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LARGE DEVIATIONS FOR THE SOLUTION OF A KAC-TYPE KINETIC EQUATION

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Abstract. The aim of this paper is to study large deviations for the self-similar solution of a Kac-type kinetic equation. Under the assumption that the initial condition belongs to the domain of normal attraction of a stable law of index $\alpha < 2$ and under suitable assumptions on the collisional kernel, precise asymptotic behavior of the large deviations probability is given.

1. Introduction.

This paper deals with the probability of large deviations for the solutions of a class of one dimensional Boltzmann-like equations. Specifically, given an initial probability distribution $\bar{\rho}_0$ on $\mathcal{B}(\mathbb{R})$, the Borel σ-field of $\mathbb{R}$, we consider a time-dependent probability measure ρ_t solution of the homogeneous kinetic equation

$$\begin{cases} \partial_t \rho_t + \rho_t = Q^+(\rho_t, \rho_t) \\ \rho_0 = \bar{\rho}_0. \end{cases} \tag{1}$$

Following [3, 11], we assume that Q^+ is the *smoothing transformation* defined by

$$Q^+(\rho, \rho) = \text{Law}(LX + RX') \tag{2}$$

where (L, R) is a given random vector of $\mathbb{R}^2$, ρ is the law of X and X', and $(L, R), X, X'$ are stochastically independent.

The first model of type (1)-(2) has been introduced by Kac [23], with collisional parameters $L = \sin \tilde{\theta}$ and $R = \cos \tilde{\theta}$ for a random angle $\tilde{\theta}$ uniformly distributed on $[0, 2\pi)$. In the original Kac equation ρ_t represents the probability distribution of the velocity of a particle in a homogeneous gas. In addition to the Kac equation, also some one dimensional dissipative Maxwell models for colliding molecules, see e.g. [8, 27, 29], can be seen as special cases of (1)-(2). Moreover, equations (1)-(2) have been used to describe socio-economical dynamics see, e.g., [5, 7, 15, 25, 28, 31] and the references therein. In this last case particles are replaced by agents in a market and velocities by some quantities of interest (money, wealth, information,...).

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Finally, it is worth recalling that, using results in [10, 11], it can be shown that the isotropic solution of the multidimensional inelastic Boltzmann equation [9] can be expressed in terms of the solution of equation (1) for a suitable choice of (L, R).

The *generalized Kac-equation* (1)-(2) has been extensively studied in many aspects. In particular, the asymptotic behavior of the solutions of (1)-(2) has been treated in details in [2, 3, 11].

As for the speed of convergence to equilibrium, explicit rates with respect to suitable probability metrics have been derived in various papers. For the Kac equation see [13, 14, 18], for the inelastic Kac equation see [4], for the solutions of the general model (1)-(2) see [2, 3, 6].

Many of the above mentioned results are based on a probabilistic representation of the solution ρ_t. In point of fact, as we will briefly explain in Section 2.2, it can be proved that the unique solution ρ_t of (1)-(2) is the law of the random variable

$$V_t = \sum_{j=1}^{\nu_t} \beta_{j,\nu_t} X_j \tag{3}$$

where ν_t is a Yule process, $[\beta_{j,n}]_{jn}$ are suitable random weights and X_j are independent identically distributed (i.i.d., for short) random variables with law $\bar{\rho}_0$. In other words, $\rho_t(A) = P\{V_t \in A\}$, for every $t > 0$ and every Borel set $A \subset \mathbb{R}$.

The aim of this paper is to study large deviations for the (eventually rescaled) solution ρ_t when the initial condition $\bar{\rho}_0$ belongs to the domain of normal attraction of an α-stable law. More precisely, we will study the large deviation probability for $e^{-t\mu(\alpha)}V_t$ when, for a suitable $\mu(\alpha)$, $e^{-t\mu(\alpha)}V_t$ converges in distribution to a scale mixture of α-stable distributions. In the following we shall assume that $\alpha < 2$, the study of the case $\alpha = 2$ is postponed to a future work since it requires completely different techniques.

In view of the probabilistic representation (3) it is not surprising that the study of the large deviation probabilities for ρ_t is strictly related to large deviations for sums of i.i.d. random variables.

Let us briefly recall these classical results. If $\alpha \in (0, 1) \cup (1, 2)$ and if $(X_n)_{n \geq 1}$ is a sequence of i.i.d. random variables in the domain of normal attraction of an α-stable law, centered if $\alpha > 1$, then, $n^{-1/\alpha} \sum_{i=1}^n X_i$ converges in distribution to an α-stable random variable. Moreover, if $x_n \to +\infty$, then

$$P\Big\{\Big|n^{-\frac{1}{\alpha}} \sum_{i=1}^n X_i\Big| > x_n\Big\} \sim P\{n^{-\frac{1}{\alpha}} \max_{j=1,\dots,n} |X_j| > x_n\} \sim \frac{c_0}{x_n^\alpha}, \tag{4}$$

where c_0 is a positive constant determined by the law of X_1 and, as usual, $a_n \sim b_n$ means $a_n/b_n \to 1$ for $n \to +\infty$. See [19, 20, 21]. For more information on large deviations for sums of i.i.d. random variables see, for example, [12, 30] and the references therein.

Our main result, which is stated in Theorem 3.1, is reminiscent of (4). It can be summarized by saying that *if the initial distribution $\bar{\rho}_0$ belongs to the domain of normal attraction of an α-stable law with $\alpha < 2$ and the collision coefficients (L, R) satisfy some additional assumptions, then*

$$\rho_t\big([-e^{t\mu(\alpha)}x_t, e^{t\mu(\alpha)}x_t]^c\big) = P\{|e^{-t\mu(\alpha)}V_t| > x_t\} \sim \frac{c_0}{x_t^\alpha} \tag{5}$$

and

$$P\{|e^{-t\mu(\alpha)}V_t| > x_t\} \sim P\{e^{-t\mu(\alpha)} \max_{j=1,\dots,\nu_t} |\beta_{j\nu_t} X_j| > x_t\} \tag{6}$$

as x_t goes to $+\infty$. As in the i.i.d. case, (6) can be interpreted by saying that the main part of probability of large deviations for V_t is generated by one large summand comparable with the whole sum process V_t.

It may be useful to give a sort of kinetic interpretation of this last statement. As we shall see, the law $\tilde{\rho}_t$ of $\max_{j=1,\dots,\nu_t} |\beta_{j\nu_t} X_j|$ is the unique solution of the kinetic equation

$$\begin{cases} \partial_t \tilde{\rho}_t + \tilde{\rho}_t = \tilde{Q}^+(\tilde{\rho}_t, \tilde{\rho}_t) \\ \tilde{\rho}_0(\cdot) = P\{|X_1| \in \cdot\} \end{cases} \tag{7}$$

with X_1 distributed according to $\bar{\rho}_0$ and

$$\tilde{Q}^+(\rho, \rho) = \mathrm{Law}(\max\{|LX|, |RX'|\}), \tag{8}$$

where ρ is the law of X and X', and (L, R) is the same random vector appearing in (2). As before, (L, R), X and X' are assumed stochastically independent.

Equations (5)-(6) state that the tail of the solution ρ_t (when $t \to +\infty$) have the same power law decay of the tail of the solution $\tilde{\rho}_t$ of the kinetic equation (7)-(8). In this last equation, one considers post-collisional velocities given by a randomly weighted maximum of the pre-collisional ones - see (8) - in place of the usual post-collisional velocities that are random linear combinations of the pre-collisional ones, see (2).

The paper is organized as follows. Section 2.1 is devoted to a brief review of some known results on the self-similar asymptotics for the solutions of (1). Section 2.2 contains the detailed description of the probabilistic representation (3). In Section 2.3 we provide some results on the process $H_t = \max_{j=1,\dots,\nu_t} |\beta_{j\nu_t} X_j|$. Section 3 contains the large deviation results for ρ_t. Section 4 deals with the study of large deviation probabilities for weighted sums of i.i.d. random variables. The proofs of the results stated in Section 2 and 3 are collected in Section 5.

2. **Self-similar asymptotics for the solutions.**

In the following, all the random elements are defined on a given probability space $(\Omega, \mathcal{F}, P)$ and $\mathbb{E}$ denotes the expectation with respect to P.

Throughout the paper we assume that

L and R are non-negative random variables such that $P\{L > 0\} + P\{R > 0\} > 1$.

As for the initial probability distribution $\bar{\rho}_0$ is concerned, we will assume that it belongs to the *domain of normal attraction* of an α-stable law. It is well-known that, provided $\alpha \neq 2$, a probability measure $\bar{\rho}_0$ belongs to the domain of normal attraction of an α-stable law if and only if its distribution function $F_0(x) := \bar{\rho}_0\{(-\infty, x]\}$ satisfies

$$\lim_{x \to +\infty} x^\alpha (1 - F_0(x)) = c_0^+ < +\infty, \qquad \lim_{x \to -\infty} |x|^\alpha F_0(x) = c_0^- < +\infty. \tag{9}$$

Typically, one also requires that $c_0^+ + c_0^- > 0$. See for example Chapter 2 of [22].

Finally, let us introduce the convex function $\mathcal{S} : [0, +\infty) \to [-1, +\infty]$ by

$$\mathcal{S}(s) := \mathbb{E}[L^s + R^s] - 1,$$

with the convention that $0^0 = 0$ and let

$$\mu(s) := \frac{\mathcal{S}(s)}{s} \qquad (s > 0)$$

be the so called spectral function of Q^+, see [2] and [11].

2.1. **Convergence to self-similar solutions.** In the study of the asymptotic behavior of the solutions of (1), a fundamental role is played by the fixed point equation for distributions

$$Z \stackrel{\mathcal{L}}{=} \Theta^{\mathcal{S}(\alpha)}(L^\alpha Z_1 + R^\alpha Z_2) \tag{10}$$

where Z, Z_1, Z_2 are i.i.d. positive random variables, Θ is a random variable with uniform distribution on $(0,1)$, (Z, Z_1, Z_2), Θ and (L, R) are stochastically independent.

As already recalled in the introduction, the unique solution ρ_t to (1)-(2) is the law of V_t defined in (3). Further details on this probabilistic representation will be given in Section 2.2. The next results, concerning the convergence in distribution of a suitable rescaling of V_t to the so-called self-similar solutions of (1), are proved in [2].

It is worth recalling that a sequence of random variables $(Y_n)_{n\geq 1}$, with probability distributions $(l_n)_{n\geq 1}$, is said to converge in distribution to a random variable Y, with law l, if $(l_n)_{n\geq 1}$ converges weakly to l, that is

$$\lim_{n\to+\infty} \int g(y) l_n(dy) = \int g(y) l(dy)$$

for every bounded and continuos real valued function g.

Theorem 2.1 (CLT when $\alpha \neq 1$, [2])**.** *Let $\alpha \in (0,1)\cup(1,2)$ and let condition (9) be satisfied for some (c_0^+, c_0^-) such that $c_0^+ + c_0^- > 0$, with $\int v\bar\rho_0(dv) = 0$ if $\alpha > 1$. If $\mu(\delta) < \mu(\alpha) < +\infty$ for some $\delta > \alpha$, then $e^{-\mu(\alpha)t}V_t$ converges in distribution, as $t \to +\infty$, to a random variable V_∞ with the following characteristic function:*

$$\mathbb{E}[e^{i\xi V_\infty}] = \mathbb{E}[\exp\{-|\xi|^\alpha \lambda Z_\infty(\alpha)(1 - i\eta \tan(\pi\alpha/2)\operatorname{sign}\xi)\}] \qquad (\xi \in \mathbb{R}) \tag{11}$$

where

$$\lambda = \frac{(c_0^+ + c_0^-)\pi}{2\Gamma(\alpha)\sin(\pi\alpha/2)}, \qquad \eta = \frac{c_0^+ - c_0^-}{c_0^+ + c_0^-} \tag{12}$$

and the law of $Z_\infty(\alpha)$ is the unique positive solution to (10) with $\mathbb{E}[Z_\infty(\alpha)] = 1$.

