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Asymptotic results for the Fourier estimator of the integrated quarticity

Giulia Livieri · Maria Elvira Mancino · Stefano Marmi

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Abstract In this paper we prove a central limit theorem for an estimator of the integrated quarticity based on Fourier analysis, strictly related to the one proposed in [Mancino and Sanfelici, 2012]. Also, a consistency result is derived. We show that the estimator reaches the parametric rate $\rho(n)^{1/2}$, where $\rho(n)$ is the discretization mesh and n the number of points of such discretization. The optimal variance is obtained, with a suitable choice of the number of frequencies employed to compute the Fourier coefficients of the volatility, while the limiting distribution has a bias. As a by-product, thanks to the Fourier methodology, we obtain consistent estimators of any even power of the volatility function as well as an estimator of the spot quarticity. We assess the finite sample performance of the Fourier quarticity estimator in a numerical simulation.

Keywords (powers of) volatility estimation · quarticity · central limit theorem · Fourier analysis · high frequency data.

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Giulia Livieri
Scuola Normale Superiore, Piazza dei Cavalieri, 7, 56126, Pisa, Italy
Tel.: +39 050 509259
E-mail: giulia.livieri@sns.it

Maria Elvira Mancino
Dept. Economics and Management, University of Firenze, Via delle Pandette, 9, 50127, Firenze, Italy
Tel.: +39 055 4374660
E-mail: mariaelvira.mancino@unifi.it

Stefano Marmi
Scuola Normale Superiore, Piazza dei Cavalieri, 7, 56126, Pisa, Italy
Tel.: +39 050 509064
E-mail: stefano.marmi@sns.it

1 Introduction

Over the past two decades many scholars have worked in the theory and application of volatility and volatility functionals measurement from high-frequency data. In particular, the (ex-post) measure of the second power of the volatility has been the focus of much attention, both in its spot (i.e., spot variance) and integrated (i.e., integrated variance) form. Nonetheless, the estimation of the fourth power of the volatility turns out to be fundamental as well because it is needed in order to produce feasible central limit theorems for all these estimators. Integrated fourth power of the volatility, dubbed *integrated quarticity* or, simply, *quarticity*, is heavily employed in practice to estimate the confidence intervals for integrated variance estimate [Andersen et al., 2014], to apply jump tests [Dumitru and Urga, 2012], to forecast volatility [Bollerslev et al., 2016] and to compute the optimal sampling frequency in the presence of market microstructure noise [Bandi and Russel, 2006]. The instantaneous fourth power of the volatility, dubbed *spot quarticity*, mainly appears in the limiting distribution of spot volatility estimators, such as in [Ogawa and Sanfelici, 2011, Zu and Boswijk, 2014], but it is still less investigated.

Since the efficient estimation of quarticity turns out to be a sensitive issue as pointed out by [Barndorff-Nielsen et al., 2008a], not so many works are exclusively focused on this topic, at least if compared to those focusing on the estimation of variance and integrated variance. A consistent estimator of quarticity – in the absence of microstructure noise – has been proposed in [Barndorff-Nielsen and Shephard, 2004] and [Barndorff-Nielsen et al., 2006]. Always in absence of microstructure noise, we mention the quarticity estimator proposed by [Mykland, 2012] which is strongly related to the one of [Barndorff-Nielsen and Shephard, 2002]. [Andersen et al., 2014] propose two (jump-robust) estimators of integrated quarticity which take the minimum or the median of two and three adjacent returns, respectively, in order to reduce the influence of jumps. These estimators perform very well in the presence of jumps, but are less efficient for diffusive processes. [Jacod and Rosenbaum, 2013, Jacod and Rosenbaum, 2015], instead, propose a general framework for the estimation of integrals of *smooth* functions of the volatility in the absence of microstructure frictions, to which the quarticity estimator belongs too. They also provide an unbiased central limit theorem for the latter estimator, which is efficient in the sense of the Hajek convolution theorem, (see [Clement et al., 2013]). More precisely, they show the optimality of the estimator both in terms of rate, that is $\rho(n)^{1/2}$, with $\rho(n)$ the (regular) discretization mesh and n the number of points of such a discretization, and of asymptotic variance, that is equal to $8\int_0^t \sigma^8(s)\,ds$. A closely related estimator is the one recently proposed by [Kolokolov and Renó, 2018], defined as a suitable linear combination of a *fixed* number m of multi-power estimators. This estimator coincides with the one of [Jacod and Rosenbaum, 2013] when m diverges to infinity but presents a smaller limiting variance when m is fixed. [Mancino and Sanfelici, 2012] develop a methodology based on Fourier analysis to estimate both spot and integrated quarticity. The authors prove the consistency of the quarticity estimator in the absence of noise and analyse its finite-sample efficiency, both theoretically and empirically, in the presence of microstructure noise. Indeed, they construct an optimal Mean Squared Error (hereafter, MSE) based Fourier quarticity estimator, which makes the methodology feasible with real data. However, the rate of convergence of the Fourier estimator of quarticity is not studied.

In the present work, we propose an estimator of the integrated quarticity based on Fourier methodology, strictly related to the previous one, and we study its asymptotic properties in the absence of microstructure noise. The properties of the estimator largely depend on two cutting frequencies: the first one, N, refers to the number of Fourier frequencies of the asset returns needed to estimate the Fourier coefficients of the variance, while the second one, M, controls the product formula between the Fourier coefficients of the variance and contributes to increase the precision of the quarticity estimator obtained by simply taking the square of the Fourier estimator of integrated variance. Under suitable asymptotic conditions on these two cutting frequencies, we prove that the proposed estimator converges (stably) in law to a non-centered normal random variable with optimal asymptotic variance equal to $8 \int_0^t \sigma^8(s)\, ds$, when the Nyquist frequency is chosen for N, and the rate of convergence is equal to $\rho(n)^{1/2}$. The bias term of the asymptotic distribution is in line with the one found by [Jacod and Rosenbaum, 2015], and depends on the volatility of volatility, on a statistical error term, and on the choice of the parameter M. Notice that it is not surprising to find such a rate of convergence in the absence of microstructure noise since, by definition, the Fourier quarticity estimator relies on the estimated Fourier coefficients of the integrated variance and the product formula. Indeed, in a non-parametric setting for integrated functionals estimation, the rate of convergence is usually $\rho(n)^{1/2}$ (see, e.g., [Bickel and Ritov, 2003]).

We confirm the theoretical findings in a realistic Monte Carlo experiment. Moreover, we make a comparative study of the performance of the Fourier quarticity estimator with that of some estimators used in the literature, extending the analysis performed in [Mancino and Sanfelici, 2012] by considering the case of non-regular sampling and with different market microstructure frictions, such as auto-correlated market microstructure noise.

The paper is organized as follows. Section 2 presents the model and the setting. In Section 3 the Fourier estimator of the quarticity is defined and the main theorems are stated. Section 4 contains the simulation results, while Section 5 concludes. Finally, the proof of the main theorems are in the Appendix 6, whereas the Appendix 7 summarizes some properties of the Dirichlet kernel.

2 Setting and notation

In this section, we recall the main results of the Fourier estimation approach by [Malliavin and Mancino, 2002]. Let $(p\,(t))_{t\geq 0}$ be the logarithmic price process, which is observed at time t over a fixed time time period $[0, T]$, with $T < \infty$ (e.g., a trading day). We assume that

(A.I) p is an Itô process satisfying

$$dp(t) = \sigma(t)\, dW(t) + b(t)\, dt,$$

where W is a Brownian motion on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ satisfying the usual conditions, and σ and b are adapted stochastic processes such that

$$\mathbb{E}\left[\int_0^T \sigma^8(t)dt\right] < \infty,\ \mathbb{E}\left[\int_0^T b^4(t)dt\right] < \infty.$$

We want to compute the fourth power of the volatility function, i.e., σ^4. Towards this aim, we use a methodology which allows to compute the Fourier coefficients of the latter from the asset prices observations only and therefore to recover its Fourier expansion. By a change of the origin of time and rescaling the unit of time we can always reduce ourselves to the case where the time window $[0, T]$ becomes $[0, 2\pi]$.

First, for any integer k, define the *Fourier coefficient* of the log-return dp as

$$\mathcal{F}(dp)(k) \doteq \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}kt}\, dp(t)$$

and the Fourier coefficients of the function σ^2 as

$$\mathcal{F}(\sigma^2)(k) \doteq \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}kt}\sigma^2(t)dt.$$

The first step of the Fourier method consists in the computation of $\mathcal{F}(\sigma^2)(k)$. The latter is obtained by means of a convolution product as in the following result.

Theorem 1 *Suppose that the log-price p satisfies assumption* (A.I)*. Then, the following convergence in probability holds*

$$\mathcal{F}(\sigma^2)(k) = \lim_{N\to\infty} \frac{2\pi}{2N+1} \sum_{|s|\le N} \mathcal{F}(dp)(s)\, \mathcal{F}(dp)(k-s), \quad \textit{for all } k \in \mathbb{Z}. \quad (1)$$

Hereafter, for simplicity, we assume that b is equal to zero since the drift term gives no contribution to the estimation of the Fourier coefficients of the volatility (see [Malliavin and Mancino, 2009] for the proof).

The second step consists in the computation of the k-th Fourier coefficient of σ^4. To this end, the product formula of Fourier series can be applied and the k-th Fourier coefficient of the stochastic function σ^4 is obtained as the following limit

$$\mathcal{F}(\sigma^4)(k) = \lim_{M\to\infty} \sum_{|s|\le M} \mathcal{F}(\sigma^2)(s)\, \mathcal{F}(\sigma^2)(k-s). \quad (2)$$

Finally, the fourth power of the volatility function can be reconstructed by means of its Fourier coefficients (2) as the following limit in probability

$$\sigma^4(t) = \lim_{N\to\infty} \sum_{|k|\le N} \mathcal{F}(\sigma^4)(k)\, e^{\mathrm{i}kt} \quad \text{for all } t \in (0, 2\pi). \tag{3}$$

Remark 1 The Fourier methodology presented in this section can be easily adapted to obtain unbiased estimators of higher power functions of volatility. Indeed, by using as building blocks the Fourier coefficients of the variance function and the product formula, we can compute the Fourier coefficients of any positive even power of the volatility function. In particular, the following formula holds

$$\mathcal{F}(\sigma^{2(p+q)})(k) = \lim_{M\to\infty} \sum_{|s|\le M} \mathcal{F}(\sigma^{2p})(s)\, \mathcal{F}(\sigma^{2q})(k-s) \tag{4}$$

for any $p, q \ge 1$. In particular, if $p = q = 1$ we get the $k-th$ Fourier coefficient of the fourth power.

Remark 2 The integrated quarticity can be obtained from (2) by setting $k = 0$. Indeed, it holds that

$$\int_0^{2\pi} \sigma^4(t)dt = 2\pi\mathcal{F}(\sigma^4)(0).$$

We stress that in order to compute the integrated fourth power of volatility function the datum of the integrated volatility (the 0^{th} Fourier coefficient) is not sufficient, but *all* the Fourier coefficients of the volatility are needed. Nonetheless, the knowledge of the instantaneous volatility is not required.

