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Stability issues in disturbance decoupling for switching linear systems

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Abstract

Disturbance decoupling — i.e., the problem of making the output of a dynamical system insensitive to undesired inputs — is a classical problem of control theory and a main concern in control applications. Hence, it has been solved for many classes of dynamical systems, considering both structural and stability requirements. As to decoupling in linear switching systems, several definitions of stability apply. The aim of this contribution is investigating different decoupling problems with progressively more stringent stability requirements: from structural decoupling to decoupling with local input-to-state stability. A convex procedure for the computation of the switching compensator is presented, based on the fact that quadratic stability under arbitrary switching guarantees global uniform asymptotic stability and the latter implies local input-to-state stability. Measurable and inaccessible disturbances are considered in a unified setting. The work is focused on discrete-time systems, although all the results hold for continuous-time systems as well, with the obvious modifications.

1 Introduction.

Disturbance decoupling, which is a main issue in control system design, was first solved for linear systems in the late sixties, within the geometric approach (see, e.g., [1, 2] and the references therein). Since then, the problem has been reformulated for different classes of dynamical systems and, due to the peculiarities of each context, it still attracts a fair amount of research effort: e.g., nonlinear systems have recently been considered in [3], descriptor systems and systems over rings in [4], time-delay systems in [5], linear parameter varying systems in [6, 7].

Lately, disturbance decoupling has been tackled for switching linear systems, that are dynamical systems described by a set of linear time-invariant systems — each of them modeling a different mode of operation — and a switching signal, designating the active mode at each time instant [8–10]. Indeed, when switching linear systems are involved, the problem of disturbance decoupling lends itself to a number of formulations, characterized by different approaches to both switching systems and disturbance decoupling.

A first distinction can be made between those works where the switching signal is considered the only manipulable variable, like [11], and those where it represents an exogenous input, possibly available in real time, while the manipulable variable is a standard control input, like [12]. In-between cases, where both the switching rule and the control input may be designed, have also been examined [13–15]. Another distinctive feature concerns the meaning of decoupling: i.e., while the objective of some studies is rendering the output totally insensitive to the disturbance [13–16], the aim of some others is achieving a certain attenuation level [11,12,17].

Apart from these characteristics, which, even alone, can account for the variety of approaches to disturbance decoupling for switching linear systems available in the literature, another key aspect is stability. As is well-known [18, 19], different definitions of stability apply to switching systems without inputs, ranging from quadratic stability to exponential stability or asymptotic stability. Moreover, a switching system may not enjoy a certain stability property under arbitrary switching, but may have this property when it is ruled by somehow restricted switching signals, such as those satisfying a minimum dwell-time or average dwell-time: i.e., the time or the average time between two consecutive switches is not smaller than a constant [20, 21].

Indeed, only few of the abovementioned works take into account stability requirements for the compensated system achieving disturbance decoupling. Mainly, these papers are [13, 14], where quadratic stability is sought for a suitable choice of the switching rule. In addition, in [22], sufficient conditions for obtaining disturbance decoupling with quadratic stability under arbitrary switching of the closed-loop dynamics are shown. However, no computational procedures are provided for the synthesis of the switching state feedback. Furthermore, in the most recent papers [23, 24], disturbance decoupling is achieved with exponential stability of the closed-loop dynamics and a synthesis procedure for the switching state feedback is illustrated, but the admissible switching signals are subject to a minimum dwell-time restriction.

In this context, it is worth stressing that switch-

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ing linear systems having some zero-input state stability properties do not automatically enjoy good input-to-state stability properties, such as bounded inputs resulting in bounded state trajectories or inputs converging to zero resulting in state trajectories converging to zero. These facts have been pointed out in [10], where arguments holding true for nonlinear systems in general (see, e.g., [25] and the references therein) are specifically referred to switching systems. This aspect is crucial in the problem of disturbance decoupling, but, to the best of the author's knowledge, it has not been addressed in any of the abovementioned papers. Hence, the scope of this work is to show how to solve disturbance decoupling problems with progressively more strict stability requirements, so as to achieve, at the last step, local input-to-state stability of the compensated system. The suggested methodology is supported by a complete computational framework, which provides the algorithms for the synthesis of the switching compensator.

2 Notation.

The symbols $\mathbb{Z}^+$ and $\mathbb{R}$ stand for the sets of nonnegative integer numbers and real numbers, respectively. Matrices and linear maps are denoted by slanted capital letters, like A. The image, the kernel, the spectrum, and the transpose of A are denoted by $\operatorname{im} A$, $\ker A$, $\lambda(A)$, and $A^\top$, respectively. Vector spaces and subspaces are denoted by calligraphic letters, like $\mathcal{V}$. The quotient space of a subspace $\mathcal{V}$ over a subspace $\mathcal{W} \subseteq \mathcal{V}$ is denoted by $\mathcal{V}/\mathcal{W}$. The restriction of a linear map A to an A-invariant subspace $\mathcal{J}$ is denoted by $A|_{\mathcal{J}}$. The dimension of a subspace $\mathcal{V}$ is denoted by $\dim \mathcal{V}$. The symbols I and O denote an identity matrix and a zero matrix of appropriate dimensions, respectively.