Further information on the mixing random variable $Z_\infty(\alpha)$ are given in Proposition 2. See also [2].

The results concerning the case $\alpha = 1$ are here stated under slightly more general assumptions than in [2]. For completeness a sketch of the proof is given in Section 5.

Theorem 2.2 (CLT when $\alpha = 1$)**.** *Let (9) holds with $\alpha = 1$ and $c_0^+ = c_0^- > 0$. Suppose, in addition, that*

$$\lim_{R\to+\infty} \int_{(-R,R)} x dF_0(x) = \gamma_0 \tag{13}$$

with $-\infty < \gamma_0 < +\infty$. If $\mu(\delta) < \mu(1) < +\infty$ for some $\delta > 1$, then $e^{-\mu(1)t}V_t$ converges in distribution, as $t \to +\infty$, to a random variable V_∞ with the following characteristic function:

$$\mathbb{E}[\exp(i\xi V_\infty)] = \mathbb{E}[e^{Z_\infty(1)(i\gamma_0\xi - c_0^+\pi|\xi|)}] \tag{14}$$

and the law of $Z_\infty(1)$ is the unique positive solution to (10) for $\alpha = 1$, with $\mathbb{E}[Z_\infty(1)] = 1$.

Remark 1. In order to study the large deviations for ρ_t, in what follows we will need to assume that $c_0^+ + c_0^- > 0$, even if both Theorem 2.1 and Theorem 2.2 hold also for $c_0^+ + c_0^- = 0$. In this last case, Theorem 2.1 is valid with $\lambda = \eta = 0$ and hence $V_\infty = 0$ with probability one, while Theorem 2.2 is valid with $V_\infty = \gamma_0 Z_\infty(1)$.

Remark 2. Let us consider a random vector (L, R) such that $\mu(\alpha) = 0$, that is $\mathbb{E}[L^\alpha + R^\alpha] = 1$. As a consequence of the previous results, if $\mathbb{E}[L^\delta + R^\delta] < 1$ for some $\delta > \alpha$, then V_t converges in distribution to V_∞. In this case $Z_\infty(\alpha)$ satisfies the fixed point equation

$$Z \stackrel{\mathcal{L}}{=} L^\alpha Z_1 + R^\alpha Z_2$$

and it is easy to see that the law ρ_∞ of V_∞ is a steady state for equation (1), i.e. $\rho_\infty = Q^+(\rho_\infty, \rho_\infty)$. This case has been extensively studied in [3].

2.2. **Probabilistic representation of the solution.** The proofs of Theorems 2.1 and 2.2 are based on the fact that V_t is a randomly weighted sum of i.i.d. random variables. In [3] it has been shown that the unique solution of (1)-(2) with initial datum $\bar{\rho}_0$ is the law of

$$V_t = \sum_{j=1}^{\nu_t} \beta_{j,\nu_t} X_j,$$

provided that

- $(X_j)_{j\geq 1}$ is a sequence of i.i.d. random variables with distribution $\bar{\rho}_0$;
- $(\nu_t)_{t\geq 0}$ is a Yule process, see e.g. [1], hence in particular
$$P\{\nu_t = n\} = e^{-t}(1 - e^{-t})^{n-1}$$
for every $n \geq 1$ and $t \geq 0$;
- $(\beta_{j,n} : j = 1, \ldots, n)_{n\geq 1}$ is an array of non-negative random weights;
- $(X_j)_{j\geq 1}$, $(\nu_t)_{t\geq 0}$ and $(\beta_{j,n} : j = 1, \ldots, n)_{n\geq 1}$ are stochastically independent.

As to the definition of the weights β_{jn}'s is concerned: $\beta_{1,1} := 1$, $(\beta_{1,2}, \beta_{2,2}) := (L_1, R_1)$ and, for any $n \geq 2$,

$$\begin{aligned}(\beta_{1,n+1}, \ldots, \beta_{n+1,n+1}) \\ := (\beta_{1,n}, \ldots, \beta_{I_n-1,n}, L_n\beta_{I_n,n}, R_n\beta_{I_n,n}, \beta_{I_n+1,n}, \ldots, \beta_{n,n}),\end{aligned} \tag{15}$$

where $(L_n, R_n)_{n\geq 1}$ is a sequence of i.i.d. random vectors distributed as (L, R), $(I_n)_{n\geq 1}$ is a sequence of independent random variables uniformly distributed on $\{1, \ldots, n\}$ for every $n \geq 1$, $(L_n, R_n)_{n\geq 1}$ and $(I_n)_{n\geq 1}$ are independent.

The idea to represent solutions to the Kac's equation in a probabilistic way dates back to McKean [26]. A complete formalization of this probabilistic representation has been obtained in [17] and later generalized to Kac-type equations in [2, 3].

2.3. **Kinetic equations with max-type collisions.** Since we shall compare the large deviations of $e^{-\mu(\alpha)t}V_t$ with the large deviations of $e^{-\mu(\alpha)t}H_t$, where

$$H_t = \max_{1\leq j\leq \nu_t} |\beta_{j,\nu_t} X_j|,$$

we start by providing some results on this last process.

Theorem 2.3. *There is a unique solution $\tilde{\rho}_t$ to equation (7)-(8). Moreover, $\tilde{\rho}_t$ is the law of H_t, i.e. $\tilde{\rho}_t(A) = P\{H_t \in A\}$ for every Borel set A.*

Following the same line of reasoning of [2, 3] we prove the next result on the asymptotic behavior of $e^{-\mu(\alpha)t}H_t$.

Theorem 2.4. *Let $\alpha \in (0,1) \cup (1,2)$ and the hypotheses of* Theorem 2.1 *be in force, or let $\alpha = 1$ and the hypotheses of* Theorem 2.2 *hold. Assume also that $c_0 = c_0^+ + c_0^- > 0$. Then $e^{-\mu(\alpha)t} H_t$ converges in distribution, as $t \to +\infty$, to a random variable H_∞ with the following probability distribution function:*

$$P\{H_\infty \leq x\} = \begin{cases} \mathbb{E}\left[e^{-\frac{c_0}{|x|^\alpha} Z_\infty(\alpha)}\right] & \text{if } x > 0 \\ P\{Z_\infty(\alpha) = 0\} & \text{if } x = 0 \\ 0 & \text{if } x < 0 \end{cases} \tag{16}$$

where the law of $Z_\infty(\alpha)$ is the unique positive solution to (10) with $\mathbb{E}[Z_\infty(\alpha)] = 1$.

It is useful to note that Theorem 2.4 states that the law of H_∞ is a scale mixture of Fréchet distributions.

3. Main results: Large deviations for ρ_t.

As a consequence of Theorems 2.1-2.2, one has that, if $x_t \to +\infty$ as $t \to +\infty$, then

$$\lim_{t \to +\infty} P\{|e^{-\mu(\alpha)t} V_t| > x_t\} = 0.$$

The main result of this paper concerns the study of the speed of convergence of such a probability to zero under suitable conditions on the function $\mu(s)$.

Theorem 3.1 (Large deviations)**.** *Let $\alpha \in (0,1) \cup (1,2)$ and the hypotheses of* Theorem 2.1 *be in force, or let $\alpha = 1$ and the hypotheses of* Theorem 2.2 *hold. Assume also that $\mathcal{S}(2\alpha) < +\infty$ and $c_0 := c_0^+ + c_0^- > 0$. If $\mu(2\alpha) < \mu(\alpha)$ and $2\mathcal{S}(\alpha) > -1$, then, for every x_t such that $x_t \to +\infty$ as $t \to +\infty$, one has*

$$\lim_{t \to +\infty} \frac{x_t^\alpha}{c_0} P\{|e^{-\mu(\alpha)t} V_t| > x_t\} = \lim_{t \to +\infty} \frac{P\{|e^{-\mu(\alpha)t} V_t| > x_t\}}{P\{|V_\infty| > x_t\}} = 1 \tag{17}$$

and

$$\lim_{t \to +\infty} \frac{P\{|e^{-\mu(\alpha)t} V_t| > x_t\}}{P\{|e^{-\mu(\alpha)t} H_t| > x_t\}} = 1. \tag{18}$$

Remark 3. Let us consider Theorem 3.1 in the particular case in which $\mathbb{E}[L^\alpha + R^\alpha] = 1$ and hence $0 = 2\mathcal{S}(\alpha) > -1$. Then, if $\mathbb{E}[L^{2\alpha} + R^{2\alpha}] < 1$ and $x_t \to +\infty$ as $t \to +\infty$, one has

$$\lim_{t \to +\infty} \frac{x_t^\alpha}{c_0} P\{|V_t| > x_t\} = \lim_{t \to +\infty} \frac{P\{|V_t| > x_t\}}{P\{|V_\infty| > x_t\}} = \lim_{t \to +\infty} \frac{P\{|V_t| > x_t\}}{P\{|H_t| > x_t\}} = 1 \tag{19}$$

where the law of V_∞ is a steady state for equation (1).

Remark 4. As pointed out in the Introduction, the results stated in the previous theorem are related to large deviations for sums of i.i.d. random variables: *Let $\alpha \in (0,1) \cup (1,2)$ and let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. random variables in the domain of normal attraction of an α-stable law, centered for $\alpha > 1$, then,*

$$\lim_{n \to +\infty} \frac{P\left\{\left|n^{-\frac{1}{\alpha}} \sum_{i=1}^n X_i\right| > x_n\right\}}{nP\{|X_1| > n^{1/\alpha} x_n\}} = \lim_{n \to +\infty} \frac{P\left\{\left|n^{-\frac{1}{\alpha}} \sum_{i=1}^n X_i\right| > x_n\right\}}{P\{\max_{j=1,\dots,n} |X_j| > n^{1/\alpha} x_n\}} = 1 \tag{20}$$

whenever $x_n \to +\infty$. See [20] and [21]. It follows from (9) that $P\{|X_1| > n^{\frac{1}{\alpha}} x_n\} \sim c_0/(n x_n^\alpha)$. Moreover, if S_α is the α-stable random variable limit of $n^{-\frac{1}{\alpha}} \sum_{i=1}^n X_i$, then, $P\{|S_\alpha| > x_n\} \sim c_0/x_n^\alpha$, since each stable random variable belongs to its own domain of normal attraction. Consequently

$$\lim_{n\to+\infty} \frac{P\Big\{\Big|n^{-\frac{1}{\alpha}} \sum_{i=1}^n X_i\Big| > x_n\Big\}}{P\{|S_\alpha| > x_n\}} = \lim_{n\to+\infty} \frac{x_n^\alpha}{c_0} P\Big\{\Big|n^{-\frac{1}{\alpha}} \sum_{i=1}^n X_i\Big| > x_n\Big\} = 1. \quad (21)$$

At this stage, it should be clear that equations (17)-(18)-(19) provide analogous results for our processes.