3 Main Results

Suppose that the asset log-price p, satisfying assumption (A.I), is observed at discrete, irregularly spaced points in time: $\{0 = t_{0,n} \le \dots t_{i,n} \dots \le t_{n,n} = 2\pi\}$. In what follows, for sake of simplicity we will omit the second index n. Denote $\rho(n) \doteq \max_{0\le h\le n-1} |t_{h+1} - t_h|$ and suppose that $\rho(n) \to 0$ as $n \to \infty$. Consider the following interpolation formula

$$p_n(t) \doteq \sum_{i=0}^{n-1} p(t_i) I_{[t_i, t_{i+1}[}(t)$$

and denote the returns by $\delta_i(p) \doteq p(t_{i+1}) - p(t_i)$. For any integer k, $|k| \le 2N$, set

$$c_k(dp_n) \doteq \frac{1}{2\pi} \sum_{i=0}^{n-1} e^{-\mathrm{i}kt_i} \delta_i(p). \tag{5}$$

For any $|k| \le N$, define

$$c_k(\sigma^2_{n,N}) \doteq \frac{2\pi}{2N+1} \sum_{|s|\le N} c_s(dp_n) c_{k-s}(dp_n) \tag{6}$$

as the discrete counterpart of (1). We are interested in the 0^{th} Fourier coefficient of σ^4. Thus, relying on (2), the *Fourier estimator of quarticity* can be defined by

$$\widehat{\sigma}^4_{n,N,M} \doteq 2\pi \sum_{|k|\leq M} c_k(\sigma^2_{n,N}) c_{-k}(\sigma^2_{n,N}), \tag{7}$$

where the $c_k(\sigma^2_{n,N})$, $|k| \leq M$ are explicit functions of the log-returns $\delta_i(p)$ $(i = 1, \dots, n)$ as defined by (6).

Remark 3 Note that the definition (7) uses only integrated quantities, that is the pre-estimated Fourier coefficients of the variance. This property ensures some computational advantages due to the numerical instability of the spot variance estimators, see also Section 4.

3.1 Consistency

In this section, we prove the consistency of the estimator defined in (7). Precisely, we prove that the latter converges in probability to the quarticity under the following conditions relating the number of data and the cutting frequencies N and M: $N/n \to c_N > 0$ and $M\,\rho\,(n)^{1/2} \to c_M > 0$[1]. In particular, the first condition ensures that (6) is a consistent and efficient estimator of the Fourier coefficients of the volatility (see, e.g. [Mancino et al., 2017] Chapter 3 and references therein), while the second one is necessary for the convergence of the Fourier series resulting from the product formula (2).

In what follows, we suppose that the process $\left(\sigma^2(t)\right)_{t\geq 0}$ satisfies to the following assumptions:

(A.II) $\operatorname{ess\,sup} \|\sigma^2\|_{L^\infty} < \infty$, where $\|\sigma^2\|_{L^\infty} \doteq \sup_t |\sigma^2(t)|$,

(A.III) the function $\sigma^2(t)$ is almost surely α-Hölder continuous with $\alpha \in (0, 1]$ in $[0, 2\pi]$.

We assume a regular sampling as in [Jacod and Rosenbaum, 2013] which makes easier the comparison between the methods. The result can be extended to the case of irregularly spaced points in time or random times independent of the price and volatility process.

Theorem 2 *Under the assumptions* (A.I) − (A.II) − (A.III) *and the parameters condition $N/n \to c_N > 0$ and $M\rho(n)^{1/2} \to c_M > 0$, it holds in probability*

$$\lim_{n,N,M\to\infty} \widehat{\sigma}^4_{n,N,M} = \int_0^{2\pi} \sigma^4(t)dt.$$

Remark 4 The condition relating the number of data and the cutting frequency N assumed in [Mancino and Sanfelici, 2012], i.e., $N/n \to 0$ does not apply here. Indeed, the main interest of [Mancino and Sanfelici, 2012] is the efficiency of the Fourier estimator in the presence of microstructure noise.

[1] We stress the point that c_N and c_M are two positive constants, the notation being in line with that used in [Mancino et al., 2017].

3.2 Asymptotic Normality

In this section, we study the asymptotic error distribution for the Fourier estimator of quarticity defined by (7). In the following we assume that

(A.III)$'$ the variance process σ^2 is the Itô process described by

$$d\sigma^2(t) = \beta(t)dt + \gamma(t)dZ(t),$$

where Z is a Brownian motion, possibly correlated with W, β and γ are continuous adapted stochastic process,

(A.IV) the volatility of volatility process γ is essentially bounded and is almost surely ν-Hölder continuous with $\nu \in (0,1]$ in $[0,2\pi]$.

For instance, a model with stochastic volatility of volatility, i.e., a model in which the volatility of the variance process is driven by a second source of randomness as in [Barndorff-Nielsen and Veraart, 2013, Sanfelici et al., 2015] could be considered.

Hereafter, for simplicity, we assume that β is equal to zero. In fact, with a similar argument as in [Malliavin and Mancino, 2009], it can be shown that the drift term gives no contribution to the estimation of the Fourier coefficients of the volatility fourth power.

Theorem 3 *Under the assumptions* (A.I) $-$ (A.II) $-$ (A.III)$'$ $-$ (A.IV), $N/n \to c_N > 0$ *and* $M\rho(n)^{1/2} \to c_M > 0$, *then the following convergence holds stably in law*

$$\rho(n)^{-1/2}\left(\widehat{\sigma}^4_{n,N,M} - \int_0^{2\pi} \sigma^4(t)dt\right) \to X_{c_M} + Y_{c_M,c_N} + Z_{c_N},$$

where

$$X_{c_M} \doteq -\frac{1}{c_M}\frac{1}{\pi}\left(\int_0^{2\pi} \gamma^2(t)dt + \left(\sigma^2(2\pi) - \sigma^2(0)\right)^2\right),$$

$$Y_{c_M,c_N} \doteq 2c_M\frac{1}{\pi}(1+2\eta(c_N))\int_0^{2\pi} \sigma^4(t)dt, \qquad (8)$$

and $\eta(c_N) = (1/4c_N^2)r(2c_N)(1-r(2c_N))$, *where* $r(x) = x - [x]$ *with* $[x]$ *the integer part of* x. Z_{c_N} *is a (conditionally) Gaussian r.v. with zero mean and variance* $8(1+2\eta(c_N))\int_0^{2\pi}\sigma^8(t)dt$.

Remark 5 A priori, the asymptotic variance depends also on c_M. However, it is worth noting that the asymptotic variance in Theorem 3 depends only on the parameter c_N, which relates the number of observations to the Fourier frequencies needed to estimate the Fourier coefficients of the squared volatility. In particular, the estimator has the optimal asymptotic variance, $8\int_0^{2\pi}\sigma^8(t)\,dt$, if one chooses the so called Nyquist frequency, which is known to be the optimal choice to estimate the volatility in the absence of noise, see [Mancino et al., 2017]. By definition, $\eta(c_N) = 0$ if one sets $c_N = (1/2)m$, with m a positive integer, and $\eta(c_N) > 0$ otherwise. This latter observation suggests the choice

$c_N = (1/2)m$ in Theorem 3, which leads to the optimal asymptotic variance. On the other hand, it is known that N must be chosen less or equal to the Nyquist frequency $n/2$ in order to avoid aliasing effects, which leads to the choice of $c_N = 1/2$ as the most suitable here. However, the choice of the frequency $N << n/2$ makes the Fourier quarticity estimator efficient in the presence of noise (see [Mancino and Sanfelici, 2012] and Section 4).

Remark 6 Concerning the parameter c_M, it plays a crucial role in controlling the two bias terms Y_{c_M,c_N} and X_{c_M}. More precisely, a choice of c_M close to 0 would make the bias Y_{c_M,c_N} vanishing, but the bias X_{c_M} exploding, as it happens for the (un-corrected) estimator of quarticity by [Jacod and Rosenbaum, 2015]. Notice that, if the Nyquist frequency is chosen, then the bias term Y_{c_M,c_N} coincides with the bias term A^2 of the (un-corrected) estimator of quarticity by [Jacod and Rosenbaum, 2015], letting $\pi/c_M = \theta$. The bias term X_{c_M}, instead, depends on a boundary term and on the integrated volatility of variance, similarly as [Jacod and Rosenbaum, 2015]. In particular, we could proceed with a bias correction of the Fourier estimator of quarticity by using a consistent estimator of the integrated volatility of variance appearing in X_{c_M}. This can be obtained with its Fourier estimator, which has been studied in [Barucci and Mancino, 2010,Sanfelici et al., 2015]. However, the simulation study conducted in Section 4 shows that the performance of the Fourier estimator (7) without any correction is quite satisfactory in comparison with other estimators and even better in the presence of microstructure noise.

Finally, notice that (4) would produce (with the choice $p = q = 2$) a consistent estimator of the integrated eighth power of volatility, that is the asymptotic variance in Theorem 3. This result plays a role if one wants to obtain a feasible version of the central limit theorem (Theorem 3).

4 Simulation results

In this section, we assess via a detailed and realistic Monte Carlo simulation the finite sample performance of the Fourier quarticity estimator, whose asymptotic properties have been investigated in Section 3. We illustrate such analysis in the case of the following model for log-price $p(t)$ and instantaneous log-variance $\log \sigma^2(t)$:

$$\begin{aligned} dp(t) &= c_\sigma\, \eta(t)\, \sigma(t)\, dW(t) + \mu\, dt \\ d\log\, \sigma^2(t) &= \gamma\, dZ(t) + (\alpha - \beta\, \log\, \sigma^2(t))\, dt, \end{aligned} \tag{9}$$

where W and Z are two correlated Brownian motions with correlation ρ. For the values of the parameters (i.e., α, β, γ, μ and ρ), we use those estimated by [Andersen et al., 2002] on S&P500 (parameters are expressed in daily units and returns are in percentage). The volatility factor $c_\sigma = 2$ corresponds to (roughly) a daily volatility of 1% and the process $\eta(t)$ is intended to capture the U-shaped pattern of the intraday volatility. The latter coincides with the process estimated on S&P500 by [Kolokolov and Renó, 2018]. Numerical integration of the stochastic volatility model (9) is performed on a one-second time grid via a standard Euler scheme and with initial conditions $p(0) = \log 100$ and $\log \sigma^2(0) = \alpha/\beta$. Once simulated, the efficient prices are sampled every minute, i.e., we have a total of $n = 390$ intraday returns.

In the first part, we numerically confirm the asymptotic result provided in Theorem 3. Precisely, we display in Figure 1 the histograms of the following error

$$\rho(n)^{-1/2} \frac{\widehat{\sigma}^4_{n,N,M} - \int_0^{2\pi} \sigma^4(t)\, dt}{\sqrt{8 \int_0^{2\pi} \sigma^8(t)\, dt}}.$$

We remind that the Fourier quarticity estimator is characterized by the cutting frequencies N and M, which satisfy the following two assumptions $N/n \to c_N > 0$ and $M\rho(n)^{1/2} \to c_M > 0$, and that in our simulation exercise $n = 390$. As discussed in Remark 5, we choose $c_N = 1/2$, that is $N = n/2$, and we vary c_M in such a way that M belongs to the following set of values $\{1, 2, 3, 5, 10, 15\}$, i.e., $c_M \in \{0.12, 0.25, 0.38, 0.63, 1.26, 1.90\}$. Histograms are computed on a basis of 10,000 replications. The plots confirm the discussion in Remark 6. More precisely, c_M can be chosen to trade-off the two bias terms X_{c_M} and $Y_{c_M,1/2}$. When c_M is small, the term that contributes most to the bias is X_{c_M}, whereas the bias caused by $Y_{c_M,1/2}$ (notice that $\eta(c_N) = 0$ with $c_N = 1/2$) is almost negligible. On the other hand, if we increase c_M the bias induced by X_{c_M} becomes negligible, whereas $Y_{c_M,1/2}$ contributes to the bias the most. Accordingly, the resulting histograms shift to the right.