3 Structural Disturbance Decoupling.

The discrete-time switching linear system

$$\Sigma_{\sigma_t} \equiv \begin{cases} x_{t+1} = A_{\sigma_t} x_t + B_{\sigma_t} u_t + H_{\sigma_t} h_t, \\ e_t = E_{\sigma_t} x_t, \end{cases} \tag{3.1}$$

is considered, where $t \in \mathbb{Z}^+$ is the time variable, $x \in \mathcal{X} = \mathbb{R}^n$ is the state, $u \in \mathcal{U} = \mathbb{R}^p$ is the control input, $h \in \mathbb{R}^m$ is the measurable disturbance input, and $e \in \mathbb{R}^q$ is the output, with $p, m, q \leq n$. As mentioned earlier, measurable and inaccessible disturbances are treated in a unified setting. A distinction between the two cases will be highlighted in Remark 5.1. For a deeper discussion about different classes of disturbance inputs in the basic, linear time-invariant case, the reader is referred to [26–28]. The switching signal σ_t is defined as the arbitrary, measurable and not a-priori known correspondence $\sigma : \mathbb{Z}^+ \to \mathcal{I}$, where $\mathcal{I} = \{1, 2, \ldots, N\}$ denotes the finite index set of the modes of Σ_{σ_t}. With a slight abuse of notation, the set of the modes of Σ_{σ_t}, which are linear time-invariant systems whose input and output distribution matrices are assumed to be full-rank, is denoted by $\{\Sigma_\sigma,\, \sigma \in \mathcal{I}\}$. The set of the admissible measurable disturbances is defined as the set of the sequences h_t, $t \in \mathbb{Z}^+$, with bounded values in $\mathbb{R}^m$. Hence, the problem of structural measurable disturbance decoupling is stated as follows.

Problem 3.1. Given the discrete-time switching linear system Σ_{σ_t}, defined by (3.1), find a switching control law

$$u_t = F_{\sigma_t} x_t + G_{\sigma_t} h_t, \quad t \in \mathbb{Z}^+, \tag{3.2}$$

such that the compensated discrete-time switching linear system $\hat{\Sigma}_{\sigma_t}$, defined by

$$\hat{\Sigma}_{\sigma_t} \equiv \begin{cases} x_{t+1} = (A_{\sigma_t} + B_{\sigma_t} F_{\sigma_t})\, x_t + (H_{\sigma_t} + B_{\sigma_t} G_{\sigma_t})\, h_t, \\ e_t = E_{\sigma_t} x_t, \end{cases} \tag{3.3}$$

satisfies the following requirement:

$\mathcal{R}$ 1. the output $e_t \to 0$ as $t \to \infty$, for any admissible measurable disturbance h_t, with $t \in \mathbb{Z}^+$, any initial state $x_0 \in \mathbb{R}^n$, and any switching signal σ_t.

4 A Geometric Approach to Switching Linear Systems.

The aim of this section is to review some basic notions of the geometric approach [1, 2] and to introduce some new ones, useful to solve the Problem 3.1. In particular, the definition of maximal robust controlled invariant subspace, first presented in [29] for a generic set of linear time-invariant systems, is herein referred to the modes of a switching linear system and enhanced with the novel definitions of outer switching dynamics and outer stabilizability under arbitrary switching.

The following symbols, which refer to the set $\{\Sigma_\sigma,\, \sigma \in \mathcal{I}\}$ of the modes of Σ_{σ_t}, will be used throughout the work: $\mathcal{B}_\sigma$, $\mathcal{H}_\sigma$, and $\mathcal{E}_\sigma$ respectively stand for $\operatorname{im} B_\sigma$, $\operatorname{im} H_\sigma$, and $\ker E_\sigma$, with $\sigma \in \mathcal{I}$. The symbol $\mathcal{E}$ stands for $\bigcap_{\sigma \in \mathcal{I}} \mathcal{E}_\sigma$. It is worth mentioning that a subspace $\mathcal{J} \subseteq \mathcal{X}$ is said to be a robust A_σ-invariant subspace if $A_\sigma \mathcal{J} \subseteq \mathcal{J}$ for all $\sigma \in \mathcal{I}$. A subspace $\mathcal{V} \subseteq \mathcal{X}$ is said to be a robust $(A_\sigma, \mathcal{B}_\sigma)$-controlled invariant subspace if $A_\sigma \mathcal{V} \subseteq \mathcal{V} + \mathcal{B}_\sigma$ for all $\sigma \in \mathcal{I}$. Moreover, a subspace $\mathcal{V} \subseteq \mathcal{X}$ is a robust $(A_\sigma, \mathcal{B}_\sigma)$-controlled invariant subspace if and only if a set of linear maps $\{F_\sigma : \mathcal{X} \to \mathcal{U},\, \sigma \in \mathcal{I}\}$ exists, such that $(A_\sigma + B_\sigma F_\sigma)\, \mathcal{V} \subseteq \mathcal{V}$ for all $\sigma \in \mathcal{I}$.

