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Default Probability Estimation via Pair Copula Constructions

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Abstract

In this paper we present a novel Bayesian approach for default probability estimation. The methodology is based on multivariate contingent claim analysis and pair copula theory. Balance sheet data are used to asses the firm value and to compute its default probability. The firm pricing function is obtained via a pair copula approach, and Monte Carlo simulations are used to calculate the default probability distribution. The methodology is illustrated through an application to defaulted firms data.

Keywords: *Bayesian analysis, Pair Copula, Default Risk, Multivariate Contingent Claim, Markov Chain Monte Carlo, Vines.*

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1 Introduction

Default risk is defined as the risk of loss when in a financial contract a debtor (in our case a firm) does not fulfil its commitments and a default event takes place. Default probability is the probability that a default happens.

Following the growing financial uncertainty, there has been intensive research by financial institutions, regulators and academics to develop models for default risk estimation. Academics and practitioners increasingly propose new methodologies able to evaluate in a pragmatic way the firm under examination.

The main differences among the existing methodologies for default risk evaluation depend on the available information and data used for assessing the firm value and its default probability. These models can be broadly classified in market data based models and accounting data based models.

Within the market based models the most popular are the structural models; see Merton (1970, 1974 and 1977) and its extensions. Stocks and bonds are used as structural variables for firm evaluation and default probability estimation. The asset value is considered to be exogenous and it is treated as the underlying asset in an option pricing framework. A common assumption is that the asset value follows a geometric Brownian motion and the drift and volatility coefficients do not depend on the capital structure of the firm. The Black and Scholes formula is applied to evaluate the asset value and therefore the probability of default of the firm.

The second class of models use accounting data and financial ratios to evaluate the firm value and to estimate the probability of default. They origin from the works of Beaver (1966) and Altman (1968), who developed univariate and multivariate models to predict the default of specific firms by using a set of financial ratios. In this class another commonly used default prediction model is based on the use of logistic regression, as proposed by Ohlson (1980).

Default probability estimation has been carried out both in a classical and a Bayesian framework. For the classical framework see e.g. McNeil et al. (2005), Schuermann (2005), De Giuli et al. (2008), Su and Huang (2010), and references herein. For the Bayesian analysis see e.g. Kiefer (2009, 2010 and 2011), Park et al. (2010) and Tasche (2011), and references herein.

Whatever the working framework is, one should impose a specific distributional assumption on the data. For a long time normality has been a standard assumption. However, in most cases a preliminary exploratory analysis of the data reveals significant departures from normality. A well known and popular solution to this problem is the use of copulas, introduced by Sklar (1959). The advantage of copulas is the possibility to combine different marginal distributions via a copula function. Unfortunately, while there is a wide range of possible alternative copula functions for the bivariate case, in the multivariate setting the use of distribution different from Gaussian and Student's t is rather scarce, due to computational and theoretical limitations. For this reason Joe (1996) introduces Pair Copula Constructions (PCCs) to represent complex structure of dependence among multivariate data. Pair Copulas are a collection of potentially different bivariate copulas, used to construct the joint distribution of interest. Different types and strengths of dependence may be easily represented via a Pair Copula model. For this reason, they constitute a flexible and very appealing tool for financial analysis, see e.g. Vaz de Melo Mendes et al. (2010). In particular, a very interesting application in this field can be found in a recent work by Bernard and Czado (2013). The authors focus on contingent claim pricing models, considering multivariate underlying indices in option contracts and use PCCs to allow for more flexibility in the estimation of dependencies.

In this paper we propose a novel semi-Bayesian approach for default probability estimation. We combine the main features of structural and accounting based models. We go beyond structural models, applying a contingent claim model based on accounting data. The equity of a firm is obtained via a Pair Copula model based on balance sheet data, and Monte Carlo simulations are used to calculate the probability of default. To our knowledge this is the first time that pair copulas are applied in this context.

The plan of the paper is the following. In Section 2 we introduce Copulas and PCCs. In Section 3 we illustrate the contingent claim model based on accounting data. Section 4 is devoted to the estimation of the marginals and of the PCCs. Section 5 applies the methodology to the stock data of four famous defaulted companies. Finally, concluding remarks are given in Section 6.

2 Background and Preliminaries

2.1 Copula Function

Copulas are a very popular statistical tools, applicable to a wide variety of fields, such as finance, economics, risk management and marketing; for a recent review see e.g. Jaworski (2010). Copulas allow to describe complex multivariate pattern of dependence binding together the marginal distributions.

By definition, the copula is a multivariate distribution function with marginals distributed according to a uniform on the interval $[0, 1]$. This function, once applied to the univariate marginal distributions, returns their multivariate joint distribution, enclosing all the information about the dependence structure of the marginals. Thus, the use of copulas allows to split the distribution of a random vector into its individual marginal components, and the dependence structure is modelled through the copula function without losing information; for more details see e.g. Joe (1997) and Nelsen (1999).

The most important result about copula theory is Sklar's theorem. This theorem states that, given $\mathbf{X} = (X_1, \ldots, X_d)$, a vector of random variables with d-dimensional joint cumulative distribution function $F(x_1, \ldots, x_d)$ and marginal cumulative distributions $F_m(x_m)$ (with $m = 1, \ldots, d$), there exist a d-dimensional copula C such that

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)).$$

Conversely, according to Nelsen's corollary, the inversion method allows to express the copula in the following way

$$C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))$$

where $F_1^{-1}, \ldots, F_d^{-1}$ are the generalised inverse functions of the marginals. The joint density function is therefore

$$f(x_1, \ldots, x_d) = c(F_1(x_1), \ldots, F_d(x_d)) \cdot f_1(x_1) \cdots f_d(x_d)$$

where $c(F_1(x_1), \ldots, F_d(x_d))$ is the d-variate copula density.

The existing literature on copulas mainly focuses on the bivariate case. In the multivariate case, the Gaussian and Student's t copula are the most popular, while the usage of other multidimensional copulas is rather limited, due to the complexity of their construction, see e.g. Aas and Berg (2007). However, Gaussian and Student's t copula are often not flexible enough to represent the dependence structure (especially the tail dependence) of many financial data. Hence, multivariate extensions of Archimedean copulas were proposed in the form of partially nested Archimedean copulas by Joe (1997) and Whelan (2004); hierarchical Archimedean copulas by Savu and Trede (2006); and multiplicative Archimedean copulas by Morillas (2005) and Liebscher (2006). Nevertheless, these multivariate extensions imply additional restrictions on the parameters that limit their flexibility.

A possible solution to this problem is provided by PCCs. PCCs were originally proposed by Joe (1996), and later discussed in detail by Bedford and Cooke (2001 and 2002), Kurowicka and Cooke (2006) and Aas et al. (2009). For some recent works see Czado (2010), Min and Czado (2010) and Nikoloulopoulos et al. (2012). A PCC represents the complex pattern of dependence of multivariate data via a cascade of bivariate copulas, and permits to construct flexible high-dimensional copulas by using only bivariate copulas as building blocks, see Aas et al. (2009). Therefore, the joint distribution is obtained on the basis of bivariate pair copulas, that may be conditional on a specific set of variables, allowing to model the dependence among the marginals.

2.2 Pair Copula Constructions

We now briefly introduce PCCs, the related notation and terminology; for more details see e.g. Czado (2010).

It is well known that the distribution $f(x_1, \ldots, x_d)$ of a random vector $\mathbf{X} = (X_1, \ldots, X_d)$ can be factorized through conditional densities as follows

$$f(x_1, \ldots, x_d) = f_d(x_d) \times f_{d-1|d}(x_{d-1}|x_d) \times \ldots \times f_{1|2 \cdots d}(x_1|x_2, \ldots, x_d). \tag{1}$$

The factorisation in (1) is unique up to re-labeling of the variables, and it can be reexpressed in terms of a product of bivariate copulas.

By Sklar's theorem, in fact, the joint distribution of the subvector (X_d, X_{d-1}) can be expressed in terms of a copula density

$$f(x_{d-1}, x_d) = c_{d-1,d}(F_{d-1}(x_{d-1}), F_d(x_d)) \times f_{d-1}(x_{d-1}) \times f_d(x_d),$$

where $c_{d-1,d}(\cdot,\cdot)$ is an arbitrary bivariate copula (pair copula) density. Hence, the conditional density of $X_{d-1}|X_d$ can be easily rewritten as

$$f_{d-1|d}(x_{d-1}|x_d) = c_{d-1,d}(F_{d-1}(x_{d-1}), F_d(x_d)) \times f_{d-1}(x_{d-1}). \tag{2}$$

Through a straightforward generalisation of equation (2), each term in (1) can be decomposed into the appropriate pair copula times a conditional marginal density.

