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Geometric representation and algebraic formalization of musical structures

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Introduction

Despite a long historical relationship between mathematics and music, the research in Mathematical Music Theory is very recent. From a superficial perspective, mathematics is considered the world of rules and rationality, while music is the world of creativity and freedom. Is it really the case? A look at the history of both disciplines clearly shows that mathematics and music are deeply related and that the relationships between these two disciplines goes back to ancient times. One way to approach the problem of possible connections between mathematics and music is to look at the physical and acoustical aspects of musical structures, such as the Pythagorean scale. Although Pythagoras' school role in the development of this scale is debatable, the relations between the length of a string and the pitch of the note that is perceived when the string is played and the subsequent link between rational numbers and consonant music intervals were certainly discovered already in classical antiquity.

Although the use of mathematics to describe music from an acoustical point of view is simple and intuitive to understand, this is not the only way to look for possible connections between these two fields. By leaving the acoustical domain and taking into account a more conceptual level, which is the level of the compositional act, it appears that music is rich of rules and structures, which are well represented and formalized via mathematical concepts. This is the aim of music theory, whose theoretical constructions are very much like the grammatical rules that govern written language. The Mathematical Music Theory offers to the contemporary music theorist and musicologist the way to properly define and describe the different musical objects as well as the transformations between them in a way that can be useful for musical analysis and composition.

More specifically, the problems studied in this thesis are part of transformational theory, an area of Mathematical Music Theory born in the 1980s with the pioneering works by David Lewin [52] [53] and Guerino Mazzola [56] [57]. It is based on the use of mathematical group structure to define musical transformations. The use of algebra in music started with several music theorists because it provides a deeper insight into the concept of musical structures and processes.

In Chapter 1 we will give an introduction to Mathematical Music Theory, in particular to transformational theory. We will focus on one of the most important transformation groups, known as the neo-Riemannian PLR-group. This group is generated by the three musical operations P, L and R, which exchange major and minor triads moving a note by a semitone or a whole tone. These operations are interesting from a musical point of view, because they are an additional tool useful to describe the property of stepwise motion in a single voice known as *parsimonious voice leading*. Voice leading is the interplay of two or more musical lines that realize chord progressions, according to the principles of the common-practice and counterpoint. It is organized according to musical rules, and one of the most important in the common-practice period being to connect one chord to another by parsimonious movements.

The PLR-group is also interesting from a mathematical point of view: it acts on the set of 24 major and minor triads generating a group isomorphic to the dihedral group of order 24. Moreover, the three neo-Riemannian operations have graphic representations in terms of graphs and simplicial complexes, the most important of which is the *Tonnetz*: a 2-dimensional simplicial complex which tiles the Euclidean plane with triangles, where 0-simplices represent pitch-classes, and 2-simplices identify major and minor triads. Given a triangle, each ‘’edge-flip” describes one the three neo-Riemannian operations. Considered as a graph, it is a note based-graph, and its dual is a chord-based graph known as *Chicken-wire Torus*, whose vertices represent major and minor triads, while edges correspond to the P, L and R operations. Since paths represent sequences through major and minor triads using P, L and R operations, the problem of finding all possible Hamiltonian cycles in the *Chicken-wire Torus* naturally arise. These cycles have been studied and classified by Giovanni Albini and Samuele Antonini, who provided the first list of these remarkable structures up to isomorphism[1]. At the same time, these classes of cycles are a useful compositional device, providing new interesting musical material for contemporary composition in art as well as popular music. Therefore, while algebra is useful to describe musical structures, geometric models have a double utility: they not only allow an easier visualization of musical structures which are useful in musical analysis, but they can also provide music-analytical tools for compositional applications.

In Chapter 2 we will present our historical investigation on the *Tonnetz* [19]. It is commonly said that the *Tonnetz* was introduced by Leonhard Euler in his 1739 *Tentamen novae theoriae musicae* [29] [20]. Indeed, a graph similar to the *Tonnetz* appeared in a the chapter with the aim to represent some intervals of the just intonation in a scheme. We observe that Euler did not define it as a graph, but instead used the term “figure”, nor did he call it *Tonnetz*. Another graph similar to the *Tonnetz*, called *Speculum musicum*, appeared in Euler's *De harmoniae veris principiis per speculum musicum repraesentatis* [30] [20]. After some observations on these two graphs, we will present some diagrams and tables similar to the *Speculum musicum* found in some works by music theorists of the XIX$^{\text{th}}$ century, such as Carl Ernst Naumann, Arthur von Oettingen and Hugo Riemann. We will continue the investigations until the 1980s and 1990s, when the *Tonnetz* was considered in the twelve-tone equal temperament and extended as an infinity graph. Moreover, following and integrating Dmitri Tymoczko's ideas [69], we will present some different generalized *Tonnetz* introduced in the last decades.

There are several studies on algebraic formalizations and geometric visualizations of transformations on major and minor triads. But, in addition to triads, seventh chords are often used in the music. In Chapter 3 we will present a generalization of the PLR-group for seventh chords. We have considered two different sets of sevenths of the twelve-tone equal temperament. The first set H includes the classical types of sevenths: dominant, minor, half-diminished, major and diminished. The second set H^* includes the sevenths with intervallic structure $[c_1, c_2, c_3, c_4]$ where $c_i \in \{1, 2, 3, 4\}$ and such that $\sum_{i=1}^{4} c_i = 12$, hence: dominant, minor, half-diminished, major, diminished, minor major, augmented major, augmented, dominant seventh flat five. In both cases, we will classify all the most parsimonious operations exchanging two types of sevenths, moving a single note by a semitone or a whole tone and fixing the other three notes. The most parsimonious operations acting on H are 17, those acting on H^* are 37. Finally, we will study the algebraic group generated by all these operations: it turns out to be isomorphic to $S_5 \ltimes \mathbb{Z}_{12}^4$ in the first case [17], to $S_9 \ltimes \mathbb{Z}_{12}^8$ in the second case.

Moreover, we will present a generalized *Chicken-wire Torus* for sevenths [18]. It is a chord-based graph associated to the parsimonious operations on the set of the 5 classical types of sevenths.

Although major and minor triads have a special place in music, they are not the only possible types of triads. Two more types, the diminished and the augmented triads, are also commonly used in music composition. In Chapter 4 we will extend the PLR-group for all these types of triads. Similarly to what done in the case of sevenths chords, we will classify the most parsimonious operations acting on the set of the four types of triads and we will prove that the group generated by them is $S_4 \ltimes \mathbb{Z}_{12}^3$. We will also construct the graphs associated to the parsimonious operations on major, minor and augmented triads, and to those on major, minor and diminished triads.

Finally, in the last chapter, we have studied operations among sevenths and triads, in which it is possible to add a note (from a triad to a seventh) or to delete a note (from a seventh to a triad). We will introduce a general framework in which to include the already known parsimonious operations among triads, the parsimonious operations among sevenths, and also parsimonious operations among seventh and triads. As a special case, we will consider the set of the 5 classical types of sevenths and the 4 types of triads. We prove that the ST-group, generated by the 17 most parsimonious operations among sevenths, the 13 most operations among triads and the new 12 operations among sevenths and triads, is isomorphic to $S_9 \ltimes \mathbb{Z}_{12}^8$.

Chapter 1

Introduction to Mathematical Music Theory

1.1 Musical set theory

One of the most important aspects that makes a musical piece interesting is the ability of the composer to control the interaction among the different melodic lines creating the harmony, which is traditionally known as *voice leading*. It is in fact the interplay of two or more musical lines that realizes chord progressions, according to the principles of common-practice and counterpoint. This corresponds to the two main possible ways of analyzing a musical score, respectively horizontally (or melodically) and vertically (or harmonically).

We recall that the largest part of the Western music from the XVII$^{\text{th}}$ to the XIX$^{\text{th}}$ century is based on four-part harmony, which means that every chord in the progression contain four tones. Therefore there are four melodic lines, conventionally known with the names of the four voices of the chant: soprano, alto, tenor, bass (see Fig 1.1).

Figure 1.1: An example of voice leading. At the top a piece for piano. Below the decomposition of the same piece into its 4 melodic lines accompanied by the chords notation.

Voice leading is organized according to musical rules, and music theory also deals with the study of these compositional strategies. We might say that music theory is fundamental for composers as grammar is for poets. The Western music of this era – also known as *common practice period* – was tonal. Music written within the tonal system is generally analyzed by defining a certain note as the primary or "tonic". The other notes are subservient to the tonic and in a strict hierarchy: the "dominant" (or fifth degree), the "sub-dominant" (or fourth degree) and so on.

In the late XIX$^{\text{th}}$ century composers began to investigate new musical possibilities abandoning tonal music that had characterized the XVII$^{\text{th}}$, XVIII$^{\text{th}}$ and XIX$^{\text{th}}$ century music. Unlike the tonal system, in post-tonal music, composers moved away from a hierarchy among tones. This is impersonated by several composers such as Richard Wagner (1813–1883), especially in his *Tristan und Isolde*, composed between 1857 and 1859. Another important and famous example is *La Fille aux cheveux de lin* (1909–1910) by Claude Debussy (1862–1918). The ways out were many. Some composers, such as Mussorgsky, Bartok, Debussy and Stravinsky, moved away from the tonal system inspired by the music of their countries of origin. On the contrary, other composers such as Schönberg, Webern and Berg, came to a complete break with the past and embraced complete atonality. In atonal compositions, the hierarchy among tones focusing on a central tone is not used. All the notes of the chromatic scale are used with functions that are independent from each other.

In tonal music, the properties of musical elements such as chords are fully described by the traditional music theory. But the latter was not able to provide appropriate tools to analyze and describe new post-tonal musical system. The role and the structural meaning of certain chords and chord sequences are different; the harmonic vocabulary of post-tonal music is much more varied and complex than that of tonal music. To define the rules and the significant relationships between the complex and heterogeneous musical structures of post-tonal music, music theorists used set theory, developed by Georg Cantor and others between 1874 and 1897, to extend traditional music theory. This idea emerged in the United States, after World War II, allowing the birth of musical set theory. This combination between the demands of compositional experimentation and the necessity of analytic interpretation has decisively influenced both the birth and the developments of this type of music theory.

The main concepts and notions of musical set theory were firstly introduced around 1960 by the composer, director and music theorist Howard Hanson (1896–1981) for tonal music and by the musicologist and music theorist Allen Forte (1926–2014) in his articles *A Theory of Set-Complexes for Music* [32] and successively systematized in the book *The Structure of Atonal Music* [33] for atonal music. The concepts developed in musical set theory are very general and are applied to tonal and atonal music in any equal temperament [4][46].

Pitch-class and pitch-class set

The application of set theory in music presupposes a substitution of the traditional music notation with a numerical representation. The starting point is to consider two equivalence relations on the set of all possible pitches.

Enharmonic equivalence : two tones are enharmonic equivalent if they have the same pitch, but named differently. Example: $C\sharp \sim D\flat$.

Octave equivalence : two pitches $x, y \in \mathbb{R}^+$ are equivalent if their interval distance is an octave. We remind the reader that if the frequency of a tone is f, the frequency of the tone one octave higher is $2f$, while the frequency of the tone one octave lower is $\frac{1}{2}f$. Therefore octave equivalence is mathematically described as follows:

$$x \sim y \qquad \Leftrightarrow \qquad y = 2^n x \qquad n \in \mathbb{Z}$$

Since we want to define a numerical representation for the twelve-tone equal temperament, from these equivalence relations we obtain twelve equivalence classes called *pitch-classes*, one for each note of the chromatic scale.

Definition 1.1 (Pitch-class)**.** *A* ***pitch-class*** *is a maximal collection of pitches related by both the enharmonic and the octave equivalence.*

We usually map the twelve pitch-classes of the twelve-tone equal temperament to $\mathbb{Z}_{12}$, starting by mapping the pitch-class C to number 0. The *tone-clock*, also known as *musical clock* or *pitch-class space*, is a circular disposition of the twelve pitch-classes and the elements of $\mathbb{Z}_{12}$ representing them bijectively. The pitch-classes are organized along the circle at a distance of semitone, each pitch-class is labeled with a number from 0 (in the position of the 12 in the clock) to 11, starting from $C = 0$ (see Fig. 1.2). In musical literature, another convention uses t (ten) for $B_\flat$ and e (eleven) for B.

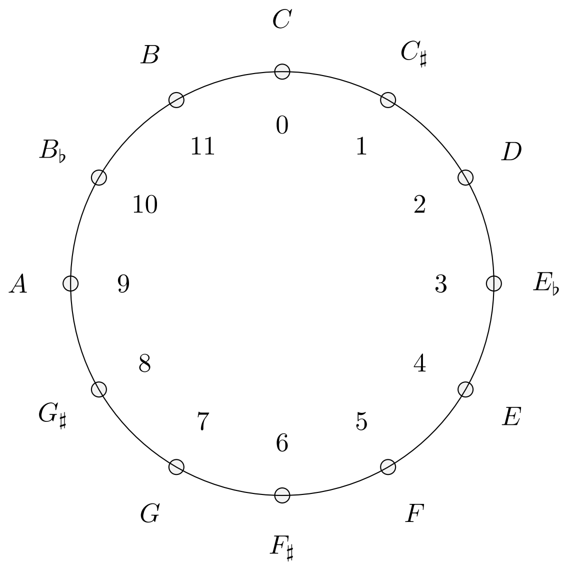


Figure 1.2: The musical clock

More generally, for any musical scale of an equal temperament constituted by n notes we can define similar equivalence relations, obtaining n pitch-classes and a bijection with $\mathbb{Z}_n$. The traditional diatonic theory has been generalized by Franck Jedrzejewski in this framework [47].

On the musical clock, we order the pitch-classes adopting the clockwise direction. We define the distance between two pitch-classes s, t as the minimum number $d(s, t)$ of steps required to reach one pitch-class from the other in clockwise or anticlockwise direction.

Like in traditional music theory where the distance between two notes is known as an *interval*, in musical set theory the distance between two pitch-classes is known as a *pitch-class interval* or an *interval class*. Two different types of interval classes are distinguished: ordered and unordered. Given two pitch-classes $x, y \in \mathbb{Z}_n$, their ordered interval class is $i\langle x, y\rangle = y - x \pmod{n}$. Their unordered interval class $i(x, y)$ is defined as the smallest between $i\langle x, y\rangle$ and $i\langle y, x\rangle$.

Example 1.2. *The ordered interval class between* 6 *and* 2 *in* $\mathbb{Z}_{12}$ *is* $i\langle 6,2\rangle = 2-6 = -4 \equiv 8$ *(mod 12). If we interchange the order of the pitch-classes their ordered interval class becomes* $i\langle 2,6\rangle = 4$ *(mod 12).*

The unordered interval class between 6 *and* 2 *is* $i(6,2) = min\{i\langle 6,2\rangle, i\langle 2,6\rangle\} = min\{8,4\} = 4$.

The main elements of voice leading are the chords. A chord is a set of two or more pitches played simultaneously. In musical set theory, a chord can be represented as a pitch-class set.

Definition 1.3 (Pitch-class set)**.** *A* ***pitch-class set*** *is an unordered collection of pitch-classes without replication.*

Following the usual convention, we will list the elements of pitch-class sets within curly brackets. Some theorists use angle brackets to denote ordered pitch-class sets.

Example 1.4.

$$C_M = \{0,4,7\} = \langle 0,4,7\rangle$$
$$A_m = \{9,0,4\} = \langle 9,0,4\rangle$$

In this work, we will use the following notation defined in [17].

Definition 1.5. *A* ***cyclicly marked chord*** $[\underline{x_1}, x_2, \ldots, x_n]$ *is a chord constituted by the* n *pitch-classes* $x_1, x_2, \ldots, x_n \in \mathbb{Z}_{12}$ *in the chosen order, where the underlined pitch-class corresponds to the root of the chord; we consider equivalent the cyclicly marked chords* $[\underline{x_1}, x_2, \ldots, x_n], [x_2, \ldots, x_n, \underline{x_1}], \ldots, [x_n, \underline{x_1}, \ldots, x_2]$.[1]

Example 1.6.

$$C_M = [\underline{0},4,7] = [7,\underline{0},4] = [4,7,\underline{0}]$$
$$A_m = [\underline{9},0,4] = [4,\underline{9},0] = [0,4,\underline{9}]$$

The structure of a set of pitch classes can be summarized using an interval vector.

Definition 1.7 (Interval vector)**.** *Given a pitch-class set* $A \subset \mathbb{Z}_n$*, its* ***interval vector*** $\mathbf{iv}(A) = \langle x_1\, x_2 \ldots x_k\rangle$ *is an array of natural numbers where each digit* $x_i = \#\{\{s,t\} \subset A | d(s,t) = i, s \neq t\}$ *represents the number of occurrences of each interval class that appears in the set.*

In the twelve-tone equal temperament, an interval vector has six digits, because the possible intervals are six: minor seconds/major sevenths, major seconds/minor sevenths, minor thirds/major sixths, major thirds/minor sixths, perfect fourths/perfect fifths, tritones.

Example 1.8. *The interval vector of* $C_M = [\underline{0},4,7]$ *is* $\langle 0\,0\,1\,1\,1\,0\rangle$.

There is another vector useful to represent a chord considering its intervals.

Definition 1.9 (Intervallic structure)**.** *Given a cyclicly marked chord* $[\underline{x_1}, x_2, \ldots, x_n]$, $x_i \in \mathbb{Z}_{12}$ *for each* $i \in \{1, \ldots, n\}$*, its* ***intervallic structure*** $(x_2 - x_1, x_3 - x_2, \ldots, x_1 - x_n)$ *is a vector where each element represents the interval class between two successive pitch-classes of the chord.*

Triads and sevenths

The most used chords, at least from the perspective of Western musical tradition, are triads (major and minor) and some seventh chords. A triad is a chord of three tones (3-chord) where the intervals between adjacent

[1]The root is the element of the chord associated to his name. For example the root of the C major chord is the pitch-class $C = 0$.

tones comprise a minor third interval (formed by three semitones) or a major third (formed by four semitones). A major triad is obtained from the superposition between a major third (bottom) and a minor third (top). Conversely, a minor triad has a minor third followed by a major third. Considering a triad as a superposition of thirds, their members are: the root (the lowest tone), the third (the middle tone, forming a third with the root), and the fifth (the higher tone, forming a fifth with the root). Using Definition 1.5, we can mathematically define them respectively as follows:

$$[\underline{x}, x+4, x+7] \pmod{12}, \quad x \in \mathbb{Z}_{12}, \quad \text{intervallic structure: } (4,3,5) \qquad \text{(major triad)}$$
$$[\underline{x}, x+3, x+7] \pmod{12}, \quad x \in \mathbb{Z}_{12}, \quad \text{intervallic structure: } (3,4,5) \qquad \text{(minor triad)}$$

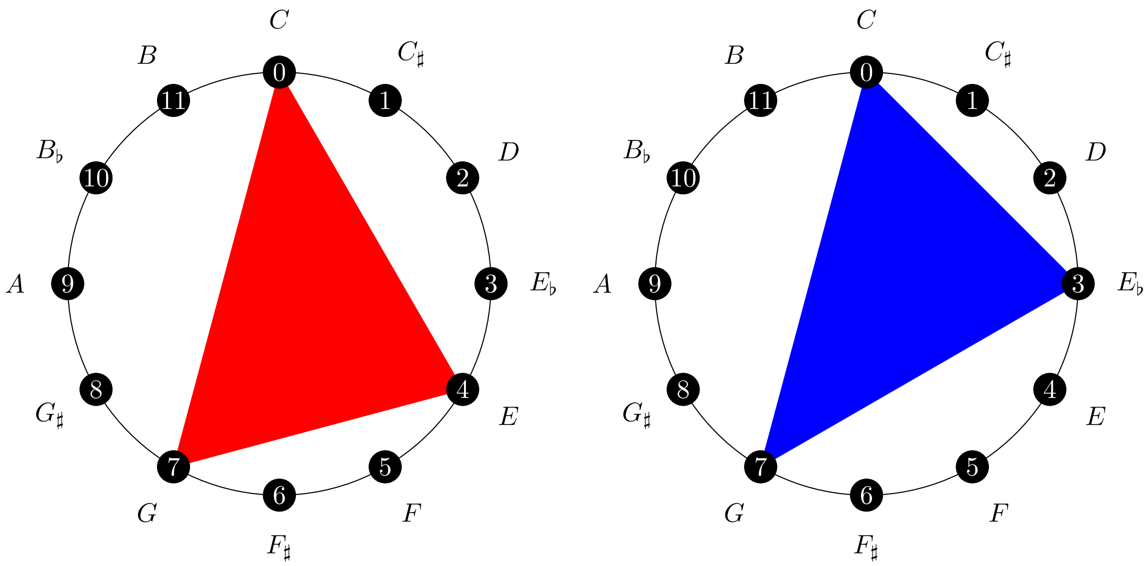


Figure 1.3: On the left C major triad, on the right C minor triad.

Other chords often used are the sevenths. A seventh is a chord of four tones (4-chord) obtained by overlapping three intervals of third. When stacked in thirds, the members of a seventh from the lowest to the highest tone are: the root, the third, the fifth and the seventh. In our work, we will consider the following types (see Chapter 3):

$$[\underline{x}, x+4, x+7, x+10] \pmod{12}, \quad x \in \mathbb{Z}_{12}, \quad \text{intervallic structure: } (4,3,3,2) \qquad \text{(dominant seventh)}$$
$$[\underline{x}, x+3, x+7, x+10] \pmod{12}, \quad x \in \mathbb{Z}_{12}, \quad \text{intervallic structure: } (3,4,3,2) \qquad \text{(minor seventh)}$$
$$[\underline{x}, x+3, x+6, x+10] \pmod{12}, \quad x \in \mathbb{Z}_{12}, \quad \text{intervallic structure: } (3,3,4,2) \qquad \text{(half-diminished seventh)}$$
$$[\underline{x}, x+4, x+7, x+11] \pmod{12}, \quad x \in \mathbb{Z}_{12}, \quad \text{intervallic structure: } (4,3,4,1) \qquad \text{(major seventh)}$$
$$[\underline{x}, x+3, x+6, x+9] \pmod{12}, \quad x \in \mathbb{Z}_{12}, \quad \text{intervallic structure: } (3,3,3,3) \qquad \text{(diminished seventh)}$$
$$[\underline{x}, x+3, x+7, x+11] \pmod{12}, \quad x \in \mathbb{Z}_{12}, \quad \text{intervallic structure: } (3,4,4,1) \qquad \text{(minor major seventh)}$$
$$[\underline{x}, x+4, x+8, x+11] \pmod{12}, \quad x \in \mathbb{Z}_{12}, \quad \text{intervallic structure: } (4,4,3,1) \qquad \text{(augmented major seventh)}$$
$$[\underline{x}, x+4, x+8, x+10] \pmod{12}, \quad x \in \mathbb{Z}_{12}, \quad \text{intervallic structure: } (4,4,2,2) \qquad \text{(augmented seventh)}$$
$$[\underline{x}, x+4, x+6, x+10] \pmod{12}, \quad x \in \mathbb{Z}_{12}, \quad \text{intervallic structure: } (4,2,4,2) \qquad \text{(dominant seventh flat five)}$$

For example, by taking the note $C = 0$ as the root $\underline{x}$ of the chord, we have the following chords expressed with the usual music notation:

$$\begin{aligned}
C^7 &= [\underline{0}, 4, 7, 10] \quad (C \text{ dominant seventh})\\
C_m &= [\underline{0}, 3, 7, 10] \quad (C \text{ minor seventh})\\
C^{ø} &= [\underline{0}, 3, 6, 10] \quad (C \text{ half-diminished seventh})\\
C^{\Delta} &= [\underline{0}, 4, 7, 11] \quad (C \text{ major seventh})\\
C^{o} &= [\underline{0}, 3, 6, 9] \quad (C \text{ diminished seventh})\\
C_m^{\Delta} &= [\underline{0}, 3, 7, 11] \quad (C \text{ minor major seventh})\\
C_+^{\Delta} &= [\underline{0}, 4, 8, 11] \quad (C \text{ augmented major seventh})\\
C_+^{7} &= [\underline{0}, 4, 8, 10] \quad (C \text{ augmented seventh})\\
C^{7\flat 5} &= [\underline{0}, 4, 6, 10] \quad (C \text{ dominant seventh flat five})
\end{aligned}$$

1.2 Transformational theory

Changing the focus from musical objects to the relations between them, transformational theory was developed since the 1980s. Its main idea is based on the use of group-theoretical concepts to define musical objects and transformations. Historically, this approach was introduced by the music theorist David Lewin in the late 1980s and systematized in his book *Generalized Musical Intervals and Transformations* [53].

The GIS structure

One of the most important examples of transformational construction is the notion of GIS.

Definition 1.10 (Generalized Interval System (GIS))**.** *A* ***Generalized Interval System (GIS)*** *is an ordered triplet* $(S, IVLS, int)$*, where* S *is a set of musical elements,* $IVLS$ *is a mathematical group called the* interval group*, and* int *is a function* $int\colon S \times S \to IVLS$*, such that:*

$$int(r,s)int(s,t) = int(r,t) \qquad \forall r,s,t \in S \tag{1.1}$$

$$\forall s \in S, \forall i \in IVLS, \exists! t \in S \text{ s.t. } int(s,t) = i. \tag{1.2}$$

Proposition 1.11. *Let* $e \in IVLS$ *be the neutral element. In any* GIS*, we have* $int(s,s) = e$ *and* $int(t,s) = int(s,t)^{-1}$ *for every* $s,t \in S$*.*

Proof. From 1.1 we have $int(s,s)int(s,s) = int(s,s)$, hence $int(s,s) = e$. Moreover $int(t,s)int(s,t) = int(t,t) = e \Rightarrow int(t,s) = int(s,t)^{-1}$. □

From the last proposition it is clear that GIS's are simply transitive group actions.

Example 1.12. $(\mathbb{Z}_{12}, (\mathbb{Z}_{12}, +), int)$*, where* $\mathbb{Z}_{12}$ *is the set of* 12 *pitch-classes,* $(\mathbb{Z}_{12}, +)$ *is the interval group, and* $int(s,t) := t - s$*, is a* GIS*. In fact, for every* $x, y, z \in \mathbb{Z}_{12}$*:*

$$int(x,y) + int(y,z) = (y-x) + (z-y) = z - x = int(x,z),$$

hence 1.1 is satisfied. Moreover given a pitch-class $s \in \mathbb{Z}_{12}$ *and given an interval* $i \in \mathbb{Z}_{12}$ *there exists a unique pitch-class* $t \in \mathbb{Z}_{12}$ *such that* $int(s,t) = i$*. In fact if there exists* $u \in S$ *such that* $int(s,u) = i$ *we would have* $int(s,t) = int(s,u)$*, then* $t - s = u - s \Rightarrow t = u$*.*

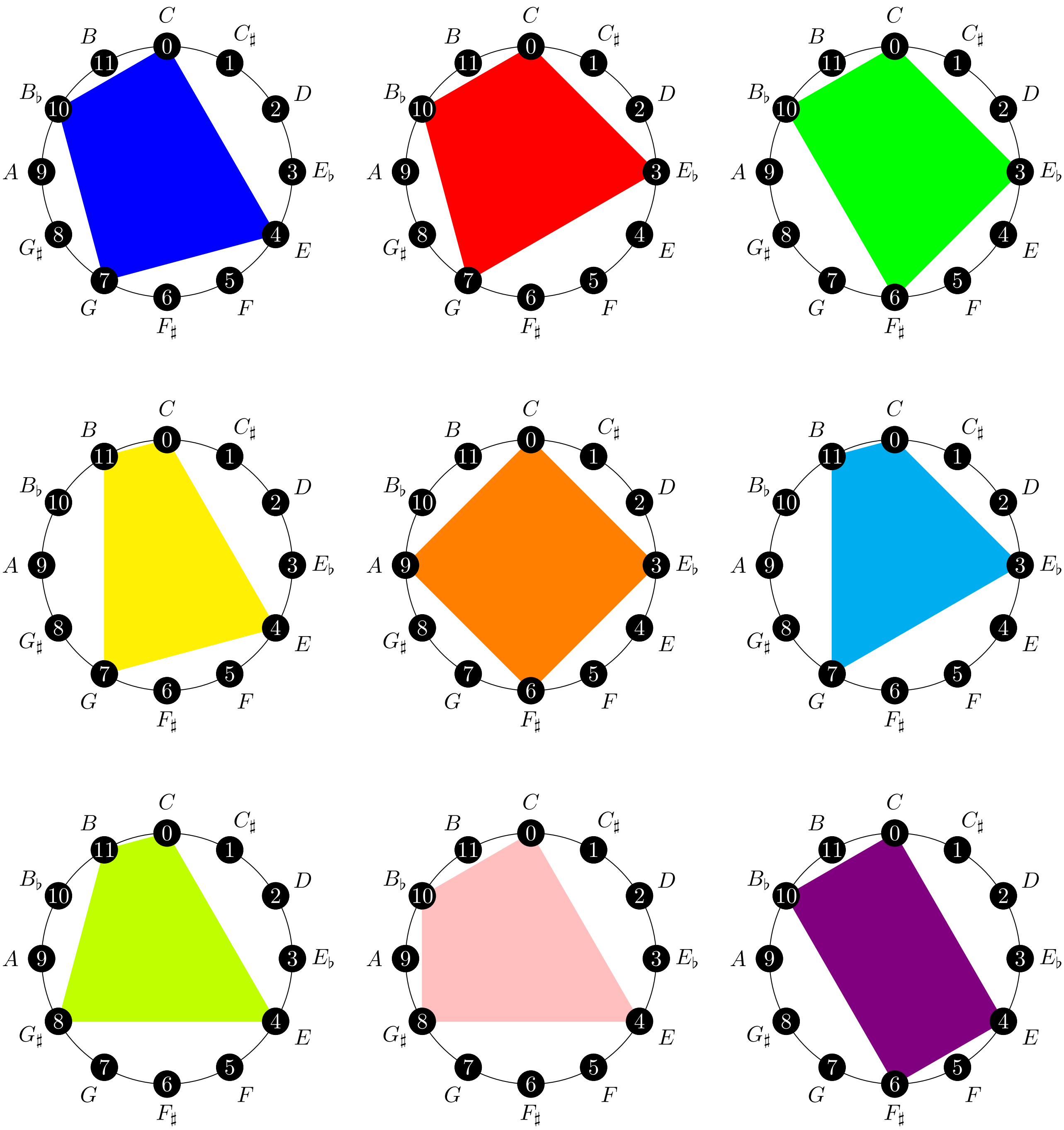


Figure 1.4: From top-left to bottom-right: dominant, minor, half-diminished, major, diminished, minor major, augmented major, augmented and dominant seventh flat five.

Transpositions and inversions

Many musical operations can be modeled as algebraic transformations. One of the most used musical operations is transposition.

Definition 1.13 (Transposition)**.** *A* ***transposition*** T_n *moves a set o pitch-classes in pitch by a constant interval* n*:*

$$\begin{aligned} T_n \colon \mathbb{Z}_{12}^k &\to \mathbb{Z}_{12}^k \\ [x_1, \ldots, x_k] &\mapsto [x_1 + n, \ldots, x_k + n] \end{aligned} \tag{1.3}$$

For musicians, being able to perform transpositions is a useful skill: if they are accompanying a singer and the musical piece is too high or low for the singer's voice, it is better to transpose it down or up in another key. Composers use transpositions very often in their works, especially in canons, fugues and for modulations in popular music.

Transpositions correspond to translations in the mathematical formula 1.3 and in the score: the given melody is shifted up or down maintaining the same tone structure (see Fig. 1.6). In the musical clock, transpositions correspond to rotations along the circle.

Example 1.14. *Given the* C *major triad* $[\underline{0}, 4, 7]$*, we apply* T_3*:*

$$T_3([\underline{0}, 4, 7]) = [0 + 3, 4 + 3, 7 + 3] = [\underline{3}, 7, 10]$$

that is $E_\flat$ *major triad.*

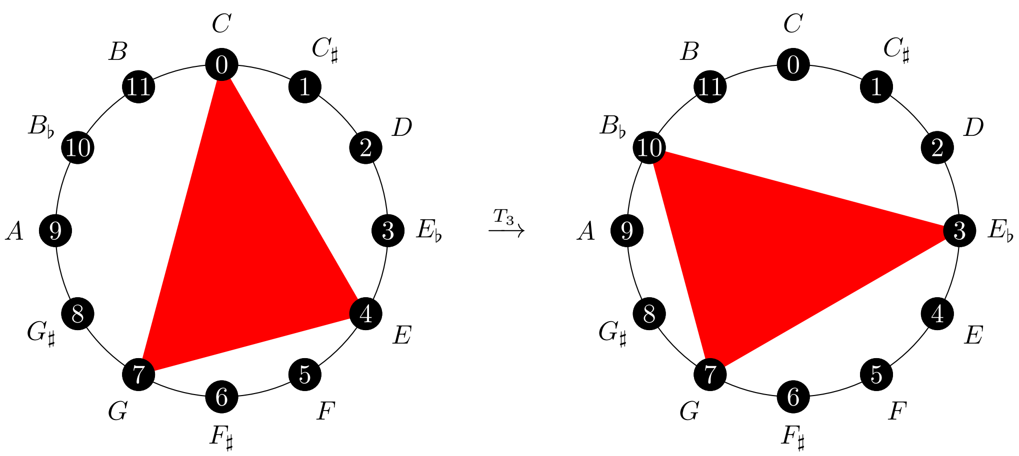


Figure 1.5: A minor third transposition of the C major triad in the musical clock.

Let $\mathcal{T} = \{T_0, T_1, \ldots, T_{11}\}$ be the set of all transpositions. The set $\mathcal{T}$ with the function composition

$$T_n \circ T_m = T_{n+m} \qquad m, n \in \mathbb{Z}_{12}$$

Figure 1.6: An example of transposition: a canonic passage in Beethoven's Symphony IV, first movement

is an abelian group. In fact:

$$\begin{aligned}
&\forall T_n, T_m \in \mathcal{T} \quad && T_n \circ T_m = T_{n+m} \in \mathcal{T} \quad && \text{(closure)}\\
&\forall T_n, T_m, T_l \in \mathcal{T} \quad && T_l \circ (T_n \circ T_m) = (T_l \circ T_n) \circ T_m \quad && \text{(associativity)}\\
&\forall T_n \in \mathcal{T}\, \exists T_0 \in \mathcal{T} \quad \text{s.t.} && T_n \circ T_0 = T_0 \circ T_n = T_n \quad && \text{(identity element)}\\
&\forall T_n \in \mathcal{T}\, \exists T_{-n} \in \mathcal{T} \quad \text{s.t.} && T_n \circ T_{-n} = T_{-n} \circ T_n = T_0 \quad && \text{(inverse element).}
\end{aligned}$$

More precisely $\mathcal{T}$ is an abelian cyclic group. Its generators are the elements T_n s.t. n and 12 are relatively prime, thus: T_1, T_5, T_7 and T_{11}.

Another operation often used in composition is inversion. Intuitively, an inversion of a melody is the melody in contrary motion.

Definition 1.15 (Inversion I)**.** *An* ***inversion*** *of a set of pitch-classes is the set of their inverses:*

$$\begin{aligned}
I \colon \mathbb{Z}_{12}^k &\to \mathbb{Z}_{12}^k \\
[x_1, \ldots, x_k] &\to [-x_1, \ldots, -x_k].
\end{aligned} \tag{1.4}$$

In the musical clock, the inversion I corresponds to a reflection through the diameter passing through 0 and 6, therefore the pitch-classes 0 and 6 remain fixed. But this is not the only inversion used in music: in many compositions, inversions are applied without fixed pitch-classes or with different fixed pitch-classes. From a mathematical point of view, they correspond to reflections through different diameters of the musical clock, identified combining I and a transposition T_n.

Definition 1.16 (Inversion I_n)**.** *An* ***inversion*** *I_n moves a set of pitch classes in contrary motion applying $T_n \circ I$:*

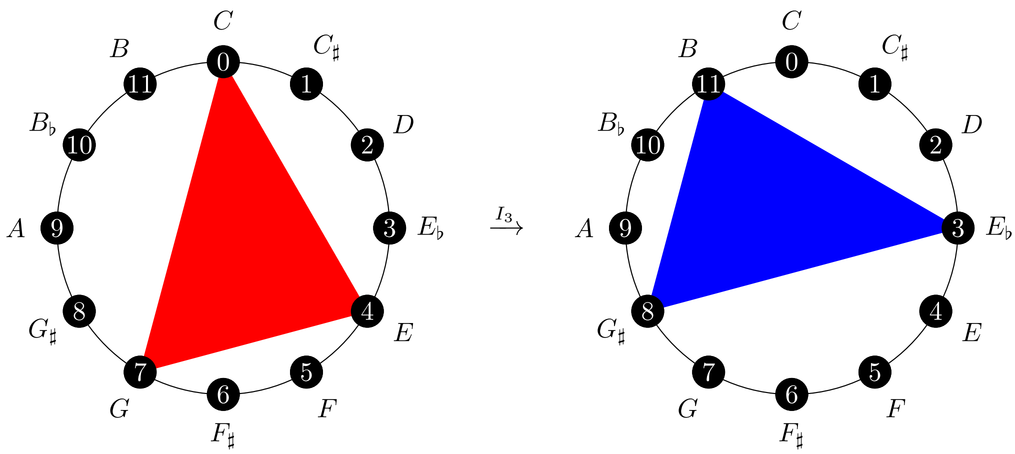
$$\begin{aligned}
I_n \colon \mathbb{Z}_{12}^k &\to \mathbb{Z}_{12}^k \\
[x_1, \ldots, x_k] &\to [-x_1 + n, \ldots, -x_k + n].
\end{aligned} \tag{1.5}$$

Since the transpositions $T_n \in \mathcal{T}$ are 12, we have 12 different inversions I_n corresponding to 12 different reflections in the musical clock. We observe that $I_0 = T_0 I = I$, something that we already know. The other inversions can be divided into two types. If n is even, I_n corresponds to a reflection through the diameter passing in $\frac{n}{2}$. If n is odd, I_n corresponds to a reflection through the diameter passing in the middle between $\frac{n-1}{2}$ and $\frac{n+1}{2}$.

Example 1.17. *Given the C major triad $[\underline{0}, 4, 7]$, we apply I_3:*

$$I_3([\underline{0}, 4, 7]) = T_3 I([\underline{0}, 4, 7]) = [0 + 3, -4 + 3, -7 + 3] = [3, 11, 8] = [\underline{8}, 11, 3],$$

that is, the $G_\sharp$ minor triad.



Remark 1.18. *Some remarks on transpositions and inversions.*

1. *By applying a transposition to a chord, we obtain a chord with the same intervallic structure. On the contrary, if we apply an inversion in general we obtain a chord whose intervallic structure is equal to the retrogradation (up to cyclic permutation) of the intervallic structure of the initial chord.*

2. *The set $\mathcal{T}$ of transpositions is a group. Conversely inversions do not form a group because the composition of two inversions is not an inversion.*

3. *Each inversion is an involution. Among transpositions, only T_0 and T_6 are involutions.*

Transpositions and inversions satisfy the following properties:

$$\begin{aligned} T_m \circ I_n &= I_{m+n} \pmod{12} \\ I_m \circ T_n &= I_{m-n} \pmod{12} \\ I_m \circ I_n &= T_{m-n} \pmod{12}. \end{aligned}$$

Theorem 1.19. *Inversions and transpositions act on the set S of all 24 major and minor triads generating the group T/I[2] that is isomorphic to the dihedral group D_{12} of order 24.*

Proof. We remind the reader that the dihedral group D_n of order $2n$ is the group of isometries of a regular n-gon in the plane, including rotations and reflections. Its representation is

$$\langle s, t | s^n = t^2 = Id, tst = s^{-1} \rangle.$$

[2] The notation does not represent the set-theoretical difference between T and I.

We observe that T_1 corresponds to a rotation of the musical clock by $\frac{1}{12}$ of a turn, while I_0 corresponds to a reflection of the musical clock through the axis passing through 0 and 6. Therefore $T_1^{12} = T_0 = Id$ and $I_0^2 = Id$. Moreover, for each $x \in S$,

$$(I_0 T_1 I_0)(x) = (I_0 I_1)(x) = I_0(-x+1) = x - 1 = T_{-1}(x) = T_1^{-1}(x).$$

Therefore the presentation of the group T/I is

$$\langle T_1, I_0 | T_1^{12} = I_0^2 = Id, I_0 T_1 I_0 = T_1^{-1} \rangle,$$

hence $T/I \simeq D_{12}$. □

Crans, Fiore and Satyendra proved that the action of the T/I-group is simply transitive [27]. Let $x \in S$ be a major triad. For each $n \in \{0, 1, \dots, 11\}$, $T_n(x)$ is a major triad, and $I_n(x)$ is a minor triad. In particular, we can obtain all the other major and minor triads applying T_n and I_n to x. Similarly, given a minor triad $y \in S$, computing all $T_n(y)$, we obtain all minor triads and computing $I_n(y)$, all major triads. More generally, given three triads $x, y, z \in S$, there exist $g_1, g_2 \in T/I$ such that $g_1 x = y$ and $g_2 x = z$. Then $g_1 g_2^{-1} z = y$. It is possible to show that $g = g_1 g_2^{-1}$ is unique using the stabilizer theorem.

Theorem 1.20 (Stabilizer theorem)**.** *Let G be a group that acts on a finite set S, let G_x be the stabilizer group of $x \in S$ and let $O(x)$ be the orbit of x. Then*

$$\frac{|G|}{|G_x|} = |O(x)|.$$

In our case S is the set of consonant triads, and $G = T/I$. By Theorem 1.19, $T/I \simeq D_{12}$, therefore it has 24 elements. Also $|O(z)|$ has 24 elements, hence $|G_z| = 1$. This proves uniqueness.

1.3 Neo-Riemannian operations and the PLR-group

David Lewin [52][53] rediscovered the P (parallel), L (leading-tone) and R (relative) operations introduced in the late XIX$^{\text{th}}$ century by the musicologist Hugo Riemann. With this rediscovery and the works of Brian Hyer [41][42] and Richard Cohn [24][25][26], a branch of transformational theory called *neo-Riemannian theory* appeared. As the name suggests, the term "neo-Riemannian" originates from Hugo Riemann.

