

ANALYTICAL EXPANSIONS FOR PARABOLIC EQUATIONS*

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Abstract. We consider the Cauchy problem associated with a general parabolic partial differential equation in d dimensions. We find a family of closed-form asymptotic approximations for the unique classical solution of this equation as well as rigorous short-time error estimates. Using a boot-strapping technique, we also provide convergence results for arbitrarily large time intervals.

Key words. parabolic PDE, asymptotic expansion, singular perturbation, analytical approximation

AMS subject classifications. 35K10, 74G10, 91G20

DOI. 10.1137/130949968

1. Introduction. Asymptotic analysis and perturbation theory have a long history in a variety of fields including quantum mechanics (cf. Sakurai (1994)), classical mechanics (cf. Goldstein (1980)), fluid mechanics (cf. Van Dyke (1975); Lagerstrom (1988); Kevorkian and Cole (1996)) and mathematical biology (cf. Murray (2002)). More recently, some of techniques from perturbation theory and heat kernel expansions have been applied to problems arising in mathematical finance: see, for instance, Hagan and Woodward (1999); Henry-Labordère (2009); Benhamou, Gobet, and Miri (2010); Cheng et al. (2011), and Fouque et al. (2011). The authors of the present manuscript have also made recent contributions in mathematical finance with a focus on finding closed-form pricing approximations for models both without jumps (cf. Corielli, Foschi, and Pascucci (2010); Pagliarani, Pascucci, and Riga (2013)) and with jumps (cf. Lorig, Pagliarani, and Pascucci (2015a); Jacquier and Lorig (2013)), as well as finding closed-form approximations for implied volatility (cf. Lorig, Pagliarani, and Pascucci (2015b, 2014); Lorig (2013)).

In this paper, we shall consider the Cauchy problem

$$(1.1) \quad \begin{cases} (\partial_t + \mathcal{A}) u(t, x) = 0, & t \in [0, T[, x \in \mathbb{R}^d, \\ u(T, x) = \varphi(x), & x \in \mathbb{R}^d, \end{cases}$$

where \mathcal{A} is the second order elliptic differential operator with variable coefficients

$$(1.2) \quad \mathcal{A} = \sum_{i,j=1}^d a_{ij}(t, x) \partial_{x_i x_j} + \sum_{i=1}^d a_i(t, x) \partial_{x_i} + a(t, x), \quad t \in \mathbb{R}, x \in \mathbb{R}^d.$$

Cauchy problems of the form (1.1) arise whenever expectations of solutions of stochastic differential equations are considered. This is the case, for example, in option pricing. Cauchy problems of the form (1.1) also arise in quantum mechanics. However,

*Received by the editors December 20, 2013; accepted for publication (in revised form) December 31, 2014; published electronically March 19, 2015.

<http://www.siam.org/journals/siap/75-2/94996.html>

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in this case, one typically considers initial rather than final data (i.e., $u(0, x) = \varphi(x)$) as well as imaginary time: $\partial_t \rightarrow i\partial_t$. Indeed, many of the techniques used for finding approximation solutions of (1.1) have been developed by mathematical physicists.

In analyzing (1.1), rather than seek a general solution u , one typically seeks the *fundamental solution* $\Gamma(t, x; T, y)$ (also referred to as the *Green's function*), which is obtained by setting the final datum equal to a Dirac delta function $\varphi = \delta_y$, and from which the general solution u can be obtained via integration.

Unfortunately, for general x -dependent coefficients (a_{ij}, a_i, a) , the fundamental solution is not available in closed-form. As such, one instead seeks an approximation of the fundamental solution. Typically, this is achieved by expressing the operator \mathcal{A} as $\mathcal{A} = \mathcal{A}_0 + \mathcal{B}_1$, where the fundamental solution Γ_0 corresponding to \mathcal{A}_0 is known in closed-form and where $\mathcal{B}_1 = \mathcal{A} - \mathcal{A}_0$. Formally, then, one obtains the fundamental solution Γ corresponding to \mathcal{A} through a Dyson (also known as Volterra) series expansion (cf. Avramidi (2007); Berline, Getzler, and Vergne (1992)).

While it is a useful tool, the Dyson series has some notable draw-backs. First, to compute the Dyson series, one must evaluate operator-valued functions of the form

$$\mathcal{V}(t_0, t_1) := \exp\left(\int_{t_0}^{t_1} \mathcal{A}_0(s) ds\right) \mathcal{B}_1(t_1) \exp\left(\int_{t_1}^{t_0} \mathcal{A}_0(s) ds\right),$$

where we have explicitly indicated the time-dependence in the operators \mathcal{A}_0 and \mathcal{B}_1 . It is rare that the operator $\mathcal{V}(t_0, t_1)$ can be computed explicitly, and it is certainly not explicitly computable in the general case. Second, the Dyson series is typically asymptotically divergent. Hence, even if the first few terms of a Dyson series expansion can be computed explicitly, one is still left to wonder how accurate the truncated series is.

In this paper, rather than expand the operator \mathcal{A} as $\mathcal{A} = \mathcal{A}_0 + \mathcal{B}_1$, we expand it as an infinite sum: $\mathcal{A} = \sum_{n \geq 0} \mathcal{A}_n$. The basic ideas of the expansion technique were introduced in Pagliarani and Pascucci (2012), where \mathcal{A} is a differential operator corresponding to the generator of a scalar diffusion. These ideas were later extended in Pagliarani, Pascucci, and Riga (2013) and Lorig, Pagliarani, and Pascucci (2015a) to the case where \mathcal{A} may be an integro-differential operator corresponding to the generator of a scalar Lévy-type process. Both papers mentioned above establish rigorous short-time error bounds for the approximate fundamental solution of $(\partial_t + \mathcal{A})$. However, the results of these papers are limited to one dimension and leave unanswered some important practical and theoretical issues, for example, (i) Is there an explicit (and fully implementable) representation for the approximate solution at any given order N ? (ii) Can the smoothness of the terminal data φ be used to establish a higher order of accuracy of the asymptotic approximation? and (iii) Can anything be said about the large-time accuracy of the approximation? We address all of these questions in this manuscript. In particular, in a multidimensional framework, we accomplish the following tasks:

1. First, we derive fully explicit approximations at any order for fundamental solution $\Gamma(t, x; T, y)$. We emphasize that, for every n , our n th order approximation of the fundamental solution Γ is explicit; no integrals or special functions are required. This is not the case for the formal Dyson series expansion.
2. Second, we show how regularity of the terminal datum φ can be used to establish a higher order of accuracy for small times.
3. Third, we prove convergence results on arbitrarily large time intervals.

On an applied level, the results proved in this manuscript serve as the foundation for some recent developments in mathematical finance. More specifically, in

Lorig, Pagliarani, and Pascucci (2015b), the authors use the small-time error bounds established here for solutions u of (1.1) in order to prove small-time error bounds for the *implied volatility* of European call options in a general *multifactor* local-stochastic volatility model. We note that proving the accuracy result for implied volatility depends on exploiting the smoothness of the terminal datum φ .

Our proofs in this manuscript are based on a combination of symmetry properties of Gaussian kernels and (very general) classical results such as Duhamel's principle, the Chapman–Kolmogorov identity, and some upper bounds for the fundamental solution of the operator $(\partial_t + \mathcal{A})$. Due to the generality of the main ingredients in the proofs, our approach opens the door to more general expansions, which may not necessarily be based on Gaussian kernels.

The analytical techniques presented in this paper were originally developed with applications to financial mathematics in mind. However, because we provide a systematic treatment of Cauchy problem (1.1), including complete and rigorous proofs of error bounds and convergence, we believe that our results are of interest in other fields in which parabolic equations arise, such as mathematical biology, chemistry, physics, engineering, and economics.

The rest of this paper proceeds as follows: in section 2 we introduce the idea of expanding the coefficients of \mathcal{A} as a sum of polynomial basis functions. We provide examples of useful basis functions and list our main assumptions. Next, in section 3, we present our main results. Theorem 3.2 provides a closed-form expression for the n th term of the asymptotic expansion of u , the solution of (1.1). The theorem is written in a very general fashion, which allows for not just a single asymptotic expansion of u but for *an entire family of asymptotic expansions* for u . In Theorem 3.4, we provide small-time error bounds for our asymptotic approximation of u . And in Theorem 3.6, we provide convergence results, which are valid on any finite time interval. Next, in section 4, we illustrate how the solution to Cauchy problem (1.1) relates to the pricing of derivatives in financial mathematics. Finally, sections 5, 6, and 7 contain the proofs of Theorems 3.2, 3.4, and 3.6, respectively.

2. General expansion basis. To begin, we will establish some notation and state our main assumptions. For any $n \in \mathbb{N}_0$, we denote by $C_b^{n,1}(\mathbb{R}^d)$ the class of bounded functions with (globally) Lipschitz continuous derivatives of order less than or equal to n , and by $\|f\|_{C_b^{n,1}}$ the sum of the L^∞ -norms of the derivatives of f up to order n and the Lipschitz constants of the derivatives of order n of f . We also denote by $C_b^{-1,1} = L^\infty$ the class of bounded and measurable functions and set $\|\cdot\|_{C_b^{-1,1}} = \|\cdot\|_{L^\infty}$.

Throughout the rest of the paper, we shall assume that $\bar{T} > 0$ and $N \in \mathbb{N}_0$ are fixed and the coefficients of the operator \mathcal{A} in (1.2) satisfy the following assumption.

Assumption 2.1. There exists a positive constant M such that the following hold:

(i) Uniform ellipticity:

$$M^{-1}|\xi|^2 \leq \sum_{i,j=1}^d a_{ij}(t,x)\xi_i\xi_j \leq M|\xi|^2, \quad t \in [0, \bar{T}], \quad x, \xi \in \mathbb{R}^d.$$

(ii) Regularity and boundedness: the coefficients $a_{ij}, a_i, a \in C([0, \bar{T}] \times \mathbb{R}^d)$ and for any $t \in [0, \bar{T}]$ we have $a_{ij}(t, \cdot), a_i(t, \cdot), a(t, \cdot) \in C_b^{N,1}(\mathbb{R}^d)$ with their $\|\cdot\|_{C_b^{N,1}}$ -norms bounded by M .

Under Assumption 2.1, it is well known that, for any $T \in]0, \bar{T}]$ and $\varphi \in C_b^{-1,1}$, the backward parabolic Cauchy problem (1.1) admits a classical solution u . However,

in general, the function u is not known in closed-form and, for practical purposes, must be computed numerically.