More precisely, for a generic element X_j of the vector $\mathbf{X}$ we obtain

$$f_{x_j|\mathbf{v}}(x_j|\mathbf{v}) = c_{x_j,v_j|\mathbf{v}_{-j}}(F_{x_j|\mathbf{v}_{-j}}(x_j|\mathbf{v}_{-j}), F_{v_j|\mathbf{v}_{-j}}(v_j|\mathbf{v}_{-j})) \times f_{x_j|\mathbf{v}_{-j}}(x_j|\mathbf{v}_{-j}), \tag{3}$$

where $\mathbf{v}$ is the conditioning vector, v_j is an generic component of $\mathbf{v}$, $\mathbf{v}_{-j}$ is the vector $\mathbf{v}$ without the component v_j, $F_{x_j|\mathbf{v}_{-j}}(\cdot|\cdot)$ is the conditional distribution of x_j given $\mathbf{v}_{-j}$, and $c_{x_j,v_j|\mathbf{v}_{-j}}(\cdot,\cdot)$ is the conditional pair copula density. The d-dimensional joint multivariate distribution function can be expressed as a product of pair copulas by recursively using equation (3) in equation (1). Since the conditional distributions of the form $F_{x|\mathbf{v}}(\cdot|\cdot)$ are not directly observable, we calculate them using Joe's (1996) result

$$F_{x|\mathbf{v}}(x|\mathbf{v}) = \frac{\partial C_{x,v_j|\mathbf{v}_{-j}}(F(x|\mathbf{v}_{-j}), F(v_j|\mathbf{v}_{-j}))}{\partial F(v_j|\mathbf{v}_{-j})}. \tag{4}$$

If the conditioning set $\mathbf{v}$ is univariate, $\mathbf{v} = v$ and expression (4) can be written as

$$F(x|v) = \frac{\partial C_{x,v}(x, y, \boldsymbol{\theta})}{\partial v} = h(x, v, \boldsymbol{\theta}),$$

where $\boldsymbol{\theta}$ denotes the set of parameters of the copula, and $F(x|v)$ is named the *h function.* The forms of the *h functions* for the main classes of copulas are given in Aas et al. (2009).

Therefore, a multivariate density can be expressed as a product of pair copulas acting on several different conditional distributions, obtaining a PCC. It is worth noting that the PCC is order dependent, and given a specific factorisation there are still many different parameterisations. We will provide more details about the different parameterisations in section 5.

2.3 D-Vines Distributions and Pair Copulas

For high-dimensional distributions, the number of possible PCCs is very high. Hence a suitable representation of all of them is necessary. Bedford and Cooke (2001 and 2002) introduced regular vines as a pictorial representation of PCCs.

Regular vines are a particular type of graphical models, that use a nested set of trees to represent the decomposition of the joint distribution into its bivariate components, incorporating the dependence structure of the variables of interest. Within the class of regular vines we consider the subset of D-vines; for more details about the different classes of vines see Kurowicka and Cooke (2006).

A vine $\mathcal{V}(d)$ on d variables is a nested set of trees (connected acyclic graphs) $T_1, ..., T_{d-1}$. The edges of tree T_τ are the nodes of tree $T_{\tau+1}$, $\tau = 1, \ldots, d-1$. In a regular vine, if two edges of tree T_τ share a common node, they are represented in tree $T_{\tau+1}$ by nodes joined by an edge. Finally, a D-vine is a regular vine where all nodes do not have degree higher than 2, that is each node is connected to no more than two other nodes. Figure 1 represents a 4-dimensional D-vine.

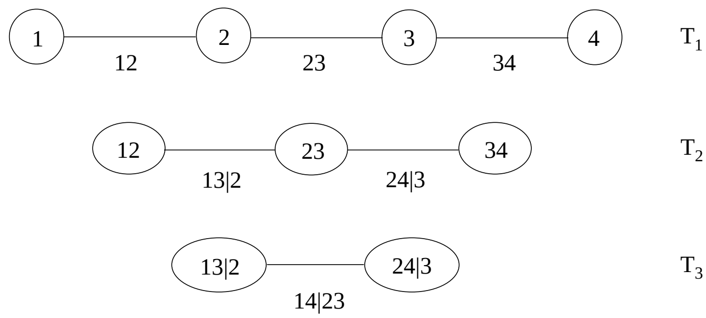

Figure 1: The graphical representation of the D-vine with 4 marginals.

Using the D-vine representation, the joint density can be decomposed in terms of conditional copula densities, identified by the labels of the edges in the considered trees, times the marginal densities of the examined variables. For the D-vine represented in Figure 1 the joint density is given by

$$f(x_1, \ldots, x_4) = \prod_{\tau=1}^{4} f_\tau(x_\tau) \times c_{12} \times c_{23} \times c_{34} \times c_{13|2} \times c_{24|3} \times c_{14|23}.$$

Note that in the previous equation we have simplified the notation, setting $c_{ab} = c_{ab}(F(x_a), F(x_b))$.

More generally, the density of a D-vine of dimension d takes the form

$$f(x_1, \ldots, x_d) = \prod_{\tau=1}^{d} f_\tau(x_\tau) \prod_{j=1}^{d-1} \prod_{i=1}^{d-j} c_{i,i+j|i+1,\ldots,i+j-1}(F(x_i|x_{i+1}, \ldots, x_{i+j-1}), F(x_{i+j}|x_{i+1}, \ldots, x_{i+j-1})),$$

which is the product of d marginal densities f_τ and $d(d-1)/2$ bivariate copulas $c_{i,i+j|i+1,\ldots,i+j-1}(\cdot, \cdot)$ evaluated at the conditional distribution functions $F(\cdot|\cdot)$.

3 An Accounting Firm Value Model

Following De Giuli et al. (2008) and Su and Huang (2010) firm value is modelled via a contingent claim on the underlings of the observed traded securities (stocks and bonds). The value of a contingent claim at time of maturity T can be written as

$$G(S_s(T)) \; s = 1, 2$$

where $G(\cdot)$ is the pay-off function, $S_1(T)$ and $S_2(T)$ are stocks and bonds prices at time to maturity T. The final value of the firm A_T can be expressed as

$$\begin{aligned} \mathsf{A}_T = G(\mathsf{E}_T, \mathsf{B}_T; T) &= max[(\mathsf{E}_T + \mathsf{B}_T); 0] \cdot \mathbb{I}_{\{(\mathsf{E}_T \geq 0), (0 \leq \mathsf{B}_T \leq \mathsf{D})\}} \\ &= (\mathsf{E}_T + \mathsf{B}_T)\mathbb{I}_{\{(\mathsf{E}_T > -\mathsf{B}_T), (\mathsf{E}_T \geq 0), (0 \leq \mathsf{B}_T \leq \mathsf{D})\}}, \end{aligned} \tag{5}$$

where E_T denotes the equity (stock value) at time of maturity T, B_T denotes the bond value, D denotes the debt value at the maturity and $\mathbb{I}$ is the indicator function.

We use the contingent claim representation to calculate the probability of default. The probability of default at generic time t is given by $Pr(\mathsf{E}_t \leq 0)$. At a generic time t the equity can be computed as $P(t,T) \cdot \mathbb{E}(\mathsf{E}_T)$, where $P(t,T)$ is the risk free discount factor and $\mathbb{E}(\mathsf{E}_T)$ is the expected value of E_T.