The neo-Riemannian P, L and R operations

To define them, let $S = \{[\underline{x_1}, x_2, x_3] | x_1, x_2, x_3 \in \mathbb{Z}_{12}, x_2 = x_1 + 3 \text{ or } x_2 = x_1 + 4, x_3 = x_1 + 7\}$ be the set of all 24 major and minor triads. The transformations P, R, $L\colon S \to S$ are defined as follows.

Definition 1.21 (Parallel)**.** *The operation P maps a triad to its parallel. If the triad is major, it moves the third a semitone down. If the triad is minor, it moves the third a semitone up.*

$$P\colon [\underline{x}, x+4, x+7] \leftrightarrow [\underline{x}, x+3, x+7] \qquad (\textit{mod } 12). \tag{1.6}$$

Definition 1.22 (Relative)**.** *The operation R maps a triad to its relative. If the triad is major, it moves the fifth a whole tone up. If the triad is minor, it moves the root a whole tone down.*

$$R\colon [\underline{x}, x+4, x+7] \leftrightarrow [x, x+4, \underline{x+9}] \qquad (\textit{mod } 12). \tag{1.7}$$

Definition 1.23 (Leading-tone). *The operation L maps a major triad into a minor one which is a major third up and a minor triad into a major one which is a major third down. If the triad is major, L moves the root a semitone down, instead, if the triad is minor, L moves the fifth a semitone up.*

$$L\colon [\underline{x}, x+4, x+7] \leftrightarrow [x-1, \underline{x+4}, x+7] \qquad \textit{(mod 12)} \tag{1.8}$$

Example 1.24.

$$\begin{aligned}
P(C_M) &= P([\underline{0},4,7]) = [\underline{0},3,7] = C_m\\
P(C_m) &= P([\underline{0},3,7]) = [\underline{0},4,7] = C_M\\
R(C_M) &= R([\underline{0},4,7]) = [0,3,\underline{9}] = A_m\\
R(A_m) &= R([\underline{9},0,3]) = [7,\underline{0},4] = C_M\\
L(C_M) &= L([\underline{0},4,7]) = [11,\underline{4},7] = E_m\\
L(E_m) &= R([\underline{4},7,11]) = [4,7,\underline{0}] = C_M.
\end{aligned}$$

Remark 1.25. *Each neo-Riemannian operation*

- *exchanges major and minor triads;*
- *fixes two notes;*
- *moves a single note by a semitone or a whole-tone.*

There exist other ways to define algebraically such transformations. Crans, Fiore and Satyendra [27] define them using the inversion operation. Let $\langle x_1, x_2, x_3\rangle$ be a triad (minor or major) represented as in Table 1.1. Then:

$$P(\langle x_1,x_2,x_3\rangle) = I_{x_1+x_3}(\langle x_1,x_2,x_3\rangle) \tag{1.9}$$
$$R(\langle x_1,x_2,x_3\rangle) = I_{x_1+x_2}(\langle x_1,x_2,x_3\rangle) \tag{1.10}$$
$$L(\langle x_1,x_2,x_3\rangle) = I_{x_2+x_3}(\langle x_1,x_2,x_3\rangle). \tag{1.11}$$

The definition in terms of inversions is very interesting, because it emphasizes the relations between major and minor triads. Since inversions are involutions, it is immediate to observe that also P, L and R are involutions. But, as we observe in the table 1.1, using this definition, we represent major and minor triads as ordered pitch-class sets, with a different order: major chords are written from left to right, minor chords from right to left.

Example 1.26.

$$\begin{aligned}
P(C_M) &= I_7(\langle 0,4,7\rangle) = \langle 7,3,0\rangle = C_m\\
P(C_m) &= I_7(\langle 7,3,0\rangle) = \langle 0,4,7\rangle = C_M\\
R(C_M) &= I_4(\langle 0,4,7\rangle) = \langle 4,0,9\rangle = A_m\\
R(A_m) &= I_4(\langle 4,0,9\rangle) = \langle 0,4,7\rangle = C_M\\
L(C_M) &= I_11(\langle 0,4,7\rangle) = \langle 11,7,4\rangle = E_m\\
L(E_m) &= I_11(\langle 11,7,4\rangle) = \langle 0,4,7\rangle = C_M.
\end{aligned}$$

Major triads	Minor triads
$C_M = \langle 0,4,7\rangle$	$\langle 0,8,5\rangle = F_m$
$C_{\sharp M} = D_{\flat M} = \langle 1,5,8\rangle$	$\langle 1,9,6\rangle = F_{\sharp m} = G_{\flat m}$
$D_M = \langle 2,6,9\rangle$	$\langle 2,10,7\rangle = G_m$
$D_{\sharp M} = E_{\flat M} = \langle 3,7,10\rangle$	$\langle 3,11,8\rangle = G_{\sharp m} = A_{\flat m}$
$E_M = \langle 4,8,11\rangle$	$\langle 4,0,9\rangle = A_m$
$F_M = \langle 5,9,0\rangle$	$\langle 5,1,10\rangle = A_{\sharp m} = B_{\flat m}$
$F_{\sharp M} = G_{\flat M} = \langle 6,10,1\rangle$	$\langle 6,2,11\rangle = B_m$
$G_M = \langle 7,11,2\rangle$	$\langle 7,3,0\rangle = C_m$
$G_{\sharp M} = A_{\flat M} = \langle 8,0,3\rangle$	$\langle 8,4,1\rangle = C_{\sharp m} = D_{\flat m}$
$A_M = \langle 9,1,4\rangle$	$\langle 9,5,2\rangle = D_m$
$A_{\sharp M} = B_{\flat M} = \langle 10,2,5\rangle$	$\langle 10,6,3\rangle = D_{\sharp m} = E_{\flat m}$
$B_M = \langle 11,3,6\rangle$	$\langle 11,7,4\rangle = E_m$

Table 1.1: The 24 major and minor triads represented by Crans, Fiore and Satyendra.

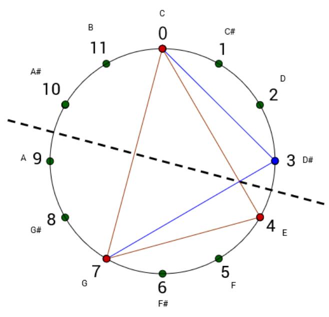


Figure 1.7: $P(C_M) = C_m$

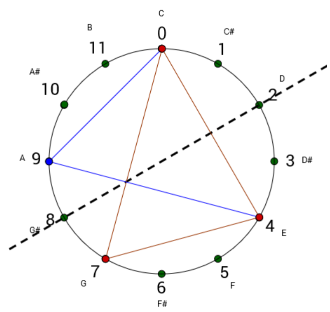


Figure 1.8: $R(C_M) = A_m$

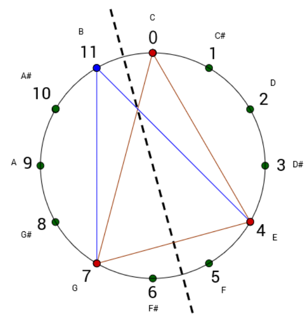


Figure 1.9: $L(C_M) = E_m$

Another way to define the P, L and R operations is proposed by Arnett and Barth [8] as follows:

$$P\colon M \leftrightarrow m \qquad P\colon [x, x+4, x+7] \leftrightarrow [x, x+3, x+7] \pmod{12} \tag{1.12}$$
$$R\colon M \leftrightarrow m-3 \qquad R\colon [x, x+4, x+7] \leftrightarrow [x, x+4, x+9] \pmod{12} \tag{1.13}$$
$$L\colon M \leftrightarrow m+4 \qquad L\colon [x, x+4, x+7] \leftrightarrow [x-1, x+4, x+7] \pmod{12} \tag{1.14}$$

where M represents a major triad and m a minor one. From this definition, we have observed that R and L can be defined as the composition of P and a transposition.

Proposition 1.27. *For any triad $[\underline{x_1}, x_2, x_3]$ we have*

$$R = \begin{cases} P \circ T_{-3} = T_{-3} \circ P & \text{if the triad is major} \\ P \circ T_3 = T_3 \circ P & \text{if the triad is minor} \end{cases}$$
$$L = \begin{cases} P \circ T_4 = T_4 \circ P & \text{if the triad is major} \\ P \circ T_{-4} = T_{-4} \circ P & \text{if the triad is minor.} \end{cases}$$

Proof. From the commutative property of the abelian group $(\mathbb{Z}_{12}, +)$ and the definition of transposition, it is easy to see that each transposition T_i commutes with the neo-Riemannian operations.
Let $X = [\underline{x}, x+4, x+7] \in S$ be a major triad. Then

$$(T_{-3} \circ P)(X) = T_{-3}([\underline{x}, x+3, x+7]) = [\underline{x+9}, x, x+4] = R(X)$$
$$(T_4 \circ P)(X) = T_4([\underline{x}, x+3, x+7]) = [\underline{x+4}, x+7, x+11] = L(X).$$

Let $Y = [\underline{y}, y+3, y+7]$ be a minor triad. Then

$$(T_3 \circ P)(Y) = T_3([\underline{y}, y+4, y+7]) = [\underline{y+3}, y+7, y+10] = R(X)$$
$$(T_{-4} \circ P)(Y) = T_{-4}([\underline{y}, y+4, y+7]) = [\underline{y+8}, y, y+3] = L(Y).$$

□

PLR-group

The neo-Riemannian operations generate a group known as PLR-group or neo-Riemannian group.

Lemma 1.28. *The neo-Riemannian operations L and R act on the set S of all 24 major and minor triads generating a group.*

Proof. First of all we observe that $P = RLRLRLR$. In fact, given a major triad $[\underline{x}, x+4, x+7] \in S$ we have

$$\begin{aligned} RLRLRLR([\underline{x}, x+4, x+7]) &= RLRLRL([x, x+4, \underline{x+9}]) = \\ &= RLRLR([x, \underline{x+5}, x+9]) = \\ &= RLRL([\underline{x+2}, x+5, x+9]) = \\ &= RLR([x+2, x+5, \underline{x+10}]) = \\ &= RL([x+2, \underline{x+7}, x+10]) = \\ &= R([\underline{x+3}, x+7, x+10]) = \\ &= ([x+3, x+7, \underline{x}]) = \\ &= P([\underline{x}, x+4, x+7]). \end{aligned}$$

We get a similar result applying $RLRLRLR$ to a minor triad.

Now we prove that L and R act on S generating a group. First of all, $L, R\colon S \to S$ and every operations obtained as compositions of them is well defined on S. The composition of functions is always associative.

We remind the reader that for every triad $X \in S$ the neo-Riemannian operations are involutions, therefore we have an identity function Id.

Finally, since L and R are involutions, each transformation can be decomposed by alternating the generating function L and R. To construct the inverse of each transformation defined as alternation of L and R, it is sufficient to invert the order of composition. □

Theorem 1.29. *The PLR-group is isomorphic to the dihedral group D_{12} of order 24.*

Proof. We observe that the elements of the PLR-group are described as $R(LR)^n$ or $(LR)^n$, $0 \le n \le 11$. In fact, from Lemma 1.28 we know that each transformation of the PLR-group is constructed by alternating the generating function L and R. Moreover the transformations described as $(LR)^k L$ are equivalent to $R(LR)^{11-k}$. Furthermore we observe that $(LR)^{12} = Id$, then we consider the transformations obtained by alternating L and R whose length is lower than 24. Therefore we have twelve elements of type $R(LR)^n$ and other twelve of type $(LR)^n$, hence the PLR-group has 24 elements. Finally we observe that we can describe the group with the following presentation

$$\langle (LR), L | (LR)^{12} = L^2 = Id, L(RL)L = RL \rangle$$

hence $PLR \simeq D_{12}$. □

Corollary 1.30. *The PLR-group acts simply transitively on the set S of 24 major and minor triads.*

Proof. From the Theorem 1.29, given any chord $X \in S$ its orbit has 24 elements. Since the PLR-group also has 24 elements, simple transitivity follows from the orbit-stabilizer theorem. □

Musical examples of neo-Riemannian operations

Although the neo-Riemannian operations highlight several interesting algebraic results, they were introduced for musical reasons, and they are useful for music analysis. Since each neo-Riemannian operation fixes two notes and moves one note by a semitone or a whole-tone, they are very useful to describe the property of stepwise motion in a single voice known as *parsimonious voice leading*. For instance, we consider the measures 270-278 of the first movement of Brahms' Concerto for violin and cello analyzed by Cohn in [24] (see Fig. 1.10).

Figure 1.10: Cohn's reduction of Brahms' Concerto for violin and cello, mm. 270-278.

First of all we observe an the alternation of major and minor triads. Moreover the root of the first four pairs of triads is the same. Finally, between the second and the third, the fourth and the fifth, and sixth and seventh pairs of triads the root is moved down a major third. We can easily summarize these observations through the P and L operations: the sequence of chords in Fig. 1.10 can be described as an application of P, L, P, L, P, L, P.

Because of this property, such operations are also known as parsimonious.

1.4 *Tonnetz*, *Chicken-wire Torus* and other parsimonious graphs

One of the most peculiar musical rules is to organize voice leading making the least possible movement. Music theorists typically use graphs to describe parsimonious voice leading. Two different kinds of graphs are used: note-based and chord-based graphs. In the former each vertex represents a note, in the latter, by contrast, each vertex represents a chord, and parsimonious voice leading corresponds to short-distance motions along edges.

Definition 1.31 (Note-based graph). *A graph $G = (V, E, L)$ is a* ***note-based graph*** *if each vertex $v \in V$ is labeled with a note $l \in L$.*

Definition 1.32 (Chord-based graph). *A graph $G = (V, E, L)$ is a* ***chord-based graph*** *if each vertex $v \in V$ is labeled with a chord $l \in L$.*

We will use the term "parsimonious graph", introduced by Douthett and Steinbach in [28], to indicate the graph associated with parsimonious operations, thus displaying parsimonious voice leading.

Tonnetz

In music, the most famous example of a note-based graph is probably the *Tonnetz* (see Fig. 1.11), first introduced by Euler in his 1739 *Tentamen novae theoriae musicae* [29] and replicated by several musicologists of the XIX$^{\text{th}}$ century, such as Hugo Riemann. It is a note-based graph in which pitch classes of the twelve-tone equal temperament are organized along intervals of fifth in the horizontal axis, major and minor thirds in the diagonal axis. In this construction each triplet of distinct vertices, which are adjacent pairwise, is a triangle representing a major or a minor triad. We observe that, given a triangle, each reflection preserving one of its edges is another triangle sharing two pitch-classes with the given one. From a musical point of view, this means that, given a triangle representing a triad, from each of its "edge-flips" we obtain another triad sharing two pitch-classes with the given triad. These three "edge-flips" represent the three neo-Riemannan operations P, R and L, commonly used in parsimonious voice leading.

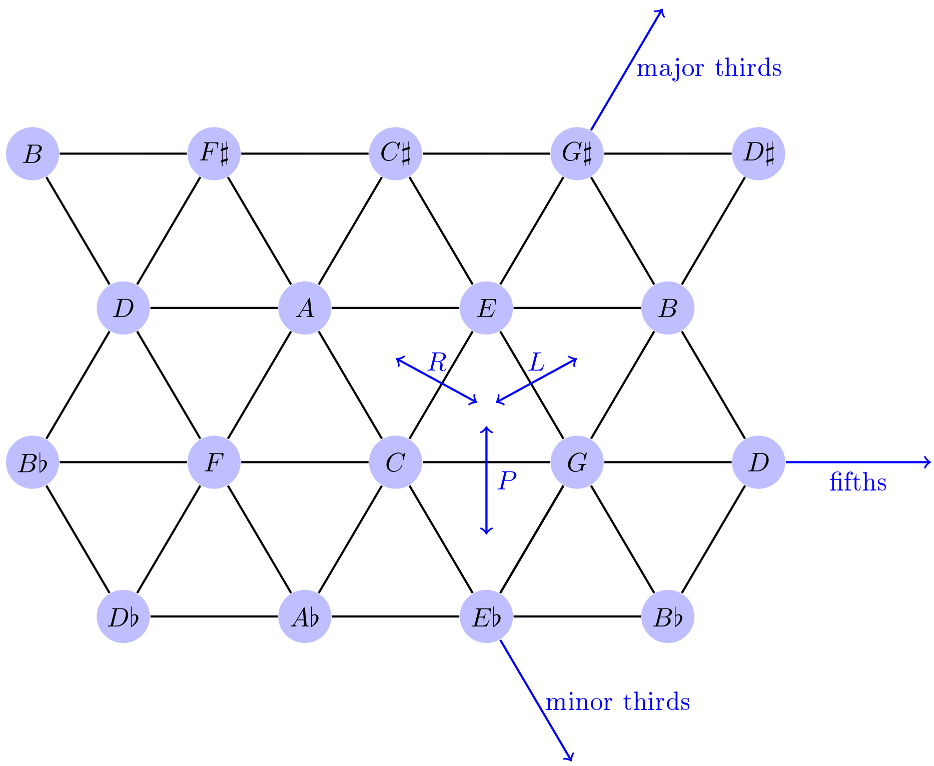


Figure 1.11: The *neo-Riemannian Tonnetz*. The edge-flips represent the P, L and R operations.

Visualizing the "edge-flips" in the *Tonnetz*, we are considering not only the edges of the triangles, but the entire polygon. Therefore, in this case it is better to define it as an infinite 2-dimensional simplicial complex which tiles the Euclidean plane with triangles, where 0-simplices represent pitch-classes, and 2-simplices identify major and minor triads.

Cohn noted that the *neo-Riemannian Tonnetz* is toroidal [25], an observation[3] which was also made by Guerino Mazzola in his book *Geometrie der Töne* [57]. In fact, because of the cyclical nature of the pitch classes in the equal temperament, the two axes have cyclic periodicities. Therefore in the infinite *Tonnetz* there is a pattern repeated infinitely often. In this pattern the vertices of the top and bottom sides represent the same pitch classes, therefore we can identify the two sides. The same feature is also present between the left and the right hand sides, so we can also identify these two sides. Thus the *Tonnetz* is mathematically isomorphic to a torus.

[3]The *Tonnetz* is called *Terztorus* in [57] since it is generated by major and minor thirds.

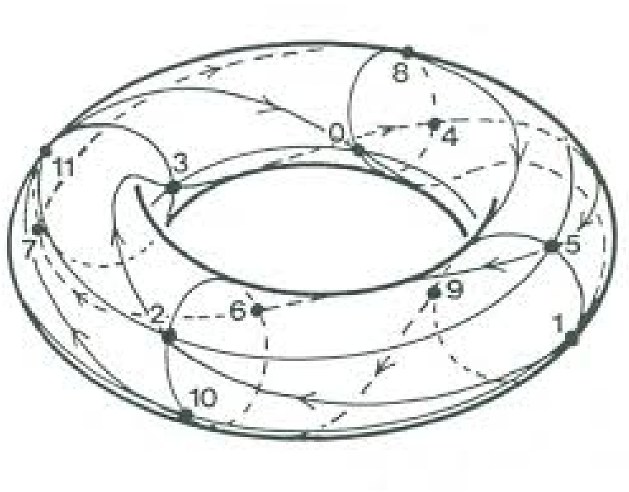


Figure 1.12: A toroidal representation of the *Tonnetz* (from [57]).

Chicken-wire Torus

The dual graph of the Tonnetz, called *Chicken-wire torus* by Douthett and Steinbach [28], is a chord-based graph in which vertices represent major and minor triads (see Fig. 1.13). We recall that given a planar graph

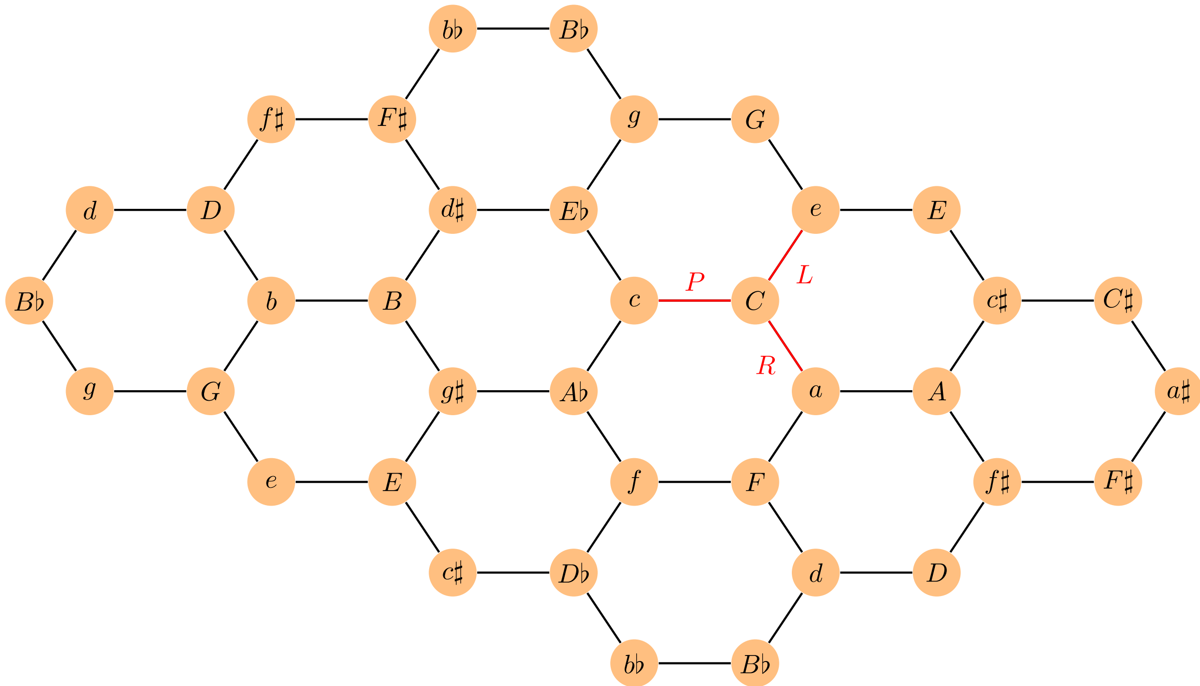


Figure 1.13: *Chicken-wire torus*. The capital letters represent major triads, the lower-case letters represent minor triads.

G, its dual G^* is a graph in which each vertex corresponds to a face of G, and each face corresponds to a vertex in G. Two vertices in G^* are connected by an edge if the corresponding faces in G have a boundary edge in common. The edges in the *Chicken-wire torus* represent the P, L and R operations, therefore paths along the edges correspond to sequences of major and minor triads. Moreover, as for the *Tonnetz* as the name suggests, also the *Chicken-wire torus* can be embedded in a torus.

Other parsimonious graphs

Douthett and Steinbach also gave a graph-theoretical approach to construct parsimonious graphs whose vertices represent triads or seventh chords. This construction is based on the *relation definition*, according to which a transformation between two chords is parsimonious if the common tones remain fixed, and the other tones are moved by a semitone or a whole tone.

Definition 1.33 (Relation definition). *Let X and Y be two pitch-class sets. We say that X and Y are $P_{m,n}$-**related** (written $XP_{m,n}Y$) if there exists a set $\{x_k | k = 0, 1, \ldots, m+n-1\}$ and a bijection $\tau \colon X \to Y$ such that $X \setminus Y = \{x_k | k = 0, 1, \ldots, m+n-1\}$, $\tau(x) = x$ if $x \in X \cap Y$, and*

$$\tau(x_k) = \begin{cases} x_k \pm 1 \pmod{12} & \text{if } 0 \leq k \leq m-1 \\ x_k \pm 2 \pmod{12} & \text{if } m \leq k \leq m+n-1. \end{cases}$$

We note that since τ is bijective, the cardinality of the two $P_{m,n}$-related chords has to be the same. This definition is similar to the P_n-relation given by Childs [23], according to which two chords are P_n-related if they differ by a half-step in n voices, while the others remain fixed.

Remark 1.34. *Let $X = [x_1, x_2, x_3], Y = [y_1, y_2, y_3]$ be two triads.*

- *If $P(X) = Y$, then X and Y are $P_{1,0}$-related. In fact $m + n - 1 = 0$, therefore there exists a unique $x_k \in X \setminus Y$ s.t. $\tau(x_k) = x_k \pm 1$, and the other two tones remain fixed. This corresponds to what we have applying the parallel P: two tones remain fixed, the other tone is moved by a semitone.*
- *If $R(X) = Y$, then X and Y are $P_{0,1}$-related. In fact $m + n - 1 = 0$, therefore there exists a unique $x_k \in X \setminus Y$ s.t. $\tau(x_k) = x_k \pm 2$, and the other two tones remain fixed. This corresponds to what we have applying the relative R: two tones remain fixed, the other tone is moved by a whole-tone.*
- *If $L(X) = Y$, then X and Y are $P_{1,0}$-related. This case is analogous to that of the parallel P.*

We can observe that different transformations may be described by the same $P_{m,n}$ relation.

Douthett and Steinbach defined *parsimonious graphs* as chord-based graphs whose edges connect vertices governed by some parsimony-related rule.

Example 1.35. *We consider the set $\mathcal{F} = \{D_M, D_m, B_{\flat M}, A_M, F^7\}$ and suppose to construct the graph induced by $P_{1,0}$ and $P_{2,0}$. Since $D_M P_{1,0} d_m P_{1,0} B_{\flat M} P_{2,0} D_M$ and A_M and F^7 are not $P_{1,0}$- or $P_{2,0}$-related to any chord in $\mathcal{F}$, the graph of $\mathcal{F}$ induced by $P_{1,0}$ and $P_{2,0}$ is that in Fig. 1.14.*

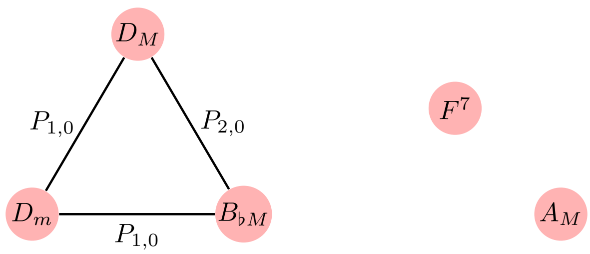


Figure 1.14: Example of graph induced by $P_{1,0}$ and $P_{2,0}$

For instance, the graph of the set S of all 24 major and minor triads induced by $P_{1,0}$ and $P_{0,1}$ is the *Chicken-wire torus.*

Another famous example given by Douthett and Steinbach was the *Cube Dance.* We consider the set $\mathcal{A} = \{[x_1, x_2, x_3] | x_1, x_2, x_3 \in \mathbb{Z}_{12}, x_2 = x_1 + 4, x_3 = x_1 + 8\}$ of the augmented triads. The graph of $S \cup \mathcal{A}$ (major, minor and augmented triads) induced by $P_{1,0}$, known as *Cube Dance*, is a circle of 4 cubes connected by shared vertices. The shared vertices represent the 4 augmented triads, the other vertices of the cubes are major and minor triads (see Fig. 2.10).

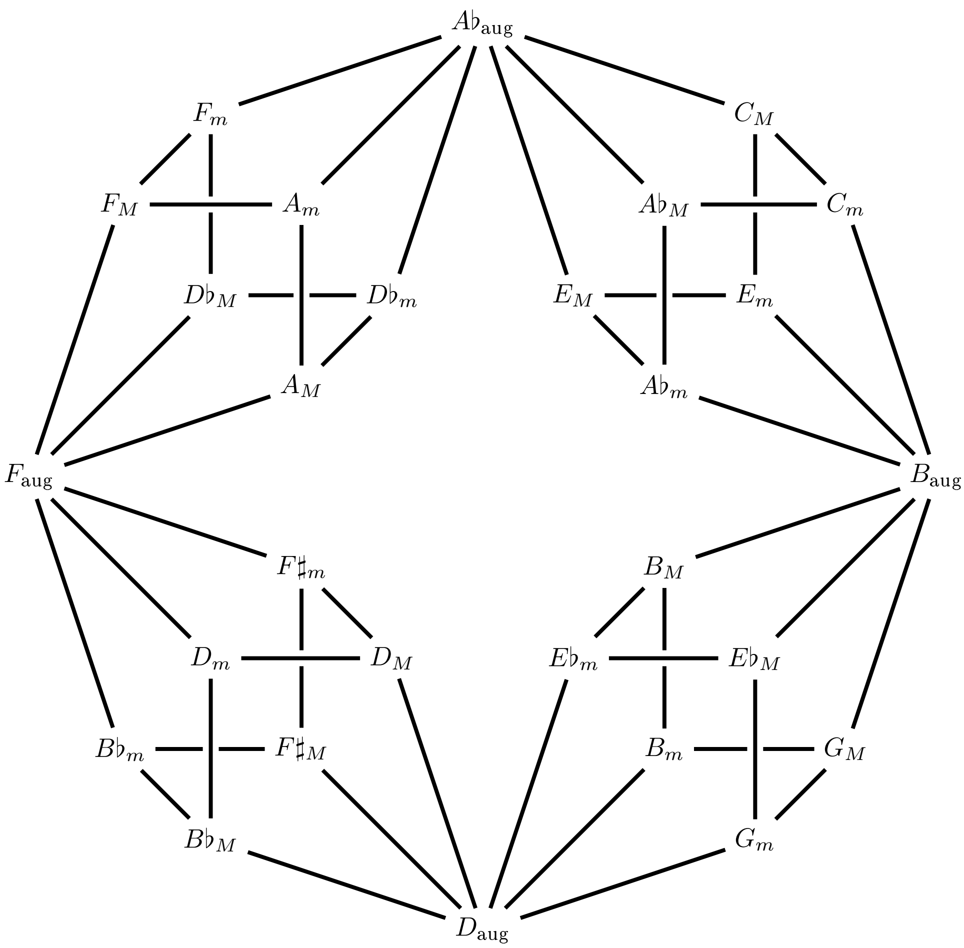


Figure 1.15: Douthett and Steinbach's *Cube Dance*. The subscripts M, m, and aug refer to major, minor, and augmented triads respectively.

1.5 Hamiltonian cycles of musical graphs in music analysis and composition

Hamiltonian cycle in the Chicken-wire Torus

Since paths in the *Chicken-wire torus* represent sequences through major and minor triads using P, L and R operations, it is interesting to study particular cycles and paths. For example Albini and Antonini enumerated, classified and studied all its Hamiltonian cycles [1] (see Table 1.2).

Definition 1.36 (Hamiltonian cycle)**.** *Given a graph $G = (V, E)$, a* ***Hamiltonian cycle*** *of G is a closed path that visits each vertex exactly once.*

The Hamiltonian cycles in the *Chicken-wire Torus* are 124, which are reduced to 62 if we do not distinguish the direction of the cycle. This number can be reduced considering the actions of the group of automorphisms of the *Chicken-wire Torus*.

L	R	L	R	L	R	L	R	L	R	L	R	L	R	L	R	L	R	L	R	L	R	L	R	#41	H1
P	R	L	R	P	R	L	R	P	R	L	R	P	R	L	R	P	R	L	R	P	R	L	R	#32	H2
L	R	P	R	L	R	P	R	L	R	P	R	L	R	P	R	L	R	P	R	L	R	P	R	#45	H2
P	L	P	L	R	L	P	L	P	L	R	L	P	L	P	L	R	L	P	L	P	L	R	L	#4	H3
P	L	R	P	L	P	L	R	L	P	L	P	L	R	L	P	L	P	L	R	L	P	L	P	#13	H3
L	P	L	P	L	R	L	P	L	P	L	R	L	P	L	P	L	R	L	P	L	P	L	R	#62	H3
P	R	P	R	P	R	L	R	P	R	P	R	P	R	L	R	P	R	P	R	P	R	L	R	#33	H4
P	R	P	R	L	R	P	R	P	R	P	R	L	R	P	R	P	R	P	R	L	R	P	R	#38	H4
P	L	R	L	P	R	P	R	P	R	L	R	P	R	P	R	P	R	L	R	P	R	P	R	#40	H4
L	R	P	R	P	R	P	R	L	R	P	R	P	R	P	R	L	R	P	R	P	R	P	R	#44	H4
P	R	P	R	L	R	L	R	P	R	P	R	L	R	L	R	P	R	P	R	L	R	L	R	#34	H5
P	R	L	R	L	R	P	R	P	R	L	R	L	R	P	R	P	R	L	R	L	R	P	R	#39	H5
L	R	P	R	P	R	L	R	L	R	P	R	P	R	L	R	L	R	P	R	P	R	L	R	#42	H5
L	R	L	R	P	R	P	R	L	R	L	R	P	R	P	R	L	R	L	R	P	R	P	R	#43	H5
P	L	P	L	R	P	L	P	L	P	R	P	L	P	L	P	R	P	L	P	L	P	R	L	#6	H6
P	L	P	R	L	P	L	P	L	R	P	L	P	L	P	R	P	L	P	L	P	R	P	L	#7	H6
P	L	P	R	P	L	P	L	P	R	L	P	L	P	L	R	P	L	P	L	P	R	P	L	#8	H6
P	L	R	P	L	P	L	P	R	P	L	P	L	P	R	P	L	P	L	P	R	L	P	L	#9	H6
P	L	P	R	P	L	P	L	P	R	P	L	P	L	P	R	L	P	L	P	L	R	P	L	#10	H6
P	R	L	P	L	P	L	R	P	L	P	L	P	R	P	L	P	L	P	R	P	L	P	L	#19	H6
P	R	P	L	P	L	P	R	L	P	L	P	L	R	P	L	P	L	P	R	P	L	P	L	#21	H6
P	R	P	L	P	L	P	R	P	L	P	L	P	R	L	P	L	P	L	R	P	L	P	L	#22	H6
P	L	P	L	P	R	L	P	L	P	L	R	P	L	P	L	P	R	P	L	P	L	P	R	#27	H6
P	L	P	L	P	R	P	L	P	L	P	R	L	P	L	P	L	R	P	L	P	L	P	R	#30	H6
P	L	P	L	P	R	P	L	P	L	P	R	P	L	P	L	P	R	L	P	L	P	L	R	#31	H6
L	P	L	P	L	R	P	L	P	L	P	R	P	L	P	L	P	R	P	L	P	L	P	R	#58	H6
P	L	R	P	L	P	R	L	P	L	R	L	P	R	L	R	P	R	L	R	P	L	R	L	#3	H7
P	L	R	L	P	R	L	R	P	R	L	R	P	L	R	L	P	L	R	P	L	P	R	L	#12	H7
P	R	L	P	L	R	L	P	R	L	R	P	R	L	R	P	L	R	L	P	L	R	P	L	#15	H7
P	R	L	R	P	R	L	R	P	L	R	L	P	L	R	P	L	P	R	L	P	L	R	L	#17	H7
P	L	P	R	L	P	L	R	L	P	R	L	R	P	R	L	R	P	L	R	L	P	L	R	#26	H7
P	L	R	L	P	L	R	P	L	P	R	L	P	L	R	L	P	R	L	R	P	R	L	R	#28	H7
P	R	L	R	P	L	R	L	P	L	R	P	L	P	R	L	P	L	R	L	P	R	L	R	#35	H7
L	R	P	R	L	R	P	L	R	L	P	L	R	P	L	P	R	L	P	L	R	L	P	R	#46	H7
L	R	P	L	R	L	P	L	R	P	L	P	R	L	P	L	R	L	P	R	L	R	P	R	#51	H7
L	P	L	R	L	P	R	L	R	P	R	L	R	P	L	R	L	P	L	R	P	L	P	R	#52	H7
L	P	R	L	R	P	R	L	R	P	L	R	L	P	L	R	P	L	P	R	L	P	L	R	#56	H7
L	P	L	R	P	L	P	R	L	P	L	R	L	P	R	L	R	P	R	L	R	P	L	R	#61	H7
P	L	R	P	L	P	R	P	L	R	P	R	L	R	L	P	L	R	L	P	R	L	R	L	#1	H8
P	L	R	L	P	R	L	R	L	P	L	R	P	L	P	R	P	L	R	P	R	L	R	L	#2	H8
P	L	R	L	R	P	R	L	P	R	P	L	P	R	L	P	L	R	L	R	P	L	R	L	#5	H8
P	L	R	L	R	P	L	R	L	P	L	R	L	R	P	R	L	P	R	P	L	P	R	L	#11	H8
P	R	L	P	L	R	L	R	P	L	R	L	P	L	R	L	R	P	R	L	P	R	P	L	#14	H8
P	R	P	L	R	P	R	L	R	L	P	L	R	L	P	R	L	R	L	P	L	R	P	L	#16	H8
P	R	L	R	L	P	L	R	P	L	P	R	P	L	R	P	R	L	R	L	P	L	R	L	#18	H8
P	R	P	L	P	R	L	P	L	R	L	R	P	L	R	L	P	L	R	L	R	P	R	L	#20	H8
P	L	P	R	P	L	R	P	R	L	R	L	P	L	R	L	P	R	L	R	L	P	L	R	#23	H8
P	L	P	R	L	P	L	R	L	R	P	L	R	L	P	L	R	L	R	P	R	L	P	R	#24	H8
P	L	R	P	R	L	R	L	P	L	R	L	P	R	L	R	L	P	L	R	P	L	P	R	#25	H8
P	L	R	L	P	L	R	L	R	P	R	L	P	R	P	L	P	R	L	P	L	R	L	R	#29	H8
P	R	L	P	R	P	L	P	R	L	P	L	R	L	R	P	L	R	L	P	L	R	L	R	#36	H8
P	R	L	R	L	P	L	R	L	P	R	L	R	L	P	L	R	P	L	P	R	P	L	R	#37	H8
L	R	P	R	L	P	R	P	L	P	R	L	P	L	R	L	R	P	L	R	L	P	L	R	#47	H8
L	R	L	P	L	R	P	L	P	R	P	L	R	P	R	L	R	L	P	L	R	L	P	R	#48	H8
L	R	P	L	R	L	P	L	R	L	R	P	R	L	P	R	P	L	P	R	L	P	L	R	#49	H8
L	R	L	P	L	R	L	P	R	L	R	L	P	L	R	P	L	P	R	P	L	R	P	R	#50	H8
L	P	L	R	L	P	R	L	R	L	P	L	R	P	L	P	R	P	L	R	P	R	L	R	#53	H8
L	P	L	R	L	R	P	L	R	L	P	L	R	L	R	P	R	L	P	R	P	L	P	R	#54	H8
L	P	R	P	L	P	R	L	P	L	R	L	R	P	L	R	L	P	L	R	L	R	P	R	#55	H8
L	P	L	R	L	R	P	R	L	P	R	P	L	P	R	L	P	L	R	L	R	P	L	R	#57	H8
L	P	L	R	P	L	P	R	P	L	R	P	R	L	R	L	P	L	R	L	P	R	L	R	#59	H8
L	P	R	L	R	L	P	L	R	P	L	P	R	P	L	R	P	R	L	R	L	P	L	R	#60	H8

Table 1.2: Classification of all Hamiltonian cycles in the *Chicken-wire Torus*.

Theorem 1.37. *The automorphism group of the* Chicken-wire Torus *is isomorphic to the dihedral group D_{12} of order* 24*.*

Proof. First of all, we recall that for Theorem 1.19 the T/I-group acts on the 24 major and minor triads and $T/I \simeq D_{12}$. We will prove that the automorphism group of the *Chicken-wire Torus* is isomorphic to the T/I-group. We will denote by $CWT = (V, E)$ the *Chicken-wire Torus*. Since the labelling are 24, the automorphisms of CWT are at least 24. Moreover, since given any two vertices $v_1, v_2 \in V(CWT)$ there exists an automorphism f such that $f(v_1) = v_2$, the automorphism group is vertex transitive. To prove the theorem it is sufficient to show that it has no more than 24 elements. Let $v \in V(CWT)$ be a vertex and let f be an automorphism such that $f(v) \neq v$. We take $g \in T/I$ such that $(g \circ f)(v) = v$. Because of the structure of CWT this composition can be only the identity, therefore f is the inverse of an element of T/I, thus it belongs to it itself. □

We observe that given a Hamiltonian cycle, if n automorphisms of CWT transform it to itself, then there are $\frac{24}{n}$ different Hamiltonian cycles sharing the model of transformation. In CWT we have 8 models.

H1 – cycle #41. It is characterized by the repetition of the model LR. All automorphisms map this cycle into itself.

H2 – cycles #32 and #45. They are characterized by the repetition of the model $PRLR$. 12 automorphisms map these cycles into themselves.

H3 – cycles #4, #13 and #62. They are characterized by the repetition of the model $LPLPLR$. 8 automorphisms map these cycles into themselves.

H4 – cycles #33, #38, #40 and #44. They are characterized by the repetition of the model $PRPRPRLR$. 6 automorphisms map these cycles into themselves.

H5 – cycles #34, #39, #42 and #43. They are characterized by the repetition of the model $PRPRLRLR$. 6 automorphisms map these cycles into themselves.

H6 – cycles #6, #7, #8, #9, #10, #19, #21, #27, #30 and #31. They are characterized by the repetition of the model $LPLPLR$. 8 automorphisms map these cycles into themselves.

H7 – cycles #3, #12, #15, #17, #26, #28, #35, #46, #51, #51, #56 and #61. They are characterized by the repetition of the model $PLRPLPRLPLRLPRLRPRLRPLRL$. 2 automorphisms map these cycles into themselves.

H8 – the remaining 24 cycles. They are characterized by the model $LRLPLRLPRLRLPLRPLPRPLRPR$. Only the identity automorphisms map these cycles into themselves.

Musical applications

From a musical point of view, each Hamiltonian cycle of the *Chicken-wire Torus* is a chord progression which modulates to all 24 major and minor triads using the three neo-Riemannian operations P, L and R. Therefore, it is not surprising to find Hamiltonian cycles, or part of them, in some musical pieces. A famous example is in the second movement of Beethoven's Ninth Symphony, between measures 143 and 176. The German composer covered 19 triads of the Hamiltonian cycle #41. Therefore, he did not make use of the entire cycle, but it is interesting to observe that such type of cycle can be found in musical, independently from the knowledge on graph theory by the composer. We observe that Beethoven lived between 1770 and 1827, graph theory was just beginning and even if he had known Euler's *Tonnetz* he would not have thought of Hamiltonian cycles on its dual and their use for composition.

At the same time, these classes of cycles are a useful compositional device for contemporary music [2], so shown by the composer Giovanni Albini.[4] Moreno Andreatta used different Hamiltonian cycles in a popular music context [14].[5] In the song *La sera non è più la tua canzone*, with lyrics by Mario Luzi, starting from $A_{\flat M}$ he applied $LRLPLP$ four times realizing the cycle #62. In the song *Aprile*, with lyrics by the Italian decadent poet Gabriele D'Annunzio, he used the cycles: #1, #60 and #53 starting from $B_{\flat M}$. We observe that these 3 cycles cannot be decomposed into sub-patterns, and the composer chose them for their "non-redundancy" characteristic. Gilles Baroin realized pedagogical collection of videos using his *Spinnen-Tonnetz* model [9], including the geometric visualizations of harmonic movement along the edges of the graph of the two aforementioned Hamiltonian songs.