The next definition ensues from the fact that, as was shown in [29], the set of all robust $(A_\sigma, \mathcal{B}_\sigma)$-controlled invariant subspaces contained in a given subspace is an upper semilattice, with the sum as binary operation and the inclusion as partial ordering relation.

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Definition 4.1. The maximum of the set of all robust $(A_\sigma, \mathcal{B}_\sigma)$-controlled invariant subspaces contained in the subspace $\mathcal{E}$ is called the maximal robust $(A_\sigma, \mathcal{B}_\sigma)$-controlled invariant subspace contained in $\mathcal{E}$ and is denoted by $\mathcal{V}_R^*$.

The subspace $\mathcal{V}_R^*$, which plays a key role in disturbance decoupling, can be computed as in [29, Algorithm 1].

The following statements are aimed at introducing the notions of outer switching dynamics and outer quadratic stabilizability under arbitrary switching of the subspace $\mathcal{V}_R^*$. Two lemmas are premised to the new definitions. For the sake of brevity, proofs are omitted.

Lemma 4.1. *Consider the discrete-time linear time-invariant systems of the set* $\{\Sigma_\sigma, \sigma \in \mathcal{I}\}$ *and the maximal robust* $(A_\sigma, \mathcal{B}_\sigma)$*-controlled invariant subspace contained in* $\mathcal{E}$, $\mathcal{V}_R^*$. *Perform a state space basis transformation* $T = [\,T_1 \; T_2\,]$, *where* $\operatorname{im} T_1 = \mathcal{V}_R^*$. *Then, with respect to the new coordinates,*

$$\begin{aligned}
A_\sigma^* &= T^{-1} A_\sigma T = \begin{bmatrix} A_{11,\sigma}^* & A_{12,\sigma}^* \\ A_{21,\sigma}^* & A_{22,\sigma}^* \end{bmatrix}, \\
B_\sigma^* &= T^{-1} B_\sigma = \begin{bmatrix} B_{1,\sigma}^* \\ B_{2,\sigma}^* \end{bmatrix}, \\
H_\sigma^* &= T^{-1} H_\sigma = \begin{bmatrix} H_{1,\sigma}^* \\ H_{2,\sigma}^* \end{bmatrix}, \\
E_\sigma^* &= E_\sigma T = \begin{bmatrix} O & E_{2,\sigma}^* \end{bmatrix},
\end{aligned}$$

for all $\sigma \in \mathcal{I}$.

Lemma 4.2. *Consider the discrete-time linear time-invariant systems of the set* $\{\Sigma_\sigma, \sigma \in \mathcal{I}\}$ *and the maximal robust* $(A_\sigma, \mathcal{B}_\sigma)$*-controlled invariant subspace contained in* $\mathcal{E}$, $\mathcal{V}_R^*$. *Let the set of linear maps* $\{F_\sigma : \mathcal{X} \to \mathcal{U}, \sigma \in \mathcal{I}\}$ *be such that* $\mathcal{V}_R^*$ *is a robust* $(A_\sigma + B_\sigma F_\sigma)$*-invariant subspace. Refer to the coordinates introduced in Lemma 4.1 and let*

$$F_\sigma^* = F_\sigma T = \begin{bmatrix} F_{1,\sigma}^* & F_{2,\sigma}^* \end{bmatrix}$$

be partitioned accordingly, for all $\sigma \in \mathcal{I}$. *Then, with respect to the new coordinates,*

$$A_\sigma^* + B_\sigma^* F_\sigma^* = \begin{bmatrix} A_{11,\sigma}^* + B_{1,\sigma}^* F_{1,\sigma}^* & A_{12,\sigma}^* + B_{1,\sigma}^* F_{2,\sigma}^* \\ O & A_{22,\sigma}^* + B_{2,\sigma}^* F_{2,\sigma}^* \end{bmatrix} \tag{4.4}$$

for all $\sigma \in \mathcal{I}$.

It is worthwhile to stress that, for any $\sigma \in \mathcal{I}$, the matrix $A_{22,\sigma}^* + B_{2,\sigma}^* F_{2,\sigma}^*$ in (4.4) represents the linear map induced by $A_\sigma + B_\sigma F_\sigma$ on the quotient space $\mathcal{X}/\mathcal{V}_R^*$ — briefly denoted by $(A_\sigma + B_\sigma F_\sigma)|_{\mathcal{X}/\mathcal{V}_R^*}$ and called the outer dynamics of $\mathcal{V}_R^*$ with respect to Σ_σ. Hence, the notion of outer switching dynamics of $\mathcal{V}_R^*$ can be introduced, quite naturally, as follows.