In what follows, it will be convenient to rewrite the differential operator (1.2) in the more compact form

$$(2.1) \quad \mathcal{A} := \sum_{|\alpha| \leq 2} a_\alpha(t, x) D_x^\alpha, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^d,$$

where by standard notation

$$\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d, \quad |\alpha| = \sum_{i=1}^d \alpha_i, \quad D_x^\alpha = D^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}.$$

Below, we will introduce a family of expansion schemes for the operator \mathcal{A} . Each of the different families of expansion schemes is based on a different expansion of the coefficients $(a_\alpha)_{|\alpha| \leq 2}$ and will result in a different approximation for the solution u of (1.1) as well as a different approximation for the fundamental solution Γ . Thus, for any $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq 2$, we fix an approximation sequence $(a_{\alpha, n})_{n \geq 0}$ of continuous functions

$$a_{\alpha, n} : [0, \overline{T}] \times \mathbb{R}^d \rightarrow \mathbb{R}.$$

More precisely, we introduce the following definition.

DEFINITION 2.2. *We say that $(a_{\alpha, n})_{0 \leq n \leq N}$ is an N th order polynomial expansion if, for any $t \in [0, \overline{T}]$, the functions $a_{\alpha, n}(t, \cdot)$ are polynomials with $a_{\alpha, 0}(t, \cdot) = a_{\alpha, 0}(t)$.*

The idea behind our approximation method is to choose a polynomial expansion such that the sequences of partial sums $\sum_{n=0}^N a_{\alpha, n}(t)$ approximate the coefficients $a_\alpha(t, z)$, either pointwise or in some norm. We conclude this section by presenting some practical examples of polynomial expansions.

Example 2.3 (Taylor polynomial expansion). Let Assumption 2.1(ii) hold true. Then, for any fixed $\bar{x} \in \mathbb{R}^d$, we define $a_{\alpha, n}$ as the n th order term of the Taylor expansion of a_α in the spatial variables around \bar{x} . That is, we set

$$a_{\alpha, n}(\cdot, x) = \sum_{|\beta|=n} \frac{D^\beta a_\alpha(\cdot, \bar{x})}{\beta!} (x - \bar{x})^\beta, \quad 0 \leq n \leq N, \quad |\alpha| \leq 2,$$

where as usual $\beta! = \beta_1! \dots \beta_d!$ and $x^\beta = x_1^{\beta_1} \dots x_d^{\beta_d}$. The expansion proposed in Lorig, Pagliarani, and Pascucci (2015b) and Lorig, Pagliarani, and Pascucci (2014) is the particular case where $d = 2$.

Example 2.4 (enhanced Taylor expansion). In the previous example, the n th order term \mathcal{A}_n of the polynomial expansion of \mathcal{A} coincides with the n order term of the Taylor expansion. More generally, we may define the n th order term \mathcal{A}_n of the polynomial expansion of \mathcal{A} so that it coincides with a higher order Taylor expansion. Specifically, assume $N \geq 1$, and let $M_0 = 0$ and $(M_n)_{1 \leq n \leq N}$ be a nondecreasing sequence of natural numbers where, in general, M_n may be greater than n . We may assume that

$$a_{\alpha, 0}(\cdot) = a_\alpha(\cdot, \bar{x}), \quad a_{\alpha, n}(\cdot, x) = \sum_{|\beta|=1+M_{n-1}}^{M_n} \frac{D^\beta a_\alpha(\cdot, \bar{x})}{\beta!} (x - \bar{x})^\beta, \quad 1 \leq n \leq N, \quad |\alpha| \leq 2.$$

The enhanced Taylor expansion is motivated by the fact that, in the limit as $M_1 \rightarrow \infty$, we have that $\mathcal{A}_1 = \mathcal{A} - \mathcal{A}_0 = \mathcal{B}_1$. Thus, in this limit, our expansion for u (given in

Theorem 3.2) provides an explicit asymptotic representation for the Dyson series expansion.

Example 2.5 (time-dependent Taylor polynomial expansion). For any fixed $\bar{x} : \mathbb{R}_+ \rightarrow \mathbb{R}^d$, we define $a_{\alpha,n}$ as the n th order term of the Taylor expansion of a_α in the spatial variables around \bar{x} . That is, we set

$$a_{\alpha,n}(\cdot, x) = \sum_{|\beta|=n} \frac{D^\beta a_\alpha(\cdot, \bar{x}(\cdot))}{\beta!} (x - \bar{x}(\cdot))^\beta, \quad 0 \leq n \leq N, \quad |\alpha| \leq 2.$$

This expansion for the coefficients allows the expansion point \bar{x} of the Taylor series to evolve in time. By construction, \mathcal{A}_0 is guaranteed to be the generator of a diffusion X^0 . It is natural, then, to choose $\bar{x}(t)$ to be $\bar{x}(t) = E[X_t^0]$, the expected value of X_t^0 . In Lorig, Pagliarani, and Pascucci (2015b), this choice results in a highly accurate approximation for option prices and implied volatility in the Heston (1993) model.

Example 2.6 (Hermite polynomial expansion). Hermite expansions can be useful when the diffusion coefficients are not smooth. A remarkable example in financial mathematics is given by the Dupire’s local volatility formula for models with jumps (see Friz, Gerhold, and Yor (2014)). In some cases, e.g., the well-known variance-gamma model, the fundamental solution (i.e., the transition density of the underlying stochastic model) has singularities. In such cases, it is natural to approximate it in some L^p norm rather than in the pointwise sense. For the Hermite expansion centered at \bar{x} , one sets

$$a_{\alpha,n}(t, x) = \sum_{|\beta|=n} \langle \mathbf{H}_\beta(\cdot - \bar{x}), a_\alpha(t, \cdot) \rangle_\Gamma \mathbf{H}_\beta(x - \bar{x}), \quad 0 \leq n \leq N, \quad |\alpha| \leq 2,$$

where the inner product $\langle \cdot, \cdot \rangle_\Gamma$ is an integral over \mathbb{R}^d with a Gaussian weighting centered at \bar{x} and the functions $\mathbf{H}_\beta(x) = H_{\beta_1}(x_1) \cdots H_{\beta_d}(x_d)$, where H_n is the n th one-dimensional Hermite polynomial (properly normalized so that $\langle \mathbf{H}_\alpha, \mathbf{H}_\beta \rangle_\Gamma = \delta_{\alpha,\beta}$ with $\delta_{\alpha,\beta}$ being the Kronecker’s delta function).

3. Main results: Closed-form solutions, local and global error bounds.

The main idea behind the construction of an approximation for the solution u of (1.1) is very intuitive. We begin this section by presenting the derivation of a formal expansion of u . Let us consider a polynomial expansion $(a_{\alpha,n})_{n \in \mathbb{N}_0}$, and let us assume that the operator \mathcal{A} in (2.1) can be formally written as

$$(3.1) \quad \mathcal{A} = \sum_{n=0}^{\infty} \mathcal{A}_n, \quad \mathcal{A}_n := \sum_{|\alpha| \leq 2} a_{\alpha,n}(t, x) D_x^\alpha.$$

We now follow the classical approach and expand the solution u of (1.1) as follows:

$$(3.2) \quad u = \sum_{n=0}^{\infty} u_n.$$

Inserting (3.1) and (3.2) into (1.1), we find that the functions $(u_n)_{n \geq 0}$ satisfy the following sequence of nested Cauchy problems:

$$(3.3) \quad \begin{cases} (\partial_t + \mathcal{A}_0)u_0(t, x) = 0, & t \in [0, T[, \ x \in \mathbb{R}^d, \\ u_0(T, x) = \varphi(x), & x \in \mathbb{R}^d, \end{cases}$$

and

$$(3.4) \quad \begin{cases} (\partial_t + \mathcal{A}_0)u_n(t, x) = - \sum_{h=1}^n \mathcal{A}_h u_{n-h}(t, x), & t \in [0, T], x \in \mathbb{R}^d, \\ u_n(T, x) = 0, & x \in \mathbb{R}^d. \end{cases}$$

Since, by assumption, the functions $a_{\alpha,0}$ depend only on t , the operator \mathcal{A}_0 is elliptic with time-dependent coefficients. It will be useful to write the operator \mathcal{A}_0 in the following form:

$$\mathcal{A}_0 = \frac{1}{2} \sum_{i,j=1}^d C_{ij}(t) \partial_{x_i x_j} + \langle m(t), \nabla_x \rangle + \gamma(t), \quad \langle m(t), \nabla_x \rangle = \sum_{i=1}^d m_i(t) \partial_{x_i}.$$

Here, the $d \times d$ -matrix C is positive definite, uniformly for $t \in [0, T]$, and m and γ are a d -dimensional vector and a scalar functions, respectively.

Example 3.1. If $d = 2$, we have

$$C = \begin{pmatrix} 2a_{(2,0),0} & a_{(1,1),0} \\ a_{(1,1),0} & 2a_{(0,2),0} \end{pmatrix}, \quad m = (a_{(1,0),0}, a_{(0,1),0}), \quad \gamma = a_{(0,0),0}.$$

It is clear that the leading term u_0 in the expansion (3.2) is explicitly given by

$$(3.5) \quad u_0(t, x) = e^{\int_t^T \gamma(s) ds} \int_{\mathbb{R}^d} \Gamma_0(t, x; T, y) \varphi(y) dy, \quad t < T, x \in \mathbb{R}^d,$$

where Γ_0 is the d -dimensional Gaussian density

$$(3.6) \quad \Gamma_0(t, x; T, y) = \frac{1}{\sqrt{(2\pi)^d |\mathbf{C}(t, T)|}} \exp \left(-\frac{1}{2} \langle \mathbf{C}^{-1}(t, T)(y - x - \mathbf{m}(t, T)), (y - x - \mathbf{m}(t, T)) \rangle \right)$$

with covariance matrix $\mathbf{C}(t, T)$ and mean vector $x + \mathbf{m}(t, T)$ given by

$$(3.7) \quad \mathbf{C}(t, T) = \int_t^T C(s) ds, \quad \mathbf{m}(t, T) = \int_t^T m(s) ds.$$

The first main result of the paper is Theorem 3.2 below. The theorem provides an explicit representation for each u_n in (3.2). Remarkably, every u_n can be written as a finite sum of spatial derivatives acting on u_0 .