In the second part, we numerically examine the performance of the Fourier estimator when no microstructure frictions are present, with a non-regular price sampling and with correlated market microstructure noise, by extending the numerical analysis carried out in [Mancino and Sanfelici, 2012]. We also compare them with those of some estimators used in the literature. To evaluate the impact of microstructure noise on these estimators, we contaminate the sampled efficient price with an additive noise of the form: $\epsilon_{j,n} = \rho_\epsilon \epsilon_{j-1,n} + \epsilon^*_{j,n}$, where $\epsilon^*_{j,n} \sim \mathcal{N}(0, \sigma^2_\epsilon)$, $\rho_\epsilon = 0.5$ and $\sigma^2_\epsilon = 0.0005\,\mathrm{IV}$, with IV denoting the daily integrated variance, see also the value proposed in [Podolskij and Vetter, 2009]. Concerning the irregular sampling, instead, we assume that at each time instant the observed return can be either zero or the efficient one depending on the outcome of a Bernoulli random variable having probability of success equal to 0.3, as in [Kolokolov and Renó, 2018]. To compare the performance, we implement the following estimators.

i) The Fourier estimator of quarticity proposed in [Mancino and Sanfelici, 2012]. The latter is defined in the following way

$$\tilde{\sigma}^4_{n,N,M} \doteq 2\pi \sum_{|s|<M} \left(1 - \frac{|s|}{M}\right) c_s\left(\sigma^2_{n,N}\right) c_{-s}\left(\sigma^2_{n,N}\right), \tag{10}$$

i.e., is obtained from (7) with the choice of the Fourier-Fejer summation by adding a Barlett kernel, which improves the behaviour of the estimator

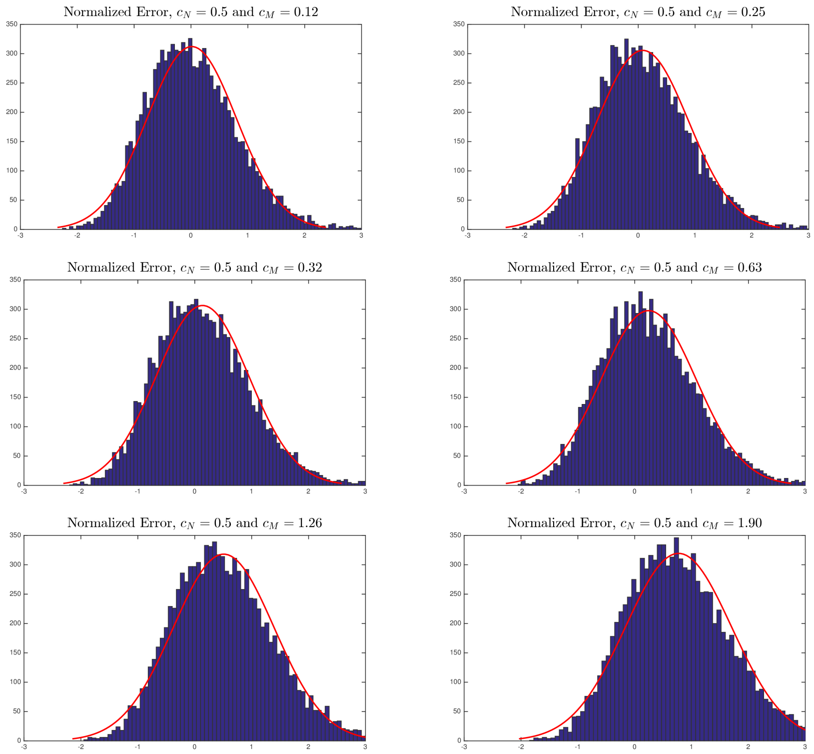

Fig. 1 **Limiting Distribution**: Figure represents the distribution of the normalized error in 4 for different choices of c_N and c_M. The distribution is computed with 10,000 replications of $n = 390$ intraday returns.

for very high observation frequencies. In what follows, we denote with $\widetilde{N}$, $\widetilde{M}$ the parameters relative to the estimator in (10).

ii) The non-truncated multi-power variation of [Barndorff-Nielsen and Shephard, 2004], defined by

$$\mathrm{MPV}([1,1,1,1]) \doteq c_{\mathbf{r}} \sum_{i=1}^{n-4+1} |\delta_i(p)||\delta_{i+1}(p)||\delta_{i+2}(p)||\delta_{i+3}(p)|,$$

where $c_{\mathbf{r}}$ is a constant meant to make the estimator unbiased in small samples under the assumption of constant volatility. Notice that, in a general setting, we may define the estimator $\mathrm{MPV}(\mathbf{r}) \doteq c_{\mathbf{r}} \sum_{i=1}^{n-m+1} \left(\prod_{j=1}^{m} |\delta_{i+j-1}(p) \right)^{r_j}$, where $\mathbf{r} = [r_1, \ldots, r_m]$, $R \doteq \sum_{j=1}^{m} r_j$ equals to the power of the volatility that we are estimating and

$$c_{\mathbf{r}} \doteq \left(\frac{n}{T}\right)^{\frac{R}{2}-1} \frac{n}{n-(m-1)} \left(\prod_{j=1}^{m} (\mu_{r_j})^{-1} \right),$$

where $\mu_r = \mathbb{E}[|u|^r]$ with u being a standard normal.

iii) The min- and med-RQ estimators proposed in [Andersen et al., 2014a] and [Andersen et al., 2014], defined by

$$\mathrm{minRQ} \doteq \frac{\pi}{3\pi - 8}\frac{n^2}{n-1}\sum_{i=1}^{n-1}\min(|\delta_i(p)|, |\delta_{i+1}(p)|)^4\,,$$

$$\mathrm{medRQ} \doteq \frac{3\pi}{9\pi + 72 - 52\sqrt{3}}\frac{n^2}{n-2}\sum_{i=1}^{n-2}\mathrm{med}\left(|\delta_i(p)|, |\delta_{i+1}(p)|, |\delta_{i+2}(p)|\right)^4.$$

iv) The estimator of [Jacod and Rosenbaum, 2013] implemented with, $\mathrm{QVeff}(k_n)$, and without, $\mathrm{QVeffB}(k_n)^2$, the bias correction proposed in [Jacod and Rosenbaum, 2015] and defined by

$$\mathrm{QVeffB}(k_n) \doteq \varDelta_n(1 - \frac{2}{k_n})\sum_{i=1}^{[1/\varDelta_n]-k_n+1}((\widehat{\sigma^2})_i^n)^2,$$

$$\mathrm{QVeff}(k_n) \doteq \varDelta_n(1 - \frac{2}{k_n})\sum_{i=1}^{[1/\varDelta_n]-k_n+1}((\widehat{\sigma^2})_i^n)^2 + (k_n - 1)\varDelta_n\frac{(\sigma^4(0) + \sigma^4(1))}{2}.$$

where

$$(\widehat{\sigma^2})_i^n \doteq \frac{1}{k_n\varDelta_n}\sum_{j=0}^{k_n-1}(\delta_{i+j}(p))^2.$$

v) The efficient multi-power estimator of [Kolokolov and Renó, 2018] which is defined as a (suitable) linear combination of multi-power estimators, denoted, by using authors' notation, with $\mathrm{GTMPV}(m)$, where m indicates both the *fixed* number of multi-power estimators involved in the linear combination and consecutive returns. It is defined as

$$\mathrm{GTMPV(m)} \doteq \frac{3}{2m+1}\mathrm{MPV}([4]) + \frac{2}{2m+1}\sum_{j=0}^{m-2}\mathrm{MPV}\left([2, \underbrace{0, \ldots, 0}_{j \text{ terms}}, 2]\right).$$

Some clarifications are in order. The estimators $\mathrm{QVeff}(k_n)$, $\mathrm{QVeffB}(k_n)$ and $\mathrm{GTMPV}(m)$ are characterized by the choice variables k_n and m respectively, and in order to compare their performance we fix $k_n = m + 1$. To fix m, and therefore k_n, in case of no market microstructure frictions, we use the recipe proposed in [Kolokolov and Renó, 2018] (3.11), by choosing m^* that minimizes the following quantity

$$\widetilde{\mathrm{MSE}}(\mathrm{GTMPV}(m)) \doteq [\mathrm{GTMPV}(m) - \hat{Q}]^2 + Var(\mathrm{GTMPV}(m)),$$

[2] The index k_n denotes the length of the window used to estimate the spot volatility.

where $\hat{Q}$ is any unbiased estimator of the quarticity (e.g., any multipower, nearest neighbour and Fourier estimator) and the variance of $Var(\text{GTMPV}(m))$ is substituted by the following approximation

$$\left(8 + \frac{8}{2m+1}\right) \text{MPV}([8/3, 8/3, 8/3]).$$

On the other hand, the Fourier estimators are characterized by the frequencies N and M. Regarding this choice, in case of no frictions, we use the recipe provided by [Mancino and Sanfelici, 2012]. More precisely, we use the upper bound provided by Corollary 3.3. Finally, when the efficient price is contaminated with market micro-structure frictions we select N, M, m and k_n by minimizing the corresponding infeasible MSE, which is defined as

$$\text{MSE}(\widehat{\sigma}^4_{\text{INT}}) = \mathbb{E}\left[\left(\widehat{\sigma}^4_{\text{INT}} - \sigma^4_{\text{INT}}\right)^2\right],$$

where $\widehat{\sigma}^4_{\text{INT}}$ is any estimator of the integrated quarticity whereas σ^4_{INT} is the true integrated quarticity. The selected values for M, N, $\widetilde{M}$, $\widetilde{N}$ and m are the following. In case of no frictions, both c_N and $c_{\widetilde{N}}$ are equal to 0.5 (i.e., $N^* = \widetilde{N}^* = n/2 = 195$), $c_M = c_{\widetilde{M}} = 0.25$ (i.e., $M^* = 2$) and $m^* = 100$. In the case of no microstructure noise but irregular sampling, we select again the Nyquist frequency $c_N = c_{\widetilde{N}} = 0.5$ (i.e., $N^* = \widetilde{N}^* = n/2 = 195$), $c_M = c_{\widetilde{M}} = 0.12$ (i.e., $M^* = 1$), according [Kolokolov and Renó, 2018] we choose $m^* = 5$. In case of autocorrelated market microstructure noise $c_N = 0.14$ (i.e., $N^* \approx n/7 = 58$), $c_M = 0.12$ (i.e., $M^* = 1$), $m^* = 100$, and $c_{\tilde{N}} = 0.17$ (i.e., $\tilde{N}^* \approx n/5.5 = 70$, $c_{\widetilde{M}} = 0.25$ (i.e., $\widetilde{M}^* = 2$). Table 1 summarises the results.

In both cases of no-frictions, the performance of the Fourier estimator in better than both the multi-power and the min-med estimators (in terms of RMSE) and in line with the corrected estimator of [Jacod and Rosenbaum, 2015].

The contamination of the efficient price with (autocorrelated) market microstructure noise induces a distortion in all the estimates, in form of an upper bias ranging from +11% ($\hat{\sigma}^4_{n,N,M}$ and $\tilde{\sigma}^4_{n,N,M}$) to (roughly) +90% (MPV([1, 1, 1, 1])). Notice that, although no estimator is corrected for the presence of market micro-structure noise, the increase in the BIAS of the Fourier estimators is approximately two times less than that of QVeffB(k_n^*) and eight times less than all the other estimators, because for the latter the noise bias is preponderant with respect to the bias of the original estimator. The observed variance is, instead, slightly greater than that of QVeff(k_n^*), QVeffB(k_n^*) and GTMPV(m^*). The fact that estimators based on Fourier methodology are robust in the presence of microstructure noise (by a suitable cutting of the highest frequency) has already been documented so far (see [Mancino and Sanfelici, 2008, Mancino and Sanfelici, 2012] and references therein). Notice that the Fourier estimator has the smallest RMSE among the estimators in the case of irregular sampling. We interpret this result as being symptomatic of the good performance of the Fourier-type estimators to irregular sampling, see also

Table 1 Table reports the relative root mean square error (RMSE), the standard deviation (STD), and the bias (BIAS) computed with 252 replications of $n = 390$ intraday returns, in case of No Frictions (*first column*), in the case of autocorrelated marketmicro structure noise (*second column*), and in the case of irregular sampling (*third column*).