If either $\mu(2\alpha) \geq \mu(\alpha)$ or $2\mathcal{S}(\alpha) \leq -1$ then (17)-(18) are still valid provided that x_t diverges to $+\infty$ at a suitable speed depending on the function $\mu(s)$. In order to state such extension in a precise way, we need some more notation. When $\mathcal{S}(2\alpha) < +\infty$ let $h(t) : [0, +\infty) \to [0, +\infty)$ be the function

$$h(t) := \begin{cases} t & \text{if } \mu(2\alpha) < \mu(\alpha) \text{ and } 2\mathcal{S}(\alpha) = -1; \\ e^{-(2\mathcal{S}(\alpha)+1)t} & \text{if } \mu(2\alpha) < \mu(\alpha) \text{ and } 2\mathcal{S}(\alpha) < -1; \\ e^{2\alpha(\mu(2\alpha)-\mu(\alpha))t} & \text{if } \mu(2\alpha) > \mu(\alpha); \\ e^{\eta t} & \text{if } \mu(2\alpha) = \mu(\alpha) \text{ and } 0 < \mathcal{S}(\alpha) \text{ for a fixed } \eta > 0; \\ te^{-(2\mathcal{S}(\alpha)+1)t} & \text{if } \mu(2\alpha) = \mu(\alpha) \text{ and } 2\mathcal{S}(\alpha) < -1; \\ t^2 & \text{if } \mu(2\alpha) = \mu(\alpha) \text{ and } 2\mathcal{S}(\alpha) = -1; \\ t & \text{if } \mu(2\alpha) = \mu(\alpha) \text{ and } -1 < 2\mathcal{S}(\alpha) \leq 0. \end{cases} \quad (22)$$

Theorem 3.2. *Let $\alpha \in (0,1)\cup(1,2)$ and the hypotheses of* Theorem 2.1 *be in force, or let $\alpha = 1$ and the hypotheses of* Theorem 2.2 *hold. Assume also that $\mathcal{S}(2\alpha) < +\infty$ and $c_0 := c_0^+ + c_0^- > 0$. If either $\mu(2\alpha) \geq \mu(\alpha)$ or $2\mathcal{S}(\alpha) \leq -1$ and x_t is such that $x_t^{\alpha-\epsilon}/h(t) \to +\infty$ as $t \to +\infty$ for some $\epsilon > 0$, with $h(t)$ as in* (22)*, then* (17)-(18) *hold true.*

We conclude this section with two examples.

Example 1. Let us consider the case in which $L = 1 - R = U$ where U is a random variable uniformly distibuted on $(0,1)$. In this special case $\mathcal{S}(s) = \frac{1-s}{1+s}$ and $\mu(s) = \frac{1-s}{s(1+s)}$. In particular, it is easy to prove that μ is a continuous function and that μ is strictly decreasing on $(0, s_0)$ and strictly increasing on $(s_0, +\infty)$ with $s_0 = 1 + \sqrt{2}$. Hence, since $\mu(\delta) < \mu(\alpha)$ for every $0 < \alpha < \delta < s_0$, Theorems 2.1-2.2 can be applied. As for the large deviation of V_t, assuming that $\bar{\rho}_0$ satisfies the assumption of Theorem 3.1, since $\mathcal{S}(\alpha) > -1/2$ for every $\alpha \in (0,2)$, it remains to study the sign of $\mu(2\alpha) - \mu(\alpha)$. Setting $\alpha_0 = (3 + \sqrt{17})/4 \approx 1.78$, it is easy to see that: $\mu(2\alpha) < \mu(\alpha)$ for $\alpha \in (0, \alpha_0)$, $\mu(2\alpha) = \mu(\alpha)$ for $\alpha = \alpha_0$ and $\mu(2\alpha) > \mu(\alpha)$ for $\alpha \in (\alpha_0, 2)$. Summarizing

(i) if $\alpha \in (0, \alpha_0)$, then (17)-(18) hold true for any x_t;
(ii) if $\alpha = \alpha_0$, then (17)-(18) hold true for any x_t such that $x_t^{\alpha-\epsilon}/t \to +\infty$ for some $\epsilon > 0$;
(iii) if $\alpha \in (\alpha_0, 2)$, then (17)-(18) hold true for any x_t such that $x_t^{\alpha-\epsilon} \exp\{-t(2\alpha^2 - 2\alpha - 1)/(2\alpha^2 + 3\alpha + 1)\} \to +\infty$ for some $\epsilon > 0$;

Example 2. An interesting example is the case of the inelastic Kac equation [29]. This equation can be reduced to a special case of equation (1)-(2) with

$L = |\cos(\tilde{\theta})|^{1+d}$ and $R = |\sin(\tilde{\theta})|^{1+d}$, $\tilde{\theta}$ being a random variable uniformly distributed on $(0, 2\pi)$ and $d > 0$. In this case

$$\begin{aligned}\mathcal{S}(s) &= \frac{1}{2\pi}\int_{(0,2\pi)} (|\sin(\theta)|^{(1+d)s} + |\cos(\theta)|^{(1+d)s})d\theta - 1 \\ &= \frac{1}{\pi}\int_{(0,2\pi)} |\sin(\theta)|^{(1+d)s} d\theta - 1 = \frac{2}{\sqrt{\pi}}\frac{\Gamma(\frac{d+1}{2}s + \frac{1}{2})}{\Gamma(\frac{d+1}{2}s + 1)} - 1\end{aligned}$$

where $\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt$. Clearly $\mathcal{S}(\alpha) = 0$ for $\alpha = 2/(d+1)$ and $\mathcal{S}(p) < 0$ for every $p > \alpha$. Hence, for this α, Theorems 2.1-2.2 can be applied. Note that in this case ρ_t converges weakly to the stationary distribution which is a stable law, in other words, $Z_\infty(\alpha) = 1$. See, also [3, 4]. Furthermore, since $\mathcal{S}(\alpha) = 0 > -1/2$ and $\mu(p) < 0 = \mu(\alpha)$ for every $p > \alpha$, Theorem 3.1 holds. If $\alpha \neq 2/(d+1)$ the situation is more involved. Since $\lim_{s\to+\infty} \mathcal{S}(s) = -1$, then $\lim_{s\to+\infty} \mu(s) = 0$ and one can prove that $\mu(s)$ has a unique minimum point in $p_0^{(d)}$. Clearly $p_0^{(d)} = 2p_0^{(1)}/(d+1)$ where $p_0^{(1)}$ is the unique minimum point of

$$s \mapsto \mu_1(s) := \frac{1}{s}\Big(\frac{2}{\sqrt{\pi}}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)} - 1\Big).$$

Numerically one sees that $p_0^{(1)} \approx 2.413$ and $\mu_1(p_0^{(1)}) \approx -0.128$. Hence, given $d > 0$, Theorems 2.1-2.2 can be applied provided that $\alpha < 2p_0^{(1)}/(d+1) \approx 4.816/(d+1)$. Moreover one can check that

$$h(t) \le e^{Ct}$$

for any $C = C(\alpha, d) > \max\{1, \alpha(d+1)|\mu_1(p_0^{(1)})|\}$. Hence, (17)-(18) hold true for any x_t such that $x_t^{\alpha-\epsilon}e^{-Ct} \to +\infty$ for some $\epsilon > 0$.

4. **Large deviation for sum of weighted i.i.d. random variables.** The present section deals with the study of the probability of large deviations for weighted sums of i.i.d. random variables. This study is a generalization of the large deviation estimates presented in [20, 21] and, besides the interest it could hold in itself, it is the first step in the proof of Theorem 3.1.

Let $(X_j)_{j\ge1}$ be a sequence of i.i.d. random variables with common distirbution function F_0 and $[b_{jn} : j = 1, \dots, n; n \ge 1]$ be an array of non-negative weights. Let

$$S_n := \sum_{j=1}^n b_{jn}X_j,$$

$b(n) := \max\{b_{jn} : j = 1, \dots, n\}$ and $b^{(1:n)} := (b_{1n}, \dots, b_{nn})$.

If F_0 satisfies (9), for every $x > 0$ define

$$\begin{aligned}\mathcal{R}(x) &:= \frac{x^\alpha}{c_0}P\{|X_1| > x\} - 1 \qquad (c_0 := c_0^+ + c_0^-) \\ \bar{\mathcal{R}}(x) &:= \sup_{y:y\ge x} |\mathcal{R}(x)|.\end{aligned}$$

Clearly

$$P\{|X_1| > x\} = c_0x^{-\alpha}(1 + \mathcal{R}(x)), \tag{23}$$

hence $\|\mathcal{R}\|_\infty := \sup_{x>0} |\mathcal{R}(x)| < +\infty$ and

$$\lim_{x\to+\infty} \bar{\mathcal{R}}(x) = \lim_{x\to+\infty} \mathcal{R}(x) = 0. \tag{24}$$

Finally, set

$$K_0 := c_0(\|\mathcal{R}\|_\infty + 1) \tag{25}$$

and

$$\Delta^{(n)}_{b(1:n)}(y) := P\{|S_n| + b(n)|X_1| \leq y\}.$$

Lemma 4.1. *Assume that F_0 satisfies (9) with $c_0 = c_0^+ + c_0^- > 0$. Moreover, if $\alpha = 1$ assume that $c_0^+ = c_0^-$ and that (13) holds, while if $\alpha > 1$ assume that $\mathbb{E}[X_1] = 0$. Then, for every $x > 0$, $n \geq 1$, $0 < \epsilon < 1$ and $\gamma > 0$, the following inequalities are valid*

$$\begin{aligned} x^\alpha P\{|S_n| > x\} \geq & \frac{\Delta^{(n)}_{b(1:n)}(\epsilon x)}{(1+\epsilon)^\alpha} c_0 \left(1 - \bar{\mathcal{R}}\left(\frac{x(1+\epsilon)}{b(n)}\right)\right) \sum_{j=1}^n b_{jn}^\alpha \\ & - \frac{K_0^2}{x^\alpha(1+\epsilon)^{2\alpha}} \left(\sum_{j=1}^n b_{jn}^\alpha\right)^2 \end{aligned} \tag{26}$$

and

$$\begin{aligned} x^\alpha P\{|S_n| > x\} \leq & \Big[\frac{c_0}{(1-\epsilon)^\alpha}\left(1 + \bar{\mathcal{R}}\left(\frac{x(1-\epsilon)}{b(n)}\right)\right) \\ & + \frac{2K_0}{\epsilon^2(2-\alpha)x^{(2-\alpha)(1-\gamma)}}\Big] \sum_{j=1}^n b_{jn}^\alpha \\ & + \left[\frac{K_0^2}{x^{\alpha(2\gamma-1)}} + \frac{K_1}{\epsilon^2 x^{2-\alpha+2(\alpha-1)\gamma}}\right] \left(\sum_{j=1}^n b_{jn}^\alpha\right)^2 \end{aligned} \tag{27}$$

where $K_1 = K_0^2/(1-\alpha)^2$ if $\alpha < 1$, $K_1 = K_0^2\alpha^2/(1-\alpha)^2$ if $\alpha > 1$ and $K_1 = (\gamma_0 + \sup_R |\int_{(-R,R)} y dF_0(y) - \gamma_0|)^2$ if $\alpha = 1$. Moreover,

$$\begin{aligned} c_0 \sum_{j=1}^n b_{jn}^\alpha \Big(1 - \bar{\mathcal{R}}\Big(\frac{x}{b(n)}\Big)\Big) - \frac{K_0^2}{x^\alpha}\left(\sum_{j=1}^n b_{jn}^\alpha\right)^2 & \leq x^\alpha P\{\max_{1\leq j\leq n} |b_{jn}X_j| > x\} \\ & \leq c_0 \sum_{j=1}^n b_{jn}^\alpha \Big(1 + \bar{\mathcal{R}}\Big(\frac{x}{b(n)}\Big)\Big). \end{aligned} \tag{28}$$

Proof. The proof of this lemma is an adaptation to the present case of the techniques used in [19, 20].