THEOREM 3.2. *For any $n \geq 1$, the n th term u_n in (3.2) is given by*

$$(3.8) \quad u_n(t, x) = \mathcal{L}_n^x(t, T) u_0(t, x), \quad t < T, x \in \mathbb{R}^d.$$

In (3.8), $\mathcal{L}_n^x(t, T)$ denotes the differential operator acting on the x -variable and defined as

$$(3.9) \quad \mathcal{L}_n^x(t, T) := \sum_{h=1}^n \int_{s_0}^T ds_1 \int_{s_1}^T ds_2 \cdots \int_{s_{h-1}}^T ds_h \sum_{i \in I_{n,h}} \mathcal{G}_{i_1}^x(s_0, s_1) \cdots \mathcal{G}_{i_h}^x(s_0, s_h),$$

where¹

$$(3.10) \quad I_{n,h} = \{i = (i_1, \dots, i_h) \in \mathbb{N}^h \mid i_1 + \dots + i_h = n\}, \quad 1 \leq h \leq n,$$

¹For instance, for $n = 3$ we have $I_{3,3} = \{(1, 1, 1)\}$, $I_{3,2} = \{(1, 2), (2, 1)\}$ and $I_{3,1} = \{(3)\}$.

and the operator $\mathcal{G}_n^x(t, s)$ is defined as

$$(3.11) \quad \mathcal{G}_n^x(t, s) := \sum_{|\alpha| \leq 2} a_{\alpha, n}(s, \mathcal{M}^x(t, s)) D_x^\alpha$$

with

$$(3.12) \quad \mathcal{M}^x(t, s) = x + \mathbf{m}(t, s) + \mathbf{C}(t, s) \nabla_x.$$

Theorem 3.2 will be proved in section 5.

Remark 3.3. Particular cases of Theorem 3.2 have been already stated, devoid of proof, in Lorig, Pagliarani, and Pascucci (2015b) and Lorig, Pagliarani, and Pascucci (2014). In Lorig, Pagliarani, and Pascucci (2015b), only time-homogeneous two-dimensional diffusions are treated. In Lorig, Pagliarani, and Pascucci (2014), only the Taylor series expansion of \mathcal{A} is treated.

Our second main result consists of local-in-time error bounds for the N th order Taylor expansion of Example 2.3. In what follows, it will be helpful to indicate explicitly the dependence on \bar{x} , the expansion point of the Taylor series. As such, we introduce the following notation: for $n \leq N$ and $\bar{x} \in \mathbb{R}^d$, we set

$$(3.13) \quad \mathcal{A}_n^{(\bar{x})} = \sum_{|\alpha| \leq 2} a_{\alpha, n}^{(\bar{x})} D_x^\alpha, \quad a_{\alpha, n}^{(\bar{x})}(t, x) = \sum_{|\beta|=n} \frac{D^\beta a_\alpha(t, \bar{x})}{\beta!} (x - \bar{x})^\beta.$$

The approximating terms $u_n = u_n^{(\bar{x})}$ in the expansion (3.2) solve

$$(3.14) \quad \begin{cases} (\partial_t + \mathcal{A}_0^{(\bar{x})}) u_0^{(\bar{x})}(t, x) = 0, & t < T, x \in \mathbb{R}^d, \\ u_0^{(\bar{x})}(T, x) = \varphi(x), & x \in \mathbb{R}^d, \end{cases}$$

and for $1 \leq n \leq N$

$$(3.15) \quad \begin{cases} (\partial_t + \mathcal{A}_0^{(\bar{x})}) u_n^{(\bar{x})}(t, x) = - \sum_{h=1}^n \mathcal{A}_h^{(\bar{x})} u_{n-h}^{(\bar{x})}(t, x), & t < T, x \in \mathbb{R}^d, \\ u_n^{(\bar{x})}(T, x) = 0, & x \in \mathbb{R}^d. \end{cases}$$

Next, we define the approximate solution at order N for the Taylor expansion centered at \bar{x} as

$$(3.16) \quad \bar{u}_N^{(\bar{x})}(t, x) := \sum_{n=0}^N u_n^{(\bar{x})}(t, x).$$

For the particular choice $\bar{x} = x$, we simply set

$$\bar{u}_N(t, x) := \bar{u}_N^{(x)}(t, x).$$

We call \bar{u}_N the N th order Taylor approximation of u . Analogously, for the fundamental solution Γ of $(\partial_t + \mathcal{A})$, we set

$$(3.17) \quad \bar{\Gamma}_N(t, x; T, y) = \bar{\Gamma}_N^{(x)}(t, x; T, y).$$

THEOREM 3.4. *Let Assumption 2.1 hold and let $0 < T \leq \bar{T}$. Assume also the initial datum $\varphi \in C_b^{k-1, 1}(\mathbb{R}^d)$ for some $0 \leq k \leq 2$. Then we have*

$$(3.18) \quad |u(t, x) - \bar{u}_N(t, x)| \leq C(T - t)^{\frac{N+k+1}{2}}, \quad 0 \leq t < T, x \in \mathbb{R}^d,$$

where the constant C only depends on M, N, \bar{T} , and $\|\varphi\|_{C_b^{k-1,1}}$. Moreover, for any $\varepsilon > 0$ we have

$$(3.19) \quad |\Gamma(t, x; T, y) - \bar{\Gamma}_N(t, x; T, y)| \leq C(T - t)^{\frac{N+1}{2}} \Gamma^{M+\varepsilon}(t, x; T, y),$$

$$0 \leq t < T, \quad x, y \in \mathbb{R}^d,$$

where $\Gamma^{M+\varepsilon}(t, x; T, y)$ is the fundamental solution of the d -dimensional heat operator

$$(3.20) \quad H^{M+\varepsilon} = (M + \varepsilon) \sum_{i=1}^d \partial_{x_i}^2 + \partial_t,$$

and C is a positive constant that depends on $M, N, \bar{T}, \varepsilon$.

Theorem 3.4 will be proved in section 6.

Remark 3.5. Theorem 3.4 can be extended by relaxing the regularity hypotheses on the terminal data φ . More precisely, if $k \in \mathbb{N}$, it is sufficient to assume that $\varphi \in C^{k-1}$ and the that derivatives are *locally* Lipschitz continuous with exponential growth at infinity. In this case, estimate (3.18) would be modified by substituting the constant C by $Ce^{C|x|}$. As we shall see in section 4, such an extension would allow for including some important functions φ commonly used in financial applications, such as the call payoff function. Even though this generalization does not change the core of the proof of Theorem 3.4, in order to avoid an excess of technicalities, we shall continue our analysis under the more restrictive hypotheses of Theorem 3.4.

We remark explicitly that (3.18) does not imply convergence as N goes to infinity because the constant C , appearing in the estimate, depends on N and, in principle, this constant can blow up in the limit as $N \rightarrow \infty$. Thus, the usefulness of Theorem 3.4 is as an asymptotic estimate for small times.

Now, we state more general convergence estimates that are valid on any time interval $[t, T]$. For any $m \in \mathbb{N}$, we consider the equispaced partition $\{t_0, \dots, t_m\}$ of $[t, T]$ defined as

$$t_k := t + k \delta_m, \quad \delta_m := \frac{T - t}{m}.$$

Moreover, we set

$$(3.21) \quad \bar{u}_{N,m}(t_0, x_0) := \int_{\mathbb{R}^{md}} \prod_{i=1}^m \bar{\Gamma}_N(t_{i-1}, x_{i-1}; t_i, x_i) \varphi(x_m) dx_1 \cdots dx_m, \quad x_0 \in \mathbb{R}^d,$$

where $\bar{\Gamma}_N$ is the N th order Taylor approximation of Γ .

THEOREM 3.6. *Assume $N \geq 1$. Under the assumptions of Theorem 3.4, we have*

$$(3.22) \quad |u(t, x) - \bar{u}_{N,m}(t, x)| \leq C \left(\frac{T - t}{m} \right)^{\frac{N+k-1}{2}}, \quad 0 \leq t < T \leq \bar{T}, \quad x \in \mathbb{R}^d,$$

where the constant C only depends on M, N, \bar{T} , and $\|\varphi\|_{C_b^{k-1,1}}$.

Theorem 3.6 will be proved in section 7. We note explicitly that, as a direct consequence of (3.22), we have that if $N \geq 2 - k$, then

$$\lim_{m \rightarrow \infty} \bar{u}_{N,m}(t, x) = u(t, x), \quad t \in [0, T], \quad x \in \mathbb{R}^d.$$

Remark 3.7. From (3.8) and (3.17), we see that

$$\bar{\Gamma}_N(t, x; T, y) = \left(1 + \sum_{i=1}^N \mathcal{L}_i^x(t, T) \right) \Gamma_0(t, x; T, y).$$

When the differential operator $(1 + \sum_i \mathcal{L}_i^x)$ hits the Gaussian kernel $\Gamma_0(t, x; T, y)$, it simply returns a polynomial of (x, y) times the Gaussian kernel $\Gamma_0(t, x; T, y)$. The coefficients $(a_{\alpha,0})_{|\alpha| \leq 2}$ of the operator $(1 + \sum_i \mathcal{L}_i^x)$ also depend on x and are smooth by Assumption 2.1, condition (ii). Thus, evaluating (3.21) involves computing an $(d \cdot m)$ -dimensional integral, where the integrand is the product of Gaussian kernels with polynomials and smooth, bounded coefficients. Since the integrand is smooth and slowly varying, these integrals can be computed numerically without major difficulties, though, clearly, there is a limit to how large $(d \cdot m)$ can be.

4. Applications to financial mathematics. In this section we motivate our analysis by illustrating how our methodology applies to the pricing derivatives in financial mathematics. To begin, we consider an arbitrage-free market. We take, as given, an equivalent martingale measure \mathbb{Q} defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\})$. All stochastic processes defined below live on this probability space, and all expectations are taken with respect to \mathbb{Q} . The risk-neutral dynamics of our market are described by the following d -dimensional Markov diffusion:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t.$$

Here, W is a standard m -dimensional Brownian motion, the function $\mu : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and the function $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$. The components of X could represent a number of things, e.g., economic factors, asset prices, economic indicators, or functions of these quantities. In particular, we assume a risk-free interest rate of the form $r(t, X_t)$, where $r : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+$. We also introduce a random time ζ , which is given by

$$\zeta = \inf \left\{ t \geq 0 : \int_0^t \gamma(s, X_s)ds \geq \mathcal{E} \right\}, \quad \gamma : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+,$$

with \mathcal{E} exponentially distributed and independent of X . The random time ζ could represent the default time of an asset, the arrival of an economic shock, etc.