Rewriting equation (5) as

$$\mathsf{E}_T = G_1(\mathsf{A}_T, \mathsf{B}_T; T) = (\mathsf{A}_T - \mathsf{B}_T)\mathbb{I}_{\{(\mathsf{A}_T > 0), (\mathsf{A}_T - \mathsf{B}_T \geq 0), (0 \leq \mathsf{B}_T \leq D)\}},$$

the equity at time t is given by

$$\mathsf{E}_t = G_1(\mathsf{A}_t, \mathsf{B}_t; t) = P(t,T) \int_0^\infty \int_0^\infty G_1(\mathsf{A}_T, \mathsf{B}_T; T) g_1(\mathsf{A}_T, \mathsf{B}_T) d\mathsf{A}_T d\mathsf{B}_T, \tag{6}$$

where G_1 and $g_1(\mathsf{A}_T, \mathsf{B}_T)$ are respectively the pay-off function and its density. The firm value and its return volatility are not directly observable, hence we use balance sheet data, A_T (activity) and B_T (liability), as a reliable proxy of the market data, see e.g. Eberhart (2005). Furthermore, we decompose A_T and B_T in terms of current (C_T) and long term components (L_T), that is $A_T = A_{C_T} + A_{L_T}$ and $B_T = B_{C_T} + B_{L_T}$. Equation (6) can then be rewritten as

$$\begin{aligned} E_t &= G_1(A_{C_t}, A_{L_t}, B_{C_t}, B_{L_t}; t) = \\ &= P(t,T) \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty G_1(A_{C_T}, A_{L_T}, B_{C_T}, B_{L_T}; T) \times \\ &\times \; g_1(A_{C_T}, A_{L_T}, B_{C_T}, B_{L_T}) dA_{C_T} dA_{L_T} dB_{C_T} dB_{L_T}. \end{aligned}$$

We now express the 4-dimensional density function $g_1(A_{C_T}, A_{L_T}, B_{C_T}, B_{L_T})$ as a copula obtaining

$$\begin{aligned} E_t &= P(t,T) \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty G_1(A_{C_T}, A_{L_T}, B_{C_T}, B_{L_T}; T) c(F_{A_C}, F_{A_L}, F_{B_C}, F_{B_L}) \times \\ &\times \; f_{A_C} f_{A_L} f_{B_C} f_{B_L} dA_{C_T} dA_{L_T} dB_{C_T} dB_{L_T}, \end{aligned} \tag{7}$$

where $c(\cdot)$ denotes the 4-dimensional copula density function, $F_{A_C}, F_{A_L}, F_{B_C}, F_{B_L}$ are the marginal cumulative distribution functions, and $f_{A_C}, f_{A_L}, f_{B_C}, f_{B_L}$ are the marginal probability density functions.

The previous firm pricing function can be approximated by Monte Carlo method as follows

$$\tilde{E}_t = P(t,T)\frac{1}{N^4}\sum_{\iota=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{r=1}^{N} G_1(\tilde{A}_{C_{T\iota}}, \tilde{A}_{L_{Tk}}, \tilde{B}_{C_{Tl}}, \tilde{B}_{L_{Tr}}; T)$$

$$= P(t,T)\frac{1}{N^4}\sum_{\iota=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{r=1}^{N}(\tilde{A}_{C_{T\iota}} + \tilde{A}_{L_{Tk}} - \tilde{B}_{C_{Tl}} - \tilde{B}_{L_{Tr}})\mathbb{I}_{\{\Upsilon\}}$$

where

$$\{\Upsilon\} = \{(\tilde{A}_{C_{T\iota}} + \tilde{A}_{L_{Tk}} > 0), (\tilde{A}_{C_{T\iota}} + \tilde{A}_{L_{Tk}} - \tilde{B}_{C_{Tl}} - \tilde{B}_{L_{Tr}} \geq 0), (0 \leq \tilde{B}_{C_{Tl}} + \tilde{B}_{L_{Tr}} \leq D)\},$$

N is the number of simulations, $\tilde{E}_t, \tilde{A}_{C_{T\iota}}, \tilde{A}_{L_{Tk}}, \tilde{B}_{C_{Tl}}$ and $\tilde{B}_{L_{Tr}}$ are the simulated values of the equity, the current assets, the fixed assets, the current liabilities and the long-term debts, respectively. The Probability of Default (PD) at time t is therefore estimated by $(PD)_t = Pr(\tilde{E}_t \leq 0)$.

4 Model Estimation

The dynamic of the equity value in equation (7) depends on two sets of parameters: the parameters of the copula and those of the marginal distributions. Let $\boldsymbol{\theta}$ denote the parameter vector of the copula function $c(F_{A_C}, F_{A_L}, F_{B_C}, F_{B_L})$, and let $\boldsymbol{\delta_m}$ denote the vector of the parameters of the m-th marginal distribution (with m assuming values in the set $\{A_C, A_L, B_C, B_L\}$). Then, $\Delta = (\boldsymbol{\delta_{A_C}}, \boldsymbol{\delta_{A_L}}, \boldsymbol{\delta_{B_C}}, \boldsymbol{\delta_{B_L}})$ contains the parameters of the marginals and $\Psi = (\Delta, \boldsymbol{\theta})$ represents the full set of parameters associated to (7).

In order to estimate Ψ we follow a two-stage procedure proposed by Joe and Xu (1996) called *Inference Functions for Margins* (IFM). The IFM method estimates the marginal parameters Δ in a first step, and in a second step estimates the copula parameters $\boldsymbol{\theta}$, given $\hat{\Delta}_{IFM}$.

We will now illustrate in detail the estimation of the parameters in the two steps of the IFM method.

4.1 Marginal Parameter Estimation

The current and long term assets and liabilities present bi-modal distributions. This behaviour can find an explanation in the effect of the managerial actions and decisions performed to save the firm from bankruptcy. These actions and decisions directly impact the dynamic of current and long term assets and liabilities, and this can intuitively explain the presence of two separated clusters of data. We estimate each marginal distribution $F_{A_C}, F_{A_L}, F_{B_C}, F_{B_L}$ via a two-component

Gaussian mixture model assuming different means but equal variances (location-shift model). The cumulative distribution function of the m-th marginal distribution at time t is given by

$$F(x_{m_t}) = \sum_{p=1}^{2} \eta_p \Phi(x_{m_t}|\mu_p, \sigma^2)$$

where η_p is the classification probability for component p ($\eta_p \geq 0$ and $\sum_{p=1}^{2} \eta_p = 1$), $\boldsymbol{x}$ is the data vector, and $\Phi(x_{m_t}|\mu_p, \sigma^2)$ is the Gaussian cumulative distribution function with mean μ_p and variance σ^2. The likelihood can be written

$$L(x_{m_t}) = \prod_{t=1}^{\mathfrak{n}} \sum_{p=1}^{2} \eta_p \phi(x_{m_t}|\mu_p, \sigma^2)$$

where $\mathfrak{n}$ is the number of observations, and ϕ is the probability density function of the Normal distribution.

Although based on standard distributions, mixture models pose highly complex computational challenges. In particular, one major obstacle is the difficulty of the estimation of the parameters. The literature about mixture models offers various solutions both in the classicaland in the Bayesian framework. Considering the classical approach, the most popular method is the EM algorithm, which is a numerical optimisation procedure allowing to calculate the maximum likelihood estimator. However, as demonstrated for example by Marin et al. (2005), this algorithm may fail to converge to the major mode of the likelihood. The Bayesian approach constitutes a more flexible and computationally convenient solution to the estimation of mixture models, allowing complex structures to be decomposed into a set of simpler structures through the use of latent variables. For this reason, we decided to adopt the Bayesian approach to estimate the parameters of the marginals. Following Bayes' theorem, the posterior distribution for the m-th marginal is given by

$$\pi(\boldsymbol{\delta_m}, \boldsymbol{\eta}|\boldsymbol{x}) \propto \left(\prod_{t=1}^{\mathfrak{n}} \sum_{p=1}^{2} \eta_p \phi(x_t|\boldsymbol{\delta_m}) \right) \times \pi(\boldsymbol{\delta_m}, \boldsymbol{\eta})$$

where $\boldsymbol{\delta_m}$ is the vector of parameters of the m-th marginal distribution, $\boldsymbol{\eta}$ is the vector of classification probabilities, $\pi(\boldsymbol{\delta_m}, \boldsymbol{\eta})$ is the joint prior distribution, and $\boldsymbol{x}$ is the data vector. The posterior $\pi(\boldsymbol{\delta_m}, \boldsymbol{\eta}|\boldsymbol{x})$ is computationally intractable to work with; hence, the data augmentation MCMC algorithm is used to estimate the parameters of the mixture distributions; see Tanner and Wong (1987). In particular, the MCMC sampling is implemented using JAGS (Just Another Gibbs Sampler; Plummer, 2003). The data augmentation algorithm introduces a vector of latent variables $\boldsymbol{z} = (z_1, \ldots, z_{\mathfrak{n}})$, that represents the allocations associated to each observation x_t. Hence, the posterior density can be expressed as

$$\pi(\boldsymbol{\delta_m}, \boldsymbol{\eta}|\boldsymbol{x}) = \int_Z \pi(\boldsymbol{\delta_m}, \boldsymbol{\eta}|\boldsymbol{z}, \boldsymbol{x}) \pi(\boldsymbol{z}|\boldsymbol{x}) d\boldsymbol{z}$$