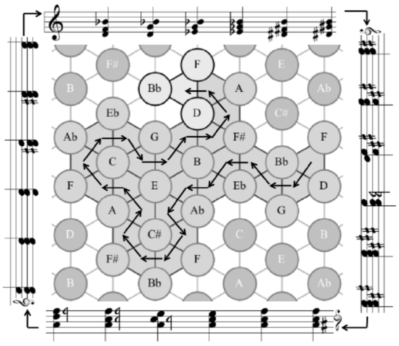


Figure 1.16: *Aprile*, Hamiltonian cycle #1 in the *Tonnetz*.

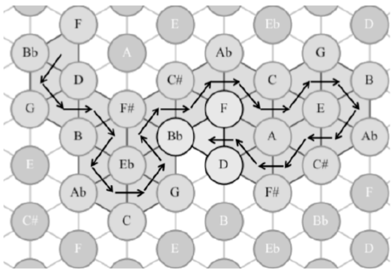


Figure 1.17: *Aprile*, Hamiltonian cycle #60 in the *Tonnetz*.

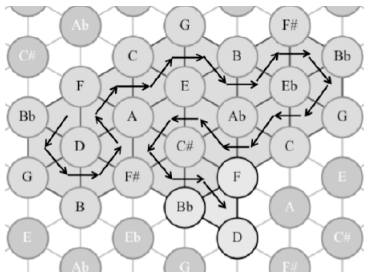


Figure 1.18: *Aprile*, Hamiltonian cycle #53 in the *Tonnetz*.

1.6 Uniform Triadic Transformations

Starting from the main works on neo-Riemannian theories introduced by David Lewin [52, 53], Richard Cohn [24, 25, 26], Brian Hyer [41, 42] and Edward Gollin [35], in 2002 Hook proposed an algebraic framework in order to have a standardized system for labeling triadic transformations and to give a precise definition of neo-Riemannian operations [43].

In this approach a triad is an ordered pair $\Delta = (r, \sigma)$, where r is the root of the triad expressed, as usual, as an element of $\mathbb{Z}_{12}$, and $\sigma = \pm$ is a sign representing its mode: "+" for major and "−" for minor triads.

Example 1.38. $C_M = (0, +)$, $A_m = (9, -)$.

The set $\Gamma = \{(r, \sigma) | r \in \mathbb{Z}_{12}, \sigma = \pm\}$ represents the collection of all 24 major and minor triads. Given two triads $\Delta_1 = (r_1, \sigma_1)$ and $\Delta_2 = (r_2, \sigma_2)$ Hook defined the Γ-interval between them as $int_\Gamma = (t, \sigma)$, where $t = r_2 - r_1$ is the *transposition level* and $\sigma = \sigma_1\sigma_2$ is the *sign factor*.

Example 1.39. *Consider the two triads* $\Delta_1 = (0, +)$ *and* $\Delta_2 = (9, -)$*. Their interval is* $(9, -)$*.*

As we observe, the Γ-interval is an element of Γ, but it does not mean that it is a triad. The following theorem makes this double nature of Γ more explicit.

Proposition 1.40. Γ *forms an abelian group isomorphic to* $\mathbb{Z}_{12} \times \mathbb{Z}_2$ *with multiplication defined by*

$$(t_1, \sigma_1)(t_2, \sigma_2) = (t_1 + t_2, \sigma_1\sigma_2) \quad (1.15)$$

[4] See http://www.giovannialbini.it/opus/

[5] See http://repmus.ircam.fr/moreno/music/

Proof. From the Formula 1.15, it is clear that the multiplication of each pair of elements in $\mathbb{Z}_{12} \times \mathbb{Z}_2$ is an element in $\mathbb{Z}_{12} \times \mathbb{Z}_2$. Since $(\mathbb{Z}_{12}, +)$ is an abelian group and $\sigma_i \sigma_j = \sigma_j \sigma_i$, commutativity is also satisfied. Moreover associativity is satisfied, for every $(t_1, \sigma_1), (t_2, \sigma_2), (t_3, \sigma_3)$

$$\begin{aligned}
&(t_1, \sigma_1)\left((t_2, \sigma_2), (t_3, \sigma_3)\right) = (t_1, \sigma_1)(t_2 + t_3, \sigma_2\sigma_3) = \\
=&(t_1 + t_2 + t_3, \sigma_1\sigma_2\sigma_3) = (t_1 + t_2, \sigma_1\sigma_2)(t_3, \sigma_3) = \\
=&\left((t_1, \sigma_1)(t_2, \sigma_2)\right)(t_3, \sigma_3).
\end{aligned}$$

For every (t, σ) the identity element is $(0, +)$ and the inverse is $(-t, \sigma)$. In fact we have $(0, +)(t, \sigma) = (t, \sigma)$ and $(t, \sigma)(-t, \sigma) = (0, +)$. Thus Γ is an abelian group. Finally, $t \in \mathbb{Z}_{12}$ and the set $\{+, -\}$ form a group isomorphic to $\mathbb{Z}_2$, hence $\Gamma \simeq \mathbb{Z}_{12} \times \mathbb{Z}_2$. □

Hook observed that Γ as a set of triads with Γ as a group of intervals and the function int_Γ form a GIS.
Hook's definitions of *triadic transformation* and *operations* are the same introduced by Lewin in [53].

Definition 1.41 (Triadic transformation)**.** *A* ***triadic transformation*** *is a map* $\Gamma \to \Gamma$.

Definition 1.42 (Operation)**.** *An* ***operation*** *is a bijective transformation* $\Gamma \to \Gamma$.

There are some important musical transformations that are not operations. For instance the transformation mapping triads to its functional dominant: both major and minor triads with the same root are mapped to the same triad.

The operations, together with the composition of maps form a group $\mathcal{G}$ of order 24!. Since the action of a transformation on a given triad is not related to its action on any other triad, most of these operations are not interesting from a musical point of view. Consequently, Hook focused on operations with a certain kind of musical coherence, intuitively he studied those that transform each triad "in the same way".

Definition 1.43 (Uniform triadic transformation (UTT))**.** *A* ***uniform triadic transformation (UTT)*** U *is an operation satisfying the following* uniformity condition*: for every triad and every transposition level* t*, if* U *transforms the triad* (r, σ) *into* (r', σ')*, then* U *transforms* $(r + t, \sigma)$ *into* $(r' + t, \sigma')$.

The behavior of a UTT is completely determined by three parameters: its sign $\sigma = \pm$, the transposition levels $m = t^+$ (for major triads) and $n = t^-$ (for minor triads). Therefore a UTT can be represented as an ordered triple $\langle \sigma, m, n \rangle$. If $\sigma = +$ the UTT is called *mode-preserving*, otherwise it is called *mode-reserving*. Many famous musical operations are UTTs.

Example 1.44.

$$\begin{aligned}
T_n &= \langle +, n, n \rangle \\
P &= \langle -, 0, 0 \rangle \\
L &= \langle -, 4, -4 \rangle \\
R &= \langle -, -3, 3 \rangle.
\end{aligned}$$

Including neo-Riemannian operations and many other operations, UTTs are very useful in the modelling sequences of triads. Nevertheless, there are noteworthy operations that are not UTTs.

Remark 1.45. *Inversions* I_n *are not UTTs.*

Now we describe the actions of UTTs on triads. Let $U = \langle \sigma_U, t^+, t^- \rangle$ be a UTT and let $\Delta = (r, \sigma_\Delta)$ a triad. U acts on Δ transposing its root by t^+ if $\sigma_\Delta = +$ or by t^- if $\sigma_\Delta = -$. Moreover, depending on the sign of σ_U,

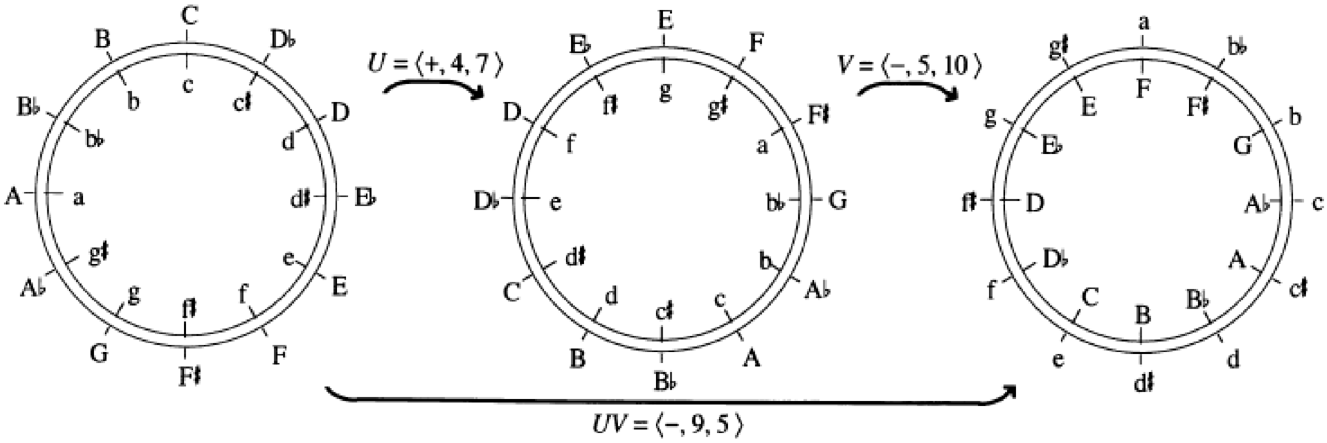


Figure 1.19: Example of dual-circle configuration (from [43]).

U can leave or change the type of the chord. More precisely:

$$U(\Delta) = \begin{cases} (r + t^+, \sigma_U) & \text{if } \sigma_\Delta = + \\ (r + t^-, -\sigma_U) & \text{if } \sigma_\Delta = - \end{cases} \quad \Rightarrow \quad U(\Delta) = (r + t^{\sigma_\Delta}, \sigma_\Delta \sigma_U). \tag{1.16}$$

Hook introduced an interesting way to visualize the actions of UTTs geometrically. He considered a *dual-circle configuration*, which consists of a pair of concentric circles: the external one to represent the major triads, and the internal one for minor triads. In each of them the triads are ordered around their own circle, ascending by semitones in the clockwise direction. A UTT $U = \langle \sigma, m, n \rangle$ acts on a dual-circle configuration rotating counterclockwise the circle representing the major triad through m positions, and the circle representing the minor triad through n positions. Moreover if $\sigma = -$, major and minor triads are exchanged (see Fig. 1.19).

Let $\mathcal{U}$ be the set of all UTTs. Since each UTT can be represented as $\langle \sigma, m, n \rangle \in \mathbb{Z}_2 \times \mathbb{Z}_{12} \times \mathbb{Z}_{12}$, the order of $\mathcal{U}$ is $2 \times 12 \times 12 = 288$. This group is a subgroup of the group of all operations $\mathcal{G}$, since $\mathcal{U}$ is the centralizer of $T_1 \in \mathcal{G}$.

Proposition 1.46. *An operation U is a UTT if and only if it commutes with the transposition T_1.*

Proof. Let $U = \langle \sigma_U, t^+, t^- \rangle$ be a UTT and let $\Delta = (r, \sigma_\Delta)$ a triad. Then

$$\begin{aligned}(T_1 U)(\Delta) = T_1(U(\Delta)) = T_1(r + t^{\sigma_\Delta}, \sigma_\Delta \sigma_U) &= \\ = (r + t^{\sigma_\Delta} + 1, \sigma_\Delta \sigma_U) &= \\ = (r + 1 + t^{\sigma_\Delta}, \sigma_\Delta \sigma_U) &= \\ = U(r + 1, \sigma_\Delta) &= \\ = U(T_1(\Delta)) = (UT_1)(\Delta).&\end{aligned}$$

Now we show that if U commutes with T_1 then U is a UTT. We have to prove that the uniformity condition is satisfied. To prove this, we observe that U commutes with every transposition T_n. In fact by induction we suppose $UT_{n-1} = T_{n-1}U$, then

$$UT_n = UT_{n-1}T_1 = T_{n-1}UT_1 = T_{n-1}T_1U = T_nU.$$

Now we assume that $U(\Delta) = U(r, \sigma_\Delta) = (r', \sigma'_\Delta) = \Delta'$. Then

$$U(r + t, \sigma) = U(T_t(\Delta)) = (UT_t)(\Delta) = (T_tU)(\Delta) = T_t(U(\Delta)) = T_t(\Delta') = (r' + t, \sigma'_\Delta).$$

□

Theorem 1.47. *$\mathcal{U}$ is a subgroup of $\mathcal{G}$.*

Proof. By Proposition 1.46 $\mathcal{U} = C_{\mathcal{G}}(T_1)$. Since the centralizer of an element of a group is its subgroup, $\mathcal{U}$ is a subgroup of $\mathcal{G}$. □

The product formula of two UTTs $U = \langle \sigma_U, t_U^+, t_U^- \rangle$, $V = \langle \sigma_V, t_V^+, t_V^- \rangle \in \mathcal{U}$ is defined as[6]

$$VU = \langle \sigma_{UV}, t_U^+ + t_V^{\sigma_U}, t_U^- + t_V^{-\sigma_U} \rangle. \tag{1.17}$$

Example 1.48. *Let $U = \langle +, 4, 7 \rangle$ and $V = \langle -, 5, 10 \rangle$ be two UTTs. Their product UV is*

$$VU = \langle -, 9, 5 \rangle \circ \langle +, 4, 7 \rangle = \langle -, 4+5, 7+10 \rangle = \langle -, 9, 5 \rangle.$$

We consider the major triad $C_M = (0, +)$ and the minor triad $C_m = (0, -)$. We apply UV to them:

$$(VU)(0,+) = V(U(0,+)) = V(4,+) = (9,-)$$
$$(VU)(0,-) = V(U(0,-)) = V(7,-) = (5,+).$$

We obtain the same result applying directly $VU = \langle -, 9, 5 \rangle$.

We observe that the group $\mathcal{U}$ is not abelian. In fact if we consider $U, V \in \mathcal{U}$ as in previous example, we have

$$UV = \langle +, 4, 7 \rangle \circ \langle -, 5, 10 \rangle = \langle -, 5+7, 10+4 \rangle = \langle -, 0, 2 \rangle \neq VU$$

Hook defined the *Riemannian UTTs* as UTTs in which the sum of the transposition levels is 0.

Definition 1.49 (Riemannian UTT)**.** *A UTT $\langle \sigma, m, n \rangle$ is **Riemannian** if $m + n \equiv 0 \pmod{12}$.*

Example 1.50. *Several UTTs are Riemannian. In addition to $P = \langle -, 0, 0 \rangle$, $L = \langle -, 4, -4 \rangle$ and $R = \langle -, -3, 3 \rangle$, $T_0 = \langle +, 0, 0 \rangle$ and $T_6 = \langle +, 6, 6 \rangle$ are among the operations satisfying the Riemannian condition.*

The Riemannian UTTs are 24: 12 mode-preserving UTTs of the form $S_n = \langle +, n, -n \rangle$ known as *Schritte* and 12 mode-reversing of the form $W_n = \langle -, n, -n \rangle$ known as *Wechsel*. Therefore $T_0 = S_0$, $T_6 = S_6$, $P = W_0$, $L = W_4$ and $R = W_9$. The set of all 24 Riemannian UTTs is denoted by $\mathcal{R}$ and form a group known as the *Riemann group*. This group is also known as the S/W-group.As it turns out, the PLR-group is isomorphic to the Riemann group.

Theorem 1.51. *The 24 Schritte S_n and Wechsel W_n form a group isomorphic to the PLR-group.*

Proof. By using Formula 1.17, it is easy to see that S_n and W_n satisfy the following properties:

$$S_n \circ S_m = S_{m+n}$$
$$W_n \circ S_m = W_{m+n}$$
$$S_n \circ W_m = W_{m-n}$$
$$W_n \circ W_m = S_{m-n}$$
$$S_n^{-1} = S_{-n}$$
$$W_n^{-1} = W_n.$$

Moreover $S_0 = T_0$ is the identity and the associativity is inherited from the group $\mathcal{U}$ of all UTTs.

[6]The product has to be read from right to left.

We denote by Q the operation $S_1 = \langle +, 1, -1 \rangle$. We observe that $S_n = Q^n$ and $W_n = PQ^n$. It suffices to prove that P and Q may be expressed as products of L and R. In Lemma 1.28 we have already proved that $P = RLRLRLR$. Since

$$LR = \langle -, 4, 8 \rangle \circ \langle -, 9, 3 \rangle = \langle +, 5, 7 \rangle = Q^5$$

and since $Q^{12} = T_0$, we have

$$(LR)^5 = (Q^5)^5 = Q^{25} = (Q^{12})^2 Q = Q$$

□

Corollary 1.52. *The Riemann group is isomorphic to the dihedral group of order* 24.

1.7 Computational approaches

In the last decades, Computational Music Analysis has become increasingly widespread, due in part to the use of different mathematical areas[7]. The first academic computational projects started in the 1960s, providing clarification of musical concepts, testing evolutionary hypotheses, contributing to music technology, creating educational music tools [73]. These projects have contributed to many different areas of music: historical musicology [74], ethnomusicology [48], research in musical performance [34], cognitive musicology [72], music theory, and music analysis [5].

It is evident that music contains many interesting structures, and music analysis is able to understand and to study them. Traditional music analysis is usually applied directly to the score using Western musical notation. But, historically, the musical language used in a score was developed according to the needs of the performer. For the music theorist Tymoczko the use of geometric models can be a good tool to solve this problem. In fact, it is a simple and immediate language to perform music, but it is less useful to study music from a theoretical point of view:

> Geometry can help to sensitize us to relationships that might not be immediately apparent in the musical score. Ultimately, this is because conventional musical notation evolved to satisfy the needs of the performer rather than the musical thinker: it is designed to facilitate the translation of musical symbols into physical action, rather than to foment conceptual clarity.[8]

New technologies based on geometric models of musical objects provide further help.

HexaChord

There are several pieces of software in which musical graphs and other geometric models used in music are integrated. One of them is HexaChord, a computer-aided music analysis environment based on spatial representation of musical objects, developed by Louis Bigo [12] [13]. The geometric models integrated in it are: the musical clock, the circle of fifths, the voice-leading space, the *Tonnetz* and other simplicial complexes isomorphic to it.

Since the *Tonnetz* represents major and minor triads and their intervallic structure is $[3, 4, 5]$, Bigo denoted it by $\mathcal{K}[3, 4, 5]$. The mathematical idea is to represent each chord as a simplex where each 0-simplex is labeled by one note of the chord. Since $card(\sigma) = dim(\sigma) + 1$, a chord of n notes is represented by an $(n-1)$-simplex. Therefore a 0-simplex represents a single note, a 1-simplex a 2-chord, a 2-simplex a 3-chord, a 3-simplex a

[7]See [59] for a comprehensive perspective on Computational Music Analysis.
[8]See [68], page 79.

4-chord, and so on. More generally, the building of a simplicial complex $\mathcal{K}[a_1, \ldots, a_i]$ associated to chords whose intervallic structure is $[a_1, \ldots, a_i]$ is described by Bigo as follows. We choose a chord with n notes with intervallic structure $[a_1, \ldots, a_n]$, and we consider the associated $(n-1)$-simplex. As in the *Tonnetz*, the 1-simplices determine axes representing particular intervals, therefore simplices are replicated along these axes.

We note that 2-simplicial complexes associated with chords of the same size are isomorphic. Therefore, from a geometrical point of view, all simplicial complexes representing trichords are isomorphic to the different *Tonnetz*: a 3-chord is a chord of 3 notes, then the associated simplex is a 2-simplex and the final simplicial complex is a tiling of the 2-dimensional Euclidean space with 2-simplices. The only differences among them are the labels. All simplicial complexes corresponding to the *Tonnetz* are integrated on HexaChord: $\mathcal{K}[1,1,10]$, $\mathcal{K}[1,2,9]$, $\mathcal{K}[1,3,8]$, $\mathcal{K}[1,4,7]$, $\mathcal{K}[1,5,6]$, $\mathcal{K}[2,2,8]$, $\mathcal{K}[2,3,7]$, $\mathcal{K}[2,4,6]$, $\mathcal{K}[2,5,5]$, $\mathcal{K}[3,3,6]$, $\mathcal{K}[4,4,4]$, $\mathcal{K}[1,1,5]$, $\mathcal{K}[1,2,4]$, $\mathcal{K}[1,3,3]$, $\mathcal{K}[2,2,3]$. These simplicial complexes are not all isomorphic, as it has been pointed out in a recent study [51]

Functionalities of HexaChord

Musical pieces are imported as MIDI files. A MIDI (Musical Instrument Digital Interface) file is a musical file with a specific protocol called MIDI, representing only player information. This is a protocol designed for recording and playing back music on digital synthesizers that is supported by many makes of personal computer sound cards.

Given a MIDI file as an input, HexaChord is able to:

- visualize chords and chord sequences.
- Compute the compactness of the trajectory of the musical piece in each simplicial complex representing a given *Tonnetz*.
- Modify the musical piece through a geometric transformation.

During the execution of the MIDI file, it is possible to visualize in real time each chord in three different geometric models: the generalized *Tonnetz*, the musical clock and the circle of fifths. With the option “intervallic structure” we can choose another simplicial complex, in which we can view the chords and the trace of the musical piece. In these simplicial complexes it is also possible to visualize the trace in real time clicking on “trace on”. More precisely, a temporal sequence of chords is represented by a static object, mathematically described as labeled subgraph or sub-complex. Let $S = [(S_0, d_0), (S_1, d_1), \ldots, (S_n, d_n)]$ be a chord sequence. It is represented by a sequence of sub-complexes, called *trajectory* by Bigo and defined as follows

$$T_{\mathcal{K}} = [(\mathcal{K}_0, d_0), (\mathcal{K}_1, d_1), \ldots, (\mathcal{K}_n, d_n)]$$

where each sub-complex $\mathcal{K}_i$, $i \in \{1, \ldots, n\}$, represents the chord S_i and it is labeled with the corresponding duration d_i in S. The simplicial complex

$$\mathcal{T} = \bigcup_{(\mathcal{K}_i, d_i) \in T_{\mathcal{K}}} \mathcal{K}_i$$

is the subcomplex of $\mathcal{K}$ representing the chords in the musical sequence S, and it is called *trace*.

Depending on the simplicial complex used, the trace of the musical piece is displayed differently. Clicking on “compute compactness” the compactness of these traces is calculated, and the results are represented in a histogram. The compactness of each trace gives us information on how much the geometric space used is suitable to represent the chosen musical piece. High compactness of the trajectory in a particular complex might be seen as a stylistic signature of the piece. Therefore, the computation of the compactness might be very helpful in music analysis for music classification.

The other great functionality of HexaChord is to modify a musical sequence through geometric transformations. Given two simplicial complexes $\mathcal{K}_1$ and $\mathcal{K}_2$, a trajectory $T_{\mathcal{K}_1}$ is embedded in $\mathcal{K}_2$ becoming $T_{\mathcal{K}_2}$. From a musical point of view it means that $T_{\mathcal{K}_2}$ is associated to a new musical sequence. Therefore, if we start from $\mathcal{K}[3,4,5]$, choosing a different simplicial complex as a destination complex and clicking on "compute transformation", HexaChord modifies the original musical sequence in a way which is compatible with the topological structure of the underlying simplicial complexes.

Other geometric transformations developed in HexaChord are translations (that correspond to a musical transpositions) and rotations. As in the previous transformation, also in these cases the musical sequence is modified.

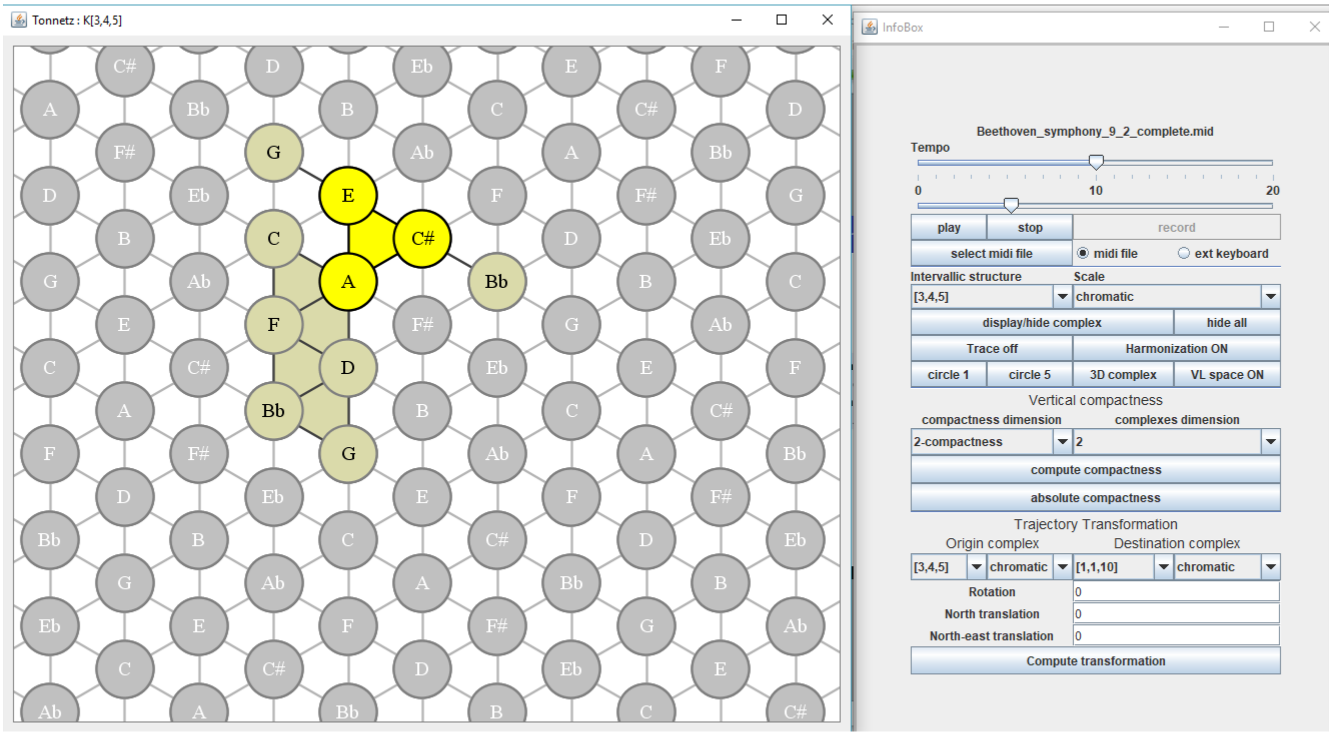


Figure 1.20: HexaChord

The main interface is simple and easy to use, and it is represented in Fig 1.20.

Moreover with HexaChord it is also possible to connect a piano keyboard and play/record a musical piece. All functionalities are available also for the performance.

Chapter 2

History of the Tonnetz[1]

2.1 Euler's *Tonnetz*

It is commonly said that the *Tonnetz* was originally introduced by Euler in 1739 in his *Tentamen novae theoriae musicae* (*Tentamen*) [29] [20]. Indeed, a graph similar to the *Tonnetz* appeared in chapter IX, "De genere diatonico-chromatico" (the diatonic-chromatic genus), to represent the scheme of a musical tuning. In chapter VII, Euler wrote about musical tuning, in particular tuning systems, i.e. the theoretical study of various systems of pitches used to tune an instrument. Starting from acoustical observations, he defined a musical genus as a division of the octave, i.e. a collection of octave-equivalent tones, mathematically of the form $2^m a$, where $m \in \mathbb{Z}$, $a \in \mathbb{N}$. Euler listed all genera up to the eighteenth, which is the one in use in his time: the diatonic-chromatic genus, represented by $2^m \cdot 3^3 \cdot 5^2$. To determine the tones contained in this genus, he considered the divisors of $3^3 \cdot 5^2$:

$$\begin{aligned}&1, \quad 3, \quad 5, \quad 3^2 = 9, \quad 3 \cdot 5 = 15, \quad 5^2 = 25, \quad 3^3 = 27, \quad 3^2 \cdot 5 = 45,\\ &3 \cdot 5^2 = 75, \quad 3^3 \cdot 5 = 135, \quad 3^2 \cdot 5^2 = 225, \quad 3^3 \cdot 5^2 = 675.\end{aligned} \tag{2.1}$$

The biggest is obviously 675: since we want 12 tones in the same octave, we find the biggest number 2^x such that $2^x < 675$, that is $2^9 = 512$, and multiply each divisor of 675 by a power of 2 until we find an octave-equivalent tone in the octave range $512--1024$. In this way we obtain the following tones of the diatonic-chromatic genus:

$$512, \quad 540, \quad 576, \quad 600, \quad 640, \quad 657, \quad 720, \quad 768, \quad 800, \quad 864, \quad 900, \quad 960, \quad 1024. \tag{2.2}$$

One would be tempted to say that this construction looks like an algorithm to build "artificial" musical scales that do not correspond to the real ones. On the contrary, the distances between the numbers in (2.2) correspond, with a good approximation, to the well-known intervallic ratios of the just intonation. A similar method was known in Greek antiquity. In particular, it is explained in Plato's *Timaeus*, who learned it from the Pythagoreans [49].

After the theoretical construction of the diatonic-chromatic genus, Euler explained how to tune the instruments with it:

1. fix the tone F, and from it get all the other F;
2. from F get C and A by forming a (perfect) fifth and a major third respectively;

[1] To appear in [19].

3. from C get G and E by forming a fifth and a major third respectively, and E will also be the fifth of A, moreover from A get $C\sharp$ by forming a major third;

4. from G get D and H by forming a fifth and a major third, and H is also the fifth of E, moreover from E get $G\sharp$ by forming a major third, it will also be the fifth of $C\sharp$;

5. from H get $F\sharp$ and $D\sharp$ by forming a fifth and a major third, and $D\sharp$ will also be the fifth of G, moreover from E and $D\sharp$ get B by forming a fifth.

It is interesting to observe that Euler considered the intervals in just intonation, but then the fifth of $D\sharp$ should be $A\sharp$, not B ($A\sharp$ and B coincide in the equal temperament, but not in the just intonation).

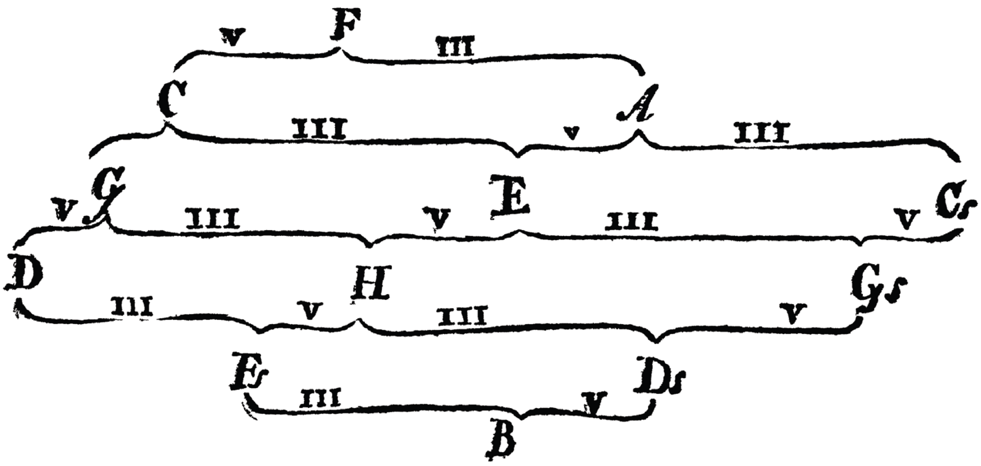


Figure 2.1: Euler's *Tonnetz* appeared in the *Tentamen*.

To help the readers in this explanation, Euler summarized it with the diagram in Fig. 2.1. He introduced it just as a scheme of his explanation on tuning for diatonic-chromatic genus; he did not define it as a graph, and did not give it a name. He used the term "figure" to introduce it: "Totus autem hic temperationis processus ex adiecta hic figura distinctius percipietur".[2]

We recall that Euler's study of the seven bridges of Königsberg, presented in 1735 and published in 1741,[3] is considered as the first one in the history of graph theory, and for this reason Euler is known as the pioneer of this subject [11]. The fact that Euler did not use the term "graph" is not surprising: as is often the case in the history of mathematics, when a theory is at the beginning many objects have no name yet. According to Howarth [44], the term "graph" was introduced by Sylvester in a paper published in 1878.[4] Furthemore it is interesting to observe that the *Tentamen*, although published in 1739, was written in 1731, and therefore the graph introduced for tuning of chromatic-diatonic genus is older than the famous graph solution of the Bridges of Königsberg.[5] Also the term *Tonnetz* came later, too. In the chapter *From matrix to map: Tonbestimmung, the Tonnetz, and Riemann's combinatorial conception of interval*, Gollin wrote: "The term *Tonnetz* was apparently introduced by Renate Imig in *Systeme der Funktionsbezeichnung in den Harmonielehren seit Hugo Riemann* (Düsseldorff: Gesellschaft zu Förderung der systematischen Musikwissenschaft, 1970) and has since been generally adopted

[2]See [29], p. 147.

[3]L. Euler, *Solutio Problematis ad Geometriam Situs Pertinentis*, Commentarii Academiae Scientiarum Imperialis Petropolitanae 8, 1741, pp.128-140.

[4]J. J. Sylvester, *Chemistry and Algebra*, Nature, 17:284, 1878.

[5]A. Papadopoulos, *Euler et le débuts de la topologie*, p. 322 in [39] Moreover see [7].

in the neo-Riemannian literature."[6]

A different version of the graph introduced in the *Tentamen* appeared in another "mathmusical" work, published by Euler in 1774: *De harmoniae veris principiis per speculum musicum repraesentatis* [30] [20]. It is a graph of 12 tones in just intonation in a 3 by 4 network, organized by major thirds vertically, and by perfect fifths horizontally (see Fig. 2.2). As the graph in the *Tentamen*, it is also used by Euler to represent the tones in the diatonic-chromatic genus but, this timer, Euler gave it a name, *Speculum Musicum*.

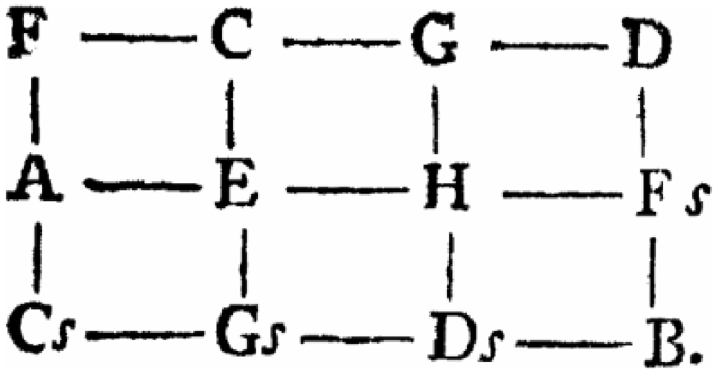


Figure 2.2: Euler's *Speculum Musicum*

Euler's idea to represent musical relations in a graph was used in 1858 by Ernst Naumann, and was disseminated some years later in a 1866 treatise by Arthur von Oettingen and, in particular, by the influential musicologist Hugo Riemann.

2.2 Naumann and Oettingen's tables

Euler's writings were well known to the German physicist and physician Hermann von Helmholtz (1821–1894), as well as to the German mathematician and philosopher Moritz Wilhem Drobisch (1802–1896). The latter wrote some monographs on tuning and temperament. In 1858, his student Carl Ernst Naumann (1832–1910) wrote the dissertation *Über die verschiedenen Bestimmungen der Tonverhältnisse und die Bedeutung des Pythagorischen oder reine Quinten-Systems für unsere heutige Musik* (*About the different determinations of the tone relations and the meaning of the Pythagorean or perfect fifths system for our music today*), in which appeared a diagram representing a tone system defined by perfect fifth and major third relations[7] (see Fig. 2.3).

The organization of this table reminds us of Euler's *Speculum Musicum*: the tones are organized along the intervals of fifth in the horizontal axis and along intervals of thirds in the vertical axis. But there is no mention to Euler's musical diagrams, there is just a citation of the *Tentamen* in a footnote on page 21.

Arthur von Oettingen (1836–1920) too knew Euler's musical works: in the 1866 treatise *Harmoniesystem in dualer Entwickelung* (*Harmony system in dual development*) [60] the Swiss mathematician is cited in the introduction, at page 19, and the *Tentamen* is mentioned in a footnote at page 45.

On page 15 a table of tones appears (see Fig. 2.4). The same table appears again on page 169. This is the same table introduced by Naumann. It is in just intonation, that was essential to Oettingen's harmonic system of tone relations. Although this table reminds us again of the structure of the *Speculum Musicum*, there is no mention of the latter or of Euler's idea to organize tones in diagrams. Moreover, we can observe that this table is infinite.

[6]See [37], p. 289.
[7]See [58], p. 19.

grosse Diesis *cis* $\frac{T^2 . 2^z}{Q^5}$	kleinere V♯ *gis* $\frac{T^2 . 2^z}{Q^4}$	kleinere II♯ *dis* $\frac{T^2 . 2^z}{Q^3}$	kleinere VI♯ *ais* $\frac{T^2 . 2^z}{Q^2}$	kleinere III♯ *eis* $\frac{T^2 . 2^z}{Q}$	VII♯ *his* $\frac{T^2 . 2^z}{Q^0}$	IV× *fisis* $T^2 . Q . 2^z$	I× *cisis* $T^2 . Q^2 . 2^z$	V× *gisis* $T^2 . Q^3 . 2^z$	II× *disis* $T^2 . Q^4 . 2^z$	VI× *aisis* $T^2 . Q^5 . 2^z$	VI.
(kleinere gr. VI) *a* $\frac{T^2 . 2^z}{Q^5}$	(kleinere gr. III) *e* $\frac{T^2 . 2^z}{Q^4}$	kleinere gr. VII *h* $\frac{T^2 . 2^z}{Q^3}$	kleinere IV♯ *fis* $\frac{T^2 . 2^z}{Q^2}$	I♯ *cis* $\frac{T^2 . 2^z}{Q}$	grössere V♯ *gis* $\frac{T^2 . 2^z}{Q^0}$	grössere II♯ *dis* $T^2 . Q . 2^z$	grössere VI♯ *ais* $T^2 . Q^2 . 2^z$	grössere III♯ *eis* $T^2 . Q^3 . 2^z$	(gröss. VII♯) *his* $T^2 . Q^4 . 2^z$	(gröss. IV×) *fisis* $T^2 . Q^5 . 2^z$	IV.
(kleinere alt. IV) *f* $\frac{T . 2^z}{Q^5}$	alterirte VIII *c* $\frac{T . 2^z}{Q^4}$	alterirte V *g* $\frac{T . 2^z}{Q^3}$	alterirte gr. II *d* $\frac{T . 2^z}{Q^2}$	gr. VI *a* $\frac{T . 2^z}{Q}$	gr. III *e* $\frac{T . 2^z}{Q^0}$	gr. VII *h* $T . Q . 2^z$	grössere IV♯ *fis* $T . Q^2 . 2^z$	kleines Limma *cis* $T . Q^3 . 2^z$	(alter. V♯) *gis* $T . Q^4 . 2^z$	(alter. II♯) *dis* $T . Q^5 . 2^z$	II.
diatonischer halber Ton *Des* $\frac{2^z}{Q^5}$	alterirte kl. VI *As* $\frac{2^z}{Q^4}$	alterirte kl. III *Es* $\frac{2^z}{Q^3}$	kl. VII *B* $\frac{2^z}{Q^2}$	IV *F* $\frac{2^z}{Q}$	I *C* $\frac{2^z}{Q^0}$	V *G* $Q . 2^z$	gr. II *D* $Q^2 . 2^z$	alterirte gr. VI *A* $Q^3 . 2^z$	alterirte gr. III *E* $Q^4 . 2^z$	alterirte gr. VII *H* $Q^5 . 2^z$	I.
kleinere VII♭ *bb'* $\frac{2^z}{T . Q^5}$	kleinere IV♭ *fes'* $\frac{2^z}{T . Q^4}$	kleinere VIII♭ *ces'* $\frac{2^z}{T . Q^3}$	kleinere V♭ *ges'* $\frac{2^z}{T . Q^2}$	kl. II *des'* $\frac{2^z}{T . Q}$	kl. VI *as'* $\frac{2^z}{T . Q^0}$	kl. III *es'* $\frac{Q . 2^z}{T}$	alterirte kl. VII *b'* $\frac{Q^2 . 2^z}{T}$	alterirte IV *f'* $\frac{Q^3 . 2^z}{T}$	synton. Comma *c'* $\frac{Q^4 . 2^z}{T}$	grössere alt. V *g'* $\frac{Q^5 . 2^z}{T}$	III.
kleinere V♭♭ *geses''* $\frac{2^z}{T^2 . Q^5}$	kleinere II♭ *deses''* $\frac{2^z}{T^2 . Q^4}$	kleinere VI♭ *asas''* $\frac{2^z}{T^2 . Q^3}$	kleinere III♭ *eses''* $\frac{2^z}{T^2 . Q^2}$	kleinere VII♭ *bb''* $\frac{2^z}{T^2 . Q}$	IV♭ *fes''* $\frac{2^z}{T . Q^0}$	grössere VIII♭ *ces''* $\frac{Q . 2^z}{T^2}$	grössere V♭ *ges''* $\frac{Q^2 . 2^z}{T^2}$	grosses Limma *des''* $\frac{Q^3 . 2^z}{T^2}$	(gröss. kl. VI) *as''* $\frac{Q^4 . 2^z}{T^2}$	(gröss. kl. III) *es''* $\frac{Q^5 . 2^z}{T^2}$	V.
III♭♭ *es♭♭'''* $\frac{2^z}{T^3 . Q^5}$	VII♭♭ *b♭♭'''* $\frac{2^z}{T^3 . Q^4}$	IV♭♭ *feses'''* $\frac{2^z}{T^3 . Q^3}$	VIII♭♭ *ceses'''* $\frac{2^z}{T^3 . Q^2}$	V♭♭ *geses'''* $\frac{2^z}{T^3 . Q}$	(kl. Diesis) II♭ *deses'''* $\frac{2^z}{T^2 . Q^0}$	grössere VI♭ *asas'''* $\frac{Q . 2^z}{T^3}$	grössere III♭ *eses'''* $\frac{Q^2 . 2^z}{T^3}$	grössere VII♭ *bb'''* $\frac{Q^3 . 2^z}{T^3}$	alterirte IV♭ *fes'''* $\frac{Q^4 . 2^z}{T^3}$	alterirte VIII♭ *ces'''* $\frac{Q^5 . 2^z}{T^3}$	VII.

