{1/n}(\xi, \zeta))) \\ &\ge Q_o(A \,|\, \mathcal{B}_{1/n}(\xi, \zeta)) \\ &> 1 - \delta \,, \end{aligned} \tag{A.12}$$

which, in view of (A.8), shows that $\sigma_\zeta^{S_n}(\zeta) \notin C_\delta$. But (A.9) and (A.11) imply that $\sigma_\zeta^{S_n}(\zeta) \in C_\delta$, which is the sought contradiction.

Therefore $A = \mathbb{Z}^{\mathbb{Z}^+} \times C \bmod \nu$, for some $C \subseteq (\mathbb{R}^+)^{\mathbb{Z}}$. We still need to prove that C is measurable. If not, by [19, Lem. A.1], there exists a measurable C' such that $C \triangle C'$ is contained in a null set, implying $A = \mathbb{Z}^{\mathbb{Z}^+} \times C' \bmod \nu$. Q.E.D.

To end the proof of Theorem 3.1 suppose, again by contradiction, that the system has an invariant set A, which, by the above lemma, must be of the form $A = \mathbb{Z}^{\mathbb{Z}^+} \times C$

mod ν, with $0 < P_o(C) < 1$. By the ergodicity of σ_ζ, there must be a subset $C' \subseteq C$, with

$$P_o(C') > 0\,, \tag{A.13}$$

such that $\sigma_\zeta(C') \subseteq C^c := (\mathbb{R}^+)^{\mathbb{Z}} \setminus C$.

For $\zeta \in C'$, set $B_{\zeta,1} := \{(\xi,\zeta) \in F_\zeta \mid \xi_1 = 1\}$. Then

$$Q_o\left(B_{\zeta,1} \,|\, F_\zeta\right) = Q_o\Big(\Big\{\xi \in \mathbb{Z}^{\mathbb{Z}^+} \;\Big|\; \xi_1 = 1\Big\}\Big) = p_1 > 0\,, \tag{A.14}$$

by the assumptions on (p_k) (symmetry, half-monotonocity, and positive variance; see, however, Remark A.3). Also, $T(B_{\zeta,1}) = F_{\sigma_\zeta(\zeta)}$ and $\sigma_\zeta(\zeta) \in C^c$. Therefore, setting $A_o := \bigsqcup_{\zeta\in C'} B_{\zeta,1}$, one has that $T(A_o) \subseteq \mathbb{Z}^{\mathbb{Z}^+} \times C^c$, with $\nu(A_o) > 0$ (the latter inequality coming from (A.13), (A.14) and Fubini's Theorem). This contradicts the invariance of A and thus proves the theorem. Q.E.D.

Proof of Lemma 3.3. The proof of Theorem 3.1 (see in particular (A.1)-(A.2) and the concluding paragraph) shows that T maps $B_{\zeta,k} := \{(\xi,\zeta) \in F_\zeta \mid \xi_1 = k\}$ onto $F_{\sigma_\zeta^k(\zeta)}$, pushing the measure $Q_o(\,\cdot\,|F_\zeta)$, restricted to $B_{\zeta,k}$, to $p_k Q_o(\,\cdot\,|F_{\sigma_\zeta^k(\zeta)})$. Since $F_\zeta = \bigsqcup_{k\in\mathbb{Z}} B_{\zeta,k}$, both statements of the lemma follow. Q.E.D.

Remark A.3 The proof of Lemma 3.3 helps to show that Theorem 3.1 holds under much weaker assumptions on the underlying random walk: it suffices to require that $v_p > 0$. This, in fact, implies that $p_k > 0$, for some $k \neq 0$. The proof of Theorem 3.1 still functions if, in the last two paragraphs, one substitutes σ_ζ with σ_ζ^k (also ergodic) and $B_{\zeta,1}$ with $B_{\zeta,k}$.

A.2 Averaging

The next lemma, which is needed in the proof of the main theorems (cf. Lemma 3.5), proves an assertion that can be roughly described as follows: given a function $a : \mathbb{Z} \longrightarrow \mathbb{R}$ and an "expanding" sequence of probability densities on $\mathbb{Z}$ that are increasing on $\mathbb{Z}^-$ and decreasing on $\mathbb{N}$, the expected value of a relative to these densities tends to its Cesaro average.