Proof of (26). Set

$$S_{n,k} := \sum_{1 \leq j \leq n, j \neq k} b_{jn} X_j \qquad k = 1, \ldots, n$$

and

$$A_j := \{|b_{jn}X_j| > (1+\epsilon)x, |S_{n,j}| \leq \epsilon x\}.$$

Clearly

$$\cup_{j=1}^n A_j \subset \{|S_n| > x\}$$

and hence, by Bonferroni inequality,

$$P\{|S_n| > x\} \geq \sum_{j=1}^n P(A_j) - \sum_{1 \leq j < k \leq n} P(A_j \cap A_k).$$

Now, from the independence of the X_j's, one obtains

$$P(A_j \cap A_k) \le P\{|b_{jn}X_j| > (1+\epsilon)x\}P\{|b_{kn}X_k| > (1+\epsilon)x\}.$$

and

$$P(A_j) = P\{|b_{jn}X_j| > (1+\epsilon)x\}P\{|S_{n,j}| \le \epsilon x\}.$$

Hence

$$\begin{aligned} P\{|S_n| > x\} \ge & \sum_{j=1}^{n} P\{|b_{jn}X_j| > (1+\epsilon)x\}P\{|S_{n,j}| \le \epsilon x\} \\ & - \left(\sum_{j=1}^{n} P\{|b_{jn}X_j| > (1+\epsilon)x\}\right)^2. \end{aligned} \tag{29}$$

Furthermore, for every $j = 1, \dots, n$,

$$P\{|S_{n,j}| \le \epsilon x\} \ge P\{|S_n| + b(n)|X_j| \le \epsilon x\} = \Delta^{(n)}_{b(1:n)}(y) \tag{30}$$

and from (23)-(25) one gets

$$\frac{c_0 b_{jn}^\alpha}{x^\alpha(1+\epsilon)^\alpha}\Big(1 - \bar{\mathcal{R}}\Big(\frac{x(1+\epsilon)}{b(n)}\Big)\Big) \le P\{|b_{jn}X_j| > (1+\epsilon)x\} \le \frac{b_{jn}^\alpha}{x^\alpha(1+\epsilon)^\alpha}K_0. \tag{31}$$

Combining (29), (30) and (31) one obtains (26).

Proof of (27). Define

$$\begin{aligned} Y_{jn} &:= b_{jn}X_j\mathbb{I}\{|b_{jn}X_j| \le x^\gamma\} \\ \tilde{S}_n &:= \sum_{j=1}^{n} Y_{jn} \\ E_n &:= \cup_{j=1}^{n}\{|b_{jn}X_j| > (1-\epsilon)x\} \\ F_n &:= \cup_{1\le i<j\le n}\{|b_{jn}X_j| > x^\gamma, |b_{in}X_i| > x^\gamma\} \\ G_n &:= \{|\tilde{S}_n| > \epsilon x\} \end{aligned}$$

It is easy to see that $\{|S_n| > x\} \subset E_n \cup F_n \cup G_n$ and hence,

$$P(|S_n| > x) \le P(E_n) + P(F_n) + P(G_n). \tag{32}$$

From (23) one obtains

$$\begin{aligned} P(E_n) \le \sum_{j=1}^{n} P(|b_{jn}X_j| > (1-\epsilon)x) &= \sum_{j=1}^{n} \frac{c_0 b_{jn}^\alpha}{x^\alpha(1-\epsilon)^\alpha}\Big(1 + \mathcal{R}\Big(\frac{x(1-\epsilon)}{b_{jn}}\Big)\Big) \\ &\le \sum_{j=1}^{n} \frac{c_0 b_{jn}^\alpha}{x^\alpha(1-\epsilon)^\alpha}\Big(1 + \bar{\mathcal{R}}\Big(\frac{x(1-\epsilon)}{b(n)}\Big)\Big) \end{aligned} \tag{33}$$

and

$$\begin{aligned} P(F_n) &\le \sum_{1\le i<j\le n} P(|b_{in}X_i| > x^\gamma)P(|b_{jn}X_j| > x^\gamma) \\ &= \sum_{1\le i<j\le n} \frac{c_0^2 b_{in}^\alpha b_{jn}^\alpha}{x^{2\gamma\alpha}}\Big(1 + \mathcal{R}\Big(\frac{x^\gamma}{b_{in}}\Big)\Big)\Big(1 + \mathcal{R}\Big(\frac{x^\gamma}{b_{jn}}\Big)\Big) \\ &\le K_0^2\Big(\sum_{j=1}^{n} b_{jn}^\alpha\Big)^2 x^{-2\alpha\gamma} \end{aligned} \tag{34}$$

where K_0 is defined in (25) and $\mathcal{R}(x^\gamma/0) := 0$. From Chebyshev inequality

$$\begin{aligned} P(G_n) &\leq \frac{1}{\epsilon^2 x^2}\mathbb{E}[\tilde{S}_n^2] \leq \frac{1}{\epsilon^2 x^2}\mathbb{E}\Big[\sum_{j=1}^n Y_{jn}^2 + \sum_{1\leq i,j\leq n} Y_{in}Y_{jn}\Big] \\ &\leq \frac{1}{\epsilon^2 x^2}\Big(\sum_{j=1}^n \mathbb{E}[Y_{jn}^2] + \Big(\sum_{j=1}^n |\mathbb{E}[Y_{jn}]|\Big)^2\Big) \end{aligned} \tag{35}$$

Note that if $b_{jn} = 0$ then $\mathbb{E}[Y_{jn}^2] = |\mathbb{E}[Y_{jn}]| = 0$, hence from now on we assume that $b_{jn} > 0$. Now

$$\mathbb{E}[Y_{jn}^2] = b_{jn}^2 \mathbb{E}[|X_j|^2 \mathbb{I}\{|X_j| \leq x^\gamma/b_{jn}\}] \leq 2b_{jn}^2 \int_0^{x^\gamma/b_{jn}} yP\{|X_1| > y\}dy.$$

Since $P\{|X_1| > y\} \leq K_0 y^{-\alpha}$, it follows that

$$\mathbb{E}[Y_{jn}^2] \leq \frac{2K_0}{2-\alpha} b_{jn}^\alpha x^{(2-\alpha)\gamma}. \tag{36}$$

It remains to consider $|\mathbb{E}[Y_{jn}]|$. If $\alpha < 1$, then

$$\begin{aligned} |\mathbb{E}[Y_{jn}]| &\leq b_{jn} \int_0^{x^\gamma/b_{jn}} P\{|X_1| > y\}dy \\ &\leq b_{jn}K_0 \int_0^{x^\gamma/b_{jn}} y^{-\alpha}dy = \frac{b_{jn}^\alpha K_0}{1-\alpha} x^{(1-\alpha)\gamma}. \end{aligned} \tag{37}$$

If $\alpha > 1$ and $\mathbb{E}[X_1] = \int y dF_0(y) = 0$, then

$$\begin{aligned} |\mathbb{E}[Y_{jn}]| &= b_{jn}\Big| \int_{\{y:|y|\leq x^\gamma/b_{jn}\}} y dF_0(y)\Big| = b_{jn}\Big| \int_{\{y:|y|> x^\gamma/b_{jn}\}} y dF_0(y)\Big| \\ &\leq b_{jn}\Big[\int_{x^\gamma/b_{jn}}^{+\infty} P\{|X_1| > y\}dy + \frac{x^\gamma}{b_{jn}} P\Big\{|X_1| > \frac{x^\gamma}{b_{jn}}\Big\}\Big] \\ &\leq b_{jn}K_0\Big[\int_{x^\gamma/b_{jn}}^{+\infty} y^{-\alpha}dy + x^{\gamma(1-\alpha)} b_{jn}^{\alpha-1}\Big] = b_{jn}^\alpha K_0 \frac{\alpha}{\alpha-1} x^{(1-\alpha)\gamma}. \end{aligned} \tag{38}$$

Finally, if $\alpha = 1$, by assumption

$$K := \sup_{R>0} |\int_{(-R,R)} yF_0(y) - \gamma_0| < +\infty.$$

Hence, in this case, one gets

$$|\mathbb{E}[Y_{jn}]| \leq b_{jn}\Big|\int_{\{y:|y|\leq x^\gamma/b_{jn}\}} y dF_0(y) - \gamma_0\Big| + b_{jn}\gamma_0 \leq b_{jn}(\gamma_0 + K). \tag{39}$$

Combining (32)-(39) one gets (27).

Proof of (28). By Bonferroni inequality, using once again (23) and (25), one gets

$$\begin{aligned} P\{\max_{1\leq j\leq n} |b_{jn}X_j| > x\} &\geq \sum_{j=1}^n P\{|b_{jn}X_j| > x\} \\ &\quad - \sum_{1\leq j<k\leq n} P\{|b_{jn}X_j| > x, |b_{kn}X_k| > x\} \\ &\geq \frac{c_0}{x^\alpha} \sum_{j=1}^n b_{jn}^\alpha \Big(1 - \bar{\mathcal{R}}\Big(\frac{x}{b(n)}\Big)\Big) - \frac{K_0^2}{x^{2\alpha}}\Big(\sum_{j=1}^n b_{jn}^\alpha\Big)^2 \end{aligned}$$

and

$$P\{\max_{1\le j\le n}|b_{j,n}X_j|>x\}\le\sum_{j=1}^n P\{|b_{jn}X_j|>x\}\le\frac{c_0}{x^\alpha}\sum_{j=1}^n b_{jn}^\alpha\Big(1+\bar{\mathcal{R}}\Big(\frac{x}{b(n)}\Big)\Big)$$

that yields (28). □

Remark 5. Notice that if $\gamma\in(1/2,1)$ and $\alpha\in(0,2)$, then $(2-\alpha)(1-\gamma)>0$ and $\alpha(2\gamma-1)>0$. Moreover, if $\gamma<1$ and $\alpha<1$, then $(2-\alpha)+2(\alpha-1)\gamma>\alpha>0$, while, if $\alpha>1$, then $(2-\alpha)+2(\alpha-1)\gamma\uparrow\alpha$ for $\gamma\uparrow 1$. Finally, $\alpha(2\gamma-1)\uparrow\alpha$ when $\gamma\uparrow 1$.

A simple consequence of Lemma 4.1 and Remark 5 is the following large deviations result for the weighted sum $S_n=\sum_{j=1}^n b_{jn}X_j$.

Corollary 1. *Assume that F_0 satisfies (9) with $c_0=c_0^++c_0^->0$. If $\alpha=1$ assume also that $c_0^+=c_0^-$ and that (13) holds, while if $\alpha>1$ assume that $\mathbb{E}[X_1]=0$. If $b(n)\to 0$, $\sum_{j=1}^n b_{jn}^\alpha\to 1$ and $x_n\to+\infty$, then*

$$\lim_{n\to+\infty}x_n^\alpha P\{|S_n|>x_n\}=c_0.$$

5. Proofs.

5.1. Preliminary results.

Let α be a given positive real number such that $\mathbb{E}[L^\alpha+R^\alpha]<+\infty$. For every integer number $n\ge 1$ set

$$M_n(\alpha):=\sum_{j=1}^n\beta_{j,n}^\alpha\qquad\text{and}\qquad\tilde M_n(\alpha):=\frac{M_n(\alpha)}{m_n(\alpha)}\tag{40}$$

where

$$m_n(\alpha):=\frac{\Gamma(n+\mathcal{S}(\alpha))}{\Gamma(n)\Gamma(\mathcal{S}(\alpha)+1)}.$$

Note that, as $n\to+\infty$, by the well-known asymptotic expansion for the ratio of Gamma functions,

$$m_n(\alpha)=n^{\mathcal{S}(\alpha)}\frac{1}{\Gamma(\mathcal{S}(\alpha)+1)}\Big(1+O\Big(\frac{1}{n}\Big)\Big).\tag{41}$$

For every $\alpha>0$, set also

$$\beta_{(n)}:=\max_{1\le j\le n}\beta_{j,n}\qquad\text{and}\qquad\tilde\beta_{(n)}:=\frac{\beta(n)}{m_n(\alpha)^{\frac{1}{\alpha}}},$$

and recall that $\mu(\alpha)=\mathcal{S}(\alpha)/\alpha$. Let us collect some results related to the sequence $(\tilde M_n(\alpha))_{n\ge1}$ proved in [2].

