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Surface operators in 5d gauge theories and duality relations

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Abstract: We study half-BPS surface operators in 5d $\mathcal{N}=1$ gauge theories compactified on a circle. Using localization methods and the twisted chiral ring relations of coupled 3d/5d quiver gauge theories, we calculate the twisted chiral superpotential that governs the infrared properties of these surface operators. We make a detailed analysis of the localization integrand, and by comparing with the results from the twisted chiral ring equations, we obtain constraints on the 3d and 5d Chern-Simons levels so that the instanton partition function does not depend on the choice of integration contour. For these values of the Chern-Simons couplings, we comment on how the distinct quiver theories that realize the same surface operator are related to each other by Aharony-Seiberg dualities.

Keywords: Supersymmetric Gauge Theory, Chern-Simons Theories, Nonperturbative Effects, Supersymmetry and Duality

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Contents

1 Introduction and summary

Surface operators were first introduced in [1, 2] as half-BPS defects of codimension two that solve the Kapustin-Witten equations in four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theories (see [3] for an overview). By giving a mass to the adjoint hypermultiplet and flowing to the infra-red (IR), these defects naturally lead to surface operators in pure $\mathcal{N} = 2$ gauge theories in four dimensions. These surface operators have been extensively studied from many different points of view [4–30].

The present paper contains a generalization of our previous work [28], in which we studied surface operators in pure Yang-Mills theories with gauge group $\mathrm{SU}(N)$ and eight supercharges in four and five dimensions, following two approaches. In the first approach, we made use of the microscopic description offered by Nekrasov localization [31, 32], suitably adapted to the case with surface operators [17, 26–28], and computed the (ramified) instanton partition function. In the second approach, we considered quiver gauge theories [4, 18] in two (or three) dimensions with an additional $\mathrm{SU}(N)$ flavour symmetry realized by a gauge theory in four (or five) dimensions. From this standpoint, one deals with combined 2d/4d (or 3d/5d) systems, whose low-energy effective action is encoded in a pair of holomorphic functions: the prepotential, which governs the dynamics in four (or five) dimensions, and the twisted chiral superpotential, which describes the massive vacua of the quiver theories in two (or three) dimensions. Following the general ideas of [18] and using a careful mapping of parameters, in [28] we were able to match the twisted superpotential computed using localization methods with the one obtained by solving the chiral ring equations in the quiver theory approach.

In the 2d/4d case there are distinct quiver descriptions for the same surface operator [27, 28] in which the corresponding 2d theories are related to each other by Seiberg-like dualities [33–35]. From the localization point of view, these distinct ultra-violet (UV) descriptions correspond to different choices of the integration contours along which one computes the integral over the (ramified) instanton moduli space to obtain the Nekrasov partition function. When these theories are lifted to 3d/5d systems, some novel features arise. Indeed, as we have shown in [28], suitable Chern-Simons terms in three dimensions are needed in order to ensure the equality of the twisted superpotentials in dual descriptions. This is not too surprising since the 3d quiver theories include bi-fundamental matter multiplets that are rendered massive by twisted masses. When one integrates out these massive chiral fields, one generates effective Chern-Simons interactions. Furthermore, since dual pairs in three dimensions are related by Aharony-Seiberg dualities [33, 36, 37] which typically act on the Chern-Simons levels, we expect that the Chern-Simons couplings of two different quiver theories describing the same surface operator must be related in a precise manner. In [28] a few examples were worked out to highlight this phenomenon. We showed that the effective twisted chiral superpotential matched only for particular values of the 3d Chern-Simons levels in the dual pairs. In this work, we perform a complete and systematic analysis of coupled 3d/5d theories that have an interpretation as supersymmetric surface operators in $\mathcal{N} = 1$ gauge theories in five dimensions, allowing for both 3d as well as 5d Chern-Simons interactions, and provide a general description of the duality relations.

We now give an overview of this paper. In section 2, we review the localization analysis of the 5d $\mathcal{N} = 1$ super Yang-Mills theory compactified on a circle and present its instanton partition function, mainly following [38] (see also [39–44]). However, instead of directly working with the Young tableaux formulation, we work with the contour integral formulation.

In section 3, we study the 5d SU(N) theories in the presence of surface operators, which we treat as monodromy defects [1, 2] labeled by the partitions of N of length M. For any given partition we present the ramified instanton partition function that is obtained by a suitable $\mathbb{Z}_M$ orbifold projection on the instanton moduli space of the theory without defects [5, 17, 26]. The integrand of this ramified instanton partition function has the same set of poles as the one presented in [28] but it has additional exponential factors that depend on M new parameters, which we denote m_I, whose sum plays the role of the Chern-Simons coupling $\mathsf{k}_{5\mathrm{d}}$ of the five dimensional SU(N) gauge theory. To obtain explicit results, one must specify the integration contours in the instanton partition function, which can be conveniently classified by a Jeffrey-Kirwan reference vector [45] (see [27, 28] for details). Here we present two choices which are complementary to each other and are simple extensions of those discussed in the pure 5d theory. For these two choices we compute the twisted chiral superpotential by explicitly evaluating the residues over the poles selected by the integration contours.

In section 4, we go on to study surface operators as coupled 3d/5d systems and identify two quiver descriptions with $(M-1)$ 3d gauge nodes and an SU(N) flavour node that is gauged in five dimensions, which are dual to each other. The identification proceeds as follows: for a given 3d/5d quiver theory, we solve the twisted chiral ring equations about a particular classical vacuum as a power series expansion in the strong coupling scales of the quiver theory. Then, we show that there is a one-to-one map between the choice of classical vacuum and the choice of discrete data that label a Gukov-Witten defect. In particular, the strong coupling scales of the 3d/5d quiver are mapped on to the Nekrasov instanton counting parameters, while the Chern-Simons levels of the 3d nodes of the quiver theory are related to the first $(M-1)$ parameters m_I of the localization calculation. However, the precise map depends on the choice of the contour prescription. In fact, with one prescription, these parameters are related to the Chern-Simons levels, but with the other they are related to the negative of the Chern-Simons levels.

In section 5, we revisit the conditions under which the two contour prescriptions yield equal results and interpret them as Aharony-Seiberg dualities [33, 36] between pairs of quiver theories. In this correspondence we find that the 3d Chern-Simons levels are integral or half-integral, depending on the ranks of the 3d/5d quiver. These constraints coincide with those derived in [46–48] by requiring the absence of a parity anomaly. Further, we find that the bounds on the 3d Chern Simons levels are the same as the ones obtained in [49] for what are called maximally chiral theories. While Aharony-Seiberg dual pairs exist for other types of 3d quivers also, it is only for the maximally chiral ones that the twisted masses (induced by the 5d Coulomb vacuum expectation values) completely lift the 3d Coulomb moduli space and render the 3d theory completely massive. This is consistent with the general analysis of [4] where it was shown that only the 2d (or 3d) massive theories can be embedded as surface operators in four (or five) dimensions. We therefore conclude that it is precisely such maximally chiral theories that have avatars as surface operators in 5d theories.

Finally, we collect some technical material in the appendices.

2 5d gauge theories

In this section we describe the derivation of the instanton partition function for a gauge theory with a Chern-Simons term in five dimensions, following the analysis of [38] that relies on the use of localization methods. This partition function has already been extensively studied in the literature (see for example [39–44]) but we review it here to set the stage for the analysis in the following sections. We then consider the resolvent of the 5d theory from the point of view of the Seiberg-Witten curve and establish a connection with the localization methods.

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2.1 Localization

Let us consider an $\mathcal{N}=1$ SU(N) gauge theory defined on $\mathbb{R}^4\times S^1$, and denote by β the length of the circumference S^1 and by $\mathsf{k}_{5\mathrm{d}}$ the coefficient of the Chern-Simons term. We study this theory in a generic point in the Coulomb branch parameterized by the vacuum expectation values a_u (with $u=1,\cdots,N$) of the adjoint scalar field Φ in the vector multiplet, that satisfy the SU(N) tracelessness condition

$$\sum_{u=1}^{N} a_u = 0 \tag{2.1}$$

but are otherwise arbitrary. Being at a generic point of the Coulomb branch, according to the analysis of [50], we must take

$$\mathsf{k}_{5\mathrm{d}}\in\mathbb{Z} \qquad \text{and} \qquad |\,\mathsf{k}_{5\mathrm{d}}\,|\leq N~. \tag{2.2}$$

The integrality constraint is a consequence of analyzing the non-compact 5d theory on the Coulomb branch and imposing gauge invariance of the resulting cubic prepotential, while the bound on $\mathsf{k}_{5\mathrm{d}}$ comes from requiring that the 5d gauge theory has an interacting UV fixed point on the entire Coulomb branch.[1]

After deforming $\mathbb{R}^4$ by an Ω-background [31, 32] parametrized by ϵ_1 and ϵ_2, we use localization methods to compute the partition function in the instanton sector. This can be written as

$$Z_{\text{inst}} = 1+\sum_{k=1}^{\infty}\frac{(-q)^k}{k!}\int_{\mathcal{C}}\prod_{\sigma=1}^{k}\left(\beta\,\frac{d\chi_\sigma}{2\pi\mathrm{i}}\right)z_k(\chi_\sigma) \tag{2.3}$$

where

$$\begin{aligned} z_k(\chi_\sigma) = \mathrm{e}^{-\beta\,\mathsf{k}_{5\mathrm{d}}\sum_\sigma\chi_\sigma}\prod_{\sigma,\tau=1}^{k}&\left[\frac{g\big(\chi_\sigma-\chi_\tau+\epsilon_1+\epsilon_2\big)}{g\big(\chi_\sigma-\chi_\tau+\epsilon_1\big)\,g\big(\chi_\sigma-\chi_\tau+\epsilon_2\big)}\right]\prod_{\substack{\sigma,\tau=1\\ \sigma\neq\tau}}^{k} g\big(\chi_\sigma-\chi_\tau\big)\\ \times\prod_{\sigma=1}^{k}\prod_{u=1}^{N}&\left[\frac{1}{g\big(\chi_\sigma-a_u+\frac{\epsilon_1+\epsilon_2}{2}\big)\,g\big(-\chi_\sigma+a_u+\frac{\epsilon_1+\epsilon_2}{2}\big)}\right]~. \end{aligned} \tag{2.4}$$

and [31, 32, 38, 53]

$$g(x) = 2\sinh\left(\frac{\beta\,x}{2}\right)~. \tag{2.5}$$

We observe that the Chern-Simons coefficient $\mathsf{k}_{5\mathrm{d}}$ only appears in the exponent of the prefactor in (2.4). The instanton counting parameter q is given by

$$q = (-1)^N(\beta\Lambda)^{2N} \tag{2.6}$$

[1] See also the recent work [51, 52] in which the results of [50] have been generalized by requiring that only a subspace of the Coulomb moduli space be physical. It would be interesting to investigate if the localization approach we are describing can be applied also to this case where novel massless degrees of freedom occur, but this is beyond the scope of this paper.

where Λ is the (complexified) strong-coupling scale. It is easy to check that in the limit $\beta \to 0$ the above expressions reduce to those appropriate for a pure $\mathcal{N} = 2$ super Yang-Mills theory in four dimensions with SU(N) gauge group and dynamically generated scale Λ.

The integral in (2.3) is performed on a closed contour $\mathcal{C}$ in the complex χ_σ-plane which has to be suitably chosen in such a way that it surrounds a finite number of singularities of the integrand function. If we make the standard choice for the imaginary part of the Ω-background parameters, namely

$$1 \gg \mathrm{Im}\,\epsilon_1 \gg \mathrm{Im}\,\epsilon_2 > 0\,, \tag{2.7}$$

and take a_u to be real for simplicity, the poles of (2.4) lie either in the upper or in the lower-half complex χ_σ-plane, and can be put in correspondence with an N-array of Young tableaux $\{Y_u\}$ such that the total number of boxes is equal to the instanton number k [31].[2] More precisely, the poles of (2.4) are located at

$$\chi_\sigma = a_u \pm \Big(i - \frac{1}{2}\Big)\epsilon_1 \pm \Big(j - \frac{1}{2}\Big)\epsilon_2 + \frac{2\pi\mathrm{i}}{\beta}\,n \tag{2.8}$$

where (i, j) run over the rows and columns of the Young tableau Y_u and the last term, proportional to the integer n, is due to the periodicity of the sinh-function of a complex variable. Notice that the Chern-Simons coupling $\mathsf{k}_{5\mathrm{d}}$ does not affect the location of the poles and it only adds additional multiplicative factors to the residue at each pole.

When we restrict to a fundamental domain by setting $n = 0$ in (2.8), we have only two sets of poles:[3] those that are just above the real axis and those that are just below it. Each of these two sets leads precisely to the results of [38, 40–44]. The poles in the region

$$0 < \mathrm{Im}\,\chi_\sigma < \frac{\pi}{\beta} \tag{2.9}$$

are selected by the contour $\mathcal{C}_+^{(\sigma)}$ as in figure 1 for the SU(3) theory at $k = 1$. Instead, the poles in the region

$$-\frac{\pi}{\beta} < \mathrm{Im}\,\chi_\sigma < 0 \tag{2.10}$$

are selected by the contour $\mathcal{C}_-^{(\sigma)}$ as in figure 2, again for the SU(3) theory at $k = 1$.

In both cases, the contours extend all the way to infinity along the horizontal direction, since the positions of the poles can have arbitrary real parts because the vacuum expectation values a_u are only subject to the condition (2.1) but are otherwise arbitrary. Another issue is the fact that the two integration contours $\mathcal{C}_\pm^{(\sigma)}$ may lead to different results. To illustrate the main ideas, it suffices to consider the 1-instanton term of the partition function, namely

$$Z_{1-\mathrm{inst}} = -q\,\frac{g(\epsilon_1+\epsilon_2)}{g(\epsilon_1)\,g(\epsilon_2)} \int_{\mathcal{C}} \Big(\beta\,\frac{d\chi}{2\pi\mathrm{i}}\Big)\,\mathrm{e}^{-\beta\,\mathsf{k}_{5\mathrm{d}}\,\chi} \prod_{u=1}^{N} \frac{1}{g\big(\chi - a_u + \frac{\epsilon_1+\epsilon_2}{2}\big)\,g\big(-\chi + a_u + \frac{\epsilon_1+\epsilon_2}{2}\big)}\,. \tag{2.11}$$

[2] For further details we refer for example to [44, 54].

[3] We observe that these two sets of poles are the same that are considered in the corresponding calculation in four dimensions [32].