Denote by V the no-arbitrage price of a European derivative expiring at time T with payoff

$$H(X_T) \mathbb{I}_{\{\zeta > T\}} + G(X_T) \mathbb{I}_{\{\zeta \leq T\}} = (H(X_T) - G(X_T)) \mathbb{I}_{\{\zeta > T\}} + G(X_T).$$

It is well known (see, for instance, Jeanblanc, Yor, and Chesney (2009)) that

$$V_t = \mathbb{E} \left[e^{-\int_t^T r(s, X_s)ds} G(X_T) | X_t \right] + \mathbb{I}_{\{\zeta > t\}} \mathbb{E} \left[e^{-\int_t^T (r(s, X_s) + \gamma(s, X_s))ds} (H(X_T) - G(X_T)) | X_t \right], t < T.$$

Then, to value a European-style option, one must compute functions of the form

$$(4.1) \quad u(t, x) := \mathbb{E} \left[e^{-\int_t^T \lambda(s, X_s)ds} \varphi(X_T) | X_t = x \right].$$

Under mild assumptions, the function u , defined by (4.1), satisfies the Kolmogorov backward equation

$$(4.2) \quad \begin{cases} (\partial_t + \mathcal{A})u(t, x) = 0, & t < T, x \in \mathbb{R}^d, \\ u(T, x) = \varphi(x), & x \in \mathbb{R}^d, \end{cases}$$

where the operator \mathcal{A} is given explicitly by

$$\mathcal{A} = \frac{1}{2} \sum_{i,j=1}^d (\sigma\sigma^T)_{ij}(t, x) \partial_{x_i} \partial_{x_j} + \sum_{i=1}^d \mu_i(t, x) \partial_{x_i} - \lambda(t, x).$$

The results of section 3 give an explicit and effective way to construct closed-form approximate solutions of problem (4.2) and therefore of closed-form approximate option prices (4.1). The rigorous error bounds prove the efficiency of the approach and confirm the high accuracy of the approximation in financial applications. For the interested reader, extensive numerical examples can be found in Pagliarani and Pascucci (2012), Pagliarani, Pascucci, and Riga (2013), Lorig, Pagliarani, and Pascucci (2015a), Lorig, Pagliarani, and Pascucci (2015b), and Lorig, Pagliarani, and Pascucci (2014).

5. Proof of Theorem 3.2: Analytical approximation formulas. The proof is based on the symmetry properties of the Gaussian fundamental solution $\Gamma_0 = \Gamma_0(t, x; s, \xi)$ as it is defined in (3.6)-(3.7), combined with an extensive use of other very general relations such as the Duhamel’s principle and the Chapman–Kolmogorov equation, which we recall for completeness.

LEMMA 5.1 (Chapman–Kolmogorov identity). *Under Assumption 2.1, for any $t < s < T$, $x, y \in \mathbb{R}^d$, we have*

$$(5.1) \quad \int_{\mathbb{R}^d} \Gamma(t, x; s, \xi) \Gamma(s, \xi; T, y) d\xi = \Gamma(t, x; T, y).$$

We start by recalling the operator

$$\mathcal{M}^x(t, s) = x + \mathbf{m}(t, s) + \mathbf{C}(t, s) \nabla_x$$

as it is defined in (3.12). Above, and throughout the proof, we use the superscript x to explicitly indicate the variables on which the operator acts. Furthermore, we define the operator

$$(5.2) \quad \bar{\mathcal{M}}^y(t, s) = y - \mathbf{m}(t, s) + \mathbf{C}(t, s) \nabla_y.$$

The following lemma illustrates how the operator ∇_x relates to ∇_y when acting on $\Gamma_0(t, x; s, y)$ and how the multiplication operators y and x relate to $\mathcal{M}^x(t, s)$ and $\bar{\mathcal{M}}^y(t, s)$, respectively, when acting on $\Gamma_0(t, x; s, y)$.

LEMMA 5.2. *For any $t < s$ and $x, y \in \mathbb{R}^d$, we have*

$$(5.3) \quad \nabla_x \Gamma_0(t, x; s, y) = -\nabla_y \Gamma_0(t, x; s, y)$$

and

$$(5.4) \quad y \Gamma_0(t, x; s, y) = \mathcal{M}^x(t, s) \Gamma_0(t, x; s, y),$$

$$(5.5) \quad x \Gamma_0(t, x; s, y) = \bar{\mathcal{M}}^y(t, s) \Gamma_0(t, x; s, y).$$

Proof. While the previous identities can be directly verified a posteriori by elementary computations, here we give an alternative “constructive” proof which shows how to find \mathcal{M}^x -like and $\bar{\mathcal{M}}^y$ -like operators, which are equivalent to multiplication by the backward and forward variables y and x , respectively, in even more general

frameworks (see Remark 5.3 below). To this end, we will require some properties of the Fourier transform

$$\mathcal{F}_x f(\xi) := \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d} e^{ix\xi} f(x) dx.$$

First, we recall that for any function f in the Schwartz space, we have

$$(5.6) \quad i\xi \mathcal{F}_x(f) = \mathcal{F}_x(-\nabla_x f), \quad \mathcal{F}_x(xf) = -i \nabla_\xi \mathcal{F}_x f.$$

Moreover, we have

$$(5.7) \quad \begin{aligned} \mathcal{F}_x \Gamma_0(t, \cdot; T, y)(\xi) &= \frac{1}{\sqrt{(2\pi)^d}} e^{i\xi(y - \mathbf{m}(t, T)) - \frac{1}{2} \langle \mathbf{C}(t, T) \xi, \xi \rangle}, \\ \mathcal{F}_y \Gamma_0(t, x; T, \cdot)(\eta) &= \frac{1}{\sqrt{(2\pi)^d}} e^{i\eta(x + \mathbf{m}(t, T)) - \frac{1}{2} \langle \mathbf{C}(t, T) \eta, \eta \rangle}. \end{aligned}$$

To obtain the identity (5.3), we simply use that $\Gamma_0(t, x; T, y) = \Gamma_0(t, x - y; T, 0)$. For (5.4), we have

$$\begin{aligned} \mathcal{F}_y(y \Gamma_0) &= -i \nabla_\eta \mathcal{F}_y(\Gamma_0) \\ &= (x + \mathbf{m}(t, s) + \mathbf{C}(t, s) i \eta) \mathcal{F}_y(\Gamma_0) && \text{(by (5.7))} \\ &= \mathcal{F}_y((x + \mathbf{m}(t, s) - \mathbf{C}(t, s) \nabla_y) \Gamma_0) && \text{(by (5.6))} \\ &= \mathcal{F}_y(\mathcal{M}^x(t, s) \Gamma_0) && \text{(by (5.3)).} \end{aligned}$$

The proof of identity (5.5) is analogous to the proof of identity (5.4). \square

Remark 5.3. It is worth noting that the argument of the above proof applies whenever the characteristic function of the stochastic process with transition density Γ_0 is explicitly known and when Γ_0 can be expressed as a function of $x - y$. Thus, \mathcal{M}^x -like and $\bar{\mathcal{M}}^y$ -like operators can be obtained, for example, when Γ_0 is the transition density of an additive (i.e., time-dependent Lévy) process. In this case, the \mathcal{M}^x -like and $\bar{\mathcal{M}}^y$ -like operators would be pseudodifferential operator rather than (usual) differential operators.

COROLLARY 5.4. *For any $t < s$, $s_1 \in [0, T]$ and $x, y \in \mathbb{R}^d$, we have*

$$(5.8) \quad a_{\alpha, n}(s_1, y) \Gamma_0(t, x; s, y) = a_{\alpha, n}(s_1, \mathcal{M}^x(t, s)) \Gamma_0(t, x; s, y),$$

$$(5.9) \quad a_{\alpha, n}(s_1, x) \Gamma_0(t, x; s, y) = a_{\alpha, n}(s_1, \bar{\mathcal{M}}^y(t, s)) \Gamma_0(t, x; s, y).$$

Proof. First, we note that the components $\mathcal{M}_i^x(t, s)$, $i = 1, \dots, d$, of the operator $\mathcal{M}^x(t, s)$ commute when applied to $\Gamma_0 = \Gamma_0(t, x; s, y)$ and to its derivatives. (Notice, however, that this is not true in general when they are applied to a generic function.) Indeed, for any multi-index β , we have

$$\begin{aligned} \mathcal{M}_i^x(t, s) \mathcal{M}_j^x(t, s) D_x^\beta \Gamma_0 &= (-1)^{|\beta|} \mathcal{M}_i^x(t, s) \mathcal{M}_j^x(t, s) D_y^\beta \Gamma_0 \quad \text{(by (5.3))} \\ &= (-1)^{|\beta|} D_y^\beta \mathcal{M}_i^x(t, s) \mathcal{M}_j^x(t, s) \Gamma_0 \\ &= (-1)^{|\beta|} D_y^\beta \mathcal{M}_i^x(t, s) y_j \Gamma_0 && \text{(by (5.4))} \\ &= (-1)^{|\beta|} D_y^\beta y_j \mathcal{M}_i^x(t, s) \Gamma_0 \\ &= (-1)^{|\beta|} D_y^\beta y_j y_i \Gamma_0 \\ &= \mathcal{M}_j^x(t, s) \mathcal{M}_i^x(t, s) D_x^\beta \Gamma_0 && \text{(by reversing the steps above).} \end{aligned}$$

Since $a_{\alpha, n}(s_1, \cdot)$ is a polynomial by construction, we therefore have that the operators $a_{\alpha, n}(s_1, \mathcal{M}^x(t, s))$ are defined unambiguously when applied to $\Gamma_0(t, x; s, y)$ and to its

derivatives. Moreover, clearly (5.8) is now a straightforward consequence of (5.4). An analogous argument shows the validity of (5.9). \square

We now recall the operators

$$(5.10) \quad \mathcal{A}_n^x(s) = \sum_{|\alpha| \leq 2} a_{\alpha,n}(s, x) D_x^\alpha, \quad \mathcal{G}_n^x(t, s) = \sum_{|\alpha| \leq 2} a_{\alpha,n}(s, \mathcal{M}^x(t, s)) D_x^\alpha, \quad n \geq 0,$$

as they are defined in (3.1) and (3.11), and we introduce the operator

$$(5.11) \quad \bar{\mathcal{G}}_n^y(t, s) = \sum_{|\alpha| \leq 2} (-1)^{|\alpha|} D_y^\alpha a_{\alpha,n}(t, \bar{\mathcal{M}}^y(t, s)), \quad n \geq 0,$$

with $\bar{\mathcal{M}}^y$ as in (5.2). We remark explicitly that, by Corollary 5.4, operators $\mathcal{G}_n^x(t, s)$ and $\bar{\mathcal{G}}_n^y(t, s)$ are defined unambiguously when applied to $\Gamma_0 = \Gamma_0(t, x; s, y)$, to its derivatives, and, more generally, by the representation formula (3.5), to solutions of the Cauchy problem (3.3).

The next proposition and its remarkable corollaries are the key of the proof of Theorem 3.2.