Proposition 1 ([2])**.** *Let $\alpha>0$ such that $\mathbb{E}[L^\alpha+R^\alpha]<+\infty$.*

(i) For every $n\ge 1$

$$\mathbb{E}[M_n(\alpha)]=m_n(\alpha).$$

(ii) $\tilde M_n(\alpha)$ is a positive martingale with respect to the filtration $(\mathcal{G}_n)_{n\ge1}$ with

$$\mathcal{G}_n=\sigma(L_1,R_1,\dots,L_{n-1},R_{n-1},I_1,\dots,I_{n-1}),$$

and $\mathbb{E}[\tilde M_n(\alpha)]=1$. Hence, $\tilde M_n(\alpha)$ converges almost surely to a random variable $\tilde M_\infty(\alpha)$ with $\mathbb{E}[\tilde M_\infty(\alpha)]\le 1$.

(iii) If for some $\delta>0$ and $\alpha>0$ one has $\mu(\delta)<\mu(\alpha)<+\infty$, then $\tilde\beta_{(n)}$ converges in probability to 0.

(iv) If $\mu(\delta) < \mu(\alpha) < +\infty$ for $\alpha < \delta$, $\tilde{M}_n(\alpha)$ converges in L^1 to $\tilde{M}_\infty(\alpha)$ and $\mathbb{E}[\tilde{M}_\infty(\alpha)] = 1$.

We shall need some more results related to M_{ν_t} and $\beta_{(\nu_t)}$.

Proposition 2. *Let $\mu(\delta) < \mu(\alpha) < +\infty$ for $\alpha < \delta$ and let $\tilde{M}_\infty(\alpha)$ be the same random variable of* Proposition 1. *Then, there exists a random variable E with exponential distribution of parameter* 1, *with E and $\tilde{M}_\infty(\alpha)$ independent, such that*

$$m_{\nu_t}(\alpha)e^{-\mathcal{S}(\alpha)t} \to \frac{E^{\mathcal{S}(\alpha)}}{\Gamma(\mathcal{S}(\alpha)+1)} \qquad a.s., \tag{42}$$

and

$$e^{-\mathcal{S}(\alpha)t}M_{\nu_t}(\alpha) \to \frac{E^{\mathcal{S}(\alpha)}\tilde{M}_\infty(\alpha)}{\Gamma(\mathcal{S}(\alpha)+1)} =: Z_\infty(\alpha) \qquad a.s. \text{ and in } L^1 \tag{43}$$

as $t \to +\infty$. Moreover, for every t,

$$\mathbb{E}[e^{-\mathcal{S}(\alpha)t}M_{\nu_t}(\alpha)] = \mathbb{E}[Z_\infty(\alpha)] = 1, \tag{44}$$

the law of $Z_\infty(\alpha)$ satisfies the fixed point equation (10) *and*

$$\mathbb{E}[Z_\infty(\alpha)^{\delta/\alpha}] < +\infty. \tag{45}$$

Finally,

$$\tilde{\beta}_{(\nu_t)} \to 0 \quad \text{and} \quad \beta_{(\nu_t)}e^{-\mu(\alpha)t} \to 0 \tag{46}$$

in probability as $t \to +\infty$.

Proof. It is well-known that if $(\nu_t)_t$ is a Yule process, then $e^{-t}\nu_t$ is a martingale and converges a.s. to an exponential random variable E of parameter 1, see e.g. [1]. Hence, by (41), $e^{-\mathcal{S}(\alpha)t}m_{\nu_t}(\alpha)$ converges a.s. to $E^{\mathcal{S}(\alpha)}/\Gamma(\mathcal{S}(\alpha)+1)$. By (iv) of Proposition 1, it follows that $\tilde{M}_{\nu_t}(\alpha)$ converges a.s. and in L^1 to $\tilde{M}_\infty(\alpha)$. Note that $\tilde{M}_\infty(\alpha)$ is measurable with respect to the σ-field generated by the β_{jn}'s and E is measurable with respect to the σ-field generated by $(\nu_t)_t$. This implies that E and $\tilde{M}_\infty(\alpha)$ are independent. Since $e^{-\mathcal{S}(\alpha)t}M_{\nu_t}(\alpha) = m_{\nu_t}(\alpha)e^{-\mathcal{S}(\alpha)t}\tilde{M}_{\nu_t}(\alpha)$, it follows that $e^{-\mathcal{S}(\alpha)t}M_{\nu_t}(\alpha)$ converges a.s. to $E^{\mathcal{S}(\alpha)}\tilde{M}_\infty(\alpha)/\Gamma(\mathcal{S}(\alpha)+1)$. Moreover, recalling that for every $\gamma > -1$ and $0 < u < 1$

$$\sum_{n=1}^{+\infty} \frac{\Gamma(\gamma+n)}{\Gamma(n)\Gamma(\gamma+1)}(1-u)^{n-1} = u^{-(\gamma+1)} \tag{47}$$

and in view of (i) of Proposition 1

$$\begin{aligned}\mathbb{E}[e^{-\mathcal{S}(\alpha)t}M_{\nu_t}(\alpha)] =& e^{-\mathcal{S}(\alpha)t}\sum_{n=1}^{+\infty} e^{-t}(1-e^{-t})^{n-1}m_n(\alpha)\\ =& e^{-(\mathcal{S}(\alpha)+1)t}\sum_{n=1}^{+\infty}(1-e^{-t})^{n-1}\frac{\Gamma(\mathcal{S}(\alpha)+n)}{\Gamma(n)\Gamma(\mathcal{S}(\alpha)+1)} = 1\end{aligned}$$

for every t. By the independence of E and $\tilde{M}_\infty(\alpha)$ and by (iv) of Proposition 1 one easily see that

$$\mathbb{E}[Z_\infty(\alpha)] = \mathbb{E}\Big[\tilde{M}_\infty(\alpha)\frac{E^{\mathcal{S}(\alpha)}}{\Gamma(\mathcal{S}(\alpha)+1)}\Big] = \mathbb{E}[\tilde{M}_\infty(\alpha)]\mathbb{E}\Big[\frac{E^{\mathcal{S}(\alpha)}}{\Gamma(\mathcal{S}(\alpha)+1)}\Big] = 1.$$

Now using (44) and the fact that $e^{-\mathcal{S}(\alpha)t}M_{\nu_t}(\alpha)$ is non-negative, it follows that the convergence of $e^{-\mathcal{S}(\alpha)t}M_{\nu_t}(\alpha)$ holds in L^1 too. In view of Propositions 5.3 and 2.1 in [2] the law of $Z_\infty(\alpha)$ is a solution of the fixed point equation (10) and (45) holds.

The proof of (46) follows immediately from (iii) of Proposition 1 and (42). □

Denote by $\mathcal{B}$ the σ–field generated by the array of random variables $[\beta_{jn}, j = 1,\dots,n; n \geq 1]$. Given $\epsilon > 0$ and $x_t \to +\infty$ as $t \to +\infty$, define the stochastic process

$$\Delta_t := \sum_{n\geq 1} \mathbb{I}\{\nu_t = n\} P\Big\{\big|\sum_{j=1}^n \beta_{j,n} X_j\big| + \beta(n)|X_1| \leq \epsilon x_t e^{\mu(\alpha)t} \Big| \mathcal{B}\Big\}.$$

Lemma 5.1. *Let the same hypotheses of* Theorem 2.1 *or* Theorem 2.2 *be in force for some* α *in* $(0, 2)$. *Then* $\Delta_t \to 1$ *in* L^1 *as* $t \to +\infty$.

Proof. Note that $0 \leq \Delta_t \leq 1$, hence

$$0 \leq \mathbb{E}[|\Delta_t - 1|] = 1 - \mathbb{E}[\Delta_t].$$

Furthermore

$$\mathbb{E}[\Delta_t] = P\{e^{-\mu(\alpha)t}(|V_t| + \beta(\nu_t)|X_1|) \leq \epsilon x_t\}.$$

From Theorems 2.1-2.2 one knows that $e^{-\mu(\alpha)t}V_t$ converges in distribution. Moreover, from (46), one gets that $e^{-\mu(\alpha)t}\beta(\nu_t)|X_1|$ converges in probability to zero. Hence, $\big(e^{-\mu(\alpha)t}(|V_t|+\beta(\nu_t)|X_1|)\big)_{t\geq 0}$ is a tight family. This means that, for every sequence $t_n \to +\infty$ and for every $\eta > 0$, there exists K such that $\inf_n P\{e^{-\mu(\alpha)t_n}(|V_{t_n}| + \beta(\nu_{t_n})|X_1|) \leq K\} \geq 1-\eta$. Since $x_{t_n} \to +\infty$, for sufficiently large n one can write

$$1 \geq \mathbb{E}[\Delta_{t_n}] \geq P\{e^{-\mu(\alpha)t_n}(|V_{t_n}| + \beta(\nu_{t_n})|X_1|) \leq K\} \geq 1 - \eta.$$

Hence $\mathbb{E}[\Delta_t] \to 1$ and $\Delta_t \to 1$ in L^1.

□

Lemma 5.2. *If* $\mathcal{S}(\alpha) < +\infty$, *one has*

$$\mathbb{E}[\tilde{M}_n(\alpha)^2] \leq C \sum_{i=1}^n i^{2\alpha(\mu(2\alpha)-\mu(\alpha))-1} \tag{48}$$

for every n, C *being a suitable constant.*

Proof. From the definition of $m_n(\alpha)$ we have $m_{n+1}(\alpha) = m_n(\alpha)(1+\frac{\mathcal{S}(\alpha)}{n})$ and from the definition of $\tilde{M}_n(\alpha)$ we obtain

$$\tilde{M}_{n+1}(\alpha) - \tilde{M}_n(\alpha) = -\frac{M_n(\alpha)}{m_n(\alpha)}\Big(\frac{\mathcal{S}(\alpha)}{n+\mathcal{S}(\alpha)}\Big) + \sum_{i=1}^n \mathbb{I}\{I_n = j\}\frac{\beta_{j,n}^\alpha(L_n + R_n - 1)}{m_{n+1}(\alpha)}.$$

Below the symbol C designates a constant, not necessarily the same at each occurrence.

$$
\begin{aligned}
&|\tilde{M}_{n+1}(\alpha)-\tilde{M}_n(\alpha)|^2\\
\le& 2\left(\tilde{M}_n(\alpha)^2\Big(\frac{\mathcal{S}(\alpha)}{n+\mathcal{S}(\alpha)}\Big)^2+\sum_{i=1}^n \mathbb{I}\{I_n=j\}\frac{\beta_{j,n}^{2\alpha}(L_n+R_n-1)^2}{m_{n+1}(\alpha)^2}\right)\\
\le& C\left[\frac{1}{n^2}\Big(\sum_{i=1}^n\frac{\beta_{j,n}^{\alpha}}{m_n(\alpha)}\Big)^2+\sum_{i=1}^n \mathbb{I}\{I_n=j\}\frac{\beta_{j,n}^{2\alpha}(L_n+R_n-1)^2}{m_{n+1}(\alpha)^2}\right]\\
\le& C\left[\frac{1}{n}\sum_{i=1}^n\frac{\beta_{j,n}^{2\alpha}}{m_n(\alpha)^2}+\sum_{i=1}^n \mathbb{I}\{I_n=j\}\frac{\beta_{j,n}^{2\alpha}(L_n+R_n-1)^2}{m_{n+1}(\alpha)^2}\right].
\end{aligned}
$$