	Absence of Frictions			With Noise			Irregular sampling		
	RMSE	STD	BIAS	RMSE	STD	BIAS	RMSE	STD	BIAS
MPV([1,1,1,1])	0.2170	0.2170	-0.0024	0.9769	0.3769	0.9013	0.7361	0.1084	-0.7281
minRQ	0.2445	0.2445	0.0026	1.0429	0.4264	0.9517	0.4198	0.2276	-0.3528
medRQ	0.2207	0.2207	0.0003	1.0096	0.3864	0.9327	0.3125	0.2182	-0.2236
QVeff(k_n^*)	0.1556	0.1446	-0.0575	0.8728	0.3023	0.8188	0.2276	0.2034	-0.1021
QVeffB(k_n^*)	0.3866	0.0982	-0.3739	0.3238	0.2031	0.2523	0.2348	0.1881	-0.1405
GTMPV(m^*)	0.1639	0.1369	-0.0901	0.8357	0.2905	0.7836	0.2298	0.1858	-0.1352
$\widehat{\sigma}^4_{n,N^*,M^*}$	0.1559	0.1535	-0.0275	0.3284	0.3090	0.1111	0.1912	0.1892	-0.0276
$\widehat{\sigma}^4_{n,\widetilde{N}^*,\widetilde{M}^*}$	0.1620	0.1610	-0.0278	0.3164	0.2822	0.1432	0.2055	0.1903	-0.0449

[Mancino and Sanfelici, 2012]. We note that the performance of our estimator is in line with the corrected estimator of [Jacod and Rosenbaum, 2015]. In particular, the de-biasing of the latter estimator can be computationally challenging (see also the discussion in [Li et al., 2019]).

Finally, let us see a purely descriptive example on the ability of the Fourier estimator in tracking the instantaneous fourth power of the volatility. The instantaneous fourth power of the volatility (usually) appears when one determines the asymptotic distribution of the spot-volatility estimator, as in [Zu and Boswijk, 2014]. The latter authors say that heuristically one could use the squared increment of the estimated spot variance to approximate the spot quarticity. In this example, we fix $N = n/2 = 195$ and $M = 2$. Figure 2 displays the simulated (*red line*), the estimated (*blue line*) instantaneous fourth power of the volatility and the estimated (*black line*) instantaneous second power of the volatility squared respectively, averaged across 252 replications of $n = 390$ intraday returns. Both the approximation of the true quantity are satisfactory, although some remarks are in orders. The heuristic estimator presents a smooth trajectory but underestimates the true instantaneous fourth power of the volatility both at the beginning and the end of the day. On the other hand, $\widehat{\sigma}^4_{n,N,M}(t)$ tracks all the trend of $\sigma^4(t)$ with satisfactory accuracy, although it appears visually noisier.

5 Conclusion

In this paper we have studied the asymptotic error distribution of the Fourier quarticity estimator providing a new consistency result and by providing asymptotically normality. More precisely, we show that the estimator reaches the optimal rate and the optimal variance through a suitable choice of the number of cutting frequencies employed to compute the Fourier coefficients of the volatility. Further, the asymptotic bias can be reduced through the choice of the second cutting frequency M, which controls the product formula between

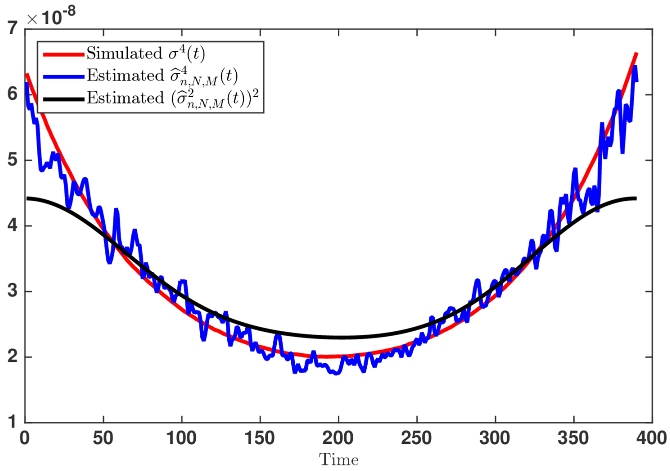

Fig. 2 Figure represents the simulated instantaneous fourth power of the volatility (*red line*), the estimated instantaneous fourth power of the volatility (*blue line*), and the estimated instantaneous second power of the volatility squared (*black line*).

the Fourier coefficient of the second power of the volatility. In a realistic Monte Carlo experiment, we show that the proposed estimator, although presenting a bias, has a quite satisfactory performance in comparison with other estimators proposed in the literature and even better in the presence of microstructure noise. The study of the asymptotic properties of the estimator when the efficient price is contaminated with market microstructure noise is left for future research.

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6 Appendix A: Proofs

We use the following notation. Let $\varphi_n(t) \doteq \sup\{t_j : t_j \leq t\}$ and denote, for $t, u \in [0, 2\pi]$, $D_{N,n}(t-u) \doteq D_N(\varphi_n(t) - \varphi_n(u))$[3], where $D_N(\cdot)$ is the rescaled Dirichlet kernel defined in (77).
In addition, we denote by $D_N(\cdot) \doteq (2N+1)D_N(\cdot)$ the non-rescaled Dirichlet kernel, and by $\bar{D}_{N,n}(t-u) \doteq (2N+1)\, D_{N,n}(t-u)$. For simplicity, we consider the case of regular sampling, i.e., $\varphi_n(t) = \frac{2\pi}{n} j$, if $\frac{2\pi}{n} j \leq t < \frac{2\pi(j+1)}{n}$.

We begin with a decomposition of $\widehat{\sigma}^4_{n,N,M}$ as in (7) in order to simplify the proofs of the asymptotic properties.

6.1 Preliminary Decomposition

From the Itô formula, the term in (6) is decomposed as

$$c_k(\sigma^2_{n,N}) = A_{k,n} + B_{k,n,N} + C_{k,n,N}, \tag{11}$$

where

$$A_{k,n} \doteq \frac{1}{2\pi} \int_0^{2\pi} e^{-\mathrm{i}k\varphi_n(s)} \sigma^2(s) ds, \tag{12}$$

$$B_{k,n,N} \doteq \frac{1}{2\pi} \int_0^{2\pi} e^{-\mathrm{i}k\varphi_n(s)} \int_0^s D_{N,n}(s-u)\sigma(u) dW_u \, \sigma(s) dW_s, \tag{13}$$

[3] We stress that with a slight abuse of notation, the discretized Dirichlet kernel $D_{N,n}(t-u)$ is used to denote $D_N(\varphi_n(t) - \varphi_n(u))$, not the kernel $D_N(\varphi_n(t-u))$. The notation is chosen to highlights the role of the convolution product, one of the key tools in the Fourier methodology.

$$C_{k,n,N} \doteq \frac{1}{2\pi}\int_0^{2\pi}\int_0^s e^{-\mathrm{i}k\varphi_n(u)} D_{N,n}(s-u)\sigma(u)dW_u\,\sigma(s)dW_s. \tag{14}$$

By exploiting decomposition (11) and the symmetry with respect to k of the summation, we rewrite the Fourier estimator of the integrated quarticity $\widehat{\sigma}^4_{n,N,M}$ as

$$2\pi\sum_{|k|\le M} A_{k,n}A_{-k,n} \tag{15}$$

$$+2\pi\sum_{|k|\le M} 2(A_{k,n}B_{-k,n,N} + A_{k,n}C_{-k,n,N}) \tag{16}$$

$$+2\pi\sum_{|k|\le M} (B_{k,n,N}B_{-k,n,N} + 2B_{k,n,N}C_{-k,n,N} + C_{k,n,N}C_{-k,n,N}). \tag{17}$$

First consider the term (16). Applying the Itô formula it has the following decomposition

$$2(AB^{(i)}_{M,n,N} + AB^{(ii)}_{M,n,N} + AC^{(i)}_{M,n,N} + AC^{(ii)}_{M,n,N}),$$

where $AB^{(i)}_{M,n,N}$ and $AB^{(ii)}_{M,n,N}$ are defined by

$$2AB^{(i)}_{M,n,N} \doteq \int_0^{2\pi} 2\int_0^u \frac{1}{2\pi}\bar{D}_{M,n}(u-s)\sigma^2(s)ds\,Y_{n,N}(u,u)\,\sigma(u)dW_u, \tag{18}$$

$$2AB^{(ii)}_{M,n,N} \doteq \int_0^{2\pi} 2\int_0^u \frac{1}{2\pi}\bar{D}_{M,n}(u-s)\,Y_{n,N}(s,u)\sigma(s)dW(s)\,\sigma^2(u)du, \tag{19}$$

where

$$Y_{n,N}(t,s) \doteq \int_0^t D_{N,n}(s-u)\sigma(u)dW_u. \tag{20}$$

Terms $2AC^{(i)}_{M,n,N}$ and $2AC^{(ii)}_{M,n,N}$ are analogous to (18) and (19) and give, asymptotically, the same contribution. Therefore, due to space constraints, they are omitted. Concerning (17), we rewrite it in the following way

$$BB^{(i)}_{M,n,N} + 2BB^{(ii)}_{M,n,N} + 2(BC^{(i)}_{M,n,N} + BC^{(ii)}_{M,n,N} + BC^{(iii)}_{M,n,N}) + CC^{(i)}_{M,n,N} + 2CC^{(ii)}_{M,n,N},$$

where, by using (20)

$$BB^{(i)}_{M,n,N} \doteq \frac{1}{2\pi}\sum_{|k|\le M}\int_0^{2\pi} Y^2_{n,N}(s,s)\sigma^2(s)ds, \tag{21}$$

$$2BB^{(ii)}_{M,n,N} \doteq \int_0^{2\pi} 2\int_0^u \frac{1}{2\pi}\bar{D}_{M,n}(u-s)Y_{n,N}(s,s)\sigma(s)dW_sY_{n,N}(u,u)\sigma(u)dW_u, \tag{22}$$

$$2BC^{(i)}_{M,n,N} \doteq \int_0^{2\pi} 2\int_0^s \frac{1}{2\pi}\bar{D}_{M,n}(s-v)D_{N,n}(s-v)\sigma(v)dW_vY_{n,N}(s,s)\sigma^2(s)ds, \tag{23}$$

$$2BC^{(ii)}_{M,n,N} \doteq \tfrac{1}{2\pi}\textstyle\sum_{|k|\leq M}\int_0^{2\pi} 2\int_0^u e^{-ik\phi_n(s)}Y_{n,N}(s,s)\sigma(s)dW_s Y_{k,n,N}(u,u)\,\sigma(u)dW_u, \tag{24}$$

$$2BC^{(iii)}_{M,n,N} \doteq \textstyle\int_0^{2\pi}\int_0^u 2\int_0^v \tfrac{1}{2\pi}\bar{D}_{M,n}(u-s)D_{N,n}(v-s)\sigma(s)dW_s\,\sigma(v)dW_v\,Y_{n,N}(u,u)\sigma(u)dW_u, \tag{25}$$

$$CC^{(i)}_{M,n,N} \doteq \frac{1}{2\pi}\sum_{|k|\leq M}\int_0^{2\pi} Y_{k,n,N}(s,s)Y_{-k,n,N}(s,s)\sigma^2(s)ds, \tag{26}$$

$$2CC^{(ii)}_{M,n,N} \doteq \frac{1}{2\pi}\sum_{|k|\leq M} 2\int_0^{2\pi}\int_0^s Y_{k,n,N}(v,v)\sigma(v)dW_v\,Y_{-k,n,N}(s,s)\sigma(s)dW_s, \tag{27}$$

where, for $|k|\leq M$

$$Y_{k,n,N}(t,s) \doteq \int_0^t e^{-\mathrm{i}k\varphi_n(u)}D_{N,n}(s-u)\sigma(u)dW_u. \tag{28}$$

6.2 Proof of Theorem 2

Relying on the decomposition obtained in the previous section, in order to prove the consistency result of Theorem 2, it is sufficient to prove that the first term in the decomposition (15) converges in probability to the quarticity, while all the remaining terms vanish.