PROPOSITION 5.5. *For any $t < s < T$, $x, y \in \mathbb{R}^d$ and $n \geq 1$, we have*

$$(5.12) \quad \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \mathcal{A}_n^\xi(s) f(\xi) d\xi = \mathcal{G}_n^x(t, s) \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) f(\xi) d\xi,$$

$$(5.13) \quad \int_{\mathbb{R}^d} f(\xi) \mathcal{A}_n^\xi(s) \Gamma_0(s, \xi; T, y) d\xi = \bar{\mathcal{G}}_n^y(s, T) \int_{\mathbb{R}^d} f(\xi) \Gamma_0(s, \xi; T, y) d\xi$$

for any $f \in C_0^2(\mathbb{R}^d)$. Furthermore, the following relation holds:

$$(5.14) \quad \mathcal{G}_n^x(t, s) \Gamma_0(t, x; T, y) = \bar{\mathcal{G}}_n^y(s, T) \Gamma_0(t, x; T, y).$$

Proof. Equation (5.12). By the definition of \mathcal{A}_n^ξ , we have

$$\begin{aligned} & \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \mathcal{A}_n^\xi(s) f(\xi) d\xi \\ &= \sum_{|\alpha| \leq 2} \int_{\mathbb{R}^d} a_{\alpha,n}(s, \xi) \Gamma_0(t, x; s, \xi) D_\xi^\alpha f(\xi) d\xi \\ &= \sum_{|\alpha| \leq 2} a_{\alpha,n}(s, \mathcal{M}^x(t, s)) \int_{\mathbb{R}^d} \Gamma_0(t, x, y; s, \xi, \omega) D_\xi^\alpha f(\xi) d\xi && \text{(by (5.8))} \\ &= \sum_{|\alpha| \leq 2} a_{\alpha,n}(s, \mathcal{M}^x(t, s)) (-1)^{|\alpha|} \int_{\mathbb{R}^d} D_\xi^\alpha \Gamma_0(t, x; s, \xi) f(\xi) d\xi && \text{(integrating by parts)} \\ &= \sum_{|\alpha| \leq 2} a_{\alpha,n}(s, \mathcal{M}^x(t, s)) D_x^\alpha \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) f(\xi) d\xi && \text{(by (5.3))} \\ &= \mathcal{G}_n^x(t, s) \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) f(\xi) d\xi && \text{(by (5.10), definition of } \mathcal{G}_n^x \text{)}. \end{aligned}$$

Similarly, for (5.13), using the definition of \mathcal{A}_n^ξ we have

$$\begin{aligned} & \int_{\mathbb{R}^d} f(\xi) \mathcal{A}_n^\xi(s) \Gamma_0(s, \xi; T, y) d\xi \\ &= \sum_{|\alpha| \leq 2} \int_{\mathbb{R}^d} f(\xi) a_{\alpha, n}(s, \xi) D_\xi^\alpha \Gamma_0(s, \xi; T, y) d\xi \\ &= \sum_{|\alpha| \leq 2} (-1)^{|\alpha|} D_y^\alpha \int_{\mathbb{R}^d} f(\xi) a_{\alpha, n}(s, \xi) \Gamma_0(s, \xi; T, y) d\xi && \text{(by (5.3))} \\ &= \sum_{|\alpha| \leq 2} (-1)^{|\alpha|} D_y^\alpha a_{\alpha, n}(s, \bar{M}^y(s, T)) \int_{\mathbb{R}^d} f(\xi) \Gamma_0(s, \xi; T, y) d\xi && \text{(by (5.9))} \\ &= \bar{\mathcal{G}}_n^y(s, T) \int_{\mathbb{R}^d} f(\xi) \Gamma_0(s, \xi; T, y) d\xi && \text{(by (5.11), definition of } \bar{\mathcal{G}}_n^y \text{).} \end{aligned}$$

Identity (5.14) follows from (5.12) and (5.13). Indeed, using the Chapman–Kolmogorov equation, we have

$$\begin{aligned} & \mathcal{G}_n^x(t, s) \Gamma_0(t, x; T, y) \\ &= \mathcal{G}_n^x(t, s) \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \Gamma_0(s, \xi; T, y) d\xi \\ &= \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \mathcal{A}_n^\xi(s) \Gamma_0(s, \xi; T, y) d\xi \\ & && \text{(applying (5.12) with } f(\xi) = \Gamma_0(s, \xi; T, y) \text{)} \\ &= \bar{\mathcal{G}}_n^y(s, T) \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \Gamma_0(s, \xi; T, y) d\xi \\ & && \text{(applying (5.13) with } f(\xi) = \Gamma_0(t, x; s, \xi) \text{)} \\ &= \bar{\mathcal{G}}_n^y(s, T) \Gamma_0(t, x; T, y) && \text{(by Chapman–Kolmogorov). } \quad \square \end{aligned}$$

COROLLARY 5.6. *For any $t < s < T$, $x, y \in \mathbb{R}$, $n \geq 1$, we have*

$$\begin{aligned} (5.15) \quad & \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \mathcal{G}_{i_1}^\xi(s, s_1) \cdots \mathcal{G}_{i_n}^\xi(s, s_n) \Gamma_0(s, \xi; T, y) d\xi \\ &= \mathcal{G}_{i_1}^x(t, s_1) \cdots \mathcal{G}_{i_n}^x(t, s_n) \Gamma_0(t, x; T, y) \end{aligned}$$

for any $i \in \mathbb{N}^n$ and $s < s_1 < \cdots < s_n < T$.

Proof. We first prove (5.15) by induction on n . For $n = 1$ and for any $i_1 \geq 1$, $t < s_1 < T$, we have

$$\begin{aligned} & \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \mathcal{G}_{i_1}^\xi(s, s_1) \Gamma_0(s, \xi; T, y) d\xi \\ &= \bar{\mathcal{G}}_{i_1}^y(s_1, T) \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \Gamma_0(s, \xi; T, y) d\xi && \text{(by (5.14))} \\ &= \bar{\mathcal{G}}_{i_1}^y(s_1, T) \Gamma_0(t, x; T, y) && \text{(by Chapman–Kolmogorov)} \\ &= \mathcal{G}_{i_1}^x(t, s_1) \Gamma_0(t, x; T, y) && \text{(by (5.14).)} \end{aligned}$$

We assume now the thesis to be true for $n \geq 1$ and for any $i \in \mathbb{N}^n$, $s < s_1, \dots, s_n < T$.

Then, for any $i_{n+1} \geq 1$, $s_n < s_{n+1} < T$, we have

$$\begin{aligned}
& \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \mathcal{G}_{i_1}^\xi(s, s_1) \cdots \mathcal{G}_{i_n}^\xi(s, s_n) \mathcal{G}_{i_{n+1}}^\xi(s, s_{n+1}) \Gamma_0(s, \xi; T, y) d\xi \\
&= \bar{\mathcal{G}}_{i_{n+1}}^y(s_{n+1}, T) \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \mathcal{G}_{i_1}^\xi(s, s_1) \cdots \mathcal{G}_{i_n}^\xi(s, s_n) \Gamma_0(s, \xi; T, y) d\xi \\
& \hspace{20em} ((5.14) \text{ on } \mathcal{G}_{i_{n+1}}^\xi(s, s_{n+1}) \Gamma_0) \\
&= \bar{\mathcal{G}}_{i_{n+1}}^y(s_{n+1}, T) \mathcal{G}_{i_1}^x(t, s_1) \cdots \mathcal{G}_{i_n}^x(t, s_n) \Gamma_0(t, x; T, y) \quad (\text{inductive hypothesis}) \\
&= \mathcal{G}_{i_1}^x(t, s_1) \cdots \mathcal{G}_{i_n}^x(t, s_n) \bar{\mathcal{G}}_{i_{n+1}}^y(s_{n+1}, T) \Gamma_0(t, x; T, y) \\
&= \mathcal{G}_{i_1}^x(t, s_1) \cdots \mathcal{G}_{i_n}^x(t, s_n) \mathcal{G}_{i_{n+1}}^x(t, s_{n+1}) \Gamma_0(t, x; T, y) \quad ((5.14) \text{ on } \bar{\mathcal{G}}_{i_{n+1}}^y(s_{n+1}, T) \Gamma_0),
\end{aligned}$$

which proves (5.15). \square

From here to the end of this section, we set $\gamma = 0$. We do this merely to save space. The general case, with $\gamma \neq 0$, is completely analogous and introduces no complications.

COROLLARY 5.7. *Let u_0 be as in (3.5) with $\gamma = 0$. For any $t < s < T$, $x, y \in \mathbb{R}$, $n \geq 1$, we have*

$$(5.16) \quad \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \mathcal{G}_{i_1}^\xi(s, s_1) \cdots \mathcal{G}_{i_n}^\xi(s, s_n) u_0(s, \xi) d\xi = \mathcal{G}_{i_1}^x(t, s_1) \cdots \mathcal{G}_{i_n}^x(t, s_n) u_0(t, x)$$

for any $i \in \mathbb{N}^n$ and $s < s_1 < \cdots < s_n < T$.

Proof. By (3.5), we have

$$\begin{aligned}
& \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \mathcal{G}_{i_1}^\xi(s, s_1) \cdots \mathcal{G}_{i_n}^\xi(s, s_n) u_0(s, \xi) d\xi \\
&= \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \mathcal{G}_{i_1}^\xi(s, s_1) \cdots \mathcal{G}_{i_n}^\xi(s, s_n) \int_{\mathbb{R}^d} \Gamma_0(s, \xi; T, y) \varphi(y) dy d\xi \\
&= \int_{\mathbb{R}^d} \varphi(y) \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \mathcal{G}_{i_1}^\xi(s, s_1) \cdots \mathcal{G}_{i_n}^\xi(s, s_n) \Gamma_0(s, \xi; T, y) d\xi dy \\
& \hspace{20em} (\text{Fubini's theorem}) \\
&= \int_{\mathbb{R}^d} \varphi(y) \mathcal{G}_{i_1}^x(t, s_1) \cdots \mathcal{G}_{i_n}^x(t, s_n) \Gamma_0(t, x; T, y) dy \\
& \hspace{20em} (\text{by Corollary 5.6}) \\
&= \mathcal{G}_{i_1}^x(t, s_1) \cdots \mathcal{G}_{i_n}^x(t, s_n) u_0(t, x) \quad (\text{by (3.5)}),
\end{aligned}$$

which concludes the proof. \square

We are now in position to prove Theorem 3.2. Proceeding by induction on n , we first prove the case $n = 1$. By definition, u_1 is the unique solution of the nonhomo-

geneous Cauchy problem (3.4) with $n = 1$. Thus, by Duhamel’s principle, we have

$$\begin{aligned}
 u_1(t, x) &= \int_t^T \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \mathcal{A}_1^\xi(s) u_0(s, \xi) d\xi ds \\
 &= \int_t^T \mathcal{G}_1^x(t, s) \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) u_0(s, \xi) d\xi ds && \text{(by (5.12) with } n = 1) \\
 &= \int_t^T \mathcal{G}_1^x(t, s) \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \int_{\mathbb{R}^d} \Gamma_0(s, \xi; T, y) \varphi(y) dy d\xi ds && \text{(by (3.5))} \\
 &= \int_t^T \mathcal{G}_1^x(t, s) \int_{\mathbb{R}^d} \varphi(y) \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \Gamma_0(s, \xi; T, y) d\xi dy ds \\
 &&& \text{(Fubini’s theorem)} \\
 &= \int_t^T \mathcal{G}_1^x(t, s) ds u_0(t, x) && \text{(Chapman–Kolmogorov and (3.5))} \\
 &= \mathcal{L}_1^x(t, T) u_0(t, x) && \text{(by (3.9)–(3.10)).}
 \end{aligned}$$