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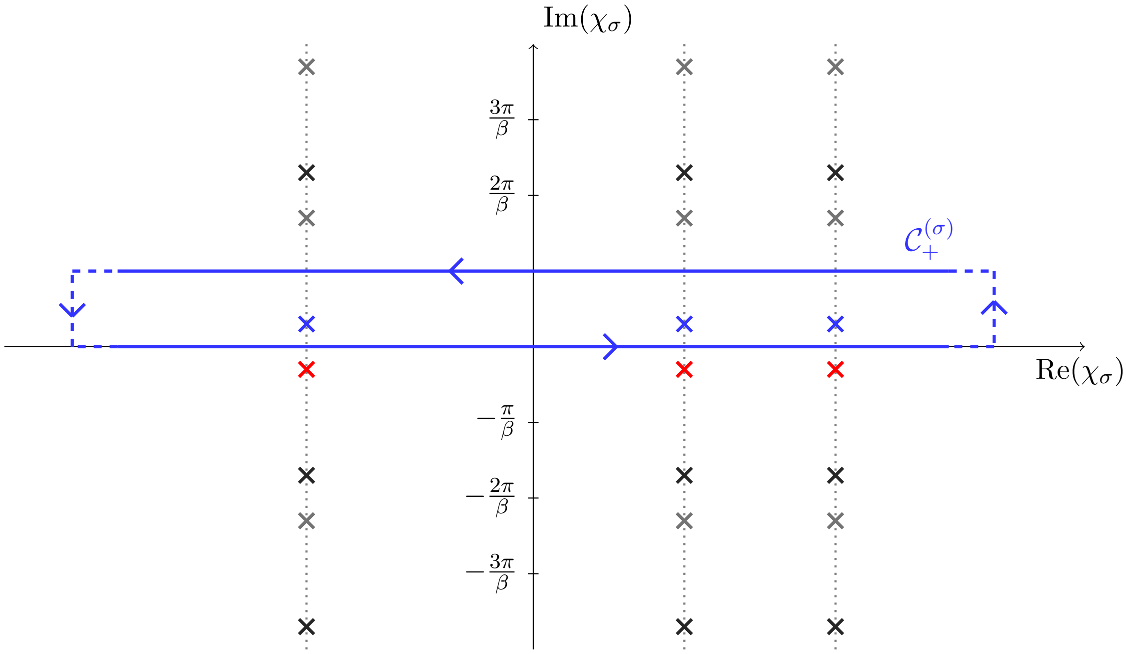

Figure 1. For each integration variable χ_σ, the fundamental domain is the region $-\infty < \mathrm{Re}\chi_\sigma < \infty$ and $-\frac{\pi}{\beta} < \mathrm{Im}\chi_\sigma < \frac{\pi}{\beta}$. The poles in the fundamental domain are shown in colour. The contour $\mathcal{C}_+^{(\sigma)}$ selects those poles in the fundamental domain that are in the upper half plane. In this picture we have explicitly shown the 1-instanton case for the SU(3) gauge theory at $k = 1$.

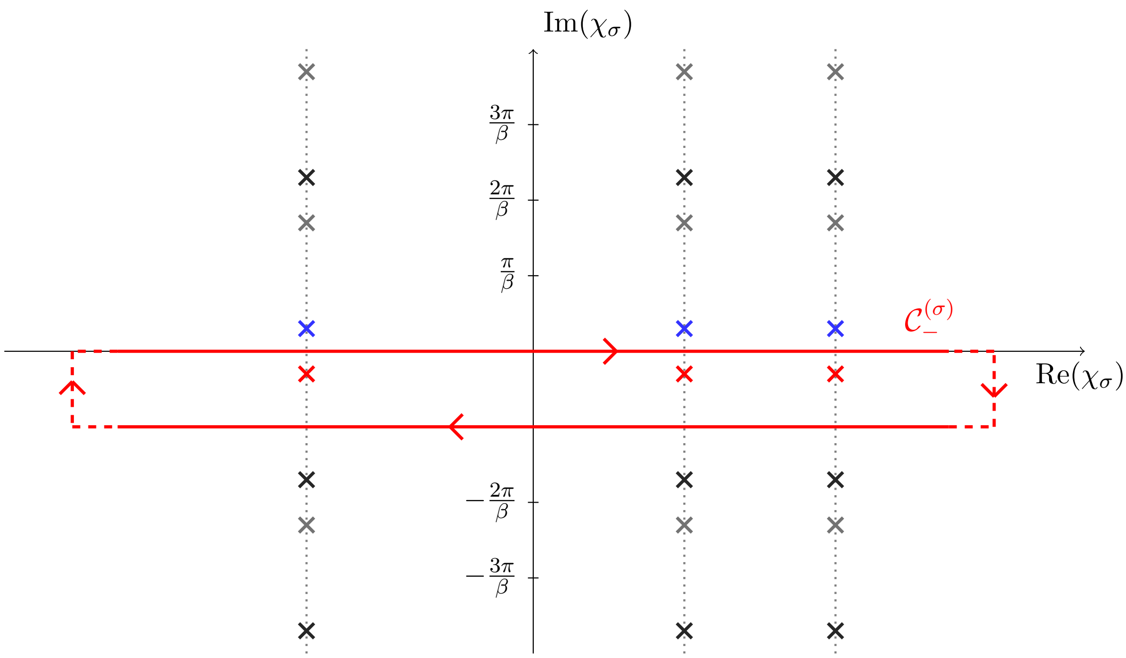

Figure 2. For each integration variable χ_σ, the contour $\mathcal{C}_-^{(\sigma)}$ selects those poles in the fundamental domain that are in the lower half plane. Once again, the poles that are shown in this picture are those for the SU(3) gauge theory at $k = 1$.

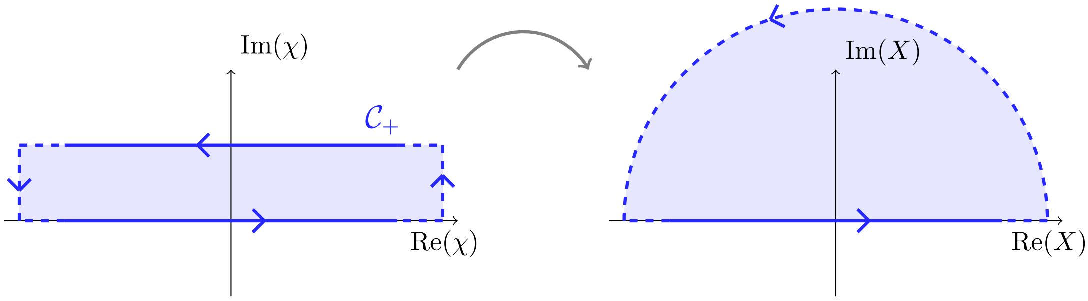

Figure 3. Map of the contour $\mathcal{C}_+$ from the χ-plane to the X-plane.

We find it convenient to perform the following change of variables

$$\chi = \frac{1}{\beta}\log X\,, \quad a_u = \frac{1}{\beta}\log A_u, \quad \epsilon_1 = \frac{1}{\beta}\log E_1\,, \quad \text{and} \;\; \epsilon_2 = \frac{1}{\beta}\log E_2\,, \tag{2.12}$$

and rewrite (2.11) as

$$Z_{1-\text{inst}} = -\,q\,\frac{E_1E_2-1}{(E_1-1)(E_2-1)}\int_{\mathcal{C}}\frac{dX}{2\pi\mathrm{i}}\,X^{N-1-\mathsf{k}_{5\mathrm{d}}}\prod_{u=1}^{N}\frac{\sqrt{E_1E_2}}{\big(X\sqrt{E_1E_2}-A_u\big)\big(A_u\sqrt{E_1E_2}-X\big)}\,. \tag{2.13}$$

Here we have exploited the tracelessness condition (2.1), which in the new variables becomes

$$\prod_{u=1}^{N} A_u = 1\,. \tag{2.14}$$

Under the map (2.12) the regions $0 < \operatorname{Im}\chi < \frac{\pi}{\beta}$ and $-\frac{\pi}{\beta} < \operatorname{Im}\chi < 0$ transform, respectively, onto the regions $\operatorname{Im} X > 0$ and $\operatorname{Im} X < 0$, and thus the fundamental domain of the χ-plane is mapped onto the entire X-plane. Furthermore, the original integration contours $\mathcal{C}_\pm$ are mapped to the (infinite) semi-circles as shown in figure 3 and figure 4. Therefore, choosing the contour $\mathcal{C}_+$ or $\mathcal{C}_-$ corresponds to choosing the poles of the integrand of (2.13), respectively in the upper- or in lower-half complex X-plane, that is

$$\begin{aligned} X &= A_u\sqrt{E_1\,E_2} && \text{for} \quad \mathcal{C}_+\,,\\ X &= \frac{A_u}{\sqrt{E_1\,E_2}} && \text{for} \quad \mathcal{C}_-\,. \end{aligned} \tag{2.15}$$

In this formulation it is evident that the constraint $\mathsf{k}_{5\mathrm{d}} \in \mathbb{Z}$ implies the absence of branch cuts in X; furthermore if

$$|\,\mathsf{k}_{5\mathrm{d}}\,| \le N-1\,, \tag{2.16}$$

one can easily see that the instanton partition function receives contributions only from the physical poles (2.15) or, equivalently, it has no contributions from $X = 0$ and $X = \infty$. Therefore, when the condition (2.16) is satisfied, the two different integration prescriptions lead to the same result for the partition function since the contours $\mathcal{C}_+$ and $\mathcal{C}_-$ can be smoothly deformed into each other.

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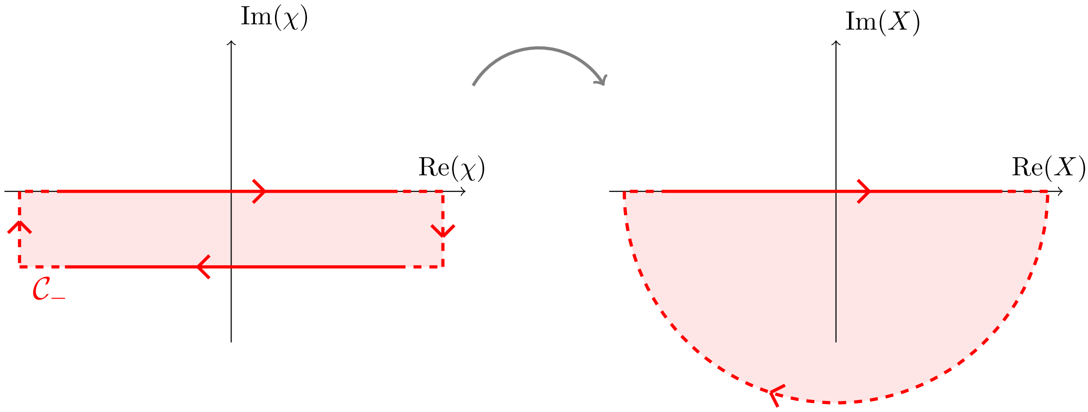

Figure 4. Map of the contour $\mathcal{C}_-$ from the χ-plane to the X-plane.

Notice that in the original χ-variable, imposing the condition $\mathsf{k}_{5\mathrm{d}} \in \mathbb{Z}$ is equivalent to requiring that the integrand function be periodic with period $\frac{2\pi}{\beta}$. When this is the case, the two contours $\mathcal{C}_\pm$ are equivalent to each other provided the contributions of the vertical segments at $\mathrm{Re}\,\chi = \pm\infty$ vanish. This happens precisely when (2.16) is satisfied. It is interesting to observe that when $|\mathsf{k}_{5\mathrm{d}}| = N$, the two contours $\mathcal{C}_\pm$ are not equivalent to each other due to the presence of a residue either at $X = 0$ for $\mathsf{k}_{5\mathrm{d}} = N$, or at $X = \infty$ for $\mathsf{k}_{5\mathrm{d}} = -N$. However, these residues are independent of A_u and N since the singularities are simple poles. They are related to the partition function of an "SU(1)" theory at level ± 1 [40, 41],[4] and thus can be interpreted as the contribution of a continuum in the Coulomb branch which has to be suitably taken into account and decoupled in order to properly define the SU(N) theory at $\mathsf{k}_{5\mathrm{d}} = \pm N$ [40–44]. In this way we recover via the contour analysis that the five dimensional Chern-Simons coupling satisfies the constraint obtained by [50]. For simplicity, in the following we will restrict ourselves to $\mathsf{k}_{5\mathrm{d}}$ as in (2.16).

2.2 Seiberg-Witten curve and resolvent

We now review the Seiberg-Witten geometry [55, 56] of an SU(N) gauge theory on $\mathbb{R}^4 \times S^1$ and propose an all-order expression for the resolvent which we shall verify using explicit localization methods. The Seiberg-Witten curve can be derived from different approaches. One way is to study M-theory on the resolution of non-compact toric Calabi-Yau spaces, the so-called $Y^{p,q}$ manifolds, which give rise to SU(p) gauge theories with $\mathsf{k}_{5\mathrm{d}} = q$ [53, 57, 58].[5] The corresponding Seiberg-Witten curve is identified with the mirror curve of the local (toric) Calabi-Yau space [53, 61]. In most of the literature, the $Y^{p,q}$ spaces are defined with $0 < q < p$ and thus only positive values of the Chern-Simons level are considered.[6] However, as we will see momentarily, the form of the resulting Seiberg-Witten curve is also valid for negative values of $\mathsf{k}_{5\mathrm{d}}$, although there are interesting subtleties that arise while comparing with localization analysis. An alternative approach is to use the NS5-D4

[4] This can be easily seen by taking (2.13) and (2.14) for $N = 1$ and $\mathsf{k}_{5\mathrm{d}} = \pm 1$.

[5] One can also study the gauge theories using 5-brane webs that are dual to the toric Calabi-Yau [59, 60].