Step I

First, using the definition of $A_{k,n}$ in (12) we have:

$$2\pi\sum_{|k|\leq M} A_{k,n}A_{-k,n} = \int_0^{2\pi} 2\int_0^s \frac{1}{2\pi}\bar{D}_{M,n}(s-u)\sigma^2(u)du\,\sigma^2(s)ds. \tag{29}$$

Exploiting (29), the fact that the following convergence in probability holds

$$\lim_{M,n\to\infty} 2\pi\sum_{|k|\leq M} A_{k,n}A_{-k,n} = \int_0^{2\pi}\sigma^4(t)dt,$$

is equivalent to prove the asymptotic negligibility of the following two differences

$$2\int_0^{2\pi}\left(\int_0^s \frac{1}{2\pi}\bar{D}_{M,n}(s-u)\sigma^2(u)du - \int_0^s \frac{1}{2\pi}\bar{D}_M(s-u)\sigma^2(u)du\right)\,\sigma^2(s)ds \tag{30}$$

$$+2\int_0^{2\pi}\int_0^s \frac{1}{2\pi}\bar{D}_M(s-u)\sigma^2(u)du\;\sigma^2(s)ds - \int_0^{2\pi}\sigma^4(s)ds. \tag{31}$$

Note that, while assumption (A.III)$'$ is needed to obtain the Central Limit theorem, in order to prove that (30) and (31) goes to zero in probability, then assumption (A.III) is enough. The convergence follows by Lemma 1 and Lemma 5.1 in [Cuchiero and Teichmann, 2015], in virtue of the fact that the function $\sigma^2(t)$ is Hölder continuous and $M^\gamma \rho(n) \to K$, with K constant and $\gamma > 1$.

Step II

For sake of brevity, we summarise results in *Step II* as a consequence of the proof of Theorem 3 in the next section. More precisely, the terms (21), (23), (26) have order equal to $O_p(\rho(n)^{1/2})$ and hence they converge to zero in probability. Terms (22), (27), (24) and (25) are $o_p(\rho(n)^{1/2})$, thus, they converge to zero in probability. Finally, the terms (18), (19) and their analogous are $O_p(\rho(n)^{1/2})$. Notice that the assumptions (A.III)$'$ and (A.IV) are necessary only for the study of the bias term in the *Step I* of the proof of Theorem 3, therefore, they are not required here.

6.3 Proof of Theorem 3

The proof consists of four steps. Throughout, C is a constant which may change from line to line.

Step I

We study the convergence in probability of the following term

$$\rho(n)^{-1/2}\left(2\pi \sum_{|k|\le M} A_{k,n}A_{-k,n} - \int_0^{2\pi} \sigma^4(t)dt\right)$$
$$= \rho(n)^{-1/2}\left(\int_0^{2\pi} 2\int_0^s \frac{1}{2\pi}\bar{D}_{M,n}(s-u)\sigma^2(u)du\ \sigma^2(s)ds - \int_0^{2\pi}\sigma^4(t)dt\right), \tag{32}$$

which can be rewritten as

$$\rho(n)^{-1/2}\left[\int_0^{2\pi} 2\int_0^s \frac{1}{2\pi}\bar{D}_M(s-u)\sigma^2(u)du\ \sigma^2(s)ds - \int_0^{2\pi}\sigma^4(t)dt\right. \tag{33}$$

$$\left.+\int_0^{2\pi} 2\left(\int_0^s \frac{1}{2\pi}\bar{D}_{M,n}(s-u)\sigma^2(u)du - \int_0^s \frac{1}{2\pi}\bar{D}_M(s-u)\sigma^2(u)du\right)\ \sigma^2(s)ds\right]. \tag{34}$$

We start from (33). First, for any $M \in \mathbb{N}$ we define

$$\sigma_M^2(s) \doteq \sum_{|k|\le M} e^{\mathrm{i}ks} c_k(\sigma^2), \tag{35}$$

where, for sake of brevity, we use the notation $c_k(f)$, instead of $\mathcal{F}(f)(k)$, to denote the k-th Fourier coefficient of a given function f. Then it holds

$\int_0^{2\pi} 2\int_0^s \frac{1}{2\pi}\bar{D}_M(s-u)\sigma^2(u)du\ \sigma^2(s)ds - \int_0^{2\pi}\sigma^4(t)dt = \int_0^{2\pi}\left(\sigma_M^2(s) - \sigma^2(s)\right)\sigma^2(s)\ ds,$

which is equal to

$$-4\pi \sum_{k>M} |c_k(\sigma^2)|^2. \tag{36}$$

Using the integration by parts formula, we have that for any $k \neq 0$

$$c_k(\sigma^2) = \frac{1}{\mathrm{i}k}\left(c_k(d\sigma^2) - \frac{\sigma^2(2\pi) - \sigma^2(0)}{2\pi}\right),$$

and, in particular, $|c_k(\sigma^2)|^2$ is equal to

$$\frac{1}{4\pi^2 k^2}\Bigg[\int_0^{2\pi} \gamma^2(s)\, ds + 2\int_0^{2\pi}\int_0^s \cos(k(s-v))\,\gamma(v)\,dZ_v\gamma(s)\,dZ_s$$
$$- 2\left(\sigma^2(2\pi) - \sigma^2(0)\right)\int_0^{2\pi}\cos(ks)\,\gamma(s)\, dZ_s + \left(\sigma^2(2\pi) - \sigma^2(0)\right)^2\Bigg].$$

We start by studying the first and the last term in the previous expression. Precisely, by plugging these terms into (36) we have to consider

$$-\rho(n)^{-1/2}\,4\pi\sum_{k>M}\frac{1}{4\pi^2k^2}\left[\int_0^{2\pi}\gamma^2(s)\, ds + \left(\sigma^2(2\pi) - \sigma^2(0)\right)^2\right]. \tag{37}$$

Noting that $\sum_{k>M}\frac{1}{k^2} = \frac{1}{M} + O\left(\frac{1}{M^2}\right)$, (37) converges to

$$-\frac{1}{\pi c_M}\left[\int_0^{2\pi}\gamma^2(s)\, ds + \left(\sigma^2(2\pi) - \sigma^2(0)\right)^2\right]. \tag{38}$$

Then, we consider the two martingale components of $|c_k(\sigma^2)|^2$ and study the sum (36). We show explicitly the computation only for the first term, since for the second a similar reasoning applies. By applying the Itô isometry, by using the orthogonality properties of the Fourier basis and the boundedness of γ, one finds that

$$\rho(n)^{-1}\,\mathbb{E}\left[\left(\sum_{k>M}\frac{1}{4\pi^2k^2}2\int_0^{2\pi}\int_0^s\cos(k(s-s_1))\,\gamma(s_1)\,dZ_{s_1}\gamma(s)\,dZ_s\right)^2\right]$$
$$\leq C\rho(n)^{-1}\sum_{k>M}\frac{1}{k^4}, \tag{39}$$

which is of order $O\left(n^{-1/2}\right)$.

We now analyse (34), i.e., the discretization error, and we show that it is asymptotically negligible. It can be rewritten as $\rho(n)^{-1/2}$ times

$$2\pi\left(\sum_{|k|\leq M}\left(c_k\left(\bar{D}_{M,n}\right) - c_k\left(\bar{D}_M\right)\right)|c_k(\sigma^2)|^2 + \sum_{|k|>M}c_k\left(\bar{D}_{M,n}\right)|c_k(\sigma^2)|^2\right). \tag{40}$$

Before proceeding[4], we compute the k-th Fourier coefficient of $\bar{D}_{M,n}(\cdot)$. To do so, we note that, for any j such that $0 \leq j \leq n-1$, it holds

$$c_k\left(I_{[\frac{2\pi}{n}j, \frac{2\pi}{n}(j+1)[}\right) = \frac{1}{2\pi \mathrm{i} k} e^{-2\pi \mathrm{i}\frac{k}{n}j}\left(1 - e^{-2\pi \mathrm{i}\frac{k}{n}}\right).$$

Therefore, we obtain

$$\begin{aligned} c_k(\bar{D}_{M,n}) &= \sum_{j=0}^{n-1} \bar{D}_M\left(\frac{2\pi}{n}j\right) c_k\left(I_{[\frac{2\pi}{n}j, \frac{2\pi}{n}(j+1)[}\right) \\ &= \frac{1}{2\pi \mathrm{i} k}\left(1 - e^{-2\pi \mathrm{i}\frac{k}{n}}\right) \sum_{|l|\leq M} \sum_{j=0}^{n-1} e^{-2\pi \mathrm{i}\frac{(l-k)}{n}j}. \end{aligned} \tag{41}$$

We note that the summation $\sum_{j=0}^{n-1} e^{-2\pi \mathrm{i}\frac{(l-k)}{n}j}$ is either equal to n or equal to zero depending on whether n divides $l-k$ or not. Moreover, if n divides k then $c_k\left(\bar{D}_{M,n}\right)$ is zero. Therefore, we only need to consider the case in which n does not divide k and n divides $l-k$. In particular, k is of the form $k = nq + r$, with $q \geq 0$ and $1 \leq r < n$. We note that if r were $r \geq M$ then n would not divide $l-k$ since $|l| < M$, thus one has $1 \leq r < M$. Moreover $l = -r$ if n divides $l-k$. Then (41) reduces to

$$c_k\left(\bar{D}_{M,n}\right) = n\frac{1}{2\pi \mathrm{i} k}\left(1 - e^{-2\pi \mathrm{i}\frac{k}{n}}\right),$$

provided that $M < n$. First, we consider the low frequencies term in (40) and we observe that

$$c_k\left(\bar{D}_{M,n}\right) - c_k\left(\bar{D}_M\right) = \left(\frac{1 - e^{-2\pi \mathrm{i}\frac{k}{n}}}{2\pi \mathrm{i}\frac{k}{n}} - 1\right) = -\frac{\pi \mathrm{i} k}{n} + O\left(\frac{k^2}{n^2}\right),\ \forall\, |k| \leq M.$$

Therefore, we need to study the order of

$$\frac{\pi}{n}\sum_{|k|\leq M} k|c_k(\sigma^2)|^2.$$

Using the definition of $|c_k(\sigma^2)|^2$ we have that the leading term is $\frac{\pi}{n}\sum_{|k|\leq M}\frac{1}{k}$, which has order $O\left(\frac{\log M}{n}\right)$. Therefore, we can conclude that

$$\rho(n)^{-1/2}\sum_{|k|\leq M}\left(c_k(\bar{D}_{M,n}) - c_k(\bar{D}_M)\right)|c_k(\sigma^2)|^2$$

converges to 0 in probability. Finally, we prove that the sum of the high-frequencies in (40) converges to zero. We explicitly compute the upper bound