Definition 4.2. The switching dynamics $(A_{\sigma_t} + B_{\sigma_t} F_{\sigma_t})|_{\mathcal{X}/\mathcal{V}_R^*}$, associated with the set of induced linear maps $\{(A_\sigma + B_\sigma F_\sigma)|_{\mathcal{X}/\mathcal{V}_R^*}, \sigma \in \mathcal{I}\}$, is called the outer switching dynamics of $\mathcal{V}_R^*$.

Moreover, the definition of outer quadratic stabilizability of $\mathcal{V}_R^*$ under arbitrary switching is stated in the following terms.

Definition 4.3. The subspace $\mathcal{V}_R^*$ is said to be outer quadratically stabilizable under arbitrary switching, if a set of linear maps $\{F_\sigma : \mathcal{X} \to \mathcal{U}, \sigma \in \mathcal{I}\}$ exists, such that $\mathcal{V}_R^*$ is a robust $(A_\sigma + B_\sigma F_\sigma)$-invariant subspace and the switching dynamics $(A_{\sigma_t} + B_{\sigma_t} F_{\sigma_t})|_{\mathcal{X}/\mathcal{V}_R^*}$ is quadratically stable under arbitrary switching.

A convex procedure to compute a set of linear maps $\{F_\sigma, \sigma \in \mathcal{I}\}$, such that $\mathcal{V}_R^*$ is a robust $(A_\sigma + B_\sigma F_\sigma)$-invariant subspace and is outer quadratically stable under arbitrary switching will be presented in the following section, as part of the solution of Problem 3.1.

5 Sufficient Conditions for Structural Disturbance Decoupling.

The main theorem of this section provides a pair of sufficient conditions for Problem 3.1 to be solvable. The proof is constructive, since it shows how to synthesize a switching control law that achieves disturbance decoupling. Arguments showing why the pair of conditions is not necessary are also given. Two lemmas are presented first, respectively expressing the two conditions, which are originally given in coordinate-free terms, with respect to the coordinates introduced in Lemma 4.1. The proofs of the lemmas are omitted.

Lemma 5.1. *Consider the discrete-time linear time-invariant systems of the set* $\{\Sigma_\sigma, \sigma \in \mathcal{I}\}$, *the maximal robust* $(A_\sigma, \mathcal{B}_\sigma)$*-controlled invariant subspace contained in* $\mathcal{E}$, $\mathcal{V}_R^*$, *the images* $\mathcal{B}_\sigma$ *of the control input matrices* B_σ, *and the images* $\mathcal{H}_\sigma$ *of the measurable disturbance input matrices* H_σ, *with* $\sigma \in \mathcal{I}$. *Let the subspaces* $\mathcal{V}_\sigma \subseteq \mathcal{V}_R^*$ *be such that* $\mathcal{V}_\sigma \cap \mathcal{B}_\sigma = \{0\}$ *and* $(\mathcal{V}_R^* \cap \mathcal{B}_\sigma) + \mathcal{V}_\sigma = \mathcal{V}_R^*$, *for all* $\sigma \in \mathcal{I}$. *Refer to the coordinates introduced in Lemma 4.1 and let* $V_\sigma^* \in \mathbb{R}^{n \times r_\sigma}$, *with* $\sigma \in \mathcal{I}$, *be basis matrices of the corresponding subspaces* $\mathcal{V}_\sigma$. *Then,*

$$\mathcal{H}_\sigma \subseteq \mathcal{B}_\sigma + \mathcal{V}_R^*, \quad \forall\, \sigma \in \mathcal{I}, \tag{5.5}$$

if and only if matrices $\Gamma_\sigma \in \mathbb{R}^{p \times m}$ *and* $\Lambda_\sigma \in \mathbb{R}^{r_\sigma \times m}$, *with* $\sigma \in \mathcal{I}$, *exist, such that*

$$H_\sigma^* = B_\sigma^* \Gamma_\sigma + V_\sigma^* \Lambda_\sigma, \quad \forall\, \sigma \in \mathcal{I}, \tag{5.6}$$

or, with respect to the partition introduced in

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Lemma 4.1,

$$\begin{bmatrix} H_{1,\sigma}^* \\ H_{2,\sigma}^* \end{bmatrix} = \begin{bmatrix} B_{1,\sigma}^* \\ B_{2,\sigma}^* \end{bmatrix} \Gamma_\sigma + \begin{bmatrix} V_{1,\sigma}^* \\ O \end{bmatrix} \Lambda_\sigma, \ \forall\, \sigma \in \mathcal{I}. \tag{5.7}$$