where $\pi(\boldsymbol{z}|\boldsymbol{x})$ denotes the predictive density of the latent data $\boldsymbol{z}$ given $\boldsymbol{x}$, with $\boldsymbol{z} = (z_1, \ldots, z_{\mathfrak{n}})$, and $\pi(\boldsymbol{\delta_m}, \boldsymbol{\eta}|\boldsymbol{z}, \boldsymbol{x})$ denotes the conditional density of the parameters given the augmented data. Moreover, $\pi(\boldsymbol{\delta_m}, \boldsymbol{\eta}|\boldsymbol{z}, \boldsymbol{x}) = \pi(\boldsymbol{\delta_m}|\boldsymbol{\eta}, \boldsymbol{z}, \boldsymbol{x})\pi(\boldsymbol{\eta}|\boldsymbol{z}, \boldsymbol{x})$, and $\pi(\boldsymbol{\eta}|\boldsymbol{z}, \boldsymbol{x}) = \pi(\boldsymbol{\eta}|\boldsymbol{z})$, since the distribution is independent of $\boldsymbol{x}$. Then, the data augmentation algorithm uses an iterative procedure, employing the Gibbs Sampler to simulate $\boldsymbol{z}$ fist, then $\boldsymbol{\eta}$ is generated from $\pi(\boldsymbol{\eta}|\boldsymbol{z})$ and finally $\boldsymbol{\delta_m}$ is generated from $\pi(\boldsymbol{\delta_m}|\boldsymbol{\eta}, \boldsymbol{z}, \boldsymbol{x})$. This approach is motivated by the fact that the densities $\pi(\boldsymbol{\eta}|\boldsymbol{z})$ and $\pi(\boldsymbol{\delta_m}|\boldsymbol{\eta}, \boldsymbol{z}, \boldsymbol{x})$, are easier to sample than the original posterior.

In the specification of the model, we assume independency between parameters *a priori* and we specify the following prior distributions

$$\begin{aligned} z_t &\sim Bernoulli(\eta_1) \\ (\eta_1, \eta_2) &\sim Dirichlet(\alpha_1, \alpha_2) \\ \mu_p &\sim Normal(b_p, B_p) \\ \sigma^2 &\sim \Gamma^{-1}(\nu/2, \nu S/2)\,. \end{aligned}$$

where the values of the hyperparameters $\alpha_1, \alpha_2, b_p, B_p, \nu, S$ are set in such a way that vague prior distributions are obtained.

For the implementation of the algorithm, we avoided unidentifiability due to label switching by specifying the constraint of unique ordering of the segments, with ascending means of the segment distributions.

4.2 Copula Parameter Estimation

In order to facilitate the estimation process, we reduced the dimensionality of the copula $c(F_{A_C}, F_{A_L}, F_{B_C}, F_{B_L}; \boldsymbol{\theta})$, using a D-vine pair copula decomposition. The estimation of the copula parameters $\boldsymbol{\theta}$ can be schematised into five phases.

In the first phase a suitable D-vine decomposition is selected. Following the approach suggested by Aas et a. (2009) for D-vines and later developed by Dißmann et al. (2011) for regular vines, we applied the maximal spanning tree algorithm to specify the first tree. This algorithm defines a tree on all nodes (named spanning tree), which maximises the sum of absolute pairwise dependencies. As a measure of pairwise dependence we used the Kendall's τ, calculated for each edge connecting two nodes in the first tree. Therefore, the strongest dependencies are captured in the first tree, allowing to obtain a more parsimonious model, with more stable parameter estimates.

In the second phase, given the selected tree, we choose pair copulas from a wide range of different families, in order to increase the flexibility of the model, where the pair copulas do not have to belong to the same family. The best fitting pair copula for each pair of variables is selected using the Akaike Information Criterion (AIC), that was chosen among other criteria, like the Vuong (1989) and Clarke (2007) goodness-of-fit test and Bayesian Information Criterion (BIC), for its good performance in simulation studies. Note that conditional independence between variables may reduce the number of levels of the pair copula decomposition, and hence simplify the construction. The conditional independence in fact would allow to remove edges in the D-vine graph, as depicted in Figure (1). In order to check for independence we performed the Genest and Favre (2007) bivariate asymptotic independence test on each pair of variables of the D-vine, before calculating the AIC.

In the third phase, the parameters of the copulas in the first tree are estimated. For each copula there is at least one parameter to be determined. The number of parameters depends on which copula type is selected in the previous phase. In order to estimate the copula parameters we employed the maximum likelihood estimation method, using the sequential updating parameter estimates as starting values; see Aas et al. (2009) for more details.

In the fourth phase, given the results of the first tree, we use formula (4) to compute pseudo-observations via the conditional distributions $F(x|\mathbf{v})$. These values are then used as input for the next trees.

In the last phase, the procedure illustrated from phase 1 to phase 4 is repeated for all trees, which are selected in a similar fashion as the first tree.

5 Analysis of Defaulted Firms Data

We applied our methodology to the analysis of four well known bankruptcy cases: Cirio (1993-2002), Enron (1997-2000), Parmalat (1990-2003), and Swissair (1988-2000). We used information and balance sheet data freely available on the World Wide Web. We considered the available semestral balance sheet, and we converted them into monthly observations assuming uniform distribution in the semester. For Swissair and Enron the balance sheets of the year of failure were not available.

We now briefly describe the main characteristics of the four examined firms. Enron was an American energy, commodities, and services company. Before its collapse in 2001 it was one of America's leading companies with a solid reputation. At the end of 2001 the real situation of the company was made public; its apparently solid financial conditions were substantially sustained by an institutionalised, systematic, accounting fraud.

Parmalat collapse was the biggest case of financial fraud and money laundering perpetrated by a private company in Europe. Although the catastrophic financial situation was disclosed and became public only in 2003 (when it collapsed), the company's financial difficulties were already detectable in the early nineties. It was the first Italian corporate crash with international implications.

Parmalat and Enron scandals present common features, hence many analysts refer to Parmalat as "Europe's Enron".

Swissair presents a different story from the previous firms. For most of its 71 years, it was one of the major international airlines with a strong financial stability. It rapidly declined from one of the major international airlines with the strongest balance into bankruptcy in 2001. This rapid decline was the consequence of inefficient alliance policies, management inability and economic turndown following the terroristic attacks of "September 11".

Finally, Cirio is an Italian Food company founded in 1856 by Francesco Cirio. During the nineties it was guided by the Cragnotti Group whose fraudulent financial policy leads the company to the bankruptcy in 2002.

5.1 Asset and Liability Data Analysis

We now proceed with the analysis of asset and liability data. The two component mixture Gaussian model described in Section 4.1 is applied to analyze the current/long term assets and liabilities of the four defaulted firms. The parameter estimates are reported in Table 1.

Table 1: Parameter estimates of the marginal distributions.

Cirio	η_1	η_2	μ_1	μ_2	σ^2
F_{A_C}	0.6350	0.3650	40.2800	111.6800	272.9400
F_{A_L}	0.5096	0.4904	18.9500	37.8400	26.2610
F_{B_C}	0.5012	0.4988	39.5400	106.4300	163.7900
F_{B_L}	0.6954	0.3046	16.7000	67.2400	100.8200

Enron	η_1	η_2	μ_1	μ_2	σ^2
F_{A_C}	0.9234	0.0766	251.3000	2531.6000	21412.0000
F_{A_L}	0.7133	0.2867	556.9000	880.7000	2834.1000
F_{B_C}	0.9233	0.0767	260.7000	2367.4000	17062.0000
F_{B_L}	0.9223	0.0777	623.7000	1843.9000	56107.0000

Parmalat	η_1	η_2	μ_1	μ_2	σ^2
F_{A_C}	0.6150	0.3850	121.9000	386.3000	3852.4000
F_{A_L}	0.5375	0.4625	63.7000	175.2000	535.4700
F_{B_C}	0.9282	0.0718	194.3000	1587.8000	23362.0000
F_{B_L}	0.9278	0.0722	194.0000	1588.8000	23228.0000

Swissair	η_1	η_2	μ_1	μ_2	σ^2
F_{A_C}	0.5442	0.4558	162.5000	297.1000	1214.3000
F_{A_L}	0.2339	0.7661	90.8300	349.6600	1373.1000
F_{B_C}	0.9127	0.0873	176.4000	350.5000	1901.9000
F_{B_L}	0.5402	0.4598	163.4000	439.5000	4259.0000

The classification probabilities η_p are quite close to 0.5 for Cirio data, for the asset marginals of Parmalat data and for the current asset and the long term liabilities of Swissair data, denoting a balanced number of observations in the two mixture components. On the contrary, Enron data, the liability marginals of Parmalat data and fixed assets and current liabilities of Swissair data show two very different classification probabilities η_1 and η_2. This means that different proportions of observations are allocated to the two components of the mixture and that one of the two components captures the greatest part of the data. The location parameters of the two Normal components of the mixture μ_p are well separated, especially for Enron and Parmalat data, denoting that the mixture model is able to express the mean difference between the two components. Regarding the dispersion parameter σ^2, we can see that it is particularly high for Enron and Parmalat, which are described by two Normal components with high variability, while it is lower for Cirio and Swissair, indicating distributions with a lower variability. Therefore, the data with more unbalanced mixture components are Enron and Parmalat (especially liability marginal data), with very different values of classification probabilities η_p and normal means μ_p and very high normal variance values σ^2. The resemblance of the structure of assets and liabilities in Enron and Parmalat may be explained by the similar behaviour of these two companies during the years before their default. Parmalat in fact has been referred to as the "Europe's Enron" by many authors.