Figure 2.3: Naumann's table

$5^m\ 3^n$.

n:	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8
m 2	c	g	d	a	e	h	fis	cis	gis	dis	ais	eis	his	fisis	cisis	gisis	disis
1	as	es	b	f	c	g	d	a	e	h	fis	cis	gis	dis	ais	eis	his
0	fes	ces	ges	des	as	es	b	f	c	g	d	a	e	h	fis	cis	gis
−1	deses	asas	eses	bb	fes	ces	ges	des	as	es	b	f	c	g	d	a	e
−2	bbb	feses	ceses	geses	deses	asas	eses	bb	fes	ces	ges	des	as	es	b	f	c

Figure 2.4: Oettingen's table

Oettingen introduced this table in order to demonstrate the derivation of intervals and tones in just intonation, not to explain or describe a function of tones in some musical context. The reason why this table was introduced is, therefore, similar to that of Euler: for the latter it was a diagram for tuning instruments in the diatonic-chromatic genus, obtained from the divisors of $2^m \cdot 3^3 \cdot 5^2$ and for Oettingen it was a way to visualize and understand the relations among the divisors of $5^m \cdot 3^n$ and the corresponding musical intervals in just intonation. In fact we observe the positive and negative numbers across the top and left hand side. They represent the values of the variables n and m in the expression $5^m \cdot 3^n$, that corresponds to relative vibrational frequencies of tones in just intonation. We need to add the factor of a power of 2 to keep tones in the same register.

Example 2.1. *The central tone* $C = 5^0 \cdot 3^0 = 1$.
The entry G in the row $m = 0$ *and* $n = 1$ *has frequency* $2^{-1} \cdot 5^0 \cdot 3^1 = \frac{3}{2}$ *relative to the central tone* C.

We can consider Oettingen's table as a multiplication matrix. It is interesting to observe that the external organization determines the internal spatial relations: each entry is obtained by the exponents m and n. In the initial exposition of the table, Oettingen studied some geometric properties related to intervals, for example he observed that the diagonal axes directed to the top left form minor thirds. This corresponds to the same geometric representation of the modern *Tonnetz*; this is probably why it is considered to be the modern version of various diagrams, including that of Oettingen.

2.3 Riemann's tables and diagrams

Hugo Riemann (1849–1919) used different tables and diagrams to represent musical relations. For him, tone relations are fundamental for harmonic relations in music such as the constitution of triads and keys. The fundamental relations between tones are represented in the *Tonverwandtschaftstabelle*, also commonly considered as an ancient version of the neo-Riemannian *Tonnetz*.

In his chapter *From matrix to map: Tonbestimmung, the Tonnetz, and Riemann's combinatorial conception of interval* in [37], Gollin examined the use of musical tables in Riemann's writings, in particular he studied the changing meaning of the *Tonnetz* over the course of his life. We find tables of tone relations in several writings of Riemann, which himself called *Tonverwandtschaftstabelle* (table of tone relations). They are diagrams with the same structure used by Euler's *Speculum Musicum*, Naumann and Oettingen: the tones are organized by fifths in the horizontal axis, and by major thirds in the vertical axis.

Gollin wrote[8] that the first Riemann table of tone relations is from his 1873 dissertation *Ueber das musikalische Hören* (*About the musical listening*), introduced as "a table of relations designed by Oettingen" (see Fig. 2.5). There are few differences between Riemann's and Oettingen's tables, the main being that Riemann did not include negative numbers. Nevertheless, analogously to what happens in Oettingen's table, the numbers in the external part determine the inner organization of the table. Furthemore, it was introduced from an acoustical perspective: the acoustical function is evidenced by the lines (*Striche* or *Kommastriche*) placed above or below the letter names on the table to reflect intonational discrepancies among tones in just intonation.

5	4	3	2	1	0	1	2	3	4	
Cis	Gis	dis	ais	eis′	his′	fisis″	cisis‴	gisis‴	disis⁗	3
,A	E	H	fis	cis′	gis′	dis″	ais″	eis‴	his‴	2
,F	C	G	d	a	e′	h′	fis″	cis‴	gis‴	1
,Des	,As	Es	B	f	c′	g′	d″	a″	e‴	0
„Bb	,Fes	Ces	Ges	des	as	es′	b′	f″	c‴	1
„G♭♭	,D♭♭	,A♭♭	E♭♭	Bb	fes	ces′	ges′	des″	as″	2
„E♭♭♭	„B♭♭	,Feses	Ceses	Geses	deses	asas	eses′	heses′	fes″	3
5	4	3	2	1	0	1	2	3	4	

Figure 2.5: Table from Riemann's dissertation *Ueber das musikalische Hören*

[8] See [37], p. 277.

But unlike Oettingen, who used his table only to demonstrate the derivation of intervals and tones, Riemann used them also to describe and explain the behavior of tones and chords in a musical context. Gollin discussed some examples in which the *Tonverwandtschaftstabelle* is a metaphorical map to represent progressions of tones, chords or keys. The first example is from the opening of the funeral march from Beethoven's Piano Sonata op.26, which was studied by Riemann in his 1880 *Skizze einer neuen Methode der Harmonielehre* (*Sketch of a new method of harmony teaching*). In the opening section there is an enharmonic nonclosure that Riemann visualized on the *Tonverwandtschaftstabelle*. But the description of enharmonic passages assumes that the tonal elements of the table are equally tempered, an assumption at odds with the original acoustical function of these tables as a diagram whose elements are tones in just intonation. The analytical problem that the acoustical view of his tables posed was nevertheless big enough that Riemann in 1880 could not reconcile the harmonic logic of the passage with its spatial manifestation in his table.

Gollin examined how Riemann later achieved this reconciliation: how the diagram as a literal matrix to represent and calculate relative frequencies of the tones in just intonation (as Oettingen) evolved into a traversable landscape of tones in his later writings. This new conception of musical diagrams is the main difference with the acoustic models previously developed.

Riemann's study *Die Natur der Harmonik* [63] suggests a shift from the initial acoustic vision. Steege's chapter *"The nature of harmony": a translation and commentary* (again in [37]) has provided an introduction and translation of this treatise. Steege pointed out Riemann's theoretical evolution, written in this text at a time when a budding psychological perspective was beginning to supersede Riemann's earlier acoustical and physiological perspective. Riemann approached music theory from three reasons: a physical (in particular acoustical), physiological, and psychological perspective. There is a path from the physical area of sounding bodies (*tönende Körper*) to the psychological one of tone sensations (*Tonempfindungen*) to the physiological one of mental representation, or imagination, of tones, represented by the *Tonvorstellungen* (tone representation). Steege compared the differences in Helmholtz and Riemann's ideas of representation (*Vorstellung*) in music: Helmholtz identifies it as the moment in which one is not hearing clearly, whereas Riemann considers it as an act before the mind's eye which corresponds to taking possession of musical objects. For Gollin, another important passage from the physical to the psychological musical meaning in Riemann occurred between the fourth (1894) and the fifth edition (1900) of *Musik Lexikon* [65]. In the fourth edition, the German music theorist wrote: "D[robisch], earlier an advocate of the twelve-tone [pitch] system, in his last writing, adopted in principle the viewpoint of Helmholtz".[9] But in the fifth edition Riemann modified the sentence in the following way: "D[robisch], earlier an advocate - based on Herbartian philosophy - of the twelve-tone [pitch] system, recognized in his last writing the importance, in principle, of [the system of] just intonation".[10]

Johann Friedrich Herbart (1776–1841) was a philosopher who proposed a theory of the mind in which physical events in the brain are secondary to logical activities. For Gollin the elimination of Helmholtz in favor of Herbart in Riemann's sentence has underlined the change in musical meaning: from physical to psychological.

The definitive move from the physical to the psychological musical meaning arrived few years later, in 1914, in Riemann's 1914 *Ideen zu einer 'Lehre von den Tonvorstellungen'* (*Ideas for a 'teaching of Tonvorstellungen'*) [66]. In this work a new *Verwandtschaftstabelle* appeared (see Fig. 2.6). Similarly to the precursory models, the horizontal axes are organized along intervals of fifths and the vertical ones along intervals of thirds. Each parallelogram can be divided into two triangles: one above and the other one below. The triangles above represent major triads and the triangle below minor triads, exactly like in the *Tonnetz*. Gollin observed that in the *Ideen* deviations in intonation had little or no bearing on a listener's understanding on the logic and function of tones. Riemann located the functional interpretation of musical tones in the path derivation of those tones

[9] See [37], p. 285. The original one is at the pages 242-243 in [65].

[10] See [37], p. 286. The original one is at page 272 in [65] (fifth edition).

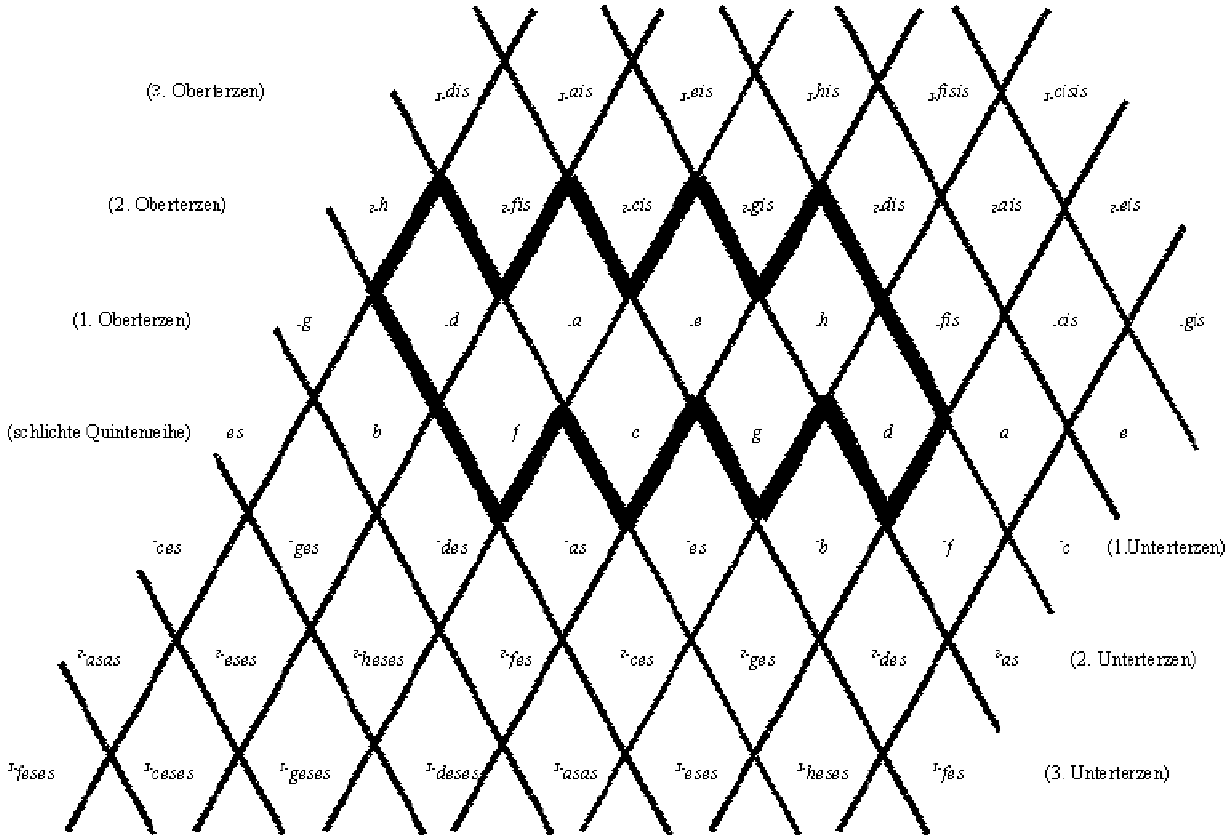


Figure 2.6: Riemann's *Verwandtschaftstabelle* in *Ideen zu einer 'Lehre von den Tonvorstellungen'*

on the musical table. Therefore, the *Verwandtschaftstabelle* of the *Ideen* is a map not of acoustically determined tones, but of logically conceivable tonal relations reckoned as pathways therein.

Gollin underlined that despite the fact that this model is close to the modern *Tonnetz*, it has some important differences. The most important is that in the *Tonnetz*, like in all modern geometric spaces, the vertices are considered as pitch classes in the twelve-tone equal temperament.

Gollin also studied the relationships between Euler's *Speculum Musicum* and the version of the *Tonnetz* that appeared in the XIX$^{\text{th}}$ century, in particular Riemann's *Tonverwandtschaftstabelle* [36]. Gollin observed that the similarity in their geometric structures was first noted by Vogel in 1960 [71]; for this reason the latter considered the *Speculum Musicum* as a precursor of the nineteenth-century diagrams. On the contrary, Gollin suggested that this conclusion is not supported by enough evidence in the works of Riemann, Oettingen, Naumann or Helmholtz. Indeed, as we have noted previously, Euler is mentioned in some works, but only briefly and there is no reference to his geometric-musical models. Moreover Euler's *Speculum Musicum* is limited, while nineteenth-century diagrams are potentially infinite.

2.4 The *Tonnetz* in Mathematical Music Theory

As mentioned in the previous chapter, in Mathematical Music Theory the *Tonnetz* is the most famous geometric model; it is also frequently used in Computational Music Analysis, and its properties offer theoretical, analytical ad well as compositional tools. It is also known as *neo-Riemannian Tonnetz*, because it was rediscovered in the neo-Riemannian theories developed from the 1980s.

A natural question arises: in what way did we arrive from nineteenth-century music theorists' tables to the *neo-Riemannian Tonnetz*?

One of the most important pioneers of neo-Riemannian theories was the music theorist David Lewin (1933–2003). In the article *Transformational techniques in atonal and other music theories* [52], Lewin proposed some musical transformations, formalized them from a mathematical point of view, but there are no references to the *Tonnetz* or to any other geometric space. Successively, in his *Generalized Musical Intervals and Transformations* [53], he gave some examples of melodic and harmonic musical spaces to visualize the intuition of distance and motion of pitches resulting from algebraic formalization. One of these examples[11] corresponds to a two-dimensional harmonic space in just intonation based on the same organization shown in Euler's *Speculum Music* and Naumann, Oettingen and in Riemann's tables. The pitches are organized horizontally along intervals of fifth, vertically along intervals of third (see Fig. 2.7). In fact, as stressed as by the author, this space is the same as a Riemann's table:

> Maps like figure 2.2 have been especially common in German theories of tonality since the eighteenth century, generally in connection with key relationships rather than root relationships (though some theories do not dwell on such a distinction). The closest precedent I can find for the actual configuration of figure 2.2 itself appears in Hugo Riemann, *Grosse Kompositionslehre*, vol. 1, *Der homophone Satz (Melodielehre und Harmonielehre)* (Berlin and Stuttgart: W. Spemann, 1902). Riemann's map is on page 479.[12]

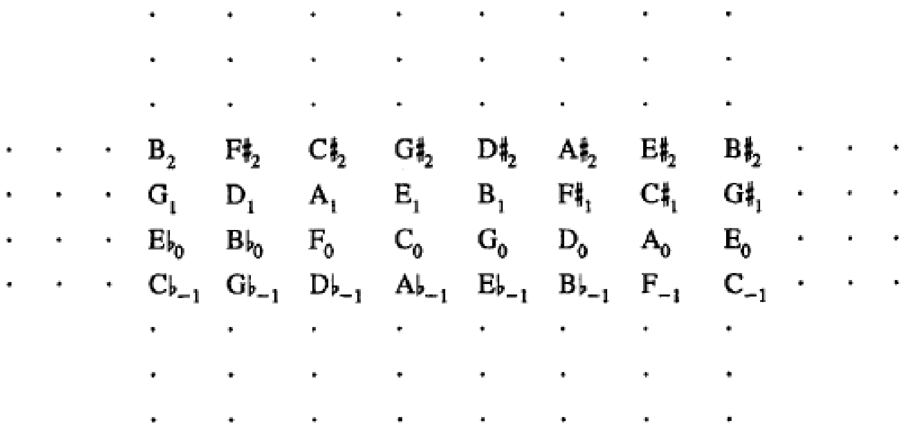


Figure 2.7: Lewin's harmonic space

Unlike Euler's *Speculum Musicum* and in accordance with Naumann, Oettingen and Riemann, Lewin's harmonic map extends to infinity.

The originality of this model is that it was not introduced for acoustic reasons or to describe the function of sounds in some musical context, but to measure the interval distances. For example, in going from C_0 to $F\sharp_1$ we move two steps right and one step up. Another originality of this model is that, despite of the fact that it is considered in just intonation, Lewin observed that it might be conceived as living in an equal temperament with an underlying equivalence between the pitches sharing the same letter name. He himself observed that in this case the geometric space is equivalent to one with identified sides: "Then $C_{-1}, C_0, C_1, C_2, \dots, C_n, \dots$ would all mean the same thing; so would $E_{-1}, E_0, E_1, E_2, \dots, E_n$. In this case, moving one square north on the [geometric space] would be functionally equivalent to moving four squares east."[13]

If he had considered equivalent all pitches with the same letter name, he might have interpreted his geometric

[11] Chapter 2, example 2.1.6, p. 21 in [53].

[12] Footnote on page 22 in [53].

[13] See [53], p. 22.

space as a torus.

In his *Introduction to Neo-Riemannian Theory: A Survey and a Historical Perspective* [26] Richard Cohn sketched the origins and the developments of neo-Riemannian theory. Starting from Lewin's works, he underlined the importance of Hyer's studies [41] [42]. In his 1989 dissertation, he extracted four Lewin triadic transformations: P, L, R and D. The last one is defined as transposition by five semitones (T_5). Hyer had the idea to recuperate the geometric models used by nineteenth-century music theorists in order to represent them. More precisely, Hyer used the geometric dual of the *Tonnetz*. Hyer's geometric model is conceived as equally tempered, and such a conception has enriched his exploration on the algebraic group structure of the transformations.

2.5 Generalized *Tonnetze*

Since the 1990s several music theorists developed generalized versions of the traditional *Tonnetz*. These models were introduced for different studies, and for this reason they are apparently not related.
What does it mean to generalize the *Tonnetz*? As the music theorist Tymoczko observed [69], the *Tonnetz* can be studied from three points of view:

- acoustical, the main point of view of the XIX$^{\text{th}}$ century music theorists;
- common-tone relationships;
- to describe parsimonious voice-leading.

The last two points of view could appear to coincide, but they do not, because in general retaining common tones does not produce the most parsimonious voice-leading between chords.
The three points of view of the *Tonnetz* are generalized in different ways.

Generalized acoustic *Tonnetz*

A first natural way to generalize the acoustic *Tonnetz* is to introduce an additional axis for each additional interval. In 1998 Gollin introduced a three-dimensional expansion of the *Tonnetz* in which horizontal planes contain copies of the traditional *Tonnetz*, while segments in a fixed direction outside the plane represent intervals of minor seventh (see Fig. 2.8).

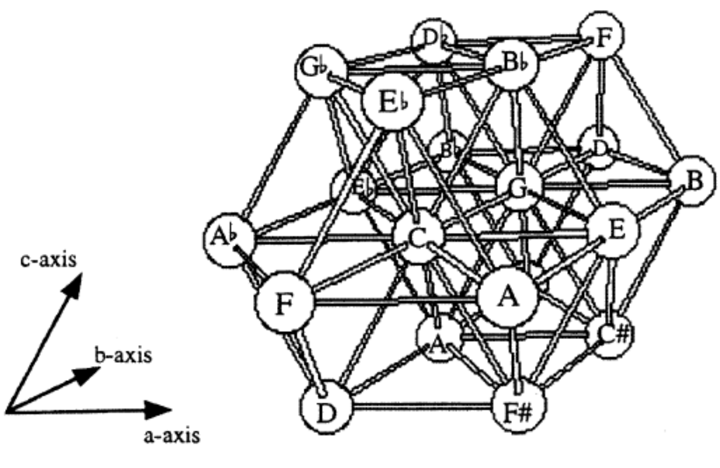


Figure 2.8: Gollin's three-dimensional Tonnetze (from [35]

While the *Tonnetz* tiles the plane with triangles, this three-dimensional expansion tiles the three-dimensional Euclidean space with tetrahedra, representing dominant and half-diminished seventh chords, and triangular prisms. Since tetrahedra sharing a common edge represent sevenths with two notes in common, this model is also used to study common-tone relationships and parsimonious voice-leading between dominant and half-diminished seventh chords.

Generalized *Tonnetz* for common-tone relationships

A generalization of the *Tonnetz* as a structure of common-tone relationships was constructed by Cohn in 1997 [25] (see Fig.2.9). The geometric structure coincides with the traditional *Tonnetz*, but the horizontal and vertical axes represent general intervals of x and y semitones, not necessarily fifths and major thirds. Then each triangle represents a trichord, not only major and minor triads.

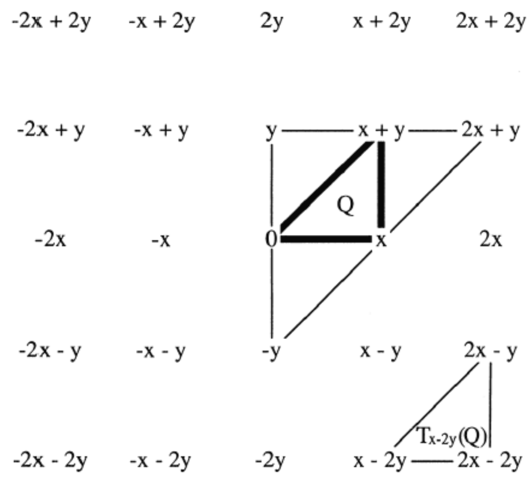


Figure 2.9: Cohn's generalized *Tonnetz*

In 1980 Balzano [10] suggested the same generalized *Tonnetz*, but starting from an algebraic point of view: he studied the cyclic groups C_{20}, C_{30} and C_{42} representing musical scales of 20, 30 and 42 tones respectively, and their geometric spaces. His idea was to consider each cyclic group as a direct product of cyclic subgroups. For example, since $C_{12} \simeq C_4 \times C_3$, we can represents the 12 tones of the twelve-tone equal temperament along intervals of 4 semitones (major thirds) in the horizontal axis, and intervals of 3 semitones (minor thirds) in the vertical one. Consequently, the diagonal axis represents intervals of 7 semitones (perfect fifths) and each triangle identifies a major or a minor triad. This space coincides with the traditional *Tonnetz* rotated by 90°. Balzano applied the same idea to the cyclic groups C_{20}, C_{30} and C_{42}, obtaining geometric spaces analogous to the traditional *Tonnetz*, but with a different number of steps in the axis. It is interesting to observe that, in this paper, Balzano did not mention previous works with the same geometric structure: he did not use the term *Tonnetz*, nor did he cite Euler or Hugo Riemann.

Other equal temperaments are also considered by Catanzaro in his article entitled *Generalized Tonnetze* [21], where he studied the generalized *Tonnetz* considering a generic N-tone equal temperaments $\mathbb{Z}_N$ and arbitrary trichords. He focused on the study of the geometric spaces from a mathematical point of view. In particular he gave a classification of all geometric space $C(n_1, n_2, n_3)$, $n_1 + n_2 + n_3 = N$ and $1 \leq n_1 \leq n_2 \leq n_3 \leq N$, where (n_1, n_2, n_3) is the intervallic structure of the trichords considered. The classification is summarized in the following theorem:

Theorem 2.2. *The connected components of the generalized* Tonnetz $C(n_1, n_2, n_3)$ *are isomorphic and are* 2*-simplices, tetrahedra boundaries, tori, cylinders, Möbius bands, or circles of tetrahedra boundaries.*

This classification has been recently refined by Lascabettes from a simplicial perspectives including homology èersistency as applied to automatic stylistic analysis [51].

Generalized *Tonnetz* to describe the parsimonious voice leading

The generalized *Tonnetz* as a graph representing parsimonious voice-leading was studied by Tymoczko: he suggested a mathematical strategy to create generalized *Tonnetze* based on the property of duality in graph theory [69]. Although this presents some problems due to the non-equivalence between musical and geometric distance, it is an interesting strategy and it includes some different models that are well known.

We recall that music theorists usually use two kinds of graphs to represent voice-leading: note-based and chord-based graphs, and in the latter parsimonious voice leading corresponds to short-distance motion along edges. We already considered the *Tonnetz* (note-based) and the *Chicken-wire torus* (chord-based).

In his treatise *A Geometry of Music* Tymoczko [68] determined a general construction for chord-based graphs: these are typically arrangements of n-dimensional cubes linked by shared faces or shared vertices. Since the *Tonnetz* is a note-based graph, his idea was to generalize note-based graphs as duals of chord-based graphs. More precisely, the duality is to be considered among their components: the note-based graph is obtained from a chord-based graph considering the dual of each n-dimensional cube.

First of all he considered chord-based graphs in the orbifold chord-space $\mathbb{T}^n/S_n$ built as follows. Let $\mathbb{R}^n$ be the set of n-tuples $(x_1, x_2, \ldots, x_n)$ such that x_i, $i = 1, \ldots, n$, represents the notes of the i-th musical part. On this space the following two equivalence relations are considered (to be applied to each component of a point in $\mathbb{R}^n$):

1. two tones that form one or more intervals of octave are equivalent

$$x_i \sim x_j \qquad \Leftrightarrow \qquad x_i \equiv x_j \pmod{12}$$

2. chords formed by the same notes are equivalent

$$x \sim y \qquad \Leftrightarrow \qquad \exists \sigma \in S_n \quad \text{s.t.} \quad \sigma(x) = y.$$

Tymoczko characterized two families of chord-based graphs. In the first one, the size of the chord evenly divides the size of the scale, and geometrically we have a circle of n-dimensional cubes linked by shared vertices. In the second one, the size of the chord is relatively prime to the size of the scale, and from a geometric point of view we have a circle of cubes linked by shared facets.[14]. The most interesting difference between the two families is the dimension of the cubes of the graph: in the first one it is determined by the size of the chords, in the second one by the number of different types of chords.

We consider the three-dimensional case of the first family. Since the dimension of the cubes corresponds to the size of the chords, geometrically we have 3-dimensional cubes in which each vertex represents a chord of 3 notes. Being in the first family, we have a circle of cubes linked by shared vertices. The well-known *Cube Dance*, introduced by Douthett and Steinbach in 1998 [28] and already mentioned in the previous chapter, realizes this construction (see Fig. 2.10). To explain the musical meaning, we consider a single 3-dimensional cube in a coordinate system such that the center coincides with the origin of the system, the edges are parallel to the axes and the coordinate vertices are $(\pm 1, \pm 1, \ldots, \pm 1)$ (see Fig. 2.11). The general idea is that each vertex represents a chord, and in the passage from a vertex to another one the movements of notes of the two chords is related to the movement of the corresponding coordinates. For example, if the vertex $(1, 1, 1)$ represents the chord $(C, E, A\flat)$, the vertex $(1, 1, -1)$ corresponds to (C, E, G): the first two notes remain fixed because

[14] In a polytope a *facet* of dimension n is a face that has dimension $n - 1$.

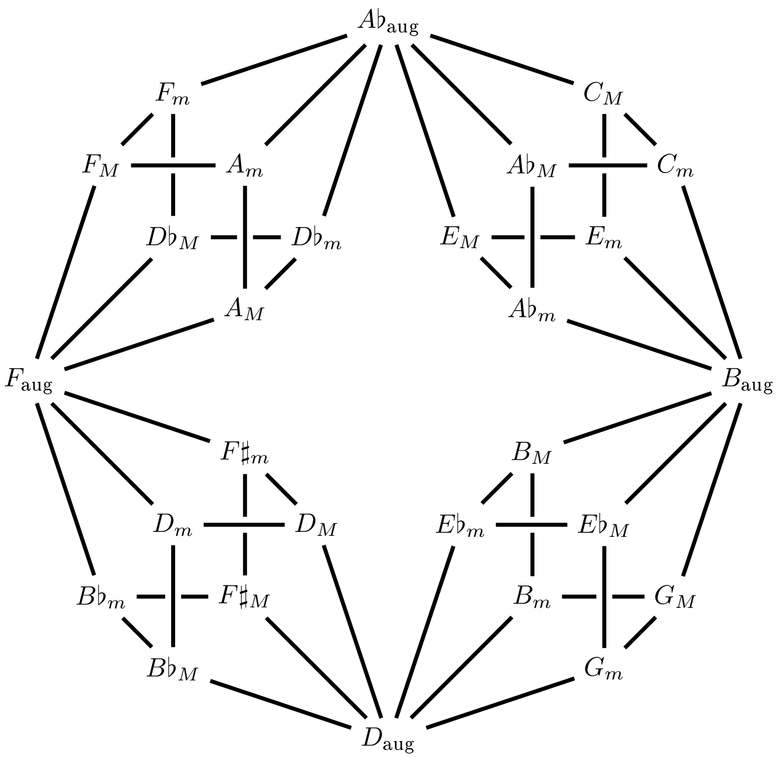


Figure 2.10: Douthett and Steinbach's *Cube Dance*. The subscripts M, m, and aug refer to major, minor, and augmented triads respectively.

the corresponding coordinates are fixed, the third note is moved down a semitone because the corresponding coordinate is decreased. To obtain the corresponding note-based graph, we replace each cube of the *Cube Dance* with its dual. The dual of a cube is a octahedron: each face of the cube is replaced by a vertex and faces sharing an edge are replaced by vertices connected by an edge. Musically speaking, the vertices of the octahedron represent the notes that in the corresponding face of the cube remain fixed (see Fig. 2.12).

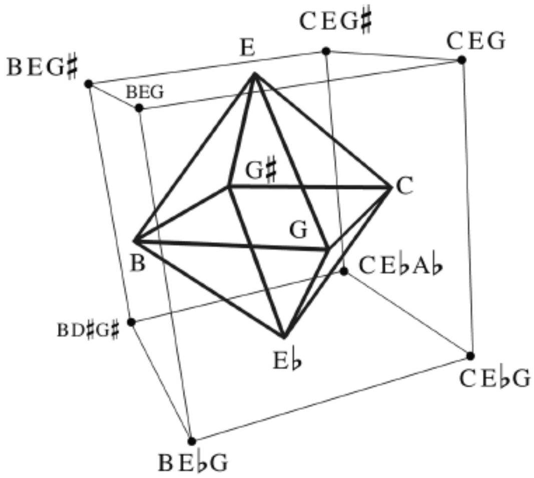


Figure 2.12: The dual of a cube is a octahedron (from [69])

Therefore the dual of the *Cube Dance* is a circle of octahedra linked by common faces (see Fig. 2.13). Musically these faces are triangles representing major, minor, and augmented triads, and edge-preserving flips represent single-semitone voice-leading. As in the traditional *Tonnetz*, in the dual of the *Cube Dance*'s dual

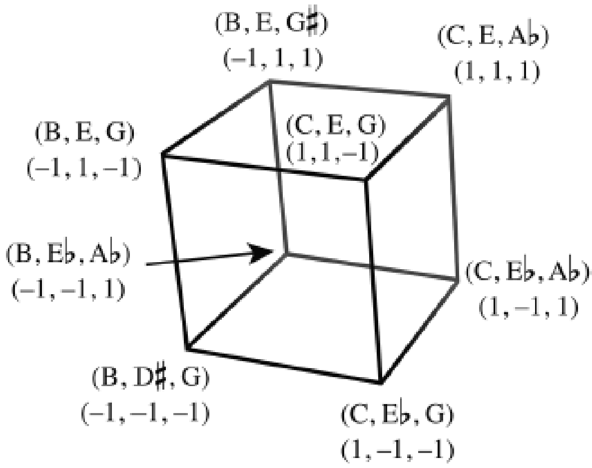


Figure 2.11: Relations between chords and coordinates of the vertices (from [69]

triangles represent triads, but in the latter there are also augmented triads, therefore the corresponding dual does not have the same geometric structure than the traditional *Tonnetz*.

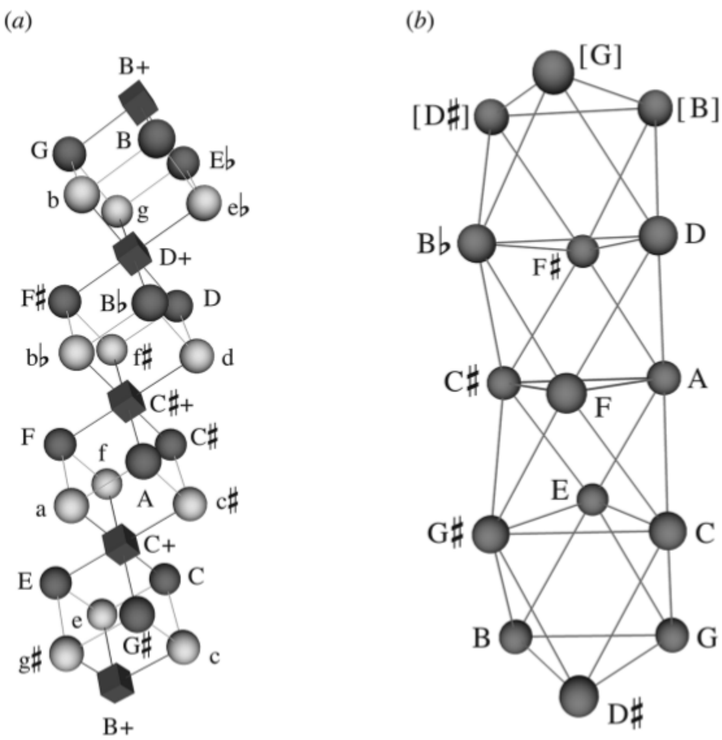


Figure 2.13: *Cube Dance* (a) and its note-based dual (b). In the *Cube Dance* each vertex represents a triad: minor (light spheres), major (dark spheres) and augmented (dark cubes). In its dual each vertex represents a single note (from [69]).

Now we consider the 4-dimensional case of the first family. The chord-based graph is a circle of 4-dimensional cubes in which each vertex represents a chord of 4 notes. In particular, the chords are dominant, minor, half-diminished, diminished sevenths and French sixths (see Fig. 2.14). The *Power Towers* graph, realized by Douthett and Steinbach in 1998 [28] (see Fig. 2.15) is a subgraph of this chord-based graph. In fact it is a chord-based graph whose vertices are dominant, minor, half-diminished and diminished seventh chords, and the edges among these types of sevenths are the same as those in Tymoczko's graph.

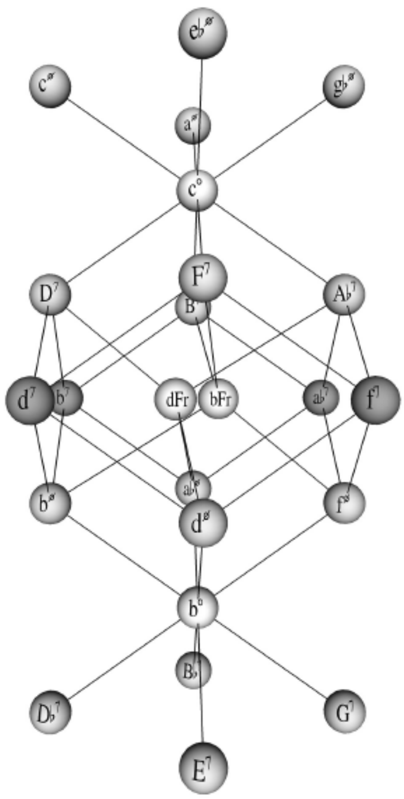


Figure 2.14: Chord-based graph of the first family, 4-dimensional case (from [69])

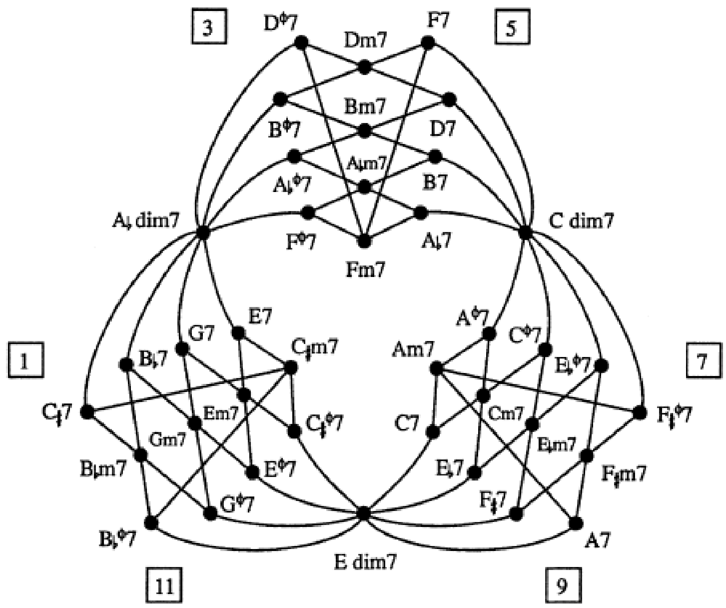


Figure 2.15: *Power Towers* by Douthett and Steinbach (from [28])

Now we construct the corresponding note-based graph of Tymoczko's graph. The 4-dimensional cubes are linked by shared vertices. The dual of each 4-dimensional cube is a 4-dimensional octahedron, whose faces are tetrahedra (see Fig. 2.16). Then the corresponding note-based graph is a circle of 4-dimensional octahedra linked by common faces, representing seventh chords or a French sixth (each note is in one of the four vertices).

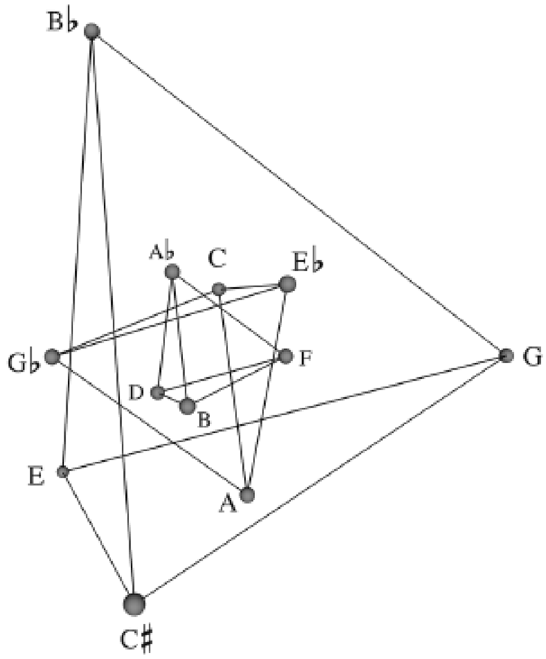


Figure 2.16: Note-based graph, dual of the chord-based graph of the first family, 4-dimensional case. The graph is a series of nested tetrahedra (from [69])

More generally, given a chord-based graph of the first family, in the n-dimensional case, the corresponding note-based graph is obtained by building the dual of each n-dimensional cube. Therefore we have a circle of n-dimensional octahedra linked by shared simplicial facets ($n-1$-simplices), each of them representing a chord of n notes. The parsimonious voice-leading is represented by flips of $n-1$-simplices through their facets ($n-2$-simplices).

Now we consider chord-based graphs of the second family. We recall that the dimension of this kind of graph does not depend on the number of notes, but on the number of types of chords we want to represent. Therefore, if we consider chords of the same type, the graph is unidimensional, independently of the dimension of the chord. In this case the chord-based graph is a "generalized circle of fifths". To construct the corresponding note-based graph we need to replace each vertex of the chord-based graph by an $n-1$-simplex, where $n-1$ is the number of notes of the chord, thus the number of vertices of the simplex. The result is a circle of simplices linked by shared facets. The unidimensional chord-based graph whose vertices represent triads in the diatonic scale[15] is a circle of vertices (see Fig. 2.17). The corresponding dual is the *diatonic Möbius strip* discovered by Mazzola in 1980 [56] and successively explored by Brower in 2008 [16]: a circle of triangles, each of which identifies a triad, linked by shared edges. More precisely the two edges B-G are linked by forming a Möbius strip.

We can construct a similar graph by considering collections of 4 notes chords in the diatonic scale. We consider a unidimensional chord-based graph in which each vertex is a seventh chord. Similarly to the previous construction, to each vertex corresponds a simplex with 4 notes in the dual, thus a tetrahedron. Furthemore, the corresponding note-based graph is a circle of tetrahedra, each of which represents a seventh chord (see Fig. 2.18).

[15]The size of the chord is 3, which is relatively prime to the size 7 of the diatonic scale. Thus the graph is in the second family. Moreover we have only triads that are one type of trichord, the dimension of the chord-based graph is 1.

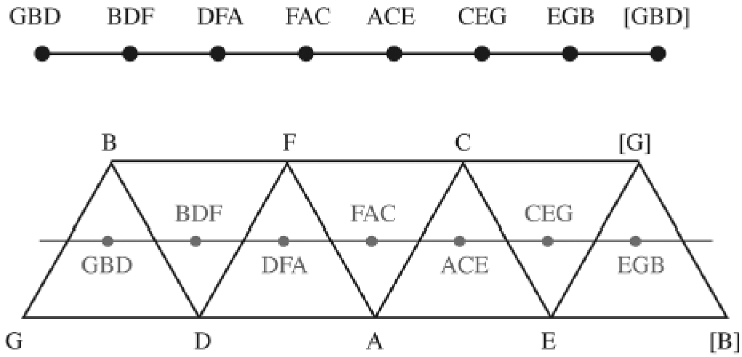


Figure 2.17: At the top a unidimensional chord-based graph of the second family. Each vertex represents a triad. At the bottom the corresponding note-based graph formally equivalent to a Möbius strip [16]

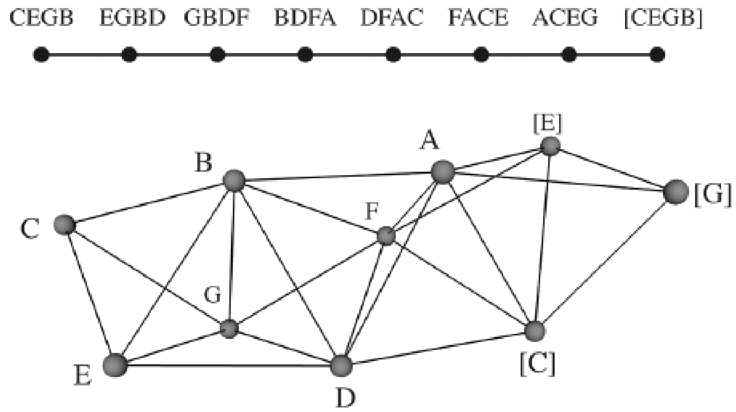


Figure 2.18: At the top a unidimensional chord-based graph of the second family. Each vertex represents a seventh chord. At the bottom the corresponding note-based graph (from [69]

Tymoczko observed that in the border of this circle of tetrahedra there is a circle of thirds that is reminiscent of Elaine Chew's *Spiral Array* [22]. From a geometric point of view this is a helical realization of the *Tonnetz*. Actually, unlike the other geometric models, the *Spiral Array* involves concentric helices representing pitches, intervals, chords and keys in the same spatial framework. The outer helix represents pitch classes, organized along intervals of fifths in the helix such that vertically the interval distance is a major third (see Fig. 2.19).