[4] With abuse of notation in equation (40) we have denoted by $c_k(\bar{D}_{M,n})$ the Fourier coefficient of $\widetilde{D}_{M,n}$, which is the kernel defined by $\widetilde{D}_{M,n}(s) \doteq \bar{D}_M(\varphi_n(s))$ (remember that $\bar{D}_M(\cdot)$ is the non-rescaled Dirichlet kernel). A straightforward but lengthy proof, which is available from the authors upon request, shows that the difference between the two kernels is negligible in estimating the quantity in (40).

for the first term, which gives a contribution of the same order of the third one, and it is the leading term

$$
\begin{aligned}
&\mathbb{E}\left[2\sum_{k>M}|c_k(\bar{D}_{M,n})|\frac{1}{k^2}\frac{1}{(2\pi)^2}\int_0^{2\pi}\gamma^2(s)ds\right] \\
&\leq C\sum_{q\geq 1}\sum_{r=1}^{M-1}\frac{1}{(nq+r)^2}\frac{1}{q+\frac{r}{n}}\,|1-e^{-2\pi \mathrm{i}\frac{r}{n}}|.
\end{aligned}
\tag{42}
$$

Using the inequality $|1-e^{-2\pi \mathrm{i}\frac{r}{n}}| \leq 2\pi\frac{r}{n}$, the term (42) is dominated by

$$
C\frac{1}{n^3}\sum_{q\geq 1}\frac{1}{q^3}\sum_{r=1}^{M-1} r\left(1+\frac{r}{nq}\right)^{-3}. \tag{43}
$$

Observe now the following fact

$$
\sum_{r=1}^{M-1}\frac{1}{(1+\frac{r}{nq})^3} \leq \int_0^M \frac{dx}{(1+\frac{x}{nq})^3} = \frac{1}{2}\left[nq - \frac{nq}{(1+\frac{M}{nq})^2}\right] = M + \frac{3}{2}\frac{M^2}{nq} + o(1),
$$

then (43) is less than CM^2/n^3. In particular, when multiplied by $\rho(n)^{-1/2}$, this term vanishes in the limit for $n \to \infty$.
Summarizing, in *Step I* we have determined the bias term X_{c_M} which is equal to

$$
X_{c_M} \doteq -\frac{1}{c_M}\frac{1}{\pi}\left(\int_0^{2\pi}\gamma^2(t)\,dt + \left(\sigma^2(2\pi) - \sigma^2(0)\right)^2\right). \tag{44}
$$

Step II. The bias
We start with the study of $BB^{(i)}_{M,n,N}$ in (21) and $BC^{(i)}_{M,n,N}$ in (23). The term $CC^{(i)}_{M,n,N}$ in (26) is analogous to $BB^{(i)}_{M,n,N}$ and gives the same contribution. We split $BB^{(i)}_{M,n,N}$ into two terms

$$
\frac{1}{2\pi}\sum_{|k|\leq M}\int_0^{2\pi}\int_0^s D^2_{N,n}(s-u)\sigma^2(u)du\,\sigma^2(s)ds \tag{45}
$$

$$
+\frac{1}{2\pi}\sum_{|k|\leq M}\int_0^{2\pi} 2\int_0^s Y_{n,N}(u,s)D_{N,n}(s-u)\sigma(u)dW_u\,\sigma^2(s)ds. \tag{46}
$$

We start from (45) and study the convergence in probability of

$$
\begin{aligned}
&\rho(n)^{-1/2}\frac{1}{2\pi}\sum_{|k|\leq M}\int_0^{2\pi}\int_0^s D^2_{N,n}(s-u)\sigma^2(u)du\,\sigma^2(s)ds \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{2\pi}\frac{(2M+1)}{\sqrt{n}}\int_0^{2\pi}\left(n\int_0^s D^2_{N,n}(s-u)\sigma^2(u)du\right)\sigma^2(s)ds.
\end{aligned}
\tag{47}
$$

By using Lemma 2 *iv)* and the fact that $M\rho\left(n\right)^{1/2} \to c_M$ the last term converges in probability to

$$\frac{1}{2\pi}c_M\left(1+2\eta\left(c_N\right)\right)\int_0^{2\pi}\sigma^4(s)ds.$$

Consider, now (46). Using the Lemma 2 *i)* and the fact that for $v < s-\varepsilon$, $D^2_{N,n}(s-v) \leq CN^{-2}$ for n large enough, thanks to the Assumption (A.II), it follows that (46) has order $o_p(\rho(n))$ along the same lines of the analogous term (41) in [Clement and Gloter, 2011]. We conclude that

$$\rho\left(n\right)^{-1/2}\frac{1}{2\pi}\sum_{|k|\leq M}\int_0^{2\pi}\int_0^s D^2_{N,n}(s-u)\sigma^2(u)du\,\sigma^2(s)ds = o_p(\rho(n)^{1/2}).$$

We consider now $BC^{(i)}_{M,n,N}$ in (23), which can be split into three terms

$$\int_0^{2\pi}2\int_0^s\frac{1}{2\pi}\bar{D}_{M,n}(s-v)D^2_{N,n}(s-v)\sigma^2(v)dv\,\sigma^2(s)ds \tag{48}$$

$$+\int_0^{2\pi}2\int_0^s\int_0^u\tfrac{1}{2\pi}\bar{D}_{M,n}(s-v)D_{N,n}(s-v)\sigma(v)dW_v\,D_{N,n}(s-u)\sigma(u)dW_u\,\sigma^2(s)ds \tag{49}$$

$$+\int_0^{2\pi}2\int_0^s\int_0^u\tfrac{1}{2\pi}D_{N,n}(s-v)\sigma(v)dW_v\,\bar{D}_{M,n}(s-u)D_{N,n}(s-u)\sigma(u)dW_u\,\sigma^2(s)ds. \tag{50}$$

Consider (48) and let $f_M\left(s,t\right) \doteq \bar{D}_M\left(s-t\right)\sigma^2\left(t\right)$. Then we study

$$\rho\left(n\right)^{-1/2}\int_0^{2\pi}2\int_0^s\frac{1}{2\pi}f_M\left(s,v\right)D^2_{N,n}\left(s-v\right)dv\,\sigma^2\left(s\right)ds.$$

By using Lemma 2 *iv)*, the previous term converges in probability to

$$\frac{1}{\pi}c_M\left(1+2\eta(c_N)\right)\int_0^{2\pi}\sigma^4(s)ds,$$

where we have used that $\bar{D}_M\left(0\right) = 2M+1$.
Thus, it remains to show that the terms (49) and (50) converge to zero. By an iterated application of Itô isometry, Assumption (A.II), Lemma 2, *ii)*, and the fact that for $v < s-\varepsilon$, $\varepsilon > 0$, then $D^2_{N,n}(s-v) \leq C\,N^{-2}$ for n large enough, as for (46), we obtain

$$\mathbb{E}\left[\left(2\int_0^s\int_0^u\frac{1}{2\pi}\bar{D}_{M,n}(s-v)D_{N,n}(s-v)\sigma(v)dW_v\,D_{N,n}(s-u)\sigma(u)dW_u\right)^2\right]$$
$$\leq C\int_0^s\int_0^u\bar{D}^2_{M,n}\left(s-v\right)D^2_{N,n}\left(s-v\right)dv\,D^2_{N,n}\left(s-u\right)du \leq CN^{-2}n^{-1}.$$

Therefore the term (49) multiplied by $\rho\left(n\right)^{-1/2}$ converges in probability to zero.

For the term (50), by an iterated application of the Itô isometry, the

Assumption (A.II), Lemma 2, *ii)* and the fact that for $v < s - \varepsilon$, $\varepsilon > 0$, then $D^2_{N,n}(s-v) \leq CN^{-2}$ for n large enough (as for (46)), and noting that $\bar{D}^2_M(0) = (2M+1)^2$, we have

$$\mathbb{E}\left[\left(2\int_0^s\int_0^u \frac{1}{2\pi}D_{N,n}(s-v)\sigma(v)dW_v\,\bar{D}_{M,n}(s-u)D_{N,n}(s-u)\sigma(u)dW_u\right)^2\right]$$
$$\leq C\int_0^s\int_0^u D^2_{N,n}\left(s-v\right)dv\,\bar{D}^2_{M,n}\left(s-u\right)\,D^2_{N,n}\left(s-u\right)du \leq Cn^{-2}.$$

Therefore the term (50) multiplied by $\rho\left(n\right)^{-1/2}$ converges in probability to zero.
Finally, the total contribution of the terms (21), (23) and (26) gives the following bias:

$$2c_M\frac{1}{\pi}(1+2\eta(c_N))\int_0^{2\pi}\sigma^4(s)ds.$$

Move now to the term $2BB^{(ii)}_{M,n,N}$ in (22) and the term $2CC^{(ii)}_{M,n,N}$ in (27). We prove that the former converges to zero. The other one is analogous. Consider

$$\rho(n)^{-1}\mathbb{E}\left[\left(\int_0^{2\pi}2\int_0^u\tfrac{1}{2\pi}\bar{D}_{M,n}(u-s)Y_{n,N}(s,s)\sigma(s)dW_s\,Y_{n,N}(u,u)\sigma(u)dW_u\right)^2\right]. \tag{51}$$

Applying first the Itô isometry, then the Itô formula we write (51) as the sum of two terms

$$\rho(n)^{-1}\mathbb{E}\left[\int_0^{2\pi}\int_0^u\frac{1}{\pi^2}\bar{D}^2_{M,n}(u-s)Y^2_{n,N}(s,s)\sigma^2(s)ds\,Y^2_{n,N}(u,u)\sigma^2(u)du\right] \tag{52}$$

and

$$\rho(n)^{-1}\mathbb{E}\left[\int_0^{2\pi}\int_0^u\int_0^s\frac{1}{\pi}\bar{D}_{M,n}(u-v)Y_{n,N}(v,v)\sigma(v)dW_v\cdot\frac{1}{\pi}\bar{D}_{M,n}(u-s)Y_{n,N}(s,s)\sigma(s)dW_s\,Y^2_{n,N}(u,u)\sigma^2(u)du\right]. \tag{53}$$

We consider first (52). Using Assumption (A.II), it is sufficient to study the following term

$$\rho(n)^{-1}\int_0^{2\pi}\int_0^u\bar{D}^2_{M,n}(u-s)\mathbb{E}\left[Y^2_{n,N}(s,s)\,Y^2_{n,N}(u,u)\right]du. \tag{54}$$

First we observe that, by applying the Burkholder-Davis-Gundy inequality and Lemma 2 the following inequality holds

$$\mathbb{E}\left[\sup_{s\leq v}Y^4_{n,N}(s,s)\right]^{1/2}\leq C\int_0^v D^2_{N,n}(v-t)dt\leq C\rho(n). \tag{55}$$

Then, by the Cauchy-Schwartz inequality and (55)

$$\mathbb{E}\left[Y^2_{n,N}(s,s)\,Y^2_{n,N}(u,u)\right]\leq C\rho(n)^2,$$

which enables us to conclude that (54) is less or equal to $C\,M\rho(n)$ and vanishes when taking the limit for $n \to \infty$. We consider now (53). Using assumption (A.II) it is sufficient to study

$$\mathbb{E}\left[\left|\int_0^u \int_0^s \frac{1}{\pi}\bar{D}_{M,n}(u-v)Y_{n,N}(v,v)\sigma(v)dW_v \frac{1}{\pi}\bar{D}_{M,n}(u-s)Y_{n,N}(s,s)\sigma(s)dW_s\, Y^2_{n,N}(u,u)\right|\right].$$