Figure 2 shows the Enron histograms of each marginal (grey bars) fitted with the location-shift model of two Gaussian components (black and grey lines) described in the previous section. In the Figure, F_{A_C} is displayed in top left panel, F_{A_L} in the top right, F_{B_C} in the bottom left and F_{B_L} in the bottom right. As an example, we focus our attention to the picture related to the fixed assets marginal (F_{A_L}) of Enron data (top right panel of Figure 2). The analysis of the histogram plot shows a strong bimodality in the distribution and suggests the use of a finite mixture model as reasonable model for our data. Similar comments arise from the analysis of the fitted histograms of the remaining marginals.

Figure 3 shows the sampled values of the μ_1 parameter on the horizontal axis and of the μ_2 parameter on the vertical axis, for the Enron data. In the Figure, F_{A_C} is displayed in top left panel, F_{A_L} in the top right, F_{B_C} in the bottom left and F_{B_L} in the bottom right. It is interesting to note that our data are not affected by label switching, since the segments are rather well separated for μ, as there are no points on the diagonal on the μ_1 versus μ_2 plots.

Focusing on the MCMC results, for lack of space we analyse here the outcomes of the Enron fixed assets data (F_{A_L}), since the results of the other marginals and data series are very similar to those presented. Figures 4, 5 and 6 depict MCMC sample paths and posterior densities for the parameters η, μ and σ^2, respectively. We run the algorithm for 4000 iterations, discarding the first 1000 iterations, as burn-in period. The sample paths show that the chains are well mixing, exploring freely the sample space and clearly reaching convergence to the target distribution. Moreover, the unidentifiability problem due to label switching, that can lead to biased estimates, in our case does not occur. Finally, the posterior density plots have regular forms and do not show multimodalities.

5.2 Pair Copula Construction for Asset and Liability Data

According to the IFM approach, once estimated the marginal parameters Δ, the second step requires the estimation of the copula parameters $\boldsymbol{\theta}$ (see section 4). However, the data feeding the copula are required to lie in the interval $[0, 1]$. These data are named pseudo-observations or *u-data* and

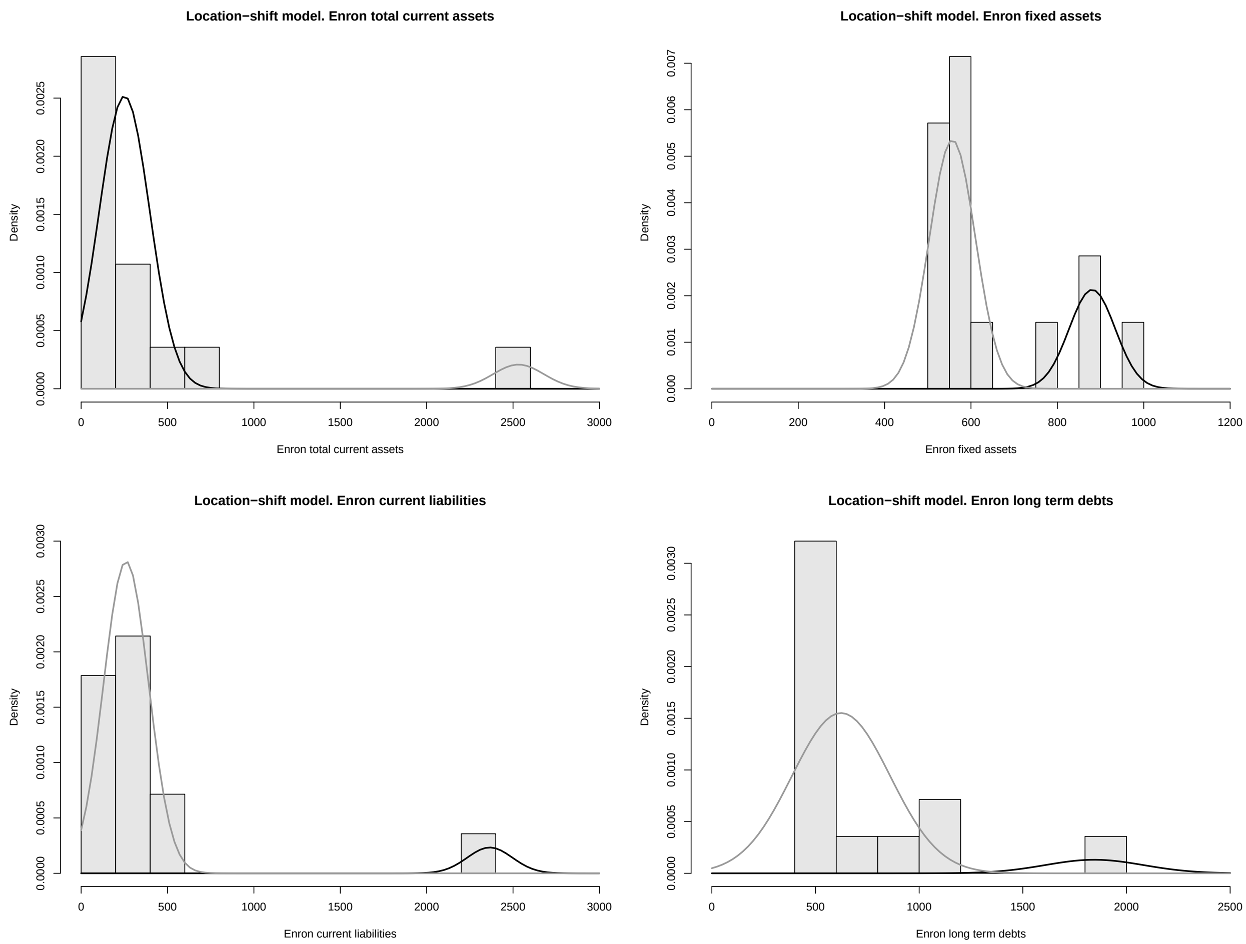

Figure 2: The Enron data fitted with a mixture of two Gaussian components: F_{A_C} (top left), F_{A_L} (top right), F_{B_C} (bottom left) and F_{B_L} (bottom right).