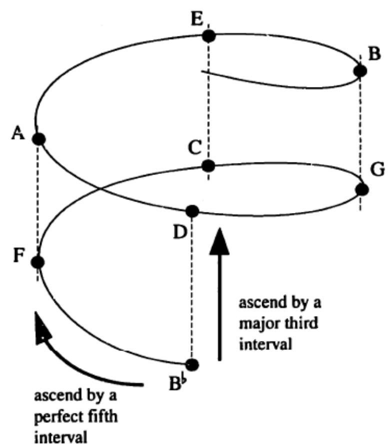


Figure 2.19: Elaine Chew's *Spiral Array*

More generally the chord-based graphs of the second family are circles of n-dimensional cubes linked by shared facets. Musically, the dimension n means that the graph contains n types of different chords. The corresponding note-based graph is a circle of n-dimensional octahedra linked by shared vertices.

Chapter 3

Parsimonious operations and graphs for seventh chords

As we have seen, there are many algebraic formalizations and geometric visualizations of transformations on major and minor triads. But in addition to triads, seventh chords are often used in the music literature. A natural question arises: can we define a group similar to the neo-Riemannian group PLR acting on the set of seventh chords (of the twelve-tone equal temperament)? More precisely: can we define a group of operations between sevenths to describe parsimonious voice leading, so that the generators fix three notes and move a single note by a semitone or a whole tone? And can we construct a generalized *Tonnetz* for sevenths?

Problems on relationships between seventh chords were studied by Childs [23], by Douthett and Steinbach [28], by Gollin [35], by Fiore and Satyendra [31], by Arnett and Barth [8] and by Kerkez [50] for some of the types of seventh chords. In this chapter, we will extend the previous studies by considering different sets of sevenths.

3.1 Previous works on transformations on seventh chords

Childs investigated transformational parsimonious voice leading between dominant and half-diminished sevenths [23]. In particular he studied their P_2-relations[1], i.e. the transformations that fix two notes and move the other two notes by a semitone or a whole tone. He found 9 transformations, classified into two types. The transformation $S_{m(n)}$ moves two notes by semitones in similar motion. Conversely, the transformation $C_{m(n)}$ moves two notes by semitones in contrary motion. For both, m represents the interval class between the two notes that remain fixed, n the interval class between the two notes that are moved. There are 6 transformations of type S and 3 of type C: $S_{2(3)}$, $S_{3(2)}$, $S_{3(4)}$, $S_{4(3)}$, $S_{5(6)}$, $S_{6(5)}$, $C_{3(2)}$, $C_{3(4)}$, $C_{6(5)}$ (see Fig. 3.1).

Figure 3.1: Child's parsimonious transformations between dominant and half-diminished sevenths (to be read vertically). The signs + and − refers to dominant and half-diminished quality.

[1] For more details on P_n-relation see Section 1.4.

Gollin also studied the relationships between the same types of sevenths chords [35] in his three-dimensional expansion of the *Tonnetz*, already mentioned in Section 2.5. We recall that each tetrahedron represents a dominant or a half-diminished seventh. There are six transformations between tetrahedra sharing a common edge: they are spatially represented as a "flip" of the two tetrahedra around their common edge, and musically they exchange the type of sevenths. Each "edge-flip" maintains at least the two notes represented by the two vertices of the shared edge, and in one case the two tetrahedra share three notes (see Fig. 3.2). The 4 "vertex-flips" musically correspond to transformations between dominant and half-diminished sevenths sharing at least one note. Gollin explained that these 10 transformations generate a group isomorphic to the S/W-group, therefore a group with the same structure of the PLR-group.

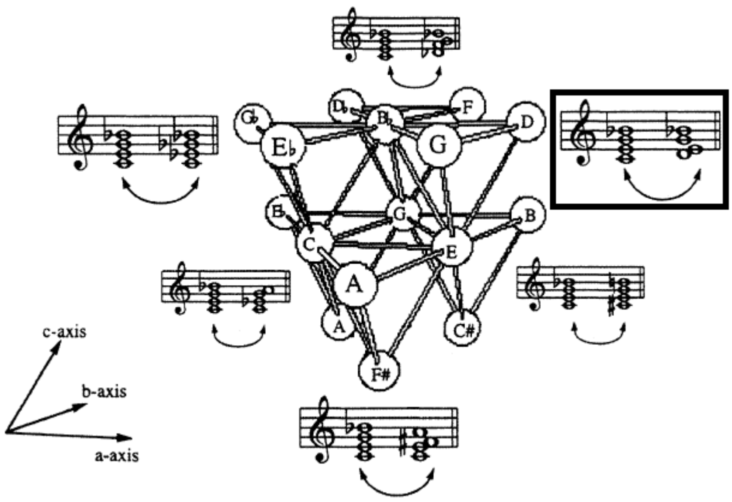


Figure 3.2: The six edge-flips between tetrahedra in the three-dimensional *Tonnetz* by Gollin. In the upper right the only flip in which the tetrahedra represent seventh chords sharing three common notes (from [35].

Arnett and Barth [8] started from the three-dimensional expansion of the *Tonnetz* introduced by Gollin and observed that Gollin's study did not include the minor seventh chords, very common in the music literature. Therefore they proposed to consider a set of 36 chords consisting of all dominant, half-diminished and minor seventh chords and to find the transformations between them that maintain three common notes. They defined the following five operations:

$$
\begin{aligned}
P1&: [x, x+4, x+7, x+10] \rightleftarrows [x, x+3, x+7, x+10]\\
P2&: [x, x+3, x+7, x+10] \rightleftarrows [x, x+3, x+6, x+10]\\
R1&: [x, x+4, x+7, x+10] \rightleftarrows [x, x+4, x+7, x+9]\\
R2&: [x, x+3, x+7, x+10] \rightleftarrows [x, x+3, x+7, x+9]\\
L&: [x, x+4, x+7, x+10] \rightleftarrows [x+2, x+4, x+7, x+10].
\end{aligned}
$$

The first four transformations move a single note by a semitone, whereas L shifts a note by a whole tone. Moreover, L is the algebraic formalization of the edge-flip between tetrahedra representing seventh chords with three common notes described in Gollin's three-dimensional Tonnetz.

Although this study includes more types of seventh chords than Childs' and Gollin's ones, other important types of seventh chords are not considered and the algebraic structure of these transformations is not studied.

Kerkez had the idea to extend the PLR-group to major and minor sevenths [50].

Let H be the set of major and minor seventh chords, that is,

$$H = \{\langle x_1, x_2, x_3, x_4\rangle \mid x_1, x_2, x_3, x_4 \in \mathbb{Z}_{12}, x_2 = x_1 + 4, x_3 = x_1 + 7, x_4 = x_1 + 11\} \cup \\ \{\langle x_1, x_2, x_3, x_4\rangle \mid x_1, x_2, x_3, x_4 \in \mathbb{Z}_{12}, x_3 = x_4 + 3, x_2 = x_4 + 7, x_1 = x_4 + 10\}.$$

Kerkez defines the following two maps $P, S \colon H \to H$:

$$P\langle a, b, c, d\rangle = \langle (type\langle a, b, c, d\rangle) \cdot 2 + d, a, b, c\rangle \\ S\langle a, b, c, d\rangle = \langle b, c, d, (-1) \cdot (type\langle a, b, c, d\rangle) \cdot 2 + a\rangle$$

where

$$type(t) = \begin{cases} 1 & \text{if } t \text{ is a minor seventh} \\ -1 & \text{if } t \text{ is a major seventh.} \end{cases}$$

P maps each major seventh to its relative minor seventh moving the seventh down a whole tone. Conversely, it maps each minor seventh to its relative major seventh moving the root up a whole tone.

S maps each major seventh to the minor seventh having root 4 semitones up, moving its root up a whole tone. On the contrary, it maps each minor seventh to the major seventh having root 4 semitones down, moving its seventh down a whole tone.

Example 3.1. *We consider the major seventh chord* $C^7 = \langle 0, 4, 7, 11\rangle$.

$$P\langle 0, 4, 7, 11\rangle = \langle 7, 4, 0, (-1) \cdot 2 + 11 \textit{ (mod 12)}\rangle = \langle 7, 4, 0, 9\rangle = A_m \\ S\langle 0, 4, 7, 11\rangle = \langle (-1) \cdot (-1) \cdot 2 + 0 \textit{ (mod 12)}, 11, 7, 4\rangle = \langle 2, 11, 7, 4\rangle = E_m.$$

We consider the minor seventh chord $C_m = \langle 10, 7, 3, 0\rangle$.

$$P\langle 10, 7, 3, 0\rangle = \langle 3, 7, 10, (+1) \cdot 2 + 0 \textit{ (mod 12)}\rangle = \langle 3, 7, 10, 2\rangle = E^7_\flat \\ S\langle 10, 7, 3, 0\rangle = \langle (-1) \cdot (+1) \cdot 2 + 10 \textit{ (mod 12)}, 0, 3, 7\rangle = \langle 8, 0, 3, 7\rangle = A^7_\flat.$$

Theorem 3.2. *The transformations P and S act on H generating a group again isomorphic to the dihedral group D_{12} of order* 24.

Proof. We note that $P^2 = S^2 = 1$, $(PS)^{12} = 1$ and $(PS)P(PS) = (PS)(PP)S = (PS)S = P(SS) = P = P^{-1}$. Therefore the group has the following presentation:

$$\langle P, PS | P^2 = 1, (PS)^{12} = 1, (PS)P(PS) = P^{-1}\rangle.$$

□

3.2 Parsimonious operations among seventh chords

Since seventh chords are often used in voice leading, our aim is to find all the most parsimonious operations among seventh chords, similar to the P, L and R operations for triads. The most parsimonious operations fix three notes and move only one note by a semitone or a whole tone. Equivalently: we want to determine all $P_{1,0}$ and $P_{0,1}$ relations among sevenths. We refine the definition of transformations and operations among sevenths, extending in a natural way Definitions 1.41 and 1.42 given by Hook for triads. We will denote the set of all possible seventh chords with Σ.

Definition 3.3 (Seventh chord transformation)**.** *A* ***seventh chord transformation*** *is a map* $\Sigma \to \Sigma$.

Definition 3.4 (Seventh chord operation)**.** *A* ***seventh chord operation*** *is a bijective transformation* $\Sigma \to \Sigma$.

As for the triads, there are some musical transformations that are not operations. For instance there are some different chords with the same secondary dominant, therefore the transformation mapping each chord into its secondary dominant is not an operation.

For our study we do not want to consider all types of sevenths in Σ. For the moment we consider the following types of seventh chords: dominant (D), minor (m), half-diminished (hd), major (M) and diminished (d), and let $H \subset \Sigma$ be the set of all cyclicly marked seventh chords of these 5 types:

$$\begin{aligned} H =&\{[\underline{x_1}, x_2, x_3, x_4] \,|\, x_1, x_2, x_3, x_4 \in \mathbb{Z}_{12}, x_2 = x_1 + 4, x_3 = x_1 + 7, x_4 = x_1 + 10\} \cup \\ &\{[\underline{x_1}, x_2, x_3, x_4] \,|\, x_1, x_2, x_3, x_4 \in \mathbb{Z}_{12}, x_2 = x_1 + 3, x_3 = x_1 + 7, x_4 = x_1 + 10\} \cup \\ &\{[\underline{x_1}, x_2, x_3, x_4] \,|\, x_1, x_2, x_3, x_4 \in \mathbb{Z}_{12}, x_2 = x_1 + 3, x_3 = x_1 + 6, x_4 = x_1 + 10\} \cup \\ &\{[\underline{x_1}, x_2, x_3, x_4] \,|\, x_1, x_2, x_3, x_4 \in \mathbb{Z}_{12}, x_2 = x_1 + 4, x_3 = x_1 + 7, x_4 = x_1 + 11\} \cup \\ &\{[\underline{x_1}, x_2, x_3, x_4] \,|\, x_1, x_2, x_3, x_4 \in \mathbb{Z}_{12}, x_2 = x_1 + 3, x_3 = x_1 + 6, x_4 = x_1 + 9\} \end{aligned}$$

As in Lewin and Hook's terminology, we define a *transformation* among seventh as a map $H \to H$, and an *operation* as a bijective transformation $H \to H$.

We first analyze all $P_{1,0}$ relations, that is the transformations moving just one note by one semitone: let us call Q_{i+} the map that sends each type of seventh chord to another type moving the i-th member up a semitone, where $i = R, T, F, S$ depending on whether the member is considered to be the root (R), the third (T), fifth (F) or seventh (S), respectively. Analogously, let Q_{i-} be the map that moves the i-th member down a semitone. By overstriking the maps that do not produce any of the classical types of seventh chord, we have:

$Q_{R+}(D) = d$	$Q_{R+}(m) = D$	$Q_{R+}(hd) = m$	$Q_{R+}(M) = hd$	$Q_{R+}(d) = hd$
~~$Q_{R-}(D)$~~	~~$Q_{R-}(m)$~~	$Q_{R-}(hd) = M$	~~$Q_{R-}(M)$~~	$Q_{R-}(d) = D$
~~$Q_{T+}(D)$~~	$Q_{T+}(m) = D$	~~$Q_{T+}(hd)$~~	~~$Q_{T+}(M)$~~	$Q_{T+}(d) = hd$
$Q_{T-}(D) = m$	~~$Q_{T-}(m)$~~	~~$Q_{T-}(hd)$~~	~~$Q_{T-}(M)$~~	$Q_{T-}(d) = D$
~~$Q_{F+}(D)$~~	~~$Q_{F+}(m)$~~	$Q_{F+}(hd) = m$	~~$Q_{F+}(M)$~~	$Q_{F+}(d) = hd$
~~$Q_{F-}(D)$~~	$Q_{F-}(m) = hd$	~~$Q_{F-}(hd)$~~	~~$Q_{F-}(M)$~~	$Q_{F-}(d) = D$
$Q_{S+}(D) = M$	~~$Q_{S+}(m)$~~	~~$Q_{S+}(hd)$~~	~~$Q_{S+}(M)$~~	$Q_{S+}(d) = hd$
$Q_{S-}(D) = m$	$Q_{S-}(m) = hd$	$Q_{S-}(hd) = d$	$Q_{S-}(M) = D$	$Q_{S-}(d) = D$.

We observe that some transformations are inverse to each other:

$Q_{R+}(M) = hd$	$Q_{R-}(hd) = M$
$Q_{R+}(m) = D$	$Q_{S-}(D) = m$
$Q_{R+}(hd) = m$	$Q_{S-}(m) = hd$
$Q_{S+}(D) = M$	$Q_{S-}(M) = D$
$Q_{T+}(m) = D$	$Q_{T-}(D) = m$
$Q_{F+}(hd) = m$	$Q_{F-}(m) = hd$.

It remains to consider the following operations:

$$Q_{R+}(D) = d \quad Q_{R-}(d) = D \quad Q_{T-}(d) = D \quad Q_{F-}(d) = D \quad Q_{S-}(d) = D$$
$$Q_{S-}(hd) = d \quad Q_{S+}(d) = hd \quad Q_{R+}(d) = hd \quad Q_{T+}(d) = hd \quad Q_{F+}(d) = sd.$$

Q_{R+} is the inverse of Q_{R-}, Q_{T-}, Q_{F-} and Q_{S-}. This is due to the particular symmetry of the intervallic structure $(3, 3, 3, 3)$ of diminished sevenths, in which the members of the chord play an identical role. From an acoustical point of view we have only 3 diminished sevenths: $C^o = E\flat^o = G\flat^o = A^o = [0, 3, 6, 9], C\sharp^o = E^o = G^o = B\flat^o = [1, 4, 7, 10], D^o = F^o = A\flat^o = B^o = [2, 5, 8, 11]$. Unlike the other four types, the diminished sevenths would be only three (and not twelve), e.g. $C, C\sharp, D$, because the other nine chords are three by three enharmonic to them. This explains why we have four transformations that have the same inverse. Since we are considering cyclicly marked chords, the diminished sevenths count as 12 distinct chords, which means that our set of transformations is well-defined. Hence we have 4 transformations between diminished and half-diminished seventh chords and 4 transformations between diminished and dominant seventh chords

$$\begin{array}{ll}
Q_{S-}(hd) = d & Q_{R-}(d) = hd \\
Q_{S-}(hd) = d & Q_{T-}(d) = hd \\
Q_{S-}(hd) = d & Q_{F-}(d) = hd \\
Q_{S-}(hd) = d & Q_{S-}(d) = hd \\
Q_{R+}(D) = d & Q_{R-}(d) = D \\
Q_{R+}(D) = d & Q_{T-}(d) = D \\
Q_{R+}(D) = d & Q_{F-}(d) = D \\
Q_{R+}(D) = d & Q_{S-}(d) = D.
\end{array}$$

Now we consider the $P_{0,1}$ relations, i.e. the transformations that move a single note by a whole tone. Analogously to what was done above, let us call Q_{i++} the map which sends each type of seventh chord in another type moving the i-th member up a whole tone, and Q_{i--} the map which moves the i-th member down a whole tone. We obtain another classical type of seventh chords only moving the root up a whole tone or the seventh down a whole tone:

$$\begin{array}{lllll}
Q_{R++}(D) = hd & Q_{R++}(m) = M & \cancel{Q_{R++}(hd)} & Q_{R++}(M) = m & \cancel{Q_{R++}(d)} \\
\cancel{Q_{S--}(D)} & Q_{S--}(m) = M & Q_{S--}(hd) = D & Q_{S--}(M) = m & \cancel{Q_{S--}(d)}.
\end{array}$$

Again, we find some transformations that are the inverse of each other:

$$\begin{array}{ll}
Q_{R++}(D) = hd & Q_{S--}(hd) = D \\
Q_{R++}(m) = M & Q_{S--}(M) = m \\
Q_{R++}(M) = m & Q_{S--}(m) = M.
\end{array}$$

Overall we have 17 transformations corresponding to a parsimonious voice leading among our 5 types of seventh chords.

We want to define these transformations similarly to the neo-Riemannian operations. Let K be a set of chords of different types and of the same cardinality (in our case $K = H$). We start by defining a generalized parallel operation P.

Definition 3.5. *Let $P_{ij} \colon K \to K$ be the maps which send an i-th type of chord to a j-th type of chord with the* *same root, $i \neq j$, and vice versa, and which are identity the other types.*

In H the types of the seventh chords are 5, and we will denote them with the numbers from 1 to 5: "1" is for dominant sevenths, "2" for minor sevenths, "3" for half-diminished sevenths, "4" for major sevenths and "5" for diminished sevenths.

4 of the 17 transformations are parallel operations:

$$\begin{array}{lll} Q_T & \rightsquigarrow & P_{12}\colon [\underline{x}, x+4, x+7, x+10] \rightleftarrows [\underline{x}, x+3, x+7, x+10] \\ Q_S & \rightsquigarrow & P_{14}\colon [\underline{x}, x+4, x+7, x+10] \rightleftarrows [\underline{x}, x+4, x+7, x+11] \\ Q_F & \rightsquigarrow & P_{23}\colon [\underline{x}, x+3, x+7, x+10] \rightleftarrows [\underline{x}, x+3, x+6, x+10] \\ Q_S & \rightsquigarrow & P_{35}\colon [\underline{x}, x+3, x+6, x+10] \rightleftarrows [\underline{x}, x+3, x+6, x+9]. \end{array}$$

Remark 3.6. *P_{12} and P_{23} coincide with $P1$ and $P2$ defined by Arnett and Barth.*

Now we define the generalized relative and leading-tone operations. We recall that, from Proposition 1.27, if the triad is major $R = P \circ T_{-3} = T_{-3} \circ P$, if it is minor $R = P \circ T_3 = T_3 \circ P$.

Definition 3.7. *Let $R_{ij} \colon K \to K$ be the maps which send an i-th type of chord to a j-th type chord whose root is transposed 3 semitones down, a j-th type of chord to an i-th type of chord transposed 3 semitones up, and which fix the other types:*

$$R_{ij} = T_{\pm 3} \circ P_{ij} = P_{ij} \circ T_{\pm 3}. \tag{3.1}$$

More precisely, we are using the usual symbol T_n, but it is not the usual transposition, because it is to be considered applied only to the i-th and j-th types of sevenths. We will continue to use this notation also in the other definitions and in the following chapters.

Now, 5 of the 17 transformations are generalized relative operations:

$$\begin{array}{lll} Q_R, Q_S & \rightsquigarrow & R_{12}\colon [\underline{x}, x+4, x+7, x+10] \rightleftarrows [x, x+4, x+7, \underline{x+9}] \\ Q_R, Q_S & \rightsquigarrow & R_{23}\colon [\underline{x}, x+3, x+7, x+10] \rightleftarrows [x, x+3, x+7, \underline{x+9}] \\ Q_R, Q_S & \rightsquigarrow & R_{42}\colon [\underline{x}, x+4, x+7, x+11] \rightleftarrows [x, x+4, x+7, \underline{x+9}] \\ Q_R, Q_S & \rightsquigarrow & R_{35}\colon [\underline{x}, x+3, x+6, x+10] \rightleftarrows [x, x+3, x+6, \underline{x+9}] \\ Q_F, Q_S & \rightsquigarrow & R_{53}\colon [\underline{x}, x+3, x+6, x+9] \rightleftarrows [x, x+3, x+7, \underline{x+9}]. \end{array}$$

Remark 3.8. *R_{12} and R_{23} coincide with $R1$ and $R2$ defined by Arnett and Barth. Moreover, R_{42} coincide with the map P defined by Kerkez.*

For the operation L we recall that, from Proposition 1.27, if the triad is major $L = P \circ T_4 = T_4 \circ P$, if it is minor $L = P \circ T_{-4} = T_{-4} \circ P$.

Definition 3.9. *Let $L_{ij} \colon H \to H$ be the maps which send an i-th type of chord to a j-th type of chord whose root is transposed 4 semitones up, a j-th type of chord to an i-th type of chord transposed 4 semitones down, and which fix the other types:*

$$L_{ij} = T_{\pm 4} \circ P_{ij} = P_{ij} \circ T_{\pm 4} \tag{3.2}$$

This time, 3 of the 17 transformations are L_{ij} operation:

$$
\begin{array}{rcl}
Q_{R++} & \rightsquigarrow & L_{13}\colon [\underline{x}, x+4, x+7, x+10] \;\rightleftarrows\; [x+2, \underline{x+4}, x+7, x+10] \\
Q_R & \rightsquigarrow & L_{15}\colon [\underline{x}, x+4, x+7, x+10] \;\rightleftarrows\; [x+1, \underline{x+4}, x+7, x+10] \\
Q_{R++} & \rightsquigarrow & L_{42}\colon [\underline{x}, x+4, x+7, x+11] \;\rightleftarrows\; [x+2, \underline{x+4}, x+7, x+11].
\end{array}
$$

Remark 3.10. *L_{13} coincides with L defined by Arnett and Barth and the "edge-flip" described by Gollin in his three-dimensional* Tonnetz*.*
L_{42} coincides with S defined by Kerkez.

We have identified 12 of the 17 transformations between seventh chords as operations similar to P, L and R. We now see that the other operations can be defined as the composition of a parallel transformation and a transposition (with a number of semitones different from 3 and 4).

Definition 3.11. *Let Q_{ij} be the maps which send an i-th type of chord to a j-th type of chord transposed* 1 *semitone up, a j-th type of chord to an i-th type of chord transposed* 1 *semitone down, and fix the other types:*

$$Q_{ij} = T_{\pm 1} \circ P_{ij} = P_{ij} \circ T_{\pm 1}. \tag{3.3}$$

Definition 3.12. *Let RR_{ij} be the maps which send an i-th type of chord to a j-th type of chord transposed* 6 *semitones, and which fix the other types:*

$$RR_{ij} = T_{\pm 6} \circ P_{ij} = P_{ij} \circ T_{\pm 6}. \tag{3.4}$$

Definition 3.13. *Let QQ_{ij} be the maps which send an i-th type of chord to a j-th type of chord transposed* 2 *semitones up, a j-th type of chord to an i-th type of chord transposed* 2 *semitones down, and fix the other types:*

$$QQ_{ij} = T_{\pm 2} \circ P_{ij} = P_{ij} \circ T_{\pm 2}. \tag{3.5}$$

Definition 3.14. *Let N_{ij} be the maps which send an i-th type of chord to a j-th type of chord transposed* 5 *semitones up, a j-th type of chord to an i-th type of chord transposed* 5 *semitones down, and fix the other types:*

$$N_{ij} = T_{\pm 5} \circ P_{ij} = P_{ij} \circ T_{\pm 5}. \tag{3.6}$$

With these transformations we can define the missing operations in the following way:

$$
\begin{array}{rcl}
Q_R, Q_S & \rightsquigarrow & Q_{43}\colon [x, x+4, x+7, x+11] \;\rightleftarrows\; [x+1, x+4, x+7, x+11] \\
Q_R & \rightsquigarrow & Q_{15}\colon [\underline{x}, x+4, x+7, x+10] \;\rightleftarrows\; [\underline{x+1}, x+4, x+7, x+10] \\
Q_T, Q_S & \rightsquigarrow & RR_{35}\colon [\underline{x}, x+3, x+6, x+10] \;\rightleftarrows\; [x, x+3, \underline{x+6}, x+9] \\
Q_R, Q_T & \rightsquigarrow & QQ_{51}\colon [\underline{x}, x+3, x+6, x+9] \;\rightleftarrows\; [x, \underline{x+2}, x+6, x+9] \\
Q_R, Q_F & \rightsquigarrow & N_{51}\colon [\underline{x}, x+3, x+6, x+9] \;\rightleftarrows\; [x, x+3, \underline{x+5}, x+9].
\end{array}
$$

Remark 3.15. *Our* 17 *operations and the neo-Riemannian P, L and R operations have some features in common.*

- *Each operation exchanges two types of chord, thus it is mode-reversing.*
- *Each operation is an involution.*
- *Each operation move only one note by a semitone or by a whole tone.*

- *Like the neo-Riemannian operations act on the set S of all 24 the major and minor triads, our 17 operations act on the set H of all 60 sevenths considered.*
- *The 17 operations we found satisfy the uniformity condition introduced by Hook for triads in Definition 1.43.*

Remark 3.16. *Crans, Fiore and Satyendra define P, L and R as inversions I_n; since inversions are isometries, they leave unchanged lengths and angles, and minor and major triads geometrically are represented by triangles which the edge lengths correspond to 3, 4 and 5 semitones. This idea could in principle also be used to define transformations between seventh chords, but it cannot be applied to all types since the lengths of the edges and the angles of the quadrilaterals that compose them are not equal. We have only 2 quadrilaterals that are isometric: the one representing the dominant sevenths and the one representing half-diminished sevenths. There exists a unique transformation between these types of seventh chords, L_{13}.*

We summarize the 17 transformations:

$$
\begin{aligned}
P_{12}\colon [\underline{x}, x+4, x+7, x+10] &\rightleftarrows [\underline{x}, x+3, x+7, x+10]\\
P_{14}\colon [\underline{x}, x+4, x+7, x+10] &\rightleftarrows [\underline{x}, x+4, x+7, x+11]\\
P_{23}\colon [\underline{x}, x+3, x+7, x+10] &\rightleftarrows [\underline{x}, x+3, x+6, x+10]\\
P_{35}\colon [\underline{x}, x+3, x+6, x+10] &\rightleftarrows [\underline{x}, x+3, x+6, x+9]\\
R_{12}\colon [\underline{x}, x+4, x+7, x+10] &\rightleftarrows [x, x+4, x+7, \underline{x+9}]\\
R_{23}\colon [\underline{x}, x+3, x+7, x+10] &\rightleftarrows [x, x+3, x+7, \underline{x+9}]\\
R_{42}\colon [\underline{x}, x+4, x+7, x+11] &\rightleftarrows [x, x+4, x+7, \underline{x+9}]\\
R_{35}\colon [\underline{x}, x+3, x+6, x+10] &\rightleftarrows [x, x+3, x+6, \underline{x+9}]\\
R_{53}\colon [\underline{x}, x+3, x+6, x+9] &\rightleftarrows [x, x+3, x+7, \underline{x+9}]\\
L_{13}\colon [\underline{x}, x+4, x+7, x+10] &\rightleftarrows [x+2, \underline{x+4}, x+7, x+10]\\
L_{15}\colon [\underline{x}, x+4, x+7, x+10] &\rightleftarrows [x+1, \underline{x+4}, x+7, x+10]\\
L_{42}\colon [\underline{x}, x+4, x+7, x+11] &\rightleftarrows [x+2, \underline{x+4}, x+7, x+11]\\
Q_{43}\colon [x, x+4, x+7, x+11] &\rightleftarrows [x+1, x+4, x+7, x+11]\\
Q_{15}\colon [\underline{x}, x+4, x+7, x+10] &\rightleftarrows [\underline{x+1}, x+4, x+7, x+10]\\
RR_{35}\colon [\underline{x}, x+3, x+6, x+10] &\rightleftarrows [x, x+3, \underline{x+6}, x+9]\\
QQ_{51}\colon [\underline{x}, x+3, x+6, x+9] &\rightleftarrows [x, \underline{x+2}, x+6, x+9]\\
N_{51}\colon [\underline{x}, x+3, x+6, x+9] &\rightleftarrows [x, x+3, \underline{x+5}, x+9].
\end{aligned}
$$

To visualize these 17 transformations we can construct a graph whose vertices represent the types of seventh chord, and the edges represent the transformations between them. The graph has 5 vertices and 17 edges.

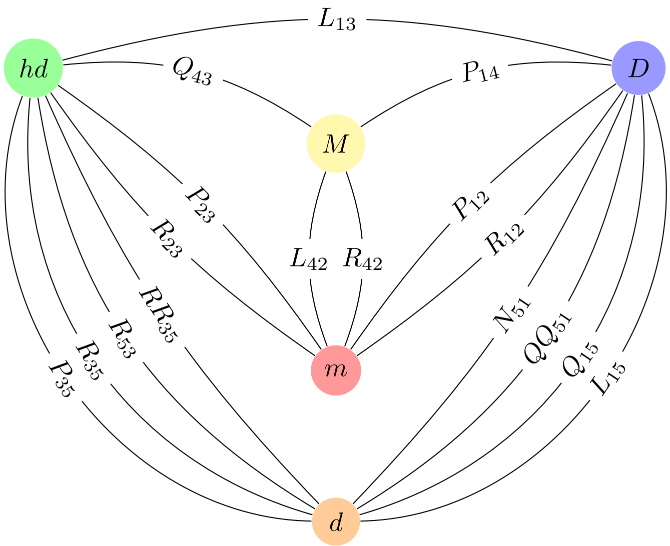


Figure 3.3: The graph representing the 17 transformations between seventh chords.

Sequences of sevenths chords are often used in jazz music. An example in which we can find some of our operations is in Ellington's *Heaven*. Between measures 11-14 there is the following passage: $E^{\Delta} \xrightarrow{Q_{43}} E \xrightarrow{R_{35}} C^{o}_{\sharp} \xrightarrow{L_{15}} A^{7}$.

Figure 3.4: Ellington, *Heaven*, mm. 11-14.

3.3 The $PLRQ$ group

We want to determine the group generated by our 17 operations. Let $PLRQ$ be the group generated by them. Each transformation $t \in PLRQ$ exchanges two types of sevenths and fixes the others, thus we can associate to it a permutation of S_5 (more precisely, a transposition). This information is not sufficient to identify the transformation, because, in general, there exist more transformations that exchange the same types of seventh (for instance P_{12} and R_{12}). Therefore, to identify it we add a vector $v \in \mathbb{Z}_{12}^5$, in which the i-th component, $i \in \{1, \ldots, 5\}$, is the number of semitones of which the root of the chord of type i has to be shifted to become the root of the chord of type j. It is easy to see that in this way no ambiguity is possible.

Example 3.17. *We consider R_{12}. It is identified by*

$$(\sigma, v)_{R_{12}} = ((12), (-3, 3, 0, 0, 0)).$$

In fact, since it exchanges dominant and minor sevenths, the associated transposition of S_5 *is* $\sigma = (12)$. *To determine the associated vector we decompose the definition of* R_{12} *for each type of seventh:*

$$
\begin{aligned}
R_{12}\colon [\underline{x}, x+4, x+7, x+10] &\mapsto [x, x+4, x+7, \underline{x+9}]\\
R_{12}\colon [\underline{x}, x+3, x+7, x+10] &\mapsto [x+1, \underline{x+3}, x+7, x+10]\\
R_{12}\colon [\underline{x}, x+3, x+6, x+10] &\mapsto [\underline{x}, x+3, x+6, x+10]\\
R_{12}\colon [\underline{x}, x+4, x+7, x+11] &\mapsto [\underline{x}, x+4, x+7, x+11]\\
R_{12}\colon [\underline{x}, x+3, x+6, x+9] &\mapsto [\underline{x}, x+3, x+6, x+9].
\end{aligned}
$$

If the seventh is dominant the root is moved 3 *semitones down. Conversely, if the seventh is minor the root is moved* 3 *semitones up. In the other cases the operations is the identity, thus the root remains fixed.*

We write the 17 transformations between seventh chords as pairs of elements $(\sigma, v) \in S_5 \times \mathbb{Z}_{12}^5$ explicitly:

$$
\begin{array}{llll}
P_{12}\colon [\underline{x}, x+4, x+7, x+10] & \rightleftarrows & [\underline{x}, x+3, x+7, x+10] & (\sigma, v) = ((12), (0,0,0,0,0))\\
P_{14}\colon [\underline{x}, x+4, x+7, x+10] & \rightleftarrows & [\underline{x}, x+4, x+7, x+11] & (\sigma, v) = ((14), (0,0,0,0,0))\\
P_{23}\colon [\underline{x}, x+3, x+7, x+10] & \rightleftarrows & [\underline{x}, x+3, x+6, x+10] & (\sigma, v) = ((23), (0,0,0,0,0))\\
P_{35}\colon [\underline{x}, x+3, x+6, x+10] & \rightleftarrows & [\underline{x}, x+3, x+6, x+9] & (\sigma, v) = ((35), (0,0,0,0,0))\\
R_{12}\colon [\underline{x}, x+4, x+7, x+10] & \rightleftarrows & [x, x+4, x+7, \underline{x+9}] & (\sigma, v) = ((12), (-3,3,0,0,0))\\
R_{23}\colon [\underline{x}, x+3, x+7, x+10] & \rightleftarrows & [x, x+3, x+7, \underline{x+9}] & (\sigma, v) = ((23), (0,-3,3,0,0))\\
R_{42}\colon [\underline{x}, x+4, x+7, x+11] & \rightleftarrows & [x, x+4, x+7, \underline{x+9}] & (\sigma, v) = ((42), (0,3,0,-3,0))\\
R_{35}\colon [\underline{x}, x+3, x+6, x+10] & \rightleftarrows & [x, x+3, x+6, \underline{x+9}] & (\sigma, v) = ((35), (0,0,-3,0,3))\\
R_{53}\colon [\underline{x}, x+3, x+6, x+9] & \rightleftarrows & [x, x+3, x+7, \underline{x+9}] & (\sigma, v) = ((53), (0,0,3,0,-3))\\
L_{13}\colon [\underline{x}, x+4, x+7, x+10] & \rightleftarrows & [x+2, \underline{x+4}, x+7, x+10] & (\sigma, v) = ((13), (4,0,-4,0,0))\\
L_{15}\colon [\underline{x}, x+4, x+7, x+10] & \rightleftarrows & [x+1, \underline{x+4}, x+7, x+10] & (\sigma, v) = ((15), (4,0,0,0,-4))\\
L_{42}\colon [\underline{x}, x+4, x+7, x+11] & \rightleftarrows & [x+2, \underline{x+4}, x+7, x+11] & (\sigma, v) = ((42), (0,-4,0,4,0))\\
Q_{43}\colon [\underline{x}, x+4, x+7, x+11] & \rightleftarrows & [\underline{x+1}, x+4, x+7, x+11] & (\sigma, v) = ((43), (0,0,-1,1,0))\\
Q_{15}\colon [\underline{x}, x+4, x+7, x+10] & \rightleftarrows & [\underline{x+1}, x+4, x+7, x+10] & (\sigma, v) = ((15), (1,0,0,0,-1))\\
RR_{35}\colon [\underline{x}, x+3, x+6, x+10] & \rightleftarrows & [x, x+3, \underline{x+6}, x+9] & (\sigma, v) = ((35), (0,0,-6,0,6))\\
QQ_{51}\colon [\underline{x}, x+3, x+6, x+9] & \rightleftarrows & [x, \underline{x+2}, x+6, x+9] & (\sigma, v) = ((51), (-2,0,0,0,2))\\
N_{51}\colon [\underline{x}, x+3, x+6, x+9] & \rightleftarrows & [x, x+3, \underline{x+5}, x+9] & (\sigma, v) = ((51), (-5,0,0,0,5)).
\end{array}
$$

More precisely, we can represent each transformation $t \in PLRQ$ as an element of

$$S_5 \times V \qquad \text{where } V = \{v \in \mathbb{Z}_{12}^5 | \sum_{i=1}^{5} v_i = 0\},$$

since this is clearly true for all the 17 generators. Extending the Hook's definition of Riemannian UTT to sevenths chords, we can observe this means that our operations are Riemannian.

The mapping thus defined becomes a group homomorphism if we define on this set the following operation:

$$
\begin{aligned}
&(\sigma_k, v_k) \circ \cdots \circ (\sigma_1, v_1) =\\
&= \big(\sigma_k \cdots \sigma_1, v_1 + \sigma_1^{-1}(v_2) + (\sigma_2\sigma_1)^{-1}(v_3) + \cdots + (\sigma_{k-1} \cdots \sigma_1)^{-1}(v_k)\big) =\\
&= \big(\sigma_k \cdots \sigma_1, v_1 + \sigma_1^{-1}(v_2) + \sigma_1^{-1}\sigma_2^{-1}(v_3) + \cdots + \sigma_1^{-1} \cdots \sigma_{k-1}^{-1}(v_k)\big).
\end{aligned}
\tag{3.7}
$$

Example 3.18.

$$\begin{aligned}Q_{43}L_{13}R_{12} &= ((43),(0,0,-1,1,0)) \circ ((13),(4,0,-4,0,0)) \circ ((12),(-3,3,0,0,0)) = \\ &= ((1243),(-3,3,0,0,0) + \sigma_1^{-1}(4,0,-4,0,0) + \sigma_1^{-1}\sigma_2^{-1}(0,0,-1,1,0)) = \\ &= ((1243),(-3,3,0,0,0) + (0,4,-4,0,0) + \sigma_1^{-1}(-1,0,0,1,0)) = \\ &= ((1243),(-3,3,0,0,0) + (0,4,-4,0,0) + (0,-1,0,1,0)) = \\ &= ((1243),(-3,6,-4,1,0)).\end{aligned}$$

We want to prove that $PLRQ$ is isomorphic to $S_5 \ltimes V$. We recall the definition of semidirect product of two subgroups.
Let G be a group. If G contains two subgroups H and K such that

i) $G = HK$;

ii) $K \trianglelefteq G$;

iii) $H \cap K = 1$;

G is the *semidirect product* of H and K. Conversely, given two groups H and K and a group homomorphism $\phi\colon H \to Aut(K)$, we can construct a new group $H \ltimes K$ defining in the cartesian product $H \times K$ the following operation:

$$(h_1,k_1)(h_2,k_2) = (h_1h_2, \phi_{h_2}(k_1)\cdot k_2).$$

Theorem 3.19. *The group $PLRQ$ is isomorphic to $S_5 \ltimes \mathbb{Z}_{12}^4$.*

Proof. First of all we prove that $PLRQ$ is isomorphic to $S_5 \ltimes V$.
We observe that the subgroup formed by the elements (Id, v) is normal. In fact, for all $(\sigma, v) \in S_5 \times V, (Id, v') \in \{Id\} \times V$, we have

$$(\sigma,v)(Id,v')(\sigma,v)^{-1} = (\sigma\sigma^{-1}, -v + \sigma(v') + \sigma(v)) = (Id, v'') \in \{Id\} \times Z.$$

On the other hand, since S_5 is generated by transpositions, it is easy to see that, calling O the origin in $\mathbb{Z}_{12}^5$, $S_5 \times \{O\} < PLRQ$, since we already have in it P_{12}, P_{14}, P_{23}, P_{35}.