LEMMA 5.2. *Consider the discrete-time switching linear system Σ_{σ_t}, with the modes $\{\Sigma_\sigma,\, \sigma \in \mathcal{I}\}$, and the maximal robust $(A_\sigma, \mathcal{B}_\sigma)$-controlled invariant subspace contained in $\mathcal{E}$, $\mathcal{V}_R^*$. Let $r = \dim \mathcal{V}_R^*$. The subspace $\mathcal{V}_R^*$ is outer quadratically stabilizable under arbitrary switching if and only if a positive-definite symmetric matrix $Q_e \in \mathbb{R}^{n-r\times n-r}$ and a set of matrices $\{Y_\sigma \in \mathbb{R}^{p\times n-r},\, \sigma \in \mathcal{I}\}$ exist, such that*

$$\begin{bmatrix} Q_e & (A_{22,\sigma}^*\, Q_e + B_{2,\sigma}^*\, Y_\sigma)^\top \\ A_{22,\sigma}^*\, Q_e + B_{2,\sigma}^*\, Y_\sigma & Q_e \end{bmatrix} > 0, \quad \forall\, \sigma \in \mathcal{I}, \tag{5.8}$$

with $A_{22,\sigma}^*$ and $B_{2,\sigma}^*$ defined as in Lemma 4.1. If this is the case, the switching dynamics $(A_{\sigma_t} + B_{\sigma_t} F_{\sigma_t})|_{\mathcal{X}/\mathcal{V}_R^*}$, associated with the set of matrices $\{A_{22,\sigma}^* + B_{2,\sigma}^* F_{2,\sigma}^*,\, \sigma \in \mathcal{I}\}$, where $F_{2,\sigma}^* = Y_\sigma Q_e^{-1}$, with $\sigma \in \mathcal{I}$, is quadratically stable under arbitrary switching.

It is worth noting that (5.8) are LMIs in the unknowns Q_e and Y_σ, with $\sigma \in \mathcal{I}$. Hence, Lemma 5.2 points out a convex procedure to derive — if the LMI problem is feasible — a set of matrices $\{F_{2,\sigma}^*, \sigma \in \mathcal{I}\}$. Furthermore, a set of matrices $\{F_{1,\sigma}^*,\, \sigma \in \mathcal{I}\}$, such that

$$A_{21,\sigma}^* + B_{2,\sigma}^*\, F_{1,\sigma}^* = 0, \quad \forall\, \sigma \in \mathcal{I}, \tag{5.9}$$

always exists, owing to the equivalence between robust controlled invariance and robust invariance by state feedback, mentioned in §4.

THEOREM 5.1. *Consider the discrete-time switching linear system Σ_{σ_t}. Problem 3.1 has a solution if the following conditions hold:*

$\mathcal{C}\,1$. $\mathcal{H}_\sigma \subseteq \mathcal{B}_\sigma + \mathcal{V}_R^*, \quad \forall\, \sigma \in \mathcal{I}$;

$\mathcal{C}\,2$. *$\mathcal{V}_R^*$ is outer quadratically stabilizable under arbitrary switching.*

Proof. Let Conditions $\mathcal{C}\,1$ and $\mathcal{C}\,2$ hold. Consider the switching control law (3.2). Let G_{σ_t} be the switching linear map associated with the set $\{G_\sigma,\, \sigma \in \mathcal{I}\}$, defined by

$$G_\sigma = -\,\Gamma_\sigma, \quad \forall\, \sigma \in \mathcal{I}, \tag{5.10}$$

where the matrices Γ_σ, with $\sigma \in \mathcal{I}$, are given by Lemma 5.1 (which is possible by virtue of Condition $\mathcal{C}\,1$). Moreover, let F_{σ_t} be the switching linear map associated with a set $\{F_\sigma,\, \sigma \in \mathcal{I}\}$ determined according to Lemmas 4.2 and 5.2 (which is feasible owing to Condition $\mathcal{C}\,2$). Then, it will be shown that the switching control law (3.2), where F_{σ_t} and G_{σ_t} are picked as specified above, solves Problem 3.1. First, note that, with respect to the coordinates introduced in Lemma 4.1, the modes $\{\hat{\Sigma}_\sigma,\, \sigma \in \mathcal{I}\}$ of the switching compensated system $\hat{\Sigma}_{\sigma_t}$ are described by

$$\hat{\Sigma}_\sigma \equiv \begin{cases} x_{1,t+1} = (A_{11,\sigma}^* + B_{1,\sigma}^*\, F_{1,\sigma}^*)\, x_{1,t} + V_{1,\sigma}^*\, \Lambda_\sigma\, h_t \\ \qquad\quad + (A_{12,\sigma}^* + B_{1,\sigma}^*\, F_{2,\sigma}^*)\, x_{2,t}, \\ x_{2,t+1} = (A_{22,\sigma}^* + B_{2,\sigma}^*\, F_{2,\sigma}^*)\, x_{2,t}, \\ e_t = E_{2,\sigma}^*\, x_{2,t}, \qquad \sigma \in \mathcal{I}, \end{cases} \tag{5.11}$$