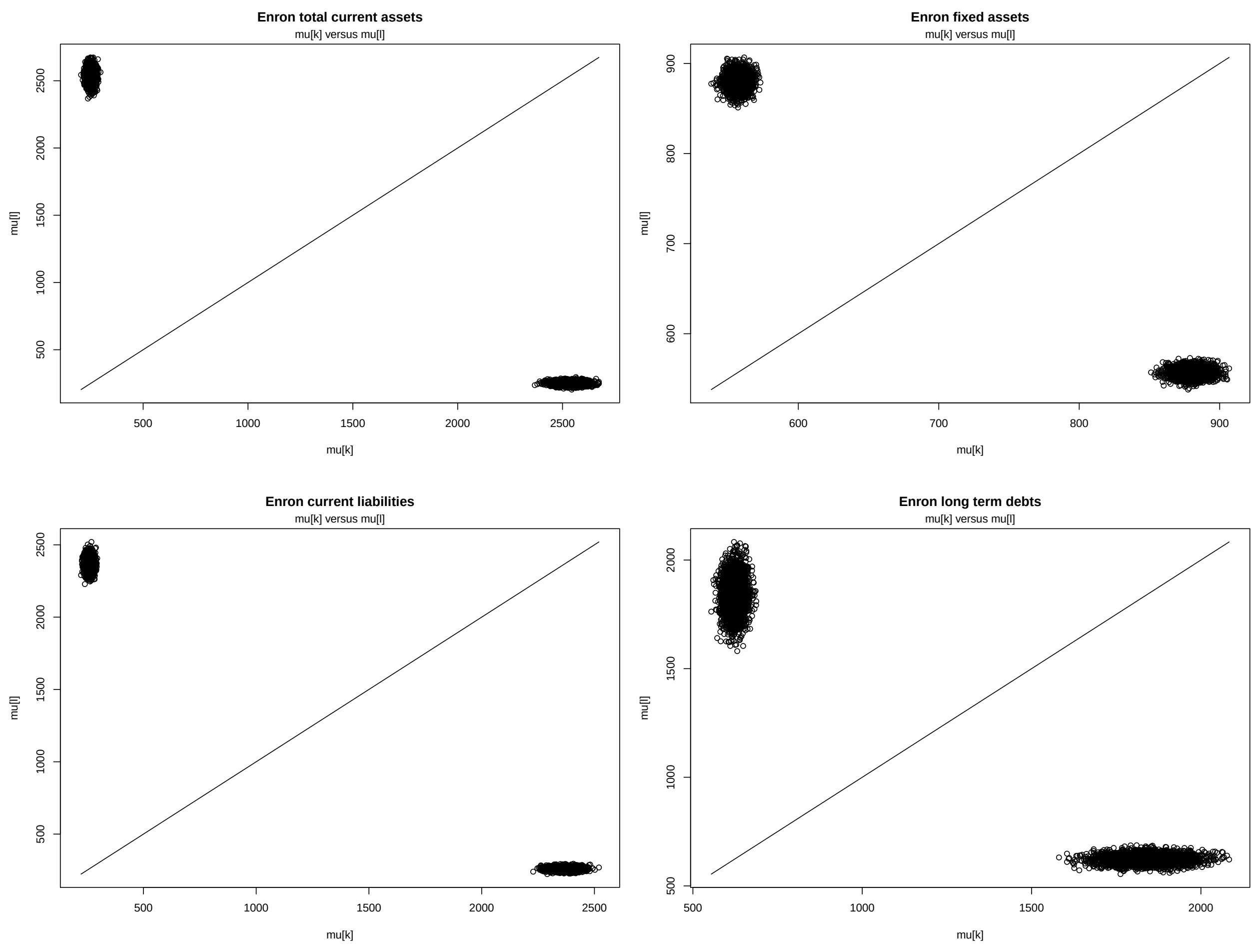

Figure 3: The Enron data: μ_1 versus μ_2. F_{A_C} is in the top left, F_{A_L} is in the top right, F_{B_C} is in the bottom left and F_{B_L} is displayed in the bottom right panel.

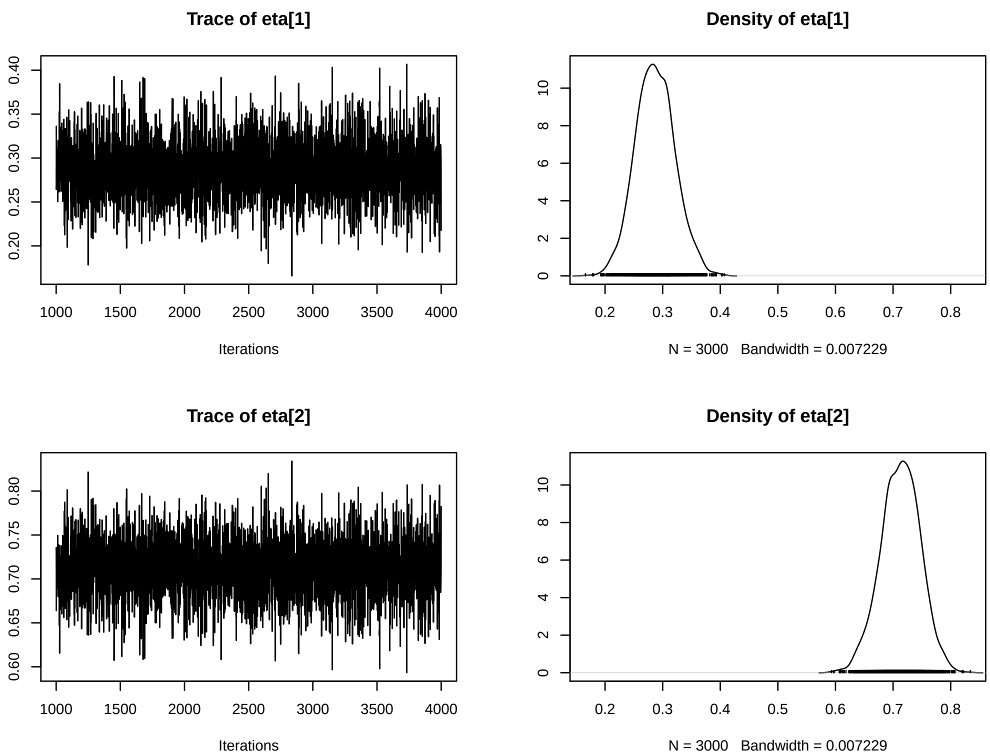

Figure 4: Enron fixed assets data: MCMC traces and posterior densities for η.

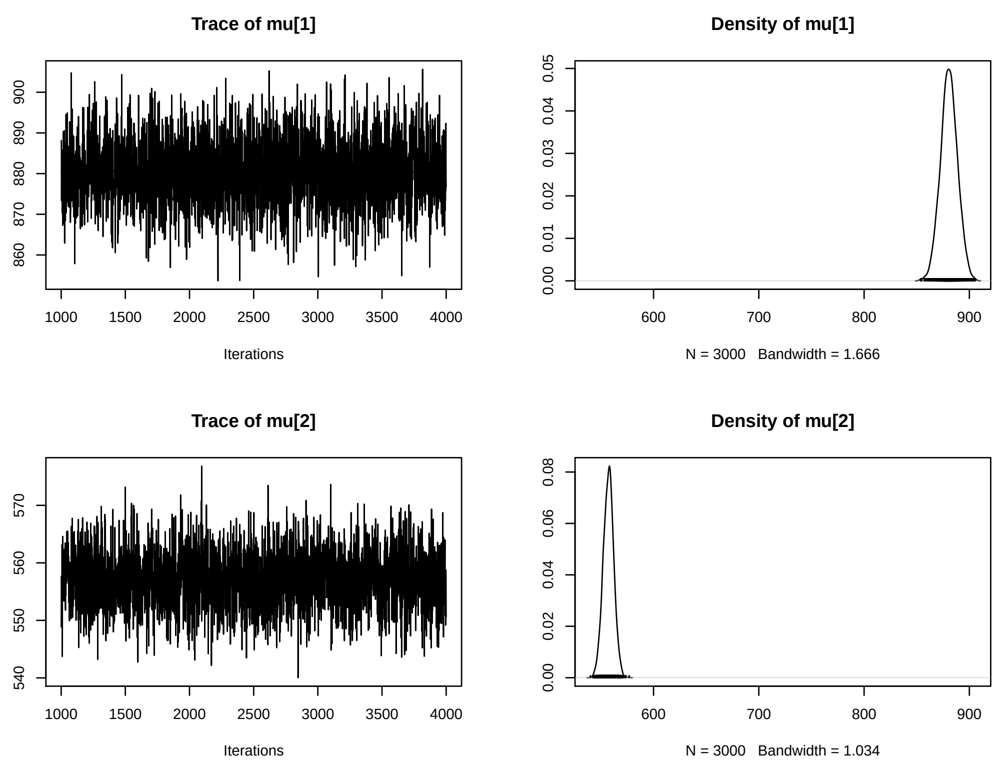

Figure 5: Enron fixed assets data: MCMC traces and posterior densities for μ.