It is clear that the two subgroups have trivial intersection. Thus, $PLRQ \simeq S_5 \ltimes V$. Now we compute the permutations and vectors associated to $R_{42}L_{42}$, $P_{14}L_{42}P_{14}R_{12}$, $P_{12}L_{13}P_{12}R_{23}$:

$$\begin{aligned}R_{42}L_{42} &= (\sigma', v') \\ \sigma' &= \sigma_2\sigma_1 = (42)(42) = Id \\ v' &= v_1 + \sigma_1^{-1}(v_2) = \\ &= (0,-4,0,4,0) + (0,-3,0,3,0) = \\ &= (0,-7,0,7,0) \\ P_{14}L_{42}P_{14}R_{12} &= (\sigma'', v'') \\ \sigma'' &= \sigma_4\sigma_3\sigma_2\sigma_1 = (14)(42)(14)(12) = Id \\ v'' &= v_1 + \sigma_1^{-1}(v_2) + \sigma_1^{-1}\sigma_2^{-1}(v_3) + \sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}(v_4) = \\ &= (-3,3,0,0,0) + (0,0,0,0,0) + (-4,4,0,0,0) + (0,0,0,0,0) = \\ &= (7,-7,0,0,0)\end{aligned}$$

$$
\begin{aligned}
P_{12}L_{13}P_{12}R_{23} = (\sigma''', v''')& \\
\sigma''' =& \sigma_4\sigma_3\sigma_2\sigma_1 = (12)(13)(12)(23) = Id \\
v''' =& v_1 + \sigma_1^{-1}(v_2) + \sigma_1^{-1}\sigma_2^{-1}(v_3) + \sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}(v_4) = \\
=& (0,-3,3,0,0) + (0,0,0,0,0) + (0,-4,4,0,0) + (0,0,0,0,0) = \\
=& (0,7,-7,0,0).
\end{aligned}
$$

With the following elements just computed

$$
\begin{aligned}
R_{42}L_{42} &= (Id, (0,-7,0,7,0)) \\
P_{14}L_{42}P_{14}R_{12} &= (Id, (7,-7,0,0,0)) \\
P_{12}L_{13}P_{12}R_{23} &= (Id, (0,7,-7,0,0))
\end{aligned}
\tag{3.8}
$$

we can generate each element $(Id, (v_1, v_2, v_3, v_4, 0))$, with $(v_1, v_2, v_3, v_4, 0) \in \mathbb{Z}_{12}^5$ such that $\sum_1^4 v_i = 0$. To see this, taken $a, b, c \in \mathbb{Z}$, we have to solve

$$
\begin{aligned}
a(0,-7,0,7,0) + b(7,-7,0,0,0) + c(0,7,-7,0,0) &\equiv (v_1, v_2, v_3, v_4, 0) \pmod{12} \\
(-7b, -7a + 7b - 7c, 7c, 7a) &\equiv (v_1, v_2, v_3, v_4, 0) \pmod{12}
\end{aligned}
$$

$$
\begin{cases}
-7b \equiv v_1 \\
-7a + 7b - 7c \equiv v_2 \\
7c \equiv v_3 \\
7a \equiv v_4
\end{cases}
\Rightarrow
\begin{cases}
-7b \equiv v_1 \\
7b - 7c \equiv v_2 + 7a \\
7c \equiv -v_3 \\
7a \equiv v_4
\end{cases}
\Rightarrow
\begin{cases}
7b \equiv -v_1 \\
-v_1 - v_3 \equiv v_2 + v_4 \\
7c \equiv -v_3 \\
7a \equiv v_4
\end{cases}
$$

which is solvable because 7 is coprime with 12.

To obtain all elements $(Id, (v_1, v_2, v_3, v_4, v_5))$, with $(v_1, v_2, v_3, v_4, v_5) \in \mathbb{Z}_{12}^5$ such that $\sum_1^5 v_i = 0$, it is sufficient to add to the 3 generators listed in 3.8 the generator $P_{12}P_{35}R_{23}P_{12}L_{15}L_{13} = (Id, (7,0,0,0,-7))$.

From this, it is evident that $V \simeq \mathbb{Z}_{12}^4$, hence $PLRQ \simeq S_5 \ltimes \mathbb{Z}_{12}^4$. □

3.4 The *Clover* graph: a generalized *Chicken-wire Torus* for sevenths

Starting from the algebraic definition of the parsimonious operations among sevenths as pairs of elements $(\sigma, v) \in S_5 \ltimes \mathbb{Z}_{12}^5$, we can construct a generalized *Chicken-wire torus* for seventh chords. It is a chord-based graph in which each vertex represents a seventh chord and each edge identifies a parsimonious musical operation (see Fig. 3.5).

For algebraic reasons we have considered 60 sevenths, 12 for each type, but we recall that some diminished sevenths are enharmonic equivalent. Similarly, some of the 17 transformations are different from a theoretical and mathematical point of view, but enharmonic equivalent from an acoustical point of view: $P_{35} \sim R_{35} \sim R_{53} \sim RR_{35}$ and $L_{15} \sim Q_{15} \sim QQ_{51} \sim N_{51}$. Since our aim is to study paths in our graph and their musical interpretation less than enharmonic equivalences, we identify vertices and edges enharmonic equivalent. Therefore our graph has $12 \cdot 4 + 3 = 51$ vertices and $12 \cdot 11 = 132$ edges.

The *Power Towers* graph (see Fig. 2.15) realized by Douthett and Stainbach [28], that we mentioned in Section 2.5, is a subgraph of our graph. The *Power Towers* is a chord-based graph representing dominant, minor, major and diminished sevenths. These vertices and their edges are also in our graph. Our *Clover* graph differs from the first one for twelve vertices representing major sevenths and the related edges.

If we observe the drawing of our graph, it is evident that a rotation of the plane by $\frac{2\pi}{3}$ induces automor-

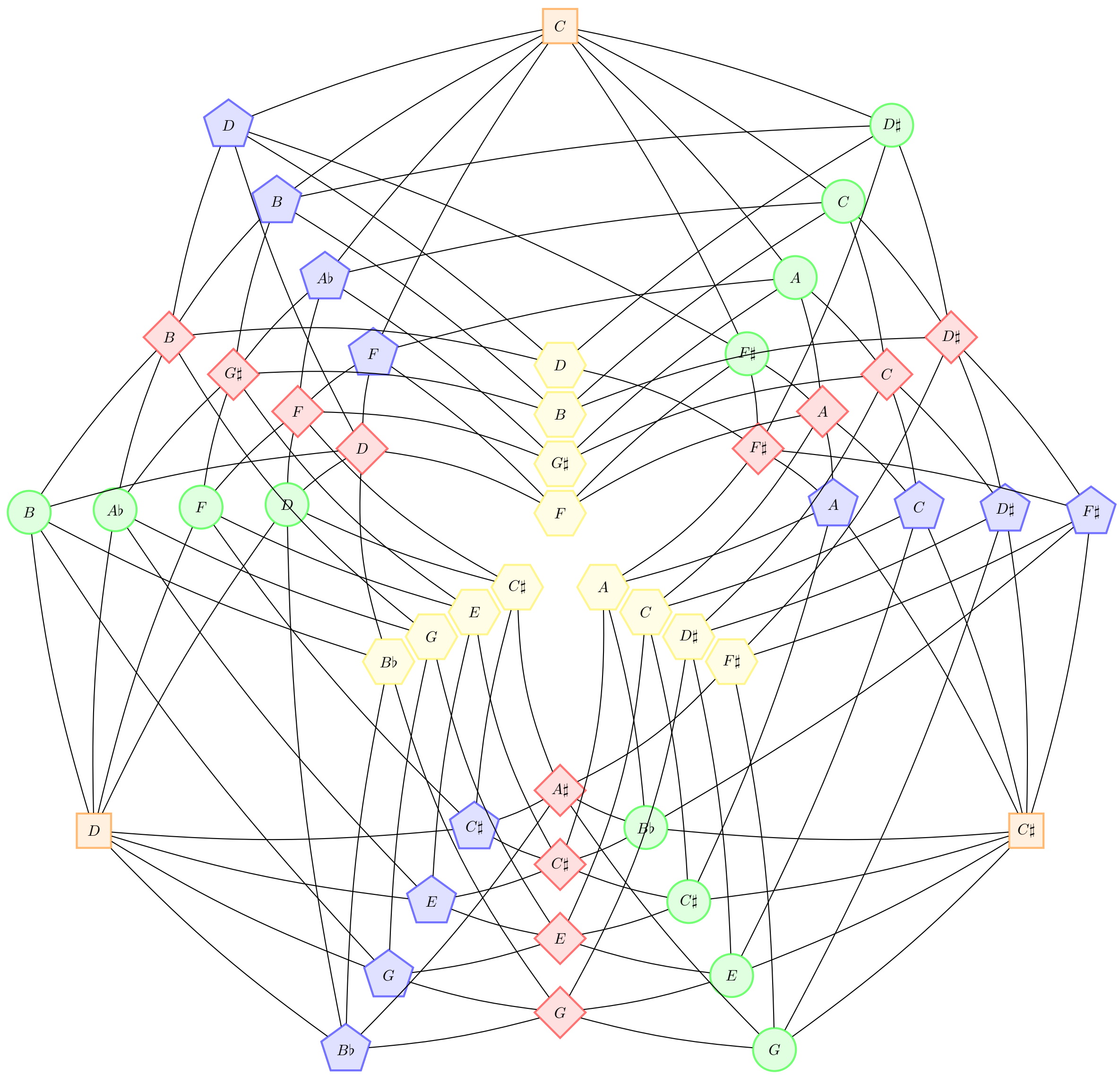


Figure 3.5: The *Clover* graph for seventh chords. Blue-pentagonal, red-rhombic, green-circular, yellow-hexagonal and orange-quadrangular vertices represent, respectively, dominant, minor, half-diminished, major and diminished sevenths.

phisms of our graph. Also reflections of the plane through the axis passing in a diminished seventh and in the center of the graph induce automorphisms.

As Albini and Bernardi observed (see Section 1.5), Hamiltonian cycles of chord-based graphs represent complete sequences through all the admitted chords which only consider certain types of transformations. A natural question arises: is our generalized *Chicken-wire Torus* Hamiltonian?

Unfortunately simple necessary and sufficient conditions for Hamiltonicity are not known in general, and proving whether a graph is Hamiltonian is an NP-complete problem. But using the backtracking algorithm we have found many Hamiltonian cycles and paths, clearly showing that our graph is Hamiltonian. An example of a Hamiltonian cycle in our generalized *Chicken-wire Torus* is the following:

$$\begin{aligned}
&C\sharp^{\Delta} \xrightarrow{L_{42}} F_m \xrightarrow{R_{42}} G\sharp^{\Delta} \xrightarrow{Q_{43}} A^{\emptyset} \xrightarrow{R_{23}} C_m \xrightarrow{P_{23}} C^{\emptyset} \xrightarrow{Q_{43}} B^{\Delta} \xrightarrow{P_{14}} B^7 \xrightarrow{L_{13}} D\sharp^{\emptyset} \xrightarrow{P_{23}} D\sharp_m \xrightarrow{R_{42}} F\sharp^{\Delta} \rightarrow \\
&\quad \xrightarrow{P_{14}} F\sharp^7 \xrightarrow{L_{13}} B\flat^{\emptyset} \xrightarrow{R_{23}} C\sharp_m \xrightarrow{P_{23}} C\sharp^{\emptyset} \xrightarrow{Q_{43}} C^{\Delta} \xrightarrow{P_{14}} C^7 \xrightarrow{L_{13}} E^{\emptyset} \xrightarrow{Q_{43}} D\sharp^{\Delta} \xrightarrow{P_{14}} D\sharp^7 \xrightarrow{L_{15}} C\sharp^o \rightarrow \\
&\quad \xrightarrow{P_{35}} G^{\emptyset} \xrightarrow{P_{23}} G_m \xrightarrow{R_{12}} B\flat^7 \xrightarrow{P_{12}} A\sharp_m \xrightarrow{R_{12}} C\sharp^7 \xrightarrow{L_{13}} F^{\emptyset} \xrightarrow{P_{35}} D^o \xrightarrow{L_{15}} G^7 \xrightarrow{R_{12}} E_m \xrightarrow{R_{42}} G^{\Delta} \rightarrow \\
&\quad \xrightarrow{Q_{43}} A\flat^{\emptyset} \xrightarrow{L_{13}} E^7 \xrightarrow{P_{14}} E^{\Delta} \xrightarrow{L_{42}} G\sharp_m \xrightarrow{P_{12}} A\flat^7 \xrightarrow{L_{15}} C^o \xrightarrow{L_{15}} F^7 \xrightarrow{P_{14}} F^{\Delta} \xrightarrow{L_{42}} A_m \xrightarrow{P_{12}} A^7 \rightarrow \\
&\quad \xrightarrow{P_{14}} A^{\Delta} \xrightarrow{R_{42}} F\sharp_m \xrightarrow{P_{23}} F\sharp^{\emptyset} \xrightarrow{L_{13}} D^7 \xrightarrow{P_{14}} D^{\Delta} \xrightarrow{R_{42}} B_m \xrightarrow{P_{23}} B^{\emptyset} \xrightarrow{Q_{43}} B\flat\Delta \xrightarrow{L_{42}} D_m \xrightarrow{P_{23}} D^{\emptyset} \xrightarrow{Q_{43}} C\sharp^{\Delta}
\end{aligned}$$

There are probably too many Hamiltonian cycles to list them all as Albini did for the traditional *Chicken-wire Torus*. But with such a symmetrical structure in the graph we expect to find sequences[2] of sevenths or other musical structures with some regularity. It is the case of the following interesting Hamiltonian path:

$$\begin{aligned}
&[B\flat^{\Delta} \xrightarrow{Q_{43}} B^{\emptyset} \xrightarrow{L_{13}} G^7 \xrightarrow{R_{12}} E_m] \xrightarrow{R_{42}} \\
&\xrightarrow{R_{42}} [G^{\Delta} \xrightarrow{Q_{43}} A\flat^{\emptyset} \xrightarrow{L_{13}} E^7 \xrightarrow{R_{12}} C\sharp_m] \rightarrow \\
&\xrightarrow{R_{42}} [E^{\Delta} \xrightarrow{Q_{43}} F^{\emptyset} \xrightarrow{L_{13}} C\sharp^7 \xrightarrow{R_{12}} A\sharp_m] \xrightarrow{R_{42}} \\
&\xrightarrow{R_{42}} (C\sharp^{\Delta} \xrightarrow{Q_{43}} D^{\emptyset} \xrightarrow{P_{35}} D^o \xrightarrow{L_{15}} B\flat^7 \xrightarrow{R_{12}} G_m) \rightarrow
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&\xrightarrow{R_{42}} [D\sharp^{\Delta} \xrightarrow{Q_{43}} E^{\emptyset} \xrightarrow{L_{13}} C^7 \xrightarrow{R_{12}} A_m] \xrightarrow{R_{42}} \\
&\xrightarrow{R_{42}} [C^{\Delta} \xrightarrow{Q_{43}} C\sharp^{\emptyset} \xrightarrow{L_{13}} A^7 \xrightarrow{R_{12}} F\sharp_m] \rightarrow \\
&\xrightarrow{R_{42}} [A^{\Delta} \xrightarrow{Q_{43}} B\flat^{\emptyset} \xrightarrow{L_{13}} F\sharp^7 \xrightarrow{R_{12}} D\sharp_m] \xrightarrow{R_{42}} \\
&\xrightarrow{R_{42}} (F\sharp^{\Delta} \xrightarrow{Q_{43}} G^{\emptyset} \xrightarrow{P_{35}} C\sharp^o \xrightarrow{L_{15}} D\sharp^7 \xrightarrow{R_{12}} C_m) \rightarrow
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&\xrightarrow{R_{42}} [G\sharp^{\Delta} \xrightarrow{Q_{43}} A^{\emptyset} \xrightarrow{L_{13}} F^7 \xrightarrow{R_{12}} D_m] \xrightarrow{R_{42}} \\
&\xrightarrow{R_{42}} [F^{\Delta} \xrightarrow{Q_{43}} F\sharp^{\emptyset} \xrightarrow{L_{13}} D^7 \xrightarrow{R_{12}} B_m] \rightarrow \\
&\xrightarrow{R_{42}} [D^{\Delta} \xrightarrow{Q_{43}} D\sharp^{\emptyset} \xrightarrow{L_{13}} B^7 \xrightarrow{R_{12}} G\sharp_m] \xrightarrow{R_{42}} \\
&\xrightarrow{R_{42}} (B^{\Delta} \xrightarrow{Q_{43}} C^{\emptyset} \xrightarrow{P_{35}} C^o \xrightarrow{L_{15}} A\flat^7 \xrightarrow{R_{12}} F_m)
\end{aligned} \tag{3.11}$$

The structure of this Hamiltonian path presents a sequence where the main motif 3.9 (from $B\flat^{\Delta}$ to G_m) is repeated twice 5 semitones up: the first repetition is from $D\sharp^{\Delta}$ to C_m, the second one from $G\sharp^{\Delta}$ to F_m. This Hamiltonian path is interesting not only because it corresponds to this sequence of sevenths, but also because within each of the three motifs we find another sequence. In fact, we consider the first of the three motifs: the

[2] In music theory a *sequence* is the repetition of a motif or melodic element at a higher or lower pitch.

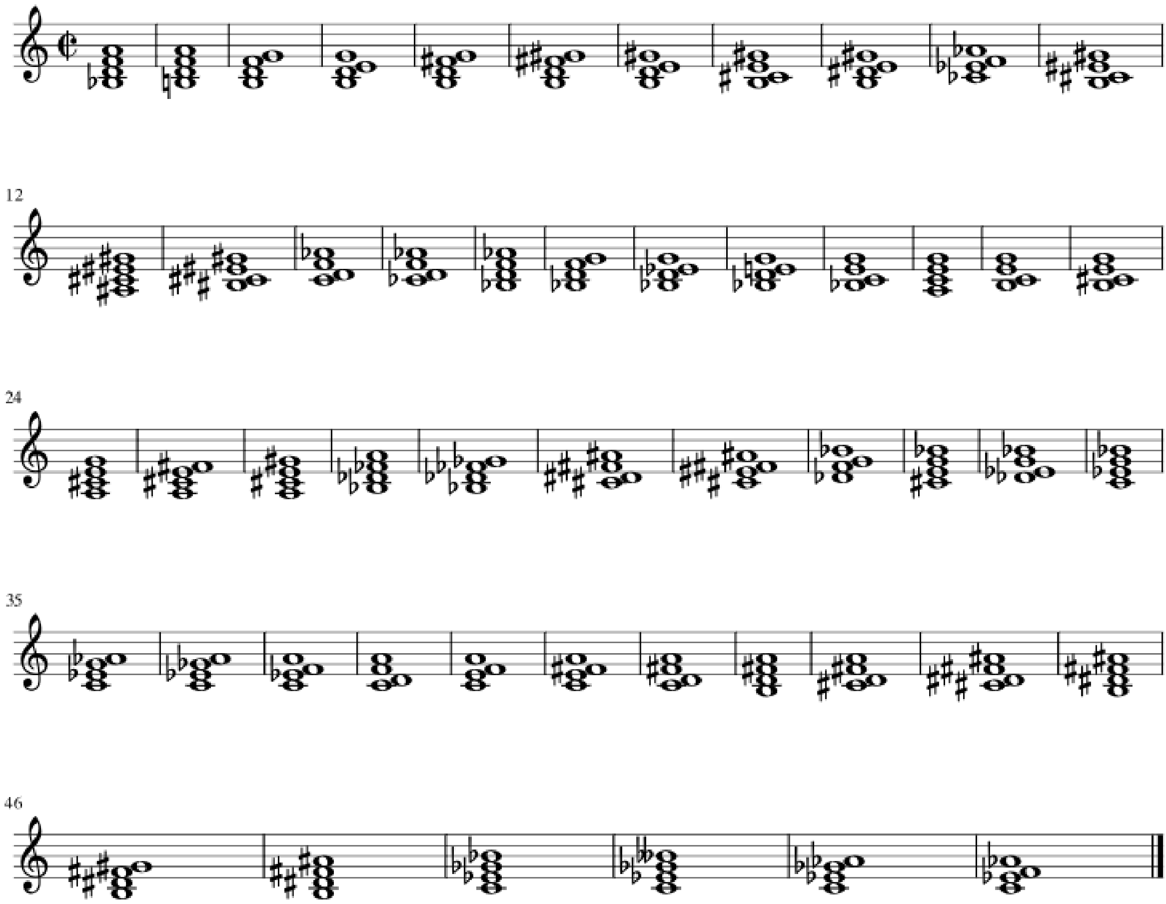


Figure 3.6: A Hamiltonian path in the *Clover* graph corresponding to a seventh chord progression.

musical phrase from $B\flat^{\triangle}$ to E_m is repeated three times 3 semitones down. And in the third repetition the repetition is modified: instead of connecting a half-diminished seventh with a dominant seventh through L_{13}, a diminished seventh is inserted and the connection L_{13} is substituted by $L_{15} \circ P_{35}$. This new phrase is indicated in parentheses. Musically it sounds like a passage that allows the composer to get to the next sequence creating a sense of variety. We used OpenMusic, a visual programming language that has some tools dedicated to this, to write a piece of music using this sequence (see Fig. 3.7), and than let HexaChord analyze its structure (see Fig. Fig. 3.8).

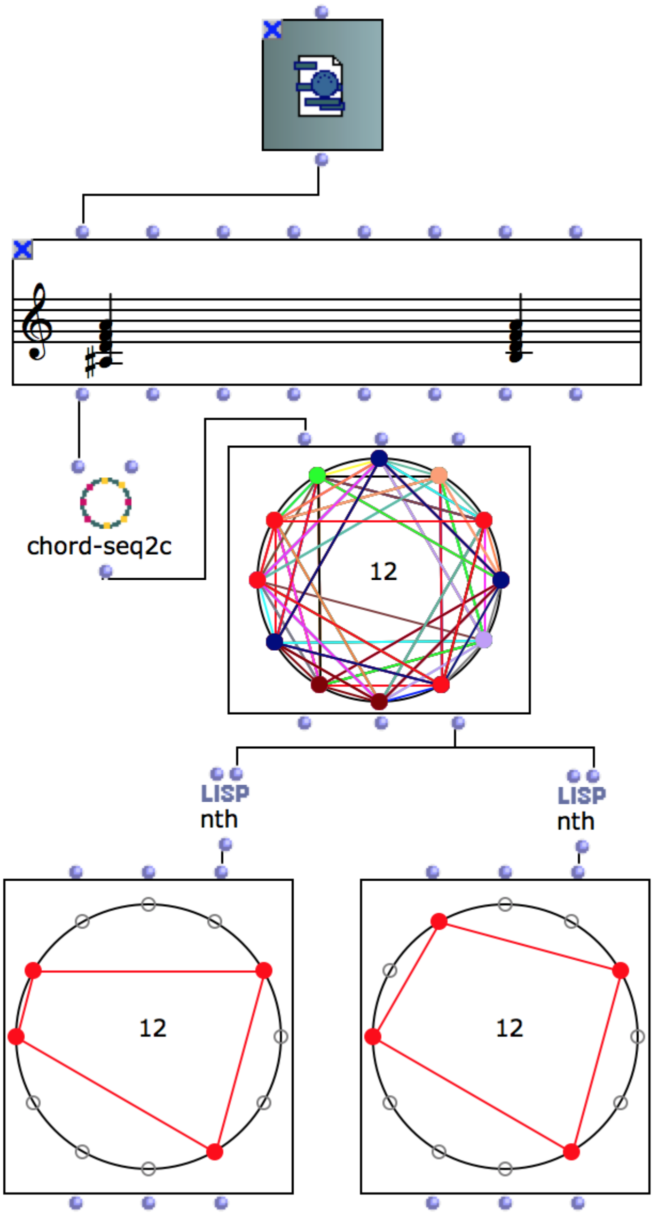


Figure 3.7: A patch in OpenMusic showing the circular representation of the entire chord progression 3.9 of the previous list together with the circular representation of the first two chords.

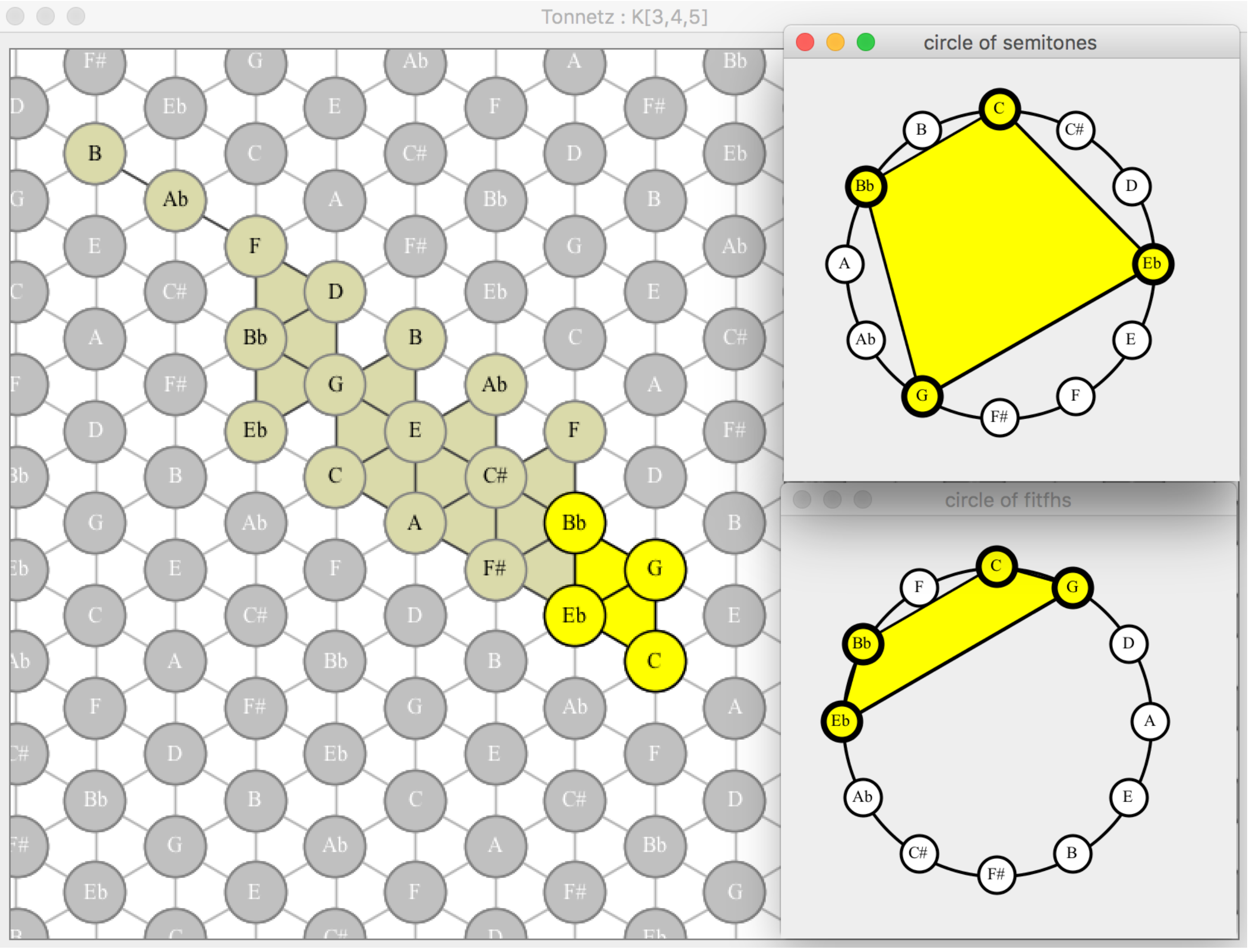


Figure 3.8: The trace of the chord progression 3.9 represented in HexaChord together with two circular representations (chromatic and circle of fifths) of two seventh chord of the same Hamiltonian path.

3.5 Extension of the $PLRQ$ group

For the generalization of the PLR-group to seventh chords, we have considered the following 5 types characterized by the following intervallic structure:

Dominant	$(4, 3, 3, 2)$
Minor	$(3, 4, 3, 2)$
Half-diminished	$(3, 3, 4, 2)$
Major	$(4, 3, 4, 1)$
Diminished	$(3, 3, 3, 3)$.

This was anyway only one of the possible choices, and we may consider also other sets of sevenths. We recall that the sevenths are 4-chords obtained by overlapping 3 thirds. By limiting ourselves to major and minor thirds, the types analyzed up to now do not exhaust all the combinatorial possibilities. In fact we also have other 2 types of sevenths with the following intervallic structure:

$$\begin{aligned} \text{Minor major} \quad & (3,4,4,1) \\ \text{Augmented major} \quad & (4,4,3,1) \end{aligned}$$

Also these types of sevenths are well known in music literature.

For combinatorial reasons, we are interested in studying the most parsimonious operations among all seventh chords with intervallic structure (c_1, c_2, c_3, c_4), where $c_i \in \{1, 2, 3, 4\}$ and such that $\sum_{i=1}^{4} c_i = 12$. The aforementioned sevenths satisfy this condition, together with the two following sevenths:

$$\begin{aligned} \text{Augmented} \quad & (4,4,2,2) \\ \text{Dominant seventh flat five} \quad & (4,2,4,2) \end{aligned}$$

We will continue the numbering of the species following the order of the list above. We summarize all the 9 species in the table 3.1.

Name	Type	Intervallic structure	Cyclic marked chord	Example
Dominant (D)	1	$(4,3,3,2)$	$[\underline{x}, x+4, x+7, x+10]$	$C^7 = [\underline{0}, 4, 7, 10]$
Minor (m)	2	$(3,4,3,2)$	$[\underline{x}, x+3, x+7, x+10]$	$C_m = [\underline{0}, 3, 7, 10]$
Half-diminished (hd)	3	$(3,3,4,2)$	$[\underline{x}, x+3, x+6, x+10]$	$C = [\underline{0}, 3, 6, 10]$
Major (M)	4	$(4,3,4,1)$	$[\underline{x}, x+4, x+7, x+11]$	$C^\Delta = [\underline{0}, 4, 7, 11]$
Diminished (d)	5	$(3,3,3,3)$	$[\underline{x}, x+3, x+6, x+9]$	$C^o = [\underline{0}, 3, 6, 9]$
Minor major (mM)	6	$(3,4,4,1)$	$[\underline{x}, x+3, x+7, x+11]$	$C_m^\Delta = [\underline{0}, 3, 7, 11]$
Augmented major (AM)	7	$(4,4,3,1)$	$[\underline{x}, x+4, x+8, x+11]$	$C_+^\Delta = [\underline{0}, 4, 8, 11]$
Augmented (A)	8	$(4,4,2,2)$	$[\underline{x}, x+4, x+8, x+10]$	$C_+^7 = [\underline{0}, 4, 8, 10]$
Dominant seventh flat five (Df)	9	$(4,2,4,2)$	$[\underline{x}, x+4, x+6, x+10]$	$C^{7\flat5} = [\underline{0}, 4, 6, 10]$

Table 3.1: Summary of the 9 types of seventh chords.

Let H^* be the set of all sevenths of the 9 types listed above:

$$\begin{aligned} H^* =& \{[\underline{x_1}, x_2, x_3, x_4] \,|\, x_1, x_2, x_3, x_4 \in \mathbb{Z}_{12}, x_2 = x_1+4, x_3 = x_1+7, x_4 = x_1+10\} \cup \\ & \{[\underline{x_1}, x_2, x_3, x_4] \,|\, x_1, x_2, x_3, x_4 \in \mathbb{Z}_{12}, x_2 = x_1+3, x_3 = x_1+7, x_4 = x_1+10\} \cup \\ & \{[\underline{x_1}, x_2, x_3, x_4] \,|\, x_1, x_2, x_3, x_4 \in \mathbb{Z}_{12}, x_2 = x_1+3, x_3 = x_1+6, x_4 = x_1+10\} \cup \\ & \{[\underline{x_1}, x_2, x_3, x_4] \,|\, x_1, x_2, x_3, x_4 \in \mathbb{Z}_{12}, x_2 = x_1+4, x_3 = x_1+7, x_4 = x_1+11\} \cup \\ & \{[\underline{x_1}, x_2, x_3, x_4] \,|\, x_1, x_2, x_3, x_4 \in \mathbb{Z}_{12}, x_2 = x_1+3, x_3 = x_1+6, x_4 = x_1+9\} \cup \\ & \{[\underline{x_1}, x_2, x_3, x_4] \,|\, x_1, x_2, x_3, x_4 \in \mathbb{Z}_{12}, x_2 = x_1+3, x_3 = x_1+7, x_4 = x_1+11\} \cup \\ & \{[\underline{x_1}, x_2, x_3, x_4] \,|\, x_1, x_2, x_3, x_4 \in \mathbb{Z}_{12}, x_2 = x_1+4, x_3 = x_1+8, x_4 = x_1+11\} \cup \\ & \{[\underline{x_1}, x_2, x_3, x_4] \,|\, x_1, x_2, x_3, x_4 \in \mathbb{Z}_{12}, x_2 = x_1+4, x_3 = x_1+8, x_4 = x_1+10\} \cup \\ & \{[\underline{x_1}, x_2, x_3, x_4] \,|\, x_1, x_2, x_3, x_4 \in \mathbb{Z}_{12}, x_2 = x_1+4, x_3 = x_1+6, x_4 = x_1+10\}. \end{aligned}$$

In order to determine all $P_{1,0}$ and $P_{0,1}$ relations on the set H^*, the first step is the exhaustive search of any seventh obtained by moving all members of each type of seventh by a semitone or a whole tone. We denote again with Q_{i+} and Q_{i++} the maps that transforms each seventh moving the i-th member up a semitone and a whole tone, where $i = R, T, F, S$, and with Q_{i-} and Q_{i--} the maps that move the i-th member down a semitone and a whole tone respectively.[3]

$Q_{R+}(D) = d$	$Q_{R+}(m) = D$	$Q_{R+}(hd) = m$	$Q_{R+}(M) = hd$	$Q_{R+}(d) = hd$
	$Q_{R+}(mM) = A$	$Q_{R+}(AM) = m$	$Q_{R+}(A) = hd$	$Q_{R+}(Df) = D$
$Q_{R-}(D)$	$Q_{R-}(m) = AM$	$Q_{R-}(d) = M$	$\cancel{Q_{R-}(M)}$	$Q_{R-}(d) = D$
	$\cancel{Q_{R-}(mM)}$	$\cancel{Q_{R-}(AM)}$	$\cancel{Q_{R-}(A)}$	$\cancel{Q_{R-}(Df)}$
$\cancel{Q_{T+}(D)}$	$Q_{T+}(m) = D$	$Q_{T+}(hd) = Df$	$\cancel{Q_{T+}(M)}$	$Q_{T+}(d) = hd$
	$Q_{T+}(mM) = M$	$\cancel{Q_{T+}(AM)}$	$\cancel{Q_{T+}(A)}$	$\cancel{Q_{T+}(Df)}$
$Q_{T-}(D) = m$	$\cancel{Q_{T-}(m)}$	$Q_{T-}(hd) = A$	$Q_{T-}(M) = mM$	$Q_{T-}(d) = D$
	$\cancel{Q_{T-}(mM)}$	$\cancel{Q_{T-}(AM)}$	$\cancel{Q_{T-}(A)}$	$Q_{T-}(Df) = hd$
$Q_{F+}(D) = A$	$\cancel{Q_{F+}(m)}$	$Q_{F+}(hd) = m$	$Q_{F+}(M) = AM$	$Q_{F+}(d) = hd$
	$\cancel{Q_{F+}(mM)}$	$\cancel{Q_{F+}(AM)}$	$\cancel{Q_{F+}(A)}$	$Q_{F+}(Df) = D$
$Q_{F-}(D) = Df$	$Q_{F-}(m) = hd$	$\cancel{Q_{F-}(hd)}$	$\cancel{Q_{F-}(M)}$	$Q_{F-}(d) = D$
	$\cancel{Q_{F-}(mM)}$	$Q_{F-}(AM) = M$	$Q_{F-}(A) = D$	$\cancel{Q_{F-}(Df)}$
$Q_{S+}(D) = M$	$Q_{S+}(m) = mM$	$\cancel{Q_{S+}(hd)}$	$\cancel{Q_{S+}(M)}$	$Q_{S+}(d) = hd$
	$\cancel{Q_{S+}(mM)}$	$\cancel{Q_{S+}(AM)}$	$Q_{S+}(A) = AM$	$\cancel{Q_{S+}(Df)}$
$Q_{S-}(D) = m$	$Q_{S-}(m) = hd$	$Q_{S-}(hd) = d$	$Q_{S-}(M) = D$	$Q_{S-}(d) = D$
	$Q_{S-}(mM) = m$	$Q_{S-}(AM) = A$	$Q_{S-}(A) = mM$	$Q_{S-}(Df) = hd$

$Q_{R++}(D) = hd$	$Q_{R++}(m) = M$	$Q_{R++}(hd) = mM$	$Q_{R++}(M) = m$	$\cancel{Q_{R++}(d)}$
	$Q_{R++}(mM) = AM$	$Q_{R++}(AM) = D$	$Q_{R++}(A) = Df$	$Q_{R++}(Df) = A$
$\cancel{Q_{R--}(D)}$	$\cancel{Q_{R--}(m)}$	$\cancel{Q_{R--}(hd)}$	$\cancel{Q_{R--}(M)}$	$\cancel{Q_{R--}(d)}$
	$Q_{R--}(mM) = AM$	$\cancel{Q_{R--}(AM)}$	$\cancel{Q_{R--}(A)}$	$\cancel{Q_{R--}(Df)}$
$\cancel{Q_{T++}(D)}$	$\cancel{Q_{T++}(m)}$	$\cancel{Q_{T++}(hd)}$	$\cancel{Q_{T++}(M)}$	$\cancel{Q_{T++}(d)}$
	$\cancel{Q_{T++}(mM)}$	$\cancel{Q_{T++}(AM)}$	$\cancel{Q_{T++}(A)}$	$\cancel{Q_{T++}(Df)}$
$\cancel{Q_{T--}(D)}$	$\cancel{Q_{T--}(m)}$	$\cancel{Q_{T--}(hd)}$	$\cancel{Q_{T--}(M)}$	$\cancel{Q_{T--}(d)}$
	$\cancel{Q_{T--}(mM)}$	$\cancel{Q_{T--}(AM)}$	$\cancel{Q_{T--}(A)}$	$Q_{T--}(Df) = A$
$\cancel{Q_{F++}(D)}$	$\cancel{Q_{F++}(m)}$	$\cancel{Q_{F++}(hd)}$	$\cancel{Q_{F++}(M)}$	$\cancel{Q_{F++}(d)}$
	$\cancel{Q_{F++}(mM)}$	$\cancel{Q_{F++}(AM)}$	$\cancel{Q_{F++}(A)}$	$Q_{F++}(Df) = A$
$\cancel{Q_{F--}(D)}$	$\cancel{Q_{F--}(m)}$	$\cancel{Q_{F--}(hd)}$	$\cancel{Q_{F--}(M)}$	$\cancel{Q_{F--}(d)}$
	$\cancel{Q_{F--}(mM)}$	$\cancel{Q_{F--}(AM)}$	$Q_{F--}(A) = Df$	$\cancel{Q_{F--}(Df)}$
$\cancel{Q_{S++}(D)}$	$\cancel{Q_{S++}(m)}$	$\cancel{Q_{S++}(hd)}$	$\cancel{Q_{S++}(M)}$	$\cancel{Q_{S++}(d)}$
	$\cancel{Q_{S++}(mM)}$	$Q_{S++}(AM) = mM$	$\cancel{Q_{S++}(A)}$	$\cancel{Q_{S++}(Df)}$
$Q_{S--}(D) = AM$	$Q_{S--}(m) = M$	$Q_{S--}(hd) = D$	$Q_{S--}(M) = m$	$\cancel{Q_{S--}(d)}$
$Q_{S--}(mM) = hd$	$Q_{S--}(AM) = mM$	$\cancel{Q_{S--}(AM)}$	$Q_{S--}(Df) = A.$	

The second step is to associate each pair of inverses and to classify the parsimonious operations. Now we

[3]The additions and changes to the previous calculations are highlighted in red.

list and classify all new parsimonious operations. We do it only for the new transformations highlighted in red.

$$
\begin{array}{rcll}
Q_{F+}(AM) = m \quad Q_{F-}(m) = AM & \rightsquigarrow & Q_{62}\colon [\underline{x}, x+4, x+8, x+11] & \rightleftarrows \; [\underline{x+1}, x+4, x+8, x+11] \\
Q_{S-}(A) = mM \quad Q_{F+}(mM) = A & \rightsquigarrow & R_{86}\colon [\underline{x}, x+4, x+8, x+10] & \rightleftarrows \; [x, x+4, x+8, \underline{x+9}] \\
Q_{T-}(hd) = A \quad Q_{F+}(A) = hd & \rightsquigarrow & QQ_{38}\colon [\underline{x}, x+3, x+6, x+10] & \rightleftarrows \; [x, \underline{x+2}, x+6, x+10] \\
Q_{Q-}(D) = Df \quad Q_{F+}(Df) = D & \rightsquigarrow & RR_{19}\colon [\underline{x}, x+4, x+7, x+10] & \rightleftarrows \; [x, x+4.\underline{x+6}, x+10] \\
Q_{Q+}(hd) = Df \quad Q_{T-}(Df) = hd & \rightsquigarrow & P_{39}\colon [\underline{x}, x+3, x+6, x+10] & \rightleftarrows \; [\underline{x}, x+4, x+6, x+10] \\
Q_{T+}(mM) = M \quad Q_{T-}(M) = mM & \rightsquigarrow & P_{64}\colon [\underline{x}, x+3, x+7, x+11] & \rightleftarrows \; [\underline{x}, x+4, x+7, x+11] \\
Q_{Q+}(D) = A \quad Q_{Q-}(A) = D & \rightsquigarrow & P_{18}\colon [\underline{x}, x+4, x+7, x+10] & \rightleftarrows \; [\underline{x}, x+4, x+8, x+10] \\
Q_{Q+}(M) = AM \quad Q_{Q-}(AM) = M & \rightsquigarrow & P_{47}\colon [\underline{x}, x+4, x+7, x+11] & \rightleftarrows \; [\underline{x}, x+4, x+8, x+11] \\
Q_{Q-}(D) = Df \quad Q_{Q+}(Df) = D & \rightsquigarrow & P_{19}\colon [\underline{x}, x+4, x+7, x+10] & \rightleftarrows \; [\underline{x}, x+4, x+6, x+10] \\
Q_{S+}(m) = mM \quad Q_{S-}(mM) = m & \rightsquigarrow & P_{26}\colon [\underline{x}, x+3, x+7, x+10] & \rightleftarrows \; [\underline{x}, x+3, x+7, x+11] \\
Q_{S+}(A) = AM \quad Q_{S-}(AM) = A & \rightsquigarrow & P_{87}\colon [\underline{x}, x+4, x+8, x+10] & \rightleftarrows \; [\underline{x}, x+4, x+8, x+11] \\
Q_{S-}(Df) = hd \quad Q_{T+}(hd) = Df & \rightsquigarrow & RR_{39}\colon [\underline{x}, x+3, x+6, x+10] & \rightleftarrows \; [x, x+4, \underline{x+6}, x+10]
\end{array}
$$

$$
\begin{array}{rcll}
Q_{S--}(mM) = hd \quad Q_{F++}(hd) = mM & \rightsquigarrow & R_{63}\colon [\underline{x}, x+3, x+7, x+11] & \rightleftarrows \; [x, x+3, x+7, \underline{x+9}] \\
Q_{S--}(AM) = mM \quad Q_{F++}(mM) = AM & \rightsquigarrow & R_{76}\colon [\underline{x}, x+4, x+8, x+11] & \rightleftarrows [x, x+4, x+8, \underline{x+9}] \\
Q_{F++}(AM) = D \quad Q_{S--}(D) = MA & \rightsquigarrow & L_{71}\colon [\underline{x}, x+4, x+8, x+11] & \rightleftarrows \; [x+2, \underline{x+4}, x+8, x+11] \\
Q_{F++}(A) = Df \quad Q_{S--}(fD) = A & \rightsquigarrow & L_{89}\colon [\underline{x}, x+4, x+8, x+10] & \rightleftarrows \; [x+2, \underline{x+4}, x+8, x+10] \\
Q_{F++}(Df) = A \quad Q_{Q--}(A) = Df & \rightsquigarrow & RR_{98}\colon [\underline{x}, x+4, x+6, x+10] & \rightleftarrows \; [x+2, x+4, \underline{x+6}, x+10] \\
Q_{F--}(mM) = AM \quad Q_{S++}(AM) = mM & \rightsquigarrow & Q_{76}\colon [\underline{x}, x+4, x+8, x+11] & \rightleftarrows \; [\underline{x+1}, x+4, x+8, x] \\
Q_{Q++}(Df) = A \quad Q_{Q--}(A) = Df & \rightsquigarrow & P_{98}\colon [\underline{x}, x+4, x+6, x+10] & \rightleftarrows \; [\underline{x}, x+4, x+8, x+10] \\
Q_{T--}(Df) = A \quad Q_{F++}(A) = Df & \rightsquigarrow & QQ_{98}\colon [\underline{x}, x+4, x+6, x+10] & \rightleftarrows \; [x, \underline{x+2}, x+6, x+10].
\end{array}
$$

We have obtained 20 new parsimonious operations. Therefore the most parsimonious operations on the 9 types of sevenths are 37 in total.