Taking the expectation on both side of the last inequality we get

$$
\begin{aligned}
\mathbb{E}(|\tilde{M}_{n+1}(\alpha)-\tilde{M}_n(\alpha)|^2) &\le \frac{C}{n}\left[\frac{m_n(2\alpha)}{m_n(\alpha)^2}+\frac{m_n(2\alpha)}{m_{n+1}(\alpha)^2}\right]\\
&\le \frac{C}{n}\Big[n^{\mathcal{S}(2\alpha)-2\mathcal{S}(\alpha)}+\frac{n^{\mathcal{S}(2\alpha)}}{(n+1)^{2\mathcal{S}(\alpha)}}\Big].
\end{aligned}
$$

Now, recalling that $(\tilde{M}_n(\alpha))_{n\ge1}$ is a martingale, we obtain

$$
\begin{aligned}
\mathbb{E}[\tilde{M}_n(\alpha)^2] &= 1+\sum_{i=1}^{n-1}\mathbb{E}(|\tilde{M}_{i+1}(\alpha)-\tilde{M}_i(\alpha)|^2)\\
&\le C\sum_{i=1}^n i^{2\alpha(\mu(2\alpha)-\mu(\alpha))-1}.
\end{aligned}
$$

□

Lemma 5.3. *If* $\mathbb{E}[L^{2\alpha}+R^{2\alpha}]<+\infty$, *one has for every* $t\ge1$

$$
e^{-2\mathcal{S}(\alpha)t}\mathbb{E}[M_{\nu_t}(\alpha)^2]\le\tilde{h}(t)
$$

where

$$
\tilde{h}(t):=\begin{cases} C & \text{if } \mu(2\alpha)<\mu(\alpha) \text{ and } 2\mathcal{S}(\alpha)>-1;\\ C\,h(t) & \text{otherwise}\end{cases} \tag{49}
$$

where $h(t)$ *is defined in* (22) *and* C *is a suitable constant.*

Proof. As above the symbol C designates a constant, not necessarily the same at each occurrence. We shall repeatedly use the following two simple facts: for any $\gamma>-1$ and any $t>0$

$$
\sum_{n\ge1}(1-e^{-t})^{n-1}n^\gamma\le Ce^{(\gamma+1)t} \tag{50}
$$

and, for every $t\ge1$,

$$
\sum_{n\ge1}(1-e^{-t})^{n-1}\frac{1}{n}=\frac{t}{1-e^{-t}}\le\frac{t}{1-e^{-1}}. \tag{51}
$$

Relation (51) follows by a simple Taylor expansion of $\log(1-x)$, while (50) follows from (47) and from the inequality

$$
n^\gamma\le C\frac{\Gamma(\gamma+n)}{\Gamma(n)\Gamma(\gamma+1)}.
$$

Since

$$I_t := \mathbb{E}[M_{\nu_t}(\alpha)^2] = e^{-t}\sum_{n\geq 1}(1-e^{-t})^{n-1}m_n(\alpha)^2\mathbb{E}[\tilde{M}_n(\alpha)^2],$$

(41) and (48) yield

$$I_t \leq Ce^{-t}\sum_{n\geq 1}(1-e^{-t})^{n-1}n^{2\mathcal{S}(\alpha)}\sum_{i=1}^{n} i^{2\alpha(\mu(2\alpha)-\mu(\alpha))-1}. \tag{52}$$

Let $t \geq 1$. We need now to distinguish among different cases.

Case 1. If $\mu(2\alpha) < \mu(\alpha)$ and $2\mathcal{S}(\alpha) > -1$, then $\sum_{i=1}^{+\infty} i^{2\alpha(\mu(2\alpha)-\mu(\alpha))-1} < +\infty$ and, by (50), one gets

$$I_t \leq Ce^{-t}\sum_{n\geq 1}(1-e^{-t})^{n-1}n^{2\mathcal{S}(\alpha)} \leq Ce^{-t+t(2\mathcal{S}(\alpha)+1)} = Ce^{2\mathcal{S}(\alpha)t}.$$

Case 2. If $\mu(2\alpha) < \mu(\alpha)$ and $2\mathcal{S}(\alpha) = -1$, then $\sum_{i=1}^{+\infty} i^{2\alpha(\mu(2\alpha)-\mu(\alpha))-1} < +\infty$ and, by (51), one gets

$$I_t \leq Ce^{-t}\sum_{n\geq 1}(1-e^{-t})^{n-1}\frac{1}{n} \leq Ct.$$

Case 3. If $\mu(2\alpha) < \mu(\alpha)$ and $2\mathcal{S}(\alpha) < -1$, then $\sum_{i=1}^{+\infty} i^{2\alpha(\mu(2\alpha)-\mu(\alpha))-1} < +\infty$ and hence

$$I_t \leq Ce^{-t}\sum_{n\geq 1}(1-e^{-t})^{n-1}n^{2\mathcal{S}(\alpha)} \leq Ce^{-t}\sum_{n\geq 1}n^{2\mathcal{S}(\alpha)} \leq Ce^{-t}.$$

Case 4. If $\mu(2\alpha) > \mu(\alpha)$, noticing that $\sum_{i=1}^{n} i^{2\alpha(\mu(2\alpha)-\mu(\alpha))-1} \leq Cn^{2\alpha(\mu(2\alpha)-\mu(\alpha))}$, one gets

$$I_t \leq Ce^{-t}\sum_{n\geq 1}(1-e^{-t})^{n-1}n^{2\alpha(\mu(2\alpha)-\mu(\alpha))},$$

and then, by (50),

$$I_t \leq Ce^{(\mathcal{S}(2\alpha)-2\mathcal{S}(\alpha))t}.$$

Case 5. If $\mu(2\alpha) = \mu(\alpha)$ and $0 < \mathcal{S}(\alpha)$

$$\begin{aligned} I_t &\leq Ce^{-t}\sum_{n\geq 1}(1-e^{-t})^{n-1}n^{2\mathcal{S}(\alpha)}\log n \\ &\leq Ce^{-t}\sum_{n\geq 1}(1-e^{-t})^{n-1}n^{2\mathcal{S}(\alpha)+\eta} = C_\eta e^{\eta t}. \end{aligned}$$

If $\mu(2\alpha) = \mu(\alpha)$ and $2\mathcal{S}(\alpha) \leq 0$, then

$$\begin{aligned} I_t &\leq Ce^{-t}\sum_{n\geq 1}(1-e^{-t})^{n-1}n^{2\mathcal{S}(\alpha)}\sum_{i=1}^{n} i^{-1} \\ &= Ce^{-t}\sum_{i\geq 1} i^{-1}\sum_{n\geq i}(1-e^{-t})^{n-1}n^{2\mathcal{S}(\alpha)} \\ &\leq Ce^{-t}\sum_{i\geq 1} i^{-1}(1-e^{-t})^{i-1}\sum_{k\geq 0}(1-e^{-t})^{k}(k+1)^{2\mathcal{S}(\alpha)} \end{aligned}$$

Hence:

Case 6. If $\mu(2\alpha) = \mu(\alpha)$ and $2\mathcal{S}(\alpha) < -1$, by (51)

$$I_t \leq Ce^{-t}\sum_{i\geq 1} i^{-1}(1-e^{-t})^{i-1}\sum_{k\geq 1}k^{2\mathcal{S}(\alpha)} = Ce^{-t}\sum_{i\geq 1} i^{-1}(1-e^{-t})^{i-1} \leq Cte^{-t}$$

Case 7. If $\mu(2\alpha) = \mu(\alpha)$ and $2\mathcal{S}(\alpha) = -1$, using (51) twice

$$I_t \leq Ct^2 e^{-t}.$$

Case 8. If $\mu(2\alpha) = \mu(\alpha)$ and $-1 < 2\mathcal{S}(\alpha) \leq 0$, by (50) and (51),

$$I_t \leq Ce^{-t} \sum_{i \geq 1} i^{-1}(1-e^{-t})^{i-1} \sum_{k \geq 0} (1-e^{-t})^k (k+1)^{2\mathcal{S}(\alpha)} = Cte^{2\mathcal{S}(\alpha)t}$$

□

5.2. **Proofs of the main theorems.**

Proof of Theorem 2.2. The proof follows the same steps of the one of Theorem 2.2 in [2], using in place of Lemma 5.1 in [2] the following simple result: *Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with common distribution function F_0. Assume that $(a_{jn})_{j\geq 1, n\geq 1}$ is an array of positive weights such that*

$$\lim_{n\to+\infty} \sum_{j=1}^{n} a_{jn} = a_\infty \qquad \textit{and} \qquad \lim_{n\to+\infty} \max_{1\leq j\leq n} a_{jn} = 0.$$

If F_0 satisfy (9) with $\alpha = 1$, $c_0^+ = c_0^- > 0$ and (13) holds, then $\sum_{j=1}^n a_{jn} X_j$ converges in distribution to a Cauchy random variable of scale parameter $\pi a_\infty c_0$ and position parameter $a_\infty \gamma_0$. To prove this claim, according to the classical general central limit theorem for array of independent random variables, it is enough to prove that

$$\lim_{n\to+\infty} \zeta_n(x) = \frac{a_\infty c_0}{|x|} \qquad (x \neq 0), \tag{53}$$

$$\lim_{\epsilon\to 0^+} \lim_{n\to+\infty} \sigma_n^2(\epsilon) = 0, \tag{54}$$

$$\lim_{n\to+\infty} \eta_n = a_\infty \gamma_0 \tag{55}$$

are simultaneously satisfied where

$$\zeta_n(x) := \mathbb{I}\{x<0\} \sum_{j=1}^{n} P\{a_{jn}X_j \leq x\} + \mathbb{I}\{x>0\} \sum_{j=1}^{n} P\{a_{jn}X_j > x\} \quad (x \in \mathbb{R}),$$

$$\sigma_n^2(\epsilon) := \sum_{j=1}^{n} \Big\{ \mathbb{E}[(a_{jn}X_j)^2 \mathbb{I}_{(-\epsilon,+\epsilon]}(a_{jn}X_j)] - \Big(\mathbb{E}[a_{jn}X_j \mathbb{I}_{(-\epsilon,+\epsilon]}(a_{jn}X_j)] \Big)^2 \Big\} \quad (\epsilon > 0),$$

$$\eta_n := \sum_{j=1}^{n} \Big\{ P\{a_{jn}X_j \geq 1\} - P\{a_{jn}X_j \leq -1\} + \mathbb{E}[a_{jn}X_j \mathbb{I}_{(-1,+1]}(a_{jn}X_j)] \Big\}.$$

See, e.g., Theorem 30 and Proposition 11 in [16]. Conditions (53) and (54) can be proved exactly as the analogous conditions of Lemma 5 in [3]. As for condition (55) note that

$$\eta_n = \sum_{j=1}^{n} a_{jn} \int_{(-1/a_{jn}, 1/a_{jn}]} x dF_0(x) + \sum_{j=1}^{n} a_{jn} \left[\Big(1 - F_0\Big(\frac{1}{a_{jn}}\Big)\Big) \frac{1}{a_{jn}} - F\Big(-\frac{1}{a_{jn}}\Big) \frac{1}{a_{jn}} \right].$$

Using the assumptions on F_0 and on $(a_{jn})_{jn}$ it follows immediately that

$$\lim_n \sum_{j=1}^n a_{jn} \int_{(-1/a_{jn},1/a_{jn}]} x dF_0(x) = a_\infty \gamma_0$$

and

$$\lim_n \sum_{j=1}^n a_{jn} \left[\Big(1 - F_0\Big(\frac{1}{a_{jn}}\Big)\Big)\frac{1}{a_{jn}} - F\Big(-\frac{1}{a_{jn}}\Big)\frac{1}{a_{jn}} \right] = a_\infty (c_0^+ - c_0^-) = 0.$$