For the general case, let us assume that (3.8) holds for $n \geq 1$ and prove it holds for $n + 1$. By definition, u_{n+1} is the unique solution of the nonhomogeneous Cauchy problem (3.4). Thus, by Duhamel’s principle, we have

$$\begin{aligned}
 u_{n+1}(t, x) &= \int_t^T \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \sum_{h=1}^{n+1} \mathcal{A}_h^\xi(s) u_{n+1-h}(s, \xi) d\xi ds \\
 &= \sum_{h=1}^{n+1} \int_t^T \mathcal{G}_h^x(t, s) \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) u_{n+1-h}(s, \xi) d\xi ds && \text{(by (5.12) with } n = h) \\
 (5.17) \quad &= \sum_{h=1}^{n+1} \int_t^T \mathcal{G}_h^x(t, s) \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \mathcal{L}_{n+1-h}^\xi(s, T) u_0(s, \xi) d\xi ds \\
 &&& \text{(by induction hypothesis).}
 \end{aligned}$$

Now, by definition (3.9)–(3.10), we have

$$\begin{aligned}
 &\int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \mathcal{L}_{n+1-h}^\xi(s, T) u_0(s, \xi) d\xi \\
 &= \sum_{j=1}^{n+1-h} \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \int_s^T ds_1 \cdots \int_{s_{j-1}}^T ds_j \sum_{i \in I_{n+1-h, j}} \mathcal{G}_{i_1}^\xi(s, s_1) \cdots \mathcal{G}_{i_j}^\xi(s, s_j) u_0(s, \xi) d\xi \\
 &= \sum_{j=1}^{n+1-h} \int_s^T ds_1 \cdots \int_{s_{j-1}}^T ds_j \sum_{i \in I_{n+1-h, j}} \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \mathcal{G}_{i_1}^\xi(s, s_1) \cdots \mathcal{G}_{i_j}^\xi(s, s_j) u_0(s, \xi) d\xi \\
 &&& \text{(Fubini’s theorem)} \\
 &= \sum_{j=1}^{n+1-h} \int_s^T ds_1 \cdots \int_{s_{j-1}}^T ds_j \sum_{i \in I_{n+1-h, j}} \mathcal{G}_{i_1}^x(t, s_1) \cdots \mathcal{G}_{i_j}^x(t, s_j) u_0(t, x) && \text{(by (5.16)).} \\
 (5.18) \quad &
 \end{aligned}$$

Next, by inserting (5.18) into (5.17), we obtain

$$u_{n+1}(t, x) = \tilde{\mathcal{L}}_n^x(t, T)u_0(t, x),$$

where

$$\begin{aligned} \tilde{\mathcal{L}}_n^x(t, T) &= \int_t^T \mathcal{G}_{n+1}^x(t, s_0) ds_0 \\ &+ \sum_{h=1}^n \sum_{j=1}^{n+1-h} \int_t^T ds_0 \int_{s_0}^T ds_1 \cdots \int_{s_{j-1}}^T ds_j \sum_{i \in I_{n+1-h,j}} \mathcal{G}_h^x(t, s_0) \mathcal{G}_{i_1}^x(t, s_1) \cdots \mathcal{G}_{i_j}^x(t, s_j). \end{aligned}$$

In order to conclude the proof, it is enough to check that $\tilde{\mathcal{L}}_n^x(t, T) = \mathcal{L}_{n+1}^x(t, T)$. By exchanging the indexes in the sums, we obtain

$$\begin{aligned} \tilde{\mathcal{L}}_n^x(t, T) &= \int_t^T \mathcal{G}_{n+1}^x(t, s_0) ds_0 \\ &+ \sum_{j=1}^n \sum_{h=1}^{n+1-j} \int_t^T ds_0 \int_{s_0}^T ds_1 \cdots \int_{s_{j-1}}^T ds_j \sum_{i \in I_{n+1-h,j}} \mathcal{G}_h^x(t, s_0) \mathcal{G}_{i_1}^x(t, s_1) \cdots \mathcal{G}_{i_j}^x(t, s_j) \end{aligned}$$

(setting $l = j + 1$)

$$\begin{aligned} &= \int_t^T \mathcal{G}_{n+1}^x(t, s_0) ds_0 \\ &+ \sum_{l=2}^{n+1} \sum_{h=1}^{n+2-l} \int_t^T ds_0 \int_{s_0}^T ds_1 \cdots \int_{s_{l-2}}^T ds_{l-1} \sum_{i \in I_{n+1-h,l-1}} \mathcal{G}_h^x(t, s_0) \mathcal{G}_{i_1}^x(t, s_1) \cdots \mathcal{G}_{i_{l-1}}^x(t, s_{l-1}) \end{aligned}$$

(replacing the integration variables: $(ds_0, ds_1, \dots, ds_{l-1}) \rightarrow (dr_1, dr_2, \dots, dr_l)$)

$$\begin{aligned} &= \int_t^T \mathcal{G}_{n+1}^x(t, s_0) ds_0 \\ &+ \sum_{l=2}^{n+1} \sum_{h=1}^{n+2-l} \int_t^T dr_1 \int_{r_1}^T dr_2 \cdots \int_{r_{l-1}}^T dr_l \sum_{i \in I_{n+1-h,l-1}} \mathcal{G}_h^x(t, r_1) \mathcal{G}_{i_1}^x(t, r_2) \cdots \mathcal{G}_{i_{l-1}}^x(t, r_l) \end{aligned}$$

$$\begin{aligned} &= \int_t^T \mathcal{G}_{n+1}^x(t, s_0) ds_0 \\ &+ \sum_{l=2}^{n+1} \int_t^T dr_1 \int_{r_1}^T dr_2 \cdots \int_{r_{l-1}}^T dr_l \sum_{h=1}^{n+2-l} \sum_{i \in I_{n+1-h,l-1}} \mathcal{G}_h^x(t, r_1) \mathcal{G}_{i_1}^x(t, r_2) \cdots \mathcal{G}_{i_{l-1}}^x(t, r_l) \end{aligned}$$

(by definition (3.10))

$$= \sum_{l=1}^{n+1} \int_t^T dr_1 \int_{r_1}^T dr_2 \cdots \int_{r_{l-1}}^T dr_l \sum_{z \in I_{n+1,l}} \mathcal{G}_{z_1}^x(t, r_1) \mathcal{G}_{z_2}^x(t, r_2) \cdots \mathcal{G}_{z_l}^x(t, r_l)$$

(by definition (3.9))

$$= \mathcal{L}_{n+1}^x(t, T),$$

which concludes the proof. \square

6. Proof of Theorem 3.4: Error bounds for small times. Throughout this section we fix M, N , and \bar{T} . All of the constants appearing in the estimates proved in this section *depend on M, N , and \bar{T}* and will not continue repeating this below. Under the main Assumption 2.1, the operator $(\partial_t + \mathcal{A})$ admits a unique fundamental solution $\Gamma = \Gamma(t, x; T, y)$, for which the following classical Gaussian estimates hold (see Friedman (1964), Chapter 1).

LEMMA 6.1. *For any $\varepsilon > 0$ and $\beta, \nu \in \mathbb{N}_0^d$ with $|\nu| \leq N + 2$, we have*

$$|(x - y)^\beta D_x^\nu \Gamma(t, x; T, y)| \leq C \cdot (T - t)^{\frac{|\beta| - |\nu|}{2}} \Gamma^{M+\varepsilon}(t, x; T, y),$$

$$0 \leq t < T \leq \bar{T}, \quad x, y \in \mathbb{R}^d,$$

where $\Gamma^{M+\varepsilon}$ is the fundamental solution of the heat operator (3.20) and C is a positive constant, only dependent on $M, N, \bar{T}, \varepsilon$, and $|\beta|$.

In order to state our theoretical results, we need some preliminary estimates on the spatial derivatives of the solution of the Cauchy problem with coefficients that may depend on t but are constant in x . The quality of such estimates depends on the regularity of the terminal data φ .

PROPOSITION 6.2. *Assume the coefficients of \mathcal{A} to be constant in space (i.e., $a_\alpha(t, \cdot) \equiv a_\alpha(t)$). Let $\beta \in \mathbb{N}_0^d$ and $\varphi \in C_b^{k-1,1}(\mathbb{R}^d)$ for some $k \in \mathbb{N}_0$. Then the solution of the Cauchy problem (1.1) satisfies*

$$|D_x^\beta u(t, x)| \leq C \cdot (T - t)^{\frac{\min\{k-|\beta|, 0\}}{2}}, \quad 0 \leq t < T \leq \bar{T}, \quad x \in \mathbb{R}^d,$$

where C only depends on $M, N, \bar{T}, |\beta|$, and $\|\varphi\|_{C_b^{k-1,1}}$.