We note that the intervallic structure of the dominant seventh flat five chord is $(4, 2, 4, 2)$. Therefore from an acoustical point of view we have only 6 chords of this type: $C^{7\flat5} = F_{\sharp}^{7\flat5} = [0, 4, 6, 10]$, $C_{\sharp}^{7\flat5} = G^{7\flat5} = [1, 5, 7, 11]$, $D^{7\flat5} = G_{\sharp}^{7\flat5} = [2, 6, 8, 0]$, $D_{\sharp}^{7\flat5} = A^{7\flat5} = [3, 7, 9, 1]$, $E^{7\flat5} = B_{\flat}^{7\flat5} = [4, 8, 10, 2]$, $F^{7\flat5} = B^{7\flat5} = [5, 9, 11, 3]$. The marking allows us to solve the problem from an algebraic point of view, we need to consider twelve different dominant seventh flat five chords. As in the case of the diminished sevenths, the transformations $Q_{i\pm}$ and $Q_{i\pm\pm}$ applied to the dominant flat seventh chords have the same inverse. More precisely, the transformations enharmonic equivalent are: P_{19} and RR_{19}, P_{39} and RR_{39}, $L89$ and QQ_{98}, P_{98} and RR_{98}.

We now represent the parsimonious operations as elements $(\sigma, v) \in S_9 \times V$, where $V = \{v = (v_1, \dots, v_9) \in \mathbb{Z}_{12}^9 | \sum_{i=1}^{9} v_i = 0\}$. In fact, similarly to what we have done for the first 17 parsimonious operations on the 5 types of sevenths, considering 9 types each operation is completely identified as a transposition of S_9 and a vector $\sigma \in \mathbb{Z}_{12}^9$, where the generic i-th component, $i \in \{1, \dots, 9\}$ represents the number of semitones of the movement of the root of the chord of type i to become the root of the chord of type j. Also in this case, our parsimonious operations are Riemannian. We summarize the 37 operations as elements of $S_9 \times V$ (in red the new operations).

$$P_{12}\colon [\underline{x}, x+4, x+7, x+10] \quad \rightleftarrows \quad [\underline{x}, x+3, x+7, x+10] \qquad (\sigma, v) = ((12), (0,0,0,0,0,0,0,0,0))$$

$$P_{14}\colon [\underline{x}, x+4, x+7, x+10] \quad \rightleftarrows \quad [\underline{x}, x+4, x+7, x+11] \qquad (\sigma, v) = ((14), (0,0,0,0,0,0,0,0,0))$$

$$P_{18}\colon [\underline{x}, x+4, x+7, x+10] \quad \rightleftarrows \quad [\underline{x}, x+4, x+8, x+10] \qquad (\sigma, v) = ((18), (0,0,0,0,0,0,0,0,0))$$

$$P_{19}\colon [\underline{x}, x+4, x+7, x+10] \quad \rightleftarrows \quad [\underline{x}, x+4, x+6, x+10] \qquad (\sigma, v) = ((19), (0,0,0,0,0,0,0,0,0))$$

$$P_{23}\colon [\underline{x}, x+3, x+7, x+10] \quad \rightleftarrows \quad [\underline{x}, x+3, x+6, x+10] \qquad (\sigma, v) = ((23), (0,0,0,0,0,0,0,0,0))$$

$$P_{26}\colon [\underline{x}, x+3, x+7, x+10] \quad \rightleftarrows \quad [\underline{x}, x+3, x+7, x+11] \qquad (\sigma, v) = ((26), (0,0,0,0,0,0,0,0,0))$$

$$P_{35}\colon [\underline{x}, x+3, x+6, x+10] \quad \rightleftarrows \quad [\underline{x}, x+3, x+6, x+9] \qquad (\sigma, v) = ((35), (0,0,0,0,0,0,0,0,0))$$

$$P_{39}\colon [\underline{x}, x+3, x+6, x+10] \quad \rightleftarrows \quad [\underline{x}, x+4, x+6, x+10] \qquad (\sigma, v) = ((39), (0,0,0,0,0,0,0,0,0))$$

$$P_{47}\colon [\underline{x}, x+4, x+7, x+11] \quad \rightleftarrows \quad [\underline{x}, x+4, x+8, x+11] \qquad (\sigma, v) = ((47), (0,0,0,0,0,0,0,0,0))$$

$$P_{64}\colon [\underline{x}, x+3, x+7, x+11] \quad \rightleftarrows \quad [\underline{x}, x+4, x+7, x+11] \qquad (\sigma, v) = ((64), (0,0,0,0,0,0,0,0,0))$$

$$P_{87}\colon [\underline{x}, x+4, x+8, x+10] \quad \rightleftarrows \quad [\underline{x}, x+4, x+8, x+11] \qquad (\sigma, v) = ((87), (0,0,0,0,0,0,0,0,0))$$

$$P_{98}\colon [\underline{x}, x+4, x+6, x+10] \quad \rightleftarrows \quad [\underline{x}, x+4, x+8, x+10] \qquad (\sigma, v) = ((98), (0,0,0,0,0,0,0,0,0))$$

$$R_{12}\colon [\underline{x}, x+4, x+7, x+10] \quad \rightleftarrows \quad [x, x+4, x+7, \underline{x+9}] \qquad (\sigma, v) = ((12), (-3,3,0,0,0,0,0,0,0))$$

$$R_{23}\colon [\underline{x}, x+3, x+7, x+10] \quad \rightleftarrows \quad [x, x+3, x+7, \underline{x+9}] \qquad (\sigma, v) = ((23), (0,-3,3,0,0,0,0,0,0))$$

$$R_{42}\colon [\underline{x}, x+4, x+7, x+11] \quad \rightleftarrows \quad [x, x+4, x+7, \underline{x+9}] \qquad (\sigma, v) = ((42), (0,3,0,-3,0,0,0,0,0))$$

$$R_{35}\colon [\underline{x}, x+3, x+6, x+10] \quad \rightleftarrows \quad [x, x+3, x+6, \underline{x+9}] \qquad (\sigma, v) = ((35), (0,0,-3,0,3,0,0,0,0))$$

$$R_{53}\colon [\underline{x}, x+3, x+6, x+9] \quad \rightleftarrows \quad [x, x+3, x+7, \underline{x+9}] \qquad (\sigma, v) = ((53), (0,0,3,0,-3,0,0,0,0))$$

$$R_{63}\colon [\underline{x}, x+3, x+7, x+11] \quad \rightleftarrows \quad [x, x+3, x+7, \underline{x+9}] \qquad (\sigma, v) = ((63), (0,0,3,0,0,-3,0,0,0))$$

$$R_{76}\colon [\underline{x}, x+4, x+8, x+11] \quad \rightleftarrows \quad [x, x+4, x+8, \underline{x+9}] \qquad (\sigma, v) = ((76), (0,0,0,0,0,3,-3,0,0))$$

$$R_{86}\colon [\underline{x}, x+4, x+8, x+10] \quad \rightleftarrows \quad [x, x+4, x+8, \underline{x+9}] \qquad (\sigma, v) = ((86), (0,0,0,0,0,3,0,-3,0))$$

$$L_{13}\colon [\underline{x}, x+4, x+7, x+10] \quad \rightleftarrows \quad [x+2, \underline{x+4}, x+7, x+10] \qquad (\sigma, v) = ((13), (4,0,-4,0,0,0,0,0,0))$$

$$L_{15}\colon [\underline{x}, x+4, x+7, x+10] \quad \rightleftarrows \quad [x+1, \underline{x+4}, x+7, x+10] \qquad (\sigma, v) = ((15), (4,0,0,0,-4,0,0,0,0))$$

$$L_{42}\colon [\underline{x}, x+4, x+7, x+11] \quad \rightleftarrows \quad [x+2, \underline{x+4}, x+7, x+11] \qquad (\sigma, v) = ((42), (0,-4,0,4,0,0,0,0,0))$$

$$L_{71}\colon [\underline{x}, x+4, x+8, x+11] \quad \rightleftarrows \quad [x+2, \underline{x+4}, x+8, x+11] \qquad (\sigma, v) = ((71), (-4,0,0,0,0,0,4,0,0))$$

$$L_{89}\colon [\underline{x}, x+4, x+8, x+10] \quad \rightleftarrows \quad [x+2, \underline{x+4}, x+8, x+10] \qquad (\sigma, v) = ((89), (0,0,0,0,0,0,0,4,-4))$$

$$Q_{43}\colon [\underline{x}, x+4, x+7, x+11] \quad \rightleftarrows \quad [\underline{x+1}, x+4, x+7, x+11] \qquad (\sigma, v) = ((43), (0,0,-1,1,0,0,0,0,0))$$

$$Q_{15}\colon [\underline{x}, x+4, x+7, x+10] \quad \rightleftarrows \quad [\underline{x+1}, x+4, x+7, x+10] \qquad (\sigma, v) = ((15), (1,0,0,0,-1,0,0,0,0))$$

$$Q_{62}\colon [\underline{x}, x+4, x+8, x+11] \quad \rightleftarrows \quad [\underline{x+1}, x+4, x+8, x+11] \qquad (\sigma, v) = ((62), (0,-1,0,0,0,1,0,0,0))$$

$$Q_{76}\colon [\underline{x}, x+4, x+8, x+11] \quad \rightleftarrows \quad [\underline{x+1}, x+4, x+8, x] \qquad (\sigma, v) = ((76), (0,0,0,0,0,-1,1,0,0))$$

$$RR_{19}\colon [\underline{x}, x+4, x+7, x+10] \quad \rightleftarrows \quad [x, x+4.\underline{x+6}, x+10] \qquad (\sigma, v) = ((19), (-6,0,0,0,0,0,0,0,6))$$

$$RR_{35}\colon [\underline{x}, x+3, x+6, x+10] \quad \rightleftarrows \quad [x, x+3, \underline{x+6}, x+9] \qquad (\sigma, v) = ((35), (0,0,6,0,-6,0,0,0,0))$$

$$RR_{39}\colon [\underline{x}, x+3, x+6, x+10] \quad \rightleftarrows \quad [x, x+4, \underline{x+6}, x+10] \qquad (\sigma, v) = ((87), (0,0,6,0,0,0,0,0,-6))$$

$$RR_{98}\colon [\underline{x}, x+4, x+6, x+10] \quad \rightleftarrows \quad [x+2, x+4, \underline{x+6}, x+10] \qquad (\sigma, v) = ((98), (0,0,0,0,0,0,0,-6,6))$$

$$QQ_{38}\colon [\underline{x}, x+3, x+6, x+10] \quad \rightleftarrows \quad [x, \underline{x+2}, x+6, x+10] \qquad (\sigma, v) = ((38), (0,0,2,0,0,0,0,-2,0))$$

$$QQ_{51}\colon [\underline{x}, x+3, x+6, x+9] \quad \rightleftarrows \quad [x, \underline{x+2}, x+6, x+9] \qquad (\sigma, v) = ((51), (-2,0,0,0,2,0,0,0,0))$$

$$QQ_{98}\colon [\underline{x}, x+4, x+6, x+10] \quad \rightleftarrows \quad [x, \underline{x+2}, x+6, x+10] \qquad (\sigma, v) = ((98), (0,0,0,0,0,0,0,-2,2))$$

$$N_{51}\colon [\underline{x}, x+3, x+6, x+9] \quad \rightleftarrows \quad [x, x+3, \underline{x+5}, x+9] \qquad (\sigma, v) = ((51), (-5,0,0,0,5,0,0,0,0)).$$

Now we determine the group generated by these 37 operations.

Theorem 3.20. *The* $PRLQ^*$*-group is isomorphic to* $S_9 \ltimes \mathbb{Z}_{12}^8$.

Proof. First of all we have to prove that the $PLRQ^*$-group is isomorphic to $S_9 \ltimes V$.

By definition $PLRQ^* \simeq S_9 \times V$. Let $\{O\}$ be the identity in V. The elements P_{12}, P_{23}, P_{14}, P_{35}, P_{26}, P_{47}, P_{18} and P_{19} generate $S_9 \times \{O\}$, since the vectors of the parallel operations are identities and the transpositions (12), (23), (14), (35), (26), (47), (18) and (19) generate S_9. Thus $S_9 \times \{O\}$ is a subgroup of $PLRQ^*$.

Moreover the subgroup formed by the elements (Id, v) is normal. In fact, for all $(\sigma, v) \in S_9 \times V$, $(Id, v') \in \{Id\} \times V$, we have

$$(\sigma, v)(Id, v')(\sigma, v)^{-1} = (\sigma\sigma^{-1}, -v + \sigma(v') + \sigma(v)) = (Id, v'') \in \{Id\} \times V.$$

Now we prove that $V = \{v = (v_1, \ldots, v_9) \in \mathbb{Z}_{12}^9 | \sum_{i=1}^{9} v_i = 0\} \simeq \mathbb{Z}_{12}^8$. As a lengthy but straightforward calculation shows, the following elements generate all elements $(Id, (v_1, \ldots, v_9))$:

$$\begin{aligned}
R_{42}L_{42} &= (Id, (0, -7, 0, 7, 0, 0, 0, 0, 0)) \\
P_{14}L_{42}P_{14}R_{12} &= (Id, (7, -7, 0, 0, 0, 0, 0, 0, 0)) \\
P_{12}L_{13}P_{12}R_{23} &= (Id, (0, 7, -7, 0, 0, 0, 0, 0, 0)) \\
P_{12}P_{35}R_{23}P_{12}L_{15}L_{13} &= (Id, (7, 0, 0, 0, -7, 0, 0, 0)) \\
P_{64}P_{14}P_{18}P_{19}P_{14}P_{64}R_{86}QQ_{98} &= (Id, (0, 0, 0, 0, 0, 3, 0, -2, -1)) \\
P_{64}P_{47}P_{64}Q_{76} &= (Id, (0, 0, 0, 0, 0, -1, 1, 0, 0)) \\
P_{64}P_{87}P_{47}P_{64}Q_{76}P_{87} &= (Id, (0, 0, 0, 0, 0, -1, 0, 1, 0)) \\
P_{26}Q_{62} &= (Id, (0, -1, 0, 0, 0, 1, 0, 0, 0).
\end{aligned}$$

Finally, it is clear that the intersection $(S_9 \times \{O\}) \cap (\{Id\} \times V)$ is trivial. Therefore, we can conclude that $PRLQ^*$-group is isomorphic to $S_9 \ltimes \mathbb{Z}_{12}^8$. □

Chapter 4

Parsimonious operations and graphs for triads

As we have already observed, there are many algebraic and geometric studies on major and minor triads. Although they are the chords which are the most used in music, they are not the only types of triads: there are also the diminished and the augmented triads. The diminished triads are formed by two minor thirds. Conversely the augmented triads are obtained by two major thirds, therefore:

$$[\underline{x}, x+3, x+6] \pmod{12}, \quad x \in \mathbb{Z}_{12}, \quad \text{intervallic structure: } [3,3,6] \qquad \text{(diminished triad)}$$

$$[\underline{x}, x+4, x+8] \pmod{12}, \quad x \in \mathbb{Z}_{12}, \quad \text{intervallic structure: } [4,4,4] \qquad \text{(augmented triad)}$$

Example 4.1.

$$C_d = [\underline{0}, 3, 6] = [6, \underline{0}, 3] = [3, 6, \underline{0}]$$
$$C_A = [\underline{0}, 4, 8] = [8, \underline{0}, 4] = [4, 8, \underline{0}]$$

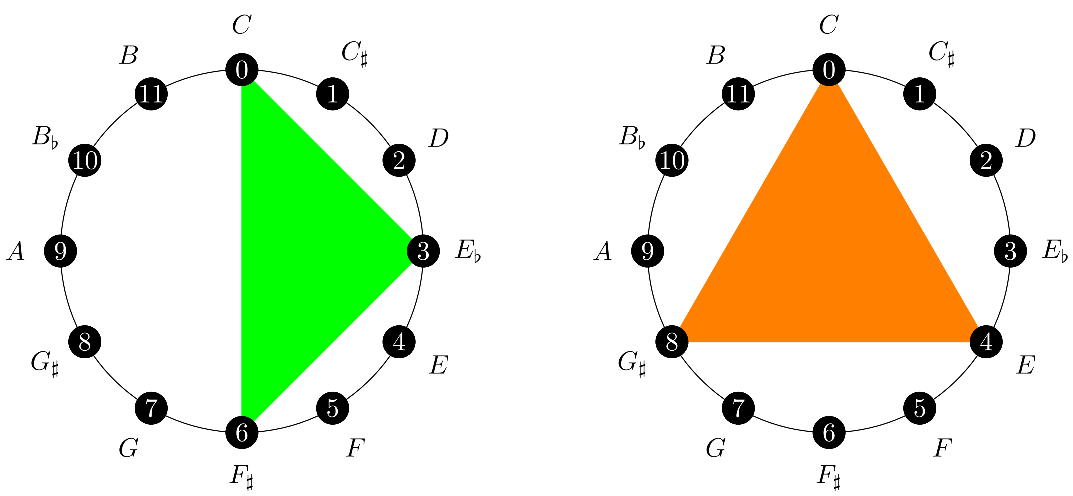


Figure 4.1: On the left C diminished triad, on the right C augmented triad.

We want to extend the studies on triadic operations including also diminished and augmented triads. We will start from Hook's work (see Section 1.6) extending the definition of UTTs. In particular, we will classify and analyze the parsimonious UTTs such that the triads of the four types are $P_{1,0}$ and $P_{0,1}$-related, and we want to determine the algebraic group generated by these new operations and the corresponding parsimonious graphs.

4.1 Extension of the UTTs

Following Hook's approach, we can denote a triad as an ordered pair $\Delta = (r, s)$, where $r \in \mathbb{Z}_{12}$ is the root of the triad and $s = s_1, s_2, s_3, s_4$ represents its type: s_1 for major, s_2 for minor, s_3 for diminished and s_4 for augmented triads.

Example 4.2. $C_M = (0, s_1)$, $A_m = (9, s_2)$, $G_d = (7, s_3)$, $E_A = (4, s_4)$.

The set $S^* = \{(r, s) | r \in \mathbb{Z}_{12}, s = s_1, s_2, s_3, s_4\}$ represents the collection of all $12 \cdot 4 = 48$ triads.

Remark 4.3. *In the set S^* there are the augmented triads, having intervallic content $(4, 4, 4)$. As for the diminished sevenths, in a chord progression it is not clear what is the root in augmented triads. Thus, from an acoustical point of view, we have not 12 different augmented triads but only 4: $(0, s_4) = (4, s_4) = (8, s_4)$, $(1, s_4) = (5, s_4) = (9, s_4)$, $(2, s_4) = (6, s_4) = (10, s_4)$ and $(3, s_4) = (7, s_4) = (11, s_4)$. But, for algebraic motivations, using the cyclicly marked chords we will consider them as 12 distinguished chords.*

The extension of the definitions of triadic transformation and operation are natural.

Definition 4.4 (Triadic transformation)**.** *A **triadic transformation** is a map $S^* \to S^*$.*

Definition 4.5 (Triadic operation)**.** *A **triadic operation** is a bijective transformation $S^* \to S^*$.*

The set of all triadic operations with the composition of maps generate a group $\mathcal{G}^*$ of order 48!. Inversions and transpositions are examples of triadic operations.

We are interested only in the uniform triadic transformations on S^*, whose definition is an extension to that given by Hook.

Definition 4.6 (Uniform triadic transformation)**.** *A **uniform triadic transformation** U is an operation satisfying the following* uniformity condition*: for every triad $(r, s) \in S^*$ and every transposition $t \in \mathcal{T}$, if U transforms the triad (r, s) in (r', s'), then U transforms $(r + t, s)$ to $(r' + t, s')$.*

Similar to what Hook did, we can represent a UTT on S^* as an ordered pair $(\sigma, (v_1, v_2, v_3, v_4))$, where:

- s is a permutation of the symmetric group S_4 representing the change of the type of the triad if the UTT is mode-reversing. If the UTT is mode-preserving, $\sigma = Id \in S_4$.
- $(v_1, v_2, v_3, v_4) \in \mathbb{Z}_{12}^4$ is a vector where each component represents the number of semitones by which the root of the corresponding triad is moved. More precisely the first component v_1 represents the movement of the root of major triads, the second component v_2 is for minor triads, the third component v_3 is for augmented triads and v_4 is for diminished triads.

Example 4.7.

$$T_n = (+, (n, n, n, n))$$
$$P = ((12), (0, 0, 0, 0))$$
$$L = ((12), (4, -4, 0, 0))$$
$$R = ((12), (-3, 3, 0, 0)).$$

We observe that we can extend the neo-Riemannian P, L and R operations to diminished and augmented triads defining it as the identity on these two types of triads. In fact, for example

$$\begin{aligned} R&\colon (s_1, x) \mapsto (s_2, x-3) && \text{if the triad is major} \\ R&\colon (s_2, x) \mapsto (s_1, x+3) && \text{if the triad is minor} \\ R&\colon (s_3, x) \mapsto (s_3, x) && \text{if the triad is diminished} \\ R&\colon (s_4, x) \mapsto (s_4, x) && \text{if the triad is augmented.} \end{aligned}$$

Let $\mathcal{U}^*$ be the set of all UTTs. Since each UTT can be represented as $(\sigma, (v_1, v_2, v_3, v_4)) \in S_5 \times \mathbb{Z}_{12}^4$, the order of $\mathcal{U}^*$ is $120 \times 12^4 = 2488320$.

Now we describe the actions of UTTs on triads. Let $U = (\sigma, (v_1, v_2, v_3, v_4))$ be a UTT and let $\Delta = (r, s)$ be a triad. U acts on Δ transposing its root by v_1 if $s = s_1$, by v_2 if $s = s_2$, by v_3 if $s = s_3$, by v_4 if $s = s_4$. Moreover, depending on the permutation $\sigma \in S_4$, U can leave or change the type of the chord. More precisely:

$$U(\Delta) = (r + v_i, s_{\sigma(i)}). \tag{4.1}$$

Example 4.8. *Let $U = ((132), (2, 5, 4, 0))$ be a UTT and let $\Delta = G_d = (7, s_3)$ be a diminished triad. We apply U to Δ:*

$$U(\Delta) = U(7, s_3) = (7 + 4, s_{\sigma(3)}) = (11, s_2) = B_m.$$

The product formula for the UTTs on S^* is the same defined in 3.7:

$$\begin{aligned} &(\sigma_k, v_k) \circ \cdots \circ (\sigma_1, v_1) = \\ &= \left(\sigma_k \cdots \sigma_1, v_1 + \sigma_1^{-1}(v_2) + (\sigma_2\sigma_1)^{-1}(v_3) + \cdots + (\sigma_{k-1} \cdots \sigma_1)^{-1}(v_k)\right) = && (4.2) \\ &= \left(\sigma_k \cdots \sigma_1, v_1 + \sigma_1^{-1}(v_2) + \sigma_1^{-1}\sigma_2^{-1}(v_3) + \cdots + \sigma_1^{-1} \cdots \right) && (4.3) \end{aligned}$$

Example 4.9. *Let $U = (Id, (4, 7, 0, 0))$, $V = ((12), (5, 10, 0, 0))$ and $W = ((23, (0, 5, 4, 0))$. Their product is*

$$\begin{aligned} WVU &= ((23), (0, 5, 4, 0)) \circ ((12), (5, 10, 0, 0)) \circ (Id, (4, 7, 0, 0)) = \\ &= ((23) \circ (12), (4, 7, 0, 0) + (5, 10, 0, 0) + (5, 0, 4, 0)) = \\ &= ((132), (2, 5, 4, 0)). \end{aligned}$$

In fact, if we consider $D_M = (2, s_1)$, $E_m = (4, s_2)$, $G_d = (7, s_3)$ and $B_A = (11, s_4)$ and if we apply U, V and W

$$\begin{aligned} W(V(U(D_M))) &= W(V(6, s_1)) = W(11, s_2) = (4, s_3) \\ W(V(U(E_m))) &= W(V(11, s_2)) = W(9, s_1) = (9, s_1) \\ W(V(U(G_d))) &= W(V(7, s_3)) = W(7, s_3) = (11, s_2) \\ W(V(U(B_A))) &= W(V(11, s_4)) = W(11, s_4) = (11, s_4). \end{aligned}$$

We obtain the same results applying directly WVU

$$\begin{aligned} (WVU)(D_M) &= (4, s_3) \\ (WVU)(E_m) &= (9, s_1) \\ (WVU)(G_d) &= (11, s_2) \\ (WVU)(B_A) &= (11, s_4). \end{aligned}$$

Finally, we extend Hook's definition of Riemannian UTT.

Definition 4.10 (Riemannian UTT)**.** *A UTT* $U = (\sigma, (v_1, v_2, v_3, v_4))$ *is* ***Riemannian*** *if* $\sum_{i=1}^{4} v_i = 0$.

Example 4.11. *The transposition* $T_0 = (Id, (0,0,0,0))$ *and the neo-Riemannian operations* $P = ((12), (0,0,0,0))$, $L = ((12), (4, -4, 0, 0))$ *and* $R = ((12), (-3, 3, 0, 0))$ *are Riemannian also with this new definition.*

Among all Riemannian UTTs on S^*, we are interested on the most parsimonious ones.

4.2 Parsimonious operations among triads

First of all we consider all possible movements by semitones or whole tone of the three members of all types of triads. Similarly to what was done in the previous chapter, we denote by Q_{i+} and Q_{i++}, where $i = R, T, F$, the maps that transform each triad moving the i-th member a semitone and a whole tone up, and with Q_{i-} and Q_{i--} the maps that move the i-th member a semitone and a whole tone down respectively.

$$
\begin{array}{llll}
Q_{R+}(M) = d & \cancel{Q_{R+}(m)} & \cancel{Q_{R+}(d)} & Q_{R+}(A) = m \\
Q_{R-}(M) = m & Q_{R-}(m) = A & Q_{R-}(d) = M & Q_{R-}(A) = M \\
\cancel{Q_{T+}(M)} & Q_{T+}(m) = M & \cancel{Q_{T+}(d)} & Q_{T+}(A) = m \\
Q_{T-}(M) = m & \cancel{Q_{T-}(m)} & \cancel{Q_{T-}(d)} & Q_{T-}(A) = M \\
Q_{F+}(M) = A & Q_{F+}(m) = M & Q_{F+}(d) = m & Q_{F+}(A) = m \\
\cancel{Q_{F-}(M)} & Q_{F-}(m) = d & \cancel{Q_{R-}(d)} & Q_{F-}(A) = M
\end{array}
$$

$$
\begin{array}{llll}
Q_{R--}(M) = d & Q_{R--}(m) = M & Q_{R--}(d) = m & \cancel{Q_{R--}(A)} \\
Q_{F++}(M) = m & Q_{F++}(m) = d & Q_{F++}(d) = M & \cancel{Q_{F++}(A)}.
\end{array}
$$

Secondly, we group each map with its inverse and we classify the parsimonious transformations using the definitions introduced in the previous chapter:

$$
\begin{array}{rclcl}
Q_{R+}(M) = d \quad Q_{R-}(d) = M & \rightsquigarrow & Q_{Md}\colon [\underline{x}, x+4, x+7] & \rightleftarrows & [\underline{x+1}, x+4, x+7] \\
Q_{R+}(A) = m \quad Q_{R-}(m) = A & \rightsquigarrow & Q_{Am}\colon [\underline{x}, x+4, x+8] & \rightleftarrows & [\underline{x+1}, x+4, x+8] \\
Q_{R-}(M) = m \quad Q_{F+}(m) = M & \rightsquigarrow & L_{Mm}\colon [\underline{x}, x+4, x+7] & \rightleftarrows & [x+1, \underline{x+4}, x+7] \\
Q_{R-}(A) = M \quad Q_{F+}(M) = A & \rightsquigarrow & L_{AM}\colon [\underline{x}, x+4, x+8] & \rightleftarrows & [x+1, \underline{x+4}, x+8] \\
Q_{T+}(A) = m \quad Q_{R-}(m) = A & \rightsquigarrow & N_{Am}\colon [\underline{x}, x+4, x+8] & \rightleftarrows & [x, \underline{x+5}, x+8] \\
Q_{T-}(M) = m \quad Q_{T+}(m) = M & \rightsquigarrow & P_{Mm}\colon [\underline{x}, x+4, x+7] & \rightleftarrows & [\underline{x}, x+3, x+7] \\
Q_{T-}(A) = M \quad Q_{F+}(M) = A & \rightsquigarrow & L_{MA}\colon [\underline{x}, x+4, x+7] & \rightleftarrows & [x, \underline{x+4}, x+8] \\
Q_{F+}(d) = m \quad Q_{F-}(m) = d & \rightsquigarrow & P_{dm}\colon [\underline{x}, x+3, x+6] & \rightleftarrows & [\underline{x}, x+3, x+7] \\
Q_{F+}(A) = m \quad Q_{R-}(m) = A & \rightsquigarrow & R_{Am}\colon [\underline{x}, x+4, x+8] & \rightleftarrows & [x, x+4, \underline{x+9}] \\
Q_{F-}(A) = M \quad Q_{F+}(M) = A & \rightsquigarrow & P_{AM}\colon [\underline{x}, x+4, x+8] & \rightleftarrows & [\underline{x}, x+4, x+7] \\
Q_{R--}(M) = d \quad Q_{F++}(d) = M & \rightsquigarrow & L_{Md}\colon [\underline{x}, x+4, x+7] & \rightleftarrows & [x+10, \underline{x+4}, x+7] \\
Q_{R--}(m) = M \quad Q_{F++}(M) = d & \rightsquigarrow & R_{Mm}\colon [\underline{x}, x+4, x+7] & \rightleftarrows & [x, x+4, \underline{x+9}] \\
Q_{R--}(d) = m \quad Q_{F++}(m) = d & \rightsquigarrow & R_{md}\colon [\underline{x}, x+3, x+7] & \rightleftarrows & [x, x+3, \underline{x+9}].
\end{array}
$$

Therefore, the most parsimonious operations among triads are 13: 10 of them are $P_{1,0}$ relations, 3 are $P_{0,1}$ relations.

Le $S^* = \{[\underline{x_1}, x_2, x_3] | x_1, x_2, x_3 \in \mathbb{Z}_{12}, x_2 = x_1 + 3 \text{ or } x_2 = x_1 + 4, x_3 = x_2 + 3 \text{ or } x_3 = x_2 + 4\}$ be the set of cyclicly marked major, minor, diminished and augmented triads. Because of the intervallic structure of augmented triads, as in the case of diminished sevenths, some of the maps $Q_{i\pm}$ above have the same inverse, thus some operations are enharmonic equivalent from a musical point of view:

$$Q_{Am} \sim N_{Am} \sim R_{Am} \qquad L_{AM} \sim L_{MA} \sim P_{AM}.$$

Remark 4.12. *Three operations of the list coincide with the three neo-Riemannian operations:* $L_{Mm} = L$*,* $P_{Mm} = P$ *and* $R_{Mm} = R$*. Moreover, they are the only most parsimonious operations between major and minor triads.*

We summarize the operations with the following graph.

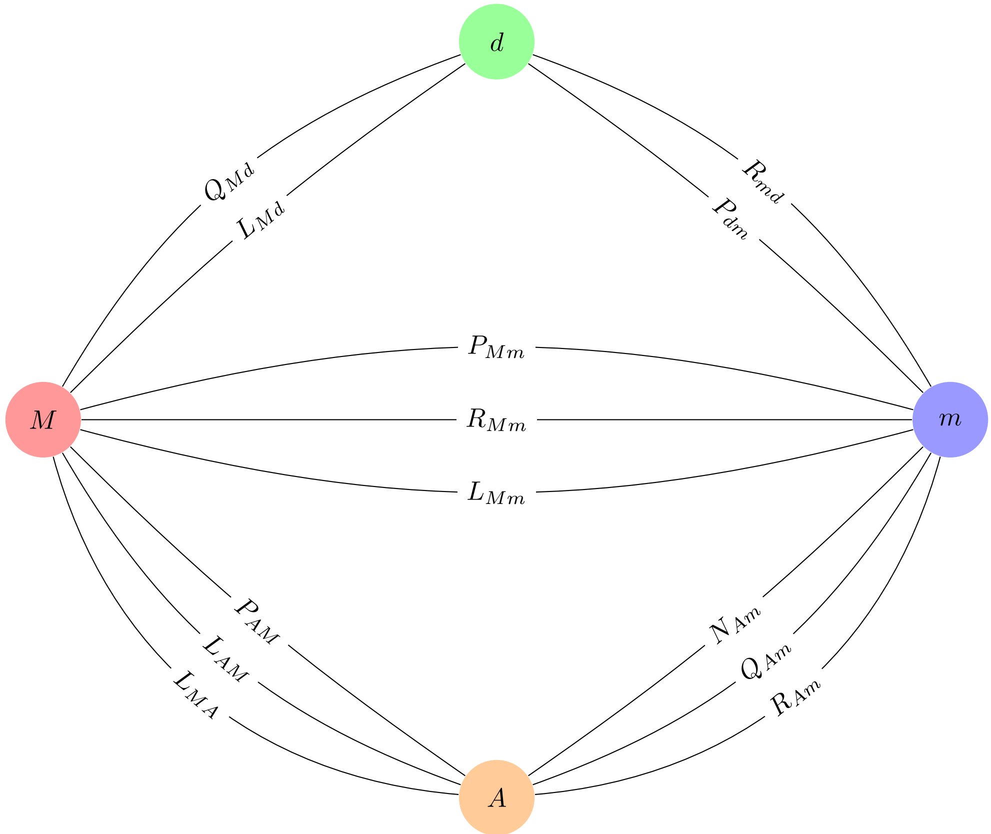


Figure 4.2: The graph representing the 13 parsimonious operations among triads.

4.3 Generalized *Cube Dance* and *Chicken-wire Torus*

We focus on the particular case of major, minor and augmented triads. We recall that $S = \{[\underline{x_1}, x_2, x_3] | x_1, x_2, x_3 \in \mathbb{Z}_{12}, x_2 = x_1 + 3 \text{ or } x_2 = x_1 + 4, x_3 = x_1 + 7\}$ is the set of 24 cyclicly marked major and minor triads. Let $S_1 = S \cup \{[\underline{x_1}, x_2, x_3] | x_1, x_2, x_3 \in \mathbb{Z}_{12}, x_2 = x_1 + 4, x_3 = x_1 + 8\}$ be the set of major, minor and augmented triads. The most parsimonious operations acting on S_1 are: Q_{Am}, $L_{Mm} = L$, L_{AM}, N_{Am}, $P_{Mm} = P$, L_{MA}, R_{Am}, P_{AM}, and $R_{Mm} = R$. Our aim is to construct the associated parsimonious graph.

As already done in the generalized *Chicken-wire Torus* for seventh chords, we identify edges and vertices that are enharmonic equivalent. Therefore we have 12 vertices representing major triads, other 12 vertices for minor triads, and only 4 vertices representing augmented triads. The associated graph is represented in Fig. 4.3. We can observe that the *Cube Dance* is a subgraph of it. This is because the *Cube Dance* is the parsimonious graph representing all $P_{1,0}$ relations among major, minor and augmented triads. In our graph there are all of them and, in addition, the relative $R_{Mm} = R$ that is the unique $P_{0,1}$-relation. Therefore we can consider this graph as a *generalized Cube Dance*. Since it represents all neo-Riemannian P, L and R operations, the *Chicken-wire Torus* is another subgraph of it.

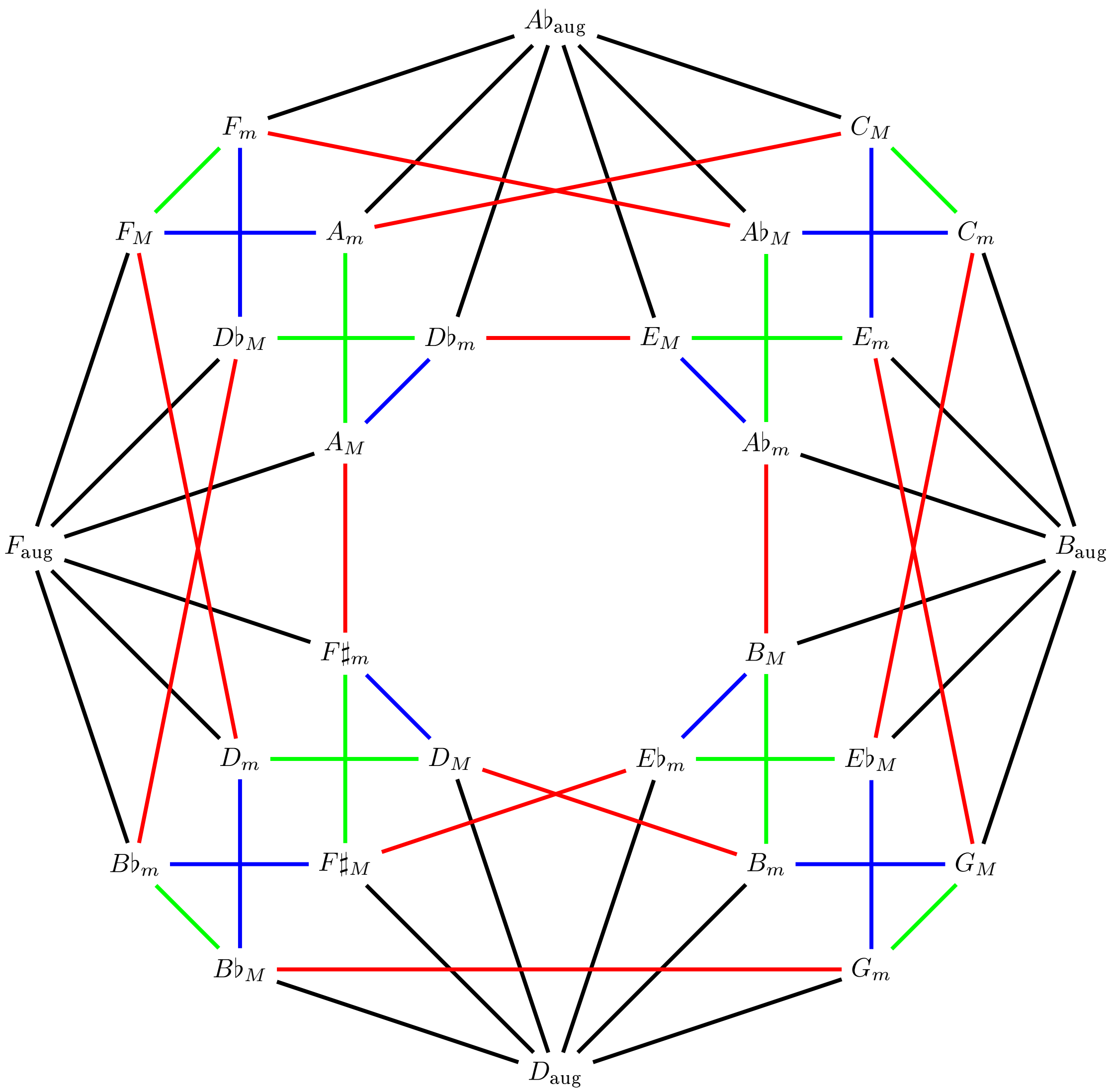


Figure 4.3: The generalized *Cube Dance* representing all the most parsimonious operations among major, minor and augmented triads. Green, blue and red edges represent, respectively, P, L and R.

Now we focus on the particular case of major, minor and diminished triads. We recall that $S = \{[\underline{x_1}, x_2, x_3] | x_1, x_2, x_3 \in \mathbb{Z}_{12}, x_2 = x_1 + 3 \text{ or } x_2 = x_1 + 4, x_3 = x_1 + 7\}$ is the set of 24 major and minor triads. Let $S_2 = S \cup \{[\underline{x_1}, x_2, x_3] | x_1, x_2, x_3 \in \mathbb{Z}_{12}, x_2 = x_1 + 3, x_3 = x_1 + 6\}$ be the set of major, minor and diminished triads. The most parsimonious operations acting on S_2 are: Q_{Md}, $L_{Mm} = L$, $P_{Mm} = P$, P_{dm}, L_{Md}, $R_{Mm} = R$ and R_{md}. As in the previous section, our aim is to construct the associated parsimonious graph. We have 12 vertices representing major triads, 12 more vertices for minor triads, and finally 12 vertices for diminished triads. The associated graph is represented in Fig. 4.4. We can observe that the *Chicken-wire Torus* is a subgraph of it. This is because it contains all $P_{1,0}$ and $P_{0,1}$ relations among major and minor triads. Therefore we can consider our graph as a *generalized Chicken-wire Torus*.