This gives (55). □

Proof of Theorem 2.3. For the sake of simplicity let us assume that $L > 0$ and $R > 0$ almost surely. Equation (7)-(8) can be written in integral form as

$$\tilde\rho_t = \tilde\rho_0 + \int_0^t [\tilde Q^+(\tilde\rho_s, \tilde\rho_s) - \tilde\rho_s] ds.$$

Observe that, if ρ is the law of X and X' and $F(x) = P\{|X| \le x\}$, one can write

$$\begin{aligned}\tilde Q^+(\rho,\rho)((-\infty,x]) &= P\{\max\{|LX|,|RX'|\} \le x\} = P\{|X| \le x/L, |X'| \le x/R\} \\ &= \mathbb{E}[F(x/L)F(x/R)].\end{aligned}$$

Hence, setting $\mathfrak{H}_t(x) = \tilde\rho_t((-\infty,x])$ and $\mathfrak{H}_0(x) = P\{|X_1| \le x\}$, we get that

$$\mathfrak{H}_t(x) = \mathfrak{H}_0(x) + \int_0^t \Big(\mathbb{E}[\mathfrak{H}_s(x/L)\mathfrak{H}_s(x/R)] - \mathfrak{H}_s(x)\Big) ds \tag{56}$$

is an equivalent formulation of (7)-(8). Now uniqueness follows by standard arguments. If $\mathfrak{H}_t'(x)$ is another solution, setting $d_t(x) = |\mathfrak{H}_t(x) - \mathfrak{H}_t'(x)|$ one immediately gets

$$d_t(x) \le d_0(x) + 3\int_0^t \sup_y d_s(y) ds.$$

Since $d_0 \equiv 0$, Gronwall's lemma (for locally bounded functions) gives $d_t = 0$. At this stage, setting $\tilde q_0(x) := P\{|X_1| \le x\}$ and

$$\tilde q_n(x) := \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}[\tilde q_i(x/L)\tilde q_{n-1-i}(x/R)] \qquad (n \ge 1), \tag{57}$$

by direct computations, or using the results in [24], one proves that

$$\mathfrak{H}_t(x) = \sum_{n \ge 1} e^{-t}(1-e^{-t})^{n-1}\tilde q_n(x)$$

is a solution of (56).

In order to complete the proof it remains to show that, for every $l \ge 1$,

$$\tilde q_{l-1}(x) = P\{\tilde H_l \le x\} \tag{58}$$

with $\tilde H_l := \max_{1 \le i \le l} |\beta_{jl} X_j|$. We shall prove (58) by induction on $l \ge 1$. First, note that $P\{\tilde H_1 \le x\} = P\{|X_1| \le x\} = \tilde q_0(x)$ and $P\{\tilde H_2 \le x\} = P\{\max(|LX_1|, |RX_2| \le x\} = \tilde q_1(x)$, which shows (58) for $l = 1$ and $l = 2$. Let $n \ge 3$, and assume that (58) holds for all $1 \le l < n$; we prove (58) for $l = n$. Recall that the weights β_{jn} are products of random variables L_i and R_i. Define the random index $K_n < n$ such that all products $\beta_{j,n}$s with $j \le K_n$ contain L_1 as a factor, while the $\beta_{j,n}$s with

$K_n+1 \le j \le n$ contain R_1. By induction it is easily seen that $P\{K_n = i\} = 1/(n-1)$ for $i = 1, \dots, n-1$. Now,

$$A_{K_n} := \max_{1\le j\le K_n} \frac{\beta_{jn}}{L_1}|X_j|, \quad B_{K_n} := \max_{K_n+1\le j\le n} \frac{\beta_{jn}}{R_1}|X_j| \quad \text{and} \quad (L_1, R_1)$$

are conditionally independent given K_n. By the recursive definition of the weights $\beta_{j,n}$ in (15), the following is easily deduced: the conditional distribution of A_{K_n}, given $\{K_n = k\}$, is the same as the (unconditional) distribution of $\tilde{H}_k = \max_{1\le j\le k} \beta_{j,k}|X_j|$. Analogously, the conditional distribution of B_{K_n}, given $\{K_n = k\}$, equals the distribution of $\tilde{H}_{n-k} \max_{1\le j\le n-k} \beta_{j\ n-k}|X_j|$. Hence,

$$\begin{aligned} P\{\tilde{H}_n \le x\} &= \frac{1}{n-1}\sum_{k=1}^{n-1} \mathbb{E}\big[P\{\max(|L_1A_k|, |R_1B_k|) \le x\big|\{K_n = k\}\}\big] \\ &= \frac{1}{n-1}\sum_{k=1}^{n-1} \mathbb{E}[P\{L_1\tilde{H}_k \le x|L_1, R_1\}P\{R_1\tilde{H}_{n-k} \le x|L_1, R_1]] \\ &= \frac{1}{n-1}\sum_{j=0}^{n-2} \mathbb{E}[\tilde{q}_{n-2-j}(x/L_1)\tilde{q}_j(x/R_1)] \end{aligned}$$

which is $\tilde{q}_{n-1}$ by the recursive definition in (57).

□

Proof of Theorem 2.4. We need to show that, for every x in $\mathbb{R}$,

$$\lim_{t\to+\infty} P\{e^{-\mu(\alpha)t}H_t \le x\} = P\{H_\infty \le x\}.$$

Let $x > 0$ and let $\mathcal{B}^*$ the σ-field generated by the array of weights $[\beta_{j,n}]_{j,n}$ and by the Yule process $[\nu_t]_{t\ge0}$. Then

$$\begin{aligned} P\{e^{-\mu(\alpha)t}H_t \le x\} &= \mathbb{E}\left[\prod_{j=1}^{\nu_t} P\{|e^{-\mu(\alpha)t}\beta_{j,\nu_t}X_j| \le x|\mathcal{B}^*\}\right] \\ &= \mathbb{E}\left[\prod_{j=1}^{\nu_t} \Big(1 - P\{|e^{-\mu(\alpha)t}\beta_{j,\nu_t}X_j| > x|\mathcal{B}^*\}\Big)\right] \\ &= \mathbb{E}\left[e^{-\frac{c_0}{x^\alpha}e^{-\mathcal{S}(\alpha)t}M_{\nu_t}(\alpha)} + \Lambda_t(x)\right] \end{aligned} \tag{59}$$

where

$$\Lambda_t(x) := \prod_{j=1}^{\nu_t}(1 - P\{|e^{-\mu(\alpha)t}\beta_{j,\nu_t}X_j| > x|\mathcal{B}^*\}) - \prod_{j=1}^{\nu_t} e^{-\frac{c_0}{x^\alpha}e^{-\mathcal{S}(\alpha)t}\beta_{j,\nu_t}^\alpha}.$$

By (23)

$$P\{|e^{-\mu(\alpha)t}\beta_{j,\nu_t}X_j| > x|\mathcal{B}^*\}) = \frac{c_0e^{-\mathcal{S}(\alpha)t}\beta_{j,\nu_t}^\alpha}{x^\alpha}\Big(1 + \mathcal{R}\Big(\frac{x}{e^{-\mu(\alpha)t}\beta_{j,\nu_t}}\Big)\Big). \tag{60}$$

Now recall that, given $2N$ complex numbers $a_1, \dots, a_N, b_1, \dots, b_N$ with $|a_i|, |b_i| \le 1$, $|\prod_{i=1}^N a_i - \prod_{i=1}^N b_i| \le \sum_{i=1}^N |a_i - b_i|$. Moreover, for every $x > 0$ and $1 \le r \le 2$ one has $|1-x-e^{-x}| \leq C_r|x|^r$. Combining these facts with (60) one gets

$$\begin{aligned}
|\Lambda_t(x)| &\leq \sum_{j=1}^{\nu_t} \Big| 1 - P\{|e^{-\mu(\alpha)t}\beta_{j,\nu_t}X_j| > x|\mathcal{B}^*\} - e^{-\frac{c_0}{x^\alpha}e^{-\mathcal{S}(\alpha)t}\beta^\alpha_{j,\nu_t}} \Big| \\
&= \sum_{j=1}^{\nu_t} \Big| 1 - \frac{c_0 e^{-\mathcal{S}(\alpha)t}\beta^\alpha_{j,\nu_t}}{x^\alpha}\Big(1+\mathcal{R}\Big(\frac{x}{e^{-\mu(\alpha)t}\beta_{j,\nu_t}}\Big)\Big) - e^{-\frac{c_0}{x^\alpha}e^{-\mathcal{S}(\alpha)t}\beta^\alpha_{j,\nu_t}} \Big| \\
&\leq C_r c_0^r e^{-r\mathcal{S}(\alpha)t} \sum_{j=1}^{\nu_t} \frac{\beta^{r\alpha}_{j,\nu_t}}{x^{r\alpha}} + \sum_{j=1}^{\nu_t} \frac{c_0 e^{-\mathcal{S}(\alpha)t}\beta^\alpha_{j,\nu_t}}{x^\alpha} \Big| \mathcal{R}\Big(\frac{x}{e^{-\mu(\alpha)t}\beta_{j,\nu_t}}\Big) \Big| \\
&\leq \frac{C_r c_0^r}{x^{r\alpha}} e^{-r\mathcal{S}(\alpha)t} M_{\nu_t}(r\alpha) + \frac{c_0}{x^\alpha} e^{-\mathcal{S}(\alpha)t} M_{\nu_t}(\alpha) \bar{\mathcal{R}}\Big(\frac{x}{e^{-\mu(\alpha)t}\beta_{(\nu_t)}}\Big) \\
&= \frac{C_r c_0^r}{x^{r\alpha}} e^{-r\alpha(\mu(\alpha)-\mu(r\alpha))t} e^{-\mathcal{S}(r\alpha)t} M_{\nu_t}(r\alpha) \\
&\quad + \frac{c_0}{x^\alpha} e^{-\mathcal{S}(\alpha)t} M_{\nu_t}(\alpha) \bar{\mathcal{R}}\Big(\frac{x}{e^{-\mu(\alpha)t}\beta_{(\nu_t)}}\Big)
\end{aligned}$$

Now choose $r = \delta/\alpha$ and notice that $r > 1$. Moreover, by the convexity of $\mathcal{S}(s)$, it is easy to see that $\mu(s) < \mu(\alpha)$ if $\alpha < s < \delta$. Hence, without loss of generality, we can suppose that $\alpha < \delta < 2$. Then, arguing as in the proof of (44) of Proposition 2, it is immediate to see that it also holds

$$\mathbb{E}[e^{-\mathcal{S}(r\alpha)t} M_{\nu_t}(r\alpha)] = 1.$$

Moreover, by assumption, $\mu(\alpha) - \mu(\delta) = \mu(\alpha) - \mu(r\alpha) > 0$, hence

$$\mathbb{E}[e^{-r\alpha(\mu(\alpha)-\mu(r\alpha))t} e^{-\mathcal{S}(r\alpha)t} M_{\nu_t}(r\alpha)] \to 0$$

when $t \to +\infty$. Combining (43) and (46) by the generalized dominated convergence theorem one gets also that

$$\mathbb{E}\left[e^{-\mathcal{S}(\alpha)t} M_{\nu_t}(\alpha) \bar{\mathcal{R}}\Big(\frac{x}{e^{-\mu(\alpha)t}\beta_{(\nu_t)}}\Big) \right] \to 0.$$

Hence $\mathbb{E}[\Lambda_t(x)] \to 0$ as $t \to +\infty$. Using once again (43) one gets

$$\mathbb{E}\left[e^{-\frac{c_0}{x^\alpha} e^{-\mathcal{S}(\alpha)t} M_{\nu_t}(\alpha)} \right] \to \mathbb{E}\left[e^{-\frac{c_0}{x^\alpha} Z_\infty(\alpha)} \right].$$

Plugging these last convergences in (59) one concludes the proof for $x > 0$. Since for $x < 0$ there is nothing to prove, let us assume that $x = 0$. By dominated convergence theorem it is easy to see that

$$\lim_{x \downarrow 0} \mathbb{E}[e^{-|x|^{-\alpha} Z_\infty(\alpha)}] = P\{Z_\infty(\alpha) = 0\}.$$

Hence, if $P\{Z_\infty(\alpha) = 0\} > 0$ there is nothing to be proved since 0 is a discontinuity point for $x \mapsto \mathbb{E}[e^{-|x|^{-\alpha} Z_\infty(\alpha)}] =: \mathfrak{H}_\infty(x)$. Assuming now $P\{Z_\infty(\alpha) = 0\} = 0$, one obtains that $\mathfrak{H}_\infty$ is continuous and that for every $\epsilon > 0$ there is $\eta = \eta(\epsilon)$ such that $\mathfrak{H}_\infty(\eta) \leq \epsilon$. So that

$$0 \leq \limsup_{t \to +\infty} P\{e^{-\mu(\alpha)t} H_t \leq 0\} \leq \limsup_{t \to +\infty} P\{e^{-\mu(\alpha)t} H_t \leq \eta\} = \mathfrak{H}_\infty(\eta) \leq \epsilon.$$

This proves that $\lim_{t \to +\infty} P\{e^{-\mu(\alpha)t} H_t \leq 0\} = 0$.