[6] The boundary values $q = 0$ and $q = p$ are discussed in [57, 58].

brane set up [62] to engineer the classical gauge theory and study its M-theory lift [63]. Both approaches give identical results and the Seiberg-Witten curve for a 5d SU(N) gauge theory with Chern-Simons level $\mathsf{k}_{5\mathrm{d}}$ takes the following form

$$Y^2 = P_N^2(Z) - 4(\beta\Lambda)^{2N} Z^{-\mathsf{k}_{5\mathrm{d}}} \tag{2.17}$$

where

$$P_N(Z) = Z^{-\frac{N}{2}} \left(Z^N + \sum_{i=1}^{N-1} (-1)^i \, Z^{N-i} \, U_i(\mathsf{k}_{5\mathrm{d}}) + (-1)^N \right) . \tag{2.18}$$

Here $U_i(\mathsf{k}_{5\mathrm{d}})$ are the gauge invariant coordinates on the Coulomb branch of the 5d theory. They are the quantum completion of the classical symmetric polynomials

$$U_i^{\mathrm{class}} = \sum_{u_1 \neq u_2 \cdots \neq u_i = 1}^{N} A_{u_1} A_{u_2} \cdots A_{u_i} \tag{2.19}$$

in the vacuum expectation values A_u subjected to the tracelessness condition (2.14), and explicitly depend on the Chern-Simons level. Notice that if we use (2.19) in (2.18), we simply obtain

$$P_N^{\mathrm{class}}(Z) = Z^{-\frac{N}{2}} \prod_{u=1}^{N} (Z - A_u) \, , \tag{2.20}$$

which is the expected classical expression for P_N. If we now impose the condition that the right hand side of (2.17) is a doubly monic Laurent polynomial in Z, it follows that the absolute value of the Chern-Simons level $|\mathsf{k}_{5\mathrm{d}}|$ has to be an integer smaller than N. We thereby recover the constraint (2.16) from the geometry of the Seiberg-Witten curve. While the curve (2.17) was derived for positive values of $\mathsf{k}_{5\mathrm{d}}$, it is easy to realize that it holds for negative values as well. Indeed, from the brane-web construction, one can show that changing the sign of the Chern-Simons coupling amounts to a π-rotation of the brane configuration. In our explicit realization this corresponds to

$$Z \to \frac{1}{Z} \qquad \text{and} \qquad A_u \to \frac{1}{A_u} \, . \tag{2.21}$$

Let us start from the curve (2.17) with a positive $\mathsf{k}_{5\mathrm{d}}$ and perform the above map. This yields

$$Y^2 = Z^{-N} \left(Z^N + \sum_{i=1}^{N-1} (-1)^i \, Z^{N-i} \, \widetilde{U}_{N-i}(\mathsf{k}_{5\mathrm{d}}) + (-1)^N \right)^2 - 4(\beta\Lambda)^{2N} Z^{\mathsf{k}_{5\mathrm{d}}} \tag{2.22}$$

where $\widetilde{U}_i$ is obtained from U_i under the inversion of A_u. By setting

$$\widetilde{U}_{N-i}(\mathsf{k}_{5\mathrm{d}}) = U_i(-\mathsf{k}_{5\mathrm{d}}) \, , \tag{2.23}$$

we can rewrite (2.22) as

$$Y^2 = Z^{-N} \left(Z^N + \sum_{i=1}^{N-1} (-1)^i \, Z^{N-i} \, U_i(-\mathsf{k}_{5\mathrm{d}}) + (-1)^N \right)^2 - 4(\beta\Lambda)^{2N} Z^{-(-\mathsf{k}_{5\mathrm{d}})} \tag{2.24}$$

and interpret it as the curve describing an SU(N) theory with Chern-Simons coupling $-\mathsf{k}_{5\mathrm{d}}$, since it has exactly the same form of (2.17). At the classical level, i.e. for $\beta\Lambda \to 0$, it is trivial to check that $\widetilde{U}_{N-i}^{\text{class}} = U_i^{\text{class}}$. Indeed, it suffices to perform the inversion of A_u in (2.19) and use the tracelessness condition (2.14). What is less obvious is to check the relation (2.23) at the quantum level, i.e. when the non-perturbative corrections are taken into account. In appendix A we explicitly verify this relation exploiting the localization calculation of the chiral correlators at 1-instanton. This provides a clear confirmation of the fact that the Seiberg-Witten curve takes the form (2.17) for negative Chern-Simons levels also. As a bonus, we see that the constraint (2.16) has a natural interpretation also from the point of view of the Seiberg-Witten curve.

We now turn to the resolvent of the 5d gauge theory. This is the generating function of all the chiral correlators and is defined as the following expectation value [64]:

$$T = \left\langle \mathrm{Tr}\ \coth \frac{\beta(z-\Phi)}{2} \right\rangle = \frac{2}{\beta}\frac{\partial}{\partial z}\left\langle \mathrm{Tr}\ \log\left(2\sinh\frac{\beta(z-\Phi)}{2}\right)\right\rangle \tag{2.25}$$

where Φ is the complex scalar field of the adjoint vector multiplet. Setting

$$z = \frac{1}{\beta}\log Z \tag{2.26}$$

and expanding for large Z, we find

$$T = N + 2\sum_{\ell=1}^{\infty}\frac{V_\ell}{Z^\ell} \tag{2.27}$$

where

$$V_\ell = \left\langle \mathrm{Tr}\, \mathrm{e}^{\ell\,\beta\,\Phi}\right\rangle . \tag{2.28}$$

Of course, due to the SU(N) condition (2.14), only the correlators V_ℓ with $\ell = 1, \cdots, N-1$ are independent of each other.

We propose that the integral of the resolvent is given by

$$\left\langle \mathrm{Tr}\ \log\left(2\sinh\frac{\beta(z-\Phi)}{2}\right)\right\rangle = \log\left(\frac{P_N(Z)+Y}{2}\right) \tag{2.29}$$

where Y satisfies the Seiberg-Witten curve in (2.17), and z is related to Z as in (2.26). This proposal is suggested by the fact that the quantity appearing on the right hand side is closely related to the Seiberg-Witten differential of the 5d gauge theory [58]. Differentiating (2.29) with respect to z, after a straightforward calculation we obtain the explicit expression for the resolvent in terms of the function appearing in the Seiberg-Witten curve, namely

$$T = \frac{2}{\beta}\frac{\partial}{\partial z}\left[\log\left(\frac{P_N(Z)+Y}{2}\right)\right] = 2\,Z\,\frac{P_N'(Z)}{Y} - \mathsf{k}_{5\mathrm{d}}\left(1-\frac{P_N(Z)}{Y}\right) \tag{2.30}$$

where $'$ stands for the derivative with respect to Z. The first term is precisely the 5d lift of the classic result from [65], which was already used in [28] for the case $\mathsf{k}_{5\mathrm{d}} = 0$. The second term is the modification due to the Chern-Simons coupling.

Inserting (2.17) and (2.18) into the right hand side of (2.30) and expanding for large Z, we obtain an expression for the resolvent in terms of the gauge invariant coordinates $U_i(\mathsf{k}_{5\mathrm{d}})$. This can then be compared with (2.27) to establish a relation with the chiral correlators V_ℓ. Proceeding this way, we find for example

$$U_1(\mathsf{k}_{5\mathrm{d}}) = V_1 - (\beta\Lambda)^{2N}\,\delta_{\mathsf{k}_{5\mathrm{d}},1-N}\ . \tag{2.31}$$

Similar relations can be found for the higher $U_i(\mathsf{k}_{5\mathrm{d}})$'s as we show explicitly in appendix A. There, we also show that the correlators V_ℓ can be calculated order by order in the instanton expansion using localization methods involving the partition function (2.3) with suitable insertions. Thus, our proposal for the resolvent provides a systematic way to obtain the explicit non-perturbative expressions for $U_i(\mathsf{k}_{5\mathrm{d}})$ in terms of the Coulomb vacuum expectation values of the 5d gauge theory. These can be used to check the relation (2.23), thus confirming the consistency of the whole construction. Furthermore, as we will see in the next section, this knowledge will prove to be an essential ingredient to study surface operators as coupled 3d/5d gauge theories.

3 5d gauge theories with surface operators

We now turn to the study of SU(N) gauge theories on $\mathbb{R}^4 \times S^1$ in the presence of a surface operator extended along a plane $\mathbb{R}^2 \subset \mathbb{R}^4$ and wrapped around S^1. We treat such surface operators as monodromy defects, also known as Gukov-Witten defects [1, 2]. The discrete data that label these defects are the partitions of N, i.e. the sets of positive integers $\vec{n} = [n_1, n_2, \ldots, n_M]$ such that $\sum_{i=1}^{M} n_i = N$. They are related to the breaking pattern (or Levi decomposition) of the gauge group near the defect as follows,

$$\mathrm{SU}(N) \quad\longrightarrow\quad \mathrm{S}\big[\mathrm{U}(n_1)\times\ldots\times\mathrm{U}(n_M)\big]\ . \tag{3.1}$$

The instanton partition function in the presence of such a defect can be obtained by generalizing the pure five-dimensional analysis presented in the previous section with the addition of a $\mathbb{Z}_M$ orbifold projection [17], along the lines discussed in [28] in the absence of Chern-Simons interactions. The result is the partition function for the so-called ramified instantons.

3.1 Ramified instantons

Let us introduce a partition of order M and, for each sector $I = 1, \cdots, M$, consider d_I ramified instantons.[7] The partition function for such a configuration can be written as

$$Z_{\mathrm{inst}}[\vec{n}] = \sum_{\{d_I\}} Z_{\{d_I\}}[\vec{n}] \tag{3.2}$$

where

$$Z_{\{d_I\}}[\vec{n}] = \prod_{I=1}^{M}\left[\frac{(-q_I)^{d_I}}{d_I!}\int_{\mathcal{C}}\prod_{\sigma=1}^{d_I}\left(\beta\,\frac{d\chi_{I,\sigma}}{2\pi\mathrm{i}}\right)\mathrm{e}^{-\beta\,\mathsf{m}_I\chi_{I,\sigma}}\right] z_{\{d_I\}} \tag{3.3}$$

[7] Here and in the following, the index I is always taken modulo M.

with

$$z_{\{d_I\}} = \prod_{I=1}^{M} \Big[\prod_{\sigma,\tau=1}^{d_I} \frac{1}{g(\chi_{I,\sigma} - \chi_{I,\tau} + \epsilon_1)} \prod_{\substack{\sigma,\tau=1\\ \sigma\neq\tau}}^{d_I} g(\chi_{I,\sigma} - \chi_{I,\tau}) \Big]$$
$$\times \prod_{I=1}^{M} \prod_{\sigma=1}^{d_I} \prod_{\rho=1}^{d_{I+1}} \frac{g\left(\chi_{I,\sigma} - \chi_{I+1,\rho} + \epsilon_1 + \hat{\epsilon}_2\right)}{g\left(\chi_{I,\sigma} - \chi_{I+1,\rho} + \hat{\epsilon}_2\right)} \tag{3.4}$$
$$\times \prod_{I=1}^{M} \Big[\prod_{\sigma=1}^{d_I} \prod_{s=1}^{n_I} \frac{1}{g\left(a_{I,s} - \chi_{I,\sigma} + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)\right)} \prod_{t=1}^{n_{I+1}} \frac{1}{g\left(\chi_{I,\sigma} - a_{I+1,t} + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)\right)} \Big] .$$

Here q_I is the instanton weight in the I-th sector, and $\hat{\epsilon}_2 = \epsilon_2/M$ as a consequence of the $\mathbb{Z}_M$ orbifold projection. Note that these expressions are the same as those in [28], apart from a minor modification in the integrand of (3.3) represented by exponential factors that introduce a coupling to $\mathrm{Tr}\,\chi_I$ with coefficient m_I. Anticipating the description of surface operators from the point of view of 3d/5d coupled theories [4, 18], we propose (3.2) and (3.3) to be the generalization of the results of [28] when Chern-Simons terms are included in the 3d gauge theories defined on the world-volume of the defects. We will provide strong evidence for this in the following sections.

We now describe how to evaluate the integrals (3.3) over $\chi_{I,\sigma}$. The procedure is quite similar to what we saw in the previous section. The first step is the choice of the integration contour and the prescription to pick up the poles of the integrand (3.4). A convenient way to classify the possible contours of interest is via the Jeffrey-Kirwan (JK) parameter η [45]. Different choices of η correspond to picking different sets of poles in (3.3), which may lead to different results for the instanton partition function. Such issues become even more subtle once we introduce the parameters m_I, since non-trivial residues at zero or infinity, and even branch cuts may appear. We now describe the two choices of contour that were already introduced in [27, 28].

Prescription $\mathbf{JK_I}$. In our first prescription for the integration contour, the JK parameter is[8]

$$\eta = -\sum_{I=1}^{M-1} \chi_I + \xi\, \chi_M \tag{3.5}$$

with ξ an arbitrary large positive number. Using (2.7), one can see that this choice is equivalent to selecting the poles for $\chi_{I,\sigma}$ as follows

$$\begin{aligned} 0 < \mathrm{Im}\,\chi_{I,\sigma} < \frac{\pi}{\beta} \qquad &\text{for} \quad I = 1,\dots,M-1 \quad \text{and} \quad \sigma = 1,\dots,d_I\,,\\ -\frac{\pi}{\beta} < \mathrm{Im}\,\chi_{M,\sigma} < 0 \qquad &\text{for} \quad \sigma = 1,\dots,d_M\ . \end{aligned} \tag{3.6}$$

This corresponds to choosing the contour $\mathcal{C}_+$ for the first $M-1$ sets of integration variables and the contour $\mathcal{C}_-$ for the Mth set. The two contours $\mathcal{C}_+$ and $\mathcal{C}_-$ are shown, respectively, in figure 5 and figure 6, for the SU(3) theory in the presence of the [1,2] surface operator, at the 1-instanton level.

[8] For details see for example [27, 28].

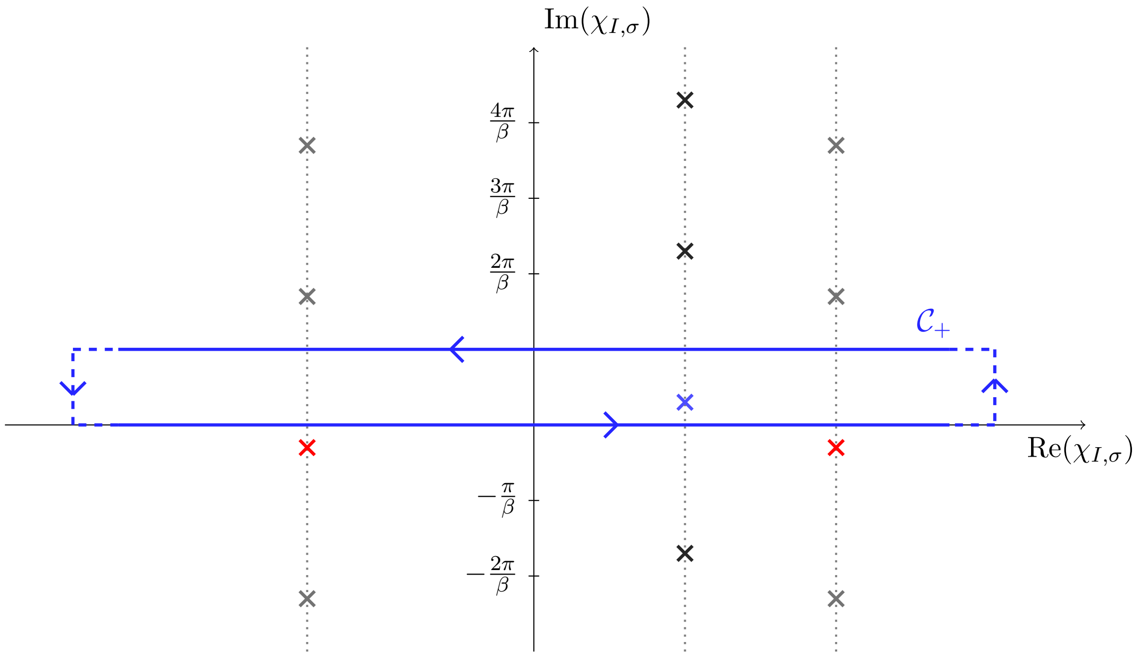

Figure 5. The contour $\mathcal{C}_+$ for the case of the $[1, 2]$ surface operator in SU(3) at 1-instanton.