Proof. As \mathcal{A} has space-independent coefficients, the fundamental solution of $(\partial_t + \mathcal{A})$ is the Gaussian function in (3.6). A direct computation shows that for any polynomial function $p = p(y)$, we have

$$\int_{\mathbb{R}^d} p(y) \Gamma_0(t, x; T, y) \, dy = \bar{p}(x),$$

where \bar{p} is a polynomial with degree $\deg(\bar{p}) = \deg(p)$. Thus, for any $\nu \in \mathbb{N}_0^d$ with $|\nu| > \deg(p)$, we have

$$\int_{\mathbb{R}^d} p(y) D_x^\nu \Gamma_0(t, x; T, y) \, dy = D_x^\nu \int_{\mathbb{R}^d} p(y) \Gamma_0(t, x; T, y) \, dy = 0.$$

In particular, let us set $h = \min\{|\beta|, k\}$ and denote by $\mathbf{T}_{\bar{x}, h}^\varphi$ the h th order Taylor polynomial of φ centered at \bar{x} , i.e.,

$$(6.1) \quad \mathbf{T}_{\bar{x}, h}^\varphi(x) = \sum_{|\nu| \leq h} \frac{D^\nu \varphi(\bar{x})}{\nu!} (x - \bar{x})^\nu,$$

where, by convention, when $h = -1$, $\mathbf{T}_{\bar{x}, -1}^\varphi \equiv 0$. Then we have

$$(6.2) \quad \int_{\mathbb{R}^d} \mathbf{T}_{\bar{x}, h-1}^\varphi(y) D_x^\beta \Gamma_0(t, x; T, y) \, dy = 0.$$

Now, by Duhamel’s principle, we have

$$u(t, x) = e^{\int_t^T \gamma(s) ds} \int_{\mathbb{R}^d} \Gamma_0(t, x; T, y) \varphi(y) \, dy, \quad t < T, \quad x \in \mathbb{R}^d.$$

Next, since $\varphi \in C_b^{k-1,1}(\mathbb{R}^d)$, by (6.2) we obtain

$$\begin{aligned} D_x^\beta u(t, x) &= e^{\int_t^T \gamma(s) ds} \int_{\mathbb{R}^d} \varphi(y) D_x^\beta \Gamma_0(t, x; T, y) dy \\ &= e^{\int_t^T \gamma(s) ds} \int_{\mathbb{R}^d} (\varphi(y) - \mathbf{T}_{x,h-1}^\varphi(y)) D_x^\beta \Gamma_0(t, x; T, y) dy. \end{aligned}$$

Thus, by the Taylor theorem with integral remainder, we obtain

$$|D_x^\beta u(t, x)| \leq C \int_{\mathbb{R}^d} |x - y|^h |D_x^\beta \Gamma_0(t, x; T, y)| dy,$$

where C depends on $\|\varphi\|_{C_b^{k-1,1}}$. The thesis follows from Lemma 6.1 and from

$$\int_{\mathbb{R}^d} \Gamma^{M+\varepsilon}(t, x; T, y) dy = 1. \quad \square$$

Hereafter, we assume that all the hypotheses of Theorem 3.4 are satisfied. The proof of Theorem 3.4 is based on the following lemmas.

LEMMA 6.3. *Under the hypotheses of Theorem 3.4, for any $\bar{x} \in \mathbb{R}^d$ and $N \in \mathbb{N}_0$, we have*

$$\begin{aligned} u(t, x) - \bar{u}_N^{(\bar{x})}(t, x) &= \int_t^T \int_{\mathbb{R}^d} \Gamma(t, x; s, \xi) \sum_{n=0}^N (\mathcal{A} - \bar{\mathcal{A}}_n^{(\bar{x})}) u_{N-n}^{(\bar{x})}(s, \xi) d\xi ds, \quad t < T, \quad x \in \mathbb{R}^d, \end{aligned}$$

where the function u is the solution of (1.1), the function $\bar{u}_N^{(\bar{x})}$ is the N th order approximation in (3.16), and

$$\bar{\mathcal{A}}_n^{(\bar{x})} = \sum_{i=0}^n \mathcal{A}_i^{(\bar{x})}.$$

Proof. We first prove the identity

$$(6.3) \quad (\partial_t + \mathcal{A}) \bar{u}_N^{(\bar{x})}(t, x) = \sum_{n=0}^N (\mathcal{A} - \bar{\mathcal{A}}_n^{(\bar{x})}) u_{N-n}^{(\bar{x})}(t, x), \quad t < T, \quad x \in \mathbb{R}^d.$$

For $N = 0$ we have

$$(\partial_t + \mathcal{A}) \bar{u}_0^{(\bar{x})} = (\mathcal{A} - \mathcal{A}_0^{(\bar{x})}) u_0^{(\bar{x})},$$

because $(\partial_t + \mathcal{A}_0^{(\bar{x})}) u_0^{(\bar{x})} = 0$ by definition (3.14). We assume now that (6.3) holds for

$N \geq 0$, and we prove it to hold for $N + 1$. We have

$$\begin{aligned}
 & (\partial_t + \mathcal{A}) \bar{u}_{N+1}^{(\bar{x})} \\
 &= (\partial_t + \mathcal{A}) \bar{u}_N^{(\bar{x})} + (\partial_t + \mathcal{A}) u_{N+1}^{(\bar{x})} \\
 &= \sum_{n=0}^N (\mathcal{A} - \bar{\mathcal{A}}_n^{(\bar{x})}) u_{N-n}^{(\bar{x})} + (\mathcal{A} - \mathcal{A}_0^{(\bar{x})}) u_{N+1}^{(\bar{x})} - \sum_{n=1}^{N+1} \mathcal{A}_n^{(\bar{x})} u_{N+1-n}^{(\bar{x})} \\
 & \hspace{15em} \text{(by inductive hypothesis and by (3.15))} \\
 &= \sum_{n=1}^{N+1} (\mathcal{A} - \bar{\mathcal{A}}_{n-1}^{(\bar{x})}) u_{N+1-n}^{(\bar{x})} + (\mathcal{A} - \mathcal{A}_0^{(\bar{x})}) u_{N+1}^{(\bar{x})} - \sum_{n=1}^{N+1} \mathcal{A}_n^{(\bar{x})} u_{N+1-n}^{(\bar{x})} \\
 & \hspace{15em} \text{(by shifting the index of the first sum)} \\
 &= \sum_{n=1}^{N+1} (\mathcal{A} - \bar{\mathcal{A}}_n^{(\bar{x})}) u_{N+1-n}^{(\bar{x})} + (\mathcal{A} - \mathcal{A}_0^{(\bar{x})}) u_{N+1}^{(\bar{x})} = \sum_{n=0}^{N+1} (\mathcal{A} - \bar{\mathcal{A}}_n^{(\bar{x})}) u_{N+1-n}^{(\bar{x})}.
 \end{aligned}$$

Now, since u is the classical solution of (1.1), we have by (6.3) that $v := u - \bar{u}_N^{(\bar{x})}$ solves the following problem:

$$\begin{cases}
 (\partial_t + \mathcal{A})v(t, x) = - \sum_{n=0}^N (\mathcal{A} - \bar{\mathcal{A}}_n^{(\bar{x})}) u_{N-n}^{(\bar{x})}(t, x), & t < T, \quad x \in \mathbb{R}^d, \\
 v(T, x) = 0, & x \in \mathbb{R}^d.
 \end{cases}$$

The thesis follows by Duhamel’s principle. \square

LEMMA 6.4. *Under the assumptions of Theorem 3.4, for any multi-index $\beta \in \mathbb{N}^d$ we have*

$$(6.4) \quad \left| D_x^\beta u_0^{(\bar{x})}(t, x) \right| \leq C \cdot (T - t)^{\frac{\min\{k-|\beta|, 0\}}{2}}, \quad 0 \leq t < T \leq \bar{T}, \quad x, \bar{x} \in \mathbb{R}^d.$$

Moreover, if $N \geq 1$, then for any $n \in \mathbb{N}$, $n \leq N$, we have

$$(6.5) \quad \left| D_x^\beta u_n^{(\bar{x})}(t, x) \right| \leq C \cdot (T - t)^{\frac{n+k-|\beta|}{2}} \left(1 + |x - \bar{x}|^n (T - t)^{-\frac{n}{2}} \right), \\
 0 \leq t < T \leq \bar{T}, \quad x, \bar{x} \in \mathbb{R}^d.$$

The constants in (6.4) and (6.5) depend only on $M, N, \bar{T}, |\beta|$, and $\|\varphi\|_{C_b^{k-1,1}}$.

Proof. In this proof, $\{C_i\}_{i \geq 1}$ denote some positive constants that depend only on M, N, \bar{T} , and $\|\varphi\|_{C_b^{k-1,1}}$. For clarity, write the operators appearing in Theorem 3.2 as $\mathcal{L}_k^{x,(\bar{x})}$ and $\mathcal{G}_k^{x,(\bar{x})}$ in order to indicate that these operators are constructed using the expansion point \bar{x} and act on the variable x .

For $n = 0$, the thesis follows directly from Proposition 6.2 since $u_0^{(\bar{x})}$ solves problem (3.14). Next, we prove the assertion for $n = 1$. By Theorem 3.2, for any $\bar{x} \in \mathbb{R}^d$ we have

$$\begin{aligned}
 u_1^{(\bar{x})}(t, x) &= \mathcal{L}_1^{x,(\bar{x})}(t, T) u_0^{(\bar{x})}(t, x) = \int_t^T \mathcal{G}_1^{x,(\bar{x})}(t, s) u_0^{(\bar{x})}(s, x) \, ds \\
 &= \sum_{|\nu| \leq 2} \int_t^T a_{\nu,1}^{(\bar{x})} \left(s, x + \mathbf{m}^{(\bar{x})}(t, s) + \mathbf{C}^{(\bar{x})}(t, s) \nabla_x \right) \, ds D_x^\nu u_0^{(\bar{x})}(s, x)
 \end{aligned}$$

(by (3.13) with $n = 1$)

$$(6.6) \quad = \sum_{|\nu| \leq 2} \int_t^T \langle \nabla_x a_\nu(s, \bar{x}), x - \bar{x} + \mathbf{m}^{(\bar{x})}(t, s) + \mathbf{C}^{(\bar{x})}(t, s) \nabla_x \rangle ds D_x^\nu u_0^{(\bar{x})}(t, x).$$

Therefore, we obtain

$$\left| D_x^\beta u_1^{(\bar{x})}(t, x) \right| \leq I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &= \sum_{|\nu| \leq 2} \int_t^T |\nabla_x a_\nu(s, \bar{x})| ds |x - \bar{x}| \left| D_x^{\beta+\nu} u_0^{(\bar{x})}(t, x) \right|, \\ I_2 &= \sum_{|\nu| \leq 2} \int_t^T |\nabla_x a_\nu(s, \bar{x})| \left| \mathbf{m}^{(\bar{x})}(t, s) \right| ds \left| D_x^{\beta+\nu} u_0^{(\bar{x})}(t, x) \right|, \\ I_3 &= \sum_{|\nu| \leq 2} \int_t^T |\nabla_x a_\nu(s, \bar{x})| \left| \mathbf{C}^{(\bar{x})}(t, s) \right| ds \left| \nabla_x D_x^{\beta+\nu} u_0^{(\bar{x})}(t, x) \right|, \\ I_4 &= \sum_{\substack{|\nu| \leq 2 \\ |\delta| \leq |\beta| - 1}} \int_t^T |\nabla_x a_\nu(s, \bar{x})| ds \left| D_x^{\nu+\delta} u_0^{(\bar{x})}(t, x) \right|. \end{aligned}$$

Now, since $a_\nu \in C_b^{1,1}$, by Proposition 6.2 we have

$$I_1 \leq C_1 \sum_{|\nu| \leq 2} |x - \bar{x}| (T - t)^{\frac{2 + \min\{k - |\beta| - |\nu|, 0\}}{2}} \leq C_2 \cdot (T - t)^{\frac{1+k-|\beta|}{2}} \frac{|x - \bar{x}|}{\sqrt{T - t}}.$$

Moreover, since $a_\nu \in C_b^{1,1}$ and $|\mathbf{m}^{(\bar{x})}(t, s)| \leq C_3(s - t)$, we have by Proposition 6.2 that

$$I_2 \leq C_4 \sum_{|\nu| \leq 2} (T - t)^{2 + \frac{\min\{k - |\beta| - |\nu|, 0\}}{2}} \leq C_5 \cdot (T - t)^{\frac{2+k-|\beta|}{2}}.$$

Next, since $a_\nu \in C_b^{1,1}$ and $|\mathbf{C}^{(\bar{x})}(t, s)| \leq C_6 \cdot (s - t)$, we have by Proposition 6.2 that

$$I_3 \leq C_7 \sum_{|\nu| \leq 2} (T - t)^{2 + \frac{\min\{k - 1 - |\beta| - |\nu|, 0\}}{2}} \leq C_8 \cdot (T - t)^{\frac{1+k-|\beta|}{2}}.$$

Finally, we have the term appearing when D_x^β applies to $x - \bar{x}$ in (6.6). Using the same arguments as above, we obtain

$$I_4 \leq C_9 \sum_{|\nu| \leq 2} (T - t)^{\frac{2 + \min\{k + 1 - |\beta| - |\nu|, 0\}}{2}} \leq C_{10} \cdot (T - t)^{\frac{1+k-|\beta|}{2}}.$$

Using all the above estimates, one deduces (6.5) for $n = 1$. The general case can be proved by analogous arguments, using repeatedly the general expression of $u_n^{(\bar{x})}$ provided by Theorem 3.2 and the estimates of Proposition 6.2. We omit the details for brevity. \square

We are now in position to prove Theorem 3.4.