□

Proof of Theorems 3.1-3.2. Recalling that $\mathcal{B}$ denotes the σ–field generated by the array of random variables $[\beta_{jn}]_{jn}$, using (26) one can write

$$x_t^\alpha P\{|e^{-\mu(\alpha)t}V_t| > x_t\} = x_t^\alpha \mathbb{E}\Big[\sum_{n\geq 1}\mathbb{I}\{\nu_t = n\}P\Big\{|\sum_{j=1}^{n} e^{-\mu(\alpha)t}\beta_{j,n}X_j| > x_t|\mathcal{B}\Big\}\Big]$$
$$\geq B_t^{(0)} - B_t^{(1)}$$

where

$$B_t^{(0)} := e^{-\mathcal{S}(\alpha)t}\mathbb{E}\Big[\Delta_t\Big[\Big(1-\bar{\mathcal{R}}\Big(\frac{x_t(1+\epsilon)}{\beta(\nu_t)e^{-\mu(\alpha)t}}\Big)\Big)\vee 0\Big]\sum_{j=1}^{\nu_t}\beta_{j,\nu_t}^\alpha\Big]\frac{c_0}{(1+\epsilon)^\alpha}$$

$$B_t^{(1)} := e^{-2\mathcal{S}(\alpha)t}\frac{K_0^2}{x_t^\alpha(1+\epsilon)^{2\alpha}}\mathbb{E}\Big[(M_{\nu_t}(\alpha)^2\Big],$$

for every $\epsilon > 0$. Setting

$$D_t := \Big[\Big(1-\bar{\mathcal{R}}\Big(\frac{x_t(1+\epsilon)}{\beta(\nu_t)e^{-\mu(\alpha)t}}\Big)\Big)\vee 0\Big]M_{\nu_t}(\alpha)e^{-\mathcal{S}(\alpha)t}$$

one gets that for every $t > 0$

$$|D_t| \leq (1+||\mathcal{R}||_\infty)M_{\nu_t}(\alpha)e^{-\mathcal{S}(\alpha)t}$$

and by (43) $M_{\nu_t}(\alpha)e^{-\mathcal{S}(\alpha)t} \to Z_\infty(\alpha)$ in L^1. Furthermore $|\Delta_t| \leq 1$ and $\Delta_t \to 1$ in probability by Lemma 5.1. Finally by (46) and by (24), one gets

$$\bar{\mathcal{R}}\left(\frac{x_t(1+\epsilon)}{\beta(\nu_t)e^{-\mu(\alpha)t}}\right) \to 0$$

in probability. Combining these facts one obtains that

$$D_t \to Z_\infty(\alpha) \qquad \text{and} \qquad \Delta_t D_t \to Z_\infty(\alpha)$$

in probability for $t \to +\infty$ and, by the generalized dominated convergence theorem, that $\Delta_t D_t \to Z_\infty(\alpha)$ in L^1. Hence, in view of (44) one obtains

$$\lim_{t\to+\infty} B_t^{(0)} = \lim_{t\to+\infty}\frac{c_0}{1+\epsilon^\alpha}\mathbb{E}(\Delta_t D_t)$$
$$= \frac{c_0}{1+\epsilon^\alpha}\mathbb{E}\big(Z_\infty(\alpha)\big)$$
$$= \frac{c_0}{1+\epsilon^\alpha}.$$

As far as the term $B_t^{(1)}$ is concerned, using Lemma 5.3, one can write

$$\limsup_{t\to+\infty} B_t^{(1)} \leq C\limsup_{t\to+\infty}\frac{\tilde{h}(t)}{x_t^\alpha}$$

for a suitable constant C and $\tilde{h}(t)$ being defined as in the same lemma. Then, in view of the assumptions on x_t according to the expression of $\tilde{h}(t)$, it follows that $\limsup_{t\to+\infty} B_t^{(1)} = 0$. Hence, one gets

$$\liminf_{t\to+\infty} x_t^\alpha P\{|V_t| > x_t\} \geq \liminf_{t\to+\infty} B_t^{(0)} - \limsup_{t\to+\infty} B_t^{(1)} = \frac{c_0}{(1+\epsilon)^\alpha}$$

and then

$$\liminf_{t\to+\infty} x_t^\alpha P\{|V_t| > x_t\} \geq c_0. \tag{61}$$

On the other hand, applying (27), one gets

$$\begin{aligned} x_t^\alpha P\{|e^{-\nu(\alpha)t}V_t|>x_t\} \le \frac{c_0}{(1-\epsilon)^\alpha}\mathbb{E}\left[\left(1+\bar{\mathcal{R}}\left(\frac{x_t(1-\epsilon)}{\beta(\nu_t)e^{-\mu(\alpha)t}}\right)\right)e^{-\mathcal{S}(\alpha)t}M_{\nu_t}(\alpha)\right] \\ + \frac{2K_0}{\epsilon^2(2-\alpha)x_t^{(2-\alpha)(1-\gamma)}}\mathbb{E}\left[e^{-\mathcal{S}(\alpha)t}M_{\nu_t}(\alpha))\right] \\ + \left[\frac{K_0^2}{x_t^{\alpha(2\gamma-1)}}+\frac{K_1}{\epsilon^2 x_t^{2-\alpha+2(\alpha-1)\gamma}}\right]\mathbb{E}\left[\left(e^{-\mathcal{S}(\alpha)t}M_{\nu_t}(\alpha)\right)^2\right] \\ =: U_t^{(0)}+U_t^{(1)}+U_t^{(2)}. \end{aligned} \tag{62}$$

As in the previous part $U_t^{(0)} \to c_0/(1-\epsilon)^\alpha$. Moreover, since $(2-\alpha)(1-\gamma)>0$ for every $\gamma<1$ and $\mathbb{E}[e^{-\mathcal{S}(\alpha)t}M_{\nu_t}(\alpha)]=1$ by (44), one has $U_t^{(1)}\to 0$ for $t\to+\infty$. Finally, in view of Lemma 5.3, Remark 5 and the assumptions on x_t, according to the value of $\mathcal{S}(\alpha)$ and $\mathcal{S}(2\alpha)$, one can choose $1/2<\gamma<1$ in order that $U_t^{(2)}\to 0$ for $t\to+\infty$. Hence, $\limsup_{t\to+\infty} x_t^\alpha P\{|e^{-\mathcal{S}(\alpha)t}V_t|>x_t\}\le c_0/(1-\epsilon)^\alpha$ for every $\epsilon>0$ which implies

$$\limsup_{t\to+\infty} x_t^\alpha P\{|e^{-\mathcal{S}(\alpha)t}V_t|>x_t\}\le c_0. \tag{63}$$

In view of (61) and (63) we obtain

$$\lim_{t\to+\infty} x_t^\alpha P\{|e^{-\mathcal{S}(\alpha)t}V_t|>x_t\}= c_0. \tag{64}$$

In order to complete the proof of (17) it is sufficient to show that

$$\lim_{t\to+\infty} x_t^\alpha P\{|V_\infty|>x_t\}= c_0. \tag{65}$$

As already noted, by convexity of $\mathcal{S}$ and the condition $\mu(\delta)<\mu(\alpha)$, it follows that $\mu(s)<\mu(\alpha)$ if $\alpha<s<\delta$. Hence, without loss of generality, we can assume that $\alpha<\delta<2\alpha$.

Let $Z_\infty(\alpha)$ be as in Theorems 2.1 and 2.2. Then

$$V_\infty \stackrel{\mathcal{L}}{=} Z_\infty(\alpha)^{1/\alpha}S_\alpha$$

where S_α is a stable r.v. with index α, $Z_\infty(\alpha)$ and S_α being independent. If F_∞ and G_α denote the distribution functions of V_∞ and S_α, respectively, then

$$F_\infty(x)=\mathbb{E}\Big[G_\alpha(Z_\infty(\alpha)^{-1/\alpha}x)\mathbb{I}\{Z_\infty(\alpha)\neq 0\}.\Big]$$

Hence

$$P\{|V_\infty|>x\}=F_\infty(-x)+1-F_\infty(x)=:\frac{c_0}{x^\alpha}\mathbb{E}(Z_\infty(\alpha))+\zeta(x)=\frac{c_0}{x^\alpha}+\zeta(x) \tag{66}$$

since $\mathbb{E}(Z_\infty(\alpha))=1$ by (44). From the properties of the tails of stable distributions one can write that

$$\left|G_\alpha(-x)+1-G_\alpha(x)-\frac{c_0}{x^\alpha}\right|\le\frac{K}{x^\delta}$$

for $x>0$, since $\alpha<\delta<2\alpha$. See, e.g., [22]. Hence

$$\zeta(x)\le C\frac{\mathbb{E}(Z_\infty(\alpha))^{\delta/\alpha}}{x^\delta}$$

with $\mathbb{E}[Z_\infty(\alpha)^{\delta/\alpha}]<+\infty$ by (45).

To prove (18), use (28) to write

$$\tilde{B}_t^{(0)} - \tilde{B}_t^{(1)} \le x_t^\alpha P\{|e^{-\mu(\alpha)t} H_t| > x_t\} \le \tilde{U}_t^{(0)}$$

where

$$\tilde{B}_t^{(0)} := c_0 \mathbb{E}\Big[\Big(1 - \bar{\mathcal{R}}\Big(\frac{x_t}{\beta(\nu_t)e^{-\mu(\alpha)t}}\Big)\Big)e^{-\mathcal{S}(\alpha)t} M_{\nu_t}(\alpha)\Big]$$

$$\tilde{B}_t^{(1)} := \frac{K_0^2}{x_t^\alpha}\mathbb{E}\Big[\big(e^{-\mathcal{S}(\alpha)t} M_{\nu_t}(\alpha)\big)^2\Big]$$

$$\tilde{U}_t^{(0)} := c_0 \mathbb{E}\left[\left(1 + \bar{\mathcal{R}}\left(\frac{x_t}{\beta(\nu_t)e^{-\mu(\alpha)t}}\right)\right) e^{-\mathcal{S}(\alpha)t} M_{\nu_t}(\alpha)\right]$$

Arguing as before, one proves that $\tilde{B}_t^{(0)} \to c_0$, $\tilde{B}_t^{(1)} \to 0$ and $\tilde{U}_t^{(0)} \to c_0$ and this completes the proof. □

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