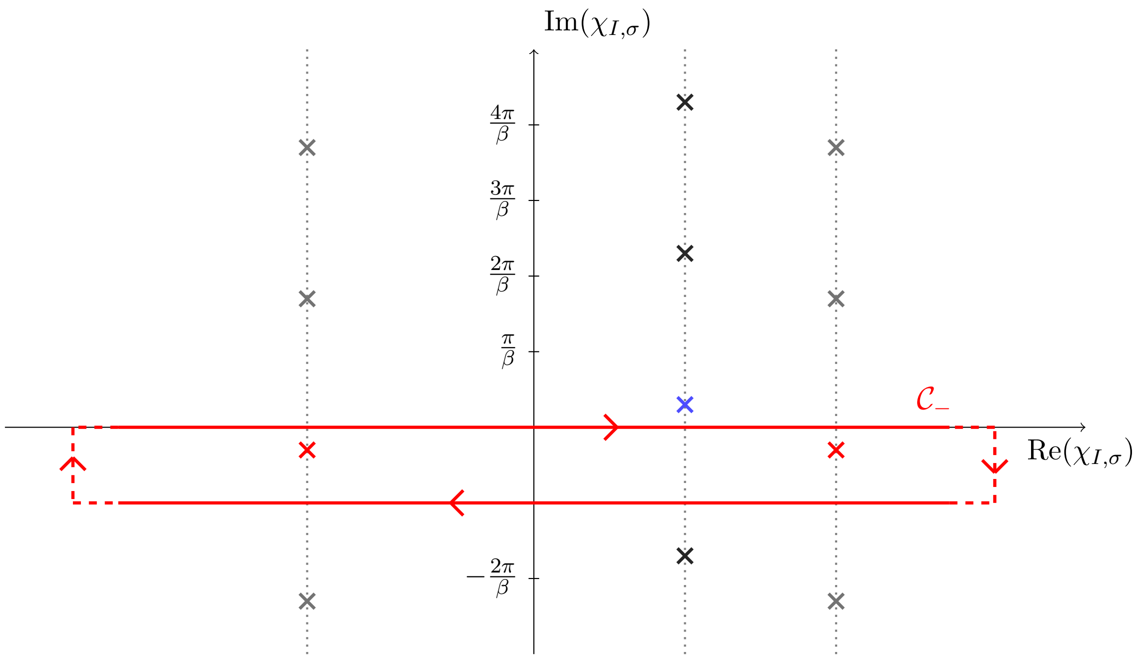

Figure 6. The contour $\mathcal{C}_-$ for the case of the $[1, 2]$ surface operator in SU(3) at 1-instanton

Prescription $\mathrm{JK_{II}}$. In our second prescription the JK parameter is given by

$$\widetilde{\eta} = \sum_{I=1}^{M-1} \chi_I - \xi\, \chi_M \tag{3.7}$$

where again ξ is an arbitrary large positive number. This corresponds to choosing the poles as follows

$$\begin{aligned} -\frac{\pi}{\beta} < \mathrm{Im}\, \chi_{I,\sigma} < 0 \qquad & \text{for} \quad I = 1, \dots, M-1 \quad \text{and} \quad \sigma = 1, \dots, d_I\,, \\ 0 < \mathrm{Im}\, \chi_{M,\sigma} < \frac{\pi}{\beta} \qquad & \text{for} \quad \sigma = 1, \dots, d_M\;. \end{aligned} \tag{3.8}$$

Equivalently we can say that one selects the contour $\mathcal{C}_-$ (see figure 6) for the first $M-1$ sets of χ-variables and the contour $\mathcal{C}_+$ (see figure 5) for the Mth set. This prescription is clearly complementary to the first one.

Our goal is to understand how and when the two prescriptions $\mathrm{JK_I}$ and $\mathrm{JK_{II}}$ can be related by contour deformation so that the partition functions obtained via the two match. To illustrate this point it suffices again to focus on the 1-instanton case.

3.2 The 1-instanton partition function

Let us consider the 1-instanton contribution to the partition function for a general surface operator of type $\vec{n} = [n_1, n_2, \dots, n_M]$. To express the formulas in a compact form, it is convenient to introduce the integers

$$r_I = \sum_{J=1}^{I} n_J \tag{3.9}$$

which will be used also in section 4. We also choose an ordering such that the Coulomb vacuum expectation values are partitioned as follows

$$\big\{a_1, \dots, a_{r_1}\big| \dots \big|a_{r_{I-1}+1}, \dots a_{r_I}\big|a_{r_I+1}, \dots a_{r_{I+1}}\big| \dots \big|a_{r_{M-1}+1}, \dots, a_N\big\}\;. \tag{3.10}$$

From the definition (3.9), it is clear that each partition is of length n_I. Compared to the notation we have used in (3.3), this ordering corresponds to

$$a_{I,s} = a_{r_{I-1}+s} \quad \text{for} \quad s = 1, \dots n_I \tag{3.11}$$

with the understanding that $r_0 = 0$. Using this notation, the 1-instanton partition function in the presence of a generic surface operator becomes

$$\begin{aligned} Z_{1-\mathrm{inst}} = -\frac{1}{g(\epsilon_1)} \sum_{I=1}^{M} q_I \int_{\mathcal{C}} \left(\beta\, \frac{d\chi_I}{2\pi \mathrm{i}}\right) \mathrm{e}^{-\beta \mathsf{m}_I \chi_I} \prod_{\ell=r_{I-1}+1}^{r_I} \frac{1}{g\big(a_\ell - \chi_I + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)\big)} \\ \times \prod_{j=r_I+1}^{r_{I+1}} \frac{1}{g\big(\chi_I - a_j + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)\big)}\;. \end{aligned} \tag{3.12}$$

We now perform a change of variables as in (2.12) to obtain X_I, E_1 and $\hat{E}_2$ from χ_I, ϵ_1 and $\hat{\epsilon}_2$ respectively. In these new variables, after some simple manipulations, (3.12) can be brought into the following form

$$\begin{aligned} Z_{1-\text{inst}} = &- \sum_{I=1}^{M} q_I \frac{E_1^{\frac{n_I+n_{I+1}}{4}+\frac{1}{2}} \hat{E}_2^{\frac{n_I+n_{I+1}}{4}}}{E_1 - 1} \int_{\mathcal{C}} \frac{dX_I}{2\pi \mathrm{i}} X_I^{\frac{n_I+n_{I+1}}{2}-\mathsf{m}_I - 1} \\ &\times \prod_{\ell=r_{I-1}+1}^{r_I} \frac{1}{\left(A_\ell \sqrt{E_1\hat{E}_2} - X_I\right)} \times \prod_{j=r_I+1}^{r_{I+1}} \frac{1}{\left(A_j - X_I\sqrt{E_1\hat{E}_2}\right)} \,. \end{aligned} \tag{3.13}$$

From this explicit expression it is clear that, besides the simple poles at

$$X_I = A_\ell \sqrt{E_1\hat{E}_2} \qquad \text{and} \qquad X_I = \frac{A_j}{\sqrt{E_1\hat{E}_2}} \,, \tag{3.14}$$

the integrand may possess branch cuts as well as singularities at $X_I = 0$ and $X_I = \infty$ depending on the value of m_I. If this is the case, the two contours are obviously not equivalent to each other. To avoid branch cuts we must require that

$$\mathsf{m}_I + \frac{n_I + n_{I+1}}{2} \in \mathbb{Z} \quad \text{for} \quad I = 1, \ldots, M \tag{3.15}$$

where $n_{M+1} = n_1$ (see footnote 7). Furthermore, to avoid contributions from the nonphysical singularities at $X_I = 0$ and $X_I = \infty$, we must constrain m_I such that

$$|\mathsf{m}_I| \leq \frac{n_I + n_{I+1}}{2} - 1 \quad \text{for} \quad I = 1, \ldots, M \,. \tag{3.16}$$

When conditions (3.15) and (3.16) are satisfied, the two JK prescriptions lead to the same result because the contours $\mathcal{C}_+$ and $\mathcal{C}_-$ can be smoothly deformed into each other.

The above analysis can be repeated at higher instanton levels, but the explicit expressions quickly become rather involved and not very illuminating. We have performed several explicit calculations up to three instantons in theories with low rank gauge groups and have encountered no other constraints on m_I other than those in (3.15) and (3.16) in order to obtain results that are independent of the prescription used to evaluate the integrals.

3.3 Parameter map

Once the instanton partition function is computed, one can extract from it the non-perturbative prepotential $\mathcal{F}_{\text{inst}}$ and the twisted superpotential $\mathcal{W}_{\text{inst}}$ according to [5, 7]

$$\log Z_{\text{inst}} = -\frac{\mathcal{F}_{\text{inst}}}{\epsilon_1 \hat{\epsilon}_2} + \frac{\mathcal{W}_{\text{inst}}}{\epsilon_1} + \text{regular terms} \,. \tag{3.17}$$

The prepotential governs the dynamics of the bulk 5d theory and depends on the parameters of this theory, namely the vacuum expectation values of the adjoint scalars, the Chern-Simons coupling $\mathsf{k}_{5\text{d}}$ and the instanton counting parameter q. The twisted superpotential, instead, controls the dynamics on the surface operator and in addition to these depends on the parameters that label the defect. From our explicit results we have verified that $\mathcal{F}_{\text{inst}}$ depends only on the vacuum expectation values, the sum of all m_I, and the product of all q_I. In particular these latter combinations play the role, respectively, of $\mathsf{k}_{5\text{d}}$ and q; thus, comparing with what we have seen in section 2, we are led to

$$\mathsf{k}_{5\text{d}} = \sum_{I=1}^{M} \mathsf{m}_I \,, \tag{3.18}$$

$$q = \prod_{I=1}^{M} q_I \;=\; (-1)^N (\beta\Lambda)^{2N} \,. \tag{3.19}$$

We recall that the instanton counting parameters q_I are related to the monodromy properties of the SU(N) gauge connection once the breaking pattern (3.1) is enforced by the presence of the defect. Building on earlier works [7, 17], this fact was explicitly shown in [26] for the $\mathcal{N} = 2^\star$ theories, and already used in [28] for the pure $\mathcal{N} = 2$ theories (see for instance, eq. (2.48) in [28]). Notice that only the product of all q_I's has a global 5d interpretation as is clear from (3.19). Similarly, the parameters m_I describe how the Chern-Simons level $\mathsf{k}_{5\text{d}}$ of the 5d SU(N) theory is split among the M factors in the Levi decomposition (3.1) and as such they are part of the data that specify the defect. From (3.4) we see that these parameters appear like Chern-Simons couplings for the U(n_I) factors, even though one should take into account that the ramified instanton partition function (3.3) is not factorizable into a product of M partition functions. Finally, we observe that the constraints (3.15) and (3.16) imply that

$$\mathsf{k}_{5\text{d}} \in \mathbb{Z} \qquad \text{and} \qquad |\,\mathsf{k}_{5\text{d}}| \leq N - M \,. \tag{3.20}$$

3.4 Simple surface operators

For the purpose of illustration, we now consider in detail the case of the simple surface operator of type $[1, N-1]$ in the SU(N) theory. This case corresponds to setting $M = 2$ and splitting the classical vacuum expectation values as $\{a_1 | a_2, \ldots, a_N\}$. Using this in (3.12), the 1-instanton contribution to the partition function becomes

$$\begin{aligned} Z_{1-\text{inst}} = &- \frac{q_1}{g(\epsilon_1)} \int_{\mathcal{C}} \Big(\beta\, \frac{d\chi_1}{2\pi \mathrm{i}}\Big) \frac{\mathrm{e}^{-\beta \mathsf{m}_1 \chi_1}}{g\big(a_1 - \chi_1 + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)\big)} \prod_{i=2}^{N} \frac{1}{g\big(\chi_1 - a_i + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)\big)} \\ &- \frac{q_2}{g(\epsilon_1)} \int_{\mathcal{C}} \Big(\beta\, \frac{d\chi_2}{2\pi \mathrm{i}}\Big) \frac{\mathrm{e}^{-\beta \mathsf{m}_2 \chi_2}}{g\big(\chi_2 - a_1 + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)\big)} \prod_{i=2}^{N} \frac{1}{g\big(a_i - \chi_2 + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2)\big)} \end{aligned} \tag{3.21}$$

We now evaluate the integrals over χ_1 and χ_2 using the two JK prescriptions described above. In the first prescription JK_I, according to (3.6), the contributing poles are located at

$$\chi_1 = a_1 + \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2) \qquad \text{and} \qquad \chi_2 = a_1 - \frac{1}{2}(\epsilon_1 + \hat{\epsilon}_2) \,. \tag{3.22}$$

Calculating the corresponding residues, extracting the twisted superpotential by means of (3.17), and expressing the results in terms of the variables (2.12), we find

$$\mathcal{W}^{(\text{I})}_{1-\text{inst}} = \frac{1}{\beta} \Big(q_1\, A_1^{\frac{N}{2} - \mathsf{m}_1 - 1} + (-1)^{N-1} q_2\, A_1^{\frac{N}{2} - \mathsf{m}_2 - 1} \Big) \prod_{i=2}^{N} (A_1 - A_i)^{-1} \,. \tag{3.23}$$

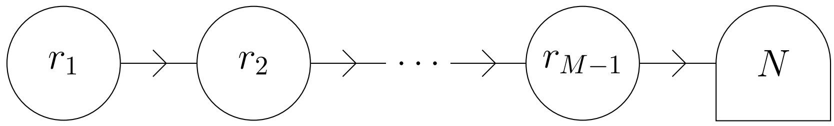

Figure 7. The quiver that describes the generic surface operator in pure SU(N) gauge theory.

Next we consider the second prescription JK_{II}; according to (3.8), the contributing poles are located at

$$\chi_1 = a_u - \frac{1}{2}(\epsilon_1+\epsilon_2) \qquad \text{and} \qquad \chi_2 = a_u + \frac{1}{2}(\epsilon_1+\epsilon_2) \qquad \text{for} \quad u = 2,3,\ldots, \, . \tag{3.24}$$

The corresponding twisted superpotential is

$$\mathcal{W}^{(\text{II})}_{1-\text{inst}} = -\frac{1}{\beta}\sum_{i=2}^{N}\left[\left(q_1\, A_i^{\frac{N}{2}-\mathsf{m}_1-1} + (-1)^{N-1} q_2\, A_i^{\frac{N}{2}-\mathsf{m}_2-1}\right)\prod_{j=1\,,j\neq i}^{N}(A_i-A_j)^{-1}\right]. \tag{3.25}$$

Comparing the two expressions (3.23) and (3.25), we see that they are very different from each other. However, if we use the SU(N) condition (2.14) and impose the constraints (3.15) and (3.16), which for $M=2$ are

$$\mathsf{m}_{1,2} + \frac{N}{2} \in \mathbb{Z} \qquad \text{and} \qquad |\,\mathsf{m}_{1,2}\,| \leq \frac{N}{2} - 1\,, \tag{3.26}$$

one can verify that $\mathcal{W}^{(\text{I})}_{1-\text{inst}}$ and $\mathcal{W}^{(\text{II})}_{1-\text{inst}}$ match. We have explicitly checked that the match continues to hold at higher instanton levels (up to three instantons for the low rank SU(N) theories).

4 3d/5d quiver theories with Chern-Simons terms

We now study surface operators from the point of view of 3d/5d coupled systems compactified on a circle of radius β, by extending the analysis of [28] to explicitly include Chern-Simons interactions.[9] We then derive and solve the resulting twisted chiral ring equations.

4.1 The linear quiver and its twisted chiral ring equations

Our proposal is that the 3d/5d system that corresponds to a surface operator labeled by the partition $\vec{n} = [n_1, n_2, \ldots, n_M]$ and treated with the first JK prescription (3.5), is the quiver theory described in figure 7. Here, the circular nodes represent 3d U(r_I) gauge theories, the rightmost node represents a 5d SU(N) gauge theory, and the arrows denote bifundamental matter fields. The ranks r_I of the 3d gauge groups are related to the surface operator data n_I as in (3.9).