Applying the Cauchy-Schwartz inequality to this term and using (55), it is enough to show that

$$\mathbb{E}\left[\left(\int_0^u \int_0^s \bar{D}_{M,n}(u-v)Y_{n,N}(v,v)\sigma(v)dW_v\, \bar{D}_{M,n}(u-s)Y_{n,N}(s,s)\sigma(s)dW_s\right)^2\right] \tag{56}$$

converges to zero. By applying the Itô formula, Assumption (A.II) and the Cauchy-Schwartz inequality, the latter is less or equal to a constant C times

$$\int_0^u \mathbb{E}\left[\left(\int_0^s \bar{D}_{M,n}(u-v)Y_{n,N}(v,v)\sigma(v)dW_v\right)^4\right]^{1/2} \mathbb{E}\left[Y^4_{n,N}(s,s)\right]^{1/2} \bar{D}^2_{M,n}(u-s)ds.$$

By using the Burkholder-Davis-Gundy inequality, inequality (55) and the relation between $\bar{D}_M(\cdot)$ and $D_M(\cdot)$, we get that (53) is less than

$$C\int_0^u \left(\int_0^s \bar{D}^4_{M,n}(u-v)\mathbb{E}\left[Y^4_{n,N}(v,v)\right]dv\right)^{1/2} \mathbb{E}\left[Y^4_{n,N}(s,s)\right]^{1/2} \bar{D}^2_{M,n}(u-s)ds$$
$$\leq C\rho(n)^2M^4 \int_0^u \left(\int_0^s D^4_{M,n}(u-v)dv\right)^{1/2} D^2_{M,n}(u-s)ds \leq C\rho(n)^2 M \to 0,$$

where, in the last step, we use Lemma 1 and the fact that, for $v < u-\varepsilon$ with $\varepsilon > 0$, then $D^2_{M,n}(u-v) \leq CM^{-2}$.

We consider now the term $2BC^{(ii)}_{M,n,N}$ defined by (24). Applying Itô formula, it splits into three terms:

$$\frac{1}{2\pi}(2M+1)\int_0^{2\pi} 2\int_0^u Y_{n,N}(s,s)D_{N,n}(u-s)\sigma^2(s)ds\,\sigma(u)dW_u \tag{57}$$

$$+\int_0^{2\pi}\int_0^u 2\int_0^v \tfrac{1}{2\pi}\bar{D}_{M,n}(v-s)Y_{n,N}(s,s)\sigma(s)dW_s\, D_{N,n}(u-v)\sigma(v)dW_v\,\sigma(u)dW_u \tag{58}$$

$$+\int_0^{2\pi}\int_0^u 2\int_0^v \tfrac{1}{2\pi}\bar{D}_{M,n}(v-s)D_{N,n}(u-s)\sigma(s)dW_s\, Y_{N,n}(v,v)\sigma(v)dW_v\,\sigma(u)dW_u. \tag{59}$$

We begin with the study of (57). In particular, we prove that

$$\rho(n)^{-1}M^2\mathbb{E}\left[\int_0^{2\pi}\left(\int_0^u Y_{n,N}(s,s)D_{N,n}(u-s)\sigma^2(s)ds\right)^2\sigma^2(u)du\right]$$

goes to 0. By Assumption (A.II) it is enough to prove that

$$\rho(n)^{-1}M^2\int_0^{2\pi}\mathbb{E}\left[\left(\int_0^u Y_{n,N}(s,s)D_{N,n}(u-s)\sigma^2(s)ds\right)^2\right]du \tag{60}$$

goes to 0. By applying the Fubini theorem and the Itô isometry, we have

$$\rho(n)^{-1}M^2\mathbb{E}\left[\left(\int_0^s\int_0^u D_{N,n}(u-s)\sigma^2(s)ds\, D_{N,n}(s-v)\sigma(v)dW_v\right)^2\right]$$

$$\leq Cn^2\int_0^s\mathbb{E}\left[\left(\int_0^u D^p_{N,n}(u-s)\sigma^{2p}(s)ds\right)^{2/p}\right]D^2_{N,n}(s-v)dv \tag{61}$$

for any $p>1$. By Lemma 2 *i)* we have, for n and N large enough,

$$n\int_0^u D^p_{N,n}(u-s)\sigma^{2p}(s)ds\leq C_p$$

for any $p>1$. Therefore, it holds that (61) has order $n^{1-\frac{2}{p}}$. Hence, it is sufficient to take $1<p<2$.

Consider now (58). We prove that

$$\rho(n)^{-1}\mathbb{E}\left[\int_0^{2\pi}\left(\int_0^u 2\int_0^v\frac{1}{2\pi}\bar{D}_{M,n}(v-s)Y_{n,N}(s,s)\sigma(s)dW_s\, D_{N,n}(u-v)\sigma(v)dW_v\right)^2\sigma^2(u)du\right] \tag{62}$$

converges to 0. By the Itô isometry and Assumption (A.II) the previous term is less or equal to

$$C\rho(n)^{-1}\int_0^{2\pi}\int_0^u\int_0^v\bar{D}^2_{M,n}(v-s)\mathbb{E}\left[Y^2_{n,N}(s,s)\right]ds\,D^2_{N,n}(u-v)dv\,du.$$

We remember now that $\mathbb{E}\left[\sup_{s\leq v}Y^2_{n,N}(s,v)\right]\leq C\rho(n)$. In addition, by using Lemma 1 and 2 this term has order $O\left(Mn^{-1}\right)$. Therefore, it converges to zero. The term (59) can be studied in a similar way.

Finally, we consider the term $2BC^{(iii)}_{M,n,N}$ defined by (25). By the Itô isometry it is enough to prove that

$$\rho(n)^{-1}\mathbb{E}\left[\int_0^{2\pi}\left(\int_0^u 2\int_0^v\tfrac{1}{2\pi}\bar{D}_{M,n}(u-s)D_{N,n}(v-s)\sigma(s)dW_s\,\sigma(v)dW_v\right)^2Y^2_{N,n}(u,u)\sigma^2(u)du\right] \tag{63}$$

converges to 0. By Assumption (A.II) and the Cauchy-Schwartz inequality, (63) is smaller than a constant C times

$$\rho(n)^{-1}\int_0^{2\pi}\mathbb{E}\left[\left(\int_0^u 2\int_0^v\tfrac{1}{2\pi}\bar{D}_{M,n}(u-s)D_{N,n}(v-s)\sigma(s)dW_s\,\sigma(v)dW_v\right)^4\right]^{1/2}\mathbb{E}\left[Y^4_{N,n}(u,u)\right]^{1/2}du.$$

Therefore, applying the Burkholder-Davis-Gundy inequality and using the inequality (55), it remains to prove that

$$\int_0^u\int_0^v\bar{D}^4_{M,n}(u-s)D^4_{N,n}(v-s)ds\,dv\to 0.$$

The above result follows by the relation between $\bar{D}_M(\cdot)$ and $D_M(\cdot)$, by the fact that for $s<u-\varepsilon$, $\varepsilon>0$, $D^2_{M,n}(u-s)\leq CM^{-2}$, and by Lemma 2 *ii)*, as

for (46).
Summarizing, in *Step II* we have determined the bias term Y_{c_M,c_N} which is equal to

$$Y_{c_M,c_N} \doteq 2c_M \frac{1}{\pi}\left(1+2\eta\left(c_N\right)\right)\int_0^{2\pi}\sigma^4\left(t\right)\,dt. \tag{64}$$

Step III: Asymptotic Variance

Following [Jacod, 1994] we determine the variance of the asymptotic distribution by studying

$$\begin{aligned}\langle\rho(n)^{-1/2}\,2(AB^{(i)}_{M,n,N}+AB^{(ii)}_{M,n,N}+AC^{(i)}_{M,n,N}+AC^{(ii)}_{M,n,N}),\\ \rho(n)^{-1/2}\,2(AB^{(i)}_{M,n,N}+AB^{(ii)}_{M,n,N}+AC^{(i)}_{M,n,N}+AC^{(ii)}_{M,n,N})\rangle,\end{aligned} \tag{65}$$

where $\langle\cdot,\cdot\rangle$ denotes the quadratic covariation in $[0,2\pi]$.
First we observe that the term $AB^{(ii)}_{M,n,N}$ (resp. $AC^{(ii)}_{M,n,N}$) can be treated as $AB^{(i)}_{M,n,N}$ (resp. $AC^{(i)}_{M,n,N}$). Indeed, by the stochastic Fubini theorem [Protter, 1990], we have that

$$2AB^{(ii)}_{M,n,N}=\int_0^{2\pi}2\int_s^{2\pi}\frac{1}{2\pi}\bar{D}_{M,n}\left(s-u\right)\sigma^2\left(u\right)\,du\,Y_{n,N}\left(s,s\right)dW_s.$$

Therefore, (65) contributes with sixteen terms, each of them leads to the same limit. We study the first in detail, being the remaining terms similar. We prove that

$$\begin{aligned}&\langle\rho\left(n\right)^{-1/2}AB^{(i)}_{M,n,N},\rho\left(n\right)^{-1/2}AB^{(i)}_{M,n,N}\rangle\\ &=\rho\left(n\right)^{-1}\int_0^{2\pi}\left(2\int_0^u\frac{1}{2\pi}\bar{D}_{M,n}\left(u-s\right)\sigma^2\left(s\right)\,ds\right)^2Y^2_{n,N}\left(u,u\right)\sigma^2\left(u\right)\,du\end{aligned}$$

converges in probability to

$$\frac{1}{2}\left(1+2\eta\left(c_N\right)\right)\int_0^{2\pi}\sigma^8\left(u\right)\,du. \tag{66}$$

By applying the Itô formula we study

$$\rho\left(n\right)^{-1}\int_0^{2\pi}\left(2\int_0^u\tfrac{1}{2\pi}\bar{D}_{M,n}\left(u-s\right)\sigma^2\left(s\right)\,ds\right)^2\int_0^u D^2_{N,n}\left(u-s\right)\sigma^2\left(s\right)ds\,\sigma^2\left(u\right)du \tag{67}$$

$$\begin{aligned}&+\rho\left(n\right)^{-1}\int_0^{2\pi}\left(2\int_0^u\frac{1}{2\pi}\bar{D}_{M,n}\left(u-s\right)\sigma^2\left(s\right)\,ds\right)^2 2\int_0^u\int_0^s D_{N,n}\left(u-v\right)\sigma\left(v\right)dW_v\cdot\\ &\cdot D_{N,n}\left(u-s\right)\sigma\left(s\right)dW_s\sigma^2\left(u\right)du.\end{aligned} \tag{68}$$

Consider (67). Using the fact that $M/n\to 0$, as shown by the study of (34) (multiplied by $\rho\left(n\right)^{-1/2}$) in *Step I*, it is enough to prove that (67) converges in probability to (66). We observe that by using *Step I*, namely (32), then (67) is equal to

$$\rho\left(n\right)^{-1}\int_0^{2\pi}\sigma^6\left(u\right)\int_0^u D_{N,n}^2\left(u-s\right)\sigma^2\left(s\right)\,ds\,du+o_p(1)$$