Lemma A.4 *Let $a := (a_j,\, j \in \mathbb{Z}) \subset \mathbb{R}$ be such that*

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} a_j = \lim_{k\to\infty} \frac{1}{k} \sum_{j=-1}^{-k} a_j = \bar{a}\,.$$

For $n \in \mathbb{N}$, let $p^{(n)} = (p_j^{(n)},\, j \in \mathbb{Z})$ be the density of a probability distribution on $\mathbb{Z}$, whose expectation we denote by $\mathcal{E}_n$, such that:

(i) $j \mapsto p_j^{(n)}$ is increasing on $\mathbb{Z}^-$ and decreasing on $\mathbb{N}$;

(ii) for all $r \in \mathbb{N}$, $\mathcal{E}_n(1_{[-r,r]}) := \sum_{j=-r}^{r} p_j^{(n)}$ *vanishes as* $n \to \infty$.

Then

$$\lim_{n\to\infty} \mathcal{E}_n(a) := \lim_{n\to\infty} \sum_{j\in\mathbb{Z}} p_j^{(n)} a_j = \bar{a}\,.$$

Proof. For $k_1, k_2 \in \mathbb{Z}$, $k_1 \le k_2$, set

$$\mathcal{U}_{k_1,k_2}(a) := \frac{1}{k_2 - k_1 + 1} \sum_{j=k_1}^{k_2} a_j\,. \tag{A.15}$$

Define also

$$\mathcal{E}_n^+(a) := \sum_{j=0}^{\infty} p_j^{(n)} a_j\,, \tag{A.16}$$

$$\mathcal{E}_n^-(a) := \sum_{j=-1}^{-\infty} p_j^{(n)} a_j\,. \tag{A.17}$$

Let us approximate (A.16) and (A.17) separately. It is not hard to see (by "slicing" the density $p^{(n)}$ horizontally) that

$$\mathcal{E}_n^+(a) = \sum_{j=0}^{\infty} \left(p_j^{(n)} - p_{j+1}^{(n)}\right) (j+1)\, \mathcal{U}_{0,j}(a)\,. \tag{A.18}$$

Fix $\varepsilon > 0$. The hypotheses on a imply that $\exists r \in \mathbb{N}$ so large that, $\forall j > r$,

$$|\mathcal{U}_{0,j}(a) - \bar{a}| \le \varepsilon/2\,; \tag{A.19}$$

$$|\mathcal{U}_{-j,-1}(a) - \bar{a}| \le \varepsilon/2\,. \tag{A.20}$$

Set

$$\mathcal{E}_{n,r}^+(a) := \sum_{j=r+1}^{\infty} \left(p_j^{(n)} - p_{j+1}^{(n)}\right) (j+1) \mathcal{U}_{0,j}(a)\,. \tag{A.21}$$

(A.19) implies that

$$\left|\mathcal{E}_{n,r}^+(a) - \bar{a}\, \mathcal{E}_{n,r}^+(1)\right| \le \frac{\varepsilon}{2}\, \mathcal{E}_{n,r}^+(1)\,, \tag{A.22}$$

with the understandable meaning that 1 also denotes the sequence that is identically equal to 1.

Analogously, the term (A.17) can be rewritten as

$$\mathcal{E}_n^-(a) = \sum_{j=1}^{\infty} \left(p_{-j}^{(n)} - p_{-j-1}^{(n)}\right) j\, \mathcal{U}_{-j,-1}(a)\,, \tag{A.23}$$

and, upon defining

$$\mathcal{E}^-_{n,r}(a) := \sum_{j=r+1}^{\infty} \left(p^{(n)}_{-j} - p^{(n)}_{-j-1}\right) j\, \mathcal{U}_{-j,-1}(a), \tag{A.24}$$

we get through (A.20) that

$$\left|\mathcal{E}^-_{n,r}(a) - \bar{a}\, \mathcal{E}^-_{n,r}(1)\right| \leq \frac{\varepsilon}{2}\, \mathcal{E}^-_{n,r}(1)\,. \tag{A.25}$$