The gauge degrees of freedom in each node can be organized in an adjoint twisted chiral multiplet $\Sigma^{(I)}$, which for notational simplicity we will often denote by its lowest

[9] The brane construction of 3d gauge theories with Chern Simons interactions has been studied in [66, 67].

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scalar component $\sigma^{(I)}$. The low-energy effective action on the Coulomb moduli space is parameterized by the diagonal components of $\sigma^{(I)}$:

$$\sigma^{(I)} = \mathrm{diag}\{\,\sigma_1^{(I)}, \sigma_2^{(I)}, \ldots, \sigma_{r_I}^{(I)}\,\}\ . \tag{4.1}$$

This can be obtained by integrating out the matter multiplets corresponding to the arrows of the quiver, which are generically massive when $\sigma^{(I)}$ and the adjoint scalar field Φ of the SU(N) theory acquire non-vanishing vacuum expectation values [68]. Supersymmetry implies that the effective action can be encoded in a twisted chiral superpotential, which takes the form (see [28] for details):

$$\mathcal{W}_0 = \sum_{I=1}^{M-1}\sum_{s=1}^{r_I} b_I \log(\beta\Lambda_I)\,\sigma_s^{(I)} - \sum_{I=1}^{M-2}\sum_{s=1}^{r_I}\sum_{t=1}^{r_{I+1}} \ell\big(\sigma_s^{(I)} - \sigma_t^{(I+1)}\big) - \sum_{s=1}^{r_{M-1}} \Big\langle \mathrm{Tr}\,\ell\big(\sigma_s^{(M-1)} - \Phi\big)\Big\rangle \tag{4.2}$$

where Λ_I is the (complexified) IR scale of the I-th node and

$$b_I = r_{I+1} - r_{I-1} \tag{4.3}$$

for $I = 1, \cdots, M-1$.[10] The expectation value in the last term of (4.2) is taken in the 5d SU(N) gauge theory and the function $\ell(z)$ obeys the relation

$$\partial_z \ell(z) = \log\left(2\sinh\frac{\beta z}{2}\right) . \tag{4.4}$$

In each 3d node of the quiver we can turn on a Chern-Simons term with coupling k_I. Upon circle compactification, these Chern-Simons terms give rise to an additional term in the superpotential which is [69][11]

$$\mathcal{W}_{\mathrm{CS}}^{(I)} = -\frac{\beta\,\mathsf{k}_I}{2}\,\mathrm{Tr}\,\big(\sigma^{(I)}\big)^2\ . \tag{4.5}$$

Thus the complete twisted superpotential governing the 3d/5d quiver theory of figure 7 is

$$\mathcal{W} = \mathcal{W}_0 + \sum_{I=1}^{M-1} \mathcal{W}_{\mathrm{CS}}^{(I)}\ . \tag{4.6}$$

The vacuum expectation values of the 5d fields appear in this twisted superpotential $\mathcal{W}$ in such a way that extremizing the latter leads to a discrete set of massive vacua, thus completely lifting the 3d Coulomb branch. We now derive the so-called twisted chiral ring equations which identify these massive vacua and study specific solutions with the aim of checking our proposal. We will show that the twisted chiral superpotential evaluated in these (massive) vacua coincides with the one obtained using the first JK prescription in the localization analysis.

The extremization equations of the superpotential $\mathcal{W}$ take the following form [70, 71]:

$$\exp\left(\frac{\partial\mathcal{W}}{\partial\sigma_s^{(I)}}\right) = 1\ . \tag{4.7}$$

[10] Here and in the following we understand that $r_0 = 0$ and $r_M = N$.

[11] This differs from the conventions in our previous paper [28] by a sign.

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These equations were analyzed in great detail in [28], and we will be brief in reviewing their derivation. We first introduce the functions

$$Q_I(z) = \prod_{s=1}^{r_I} \left(2 \sinh \frac{\beta(z - \sigma_s^{(I)})}{2} \right), \tag{4.8}$$

or, equivalently,

$$Q_I(Z) = Z^{-\frac{r_I}{2}} \prod_{s=1}^{r_I} \big(S_s^{(I)}\big)^{-\frac{1}{2}} \big(Z - S_s^{(I)}\big) \tag{4.9}$$

where

$$\sigma_s^{(I)} = \frac{1}{\beta} \log S_s^{(I)} \quad \text{and} \quad z = \frac{1}{\beta} \log Z\,. \tag{4.10}$$

Then, for $I = 1, \ldots, M-2$, the twisted chiral ring equations (4.7) become

$$Q_{I+1}(Z) = (-1)^{r_{I-1}} \, (\beta\Lambda_I)^{b_I} Z^{-\mathsf{k}_I} \, Q_{I-1}(Z) \tag{4.11}$$

for $Z = S_s^{(I)}$. Here we understand that $Q_0 = 1$. For the last 3d gauge node in the quiver, i.e. for $I = M-1$, we obtain

$$\exp \left\langle \operatorname{Tr} \log \left(2 \sinh \frac{\beta(z - \Phi)}{2} \right) \right\rangle = (-1)^{r_{M-2}} \, (\beta\Lambda_{M-1})^{b_{M-1}} \, Z^{-\mathsf{k}_{M-1}} \, Q_{M-2}(Z)\,, \tag{4.12}$$

for $z = \sigma_s^{(M-1)}$ or, equivalently, $Z = S_s^{(M-1)}$. This equation clearly shows that the coupling between the 3d and the 5d theories occurs via the integral of the resolvent of the SU(N) gauge theory (see (2.25)). Using (2.29), after some simple algebraic manipulations, we can rewrite (4.12) as

$$P_N(Z) = (-1)^{r_{M-2}} \, (\beta\Lambda_{M-1})^{b_{M-1}} \, Z^{-\mathsf{k}_{M-1}} Q_{M-2}(Z) \; + (-1)^{r_{M-2}} \, \frac{(\beta\Lambda)^{2N} Z^{-\mathsf{k}_{5d} + \mathsf{k}_{M-1}}}{(\beta\Lambda_{M-1})^{b_{M-1}} \, Q_{M-2}(Z)} \tag{4.13}$$

where P_N is defined in (2.18).

We now follow the same method described in [28] and recursively solve the chiral ring equations (4.11) and (4.13) in a semi-classical expansion around

$$S_{\star,\text{class}}^{(I)} = \operatorname{diag}(A_1, \ldots, A_{r_I})\,, \tag{4.14}$$

using the perturbative ansatz

$$S_\star^{(I)} = S_{\star,\text{class}}^{(I)} + \delta S_\star^{(I)} = \operatorname{diag}\left(A_1 + \sum_\ell \delta S_{1,\ell}^{(I)}, \cdots, A_{r_I} + \sum_\ell \delta S_{r_I,\ell}^{(I)} \right) \tag{4.15}$$

where the increasing values of the index ℓ in (4.15) correspond to corrections of increasing order in the compactification radius β. Inserting this ansatz into (4.11) and (4.13), we can explicitly work out the solution $S_\star^{(I)}$ to the desired perturbative order in β, and show that the twisted superpotential evaluated on this solution, which we denote by $\mathcal{W}_\star$, can be matched with the twisted superpotential for the corresponding surface defect obtained via

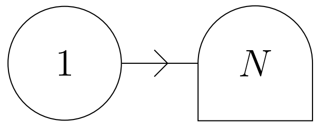

Figure 8. The quiver diagram for the simple surface operator of type $[1, N-1]$ in the $\mathrm{SU}(N)$ theory.

localization using the JK_I prescription. For this purpose, it is more convenient to consider the logarithmic derivatives of $\mathcal{W}_\star$ with respect to Λ_I which have a simple expression, namely

$$\Lambda_I \frac{d\mathcal{W}_\star}{d\Lambda_I} = \frac{b_I}{\beta} \operatorname{tr} \log S_\star^{(I)} \ . \tag{4.16}$$

As we will see, in order to obtain agreement we need a precise map between the IR parameters Λ_I and Λ of the 3d/5d coupled system and the instanton counting parameters q_I and q_M, and also a specific identification between the Chern-Simons levels of the 3d and 5d nodes with the parameters m_I introduced in the localization integrand.

We now give some details, starting from the case of simple operators, which were already analyzed in section 3.4 from the localization point of view.

Simple surface operators. In this case there is only one 3d gauge node, and the quiver diagram is represented in figure 8.

Correspondingly, we have just one variable $\sigma^{(1)}$, or $S^{(1)}$, and one chiral ring equation which is

$$P_N(S^{(1)}) = (\beta\Lambda_1)^N \big(S^{(1)}\big)^{-\mathsf{k}_1} + \frac{(\beta\Lambda)^{2N}}{(\beta\Lambda_1)^N} \big(S^{(1)}\big)^{-\mathsf{k}_{5\mathsf{d}}+\mathsf{k}_1} \ . \tag{4.17}$$

This follows from (4.13) with $M = 2$, which implies $b_1 = N$. The first non-trivial order is easy to extract. Indeed, we can start from the ansatz (4.14), namely from $S^{(1)}_{\star,\mathrm{class}} = A_1$, and use the classical approximation of P_N given in (2.20) to write

$$P_N(S^{(1)}) = \big(S^{(1)}\big)^{-\frac{N}{2}} \prod_{u=1}^{N} \big(S^{(1)} - A_u\big) + O\big((\beta\Lambda)^{2N}\big) \ . \tag{4.18}$$

Inserting this in (4.17), we find

$$S_\star^{(1)} = A_1\bigg[1 + \bigg((\beta\Lambda_1)^N A_1^{\frac{N}{2}-\mathsf{k}_1-1} + \frac{(\beta\Lambda)^{2N}}{(\beta\Lambda_1)^N} A_1^{\frac{N}{2}-\mathsf{k}_{5\mathsf{d}}+\mathsf{k}_1-1}\bigg) \prod_{i=2}^{N} \frac{1}{(A_1 - A_i)} + \cdots\bigg] \tag{4.19}$$

where the ellipses stand for terms of order $(\beta\Lambda)^{4N}$ and higher. Finally, from (4.16) we obtain

$$\begin{aligned} \frac{1}{N}\Lambda_1 \frac{d\mathcal{W}_\star}{d\Lambda_1} &= \frac{1}{\beta} \log S_\star^{(1)} \\ &= \frac{1}{\beta} \log A_1 + \frac{1}{\beta}\bigg((\beta\Lambda_1)^N A_1^{\frac{N}{2}-\mathsf{k}_1-1} + \frac{(\beta\Lambda)^{2N}}{(\beta\Lambda_1)^N} A_1^{\frac{N}{2}-\mathsf{k}_{5\mathsf{d}}+\mathsf{k}_1-1}\bigg) \\ &\quad \times \prod_{i=2}^{N} \frac{1}{(A_1 - A_i)} + \ldots \ . \end{aligned} \tag{4.20}$$

The non-perturbative part of this expression can be related to the superpotential $\mathcal{W}^{(\mathrm{I})}_{1-\mathrm{inst}}$ obtained via localization with the $\mathrm{JK_I}$ prescription and given in (3.23). Indeed, upon making the following identifications

$$q_1 = (\beta\Lambda_1)^N\,, \qquad q_2 = (-1)^N \frac{(\beta\Lambda)^{2N}}{(\beta\Lambda_1)^N}\,, \tag{4.21}$$

and

$$\mathsf{m}_1 = \mathsf{k}_1\,, \qquad \mathsf{m}_2 = \mathsf{k}_{5\mathrm{d}} - \mathsf{k}_1\,, \tag{4.22}$$

we find

$$\frac{1}{N}\,\Lambda_1 \frac{d\mathcal{W}_\star}{d\Lambda_1} = \frac{1}{\beta}\,\log A_1 + q_1 \frac{d\mathcal{W}^{(\mathrm{I})}_{1-\mathrm{inst}}}{dq_1} + \cdots\,, \tag{4.23}$$

where on the right hand side the derivative with respect to q_1 is taken by keeping the product $q_1 q_2$ fixed, i.e. at a fixed 5d scale Λ. We remark that the identifications (4.21) and (4.22) imply

$$q_1 q_2 = (-1)^N (\beta\Lambda)^{2N} \tag{4.24}$$

and

$$\mathsf{k}_{5\mathrm{d}} = \mathsf{m}_1 + \mathsf{m}_2\,, \tag{4.25}$$

in perfect agreement, respectively, with (3.19) and (3.18) for $M = 2$.

A similar analysis can be carried out at higher instanton levels. For instance, the two-instanton correction to (4.20) reads

$$\begin{aligned}
&\frac{1}{\beta}\left[A_1^{N-2(\mathsf{k}_1+1)}\left(\frac{N-1}{2} - \mathsf{k}_1 - \sum_{j=2}^{N}\frac{A_1}{A_1 - A_j}\right)\prod_{i=2}^{N}\frac{1}{(A_1-A_i)^2}\right](\beta\Lambda_1)^{2N} \\
&\quad + \frac{1}{\beta}\left[A_1^{N-2(\mathsf{k}_{5\mathrm{d}}-\mathsf{k}_1+1)}\left(\frac{N-1}{2} - \mathsf{k}_{5\mathrm{d}} + \mathsf{k}_1 - \sum_{j=2}^{N}\frac{A_1}{A_{1j}}\right)\prod_{i=2}^{N}\frac{1}{(A_1-A_i)^2}\right]\frac{(\beta\Lambda)^{4N}}{(\beta\Lambda_1)^{2N}}\,,
\end{aligned} \tag{4.26}$$

and agrees with the two-instanton term of the logarithmic q_1-derivative of the superpotential $\mathcal{W}^{(\mathrm{I})}_{\mathrm{inst}}$ computed using localization methods with the first JK prescription described in section 3, provided $|\,\mathsf{k}_{5\mathrm{d}}\,| < N$.

We have made numerous checks at higher instanton numbers and for various values of N, always finding a perfect match between localization and chiral ring analysis provided the relations (4.21) and (4.22) are used.

Generic surface operators. We now consider a generic surface operator. In order to test the correspondence between the solution of the chiral ring equations and the localization results, it is crucial to connect the parameters used in the two descriptions and generalize (4.21) and (4.22). To this purpose, it is useful to recall that in deriving these relations it is was sufficient to consider the 1-instanton result. Moreover, in comparing (4.20) and the q_1-logarithmic derivative of the superpotential (3.23), we kept fixed the scale of the 5d theory. If we temporarily set $\Lambda = 0$, and thus freeze the 5d dynamics, it becomes feasible to explicitly compute the 1-instanton contribution to the solution of the chiral ring equations for a generic surface operator and then compare with the localization results.

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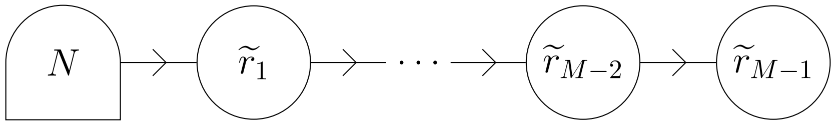


Figure 9. The quiver theory which is dual to the one in figure 7.