where (5.7), (5.9), and (5.10) have been taken into account. Equation (5.11) shows that quadratic stability under arbitrary switching of the dynamics $(A_{\sigma_t} + B_{\sigma_t} F_{\sigma_t})|_{\mathcal{X}/\mathcal{V}_R^*}$, associated with the set $\{A_{22,\sigma}^* + B_{2,\sigma}^* F_{2,\sigma}^*,\, \sigma \in \mathcal{I}\}$, implies that the state trajectory $x_{2,t}$, with $t \in \mathbb{Z}^+$, converges to 0, for any admissible measurable disturbance h_t, with $t \in \mathbb{Z}^+$ (indeed, h_t does not appear in the equation of $x_{2,t}$), any initial state $x_0 \in \mathbb{R}^n$, and any switching signal σ_t. Therefore, the same is true for the output e_t, with $t \in \mathbb{Z}^+$, which only depends on $x_{2,t}$.

It is worth noting that Condition $\mathcal{C}\,1$ is also necessary to solve Problem 3.1. In fact, if Condition $\mathcal{C}\,1$ is not met, no other robust $(A_\sigma, \mathcal{B}_\sigma)$-controlled invariant subspace contained in $\mathcal{E}$, say $\mathcal{V}_R$, exists, such that $\mathcal{H}_\sigma \subseteq \mathcal{B}_\sigma + \mathcal{V}_R$, for all $\sigma \in \mathcal{I}$, because the set of all robust $(A_\sigma, \mathcal{B}_\sigma)$-controlled invariant subspaces contained in $\mathcal{E}$ is an upper semilattice and $\mathcal{V}_R^*$ is its maximum.

Instead, Condition $\mathcal{C}\,2$ is not necessary to solve Problem 3.1. In fact, a less conservative form of stability for the switching dynamics $(A_{\sigma_t} + B_{\sigma_t} F_{\sigma_t})|_{\mathcal{X}/\mathcal{V}_R^*}$, such as asymptotic stability under arbitrary switching, would be sufficient to guarantee convergence to zero of the output as the time approaches infinity. However, as mentioned earlier, focusing on quadratic stability under arbitrary switching has the advantage of leading to a convex procedure for determining the switching state feedback to be included in the control law, thus providing a straightforward computational tool for the problem solution. In this regard, it is also mentioning that specific degrees of freedom are available in the determination of the set of linear maps $\{F_\sigma,\, \sigma \in \mathcal{I}\}$ and that these degrees of freedom are implicitly exploited in the solution of the LMI problem of Lemma 5.2.

This section is concluded by the following two remarks, which point out that inaccessible disturbance decoupling and perfect disturbance decoupling can be regarded as respective special cases of Problem 3.1.

REMARK 5.1. If the more restrictive condition

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$\mathcal{C}\,1'$. $\mathcal{H}_\sigma \subseteq \mathcal{V}_R^*$, $\quad \forall\, \sigma \in \mathcal{I}$,

holds in place of Condition $\mathcal{C}\,1$, the feedforward action applied through G_{σ_t} is not required anymore. Thus, the disturbance input h_t, with $t \in \mathbb{Z}^+$, is no longer needed to be measurable. In fact, if Condition $\mathcal{C}\,1'$ holds, Lemma 5.1 can be modified by replacing (5.6) with $H_\sigma^* = T_1^*\,\Lambda_\sigma$, for all $\sigma \in \mathcal{I}$, where $T_1^* = [\,I\;O\,]^\top$ is a basis matrix of $\mathcal{V}_R^*$ with respect to the coordinates introduced in Lemma 4.1. Consequently, the control law (3.2) reduces to $u_t = F_{\sigma_t}\, x_t$, with $t \in \mathbb{Z}^+$, where F_{σ_t} is still determined according to Lemmas 4.2 and 5.2. Namely, the switching state feedback alone achieves asymptotic decoupling of any admissible inaccessible disturbance, for any initial state $x_0 \in \mathbb{R}^n$ and any switching signal σ_t.

Remark 5.2. If the initial state x_0 belongs to the subspace $\mathcal{V}_R^*$, then the output e_t is zero for all $t \in \mathbb{Z}^+$, for any admissible measurable (or, inaccessible — if Condition $\mathcal{C}\,1'$ holds) disturbance h_t, with $t \in \mathbb{Z}^+$, and any switching signal σ_t. In fact, in light of (5.11), if $x_0 \in \mathcal{V}_R^*$, the sole state component $x_{1,0}$ may be different from zero. Therefore, $x_{2,0} = 0$ implies $x_{2,t} = 0$ for all $t \in \mathbb{Z}^+$, which, in turn, implies $e_t = 0$ for all $t \in \mathbb{Z}^+$. Namely, when $x_0 \in \mathcal{V}_R^*$, perfect decoupling of any admissible measurable (or, respectively, inaccessible) disturbance is achieved for any switching signal σ_t.