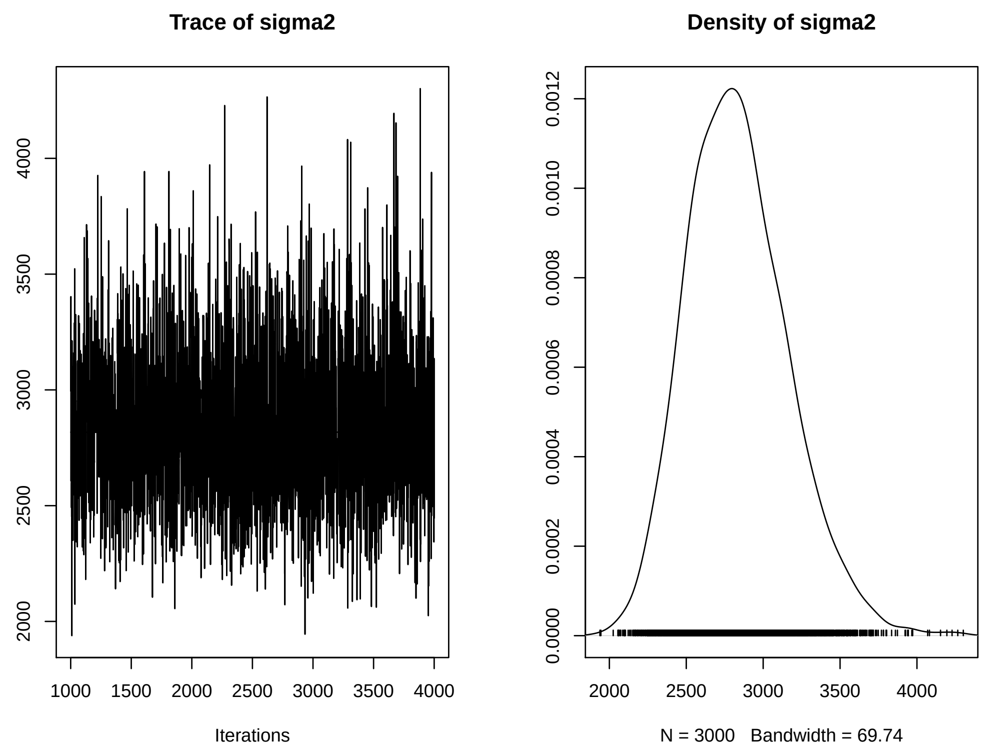

Figure 6: Enron fixed assets data: MCMC trace and posterior density for σ^2.

can be calculated via the inverse transformation method. Thus, for each marginal we simulated the distributions of the mixtures, described in section 4.1, using the Monte Carlo method, and then we inverted the data using the empirical cumulative distribution functions of each marginal. In this way, we obtain the so-called *u-data*, with values lying in the interval $[0, 1]$, that are used as input to the copula.

In order to select an appropriate pair-copula decomposition for the D-vine, we followed the phases described in section 4.2. We ordered the marginals so that the copulas to be fitted in tree 1, in the pair copula decomposition, are those corresponding to the strongest pairwise dependence among the marginals (see Figure 1). We measured the marginal dependences with the Kendall's τ and we chose the D-vine that maximizes the pairwise Kendall's τs in the first tree (Aas et al., 2009). Therefore, for all the four considered datasets, the order of the nodes in the first tree is: $A_{C_T} - B_{C_T} - B_{L_T} - A_{L_T}$.

For each of the four defaulted stock data we specified a D-vine copula model. Then, for each variable pair we performed the bivariate asymptotic independence test (Genest and Favre, 2007), where the rejection of the null hypothesis means a strong dependence between the selected marginals. Subsequently, we selected the appropriate pair copula family for given bivariate copula data using the AIC. Finally, we estimated the parameters for each copula and we evaluated the final D-vine structure.

The parameters of the D-vine are estimated using the approach described in Section 4.2 and the results are listed in Tables 2, 3, 4 and 5. Each Table displays the list of pair copulas in the trees of the D-vine, the selected copula family and the copula parameters (they can be one or two according to the type of copula). From the copula families selected, we see evidence of different types of asymmetric dependence. This demonstrates that the choice of PCCs was appropriate, since it guarantees enough flexibility to model the dependence structure of the marginals. Note that only the Cirio D-vine (Table 2) has none conditional independent variable pairs. For these data the Genest and Favre (2007) independence test rejected independency for all the copulas involved. An independent copula has been selected instead for $c_{A_{C_T},B_{L_T}|B_{C_T}}$ in the second tree for Parmalat and Swissair (Tables 4 and 5), while $c_{A_{L_T},B_{C_T}|B_{L_T}}$ has been identified as an independent copula for Enron (Table 3). In these cases the D-vine structure is simplified and we do not need to estimate the parameters of the copula $c_{A_{C_T},A_{L_T}|B_{C_T},B_{L_T}}$ in the third tree. The presence of conditional independence in this last case suggests a weak relationship between the current and fixed assets, given the values of the liabilities. From the unconditional pair copulas, we note an existing dependence between current and fixed assets or liabilities, and also a dependence between the two different types of debts. The conditional copulas instead, especially those characterized by strong dependence, may suggest imbalance, when current assets are financed by long-term debts, or a serious liquidity problem, when fixed assets are financed by short-term debts. These situations need particular attention, because they may prelude to the default of the firm.

Figure 7, obtained with the R package CDVine by Brechmann and Schepsmeier (2013), shows the D-vine tree plot for the Enron data and contains the trees of the D-vine. The blue squares represent the nodes, while the grey lines represent the arcs. The names of the nodes can be read in the blue squares, and the pair copula families and Kendall's τ values corresponding to pair copula parameters can be read in the edge labels. The thicker the grey line the higher the dependence between the variables represented by the nodes. Only the Cirio D-vine contains all the three trees, while the D-vines of the remaining data contain two trees only, because of simplification derived by conditional independence.

Table 2: Cirio data: selected copulas and D-vine PCC parameters. SBB1, BB7 and BB8 are, respectively, the Survival Clayton-Gumbel, the Joe-Clayton and the Joe-Frank copulas, that are Archimedean copula families with two parameters.

Cirio: Pair Copulas Parameters of the D-Vine			
Copulas	*family*	*parameter 1*	*parameter 2*
$c_{A_{C_T},B_{C_T}}$	SBB1	0.0010	3.3814
$c_{B_{C_T},B_{L_T}}$	BB8	1.2579	0.9902
$c_{B_{L_T},A_{L_T}}$	BB7	1.1195	4.7016
$c_{A_{C_T},B_{L_T}\|B_{C_T}}$	Frank	7.2222	0
$c_{A_{L_T},B_{C_T}\|B_{L_T}}$	Normal	-0.0337	0
$c_{A_{C_T},A_{L_T}\|B_{C_T},B_{L_T}}$	Frank	-8.9557	0

Table 3: Enron data: selected copulas and D-vine PCC parameters. SBB8 is the Survival Joe-Frank copula, Archimedean copula family with two parameters.

Enron: Pair Copulas Parameters of the D-Vine			
Copulas	*family*	*parameter 1*	*parameter 2*
$c_{A_{C_T},B_{C_T}}$	Student's t	0.9868	7.6539
$c_{B_{C_T},B_{L_T}}$	SBB8	6.0000	0.3924
$c_{B_{L_T},A_{L_T}}$	BB8	5.9831	0.9979
$c_{A_{C_T},B_{L_T}\|B_{C_T}}$	Rotated Clayton	-1.4730	0
$c_{A_{L_T},B_{C_T}\|B_{L_T}}$	Independent	0	0

5.3 Probability of Default Estimation

After the estimation of the parameters of the model, we can now calculate the PD for the four considered defaulted stock data. As illustrated in Section 3, we obtain the equity distribution by simulation and finally we compute the default probability value. In particular, for each stock data, we obtain the equity scenario by simulating from the D-vine, that, after the estimation

Table 4: Parmalat data: selected copulas and D-vine PCC parameters. BB1 is the Clayton-Gumbel copula, Archimedean copula family with two parameters.

Parmalat: Pair Copulas Parameters of the D-Vine			
Copulas	*family*	*parameter 1*	*parameter 2*
$c_{A_{C_T},B_{C_T}}$	BB1	0.4325	4.2015
$c_{B_{C_T},B_{L_T}}$	Gaussian	0.9998	0
$c_{B_{L_T},A_{L_T}}$	Clayton	1.2256	0
$c_{A_{C_T},B_{L_T}\|B_{C_T}}$	Independent	0	0
$c_{A_{L_T},B_{C_T}\|B_{L_T}}$	Frank	-7.3657	0

Table 5: Swissair data: selected copulas and D-vine PCC parameters.