Once this is done, it is possible to reinstate the dependence on Λ in a rather simple manner, and find the generalization of the maps (4.21) and (4.22). Since the derivation is a bit lengthy, we discuss it in appendix B. Here we simply report the final result which is quite simple:

$$\begin{aligned} q_I &= -(-1)^{r_I}\,(\beta\Lambda_I)^{b_I} \qquad \text{for} \quad I=1,\ldots,M-1\,,\\ q_M &= (-1)^N\,(\beta\Lambda)^{2N}\bigg(\prod_{I=1}^{M-1} q_I\bigg)^{-1}\,, \end{aligned} \tag{4.27}$$

and

$$\begin{aligned} \mathsf{m}_I &= \mathsf{k}_I \qquad \text{for} \quad I=1,\ldots,M-1\,,\\ \mathsf{m}_M &= \mathsf{k}_{5\text{d}} - \sum_{I=1}^{M-1}\mathsf{k}_I\ . \end{aligned} \tag{4.28}$$

Using these maps, we have investigated many different surface operators at the first few instanton orders and found that the relation

$$\frac{1}{b_I}\,\Lambda_I\frac{d\mathcal{W}_\star}{d\Lambda_I} = \frac{1}{\beta}\,\text{tr}\,\log S^{(I)}_{\star,\text{class}} + q_I\,\frac{d\mathcal{W}^{(\text{I})}_{\text{inst}}}{dq_I}\,, \tag{4.29}$$

which generalizes (4.23), is always obeyed if $|\,\mathsf{k}_{5\text{d}}\,| < N$.

4.2 The dual linear quiver and its twisted chiral ring equations

We now address the question of whether it is possible to establish a connection between the chiral ring equations and the localization results for the other JK prescription. This analysis will allow us to clarify the map between different residue prescriptions and distinct quiver realizations of the same surface operator.

Building on the results of [28], we propose that the quiver theory that is relevant to match the localization prescription with the JK parameter $\widetilde{\eta}$ given in (3.7) is the one represented in figure 9. Here the ranks $\widetilde{r}_I$ of the 3d gauge groups are related to the partition $\vec{n} = [n_1, n_2, \ldots, n_M]$ that labels the surface operator according to [27, 28]

$$\widetilde{r}_I = N - \sum_{J=1}^{I} n_J = N - r_I\ . \tag{4.30}$$

As for the original quiver of figure 7, in the present case the low-energy effective theory on the Coulomb moduli space is parameterized by the diagonal components of the complex scalar fields in the adjoint twisted chiral multiplets, which we denote as

$$\widetilde{\sigma}^{(I)} = \text{diag}\{\,\widetilde{\sigma}^{(I)}_1, \widetilde{\sigma}^{(I)}_2, \ldots, \widetilde{\sigma}^{(I)}_{\widetilde{r}_I}\,\} \tag{4.31}$$

for $I = 1, \cdots, M-1$. The twisted chiral superpotential corresponding to this quiver takes the form

$$\widetilde{W} = \sum_{I=1}^{M-1} \sum_{s=1}^{\widetilde{r}_I} \widetilde{b}_I \log(\beta\widetilde{\Lambda}_I)\, \widetilde{\sigma}_s^{(I)} - \sum_{I=2}^{M-1} \sum_{s=1}^{\widetilde{r}_I} \sum_{t=1}^{\widetilde{r}_{I-1}} \ell\big(\widetilde{\sigma}_t^{(I-1)} - \widetilde{\sigma}_s^{(I)}\big) - \sum_{s=1}^{\widetilde{r}_1} \Big\langle \mathrm{Tr}\,\ell\big(\Phi - \widetilde{\sigma}_s^{(1)}\big) \Big\rangle$$
$$- \sum_{I=1}^{M-1} \frac{\beta\,\widetilde{\mathsf{k}}_I}{2}\, \mathrm{Tr}\,\big(\widetilde{\sigma}^{(I)}\big)^2 \tag{4.32}$$

where $\widetilde{\Lambda}_I$ is the (complexified) strong-coupling scale of the I-th node and the last term accounts for the Chern-Simons interactions on the 3d nodes with couplings $\widetilde{\mathsf{k}}_I$. The parameters $\widetilde{b}_I$ are defined as[12]

$$\widetilde{b}_I = \widetilde{r}_{I+1} - \widetilde{r}_{I-1} \tag{4.33}$$

and, because of (4.30), they are related to the analogous parameters b_I introduced in the earlier quiver as:

$$\widetilde{b}_I = -b_I\ . \tag{4.34}$$

The chiral ring equations, obtained by extremizing $\widetilde{\mathcal{W}}$, can be concisely expressed in terms of the functions

$$\widetilde{Q}_I(Z) = Z^{-\frac{\widetilde{r}_I}{2}} \prod_{s=1}^{\widetilde{r}_I} \big(\widetilde{S}_s^{(I)}\big)^{-\frac{1}{2}} \big(Z - \widetilde{S}_s^{(I)}\big) \tag{4.35}$$

where $\widetilde{S}_s^{(I)} = \mathrm{e}^{\beta\,\widetilde{\sigma}_s^{(I)}}$ in complete analogy with (4.9). Indeed, for $I = 2, \cdots, M-1$ we find

$$\widetilde{Q}_{I-1}(Z) = (-1)^{\widetilde{r}_{I-1}}\, (\beta\widetilde{\Lambda}_I)^{-\widetilde{b}_I}\, Z^{\widetilde{\mathsf{k}}_I}\, \widetilde{Q}_{I+1}(Z)\,, \tag{4.36}$$

for $Z = \widetilde{S}_s^{(I)}$, while the chiral ring equation of the first node ($I = 1$) involves the resolvent of the 5d $\mathrm{SU}(N)$ theory and reads

$$P_N(Z) = (-1)^N (\beta\widetilde{\Lambda}_1)^{-\widetilde{b}_1}\, Z^{\widetilde{\mathsf{k}}_1}\, \widetilde{Q}_2(Z) + (-1)^N \frac{(\beta\Lambda)^{2N}}{(\beta\widetilde{\Lambda}_1)^{-\widetilde{b}_1}}\, \frac{Z^{-\mathsf{k}_{5\mathrm{d}}-\widetilde{\mathsf{k}}_1}}{\widetilde{Q}_2(z)} \tag{4.37}$$

for $Z = \widetilde{S}_s^{(1)}$.

The classical vacuum around which we perturbatively solve the above equations is

$$\widetilde{S}^{(I)}_{\star,\mathrm{class}} = \mathrm{diag}\,(A_{N-\widetilde{r}_I+1}, \ldots, A_N)\ . \tag{4.38}$$

This corresponds to simply choosing for each node I the complement set of A_u that appear in the classical vacuum (4.14) for the corresponding node in the original quiver. Using an ansatz analogous to the one in (4.15) and expanding in powers of β around (4.38), we can obtain the solution $\widetilde{S}_\star^{(I)}$ of the chiral ring equations to the desired perturbative order and, in analogy with (4.16), relate it to the logarithmic derivative with respect to $\widetilde{\Lambda}_I$ of the twisted superpotential evaluated on this solution, namely

$$\widetilde{\Lambda}_I \frac{d\widetilde{\mathcal{W}}_\star}{d\widetilde{\Lambda}_I} = \frac{\widetilde{b}_I}{\beta}\, \mathrm{tr}\,\log \widetilde{S}_\star^{(I)}\ . \tag{4.39}$$

We now give some details in the case of the simple operators of type $[1, N-1]$.

[12] Here and in the following we understand that $\widetilde{r}_0 = N$ and $\widetilde{r}_M = 0$.

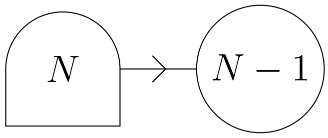


Figure 10. The dual quiver for the $[1, N-1]$ defect in the SU(N) theory.

Simple surface operators. In this case the quiver has a single 3d gauge node and is as depicted in figure 10. Correspondingly, using $\widetilde{b}_1 = -N$ and $\widetilde{Q}_2 = 1$, we see that the twisted chiral ring equations (4.37) take the following form:

$$P_N(Z) = (-1)^N(\beta\widetilde{\Lambda}_1)^N\, Z^{\widetilde{\mathsf{k}}_1} + (-1)^N\frac{(\beta\Lambda)^{2N}}{(\beta\widetilde{\Lambda}_1)^N}\, Z^{-\mathsf{k}_{5\mathrm{d}}-\widetilde{\mathsf{k}}_1} \tag{4.40}$$

for $Z = \widetilde{S}^{(1)}_s$ with $s = 1, \cdots, N-1$. To leading order these equations are solved by

$$\begin{aligned}\widetilde{S}^{(1)}_{\star,s} = A_{s+1}\Bigg[1 + (-1)^N\bigg((\beta\widetilde{\Lambda}_1)^N A_{s+1}^{\frac{N}{2}+\widetilde{\mathsf{k}}_1-1} + \frac{(\beta\Lambda)^{2N}}{(\beta\widetilde{\Lambda}_1)^N}\, A_{s+1}^{\frac{N}{2}-\mathsf{k}_{5\mathrm{d}}-\widetilde{\mathsf{k}}_1-1}\bigg)\\ \times\prod_{\substack{j=1\\ j\neq s+1}}^{N}\frac{1}{(A_{s+1}-A_j)} + \dots\Bigg]\,.\end{aligned} \tag{4.41}$$

Exploiting (4.39), we get

$$\begin{aligned}\frac{1}{N}\,\widetilde{\Lambda}_1\frac{d\widetilde{\mathcal{W}}_\star}{d\widetilde{\Lambda}_1} &= -\frac{1}{\beta}\mathrm{tr}\ \log\widetilde{S}^{(1)}_\star\\ &= -\frac{1}{\beta}\sum_{i=2}^{N}\log A_i - \frac{1}{\beta}\sum_{i=2}^{N}\Bigg[(-1)^N\bigg((\beta\widetilde{\Lambda}_1)^N A_i^{\frac{N}{2}+\widetilde{\mathsf{k}}_1-1} + \frac{(\beta\Lambda)^{2N}}{(\beta\widetilde{\Lambda}_1)^N}\, A_i^{\frac{N}{2}-\mathsf{k}_{5\mathrm{d}}-\widetilde{\mathsf{k}}_1-1}\bigg)\\ &\qquad\times\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{(A_i-A_j)}\Bigg] + \dots\,.\end{aligned} \tag{4.42}$$

The quantity in square brackets has the same structure of the (logarithmic derivative of the) twisted superpotential (3.25) computed using the second JK prescription. Indeed, if make the following identifications

$$q_1 = (-1)^N(\beta\widetilde{\Lambda}_1)^N\,, \qquad q_2 = \frac{(\beta\Lambda)^{2N}}{(\beta\widetilde{\Lambda}_1)^N}\,, \tag{4.43}$$

and

$$\mathsf{m}_1 = -\widetilde{\mathsf{k}}_1\,, \qquad \mathsf{m}_2 = \mathsf{k}_{5\mathrm{d}} + \widetilde{\mathsf{k}}_1\,, \tag{4.44}$$

we obtain

$$\frac{1}{N}\,\widetilde{\Lambda}_1\frac{d\widetilde{\mathcal{W}}_\star}{d\widetilde{\Lambda}_1} = -\frac{1}{\beta}\sum_{i=2}^{N}\log A_i + q_1\frac{d\widetilde{\mathcal{W}}^{(\mathrm{II})}_{1-\mathrm{inst}}}{dq_1} + \cdots\,, \tag{4.45}$$

where on the right hand side the derivative with respect to q_1 is taken by keeping the product $q_1 q_2$ fixed. This is clearly the counterpart in the dual quiver of the relation (4.23) that we found in the original theory. Notice that the identifications (4.21) and (4.22) continue to hold, but the map between the localization parameter m_1 and the 3d Chern-Simons coupling has an opposite sign as compared to the original quiver. We have checked in several examples that the higher-instanton corrections to the left hand side of (4.45) fully agree with those of the logarithmic derivative of $\mathcal{W}_{\text{inst}}^{(\text{II})}$, computed using localization with the second JK prescription, provided $|\,\mathsf{k}_{5\text{d}}\,| < N$.

Generic surface operators. The above procedure can be applied to a generic surface operator. The details are given in appendix B. Here we merely report the maps between the parameters used in the localization calculations and those appearing in the chiral ring equations:

$$\begin{aligned} q_I &= -(-1)^{\widetilde{r}_I}\,(\beta\widetilde{\Lambda}_I)^{-\widetilde{b}_I} \qquad \text{for} \quad I = 1, \ldots, M-1\,, \\ q_M &= (-1)^N\,(\beta\Lambda)^{2N}\left(\prod_{I=1}^{M-1} q_I\right)^{-1}\,, \end{aligned} \tag{4.46}$$

and

$$\begin{aligned} \mathsf{m}_I &= -\widetilde{\mathsf{k}}_I \qquad \text{for} \quad I = 1, \ldots, M-1\,, \\ \mathsf{m}_M &= \mathsf{k}_{5\text{d}} + \sum_{I=1}^{M-1} \widetilde{\mathsf{k}}_I\,. \end{aligned} \tag{4.47}$$

Using these maps, we have checked in several examples at the first few instanton orders that the relation

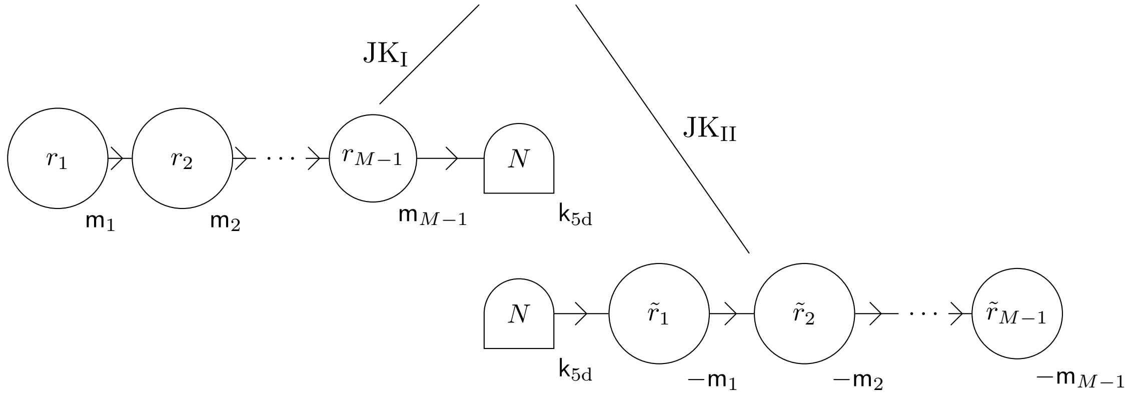
$$\frac{1}{\widetilde{b}_I}\,\widetilde{\Lambda}_I\frac{d\widetilde{\mathcal{W}}_\star}{d\widetilde{\Lambda}_I} = \frac{1}{\beta}\,\text{tr}\,\log S^{(I)}_{\star,\text{class}} - q_I\,\frac{d\mathcal{W}^{(\text{II})}_{\text{inst}}}{dq_I}\,, \tag{4.48}$$

which generalizes (4.45) to a generic surface operator, is always satisfied provided $|\mathsf{k}_{5\text{d}}| < N$.