If we name $\mathcal{E}_{n,r}(\cdot) := \mathcal{E}^+_{n,r}(\cdot) + \mathcal{E}^-_{n,r}(\cdot)$, we obtain from (A.22) and (A.25) that

$$|\mathcal{E}_{n,r}(a) - \bar{a}\, \mathcal{E}_{n,r}(1)| \leq \frac{\varepsilon}{2}\, \mathcal{E}_{n,r}(1) \leq \frac{\varepsilon}{2}\,. \tag{A.26}$$

On the other hand, it is clear from the above arguments that $1 - \mathcal{E}_{n,r}(1) = \mathcal{E}_n(1) - \mathcal{E}_{n,r}(1)$ is a portion of the mass of $p^{(n)}$ contained in $[-r, r]$, which is measured by $\mathcal{E}_n(1_{[-r,r]})$. Therefore, defining $\mathcal{A} := \max_{|j|\leq r} |a_j|$ and using *(ii)*, there exists $N = N(\varepsilon, \bar{a}, \mathcal{A}, r)$ such that, for all $n \geq N$,

$$1 - \mathcal{E}_{n,r}(1) \leq \mathcal{E}_n(1_{[-r,r]}) \leq \frac{\varepsilon}{2(\mathcal{A} + |\bar{a}|)}\,. \tag{A.27}$$

Notice that N can be thought of as a function of ε and the sequence a (for r is also a function of ε and a). Finally, $\forall n \geq N$,

$$\begin{aligned} |\mathcal{E}_n(a) - \bar{a}| &\leq |\mathcal{E}_n(a) - \mathcal{E}_{n,r}(a)| + |\mathcal{E}_{n,r}(a) - \bar{a}\, \mathcal{E}_{n,r}(1)| + |\bar{a}\, \mathcal{E}_{n,r}(1) - \bar{a}| \\ &\leq (\mathcal{A} + |\bar{a}|)\, (1 - \mathcal{E}_{n,r}(1)) + |\mathcal{E}_{n,r}(a) - \bar{a}\, \mathcal{E}_{n,r}(1)| \\ &\leq \varepsilon/2 + \varepsilon/2\,, \end{aligned} \tag{A.28}$$

by (A.26) and (A.27). This completes the proof. Q.E.D.

In the main body of the paper, Lemma A.4 is used with $p^{(n)}$ being the probability density of the underlying random walk at time n. In order to show that such densities verify condition *(i)* above, we need another simple lemma.

Lemma A.5 *If p and p' are symmetric and half-monotonic densities on $\mathbb{Z}$ (see definitions in Section 2.1), so is their convolution $p * p'$.*

Proof. Let us first treat the special case $p = q^{(r)}$ and $p' = q^{(r')}$, with $r, r' \in \mathbb{N}$, where

$$q^{(r)}_k = \begin{cases} (2r+1)^{-1}\,, & -r \leq k \leq r\,; \\ 0\,, & |k| > r\,. \end{cases} \tag{A.29}$$

Assume $r \geq r'$: this is no loss of generality as the convolution is symmetric. It is easy to calculate that

$$\left(q^{(r)} * q^{(r')}\right)_j = \frac{1}{2r+1} \cdot \begin{cases} 1\,, & |j| \leq r - r'\,; \\ \dfrac{r + r' + 1 - j}{2r' + 1}\,, & r - r' < |j| \leq r + r'\,; \\ 0\,, & |j| > r + r'\,, \end{cases} \tag{A.30}$$

which is symmetric and half-monotonic. Now, a general p as in the statement of the lemma can be rewritten as $p = \sum_{r=0}^{\infty}(p_r - p_{r+1})(2r+1)q^{(r)}$, and analogously for p'. Hence

$$p * p' = \sum_{r=0}^{\infty}\sum_{r'=0}^{\infty}(p_r - p_{r+1})(p'_{r'} - p'_{r'+1})(2r+1)(2r'+1)\, q^{(r)} * q^{(r')}, \qquad \text{(A.31)}$$

which is symmetric and half-monotonic because it is a countable linear combination of symmetric and half-monotonic densities, with positive coefficients. Q.E.D.

Compliance with ethical standards

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