Swissair: Pair Copulas Parameters of the D-Vine			
Copulas	*family*	*parameter 1*	*parameter 2*
$c_{A_{C_T},B_{C_T}}$	BB7	2.4309	5.3880
$c_{B_{C_T},B_{L_T}}$	SBB8	1.0081	1.0000
$c_{B_{L_T},A_{L_T}}$	BB7	1.0010	2.9494
$c_{A_{C_T},B_{L_T}\|B_{C_T}}$	Independent	0	0
$c_{A_{L_T},B_{C_T}\|B_{L_T}}$	Rotated Joe	-2.3405	0

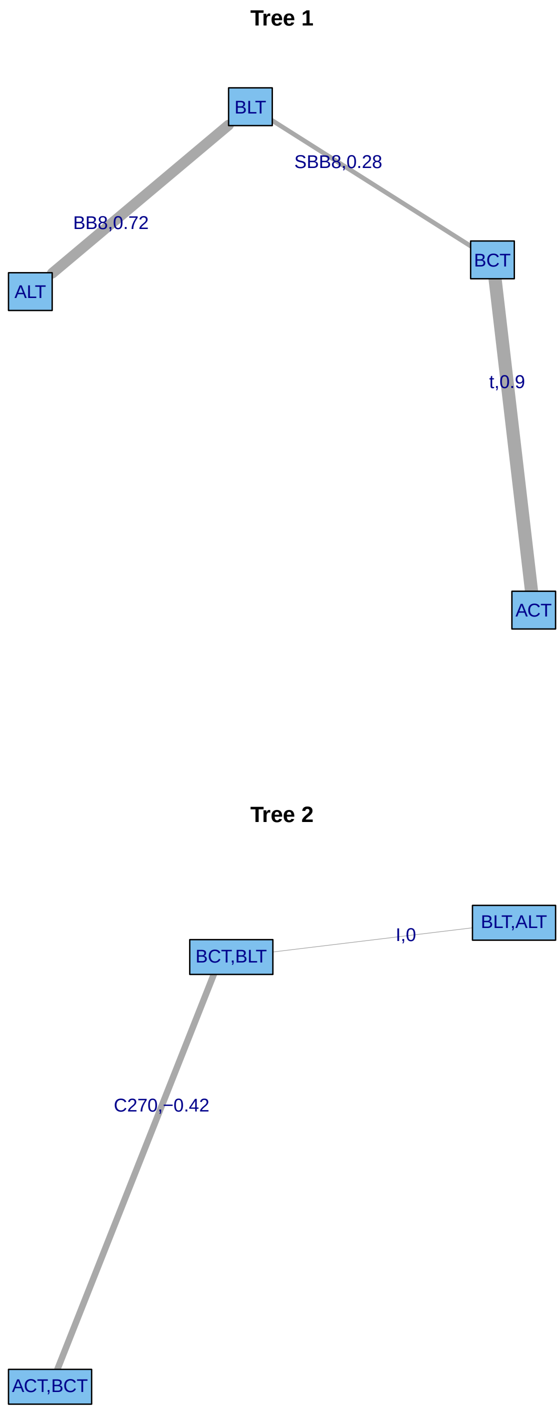

Figure 7: Enron data: D-vine PCC tree.

of the parameters, is completely specified. Then, from the distribution of the equity, the PD is determined as the probability that the simulated equity is zero or negative. Figure 8 depicts the equity distributions of Cirio, Enron, Parmalat and Swissair, respectively on the top left, top right, bottom left and bottom right panel. The value of the PD is written on the relevant distribution and corresponds to the area under the curve where the equity is zero or negative. The PD values are very high for all defaulted stocks, but for Enron and Swissair these probabilities are slightly lower than the remaining two firms. However, we need to point out that the available time series did not include the last year of activity for Enron and Swissair. This might have affected the final results, since the inclusion of the last year's data would certainly have increased the PD values.

6 Summary and Conclusions

The aim of this paper was to propose a new methodology for default risk measurement. Our final goal was to calculate the default probability of large firms using their balance sheet data. We measured the firm value via a contingent claim, whose pricing function can be expressed using copulas. The marginals are given by the current and fixed assets and the short-term and long-term debts. Hence, the pricing function is expressed by a 4-dimensional copula. To test the performance of our methodology we applied it to four famous defaulted stocks. In order to estimate the marginals we employed a Bayesian mixture model, able to model the presence of two clusters in the asset as well as in the liability data. This structure of the marginals reflects the choices of the management, trying to balance high and low accounting items during the period before the default. Considering the copula, we chose to employ PCCs, because they allow for a great flexibility in modelling the dependence structure of the marginals. As demonstrated by the results, the pair copulas selected in the D-vines belong to different families and describe various types of dependence. The analysis of these dependences already reveals substandard loans and situations of serious imbalance due to liquidity problems, especially when the firm tries to balance fixed assets with current liabilities. Finally, we calculated the default probability of the four considered firms, simulating from the D-vines and obtaining the equity distribution. The final results show a high probability of default, suggesting the forthcoming bankrupt of the firms.

The proposed methodology has proved to be successful in the evaluation of default probability and would certainly benefit analysts and managers, advising them to take actions against a potential bankruptcy.

Possible extensions of our work would be the estimation of the whole model in a full Bayesian framework, the use of balance indicators instead of accounting items, and the use of a similar methodology to analyze the contagion in sectors of activity.

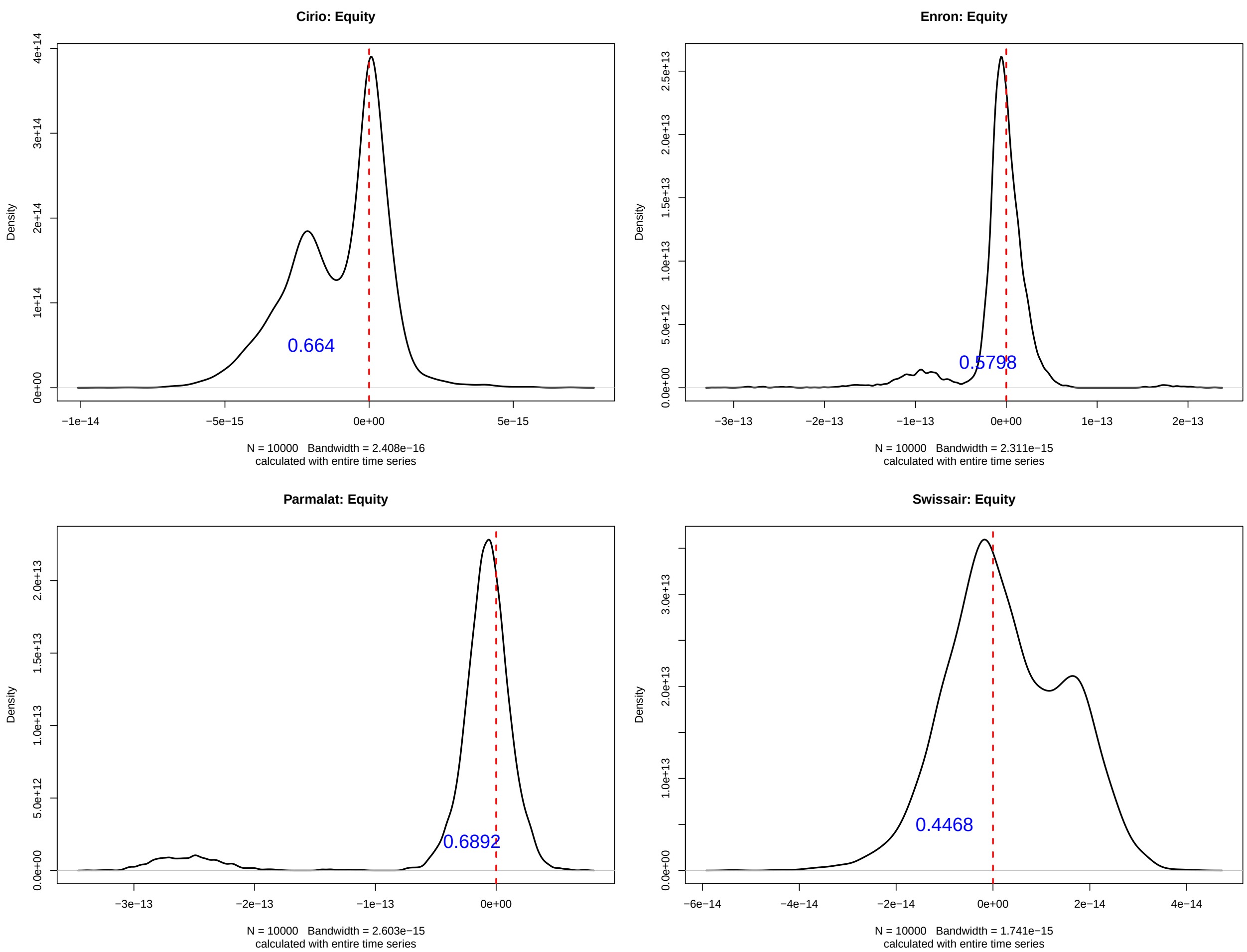

Figure 8: Equity distributions of Cirio (top left), Enron (top right), Parmalat (bottom left) and Swissair (bottom right). The value of the PD is written on the corresponding distribution.

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