4.3 Summary

We have discussed in detail how two different realizations of a surface defect encoded in the two quiver diagrams of figure 7 and figure 9 correspond, respectively, to the two different JK prescriptions used in the localization approach. We stress that the integrand in the ramified instanton partition function remains the same, and in particular that the parameters m_I do not change; what changes is the map between these parameters and the Chern-Simons coefficients of the 3d nodes in the two quiver theories. Our results can be summarized in the following diagram.

$$\prod_{I=1}^{M}\left[\frac{(-q_I)^{d_I}}{d_I!}\int_{\mathcal{C}}\prod_{\sigma=1}^{d_I}\left(\beta\,\frac{d\chi_{I,\sigma}}{2\pi\mathrm{i}}\right)\mathrm{e}^{-\beta\,\mathsf{m}_I\chi_{I,\sigma}}\right]z_{\{d_I\}}$$



(4.49)

In the next section we discuss how the two quiver theories are related to each other by IR Aharony-Seiberg dualities.

5 Relation to Aharony-Seiberg dualities

In section 3.1, we studied surface operators realized as Gukov-Witten defects by means of localization techniques and computed the ramified instanton partition function from which the twisted chiral superpotential can be extracted. Besides the instanton counting parameters q_I, our results depend on the parameters m_I that were introduced as counterparts of the Chern-Simons couplings that may appear when the surface defects are represented as coupled 3d/5d systems. Localization requires a residue prescription, usually specified by means of a Jeffrey-Kirwan parameter, in order to select the poles contributing to the integral over the instanton moduli space. We have computed the twisted superpotential using two different (and complementary) prescriptions and shown that only when the parameters m_I satisfy the constraints (3.15) and (3.16) the two results agree.

On the other hand, in section 4, we considered the realization of the defect by means of two different coupled 3d/5d quiver theories. They give rise to twisted chiral superpotentials that exactly match those arising from the two localization residue prescriptions, provided the parameters m_I are mapped to the 3d and 5d Chern-Simons levels k_I and $\mathsf{k}_{5\mathrm{d}}$ according to (4.28) or (4.47). Therefore, the conditions on m_I under which the two localization prescriptions yield the same result must correspond to the conditions that the Chern-Simons parameters must obey in order for the two quiver theories to be dual to each other. In the following, we explore the physical content of these constraints and their connection with related work in the literature.

Let us first consider the quiver theory of figure 7, and for simplicity turn off the 5d dynamics on the rightmost node in order to have a purely 3d theory. This corresponds to setting the 5d scale Λ to zero and to considering $\mathrm{SU}(N)$ as a global flavour symmetry.[13] For $I = 1, \ldots, M-1$ the constraints (3.15) and (3.16) become

$$\mathsf{k}_I + \frac{b_I}{2} \in \mathbb{Z}\,, \tag{5.1}$$

and

$$|\,\mathsf{k}_I\,| \leq \frac{b_I}{2} - 1\,. \tag{5.2}$$

Here we have used the fact that $b_I = r_{I+1} - r_{I-1}$ and, as before, understood that $r_0 = 0$ and $r_M = N$. These constraints and their physical interpretation are well known. The integrality condition (5.1) is a requirement on the absence of the $\mathbb{Z}_2$ parity anomaly in three dimensions [47, 48], and is related to the fact that integrating out an odd number of chiral fermions leads to a half-integer Chern-Simons term at one-loop. Indeed, b_I is the effective number of chiral (fundamental) matter at the I-th node. The inequality (5.2) is the constraint found in [49] (see in particular eq. (3.51) of this reference) for the so-called "maximally chiral theories". Notice that the 3d gauge theory at each node of the quiver belongs to this class, since the ranks r_I are monotonically increasing with I.

When the constraint (5.2) is satisfied, the $U(r_I)_{\mathsf{k}_I}$ theory at the I-th node admits an Aharony-Seiberg dual, which is a $U(r_{I+1}-r_I)_{-\mathsf{k}_I}$ theory[14] possessing additional (mesonic) fields with a superpotential term [33, 36, 37]. By performing subsequent duality transformations, one may obtain many distinct dual quiver theories. In particular, one can check that by successively applying such dualities to the quiver of figure 7, starting from the node with $I = M-1$ and proceeding all the way to the left-most node with $I = 1$, one ends up with precisely the linear quiver of figure 9 without any additional mesonic fields and superpotential terms. In fact, with the first duality transformation the $U(r_{M-1})_{\mathsf{k}_{M-1}}$ node becomes a $U(N-r_{M-1})_{-\mathsf{k}_{M-1}}$ theory with mesons that behave as N multiplets in the fundamental of $U(r_{M-2})_{\mathsf{k}_{M-2}}$. Dualizing the latter, we obtain a $U(N-r_{M-2})_{-\mathsf{k}_{M-2}}$ theory along with N mesons that transform in the fundamental of $U(r_{M-3})_{\mathsf{k}_{M-3}}$. Continuing in this dualization process, all superpotential terms cancel and we obtain the linear quiver of figure 9.

According to the analysis of [49], theories with 3d Chern-Simons levels outside the range (5.2) still admit Aharony-Seiberg duals, but the ranks of the gauge groups for the latter depend on the Chern-Simons levels and in certain cases exceed the rank of the global flavour symmetry. If this is the case, turning on twisted masses for the flavours would not completely lift the Coulomb branch and the resulting 3d low-energy effective theory is not massive. Thus, these dual models cannot represent Gukov-Witten defects in a higher dimensional theory since the general picture of surface operators as coupled gauge theories proposed in [4] necessarily assumes the fibration of a discrete set of vacua, namely the solutions to the twisted chiral rings of the lower dimensional theory, over the Coulomb moduli space of the higher dimensional theory.

[13] From the localization point of view, setting the 5d scale to zero reduces the ramified instanton partition function to a 3d vortex partition function.

[14] In more general situations, the dual rank is $\max(s, s') - r_I$, where s and s' are the numbers of chiral and anti-chiral matter multiplets charged with respect to the I-th gauge group.

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Let us now consider the case when the five dimensional gauge coupling is turned on. From the localization point of view, we now have to take into account the case $I = M$ in (3.15) and (3.16). This leads to the condition (3.20) on the 5d Chern-Simons coupling, i.e.

$$|\,\mathsf{k}_{5\mathrm{d}}\,| \leq N - M\ . \tag{5.3}$$

The same bound can be derived from the twisted chiral ring relations. Consider for simplicity the surface operator of type $[1, N-1]$, corresponding to $M = 2$, for which the twisted chiral ring equation (see (4.17)) is

$$\begin{aligned} Z^{\frac{N}{2}} P_N(Z) &\equiv Z^N + \sum_{i=1}^{N-1} (-1)^i\, Z^{N-i}\, U_{N-i}(\mathsf{k}_{5\mathrm{d}}) + (-1)^N \\ &= (\beta\Lambda_1)^N Z^{\frac{N}{2}-\mathsf{k}_1} + \frac{(\beta\Lambda)^{2N}}{(\beta\Lambda_1)^N} Z^{\frac{N}{2}-\mathsf{k}_{5\mathrm{d}}+\mathsf{k}_1}\,, \end{aligned} \tag{5.4}$$

with $Z = S^{(1)}$. In our analysis we assumed that it was possible to find a solution of this equation as a power series expansion around a classical vacuum specified by $S^{(1)}_{\star,\mathrm{class}} = \mathrm{e}^{\beta\, a_u}$, with a_u being one of the N vacuum expectation values of the 5d adjoint scalar Φ. Following the discussion in [4] for the (dimensionally reduced) 2d/4d case, one can analyze the fibration of these discrete solutions over the moduli space of the higher dimensional gauge theory and, from the geometry of the total space, recover the form of the Seiberg-Witten curve of the compactified 5d theory. This can be seen by defining [58]

$$Y = (\beta\Lambda_1)^N Z^{-\mathsf{k}_1} - \frac{(\beta\Lambda)^{2N}}{(\beta\Lambda_1)^N} Z^{-\mathsf{k}_{5\mathrm{d}}+\mathsf{k}_1}\,, \tag{5.5}$$

and noting that from (5.4) we have $Y^2 = P_N^2 - 4(\beta\Lambda)^{2N} Z^{-\mathsf{k}_{5\mathrm{d}}}$.

However, the chiral ring equations are related to the twisted superpotential that arises in presence of the defect, and contain more information than the Seiberg-Witten curve, which encodes the prepotential of the pure 5d theory. It is easy to check that demanding (5.4) to be a monic polynomial in Z of degree N, whose constant term is set to be $(-1)^N$ by the $\mathrm{SU}(N)$ condition, implies, beside the conditions (5.1) and (5.2), also the relation $|\,\mathsf{k}_{5\mathrm{d}}\,| \leq N-2$, which is the bound (5.3) for $M = 2$. The same kind of analysis in the case with no defect, i.e. $M = 1$, leads to the standard relation $|\,\mathsf{k}_{5\mathrm{d}}\,| \leq N-1$ in agreement with [50] (see the discussion after (2.20)).

We pause to remark that for $|\mathsf{k}_{5\mathrm{d}}| < N$, there is perfect agreement between the twisted chiral superpotentials calculated using localization and the chiral ring analysis. Thus in this range of the 5d Chern-Simons level, one can study the surface operator either as a monodromy defect or as a coupled 3d/5d system. However, what the constraint (5.3) implies is that, for $N - M < |\,\mathsf{k}_{5\mathrm{d}}\,| < N$, due to a non vanishing contribution from the residue at zero or infinity, the superpotentials calculated using the two contour prescriptions differ. It is possible that in this range of $\mathsf{k}_{5\mathrm{d}}$ one might need to modify the contour integral description of the defect and/or take into account extra light degrees of freedom in order to relate the two contour prescriptions. On the quiver side, this would require a more detailed

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understanding of Aharony-Seiberg dual theories. It would be very interesting to explore these possibilities.

In this work we have focused on the two linear quivers at the end of a chain of duality transformations. It would be nice to better understand the twisted chiral rings and the superpotentials of the intermediate quivers obtained along the way. It would also be important to understand the map between such 3d/5d theories and the different residue prescriptions that can be considered in the localization integral. We leave these issues to future work.

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A Chiral correlators in 5d gauge theories

In this appendix we outline the method of calculating the quantum chiral correlators in a 5d gauge theory, generalizing the discussion in [28] to include a non-zero Chern-Simons coupling. The key idea is to start from the formula for the chiral correlators in 4d theories [72–75], and suitably generalize it to the 5d case, namely

$$V_\ell = \left\langle \mathrm{Tr}\, \mathrm{e}^{\ell\beta\Phi} \right\rangle = \sum_{u=1}^{N} A_u^\ell - \frac{1}{Z_{\mathrm{inst}}} \sum_{k=1}^{\infty} \frac{q^k}{k!} \int_{\mathcal{C}} \left(\prod_{\sigma=1}^{k} \frac{d\chi_\sigma}{2\pi \mathrm{i}} \right) z_k(\chi_\sigma)\, \mathcal{O}_\ell(\chi_\sigma)\ . \tag{A.1}$$

Here, Z_{inst} is the instanton partition function defined in (2.3), $z_k(\chi_\sigma)$ is the integrand (2.4), and $\mathcal{O}_\ell$ is the following combination [28]

$$\mathcal{O}_\ell(\chi_\sigma) = \sum_{\sigma=1}^{k} \mathrm{e}^{\ell\,\beta\,\chi_\sigma} \big(1 - \mathrm{e}^{\ell\,\beta\,\epsilon_1}\big)\big(1 - \mathrm{e}^{\ell\,\beta\,\epsilon_2}\big)\ . \tag{A.2}$$

At the 1-instanton level, by performing explicitly the integral over χ in (A.1) we find

$$V_\ell = \sum_{u=1}^{N} A_u^\ell + \ell^2 (\beta\Lambda)^{2N} \sum_{u=1}^{N} \left[\frac{A_u^{N-2+\ell-\mathrm{k}_{5\mathrm{d}}}}{\prod\limits_{u\neq v} (A_u - A_v)^2} \right] + O\left((\beta\Lambda)^{4N} \right)\ . \tag{A.3}$$

The generating function for the V_ℓ is the resolvent of the SU(N) gauge theory:

$$T = N + 2 \sum_{\ell} \frac{V_\ell}{Z^\ell} \tag{A.4}$$

for which in section 2 we proposed an explicit formula in terms of the functions appearing in the Seiberg-Witten curve of the theory (see (2.30)). Working out the large Z expansion,

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we obtain

$$
\begin{aligned}
T = N &+ 2\,\frac{U_1(\mathsf{k}_{5\mathrm{d}})+(\beta\Lambda)^{2N}\delta_{\mathsf{k}_{5\mathrm{d}},1-N}}{Z}\\
&+ 2\,\frac{U_1^2(\mathsf{k}_{5\mathrm{d}})-2\,U_2(\mathsf{k}_{5\mathrm{d}})+\big(4(\beta\Lambda)^{2N}U_1(\mathsf{k}_{5\mathrm{d}})+3(\beta\Lambda)^{4N}\big)\delta_{\mathsf{k}_{5\mathrm{d}},1-N}+2(\beta\Lambda)^{2N}\delta_{\mathsf{k}_{5\mathrm{d}},2-N}}{Z^2}\\
&+ O(Z^{-3})
\end{aligned}
\tag{A.5}
$$

where $U_i(\mathsf{k}_{5\mathrm{d}})$ are the gauge invariant coordinates on the Coulomb branch of the 5d theory with Chern-Simons coupling $\mathsf{k}_{5\mathrm{d}}$. Comparing with (A.4), we deduce

$$
\begin{aligned}
U_1(\mathsf{k}_{5\mathrm{d}}) &= V_1 - (\beta\Lambda)^{2N}\delta_{\mathsf{k}_{5\mathrm{d}},1-N}\,,\\
U_2(\mathsf{k}_{5\mathrm{d}}) &= \frac{1}{2}\big(V_1^2 - V_2\big) + (\beta\Lambda)^{2N}\big(V_1\,\delta_{\mathsf{k}_{5\mathrm{d}},1-N} + \delta_{\mathsf{k}_{5\mathrm{d}},2-N}\big)\,.
\end{aligned}
\tag{A.6}
$$

Similar formulas can be easily worked out for higher $U_i(\mathsf{k}_{5\mathrm{d}})$ without any difficulty. However, since they are a bit involved we do not report them here. Instead, as an illustrative example, we consider the explicit expression of the above formulas in the case of the SU(3) theory for which $U_1(\mathsf{k}_{5\mathrm{d}})$ and $U_2(\mathsf{k}_{5\mathrm{d}})$ are the two independent coordinates of the quantum Coulomb branch. In this case, using (A.3) into (A.6) and taking into account the SU(3) condition, we find

$$
\begin{aligned}
U_1(\mathsf{k}_{5\mathrm{d}}) &= \sum_{u=1}^{3} A_u + (\beta\Lambda)^6\left[\sum_{u=1}^{3}\frac{A_u^{2-\mathsf{k}_{5\mathrm{d}}}}{\prod\limits_{u\neq v\neq w}(A_u-A_v)^2\,(A_u-A_w)^2} - \delta_{\mathsf{k}_{5\mathrm{d}},-2}\right] + O\big((\beta\Lambda)^{12}\big)\,,\\
U_2(\mathsf{k}_{5\mathrm{d}}) &= \sum_{u\neq v=1}^{3} A_uA_v + (\beta\Lambda)^6\left[\sum_{u=1}^{3}\frac{A_u^{-\mathsf{k}_{5\mathrm{d}}}}{\prod\limits_{u\neq v\neq w}(A_u-A_v)^2\,(A_u-A_w)^2} - \delta_{\mathsf{k}_{5\mathrm{d}},2}\right] + O\big((\beta\Lambda)^{12}\big)\,.
\end{aligned}
\tag{A.7}
$$

From these expressions, one can check that

$$
U_i(-\mathsf{k}_{5\mathrm{d}}) = \widetilde{U}_{3-i}(\mathsf{k}_{5\mathrm{d}})\,,
\tag{A.8}
$$

where $\widetilde{U}_i$ is obtained from U_i through the inversion $A_u \to 1/A_u$. This is a particular case of the relation (2.23) discussed in section 2.2. We have checked this relation also for groups of higher rank at the 1-instanton level, confirming its validity.