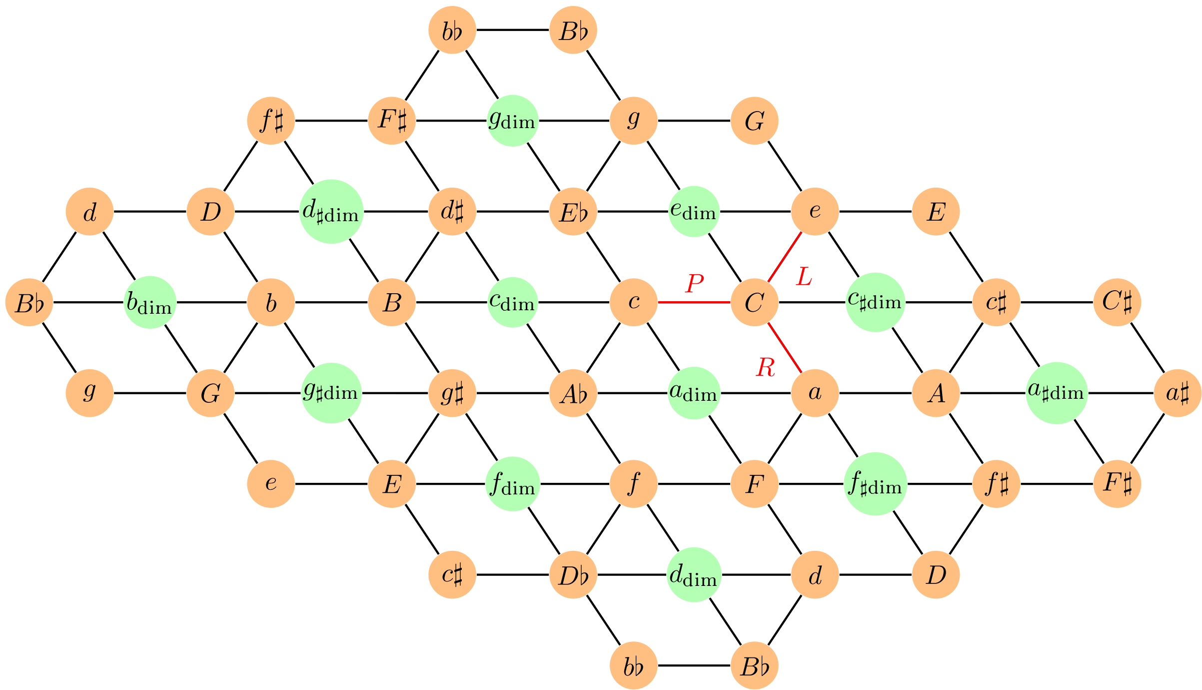


Figure 4.4: *Generalized Chicken-wire torus* for major, minor and diminished triads.

4.4 The PLR^*-group

We want to determine the group generated by the 13 most parsimonious transformations acting on S^*. We will use the same strategy introduced in the previous chapter for seventh chords.

We call PLR^* the group generated by the 13 transformations among major, minor, augmented and diminished triads. We have already seen that each UTT $t \in PLR^*$ can be expressed as an ordered pair $(\sigma, (v_1, v_2, v_3, v_4)) \in S_4 \times \mathbb{Z}_{12}^4$. We list the 13 parsimonious operations as elements of $S_4 \times \mathbb{Z}_{12}^4$:

$$\begin{array}{llll}
Q_{Md}\colon [\underline{x}, x+4, x+7] & \rightleftarrows & [\underline{x+1}, x+4, x+7] & (\sigma, v) = ((13), (1, 0, 0, -1)) \\
Q_{Am}\colon [\underline{x}, x+4, x+8] & \rightleftarrows & [\underline{x+1}, x+4, x+8] & (\sigma, v) = ((42), (0, -1, 1, 0)) \\
L_{Mm}\colon [\underline{x}, x+4, x+7] & \rightleftarrows & [x+1, \underline{x+4}, x+7] & (\sigma, v) = ((12), (4, -4, 0, 0)) \\
L_{AM}\colon [\underline{x}, x+4, x+8] & \rightleftarrows & [x+1, \underline{x+4}, x+8] & (\sigma, v) = ((41), (4, 0, -4, 0)) \\
N_{Am}\colon [\underline{x}, x+4, x+8] & \rightleftarrows & [x, \underline{x+5}, x+8] & (\sigma, v) = ((42), (0, -5, 5, 0)) \\
P_{Mm}\colon [\underline{x}, x+4, x+7] & \rightleftarrows & [\underline{x}, x+3, x+7] & (\sigma, v) = ((12), (0, 0, 0, 0)) \\
L_{MA}\colon [\underline{x}, x+4, x+7] & \rightleftarrows & [x, \underline{x+4}, x+8] & (\sigma, v) = ((14), (4, 0, -4, 0)) \\
P_{dm}\colon [\underline{x}, x+3, x+6] & \rightleftarrows & [\underline{x}, x+3, x+7] & (\sigma, v) = ((32), (0, 0, 0, 0)) \\
R_{Am}\colon [\underline{x}, x+4, x+8] & \rightleftarrows & [x, x+4, \underline{x+9}] & (\sigma, v) = ((42), (0, 3, -3, 0)) \\
P_{AM}\colon [\underline{x}, x+4, x+8] & \rightleftarrows & [\underline{x}, x+4, x+7] & (\sigma, v) = ((41), (0, 0, 0, 0)) \\
L_{Md}\colon [\underline{x}, x+4, x+7] & \rightleftarrows & [x+10, \underline{x+4}, x+7] & (\sigma, v) = ((13), (4, 0, 0, -4)) \\
R_{Mm}\colon [\underline{x}, x+4, x+7] & \rightleftarrows & [x, x+4, \underline{x+9}] & (\sigma, v) = ((12), (-3, 3, 0, 0)) \\
R_{md}\colon [\underline{x}, x+3, x+7] & \rightleftarrows & [x, x+3, \underline{x+9}] & (\sigma, v) = ((23), (0, 3, -3, 0)).
\end{array}$$

More precisely, we can represent each operation $t \in PLR^*$ as an element of $S_4 \times V$ where

$$V = \{(v_1, \dots, v_4) \in \mathbb{Z}_{12}^4 | \sum_{i=1}^{4} v_i = 0\}.$$

We recall that the composition law is defined as follows:

$$\begin{aligned}
&(\sigma_k, v_k) \circ \dots \circ (\sigma_1, v_1) = \\
&= \big(\sigma_k \cdots \sigma_1, v_1 + \sigma_1^{-1}(v_2) + (\sigma_2\sigma_1)^{-1}(v_3) + \dots + (\sigma_{k-1} \cdots \sigma_1)^{-1}(v_k)\big) = \\
&= \big(\sigma_k \cdots \sigma_1, v_1 + \sigma_1^{-1}(v_2) + \sigma_1^{-1}\sigma_2^{-1}(v_3) + \dots + \sigma_1^{-1} \cdots \sigma_{k-1}^{-1}(v_k)\big).
\end{aligned}$$

Theorem 4.13. *The PLR^*-group is isomorphic to $S_4 \ltimes \mathbb{Z}_{12}^3$.*

Proof. First of all we prove that $PLR^* \simeq S_4 \ltimes V$.

Let $\{O\}$ be the identity in V. Then $S_4 \times \{O\}$ is a subgroup of PLR^*. In fact, the elements P_{Mm}, P_{dm} and P_{AM} generate $S_4 \times \{O\}$. Moreover, the subgroup formed by the elements (Id, v) is normal, because for all $(\sigma, v) \in S_4 \times V$, $(Id, v') \in \{Id\} \times V$

$$(\sigma, v)(Id, v')(\sigma, v)^{-1} = (\sigma\sigma^{-1}, -v + \sigma(v') + \sigma(v)) = (Id, v'') \in \{Id\} \times V.$$

Furthermore, the two subgroups have trivial intersection. Hence $PLR^* \simeq S_4 \ltimes V$.

Now we observe that the normal subgroup of elements (Id, v) is isomorphic to $\{Id\} \times \mathbb{Z}_{12}^3$. In fact

$$\begin{aligned}
L_{Mm}R_{Mm} &= (Id, (-7, 7, 0, 0)) \\
P_{dm}Q_{Md}P_{Mm}Q_{Md} &= (Id, (0, 1, 0, -1)) \\
Q_{AM}P_{AM}P_{Mm}P_{AM} &= (Id, (0, 1, -1, 0)).
\end{aligned}$$

generate all elements $(Id, (v_1, v_2, v_3, v_4))$, with $(v_1, \dots, v_4) \in \mathbb{Z}_{12}^4$ such that $\sum_{i=1}^4 v_i = 0$. In fact, given $a, b, c \in \mathbb{Z}$ we have

$$a(-7, 7, 0, 0) + b(0, 1, 0, -1) + c(0, 1, -1, 0) \equiv (v_1, v_2, v_3, v_4) \pmod{12}$$

$$(-7a, 7a + b + c, -c, -b) \equiv (v_1, v_2, v_3, v_4) \pmod{12}$$

$$\begin{cases} -7a \equiv v_1 \\ 7a + b + c \equiv v_2 \\ -c \equiv v_3 \\ -b \equiv v_4 \end{cases} \Rightarrow \begin{cases} -7a \equiv v_1 \\ 7a - v_3 - v_4 \equiv v_2 \\ c \equiv -v_3 \\ b \equiv -v_4 \end{cases} \Rightarrow \begin{cases} -7a \equiv v_1 \\ -v_1 - v_4 \equiv v_2 + v_3 \\ c \equiv -v_3 \\ b \equiv -v_4 \end{cases}$$

which has solution since 7 and 12 are coprime.

Hence $PLR^* \simeq S_4 \ltimes \mathbb{Z}_{12}^3$. □

Chapter 5

Operations among sevenths and triads

In the previous chapters we have determined parsimonious operations acting on different sets of triads or sevenths. But in music literature the most common sequences of chords include both triads and sevenths. With respect to common practice period in particular, there are not so many long sequences of seventh chords, because they are dissonant in nature. But it is usual to find sequences of chords in which there are both sevenths and triads. Therefore, we want to define new musical operations among sevenths and triads. Our aim is to find a general framework in which to include the already known parsimonious operations among triads, the parsimonious operations among sevenths, and also parsimonious operations among seventh and triads.

5.1 Parsimonious operations among sevenths and triads

There are several possibilities to define parsimonious operations among triads and sevenths. For the moment, we will focus on very simple operations, that is those in which a single note is added or deleted, which is possible since some triads are embedded in some sevenths.

Similarly to what we have done in previous chapters, we will define some new parsimonious operations capable of classifying our new operations among sevenths and triads. Let K be a set of sevenths and triads.

Definition 5.1. *Let $\mathcal{P}_{ij}\colon K \to K$ be the map which sends an i-th type of seventh to a j-th type of triad with the same root, and which fixes the other types.*

Definition 5.2. *Let $\mathcal{R}_{ij}\colon K \to K$ be the map which sends an i-th type of seventh to a j-th type of triad whose root is transposed 3 semitones up, a j-th type of triad to an i-th type of seventh whose root is transposed 3 semitones down, and which fixes the other types:*

$$\mathcal{R}_{ij} = T_{\pm 3} \circ \mathcal{P}_{ij} = \mathcal{P}_{ij} \circ T_{\pm 3}.$$

Definition 5.3. *Let $\mathcal{L}_{ij}\colon K \to K$ be the map which sends an i-th type of seventh to a j-th type of triad whose root is transposed 4 semitones up, a j-th type of triad to an i-th type of seventh whose root is transposed 4 semitones down, and which fixes the other types:*

$$\mathcal{L}_{ij} = T_{\pm 4} \circ \mathcal{P}_{ij} = \mathcal{P}_{ij} \circ T_{\pm 4}.$$

We will use the notation used in the previous chapters to represent the set of chords. For instance, we consider $K_1 = H \cup S$ be the set containing the 5 classical types of sevenths (dominant, minor, half-diminished, major and diminished) and major and minor triads. The parsimonious operations among these sevenths and triads are the following:

$$\begin{aligned}
\mathcal{P}_{1M}&: [\underline{x}, x+4, x+7, x+10] \rightleftarrows [\underline{x}, x+4, x+7]\\
\mathcal{P}_{2m}&: [\underline{x}, x+3, x+7, x+10] \rightleftarrows [\underline{x}, x+3, x+7]\\
\mathcal{R}_{2M}&: [\underline{x}, x+3, x+7, x+10] \rightleftarrows [\underline{x+3}, x+7, x+10]\\
\mathcal{R}_{3m}&: [\underline{x}, x+3, x+6, x+10] \rightleftarrows [\underline{x+3}, x+6, x+10]\\
\mathcal{L}_{4M}&: [\underline{x}, x+4, x+7, x+11] \rightleftarrows [\underline{x}, x+4, x+7]\\
\mathcal{L}_{4m}&: [\underline{x}, x+4, x+7, x+11] \rightleftarrows [\underline{x+4}, x+7, x+11].
\end{aligned}$$

We observe that considering this set of sevenths and triads we have not operations including diminished sevenths. Since the intervallic structure of diminished sevenths is $(3,3,3,3)$, the only possibility to have parsimonious operations transforming them into triads is to include diminished triads. In fact, if we consider the set $K_2 = H \cup S_1$ of the classical 5 types of sevenths and major, minor and diminished triads, the parsimonious operations are the following:[1]

$$\begin{aligned}
\mathcal{P}_{1M}&: [\underline{x}, x+4, x+7, x+10] \rightleftarrows [\underline{x}, x+4, x+7]\\
\mathcal{L}_{1d}&: [\underline{x}, x+4, x+7, x+10] \rightleftarrows [\underline{x+4}, x+7, x+10]\\
\mathcal{P}_{2m}&: [\underline{x}, x+3, x+7, x+10] \rightleftarrows [\underline{x}, x+3, x+7]\\
\mathcal{R}_{2M}&: [\underline{x}, x+3, x+7, x+10] \rightleftarrows [\underline{x+3}, x+7, x+10]\\
\mathcal{P}_{3d}&: [\underline{x}, x+3, x+6, x+10] \rightleftarrows [\underline{x}, x+3, x+6]\\
\mathcal{R}_{3m}&: [\underline{x}, x+3, x+6, x+10] \rightleftarrows [\underline{x+3}, x+6, x+10]\\
\mathcal{P}_{4M}&: [\underline{x}, x+4, x+7, x+11] \rightleftarrows [\underline{x}, x+4, x+7]\\
\mathcal{L}_{4m}&: [\underline{x}, x+4, x+7, x+11] \rightleftarrows [\underline{x+4}, x+7, x+11]\\
\mathcal{P}_{5d}&: [\underline{x}, x+3, x+6, x+9] \rightleftarrows [\underline{x}, x+3, x+6]\\
\mathcal{R}_{5d}&: [\underline{x}, x+3, x+6, x+9] \rightleftarrows [\underline{x+3}, x+6, x+9]\\
\mathcal{RR}_{5d}&: [\underline{x}, x+3, x+6, x+9] \rightleftarrows [x, \underline{x+6}, x+9]\\
\mathcal{Z}_{5d}&: [\underline{x}, x+3, x+6, x+9] \rightleftarrows [x, x+3, \underline{x+9}].
\end{aligned}$$

To classify all parsimonious operations on K_2, we have introduced two new operations.

Definition 5.4. *Let $\mathcal{RR}_{ij}\colon K \to K$ be the map which sends an i-th type of seventh to a j-th type of triad transposed 6 semitones up, a j-th type of triad to an i-th type of seventh transposed 6 semitones down, and which fixes the other types:*

$$\mathcal{RR}_{ij} = T_{\pm 6} \circ \mathcal{P}_{ij} = \mathcal{P}_{ij} \circ T_{\pm 6}.$$

Definition 5.5. *Let $\mathcal{Z}_{ij}\colon K \to K$ be the map which sends an i-th type of seventh to a j-th type of triad transposed 9 semitones up, a j-th type of triad to an i-th type of seventh transposed 9 semitones down, and which fixes the other types:*

$$\mathcal{Z}_{ij} = T_{\pm 9} \circ \mathcal{P}_{ij} = \mathcal{P}_{ij} \circ T_{\pm 9}.$$

Remark 5.6. *Let $K_3 = H \cup S^*$. The parsimonious operations exchanging sevenths and triads acting on K_2 and K_3 are the same. This is because none of the 5 classical types of sevenths has intervallic structure $[c_1, c_2, c_3, c_4]$*

[1] In red the new operations with respect of those acting on K_1.

where $c_i = c_{i+1} = 4$ for any c_i.

5.2 The ST-group

Let $K = H \cup S^*$ be the set of all sevenths of the five classical types (dominant, minor, half-diminished, major and diminished) and triads (major, minor, augmented and diminished). We want to introduce the following general notation able to represent operations on K among sevenths, among triads and also operations among sevenths and triads:

$$(\theta, (v, w))$$

- θ is a permutation representing the change of the type among the chords.
- $v \in \mathbb{Z}_{12}^5$ is a vector playing a role similar to those of chapters 3 and 4: each v_i represents the movement of the (marked) root of the chords of type i (both if it is transformed into another seventh, or into a triad);
- $w \in \mathbb{Z}_{12}^4$, analogously, is a vector in which each w_i represents the movement of the (marked) root of the chords of type i (both if it is transformed into another triad, or into a seventh).

Example 5.7.

$$\begin{aligned}
P_{12} &= ((12), ((0,0,0,0,0), (0,0,0,0)))\\
R_{42} &= ((42), ((0,3,0,-3,0), (0,0,0,0)))\\
L = L_{Mm} &= ((Mm), ((0,0,0,0,0), (4,-4,0,0)))\\
\mathcal{P}_{1M} &= ((1M), ((0,0,0,0,0), (0,0,0,0)))\\
\mathcal{L}_{4m} &= ((4m), ((0,0,0,4,0), (0,-4,0,0)))
\end{aligned}$$

Remark 5.8. *Considering only transformations acting on sevenths (or triads), the vector $v = (v_1, v_2, v_3, v_4, v_5)$ (respectively $w = (w_1, w_2, w_3, w_4)$) is such that $\sum_{i=1}^{5} v_i = 0$ ($\sum_{i=1}^{4} w_i = 0$). As we see in the previous example, if we consider also transformations among sevenths and triads this property fails; anyway, it is easy to see that we always have that $\sum_{i=1}^{5} v_i + \sum_{i=1}^{4} w_i = 0$.*

Let ST be the group generated by the following parsimonious operations acting on $K = H \cup S^*$:

- the 17 parsimonious operations among the 5 types of sevenths;
- the 13 parsimonious operations among the 4 types of triads;
- the 12 parsimonious operations among the 5 types of sevenths and the 4 types of triads.

Given two musical operations $U_1 = (\theta_1, (v_1, w_1))$, $U_2 = (\theta_2, (v_2, w_2))$ on K, the composition law is defined as follows:

$$(\theta_2, (v_2, w_2)) \circ (\theta_1, (v_1, w_1)) = (\theta_2 \cdot \theta_1, (v_1, w_1) + \theta_1^{-1}(v_2, w_2)) \tag{5.1}$$

It is evident that given two musical operations acting on the set H of the classical types of sevenths, the Formula 5.1 corresponds to the Formula 3.7. Similarly, if two musical operations act on the set S^* of all triads, the Formula 5.1 corresponds to the Formula 4.2. In fact, let $U_1 = (\theta_1, (v_1, w_1))$ and $U_2 = (\theta_2, (v_2, w_2))$ be two musical operations exchanging two types of sevenths in K, then

$$\begin{aligned}
U_1 &= (\theta_1, (v_1, (0,0,0,0)))\\
U_2 &= (\theta_2, (v_2, (0,0,0,0)))
\end{aligned}$$

that we can easily represent as $U_{1|H} = (\theta_1, v_1)$ and $U_{2|H} = (\theta_2, v_2)$. Applying the Formula 5.1 we obtain

$$\begin{aligned} U_2 \circ U_1 &= (\theta_2, (v_2, (0,0,0,0,0))) \circ (\theta_1, (v_1, (0,0,0,0,0))) = \\ &= (\theta_2\theta_1, (v_1 + \theta_1^{-1}(v_2), (0,0,0,0))) \end{aligned}$$

that we can easily represent as $U_{2|H}U_{1|H} = (\theta_2\theta_1, v_1 + \theta_1^{-1}(v_2))$. But this is exactly the same result that we obtain applying the Formula 3.7. For triads the argument is analogous.

Now we show an example of applications if one operation exchanges only triads and the other one only sevenths.

Example 5.9. *We consider the operations*

$$L_{13}\colon [\underline{x}, x+4, x+7, x+10] \quad \rightleftarrows \quad [x+2, \underline{x+4}, x+7, x+10] \quad (\theta, (v, w)) = (Id, (13), Id, ((4,0,-4,0,0), (0,0,0,0)))$$

$$R_{Mm}\colon [\underline{x}, x+4, x+7] \quad \rightleftarrows \quad [x, x+4, \underline{x+9}] \qquad (\theta, \sigma, \tau, (v, w)) = ((12), ((0,0,0,0,0), (-3,3,0,0)))$$

exchanging, respectively, dominant and half-diminished sevenths and major and minor triads. Applying the formula 5.1, we obtain:

$$\begin{aligned} R_{Mm} \circ L_{13} &= ((Mm), ((0,0,0,0,0), (-3,3,0,0))) \circ ((13), ((4,0,-4,0,0), (0,0,0,0))) = \\ &= ((Mm)(13), ((4,0,-4,0,0), (0,0,0,0) + ((0,0,0,0,0), (-3,3,0,0))) = \\ &= ((Mm)(13), ((4,0,-4,0,0), (-3,3,0,0))). \end{aligned}$$

Finally, we show some example of application with two operations exchanging sevenths and triads.

Example 5.10. *We consider the operations*

$$\mathcal{L}_{4m}\colon [\underline{x}, x+4, x+7, x+11] \quad \rightleftarrows \quad [\underline{x+4}, x+7, x+11] \quad (\theta, v, w) = ((4m), ((0,0,0,4,0), (0,-4,0,0)))$$

$$\mathcal{R}_{3m}\colon [\underline{x}, x+3, x+6, x+10] \quad \rightleftarrows \quad [\underline{x+3}, x+6, x+10] \quad (\theta, v, w) = ((3m), ((0,0,3,0,0), (0,-3,0,0)))$$

exchanging, respectively, minor sevenths and minor triads, and half-diminished sevenths and minor triads. We apply the formula 5.1:

$$\begin{aligned} \mathcal{L}_{4m} \circ \mathcal{R}_{3m} &= ((4m), ((0,0,0,4,0), (0,-4,0,0))) \circ ((3m), ((0,0,3,0,0), (0,-3,0,0))) = \\ &= ((4m)(3m), ((0,0,3,0,0), (0,-3,0,0)) + (3m)((0,0,0,4,0), (0,-4,0,0))) = \\ &= ((34m), ((0,0,3,0,0), (0,-3,0,0)) + ((0,0,-4,4,0), (0,0,0,0))) = \\ &= ((34m), ((0,0,-1,4,0), (0,-3,0,0))) \end{aligned}$$

Now we list the parsimonious operations generating the ST-group as elements $(\theta, (v, w))$:

$$P_{12}\colon [\underline{x}, x+4, x+7, x+10] \quad \rightleftarrows \quad [\underline{x}, x+3, x+7, x+10] \qquad (\theta, (v, w)) = ((12), ((0,0,0,0,0), (0,0,0,0)))$$

$$P_{14}\colon [\underline{x}, x+4, x+7, x+10] \quad \rightleftarrows \quad [\underline{x}, x+4, x+7, x+11] \qquad (\theta, (v, w)) = ((14), ((0,0,0,0,0), (0,0,0,0)))$$

$$P_{23}\colon [\underline{x}, x+3, x+7, x+10] \quad \rightleftarrows \quad [\underline{x}, x+3, x+6, x+10] \qquad (\theta, (v, w)) = ((23), ((0,0,0,0,0), (0,0,0,0)))$$

$$P_{35}\colon [\underline{x}, x+3, x+6, x+10] \quad \rightleftarrows \quad [\underline{x}, x+3, x+6, x+9] \qquad (\theta, (v, w)) = ((35), ((0,0,0,0,0), (0,0,0,0)))$$

$$R_{12}\colon [\underline{x}, x+4, x+7, x+10] \quad \rightleftarrows \quad [x, x+4, x+7, \underline{x+9}] \qquad (\theta, (v, w)) = ((12), ((-3,3,0,0,0), (0,0,0,0)))$$

$$R_{23}\colon [\underline{x}, x+3, x+7, x+10] \quad \rightleftarrows \quad [x, x+3, x+7, \underline{x+9}] \qquad (\theta, (v, w)) = ((23), ((0,-3,3,0,0), (0,0,0,0)))$$

$R_{42}\colon [\underline{x}, x+4, x+7, x+11] \quad \rightleftarrows \quad [x, x+4, x+7, \underline{x+9}] \qquad (\theta,(v,w)) = ((42),((0,3,0,-3,0),(0,0,0,0)))$

$R_{35}\colon [\underline{x}, x+3, x+6, x+10] \quad \rightleftarrows \quad [x, x+3, x+6, \underline{x+9}] \qquad (\theta,(v,w)) = ((35),((0,0,-3,0,3),(0,0,0,0)))$

$R_{53}\colon [\underline{x}, x+3, x+6, x+9] \quad \rightleftarrows \quad [x, x+3, x+7, \underline{x+9}] \qquad (\theta,(v,w)) = ((53),((0,0,3,0,-3),(0,0,0,0)))$

$L_{13}\colon [\underline{x}, x+4, x+7, x+10] \quad \rightleftarrows \quad [x+2, \underline{x+4}, x+7, x+10] \qquad (\sigma,v) = ((13),((4,0,-4,0,0),(0,0,0,0)))$

$L_{15}\colon [\underline{x}, x+4, x+7, x+10] \quad \rightleftarrows \quad [x+1, \underline{x+4}, x+7, x+10] \qquad (\sigma,v) = ((15),((4,0,0,0,-4),(0,0,0,0)))$

$L_{42}\colon [\underline{x}, x+4, x+7, x+11] \quad \rightleftarrows \quad [x+2, \underline{x+4}, x+7, x+11] \qquad (\sigma,v) = ((42),((0,-4,0,4,0),(0,0,0,0)))$

$Q_{43}\colon [\underline{x}, x+4, x+7, x+11] \quad \rightleftarrows \quad [\underline{x+1}, x+4, x+7, x+11] \qquad (\sigma,v) = ((43),((0,0,-1,1,0),(0,0,0,0)))$

$Q_{15}\colon [\underline{x}, x+4, x+7, x+10] \quad \rightleftarrows \quad [\underline{x+1}, x+4, x+7, x+10] \qquad (\sigma,v) = ((15),((1,0,0,0,-1),(0,0,0,0)))$

$RR_{35}\colon [\underline{x}, x+3, x+6, x+10] \quad \rightleftarrows \quad [x, x+3, \underline{x+6}, x+9] \qquad (\sigma,v) = ((35),((0,0,-6,0,6),(0,0,0,0)))$

$QQ_{51}\colon [\underline{x}, x+3, x+6, x+9] \quad \rightleftarrows \quad [x, \underline{x+2}, x+6, x+9] \qquad (\sigma,v) = ((51),((-2,0,0,0,2),(0,0,0,0))$

$N_{51}\colon [\underline{x}, x+3, x+6, x+9] \quad \rightleftarrows \quad [x, x+3, \underline{x+5}, x+9] \qquad (\sigma,v) = ((51),((-5,0,0,0,5),(0,0,0,0))$

$Q_{Md}\colon [\underline{x}, x+4, x+7] \quad \rightleftarrows \quad [\underline{x+1}, x+4, x+7] \qquad (\theta,(v,w)) = ((13),((0,0,0,0,0),(1,0,0,-1)))$

$Q_{Am}\colon [\underline{x}, x+4, x+8] \quad \rightleftarrows \quad [\underline{x+1}, x+4, x+8] \qquad (\theta,(v,w)) = ((42),((0,0,0,0,0),(0,-1,1,0)))$

$L_{Mm}\colon [\underline{x}, x+4, x+7] \quad \rightleftarrows \quad [x+1, \underline{x+4}, x+7] \qquad (\theta,(v,w)) = ((12),((0,0,0,0,0),(4,-4,0,0)))$

$L_{AM}\colon [\underline{x}, x+4, x+8] \quad \rightleftarrows \quad [x+1, \underline{x+4}, x+8] \qquad (\theta,(v,w)) = ((41),((0,0,0,0,0),(4,0,-4,0)))$

$N_{Am}\colon [\underline{x}, x+4, x+8] \quad \rightleftarrows \quad [x, \underline{x+5}, x+8] \qquad (\theta,(v,w)) = ((42),((0,0,0,0,0),(0,-5,5,0)))$

$P_{Mm}\colon [\underline{x}, x+4, x+7] \quad \rightleftarrows \quad [\underline{x}, x+3, x+7] \qquad (\theta,(v,w)) = ((12),((0,0,0,0,0),(0,0,0,0)))$

$L_{MA}\colon [\underline{x}, x+4, x+7] \quad \rightleftarrows \quad [x, \underline{x+4}, x+8] \qquad (\theta,(v,w)) = ((14),((0,0,0,0,0),(4,0,-4,0)))$

$P_{dm}\colon [\underline{x}, x+3, x+6] \quad \rightleftarrows \quad [\underline{x}, x+3, x+7] \qquad (\theta,(v,w)) = ((32),((0,0,0,0,0),(0,0,0,0)))$

$R_{Am}\colon [\underline{x}, x+4, x+8] \quad \rightleftarrows \quad [x, x+4, \underline{x+9}] \qquad (\theta,(v,w)) = ((42),((0,0,0,0,0),(0,3,-3,0)))$

$P_{AM}\colon [\underline{x}, x+4, x+8] \quad \rightleftarrows \quad [\underline{x}, x+4, x+7] \qquad (\theta,(v,w)) = ((41),((0,0,0,0,0),(0,0,0,0)))$

$L_{Md}\colon [\underline{x}, x+4, x+7] \quad \rightleftarrows \quad [x+10, \underline{x+4}, x+7] \qquad (\theta,(v,w)) = ((13),((0,0,0,0,0),(4,0,0,-4)))$

$R_{Mm}\colon [\underline{x}, x+4, x+7] \quad \rightleftarrows \quad [x, x+4, \underline{x+9}] \qquad (\theta,(v,w)) = ((12),((0,0,0,0,0),(-3,3,0,0)))$

$R_{md}\colon [\underline{x}, x+3, x+7] \quad \rightleftarrows \quad [x, x+3, \underline{x+9}] \qquad (\theta,(v,w)) = ((23),((0,0,0,0,0),(0,3,-3,0)))$

$\mathcal{P}_{1M}\colon [\underline{x}, x+4, x+7, x+10] \quad \rightleftarrows \quad [\underline{x}, x+4, x+7] \qquad (\theta,(v,w)) = ((1M),((0,0,0,0,0),(0,0,0,0)))$

$\mathcal{L}_{1d}\colon [\underline{x}, x+4, x+7, x+10] \quad \rightleftarrows \quad [\underline{x+4}, x+7, x+10] \qquad (\theta,(v,w)) = ((1d),((4,0,0,0,0),(0,0,-4,0)))$

$\mathcal{P}_{2m}\colon [\underline{x}, x+3, x+7, x+10] \quad \rightleftarrows \quad [\underline{x}, x+3, x+7] \qquad (\theta,(v,w)) = ((2m),((0,0,0,0,0),(0,0,0,0)))$

$\mathcal{R}_{2M}\colon [\underline{x}, x+3, x+7, x+10] \quad \rightleftarrows \quad [\underline{x+3}, x+7, x+10] \qquad (\theta,(v,w)) = ((2M),((0,3,0,0,0),(-3,0,0,0)))$

$\mathcal{P}_{3d}\colon [\underline{x}, x+3, x+6, x+10] \quad \rightleftarrows \quad [\underline{x}, x+3, x+6] \qquad (\theta,(v,w)) = ((3d),((0,0,0,0,0),(0,0,0,0)))$

$\mathcal{R}_{3m}\colon [\underline{x}, x+3, x+6, x+10] \quad \rightleftarrows \quad [\underline{x+3}, x+6, x+10] \qquad (\theta,(v,w)) = ((3m),((0,0,3,0,0),(0,-3,0,0)))$

$\mathcal{P}_{4M}\colon [\underline{x}, x+4, x+7, x+11] \quad \rightleftarrows \quad [\underline{x}, x+4, x+7] \qquad (\theta,(v,w)) = ((4M),((0,0,0,0,0),(0,0,0,0)))$

$\mathcal{L}_{4m}\colon [\underline{x}, x+4, x+7, x+11] \quad \rightleftarrows \quad [\underline{x+4}, x+7, x+11] \qquad (\theta,(v,w)) = ((4m),((0,0,0,4,0),(0,-4,0,0)))$

$\mathcal{P}_{5d}\colon [\underline{x}, x+3, x+6, x+9] \quad \rightleftarrows \quad [\underline{x}, x+3, x+6] \qquad (\theta,(v,w)) = ((5d),((0,0,0,0,0),(0,0,0,0)))$

$\mathcal{R}_{5d}\colon [\underline{x}, x+3, x+6, x+9] \quad \rightleftarrows \quad [\underline{x+3}, x+6, x+9] \qquad (\theta,(v,w)) = ((5d),((0,0,0,0,3),(0,0,-3,0)))$

$\mathcal{RR}_{5d}\colon [\underline{x}, x+3, x+6, x+9] \quad \rightleftarrows \quad [x, \underline{x+6}, x+9] \qquad (\theta,(v,w)) = ((5d),((0,0,0,0,6),(0,0,-6,0)))$

$\mathcal{Z}_{5d}\colon [\underline{x}, x+3, x+6, x+9] \quad \rightleftarrows \quad [x, x+3, \underline{x+9}] \qquad (\theta,(v,w)) = ((4m),((0,0,0,0,9),(0,0,-9,0))).$

Theorem 5.11. *The ST-group is isomorphic to* $S_9 \ltimes \mathbb{Z}_{12}^8$.

Proof. First of all we prove that $ST \simeq S_9 \ltimes V$, where $V = \{(v,w) \in \mathbb{Z}_{12}^5 \times \mathbb{Z}_{12}^4 | \sum_{i=1}^5 v_i + \sum_{j=1}^4 w_j = 0\}$.

Let $\{O\} = ((0,0,0,0,0),(0,0,0,0))$ be the identity in V. Then $S_9 \times \{O\}$ is a subgroup of ST. In fact, the elements $P_{12}, P_{14}, P_{23}, P_{35}$ (generators of $S_5 \times \{O\}$ in the $PLRQ$-group), P_{Mm}, P_{dm} and P_{AM} (generator of $S_4 \times \{O\}$ in the PLR^*-group) and, in addition, P_{16} generate $S_9 \times \{O\}$. Moreover, the subgroup formed by the elements (Id, v) is normal, because for all $(\theta,(v,w)) \in S_9 \times V$, and all $(Id,(v',w')) \in \{Id\} \times V$

$$(\theta,(v,w))(Id,(v'.w'))(\theta,(v,w))^{-1} = (\theta\theta^{-1}, -(v,w) + \theta(v',w') + \theta(v,w)) = (Id,(v'',w'')) \in \{Id\} \times V.$$

Furthermore, the two subgroups has trivial intersection. Hence $ST \simeq S_9 \ltimes V$.

Now we observe that the normal subgroup of elements $(Id,(v,w))$ is isomorphic to $\{Id\} \times \mathbb{Z}_{12}^8$. In fact

$$\begin{aligned}
R_{42}L_{42} &= (Id,((0,-7,0,7,0),(0,0,0,0)))\\
P_{14}L_{42}P_{14}R_{12} &= (Id,((7,-7,0,0,0),(0,0,0,0)))\\
P_{12}L_{13}P_{12}R_{23} &= (Id,((0,7,-7,0,0),(0,0,0,0)))\\
P_{12}P_{35}R_{23}P_{12}L_{15}L_{13} &= (Id,((7,0,0,0,-7),(0,0,0,0)))\\
L_{Mm}R_{Mm} &= (Id,((0,0,0,0,0),(-7,7,0,0)))\\
P_{dm}Q_{Md}P_{Mm}Q_{Md} &= (Id,((0,0,0,0,0),(0,1,0,-1)))\\
Q_{AM}P_{AM}P_{Mm}P_{AM} &= (Id,((0,0,0,0,0),(0,1,0,1)))\\
\mathcal{R}_{3m}\mathcal{L}_{4m}P_{14}P_{12}P_{23}P_{12}P_{14}\mathcal{L}_{4m} &= (Id,((0,0,1,0,0),(0,-1,0,0)))
\end{aligned}$$

generate all elements $(Id,((v_1,v_2,v_3,v_4,v_5),(w_1,w_2,w_3,w_4)))$, with $((v_1,\dots,v_5),(w_1,\dots,w_4)) \in \mathbb{Z}_{12}^9$ such that $\sum_{i=1}^5 v_i + \sum_{j=1}^4 w_j = 0$. In fact, for any suc (v,w) we have to find $a,b,c,d,e,f,g,h \in \mathbb{Z}$ such that

$$\begin{aligned}
&a((0,-7,0,7,0),(0,0,0,0)) + b((7,-7,0,0,0),(0,0,0,0)) + c((0,7,-7,0,0),(0,0,0,0))\\
&+d(7,0,0,0,-7),(0,0,0,0)) + e((0,0,0,0,0),(-7,7,0,0)) + f((0,0,0,0,0),(0,1,0,-1))\\
&+g((0,0,0,0,0),(0,1,-1,0)) + h((0,0,1,0,0),(0,-1,0,0)) \equiv ((v_1,v_2,v_3,v_4,v_5),(w_1,w_2,w_3,w_4)) \pmod{12}
\end{aligned}$$

which amounts to solve the system

$$\begin{cases}
7b + 7d \equiv v_1\\
-7a - 7b + 7c \equiv v_2\\
-7c + h \equiv v_3\\
7a \equiv v_4\\
-7d \equiv v_5\\
-7e \equiv w_1\\
7e + f + g - h \equiv w_2\\
-g \equiv w_3\\
-f \equiv w_4
\end{cases}.$$

Calculations are simple and, as we show below, the system is solvable for all zero-sum vectors:

$$\begin{cases} 7b \equiv v_1 + v_5 \\ 7c \equiv v_2 + v_4 + v_1 + v_5 \\ h \equiv v_3 + v_2 + v_4 + v_1 + v_5 \\ 7a \equiv v_4 \\ -7d \equiv v_5 \\ -7e \equiv w_1 \\ -h \equiv w_2 + w_1 + w_3 + w_4 \\ -g \equiv w_3 \\ -f \equiv w_4 \end{cases} \Leftrightarrow \begin{cases} 7b \equiv v_1 + v_5 \\ 7c \equiv v_1 + v_2 + v_4 + v_5 \\ h \equiv v_1 + v_2 + v_3 + v_4 + v_5 \\ 7a \equiv v_4 \\ -7d \equiv v_5 \\ -7e \equiv w_1 \\ 0 \equiv w_1 + w_2 + w_3 + w_4 + v_1 + v_2 + v_3 + v_4 + v_5 \\ -g \equiv w_3 \\ -f \equiv w_4 \end{cases}.$$

Hence $ST \simeq S_9 \ltimes \mathbb{Z}_{12}^8$. □

Conclusions and future perspectives

We have studied the historical development of the *Tonnetz*, including investigations on possible relations with other musical-geometric representations sharing the same properties, and discussing in which way we arrived at the traditional graph used in Mathematical Music Theory. It would be interesting to study in more detail Vogel and Imig's works, because it seems that they might have an important role on the developments of the *Tonnetz* and its name during the XX^{th} century, at least according to Gollin's suggestions [36, 37]. It would be also interesting to better analyze Euler's mathemusical work, looking at the strategic place occupied by music in his theoretical writings.

Furthermore, we have extended and generalized the studies on the neo-Riemannian operations P, L and R for major and minor triads to other sets of chords: two sets of sevenths and the set of all 4 traditional triads. We have classified the most parsimonious operations acting on these sets and studied the group generated by them. Moreover, we have found the parsimonious graph associated to the parsimonious operations on the set of the 5 classical types of sevenths (which we have proposed to name the *Clover* graph) and on the set of major, minor and diminished triads (generalized *Chicken-wire Torus*), and on the collection of major, minor and augmented triads (generalized *Cube Dance*). It could be interesting to create the parsimonious graphs associated to the other sets analyzed from an algebraic point of view, and to study Hamiltonian paths and cycles on them (up to isomomorphisms).

We have introduced a new notation, enabling us to describe every transformation acting on different sets of chords, including sets of chords with different cardinalities. It would be interesting to study groups generated by other transformations on sets of chords of different cardinalities as well as the parsimonious graph associated to the operations on other sets of chords, in particular those including chords of different cardinalities.

It would also be interesting to extend these algebraic studies through the point of view of non-contextual transformations, introduced by Franck Jedrzejewski [45]: in fact, the P, L and R operations, once defined as inversions, are contextual transformations, since the corresponding inversions defined depend on the given chord. Jedrzejewski had the idea to re-defined them as products of permutations (more precisely transpositions), in order to define non-contextual transformations. The new transformations he defined are different from the original neo-Riemannian operations, but despite the change of perspective the main properties are maintained. Moreover, there are musical examples in which they are better adapted to describe isographies than contextual operations. The same perspective could be applied by considering seventh chords, including the comparison of advantages and disadvantages of the two approaches with relevant musical examples.

Moreover, it would be interesting to study the models presented in this thesis through the Discrete Fourier Transform (DFT), especially developed by Emmanuel Amiot, and summarized in his monograph *Music Through Fourier Space* [4]. The recent applications of the DFT in music theory take up the phases of Fourier coefficients. In particular, in [3] Amiot introduced a bidimensional toroidal space, defined by pairs of Fourier coefficients. This formalization is useful to represent simultaneously single notes, triads and any chord necessary for the analysis of a musical piece. In this space triads are disposed with the same topology as the traditional *Tonnetz*, which asks for a possible generalization of these constructions in the case of seventh chords.

Finally, since this work is within a PhD in Mathematics, we focused more on the mathematical part, leaving aside the musical application of all theoretical constructions we presented. We are aware there is still much work to be done to show that the formal and theoretical aspects that have been discussed are not only interesting from a mathematical point of view, but also as a way to suggest new original applications in music analysis and composition.

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