Finally, using Lemma 2, this last term converges to (66). Consider now (68). Using the same argument applied for (67), it is enough to prove that

$$\rho\left(n\right)^{-1}\int_0^{2\pi}\left(2\int_0^u\frac{1}{2\pi}\bar{D}_{M,n}\left(u-s\right)\sigma^2\left(s\right)\,ds\right)^2 2\int_0^u\int_0^s D_{N,n}\left(u-v\right)\sigma\left(v\right)dW_v\cdot$$
$$\cdot\, D_{N,n}\left(u-s\right)\sigma\left(s\right)dW_s\,\sigma^2\left(u\right)du.$$

converges in probability to zero. By using the convergence in *Step I*, the previous term is equal to

$$\rho\left(n\right)^{-1}\int_0^{2\pi}\sigma^6\left(u\right)\,2\int_0^u\int_0^s D_{N,n}\left(u-v\right)\sigma\left(v\right)\,dW_v D_{N,n}\left(u-s\right)\sigma\left(s\right)dW_s\,du \tag{69}$$

$$+\,\rho\left(n\right)^{-1}\int_0^{2\pi}2\int_0^u\int_0^s D_{N,n}\left(u-v\right)\sigma\left(v\right)\,dW_v\,D_{N,n}\left(u-s\right)\sigma\left(s\right)\,dW_s\,du\cdot$$
$$\cdot o_p(1). \tag{70}$$

Consider the term (69). It holds

$$\mathbb{E}\left[\left|\rho\left(n\right)^{-1}\int_0^{2\pi}\sigma^6\left(u\right)2\int_0^u\int_0^s D_{N,n}\left(u-v\right)\sigma\left(v\right)dW_v D_{N,n}\left(u-s\right)\sigma\left(s\right)\,dW_s\,ds\,du\right|\right]$$
$$\leq C\int_0^{2\pi}\mathbb{E}\left[\left(\rho\left(n\right)^{-1}2\int_0^u\int_0^s D_{N,n}\left(u-v\right)\sigma\left(v\right)dW_v D_{N,n}\left(u-s\right)\sigma\left(s\right)dW_s\right)^2\right]^{1/2}du.$$

By iterated application of Itô formula and Assumption (A.II), this is less or equal to

$$C\int_0^{2\pi}\left(\rho\left(n\right)^{-2}\int_0^u\int_0^s D_{N,n}^2\left(u-v\right)\,dv\,D_{N,n}^2\left(u-s\right)\,ds\right)^{1/2}du.$$

By using Lemma 1 and the fact that for $\varepsilon>0$ and $v<u-\varepsilon$ we have $D_{N,n}^2\left(u-v\right)\leq CN^{-2}$ for n large enough (as for (46)), the above term is of order N^{-1}. As a consequence both (69) and (70) converge to zero.

Step IV: Orthogonality

We prove that the following convergence holds in probability

$$\langle\rho(n)^{-1/2}2(AB^{(i)}_{M,n,N}+AB^{(ii)}_{M,n,N}+AC^{(i)}_{M,n,N}+AC^{(ii)}_{M,n,N}),W\rangle\longrightarrow 0.$$

We consider the term $2AB^{(i)}_{M,n,N}$. The others are analogous. By Itô formula we have

$$
\begin{aligned}
&\langle \rho(n)^{-1/2} 2AB^{(i)}_{M,n,N}\,,\, W\rangle \\
&= \rho(n)^{-1/2} \int_0^{2\pi} 2 \int_0^u \int_0^s \frac{1}{2\pi} \bar{D}_{M,n}(s-v)\sigma^2(v)dv D_{N,n}(u-s)\sigma(s)dW_s \sigma(u)du \\
&+ \rho(n)^{-1/2} \int_0^{2\pi} 2 \int_0^u \int_0^s D_{N,n}(s-v)\sigma(v)dW_v \frac{1}{2\pi} \bar{D}_{M,n}(u-s)\sigma^2(s)ds\,\sigma(u)du.
\end{aligned}
$$

In the following, we study in detail the first term since the second is analogous. Let $F_M\,(s,s)$ be defined as

$$
F_M\,(s,s) \doteq 2\int_0^s \frac{1}{2\pi} \bar{D}_M\,(s-v)\,\sigma^2\,(v)\ dv.
$$

Then, by using the fact that $M/n \to 0$ and Assumption (A.II), it is enough to prove that

$$
\rho\,(n)^{-1} \int_0^{2\pi}\int_0^{2\pi} \mathbb{E}\left[\left|\int_0^u F_M\,(s,s)\,D_{N,n}\,(u-s)\,\sigma\,(s)\,dW_s \int_0^{u'} F_M\,(s',s')\,D_{N,n}(u'-s')\sigma(s')dW_{s'}\right|\right] du\,du' \to 0. \tag{71}
$$

Without loss of generality consider the case $u \le u'$. First consider, for any $\varepsilon > 0$, the case $u < u' - \varepsilon$ and split (71) into

$$
\rho\,(n)^{-1} \int_0^{2\pi}\int_0^{2\pi} \mathbb{E}\left[\left|\int_0^u F_M\,(s,s)\,D_{N,n}\,(u-s)\,\sigma\,(s)\ dW_s \int_0^{u'-\varepsilon} F_M(s',s')D_{N,n}(u'-s')\sigma(s^{'})dW_{s'}\right|\right] du\,du' \tag{72}
$$

$$
+\rho\,(n)^{-1} \int_0^{2\pi}\int_0^{2\pi} \mathbb{E}\left[\left|\int_0^u F_M\,(s,s)\,D_{N,n}\,(u-s)\,\sigma\,(s)\ dW_s \int_{u'-\varepsilon}^{u'} F_M\,(s',s')\,D_{N,n}(u'-s')\sigma(s')dW_{s'}\right|\right] du\,du'. \tag{73}
$$

We consider first (72). By applying the Cauchy Schwartz inequality, the expectation multiplied by $\rho(n)^{-1}$, is less or equal than

$$
C\rho\,(n)^{-1}\,\mathbb{E}\left[\int_0^u F_M^2\,(s,s)\,D_{N,n}^2\,(u-s)\,\sigma^2\,(s)\ ds\right]^{1/2} \mathbb{E}\left[\int_0^{u'-\varepsilon} F_M^2\,(s',s')\,D_{N,n}^2\,(u'-s')\,\sigma^2\,(s')\ ds'\right]^{1/2},
$$

which is of order $O\left(n^{-1/2}\right)$ thanks to *Step I* and 2. Consider now (73). By applying Itô formula to the product of the two stochastic integrals, it splits into three terms

$$
\int_0^{2\pi} 1_{[0,u]\cap[u'-\varepsilon,u']}(s) F_M^2\,(s,s)\,D_{N,n}\,(u-s)\,D_{N,n}\,(u'-s)\,\sigma^2\,(s)\ ds \tag{74}
$$

$$
+ \textstyle\int_0^{2\pi}\int_0^s 1_{[0,u]}(s')F_M\,(s',s')\,D_{N,n}\,(u-s')\,\sigma\,(s')\,dW_{s'} 1_{[u'-\varepsilon,u']}(s)F_M(s,s)D_{N,n}\,(u'-s)\,\sigma\,(s)\ dW_s \tag{75}
$$

$$
+ \textstyle\int_0^{2\pi}\int_0^s 1_{[u'-\varepsilon,u']}(s')F_M(s',s')D_{N,n}\,(u'-s')\,\sigma(s')dW_{s'} 1_{[0,u]}(s)F_M\,(s,s)\ D_{N,n}\,(u-s)\,\sigma\,(s)\ dW_s. \tag{76}
$$

In particular, it is easily seen that all the terms are identically zero as $u < u' - \varepsilon$. Finally, we consider the case $u' - \varepsilon \leq u \leq u'$ and prove that the following term converges to zero

$$\rho(n)^{-1} \int_0^{2\pi} \int_0^{2\pi} 1_{[u'-\varepsilon,u']}(u) \mathbb{E}\left[\left| \int_0^u F_M(s,s) D_{N,n}(u-s) \sigma(s) dW_s \int_0^{u'} F_M(s,s) D_{N,n}(u'-s') \sigma(s') dW_{s'} \right|\right] du du'.$$

By applying, again, the Cauchy-Schwartz inequality and by using the result in *Step I*, the previous term is less or equal to

$$\int_0^{2\pi} du \int_0^{2\pi} du' 1_{[u'-\varepsilon,u']}(u) \left(\mathbb{E}\left[\rho(n)^{-1} \int_0^u D^2_{N,n}(u-s) \sigma^6(s) ds\right]^{1/2} \mathbb{E}\left[\rho(n)^{-1} \int_0^{u'} D^2_{N,n}(u'-s') \sigma^6(s') ds'\right]^{1/2} + o(1) \right) \leq 2\varepsilon$$

for M large enough. The proof is now complete.

7 Appendix B: Auxiliary Lemmas

This Appendix contains some results about the Dirichlet and the rescaled Dirichlet kernel. We keep the notation used by [Malliavin and Mancino, 2009] and we denote with $D_N(\cdot)$ the *rescaled* Dirichlet kernel

$$D_N(x) = \frac{1}{2N+1} \sum_{|k| \leq N} e^{\mathrm{i}kx}. \tag{77}$$

We let $\bar{D}_N(\cdot) \doteq (2N+1) D_N(\cdot)$, that is the Dirichlet kernel.

Lemma 1 *Let $\bar{D}_M(\cdot)$ be the Dirichlet kernel. The following results hold.*

i) For any M

$$\int_{-\pi}^{\pi} \bar{D}_M(x)\, dx = 2\pi,$$

ii) Let $\gamma > 1$ and suppose that $\lim \frac{n}{M^\gamma} = K$ for some constant $K > 0$, then

$$\lim_{n,M\to\infty} \int_{-\pi}^{\pi} \bar{D}_M(\varphi_n(x))\, dx = \lim_{M\to\infty} \int_{-\pi}^{\pi} \bar{D}_M(x)\, dx = 2\pi,$$
$$\lim_{n,M\to\infty} \int_{-\pi}^{\pi} \frac{1}{2M+1} \bar{D}^2_M(\varphi_n(x))\, dx = 2\pi.$$

iii) Let $\gamma > 1$ and suppose that $\lim \frac{n}{M^\gamma} = K$ for some constant $K > 0$ and let f be a ν-Hölder continuous function with $\nu \in (0,1]$. Then

$$\lim_{n,M\to\infty} \int_{-\pi}^{\pi} \frac{1}{2M+1} \bar{D}^2_M(y - \varphi_n(x)) f(x)\, dx = f(y). \tag{78}$$

Proof The proof follows the lines of Lemma 5.1 of [Cuchiero and Teichmann, 2015].

Lemma 2 *Let $D_N(\cdot)$ be the rescaled Dirichlet kernel. Under the assumption that $\lim_{n,N\to\infty} \frac{N}{n} = c_N > 0$, the following results hold.*

i) For any $p > 1$ *there exists a constant* C_p *such that:*

$$\lim_{n,N\to\infty} n \sup_{x\in[0,2\pi]} \int_0^{2\pi} D^p_{N,n}(x-y)dy \leq C_p,$$

ii)

$$\lim_{n,N\to\infty} n \int_0^x D^2_{N,n}(x-y)dy = \pi(1+2\eta(c_N))$$

where

$$\eta(c_N) \doteq \frac{1}{4c_N^2} r(2c_N)(1-r(2c_N)) \tag{79}$$

and $r(x) = x - [x]$, *with* $[x]$ *the integer part of* x.

iii) For any $x < t$ *we have*

$$\lim_{N,n\to\infty} n \int_0^x D^2_{N,n}(t-y)\,dy = 0.$$

iv) For any $x < t$ *and* f *a Hölder continuous with parameter* $\nu \in (0,1]$ *we have*

$$\lim_{N,n\to\infty} n \int_0^t \int_0^x D^2_{N,n}(x-y)f(x,y)dy\,dx = \pi(1+2\,\eta(c_N)) \int_0^t f(x,x)dx.$$

Proof See [Clement and Gloter, 2011] Lemma 1 and Lemma 4.