B Map of parameters for the generic surface operator

In this appendix we consider a generic surface operator and calculate the 1-instanton contribution to its twisted chiral superpotential using the two JK prescriptions discussed in the main text, with the purpose of finding the map between the parameters introduced in the localization calculations and those appearing in the quiver theory, focusing in particular on the 3d gauge nodes.

The twisted superpotential. We start from the 1-instanton partition function $Z_{1-\text{inst.}}$. This is given in (3.13), which we rewrite here for convenience

$$\begin{aligned} Z_{1-\text{inst}} = & -\sum_{I=1}^{M} q_I \frac{E_1^{\frac{n_I+n_{I+1}}{4}+\frac{1}{2}}\hat{E}_2^{\frac{n_I+n_{I+1}}{4}}}{E_1-1}\int_{\mathcal{C}}\frac{dX_I}{2\pi\mathrm{i}}\,X_I^{\frac{n_I+n_{I+1}}{2}-\mathsf{m}_I-1} \\ & \times\prod_{\ell=r_{I-1}+1}^{r_I}\frac{1}{\left(A_\ell\sqrt{E_1\hat{E}_2}-X_I\right)}\times\prod_{j=r_I+1}^{r_{I+1}}\frac{1}{\left(A_j-X_I\sqrt{E_1\hat{E}_2}\right)}\,. \end{aligned} \tag{B.1}$$

Since our main goal is to find the 3d interpretation of the parameters, we can set $q_M = 0$, which in view of (4.27) and (4.46) is equivalent to put $\Lambda = 0$ and hence freeze out the quantum dynamical effects in the 5d theory. Then we remain only with the integrals over X_I with $I = 1, \cdots, M-1$.

In the $\mathrm{JK_I}$ prescription only poles in the upper-half complex plane of X_I are chosen. In our case they are

$$X_I = A_\ell\sqrt{E_1\hat{E}_2} \tag{B.2}$$

for $\ell = r_{I-1}+1, \ldots r_I$. Evaluating the residues and extracting the twisted chiral superpotential according to (3.17), we obtain

$$\begin{aligned} \mathcal{W}^{(\mathrm{I})}_{\text{1-inst}} = & -\frac{1}{\beta}\sum_{I=1}^{M-1}(-1)^{n_I}q_I\sum_{i=1}^{n_I}\Bigg[(A_{r_{I-1}+i})^{\frac{n_I+n_{I+1}}{2}-\mathsf{m}_I-\frac{1}{2}}\prod_{\substack{j=1\\ j\neq i}}^{n_I}A^{\frac{1}{2}}_{r_{I-1}+j}\prod_{s=1}^{n_{I+1}}A^{\frac{1}{2}}_{r_I+s} \\ & \times\prod_{\substack{\ell=1\\ \ell\neq i}}^{n_I}\frac{1}{(A_{r_{I-1}+i}-A_{r_{I-1}+\ell})}\prod_{t=1}^{n_{I+1}}\frac{1}{(A_{r_{I-1}+i}-A_{r_{I+1}-t+1})}\Bigg]\,. \end{aligned} \tag{B.3}$$

With the $\mathrm{JK_{II}}$ prescription, one makes the complementary choice of poles, namely those located at

$$X_I = \frac{A_j}{\sqrt{E_1\hat{E}_2}} \tag{B.4}$$

for $j = r_I+1, \ldots, r_{I+1}$. Computing the corresponding residues yields

$$\begin{aligned} \mathcal{W}^{(\mathrm{II})}_{\text{1-inst}} = & \frac{1}{\beta}\sum_{I=1}^{M-1}(-1)^{n_I}q_I\sum_{i=1}^{n_{I+1}}\Bigg[(A_{r_I+i})^{\frac{n_I+n_{I+1}}{2}-\mathsf{m}_I-\frac{1}{2}}\prod_{j=1}^{n_I}A^{\frac{1}{2}}_{r_{I-1}+j}\prod_{\substack{s=1\\ s\neq i}}^{n_{I+1}}A^{\frac{1}{2}}_{r_I+s} \\ & \times\prod_{\ell=1}^{n_I}\frac{1}{(A_{r_I+i}-A_{r_{I-1}+\ell})}\prod_{\substack{t=1\\ t\neq i}}^{n_{I+1}}\frac{1}{(A_{r_I+i}-A_{r_I+t})}\Bigg]\,. \end{aligned} \tag{B.5}$$

As we have seen in section 3.2, these two expressions are in general different, unless the parameters m_I satisfy the conditions (3.15) and (3.16).

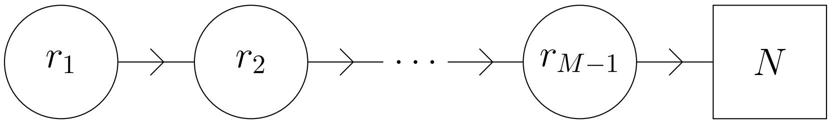

Figure 11. The 3d quiver which lifts to the generic surface operator upon gauging the SU(N) flavour symmetry represented by the square node at the right.

Linear quiver. We now study the twisted chiral ring equations whose solutions are the vacua of the 3d quiver represented in figure 11. Here the SU(N) node on the right is not gauged, since our objective is to find the map between the 3d parameters that include the Chern-Simons levels and the strong coupling scales of the gauge theory. In particular, this means that the 5d scale Λ is set to 0, as we did before in the localization calculations. All chiral ring equations for this quiver are given by

$$Q_{I+1}(S_s^{(I)}) = (-1)^{r_{I-1}}(\beta\Lambda_I)^{b_I}\big(S_s^{(I)}\big)^{-\mathsf{k}_I}Q_{I-1}(S_s^{(I)}) \tag{B.6}$$

for $I = 1, \cdots, M-1$. Here we understand that $Q_0(Z) = 1$ and $Q_M(Z) = P_N(Z)$ where P_N is defined in (2.18). Therefore, for $I = M-1$ the above expression gives the chiral ring equation (4.13) in the limit when $\Lambda = 0$. Using the explicit form of the functions Q_I given in (4.9), we can rewrite (B.6) as

$$\begin{aligned}\prod_{t=1}^{r_{I+1}}\big(S_s^{(I)} - S_t^{(I+1)}\big) &= (-1)^{r_{I-1}}(\beta\Lambda_I)^{b_I}\big(S_s^{(I)}\big)^{\frac{b_I}{2}-\mathsf{k}_I} \\ &\quad\times\prod_{t=1}^{r_{I+1}}\big(S_t^{(I+1)}\big)^{\frac{1}{2}}\prod_{u=1}^{r_{I-1}}\big(S_u^{(I-1)}\big)^{-\frac{1}{2}}\prod_{u=1}^{r_{I-1}}\big(S_s^{(I)} - S_u^{(I-1)}\big)\end{aligned} \tag{B.7}$$

where we have used $b_I = r_{I+1} - r_{I-1}$. We now solve these equations for $S^{(I)}$ using the ansatz

$$S_\star^{(I)} = \mathrm{diag}\,\big(A_1, \,\ldots\,, A_{r_{I-1}}, \, A_{r_{I-1}+1} + \delta A_{r_{I-1}+1}, \,\ldots\,, A_{r_I} + \delta A_{r_I}\big)\,. \tag{B.8}$$

Inserting this into (B.7), after some simple algebra we get

$$\delta A_s \prod_{\substack{t=1\\t\neq s}}^{r_{I+1}}(A_s - A_t) = (-1)^{r_{I-1}}(\beta\Lambda_I)^{b_I}(A_s)^{\frac{b_I}{2}-\mathsf{k}_I}\prod_{t=1}^{r_{I+1}}(A_t)^{\frac{1}{2}}\prod_{u=1}^{r_{I-1}}(A_u)^{-\frac{1}{2}}\prod_{u=1}^{r_{I-1}}(A_s - A_u) \tag{B.9}$$

for $s = r_{I-1}+1, \ldots, r_I$. This leads to

$$\delta A_s = (-1)^{r_{I-1}}(\beta\Lambda_I)^{b_I}(A_s)^{\frac{b_I}{2}-\mathsf{k}_I}\prod_{t=r_{I-1}+1}^{r_{I+1}}(A_t)^{\frac{1}{2}}\prod_{\substack{t=r_{I-1}+1\\t\neq s}}^{r_{I+1}}\frac{1}{(A_s - A_t)}\,. \tag{B.10}$$

Using this in (B.8), we find that $\mathrm{tr}\,\log S_\star^{(I)}$ is a sum of n_I terms, each of which looks very similar to the q_I-derivative of the localization result (B.3). Notice that the denominator of the latter is split into two products with $n_I - 1$ and n_{I+1} factors respectively, while

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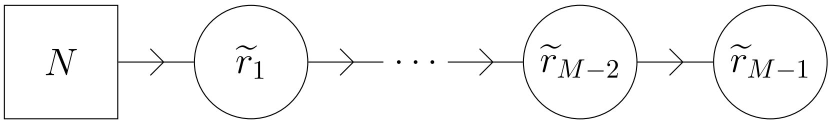

Figure 12. The quiver which is dual to the one in figure 11.

the last product in (B.10) is written in terms of the ranks of the adjacent nodes and contains $r_{I+1} - r_{I-1} - 1$ terms. However, using the relation between n_I and r_I, we see that $r_{I+1} - r_{I-1} = n_I + n_{I+1}$, and thus the two structures agree. Actually, one can explicitly check that the chiral ring results fully match those from localization with the first JK prescription if

$$q_I = -(-1)^{r_I}(\beta\Lambda_I)^{b_I} \quad \text{and} \quad \mathsf{m}_I = \mathsf{k}_I \tag{B.11}$$

for $I = 1, \cdots, M-1$.

Dual quiver. In a similar vein, we can treat the twisted chiral ring equations of the dual quiver which is represented in figure 12.

In this case, the chiral ring equations take the form

$$\widetilde{Q}_{I-1}(\widetilde{S}_s^{(I)}) = (-1)^{\widetilde{r}_{I-1}}\,(\beta\widetilde{\Lambda}_I)^{-\widetilde{b}_I}\,\big(\widetilde{S}_s^{(I)}\big)^{\widetilde{\mathsf{k}}_I}\widetilde{Q}_{I+1}(\widetilde{S}_s^{(I)}) \tag{B.12}$$

for $I = 1, \cdots, M-1$, where we understand that $\widetilde{Q}_0(Z) = P_N(Z)$ and $\widetilde{Q}_M(Z) = 1$. Again we notice that for $I = 1$, the above formula reproduces the chiral ring equation (4.37) when $\Lambda = 0$.

The analysis proceeds along the same lines as before. We first use the explicit expression of the functions $\widetilde{Q}_I$ and get

$$\begin{aligned}\prod_{t=1}^{\widetilde{r}_{I-1}}\big(\widetilde{S}_s^{(I)} - \widetilde{S}_t^{(I-1)}\big) &= (-1)^{\widetilde{r}_{I-1}}(\beta\widetilde{\Lambda}_I)^{-\widetilde{b}_I}\big(\widetilde{S}_s^{(I)}\big)^{\frac{b_I}{2}+\widetilde{\mathsf{k}}_I} \\ &\quad\times\prod_{t=1}^{\widetilde{r}_{I-1}}\big(\widetilde{S}_t^{(I-1)}\big)^{\frac{1}{2}}\prod_{u=1}^{\widetilde{r}_{I+1}}\big(\widetilde{S}_u^{(I+1)}\big)^{-\frac{1}{2}}\prod_{u=1}^{\widetilde{r}_{I+1}}\big(\widetilde{S}_s^{(I)} - \widetilde{S}_u^{(I+1)}\big)\,.\end{aligned} \tag{B.13}$$

Next, using the fact that $\widetilde{r}_I = N - r_I$, we solve this equation for $\widetilde{S}_s^{(I)}$ with the ansatz

$$\widetilde{S}_\star^{(I)} = \mathrm{diag}\left(A_{r_I+1} + \delta A_{r_I+1}\,, \ldots A_{r_{I+1}} + \delta A_{r_{I+1}}, A_{r_{I+1}+1}\,\ldots A_{r_I+\widetilde{r}_I}\right) \tag{B.14}$$

and find

$$\begin{aligned}\delta A_{r_I+s} &= (-1)^{\widetilde{r}_{I-1}}(\beta\widetilde{\Lambda}_I)^{-\widetilde{b}_I}(A_{r_I+s})^{\frac{b_I}{2}+\widetilde{\mathsf{k}}_I}\prod_{t=1}^{\widetilde{r}_{I-1}}(A_{r_{I-1}+t})^{\frac{1}{2}}\prod_{u=1}^{\widetilde{r}_{I+1}}(A_{r_{I+1}+u})^{-\frac{1}{2}} \\ &\quad\times\prod_{u=1}^{\widetilde{r}_{I+1}}(A_{r_I+s} - A_{r_{I+1}+u})\prod_{\substack{t=1\\ t\neq s+n_I}}^{\widetilde{r}_{I-1}}\frac{1}{(A_{r_I+s} - A_{r_{I-1}+t})}\,.\end{aligned} \tag{B.15}$$

There are lot of cancellations that take place between the products in the second line above, and in the end only $n_I + n_{I+1} - 1$ of the terms survive as one can check by a careful analysis. Thus, we finally obtain

$$\delta A_{r_I+s} = (-1)^{\widetilde{r}_{I-1}} (\beta \widetilde{\Lambda}_I)^{-\widetilde{b}_I} (A_{r_I+s})^{\frac{b_I}{2}+\widetilde{\mathsf{k}}_I} \prod_{t=r_{I-1}+1}^{r_{I+1}} (A_t)^{\frac{1}{2}} \prod_{\substack{t=r_{I-1}+1 \\ t\neq s+n_I}}^{r_{I+1}} \frac{1}{(A_{r_I+s} - A_t)} \,. \tag{B.16}$$

Computing tr log $\widetilde{S}_\star^{(I)}$, we find it agrees with (negative of) the q_I-derivative of the twisted chiral superpotential (B.5) obtained using the second JK prescription, provided we identify

$$q_I = -(-1)^{\widetilde{r}_I} (\beta \widetilde{\Lambda}_I)^{-\widetilde{b}_I} \quad \text{and} \quad \mathsf{m}_I = -\widetilde{\mathsf{k}}_I \tag{B.17}$$

for $I = 1, \cdots, M-1$.

This completes the identification of the parameters m_I with the Chern-Simons levels of the dual pair of 3d quiver gauge theories studied in this work.

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