6 Disturbance Decoupling with State Stability

This section is focused on an improved version of the disturbance decoupling problem, where, in addition to the original specification, the dynamics of the compensated system, in the absence of the disturbance, is required to be stable under arbitrary switching. It will be shown that the new requirement is compatible with the existing one and a new convex procedure, independent of that discussed in §5 and to be applied besides that in the synthesis of the control law, will be illustrated.

Problem 6.1. Given the discrete-time switching linear system Σ_{σ_t}, defined by (3.1), find a switching control law (3.2), such that the compensated system $\hat{\Sigma}_{\sigma_t}$, defined by (3.3), satisfies Requirements $\mathcal{R}\,1$ and

$\mathcal{R}\,2$. the switching dynamics $A_{\sigma_t} + B_{\sigma_t} F_{\sigma_t}$ is quadratically stable under arbitrary switching.

Along the same lines developed in §4, it is worth noting that, for any $\sigma \in \mathcal{I}$, the matrix $A_{11,\sigma}^* + B_{1,\sigma}^* F_{1,\sigma}^*$ in (4.4) represents the restriction of the linear map $A_\sigma + B_\sigma F_\sigma$ to the subspace $\mathcal{V}_R^*$, with respect to the coordinates of Lemma 4.2. The restricted dynamics is denoted by $(A_\sigma + B_\sigma F_\sigma)|_{\mathcal{V}_R^*}$ and called the inner dynamics of $\mathcal{V}_R^*$ with respect to Σ_σ. Hence, the definitions of inner switching dynamics and inner quadratic stabilizability under arbitrary switching of the subspace $\mathcal{V}_R^*$ follow plainly.

Definition 6.1. The switching dynamics $(A_{\sigma_t} + B_{\sigma_t} F_{\sigma_t})|_{\mathcal{V}_R^*}$, associated with the set of restrictions $\{(A_\sigma + B_\sigma F_\sigma)|_{\mathcal{V}_R^*},\, \sigma \in \mathcal{I}\}$, is called the inner switching dynamics of $\mathcal{V}_R^*$.

Definition 6.2. The subspace $\mathcal{V}_R^*$ is said to be inner quadratically stabilizable under arbitrary switching, if a set of linear maps $\{F_\sigma : \mathcal{X} \to \mathcal{U},\, \sigma \in \mathcal{I}\}$ exists, such that $\mathcal{V}_R^*$ is a robust $(A_\sigma + B_\sigma F_\sigma)$-invariant subspace and the switching dynamics $(A_{\sigma_t} + B_{\sigma_t} F_{\sigma_t})|_{\mathcal{V}_R^*}$ is quadratically stable under arbitrary switching.

By inspecting the partition, shown in (4.4), of the matrices representing the linear maps of the set $\{A_\sigma + B_\sigma F_\sigma,\, \sigma \in \mathcal{I}\}$ with respect to the coordinates of Lemma 4.2, one can see that the properties of outer and inner quadratic stabilizability of $\mathcal{V}_R^*$ are independent of each other. The former only depends on the set $\{F_{1,\sigma}^*,\, \sigma \in \mathcal{I}\}$, while the latter only depends $\{F_{2,\sigma}^*,\, \sigma \in \mathcal{I}\}$. Hence, the next lemma shows how a set of matrices $\{F_{1,\sigma}^*,\, \sigma \in \mathcal{I}\}$, such that $\mathcal{V}_R^*$ is a robust $(A_\sigma + B_\sigma F_\sigma)$-invariant subspace with the property of being inner quadratically stable under arbitrary switching, can be obtained as a solution, if it exists, of a convex problem with a linear constraint.

Lemma 6.1. *The subspace $\mathcal{V}_R^*$ is inner quadratically stabilizable under arbitrary switching if and only if a positive-definite symmetric matrix $Q_i \in \mathbb{R}^{r \times r}$, where $r = \dim \mathcal{V}_R^*$, and a set of matrices $\{W_\sigma \in \mathbb{R}^{p \times r},\, \sigma \in \mathcal{I}\}$ exist, such that*

$$\begin{cases} \begin{bmatrix} Q_i & (A_{11,\sigma}^*\, Q_i + B_{1,\sigma}^*\, W_\sigma)^\top \\ A_{11,\sigma}^*\, Q_i + B_{1,\sigma}^*\, W_\sigma & Q_i \end{bmatrix} > 0, \\ A_{21,\sigma}^*\, Q_i + B_{2,\sigma}^*\, W_\sigma = 0, \qquad \forall\, \sigma \in \mathcal{I}, \end{cases} \tag{6.12}$$