Proof of Theorem 3.4. In this proof, $\{C_i\}_{i \geq 1}$ denote some positive constants dependent only on M, N, \bar{T} , and $\|\varphi\|_{C_b^{k-1,1}}$. By Lemma 6.3, we have

$$u - \bar{u}_N = \sum_{n=0}^N I_n, \quad I_n(t, x) = \int_t^T \int_{\mathbb{R}^d} \Gamma(t, x; s, \xi) \left(\mathcal{A} - \sum_{i=0}^n \mathcal{A}_i^x \right) u_{N-n}^x(s, \xi) \, d\xi ds.$$

Moreover, $I_n = I_{n,1} + I_{n,2}$ with (cf. (6.1))

$$I_{n,1}(t, x) = \sum_{|\alpha| \leq 1} \int_t^T \int_{\mathbb{R}^d} \left(a_\alpha(s, \xi) - \mathbf{T}_{x,n}^{a_\alpha(s, \cdot)}(\xi) \right) \Gamma(t, x; s, \xi) D_\xi^\alpha u_{N-n}^x(s, \xi) \, d\xi ds,$$

$$I_{n,2}(t, x) = \sum_{|\alpha|=2} \int_t^T \int_{\mathbb{R}^d} \left(a_\alpha(s, \xi) - \mathbf{T}_{x,n}^{a_\alpha(s, \cdot)}(\xi) \right) \Gamma(t, x; s, \xi) D_\xi^\alpha u_{N-n}^x(s, \xi) \, d\xi ds.$$

Now by Lemma 6.4, we have

$$\begin{aligned} |I_{n,1}(t, x)| &\leq C_1 \sum_{|\alpha| \leq 1} \int_t^T \int_{\mathbb{R}^d} |\xi - x|^{n+1} \Gamma(t, x; s, \xi) (T - s)^{\frac{N-n-|\alpha|+k}{2}} \\ &\quad \left(1 + (T - s)^{-\frac{N-n}{2}} |x - \xi|^{N-n} \right) \, d\xi ds \\ &\leq C_2 \sum_{|\alpha| \leq 1} \int_t^T \left((T - s)^{\frac{N-n+|\alpha|+k}{2}} (s - t)^{\frac{n+1}{2}} + (T - s)^{\frac{-|\alpha|+k}{2}} (s - t)^{\frac{N+1}{2}} \right) \\ &\quad \int_{\mathbb{R}^d} \Gamma^{M+\varepsilon}(t, x; s, \xi) \, d\xi ds \\ &\leq C_3 \cdot (T - t)^{\frac{N+k+2}{2}}, \end{aligned}$$

where we have used Lemma 6.1 and the identity

$$\int_t^T (T - s)^n (s - t)^k \, ds = \frac{\Gamma_E(k + 1) \Gamma_E(n + 1)}{\Gamma_E(k + n + 2)} (T - t)^{k+n+1}$$

with Γ_E denoting the Euler Gamma function. To estimate $I_{n,2}$, we first integrate by parts and obtain

$$\begin{aligned} I_{n,2}(t, x) &= - \sum_{|\alpha_1|=1} \sum_{|\alpha_2|=1} \int_t^T \int_{\mathbb{R}^d} D_\xi^{\alpha_1} \\ &\quad \left(\left(a_{\alpha_1+\alpha_2}(t, \xi) - \mathbf{T}_{x,n}^{a_{\alpha_1+\alpha_2}(t, \cdot)}(\xi) \right) \Gamma(t, x; s, \xi) \right) D_\xi^{\alpha_2} u_{N-n}^x(s, \xi) \, d\xi ds. \end{aligned}$$

Using the same arguments as above, one can show that

$$|I_{n,2}(t, x)| \leq C_4 \cdot (T - t)^{\frac{N+k+1}{2}}.$$

Finally, estimate (3.19) is obtained by a straightforward modification of the proof of (3.18) for $k = 0$, by means of the application of Lemma 6.1 and the Chapman-Kolmogorov equation. We omit the details for simplicity. \square

7. Proof of Theorem 3.6: Error bounds for large times. In agreement with the hypothesis of Theorem 3.6, throughout this section we will assume $N \geq 1$. The proof of Theorem 3.6 is based on the Chapman–Kolmogorov identity (5.1) and on the following classical Schauder estimate (see, for instance, Friedman (1964), Chapter 3).

LEMMA 7.1. *Let u be the solution of problem (1.1) under Assumption 2.1. Then for $0 \leq k \leq 2$, we have*

$$\|u(t, \cdot)\|_{C_b^{k-1,1}(\mathbb{R}^d)} \leq C \|\varphi\|_{C_b^{k-1,1}(\mathbb{R}^d)}, \quad 0 \leq t \leq T \leq \bar{T},$$

where C is a positive constant that depends only on M and \bar{T} .

Proof of Theorem 3.6. In this proof, $\{C_i\}_{i \geq 1}$ denote some positive constants that depend only on M, N, \bar{T} , and $\|\varphi\|_{C_b^{k-1,1}}$. By an iterative use of (5.1), the Chapman–Kolmogorov identity, we have

$$u(t_0, x_0) = \int_{\mathbb{R}^{md}} \prod_{i=1}^m \Gamma(t_{i-1}, x_{i-1}; t_i, x_i) \varphi(x_m) dx_m \cdots dx_1, \quad t_0 < T, \quad x_0 \in \mathbb{R}^d.$$

Then, by definition (3.21), we obtain

$$(7.1) \quad u - \bar{u}_{N,m} = \sum_{j=1}^m I_j,$$

where

$$\begin{aligned} I_j(t_0, x_0) &= \int_{\mathbb{R}^{md}} \prod_{i=1}^{j-1} \bar{\Gamma}_N(t_{i-1}, x_{i-1}; t_i, x_i) (\bar{\Gamma}_N - \Gamma)(t_{j-1}, x_{j-1}; t_j, x_j) \\ &\quad \times \prod_{i=j+1}^m \Gamma(t_{i-1}, x_{i-1}; t_i, x_i) \varphi(x_m) dx_m \cdots dx_1 \\ &= \int_{\mathbb{R}^{(j-1)d}} \prod_{i=1}^{j-1} \bar{\Gamma}_N(t_{i-1}, x_{i-1}; t_i, x_i) \\ &\quad \int_{\mathbb{R}^d} (\bar{\Gamma}_N - \Gamma)(t_{j-1}, x_{j-1}; t_j, x_j) u(t_j, x_j) dx_j dx_{j-1} \cdots dx_1, \end{aligned}$$

where we have used Fubini’s theorem and the Chapman–Kolmogorov identity. Now by Lemma 7.1 and Theorem 3.4, we obtain

$$\left| \int_{\mathbb{R}^d} (\bar{\Gamma}_N - \Gamma)(t_{j-1}, x_{j-1}; t_j, x_j) u(t_j, x_j) dx_j \right| \leq C_1 \delta_m^{\frac{N+k+1}{2}}.$$

Thus, we have

$$\begin{aligned} |I_j(t_0, x_0)| &\leq C_1 \delta_m^{\frac{N+k+1}{2}} \int_{\mathbb{R}^{(j-1)d}} \prod_{i=1}^{j-1} |\bar{\Gamma}_N(t_{i-1}, x_{i-1}; t_i, x_i)| dx_{j-1} \cdots dx_1 \\ &\leq C_1 \delta_m^{\frac{N+k+1}{2}} \int_{\mathbb{R}^{(j-1)d}} \prod_{i=1}^{j-1} (|\bar{\Gamma}_N - \Gamma| + \Gamma)(t_{i-1}, x_{i-1}; t_i, x_i) dx_{j-1} \cdots dx_1 \\ &\leq C_1 \delta_m^{\frac{N+k+1}{2}} \int_{\mathbb{R}^{(j-1)d}} \prod_{i=1}^{j-1} (C_2 \delta_m^{\frac{N+1}{2}} \Gamma^{M+1} + \Gamma)(t_{i-1}, x_{i-1}; t_i, x_i) dx_{j-1} \cdots dx_1, \end{aligned}$$

where, in the last step, we used (3.19) in Theorem 3.4 with Γ^{M+1} being the funda-

mental solution of the heat-type operator (3.20) with $\varepsilon = 1$. Therefore, by applying repeatedly the properties

$$\int_{\mathbb{R}^d} \Gamma^{M+1}(t, x; s, y) dy = 1, \quad \int_{\mathbb{R}^d} \Gamma(t, x; s, y) dy \leq 1,$$

we obtain

$$|I_j(t_0, x_0)| \leq C_1 \delta_m^{\frac{N+k+1}{2}} \left(C_2 \delta_m^{\frac{N+1}{2}} + 1 \right)^{j-1}.$$

Eventually, since $N \geq 1$, we find by (7.1) that

$$|u(t, x) - \bar{u}_{N,m}(t, x)| \leq C_1 \left(C_2 \delta_m^{\frac{N+1}{2}} + 1 \right)^m m \delta_m^{\frac{N+k+1}{2}} \leq C_3 e^{C_2(T-t)\frac{N+1}{2}} \delta_m^{\frac{N+k-1}{2}},$$

which proves (3.22). \square

Acknowledgment. The authors would like to thank two anonymous referees, whose suggestions improved both the readability and mathematical quality of this manuscript.

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