with $A_{11,\sigma}^*$, $A_{21,\sigma}^*$, $B_{1,\sigma}^*$, $B_{2,\sigma}^*$ defined as in Lemma 4.1. If this is the case, the switching dynamics $(A_{\sigma_t} + B_{\sigma_t} F_{\sigma_t})|_{\mathcal{V}_R^*}$, associated with the set of matrices $\{A_{11,\sigma}^* + B_{1,\sigma}^* F_{1,\sigma}^*,\, \sigma \in \mathcal{I}\}$, where $F_{1,\sigma}^* = W_\sigma\, Q_i^{-1}$, with $\sigma \in \mathcal{I}$, is quadratically stable under arbitrary switching.

Hence, the following theorem provides a set of sufficient conditions for Problem 6.1 to have a solution.

Theorem 6.1. *Consider the discrete-time switching linear system Σ_{σ_t}. Problem 6.1 has a solution if Conditions $\mathcal{C}\,1$, $\mathcal{C}\,2$, and the following hold:*

$\mathcal{C}\,3$. *$\mathcal{V}_R^*$ is inner quadratically stabilizable under arbitrary switching.*

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Proof. Let Conditions $\mathcal{C}\,1$–$\mathcal{C}\,3$ hold. Consider the switching control law (3.2). Let the switching linear map G_{σ_t} be determined as in the proof of Theorem 5.1. Let the switching linear map F_{σ_t} be such that, with respect to the partition of the matrices F^*_σ, with $\sigma \in \mathcal{I}$, considered in Lemma 4.2, the matrices $F^*_{2,\sigma}$, with $\sigma \in \mathcal{I}$, are derived according to Lemma 5.2, as in the proof of Theorem 5.1, while the matrices $F^*_{1,\sigma}$, with $\sigma \in \mathcal{I}$, are derived according to Lemma 6.1 (which is feasible owing to Condition $\mathcal{C}\,3$). Hence, the proof that the control law thus devised satisfies Requirement $\mathcal{R}\,1$ is the same of Theorem 5.1. As to the proof that Requirement $\mathcal{R}\,2$ is also met, this is an immediate consequence of quadratic stability under arbitrary switching of the switching dynamics $(A_{\sigma_t} + B_{\sigma_t} F_{\sigma_t})|_{\mathcal{V}^*_R}$ and $(A_{\sigma_t} + B_{\sigma_t} F_{\sigma_t})|_{\mathcal{X}/\mathcal{V}^*_R}$ and of the block triangular structure of the dynamics of the compensated system Σ_{σ_t} shown in (5.11) (see, e.g., [30, Chapter 7]).

7 Disturbance Decoupling with Input-to-State Stability

To summarize the reasoning developed so far, the plain structural disturbance decoupling problem has been approached first. In the solution of that problem, only outer stabilization of the subspace $\mathcal{V}^*_R$ has been required, in order to guarantee asymptotic convergence to zero of the output, due to a possible nonzero initial state. Then, the problem of disturbance decoupling with state stability has been investigated and solved, which has required stabilization of both outer and inner switching dynamics of the subspace $\mathcal{V}^*_R$, in order to guarantee state stability of the closed-loop switching system.

In particular, quadratic stability under arbitrary switching has been considered. This choice has a main motivation. Indeed, as reviewed in [19], the existence of a common quadratic Lyapunov function is a sufficient condition for global uniform asymptotic stability of a switching system. On the other hand, asymptotic stability under zero input of a nonlinear system implies local input-to-state stability, as was shown in [31]. This fact was considered with specific reference to switching systems in [10, Appendix A]. Thus, in light of these results, available from the literature, the outer and inner quadratic stabilization under arbitrary switching of the subspace $\mathcal{V}^*_R$, discussed in this paper, guarantees not only disturbance decoupling with global uniform asymptotic stability of the switching system under zero input, as was shown in Theorem 6.1, but also local input-to-state stability.

To the best of the author's knowledge, input-to-state stability has never been mentioned in the literature on disturbance decoupling in switching linear systems. Indeed, stability, whenever considered, was meant to be state stability under zero input. Moreover, it is worth noting that, due to the key role played by uniform asymptotic stability, input-to-state stability cannot be achieved, in general, under less restrictive stability conditions, like, e.g., asymptotic stability under dwell-time switching.

8 Conclusions

A methodology to solve different formulations of the disturbance decoupling problem, from the mere structural decoupling to the problem with state stability under arbitrary switching, so as to guarantee also local input-to-state stability properties of the compensated system, has been shown. The synthesis procedure has been supported by a complete computational framework. The geometric approach, with some new notions specifically addressed to switching linear systems, and linear matrix inequalities have provided the theoretical and computational background.

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